Review Session
Stable Matching

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February 7, 2024

Crucial Concepts

When talking about matching, we usually talk about 2 sets **A** and **B** of the same size n

- **Matching**: A matching S is a set of pairs (a, b) where $a \in A$ and $b \in B$, and no two pairs share the same a or b.

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- **Stable Matching Problem**: Given the preference list of A and B, find a perfect matching S that is not unstable.

G-S Algorithm

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Determined Result

G-S algorithm returns a determined solution when we fix the proposing side. Therefore G-S algorithm will return at most 2 solutions. There can be different solutions other than the ones we obtain from running G-S algorithm.

Question 1

Find an instance of stable matching problem where there are multiple solutions and point out the solution that G-S algorithm will return. (Assume men proposing).

Answer 1

	1st	2nd
M1	W1	W2
M2	W2	W1
W1	M2	M1
W2	M1	M2

Table: Table caption.

1. (M1,W1), (M2,W2)

Answer 1

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W1	M2	M1
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Table: Table caption.

1. (M1,W1), (M2,W2)
2. (M1,W2), (M2,W1)

Question 2

Find an instance of statble matching problem of size n, such that G-S algorithm terminates in $O(n)$ iteration.

Answer 2

Simply assign each man with different most preferred woman. E.g. m_i prefers w_i the most. In this case G-S algorithm will run exactly n iterations as each man will propose to different woman.

Question 3

If every man has identical preference list, how many iteration does it take for G-S algorithm to terminate, give the precise answer in n.

Answer 3

With out lose of generality, let's assume that every man's preference list is exactly $(w_1, w_2, ..., w_n)$. G-S algorithm returns a stable matching $S = \{(m_1', w_1), (m_2', w_2), ..., (m_n', w_n)\}$. Since every man has the same preference list. m_i' must have proposed exactly i times. Then the total number of iteration is $\sum_{i=1}^{n} i = \frac{(n+1)n}{2}$.

Question 4

Is it true that for every $n \geq 2$, there exists an instance of stable matching problem that has only one solution?

Answer 4

Yes, simply make m_i's i-th preferred woman w_i and vice versa. The solution can only be $S = \{(m_i, w_i) | \forall i \in [1, n]\}$. The proof is just the extended version of HW1 Q4.

Question 5

Design an algorithm that determine whether an instance of stable matching has only one solution. Your algorithm should run in $O(n^2)$ time.

Answer 5

Run G-S algorithm twice, one with men proposing and the other with women proposing. The solution is unique if and only if the two runs return the same result.

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- $\leftarrow$: If two runs has the same result, this means that everyone's best and worst valid partners are the same, thus everyone can be matched with only one partner and the stable matching is unique.
- $\rightarrow$: Trivially true.