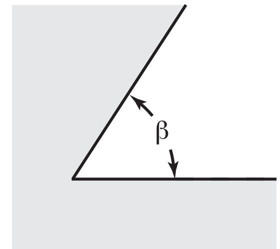
Problems

1. **A corner:**
Consider the potential in the region bounded by two half-planes, which meet at the origin at an angle β as shown in the figure. Near the origin there are no charges, although we presume there are other charges (not shown) away from the origin. For simplicity, we shall also assume that nothing depends on the variable z. Then the potential near the origin can be expressed in cylindrical coordinates, and it satisfies the Laplace equation, $\Delta\phi = 0$, with a solution of the form



$$\begin{aligned}\phi(\rho,\varphi) &= a_0 + b_0\varphi + (c_0 + d_0\varphi)\ln\rho + \sum_{n=1}^{\infty}\Big[\rho^{\nu_n}(a_n\cos\nu_n\varphi + b_n\sin\nu_n\varphi) \\ &\qquad + \rho^{-\nu_n}(c_n\cos\nu_n\varphi + d_n\sin\nu_n\varphi)\Big] .\end{aligned}$$

In this case, however, the parameters ν_n are not integers, since φ doesn't run periodically from 0 to 2π in the charge-free region. We have also included terms linear in φ, which may appear if the solution is not assumed periodic in φ.

(a) (20 pts) Assume that the half-planes are grounded, so that $\phi(\rho, 0) = \phi(\rho, \beta) = 0$. Use this to obtain the possible values of ν_n and to determine which of the coefficients are nonzero.

(b) (20 pts) Since we are interested in the field near the corner, we include the point $\rho = 0$ in the region and assume that the potential is finite as $\rho \to 0$ (no charge singularities). Write the general solution for the potential near the corner. What is the leading behavior of the potential as $\rho \to 0$? (We can assume that the coefficient of the leading term is nonzero due to the presence of the other charges away from the origin.)

(c) (20 pts) Keeping only the leading term of the solution near $\rho = 0$, determine the $\vec{E}$ field and surface charge density. Discuss the behavior of the field and surface charge near the corner as $\rho \to 0$ for the values of $\beta \approx 0$, $\beta = \pi$ and $\beta \approx 2\pi$.

2. **A cylinder (Jackson, 3.9, 40 pts):**
A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\varphi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

Problem 1

(a) We want $\phi(\rho, 0) = \phi(\rho, \beta) = 0$

$$\Rightarrow a_n = c_n = 0 \quad \forall n.$$

We are left with:

$$\phi(\rho, \varphi) = b_0 \varphi + d_0 \varphi \ln\rho + \sum_{n\geq 1}^{\infty} [\rho^{\nu_n} b_n \sin(\nu_n \varphi) + \rho^{-\nu_n} d_n \sin(\nu_n \varphi)]$$

$$\phi|_{\varphi=\beta} \stackrel{!}{=} 0 \Rightarrow d_0 = b_0 = 0$$

Left with $\boxed{\phi = \sum_{n=1}^{\infty} (\rho^{\nu_n} b_n + \rho^{-\nu_n} d_n) \sin(\nu_n \varphi)}$

So for $\phi|_{\varphi=\beta} = 0$: $\sin(\nu_n \beta) \stackrel{!}{=} 0$

$$\Rightarrow \boxed{\nu_n = \frac{n\pi}{\beta}}$$

(b) $$\phi = \sum_{n=1}^{\infty} (\rho^{\nu_n} b_n + \rho^{-\nu_n} d_n) \sin(\nu_n \varphi)$$

blows up for $\rho \to 0$

$\Rightarrow d_n = 0 \ \forall n$, so: $\boxed{\phi = \sum_{n=1}^{\infty} \rho^{\nu_n} b_n \sin(\nu_n \varphi)}$

$\nu_n = \frac{n\pi}{\beta}$ still

The leading term is $n=1$ because $\rho^{\nu_1} > \rho^{\nu_{n>1}}$

as $\rho \to 0$. so:

as $\rho \to 0$: $\phi \approx \rho^{\pi/\beta} b_1 \sin(\frac{\pi}{\beta}\varphi)$

(c)

For $\phi = \rho^{\pi/\beta} b_1 \sin(\frac{\pi}{\beta}\varphi)$

$$\vec{E} = -\,\mathrm{grad}\,\phi = \frac{\partial \varphi}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial \phi}{\partial \varphi}\vec{e}_\varphi$$

$$= -\frac{\pi}{\beta}\rho^{\frac{\pi}{\beta}-1} b_1 \left[\sin(\tfrac{\pi}{\beta}\varphi)\,\vec{e}_\rho + \cos(\tfrac{\pi}{\beta}\varphi)\,\vec{e}_\varphi\right]$$

$$\sigma = \varepsilon_0 \vec{E}\cdot\vec{n}\Big|_{\varphi=0} = -\frac{\pi b_1 \varepsilon_0}{\beta}\rho^{\frac{\pi}{\beta}-1}$$

For $\rho \to 0$:

$\beta \gtrsim 0$: $\vec{E} = \vec{0}$, $\sigma = 0$

$\beta = \pi$: $\vec{E} = -b_1[\sin\varphi\cdot\vec{e}_\rho + \cos\varphi\,\vec{e}_\varphi]$

$\sigma = -\varepsilon_0 b_1$

$\beta = 2\pi$: $\vec{E} = -\frac{1}{2}\rho^{-1/2} b_1 \left(\sin\frac{\varphi}{2}\vec{e}_\rho + \cos\frac{\varphi}{2}\vec{e}_\varphi\right)$

$\sigma = -\frac{\varepsilon_0}{2} b_1 \rho^{-1/2}$

Problem 2

3.7 Laplace Equation in Cylindrical Coordinates, Bessel Functions

In cylindrical coordinates (ρ, ϕ, z), as shown in Fig. 3.8, the Laplace equation takes the form:

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (3.71)$$

The separation of variables is accomplished by the substitution:

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z) \qquad (3.72)$$

The general solution is

$$\phi(\rho, \varphi, z) = \sum \underbrace{\left[J_m(k\rho) \text{ or } N_m(k\rho) \right]}_{R(\rho)} \cdot \underbrace{\exp(\pm i m \varphi)}_{Q(\varphi)} \cdot \underbrace{\exp(\pm k z)}_{Z(z)}$$

Since the potential is supposed to be 0 on the caps we take

$$Z_n(z) = \sin\left(\frac{n\pi}{L} z\right)$$

and $I_m(kr)$, $K_m(kr)$ instead of J_m and N_m.

And since we want a solution inside the cylinder we can discard K_m because

$$K_m(x) \xrightarrow{x \to 0} \infty .$$

So we're left with:

$$\boxed{\phi = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} I_m\left(\frac{n\pi}{L}\rho\right)\left[a_{mn}\sin(m\varphi) + b_{mn}\cos(m\varphi)\right]\sin\left(\frac{n\pi}{L}z\right)}$$

To satisfy the surface boundary condition we expand V:

$$V(\varphi, z) = \sum_{m,n} I_m\left(\frac{n\pi b}{L}\right)\left[a_{mn}\sin(m\varphi) + b_{mn}\cos(m\varphi)\right]\sin\left(\frac{n\pi}{L}z\right)$$

This is a Fourier series in φ and z of V.

Therefore:

$$\left.\begin{matrix} a_{mn} \\ \tilde{b}_{mn} \end{matrix}\right\} = \frac{2}{\pi L I_m\left(\frac{n\pi b}{L}\right)} \int_0^{2\pi} d\varphi \int_0^L dz\, V(\varphi, z) \cdot \sin\left(\frac{n\pi}{L}z\right) \cdot \left\{\begin{matrix} \sin(m\varphi) \\ \cos(m\varphi) \end{matrix}\right.$$

with $b_{0,n} = \dfrac{\tilde{b}_{0,n}}{2}$, $b_{mn} = \tilde{b}_{m,n}$ for all others.