Conic Optimization Refresher

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1 Types of Cones

Conic optimization is a class of convex optimization problems. The geneneral form of a conic optimization problem is:

$$\text{maximize } \mathbf{c}^T\mathbf{x}$$
$$\text{subject to } \mathbf{Ax} + \mathbf{b} \in \mathcal{K}$$

where $\mathcal{K}$ is a product of the following basic types of cones:

- **Linear cone:**

$$\mathbb{R}, \mathbb{R}^n_+, 0$$

- **Quadratic cone and rotated quadratic cone:**

The quadratic cone is the set

$$\mathcal{Q}^n = \left\{ x \in \mathbb{R} \middle| x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2} \right\}$$

The rotated quadratic cone is the set

$$\mathcal{Q}^n_r = \left\{ x \in \mathbb{R} \middle| 2x_1x_2 \geq x_3^2 + \cdots + x_n^2, x_1, x_2 \geq 0 \right\}$$

Together the union of these two cones covers the class of SOCO (second-order cone optimization) problems which includes all QO (quadratic optimization) and QCQO (quadratically constrained quadratic optimization) problems as well.

- **Primal power cone:**

$$\mathcal{P}_n^{\alpha,1-\alpha} = \left\{ x \in \mathcal{R}^n \middle| x_1^\alpha x_2^{1-\alpha} \geq \sqrt{x_3^2 + \cdots + x_n^2}, x_1, x_2 \geq 0 \right\}$$

- **Primal exponential cone:**

$$K_{\exp} = \left\{ x \in \mathcal{R}^3 \middle| x_1 \geq x_2 \exp\left(\frac{x_3}{x_2}\right), x_1, x_2 \geq 0 \right\}$$

- **Semidefinite cone:**

$$\mathcal{S}_+^n = \{X \in \mathbb{R}^{n\times n} | X \text{ is symmetric positive semidefinite}$$

Semidefinite cones model SDO problems.

Each of these cones allow formulating different types of convex constraints.

2 Selection of Conic Constraints

Examples of real world constraints (financial)and how to convert them to conic form.

2.1 Maximum function

Model the maximum constraint $\max(x_1, x_2, ..., x_n) \leq c$ using n linear constraints introduces with an auxiliary variable t:

$$\begin{aligned} t &\leq c, \\ t &\geq x_1, \\ &\vdots \\ t &\geq x_n. \end{aligned} \tag{1}$$

For example we can write the constraints $\max(x_i, 0) \leq c_i, 1, ..., n$ as

$$\mathbf{t} \leq \mathbf{c}, \mathbf{t} \geq \mathbf{x}, \mathbf{t} \geq \mathbf{0},$$

where $\mathbf{t}$ is an n-dimensional vector.