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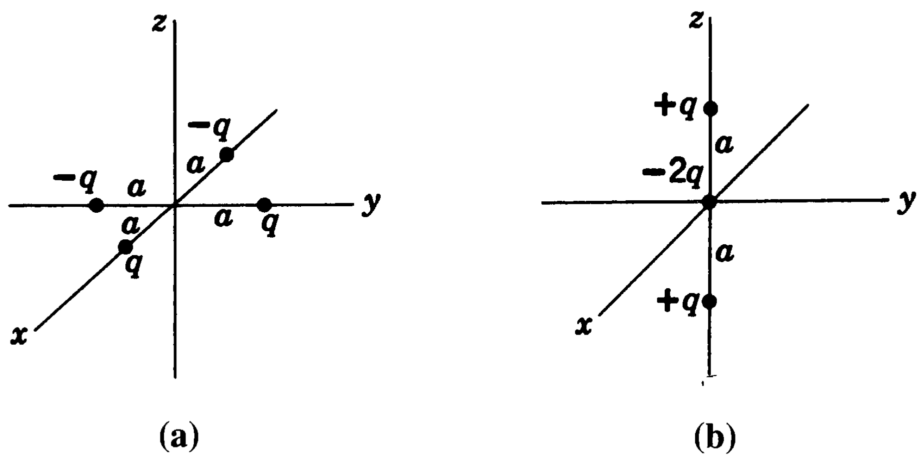
Physics 841 - Homework 7 **due Fri., Feb. 28, 2020**

Problems

1. **Spherical multipoles, Jackson, 4.1:** Calculate the multipole moments q_{lm} of the charge distributions shown as parts (a) (15 pts) and (b) (15 pts). Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first *two* sets of nonvanishing moments at the very least.



   (c) (10 pts) For the charge distribution of the second set (b) write down the multipole expansion for the potential. Keeping only the lowest-order term in the expansion, plot the potential in the $x - y$ plane as a function of distance from the origin for distances greater than a.

   (d) (10 pts) Calculate directly from Coulomb's law the exact potential for (b) in the $x - y$ plane. Plot it as a function of distance and compare with the result found in part (c).

   Divide out the asymptotic form in parts (c) and (d) to see the behavior at large distances more clearly.

2. **The potential outside a charged disk, Zangwill 4.22:** The z-axis is the symmetry axis of a disk of radius R which lies in the $x - y$ plane and carries a uniform charge per unit area σ. Let Q be the total charge on the disk.

   (a) (15 pts) Evaluate the exterior multipole moments and show that

$$\phi(r, \theta) = \frac{Q}{4\pi\epsilon_0 r} \sum_{l=0}^{\infty} \left(\frac{R}{r}\right)^l \frac{2}{l+2} P_l(0) P_l(\cos\theta) \qquad r > R. \tag{1}$$

   (b) (10 pts) Compute the potential at any point on the z-axis by elementary means and confirm that your answer agrees with the part (a) when $z > R$. Note: $P_l(1) = 1$.

Problem 1

(a)

$$q_{lm} = \int Y^*_{lm}(\theta', \phi') r'^l \rho(\mathbf{x}')\, d^3x' \qquad (4.3)$$

$$= q a^l \left(Y^*_{lm}\left(\tfrac{\pi}{2}, 0\right) + Y^*_{lm}\left(\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) - Y^*_{lm}\left(\tfrac{\pi}{2}, \pi\right) - Y^*_{lm}\left(\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)\right)$$

$$= q a^l \left(\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2} P_l^m(0)\left(1 + (-i)^m - (-i)^m - i^m\right)$$

indices l, m. From the normalization condition (3.52) it is clear that the suitably normalized functions, denoted by $Y_{lm}(\theta, \phi)$, are:

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \qquad (3.53)$$

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \tfrac{1}{2}(3x^2 - 1) \\ P_3(x) &= \tfrac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}\right\}$$

$\Rightarrow$ Those are 0 for even m: $q_{l,2k} = 0$

$$q_{l,2k+1} = 2qa^l\left(1 - i(-1)^k\right)\left(\frac{2l+1}{4\pi}\frac{(l-2k-1)!}{(l+2k+1)!}\right)^{1/2} P_l^{2k+1}(0)$$

Vanishes for even l too!

$$\Rightarrow \quad q_{1,1} = -q^*_{1,-1} = -2qa(1-i)\sqrt{\frac{3}{8\pi}}$$

$$q_{3,3} = -q^*_{3,-3} = -\frac{qa^3}{2}(1+i)\sqrt{\frac{35}{4\pi}}$$

$$q_{3,1} = -q^*_{3,-1} = \frac{qa^3}{2}(1-i)\sqrt{\frac{21}{4\pi}}$$

(b)

$$q_{lm} = q a^l \left(Y^*_{lm}(0,0) + Y^*_{lm}(\pi,0) \right), \quad l>0, \quad q_{00} = 0$$

The charge distribution is invariant under rotations in the azimuthal plane $\Rightarrow q_{lm} = 0, \ \forall m \neq 0$

The non-trivial moments remaining are:

$$q_{l0} = q a^l \sqrt{\frac{2l+1}{4\pi}} \left(P_l(1) + P_l(-1) \right)$$

$$= q a^l \left(1 + (-1)^l \right) \sqrt{\frac{2l+1}{4\pi}}, \quad l>0$$

$$\Rightarrow q_{l0} = q a^l \sqrt{\frac{2l+1}{\pi}}, \quad l = 2k, \ k \in \mathbb{N}^+$$

Lowest non-trivial:

$$q_{20} = q a^2 \sqrt{\frac{5}{\pi}}, \quad q_{40} = q a^4 \frac{3}{\sqrt{\pi}}$$

(c)

$$4\pi\varepsilon_0 \cdot \Phi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \qquad (4.1)$$

$$\Rightarrow \phi = \frac{q}{2\pi\varepsilon_0} \sum_{k=1}^{+\infty} \frac{a^{2k}}{r^{2k+1}} P_{2k}(\cos\theta)$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{a^2}{r^3} \left(3\cos^2\theta - 1 \right) + O\left(\frac{1}{r^5} \right)$$

In the azimuthal plane: $\cos\theta = 0$

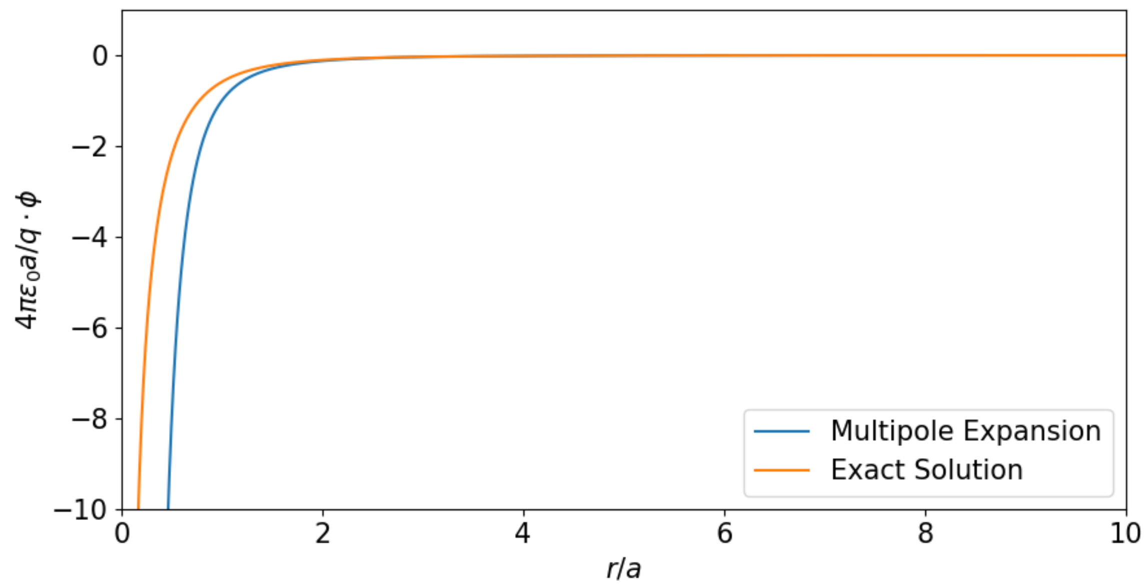
$$\Rightarrow \phi\big|_{\theta=0} = -\frac{q}{4\pi\varepsilon_0 a} \left(\frac{a}{r} \right)^3 + O(r^{-5})$$

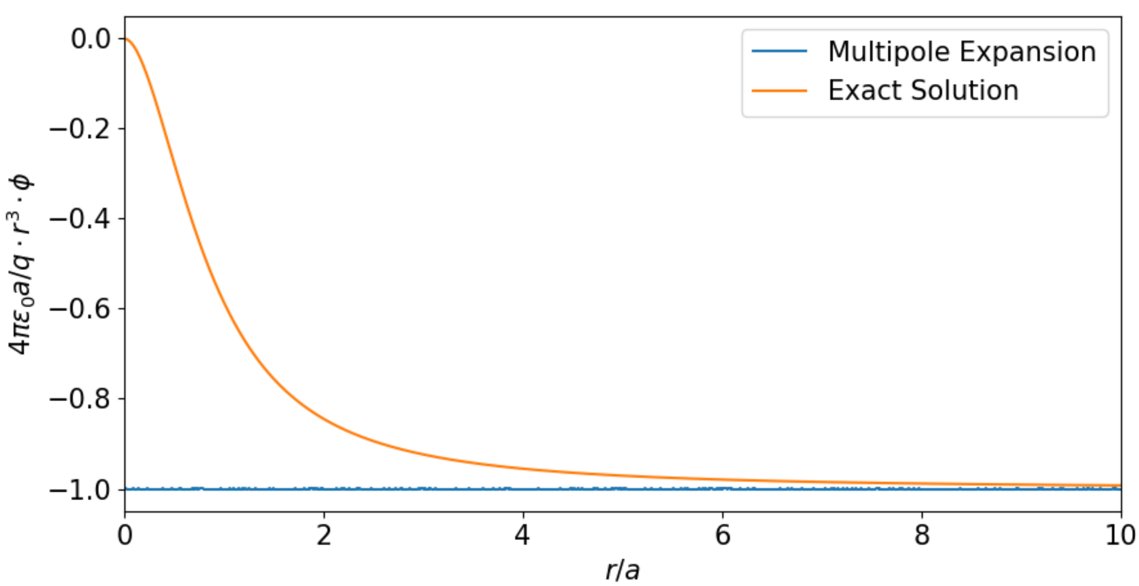
(d) Simple superposition of point charges:

$$\phi\Big|_{\theta=0} = \frac{q}{4\pi\epsilon_0}\left((r^2+a^2)^{-1/2} - \frac{1}{r} + (r^2+a^2)^{-1/2}\right)$$

$$= -\frac{q}{4\pi\epsilon_0 a}\, 2\left(\frac{a}{r} - \left(1+r^2/a^2\right)^{-1/2}\right)$$

Plots:





Problem 2

(a) From the text we know:

$$\rho(r) = \frac{\sigma}{r}\,\Theta(R-r)\,\delta(\cos\theta) \quad , \quad \sigma = \frac{Q}{4\pi R^2}$$

$$q_{lm} = \int Y^*_{lm}(\theta', \phi') r'^{l} \rho(\mathbf{x}')\, d^3x' \tag{4.3}$$

$$= \sigma \int_{-1}^{1} d\cos\theta \cdot \delta(\cos\theta) \int_0^{2\pi} d\varphi\, Y^*_{lm}(\theta,\varphi) \int_0^{\infty} dr\, r^{l+1}\, \Theta(R-r)$$

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta) e^{im\phi} \tag{3.53}$$

$$\Rightarrow \int_0^{2\pi} Y^*_{lm}\, d\varphi = 2\pi \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)\, \delta_{m,0}$$

$$\Rightarrow q_{lm} = \sqrt{(2l+1)\pi}\, \frac{\sigma R^{l+2}}{l+2}\, P_l(0)\, \delta_{m,0}$$

Now we just need to plug it into

$$4\pi\epsilon_0 \cdot \Phi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \tag{4.1}$$

$$\Rightarrow 4\pi\epsilon_0 \cdot \phi(r,\theta) = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sqrt{(2l+1)\pi}\, \frac{\overset{Q/4\pi R^2}{\sigma} R^{l+2}}{l+2}\, P_l(0)\, \frac{\overset{\sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)}{Y_{l0}(\theta,\varphi)}}{r^{l+1}}$$

$$\Rightarrow \phi = \frac{Q}{4\pi\epsilon_0 r} \sum_{l=0}^{\infty} \left(\frac{R}{r}\right)^l \frac{2}{l+2}\, P_l(0)\, P_l(\cos\theta) \quad , \quad r > R$$

(b)

$$\varphi(z) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^R \frac{r\,dr}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2}-|z|\right)$$

$$= \frac{Q}{8\pi\varepsilon_0 R^2}\left(\sqrt{R^2+z^2}-\sqrt{z^2}\right)$$

$$= \frac{Q\,|z|}{8\pi\varepsilon_0 R^2}\left(\sqrt{1+R^2/z^2}-1\right)$$

$$\phi = \frac{Q}{4\pi\varepsilon_0 r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l \frac{2}{l+2} P_l(0)\, P_l(\cos\theta) \quad , r > R$$

$$\Rightarrow \phi|_{x=y=0} = \phi|_{\cos\theta=1} = \frac{Q}{4\pi\varepsilon_0 z}\sum_{l=0}^{\infty}\left(\frac{R}{z}\right)^l \frac{2}{l+2} P_l(0)\cdot 1$$

with $\tilde{z} = R/z$ we know that:

$$\sqrt{1+\tilde{z}^2}-1 = \int_0^{\tilde{z}} \frac{u}{\sqrt{1+u^2}}\,du$$

$$= \sum_{l=0}^{\infty} P_l(0)\int_0^{\tilde{z}} u^{l+1}\,du$$

$$= \sum_{l=0}^{\infty} \frac{\tilde{z}^{l+2}}{l+2} P_l(0)$$

$\Rightarrow$ They identical on the z-axis!

3. **Potential in a box:** Consider a rectangular empty box with lengths (a, b, c) in (x, y, z) direction. All surfaces of the box have zero potential, except for the side at $z = c$, where the potential is $V(x, y) = d_0\, x\, y$ with a constant d_0.

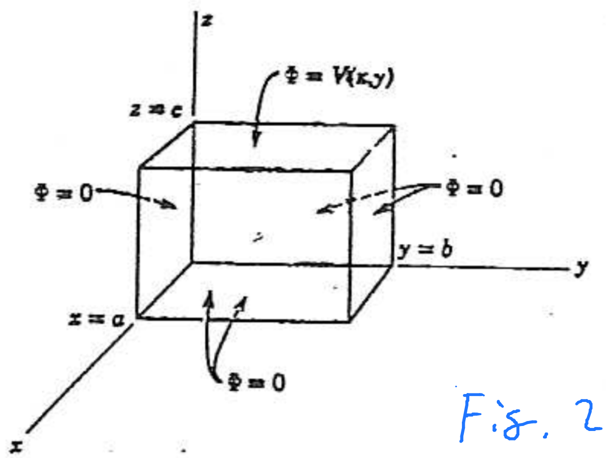


Fig. 2.9, Jackson p.70

(a) (20 pts) Solve the Laplace equation in Cartesian coordinates using a product ansatz and separation of variables. Derive a general solution for the potential with generic boundary constants.

(b) (5 pts) Determine the boundary constants by requiring the potential to take the prescribed values on the surfaces of the box.

Problem 3

(a) Once again: Can be found in Jackson p. 68-71:

The Laplace equation in rectangular coordinates is

p. 68

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \tag{2.48}$$

A solution of this *partial* differential equation can be found in terms of three *ordinary* differential equations, all of the same form, by the assumption that the potential can be represented by a product of three functions, one for each coordinate:

p. 69

$$\Phi(x, y, z)=X(x)Y(y)Z(z) \tag{2.49}$$

Substitution into (2.48) and division of the result by (2.49) yields

$$\frac{1}{X(x)}\frac{d^2X}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z}{dz^2}=0 \tag{2.50}$$

where total derivatives have replaced partial derivatives, since each term involves a function of one variable only. If (2.50) is to hold for arbitrary values of the independent coordinates, each of the three terms must be separately constant:

$$\left.\begin{aligned}\frac{1}{X}\frac{d^2X}{dx^2}&=-\alpha^2\\ \frac{1}{Y}\frac{d^2Y}{dy^2}&=-\beta^2\\ \frac{1}{Z}\frac{d^2Z}{dz^2}&=\gamma^2\\ \text{where}\quad \alpha^2+\beta^2&=\gamma^2\end{aligned}\right\} \tag{2.51}$$

If we arbitrarily choose α^2 and β^2 to be positive, then the solutions of the three ordinary differential equations (2.51) are $\exp(\pm i\alpha x)$, $\exp(\pm i\beta y)$, $\exp(\pm\sqrt{\alpha^2+\beta^2}z)$. The potential (2.49) can thus be built up from the product solutions:

p. 69

$$\Phi=e^{\pm i\alpha x}e^{\pm i\beta y}e^{\pm\sqrt{\alpha^2+\beta^2}z} \tag{2.52}$$

At this stage α and β are completely arbitrary. Consequently (2.52), by linear superposition, represents a very large class of solutions to the Laplace equation.

To determine α and β it is necessary to impose specific boundary conditions on the potential. As an example, consider a rectangular box, located as shown in Fig. 2.9, with dimensions (a, b, c) in the (x, y, z) directions. All surfaces of the box are kept at zero potential, except the surface $z=c$, which is at a potential $V(x, y)$. It is required to find the potential everywhere inside the box. Starting with the requirement that $\Phi=0$ for $x=0$, $y=0$, $z=0$, it is easy to see that the required forms of X, Y, Z are

$$\left.\begin{aligned}X&=\sin\alpha x\\ Y&=\sin\beta y\\ Z&=\sinh(\sqrt{\alpha^2+\beta^2}z)\end{aligned}\right\} \tag{2.53}$$

In order that $\Phi=0$ at $x=a$ and $y=b$, it is necessary that $\alpha a=n\pi$ and $\beta b=m\pi$. With the definitions,

$$\left.\begin{aligned}\alpha_n&=\frac{n\pi}{a}\\ \beta_m&=\frac{m\pi}{b}\\ \gamma_{nm}&=\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\end{aligned}\right\}\tag{2.54}$$

we can write the partial potential Φ_{nm}, satisfying all the boundary conditions except one,

$$\Phi_{nm}=\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}z)\tag{2.55}$$

The potential can be expanded in terms of these Φ_{nm} with initially arbitrary coefficients (to be chosen to satisfy the final boundary condition):

$$\Phi(x,y,z)=\sum_{n,m=1}^{\infty}A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}z)\tag{2.56}$$

p. 70

The potential can be expanded in terms of these Φ_{nm} with initially arbitrary coefficients (to be chosen to satisfy the final boundary condition):

p. 70

Solution for this assignment:

$$\Phi(x,y,z)=\sum_{n,m=1}^{\infty}A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}z)\tag{2.56}$$

There remains only the boundary condition $\Phi=V(x,y)$ at $z=c$:

$$V(x,y)=\sum_{n,m=1}^{\infty}A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}c)\tag{2.57}$$

This is just a double Fourier series for the function $V(x,y)$. Consequently the coefficients A_{nm} are given by:

$$A_{nm}=\frac{4}{ab\sinh(\gamma_{nm}c)}\int_0^a dx\int_0^b dy\,V(x,y)\sin(\alpha_n x)\sin(\beta_m y)\tag{2.58}$$

(b)

$$A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy \, V(x, y) \sin(\alpha_n x) \sin(\beta_m y) \qquad (2.58)$$

,

$$\alpha_n = \frac{n\pi}{a}$$

$$\beta_m = \frac{m\pi}{b}$$

$$\gamma_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

For $V(x, y) = d_0 \, x \, y$ we get:

$$I_{nm} := \int_0^a dx \int_0^b dy \; d_0 \, x y \sin(\alpha_n x) \sin(\beta_m y)$$

$$= d_0 \int_0^a x \sin(\alpha_n x) \, dx \int_0^b y \sin(\beta_m y) \, dy$$

$$= \frac{d_0}{\alpha_n^2 \beta_m^2} \left[\sin(\alpha_n a) - \alpha_n a \cos(\alpha_n a) \right] \left[\sin(\beta_m b) - \beta_m b \cos(\beta_m b) \right]$$

$$\boxed{A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \cdot I_{nm} \qquad \Phi(x, y, z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)}$$

Solution!