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Physics 841 - Midterm 1 extra problem **due Fri., Apr. 3, 2020**

Problems

1. **A charged cone:** A right circular cone of height h and base of radius R is positioned such that its symmetry axis coincides with the z-axis and the base is in the x-y plane. The cone is charged with uniform charge density ρ_0. Consider two limiting cases:

   - $R \to 0$ (segment),
   - $h \to 0$ (disk),

   taken in such a way that the total charge Q is held fixed. Find:

   (a) (1 pts) The total charge Q of the cone.

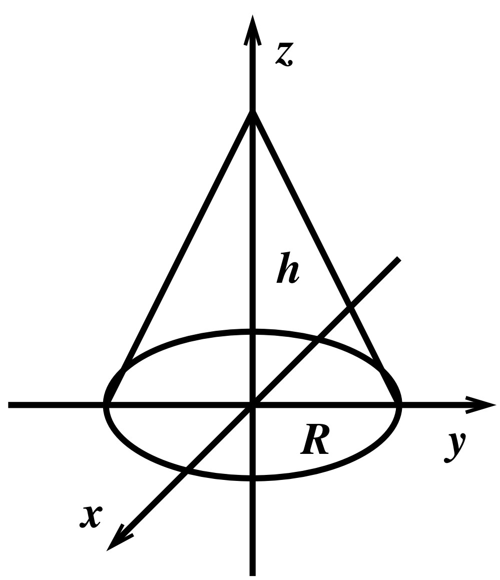
   (b) (4 pts) The charge density corresponding to the segment and the disk.

   (c) (2 pts) The total charge for the segment and the disk (*i.e.* the integral over the density found in (b)) and compare with the total charge for the cone.

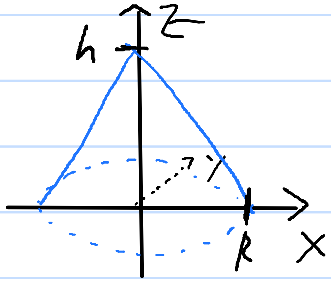
   (d) (5 pts) The dipole moment of the segment and the disk.

   (e) (6 pts) The quadrupole moment tensor of the segment and the disk.

   (f) (2 pts) Compare your results to the general case of the cone dipole and quadrupole moment and demonstrate that the general result reduces to your results in the corresponding limits.

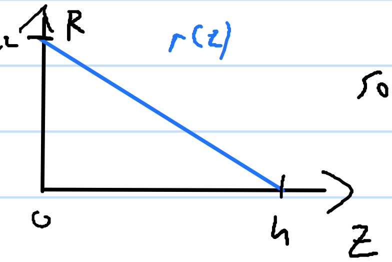


(a)



$\rho = x^2+y^2$

$\Rightarrow$



so $r(z) = R\left(1 - \frac{z}{h}\right)$

$$Q = \rho_0 \int_0^{2\pi}\int_0^{h}\int_0^{r(z)} \rho\, d\rho\, dz\, d\varphi = \pi \rho_0 \int_0^h R^2\left(1-\frac{z}{h}\right)^2 dz$$

$$= \pi \rho_0 R^2 h \int_0^1 t^{1-1}(1-t)^{3-1}\, dt$$

$$= \rho_0 \pi R^2 h \frac{\Gamma(1)\Gamma(3)}{\Gamma(4)} = \rho_0 \pi R^2 h\, \frac{2}{6} = \boxed{\rho_0\, \frac{\pi}{3} R^2 h}$$

$$\Rightarrow \rho_0 = \frac{3Q}{\pi R^2 h}$$

(b) For $R \to 0$ (segment)

$$\rho_s = \rho_0\, \pi\, r^2(z)\, \delta(x)\, \delta(y) = \frac{3Q}{\cancel{\pi R^2} h} \cancel{\pi R^2}\left(1-\frac{z}{h}\right)^2 \delta(x)\,\delta(y)$$

$$\boxed{= \frac{3Q}{h}\left(1-\frac{z}{h}\right)^2 \delta(x)\,\delta(y)}$$

For $h \to 0$ (disk)

$$\rho_d = \rho_0\, z(r)\, \delta(z)\, \Theta(R-r) \quad \text{with } z(r) = h\left(1-\frac{r}{R}\right)$$

like $r(z)$ from (a)

$$= \frac{3Q}{\pi R^2 \cancel{h}} \cancel{h}\left(1-\frac{r}{R}\right) \delta(z)\, \Theta(R-r)$$

$$= \boxed{\frac{3Q}{\pi R^2}\left(1-\frac{r}{R}\right) \delta(z)\, \Theta(R-r)}$$

(c) Well, that's kind of redundant because $\boxed{Q = Q_s = Q_d}$ is precisely the condition we used to derive ρ_s and ρ_d.

But well...

$$Q_s = \int_0^h \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{3Q}{h}\left(1-\frac{z}{h}\right)^2 dx\,dy\,dz$$

$$= \frac{3Q}{h} \int_0^h \left(1-\frac{z}{h}\right)^2 dz$$

$$= \frac{3Q}{\cancel{h}} \cancel{h} \int_0^1 t^{1-1}(1-t)^{3-1}\,dt$$

$$= 3Q \frac{\Gamma(1)\Gamma(3)}{\Gamma(4)} = 3Q\frac{2}{6} = Q \quad \text{Surprise!}$$

$$Q_d = \frac{3Q}{\pi R^2} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^R \left(1-\frac{\rho}{R}\right)\delta(z)\,\rho\,d\rho\,d\varphi\,dz$$

$$= \frac{6Q}{R^2}\int_0^R \left(\rho - \frac{\rho^2}{R}\right)d\rho = \frac{6Q}{R^2}\left(\frac{\rho^2}{2} - \frac{\rho^3}{3R}\right)\Bigg|_{\rho=0}^{\rho=R}$$

$$= \frac{6Q}{R^2}\left(\frac{R^2}{2} - \frac{R^2}{3}\right) = Q$$

d) $$\vec{P} = \int \vec{r}\rho(\vec{r})\,d^3r$$

__For the segment__

$$\vec{p}_s = \frac{3Q}{h}\int_0^h \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \vec{r}\left(1+\frac{z}{h}\right)^2 \delta(x)\delta(y)\,dx\,dy\,dz$$

$$= \frac{3Q}{h}\int_0^h z\,\vec{e}_z\left(1+\frac{z}{h}\right)^2 dz = \frac{3Q}{\cancel{h}} h^{\cancel{2}} \int_0^1 t^{2-1}(1-t)^{3-1}\,dt\,\vec{e}_z$$

$$= 3Qh\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)}\vec{e}_z = 3Qh\frac{2}{24}\vec{e}_z = \boxed{\frac{Qh}{4}\vec{e}_z}$$

For the disk

$$\vec{p}_d = \frac{3Q}{\pi R^2}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^R \vec{r}\rho\left(1-\frac{\rho}{R}\right)\delta(z)\,d\rho\,d\varphi\,dz$$

$$= \frac{3Q}{\pi R^2}\int_0^{2\pi}\int_0^R \rho^2\left(\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y\right)\left(1+\frac{\rho}{R}\right)d\rho\,d\varphi$$

Both integrals are 0! $\left(\int_0^{2\pi}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}d\varphi = \begin{pmatrix}0\\0\end{pmatrix}\right)$

$$= \boxed{\vec{0}}$$

to distinguish from charge Q

(e) $\tilde{Q}_{ij} = \int d^3r\,\rho(r)\left(3r_ir_j - r^2\delta_{ij}\right)$

For the segment

$$\tilde{Q}_{xx} = \tilde{Q}_{yy} = \frac{3Q}{h}\int_0^h\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\frac{z}{h}\right)^2\delta(x)\,\delta(y)\left(3x^2-x^2-y^2-z^2\right)dx\,dy\,dz$$

$$= \frac{3Q}{h}\int_0^h -z^2\left(1+\frac{z}{h}\right)^2 dz = -3Qh^2\int_0^1 t^{3-1}(1-t)^{3-1}\,dt$$

$$= -3Qh^2\frac{4}{120} = -\frac{Qh^2}{10}, \quad Tr(Q) = 0 \Rightarrow \tilde{Q}_{zz} = \frac{Qh^2}{5}$$

And $\tilde{Q}_{ij} = 0$ for $i \neq j$ because $\delta_{ij} = 0$ and $\int\delta(x)\delta(y)\,3r_ir_j = 0$ for $i \neq j$

So overall:

$$\boxed{\tilde{Q}_s = h^2 Q\,\mathrm{diag}\left(-\frac{1}{10}, -\frac{1}{10}, \frac{1}{5}\right)}$$

For the disk

$$\tilde{Q}_{zz} = \frac{3Q}{\pi R^2}\cancel{\int_{-\infty}^{\infty}}\int_0^{2\pi}\int_0^R \left(1-\frac{\rho}{R}\right)\delta(\cancel{z})\left(3\cancel{z^2}-\rho^2-\cancel{z^2}\right)\rho\, d\rho\, d\varphi\, \cancel{dz}$$

$$= -\frac{3Q}{\cancel{\pi} R^2}\, 2\cancel{\pi}\int_0^R \rho^3 - \frac{\rho^4}{R}\, d\rho = -\frac{6Q}{R^2}\left(\frac{R^4}{4}-\frac{R^4}{5}\right) = -\frac{3QR^2}{10}$$

Again: $\tilde{Q}_{xx} = \tilde{Q}_{yy}$ and $\tilde{Q}_{ij} = 0$ for $i \neq j$ because of symmetry.

So, since $\mathrm{Tr}(\tilde{Q}) = 0$: $\tilde{Q}_{xx} = \tilde{Q}_{yy} = -\frac{1}{2}\tilde{Q}_{zz} = \frac{3}{20} QR^2$

In total: $\boxed{\tilde{Q} = \frac{3}{20} QR^2\, \mathrm{diag}\left(1, 1, -\frac{1}{2}\right)}$

(f) For the cone we found:

$$\vec{p}_c = \frac{Qh}{4}\vec{e}_z\,, \quad \tilde{Q}_{c,zz} = \frac{3}{5}Q\left(\frac{h^2}{3}-\frac{R^2}{2}\right)$$

Dipole moment

$$\lim_{R\to 0} \vec{p}_c = \vec{p}_c = \vec{p}_s \quad \checkmark$$

$$\lim_{h\to 0} \vec{p}_c = \vec{0} = \vec{p}_d \quad \checkmark$$

Quadrupole moment

In all cases the symmetry is the same in the sense that $\tilde{Q}_{ij} = 0$ for $i \neq j$ and $\tilde{Q}_{xx} = \tilde{Q}_{yy}$. So it is sufficient to compute $\tilde{Q}_{zz}$ and all other components follow from $\mathrm{tr}(\tilde{Q}) = 0$.

$$\lim_{R \to 0} \tilde{Q}_{c,zz} = \lim_{R \to 0} \frac{3}{5} Q \left(\frac{h^2}{3} - \frac{R^2}{2} \right) = \frac{Qh^2}{5} = \tilde{Q}_{s,zz}$$ ✓

$$\lim_{h \to 0} \tilde{Q}_{c,zz} = -\frac{3}{10} Q R^2 = \tilde{Q}_{d,zz}$$ ✓

As expected, the results are in agreement!