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Physics 841 - Midterm 2 **due Sun., Apr. 12, 2020**

Problems

Expectations:

- **You present your independent work.**
- **Possibilities of plagiarism will be thoroughly checked for.**
- **All steps in the solution are clearly explained.**
- **If existing results are used (e.g. expansion in orthogonal functions), you may want to cite the literature as e.g. Jackson, Eq. (3.70).**
- **The solution is written or typed, with all equations written or typed by you (e.g. if you need a formula, write it down, do not paste a picture of it from Jackson's book).**

1. **A hemisphere:** Consider a cavity that has a shape of a hemisphere of radius R closed at the bottom, as shown in the figure. With our convention on the inclination angle it occupies the region of space with $\theta \in [0, \pi/2]$. The end goal is to calculate the electrostatic potential in the cavity.

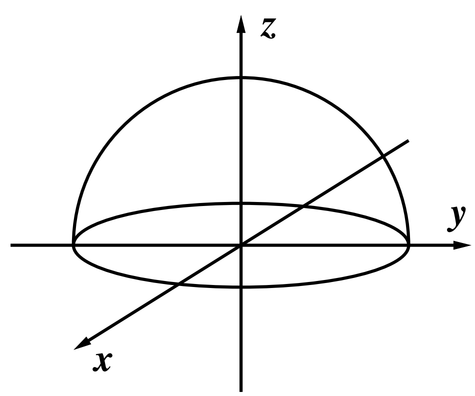
   Dirichlet interior only

   (a) (30 pts) Write down the Green's function for this problem. Explain how you arrive at it.

   (b) (40 pts) The spherical part of the cavity is maintained at the potential

   $$V(\theta, \phi) = V_0 \cos\theta,$$

   and the flat bottom part at $V = 0$. Evaluate the potential inside the cavity.

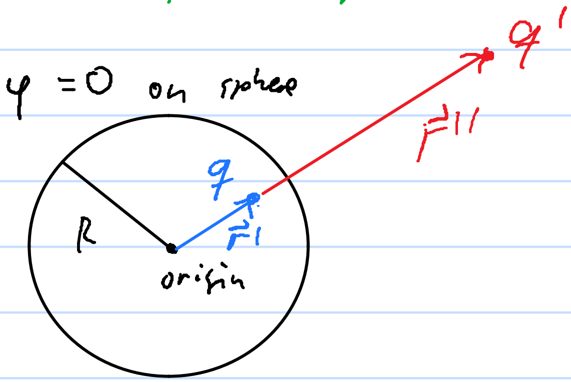


Do not try to evaluate the integrals on θ' in closed form, they may be too complicated. Rather, use properties of the spherical harmonics and the input in the problem to simplify as much as possible, so that you could make a statement e.g. like this: the result is presented as an expansion, the expansion coefficients A_{lm} are given in terms of the associated Legendre functions as $A_{lm} = \int_0^1 dx x P_l^m(x)$.

Problem 1

(a) After thinking about the problem for a while, I believe the easiest approach is to exploid the method of images.

Step 1: Find the (Interior Dirichlet) Green function for a full sphere



Using the method of images, as we have done a few times for this case:

$$q' = -\frac{R}{r'} q \qquad \vec{r}'' = \frac{R^2}{r'^2} \vec{r}'$$

As discussed in homework before, these two charges together produce a potential that vanishes on the sphere's surface:

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/r'}{|\vec{r}-\vec{r}''|}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\left(r^2+r'^2-2rr'\cos\gamma\right)^{-1/2} - \left(\left(\frac{rr'}{R}\right)^2+R^2-2rr'\cos\gamma\right)^{-1/2}\right]$$

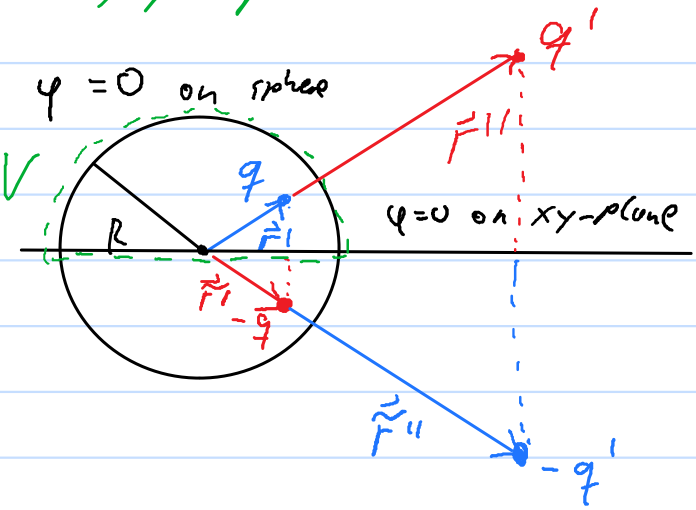
Where γ is the angle between $\vec{r}$ and $\vec{r}'$. which yields the Dirichlet Green function of a sphere:

$$\mathcal{G}_{FS}(\vec{r},\vec{r}') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/r'}{|\vec{r}-\vec{r}''|}\right], \qquad \vec{r}'' = \frac{R^2}{r'^2}\vec{r}'$$

Full Sphere
$$= \frac{1}{4\pi\varepsilon_0}\left[\left(r^2+r'^2-2rr'\cos\gamma\right)^{-1/2} - \left(\left(\frac{rr'}{R}\right)^2+R^2-2rr'\cos\gamma\right)^{-1/2}\right]$$

(We can note that for this case the exterior and interior Green function are the same.)

Step 2: Construct the Green function for the closed hemisphere V by reflecting the Green function of the full sphere on the xy-plane



It is immediately obvious, that this setup leads to $\varphi = 0$ on the xy-plane. Additionally, since both pairs of charges result in $\varphi = 0$ on the sphere, the superposition of all 4 charges still has $\varphi = 0$ on the sphere's surface. Therefore, this is just what we are looking for. However, it has to be noted that because the <u>entire</u> sphere has $\varphi = 0$, this only leads to the correct <u>interior</u> Green function. In our case, that's sufficient.

So the Green function we are looking for is:

$$\boxed{\begin{gathered} G(\vec{r}, \vec{r}') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{R/r'}{|\vec{r} - \vec{r}''|} - \frac{1}{|\vec{r} - \tilde{\vec{r}}'|} + \frac{R/r'}{|\vec{r} - \tilde{\vec{r}}''|}\right] \qquad \left(\begin{gathered}|\tilde{\vec{r}}'| = \tilde{r}' = r' \\ \tilde{r}'' = r''\end{gathered}\right) \\ \text{Where} \quad \vec{r}'' = \frac{R^2}{r'^2}\vec{r}', \quad \tilde{\vec{r}}' = (r'_x, r'_y, -r'_z)^T, \quad \tilde{\vec{r}}'' = (r''_x, r''_y, -r''_z)^T \end{gathered}}$$

This can also again be written in spherical coordinates (next page).

With $\vec{r} = (r, \theta, \varphi)$, $\vec{r}' = (r', \theta', \varphi')$ in spherical coordinates, we get:

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$$

for the angle γ between $\vec{r}$ and $\vec{r}'$.
It follows for the angle $\tilde{\gamma}$ betwen $\vec{r}$ and $\tilde{\vec{r}}'$:

$$\cos\tilde{\gamma} = \cos\theta\cos\tilde{\theta}' + \sin\theta\sin\tilde{\theta}'\cos(\varphi - \tilde{\varphi}')$$

$\tilde{\varphi}' = \varphi'$, $\tilde{\theta}' = \pi - \theta'$, $\theta' \in [0, \frac{\pi}{2}]$

$$= -\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$$

And the Green function becomes:

$$\boxed{\begin{aligned} G(\vec{r}, \vec{r}') = \frac{1}{4\pi\varepsilon_0}\Big[& (r^2 + r'^2 - 2rr'\cos\gamma)^{-1/2} - \left(\left(\tfrac{rr'}{R}\right)^2 + R^2 - 2rr'\cos\gamma\right)^{-1/2} \\ & - (r^2 r'^2 - 2rr'\cos\tilde{\gamma})^{-1/2} + \left(\left(\tfrac{rr'}{R}\right)^2 + R^2 - 2rr'\cos\tilde{\gamma}\right)^{-1/2}\Big] \end{aligned}}$$

However, using this form can hopefully be avoided by expanding $\frac{1}{|\vec{r} - \vec{r}'|}$ in spherical harmonics.

(b) Since we found the Green function for this problem, it seems to make sense to use it here by applying the "magic rule". An alternative approach could be to use the general solution of Laplace's equation in spherical coordinates and match the boundary conditions.

The "magic rule": (Zangwill 8.59)

$$\varphi(\vec{r} \in V) = \int_V d^3r' \, G(\vec{r},\vec{r}') \rho(\vec{r}') - \varepsilon_0 \int_S dS' \varphi_S(\vec{r}') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'}$$

Since here we have no charges, $\rho = 0$:

$$\varphi(\vec{r}) = -\varepsilon_0 \int_S dS' \varphi_S(\vec{r}') \frac{\partial G(\vec{r},\vec{r}')}{\partial n'}$$

And since the boundary condition for the bottom of the hemisphere is $\varphi_S(\vec{r}) = 0$, $\vec{r} = (x, y, 0)^T$, this simplifies to a spherical integral:

$$\varphi(\vec{r}) = -\varepsilon_0 \int_0^{2\pi} d\varphi' \int_0^{\pi/2} d\theta' \, R^2 \sin\theta' \underbrace{V_0 \cos\theta'}_{\varphi_S(\vec{r}')} \left. \frac{\partial G(\vec{r},\vec{r}')}{\partial r'} \right|_{r'=R}$$

$$= -\varepsilon_0 R^2 V_0 \int_0^{2\pi} d\varphi' \int_0^{\pi/2} d\theta' \sin\theta' \cos\theta' \left. \frac{\partial G(\vec{r},\vec{r}')}{\partial r'} \right|_{r'=R}$$

Finding the normal derivative of G:

As already applied, the normal component of G on the sphere's surface is r', thus

$$\frac{\partial G}{\partial n'} = \left. \frac{\partial G}{\partial r'} \right|_{r'=R}$$

using $$\frac{\partial}{\partial r'} \frac{1}{|\vec{r}-\vec{r}'|}\Big|_{r'=R} = \frac{\partial}{\partial r'} \left(r^2+r'^2-2rr'\cos\gamma\right)^{-1/2}\Big|_{r'=R}$$

$$= \left(2r'-2r\cos\gamma\right) \cdot \left\{-\frac{1}{2}\left(r^2+r'^2-2rr'\cos\gamma\right)^{-3/2}\right\}\Big|_{r'=R}$$

$$= \frac{r\cos\gamma - r'}{|\vec{r}-\vec{r}'|^3}\Big|_{r'=R} = \frac{r\cos\gamma - R}{|\vec{r} - R\vec{e}_{r'}|^3}$$

So: $$\frac{\partial}{\partial r'}\left[\frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/r'}{|\vec{r}-\vec{r}''|}\right]_{r'=R}$$ recall $\vec{r}'' = \frac{R^2}{r'^2}\vec{r}'$

$$= \frac{r\cos\gamma - R + \frac{r^2}{R} - r\cos\gamma}{|\vec{r}-R\vec{e}_{r'}|^3} = \frac{\frac{r^2}{R} - R}{|\vec{r} - R\vec{e}_{r'}|^3}$$

It is easy to see that we get the same result (except for a minus sign) for the other 2 terms of G, with the difference that $\vec{e}_r$ is reflected at the xy-plane.

So we find:

$$\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial r'}\Big|_{r'=R} = \frac{1}{4\pi\varepsilon_0}\left[\frac{\frac{r^2}{R}-R}{|\vec{r}-R\vec{e}_{r'}|^3} - \frac{\frac{r^2}{R}-R}{|\vec{r}-R\vec{e}_{\tilde{r}'}|^3}\right]$$

!

Where $\vec{e}_{\tilde{r}'}$ is $\vec{e}_{r'}$ reflected at the xy-plane. Which means $\tilde{\Theta}' = \pi - \Theta'$ and can easily achieved by adjusting the limits of the integral for the second term.

We will now plug that back into the integral expression for φ we derived before. Note that the φ' integration is still the same for all terms.

This is page 7!

$$\varphi(\vec{r}) = -\varepsilon_0 R^2 V_0 \int_0^{2\pi} d\varphi' \int_0^{\pi/2} d\theta' \sin\theta' \cos\theta' \left.\frac{\partial G(\vec{r}, \vec{r}')}{\partial r'}\right|_{r'=1}$$

$$= \frac{R(R^2 - r^2)V_0}{4\pi} \int_0^{2\pi} d\varphi' \int_0^{\pi/2} d\theta' \sin\theta' \cos\theta' \left[\frac{1}{|\vec{r} - R\vec{e}_{r'}|^3} - \frac{1}{|\vec{r} - R\vec{e}_{\tilde{r}'}|^3}\right]$$

Now I have to watch out with the signs, but I'm pretty sure this will simplify to a single Integral $\int_0^{\pi} d\theta'$.

Let's look at the second term:

$$-\int_0^{\pi/2} d\theta' \sin\theta' \cos\theta' \, |\vec{r} - R\vec{e}_{\tilde{r}'}|^{-3} := I_2$$

Substitute: $\tilde{\Theta} = \pi - \Theta' \Rightarrow \Theta' = \pi - \tilde{\Theta}$

$\Rightarrow \sin\Theta' = \sin(\pi - \tilde{\Theta}) = \sin\tilde{\Theta}$ and of course: $\vec{e}_{\tilde{r}'} \to \vec{e}_{r'}$

$\cos\Theta' = \cos(\pi - \tilde{\Theta}) = -\cos\tilde{\Theta}$

So we get $I_2 = -\int_{\pi-0}^{\pi-\frac{\pi}{2}} d\tilde{\Theta} \sin\tilde{\Theta} (-\cos\tilde{\Theta}) |\vec{r} - R\vec{e}_{r'}|^{-3}$

and <u>renaming</u> $\tilde{\Theta}$ back to Θ'

$$I_2 = - \int_{\pi/2}^{\pi} d\Theta' \sin\Theta' \cos\Theta' \, |\vec{r} - R\vec{e}_{r'}|^{-3}$$ Tada!

can't be unified to one integral. Which actually makes sense.

Looking at this math I realized: The solution can also be found by taking the solution for a full sphere with boundary condition $\varphi_S = \cos\Theta$ on the surface and subtracting the xy-plane reflection of that solution from itself and dividing the result by 2!

I want to expand that idea because I think it's actually a slight shortcut. Obviously the result will be the same (the solution is unique). And it's funny how I saw it in the math, usually it's the other way around (idea => math). So, let me explain:

Problem: Consider a sphere of radius R. Find the electric potential inside of the sphere with boundary condition $\varphi_{FS}(\vec{r}) = V_0 \cos\theta$, $\vec{r}$ on surface

FS = Full sphere

Let's assume we found the solution φ_{FS} to that problem. The solution for our problem here is then:

$$\varphi(\vec{r}) = \frac{\varphi_{FS}(\vec{r}) - \varphi_{FS}(\tilde{\vec{r}})}{2}$$

Where $\tilde{\vec{r}}$ is $\vec{r}$ reflected on the xy-plane: $\tilde{\vec{r}} = (r_x, r_y, -r_z)^T$.

Proof: Since for $z=0$ $\vec{r} = \tilde{\vec{r}}$ we have $\varphi_{FS}(\vec{r}) = \varphi_{FS}(\tilde{\vec{r}})$ on the xy-plane and therefore $\varphi(z=0) = 0$, which is our boundary condition for the flat bottom part of the upper hemisphere.

Additionally: $\cos\tilde{\theta} = -\cos\theta$, $\tilde{\theta} = \pi - \theta$
and thus $\varphi_{FS}(\vec{r}) = V_0 \cos\theta = -V_0 \cos\tilde{\theta} = -\varphi_{FS}(\tilde{\vec{r}})$
for $\vec{r}$ on surface of upper hemisphere.

So $\varphi(\vec{r}) = \frac{1}{2}\left(\varphi_{FS}(\vec{r}) - \varphi_{FS}(\tilde{\vec{r}})\right) = \frac{1}{2}\left[V_0 \cos\theta - (-V_0 \cos\theta)\right)$
$= V_0 \cos\theta$, on the surface of the upper hemisphere.

So φ is the unique solution we are looking for. $\square$

So let's try to find the solution φ_{FS} and then we can stich φ together from there.

In (a) I already discussed the Green function for the full sphere:

$$g_{FS}(\vec{r},\vec{r}') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/r'}{|\vec{r}-\vec{r}''|}\right], \qquad \vec{r}'' = \frac{R^2}{r'^2}\vec{r}'$$

Using this Green function we can again apply the magic rule. Again, there are no charges. So all that's left is:

$$\varphi_{FS} = -\varepsilon_0 V_0 \oint_{FS} dS' \cos\Theta' \frac{\partial g_{FS}}{\partial n'}$$

and inside of the sphere the normal derivative of g is:

$$\frac{\partial g_{FS}}{\partial n'} = + \left.\frac{\partial g_{FS}}{\partial r'}\right|_{r'=R}$$

Then:

$$\varphi_{FS} = -\epsilon_0 \oint_{FS} ds'\, V_0 \cos\theta' \frac{\partial g_{FS}}{\partial n'}$$

$$= -\epsilon_0 \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta'\, R^2 \sin\theta'\, V_0 \cos\theta' \left.\frac{\partial g_{FS}}{\partial r'}\right|_{r'=R}$$

$\vec{r}'' = \frac{R^2}{r'^2}\vec{r}'$

$$= -\epsilon_0 V_0 R^2 \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta' \left[\frac{\partial}{\partial r'} \frac{1}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/r'}{|\vec{r}-\vec{r}''|}\right)\right]_{r'=R}$$

$$= -\frac{V_0 R^2}{4\pi}\left[\frac{\partial}{\partial r'} \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta' \left(\frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/r'}{|\vec{r} - \frac{R^2}{r'^2}\vec{r}'|}\right)\right]_{r'=R}$$

Using (Zangwill 4.84) for $r < r'$:

inside $\vec{r}$: $r<R$; on surface $r'=R$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{r^l}{r'^{l+1}} \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega')$$

$\left|\frac{R^2}{r'^2}\vec{r}'\right| = \frac{R^2}{r'}$; on surface R

$$\frac{R/r'}{|\vec{r} - \frac{R^2}{r'^2}\vec{r}'|} = \underbrace{\frac{1}{r'}\cdot\frac{R}{r'}}_{R/r'^2} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1}\left(\frac{r}{R^2/r'}\right)^l \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega'), \quad \frac{R^2}{r'^2} r' > r$$

$$= \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} R\, r'^{\,l-2}\left(\frac{r}{R^2}\right)^l \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega')$$

So
$$\frac{\partial}{\partial r'}\left[\frac{1}{|\vec{r}-\vec{r}'|} - \frac{R/r'}{|\vec{r} - \frac{R^2}{r'^2}\vec{r}'|}\right]_{r'=R}$$

$$= \sum_{l=0}^{\infty} \frac{4\pi}{2l+1}\left[\frac{\partial}{\partial r'}\left(\frac{r^l}{r'^{l+1}} - R\, r'^{\,l-2}\left(\frac{r}{R^2}\right)^l\right)\right]_{r'=R} \cdot \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega')$$

$$= \sum_{l=0}^{\infty} \frac{4\pi}{2l+1}\left[\frac{(l+1)\, r^l}{r'^{l+}} - R\,(l-2)\, r'^{\,l-3}\left(\frac{r}{R^2}\right)^l\right]_{r'=R} \cdot \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega')$$

$$= \sum_{l=0}^{\infty} \frac{4\pi}{2l+1}\left[\frac{-(l+1)\, r^l}{R^{l+2}} - (l-2)\, R^{l-2}\, \frac{r^l}{R^{2l}}\right] \cdot \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega')$$

$$= \sum_{l=0}^{\infty} \frac{4\pi}{\cancel{2l+1}}\left[-(\cancel{2l+1})\frac{r^l}{R^{l+2}}\right] \cdot \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega')$$

Plugging that back into φ_{FS}:

$$\varphi_{FS} = -\frac{V_0 R^2}{4\pi}\left[\frac{\partial}{\partial r'}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta' \left(\frac{1}{|\vec r - \vec r'|} - \frac{R/r'}{|\vec r - \frac{R^2}{r'^2}\vec r'|}\right)\right]_{r'=R}$$

$$= + \frac{V_0 \cancel{R^2}}{\cancel{4\pi}} \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta' \left(+ \sum_{l=0}^{\infty} \cancel{4\pi} \frac{r^l}{R^{l\cancel{+2}}}\right) \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega')$$

$$= V_0 \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\frac{r}{R}\right)^l Y^*_{lm}(\Omega) \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta'\, Y_{lm}(\Omega')$$

This is an interior spherical expansion
(see Zangwill chapter 4.6.2, especially equation 4.89)

$$\varphi_{FS}(\vec r) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm}\, r^l\, Y^*_{lm}(\Omega), \quad r < R$$

With coefficients

$$B_{lm} = 4\pi\varepsilon_0 \frac{V_0}{R^l} \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta'\, Y_{lm}(\theta', \varphi')$$

With that we found our solution:

$$\varphi(\vec r) = \frac{\varphi_{FS}(\vec r) - \varphi_{FS}(\tilde{\vec r})}{2} = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{B_{lm}}{2} r^l \left[Y^*_{lm}(\Omega) - Y^*_{lm}(\tilde\Omega)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{B_{lm}}{2} r^l \left[Y^*_{lm}(\theta, \varphi) - Y^*_{lm}(\pi - \theta, \varphi)\right]$$

for $\theta \in [0, \frac{\pi}{2}]$

This can be further simplified by exploiting the azimuthal symmetry using (Zangwill 4.91) and (Zangwill 4.92):

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta) \qquad (Z\ 4.91)$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\, Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)\,\delta_{m,0} \qquad (Z\ 4.92)$$

With that we get

$$B_{lm} = 4\pi\varepsilon_0 \frac{V_0}{R^l}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta'\, Y_{lm}(\theta',\varphi')$$

$$= 4\pi\varepsilon_0 \frac{V_0}{R^l}\, 2\pi\, \delta_{m,0} \int_0^{\pi} d\theta' \sin\theta' \cos\theta' \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta')$$

$$= \delta_{m,0}\, 8\pi^2 \sqrt{\frac{2l+1}{4\pi}}\, \frac{V_0}{R^l} \int_0^{\pi} d\theta' \sin\theta' \cos\theta'\, P_l(\cos\theta')$$

With that the solution simplifies to:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{B_{lm}}{2} r^l \left[Y^*_{lm}(\theta,\varphi) - Y^*_{lm}(\pi-\theta,\varphi)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} \frac{B_{l0}}{2} r^l \left[Y^*_{l0}(\theta,\varphi) - Y^*_{l0}(\pi-\theta,\varphi)\right]$$

From (Z 4.91) it follows that $Y^*_{l0} = Y_{l0}$

And using (Jackson 3.16) we see that

$$P_l(-x) = (-1)^l P_l(x)$$

And since $\cos(\pi-\theta) = -\cos(\theta)$ we conclude that all terms with uneven l vanish and only even terms survive.

So we can further simplify the result:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{B_{l0}}{2} r^l \left[Y_{l0}^*(\theta,\varphi) - Y_{l0}^*(\pi-\theta,\varphi) \right]$$

(2.4.91)

$$= \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{B_{l0}}{2} r^l \sqrt{\frac{2l+1}{4\pi}} \left[P_l(\cos\theta) - P_l(-\cos\theta) \right]$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \delta_{-1,(-1)^l} B_{l0} r^l \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Which is an expansion in Legendre polynomials with coefficients:

$$A_l = \delta_{-1,(-1)^l} \sqrt{\frac{2l+1}{4\pi}} B_{l0}$$

$$= \delta_{-1,(-1)^l} \sqrt{\frac{2l+1}{4\pi}} 8\pi^2 \sqrt{\frac{2l+1}{4\pi}} \frac{V_0}{R^l} \int_0^\pi d\theta' \sin\theta' \cos\theta' P_l(\cos\theta')$$

$$= \delta_{-1,(-1)^l} 2\pi(2l+1) \frac{V_0}{R^l} \underbrace{\int_0^\pi d\theta' \sin\theta' \cos\theta' P_l(\cos\theta')}_{\text{can be solved with } x=\cos\theta'}$$

With all A_l with even l vanishing.

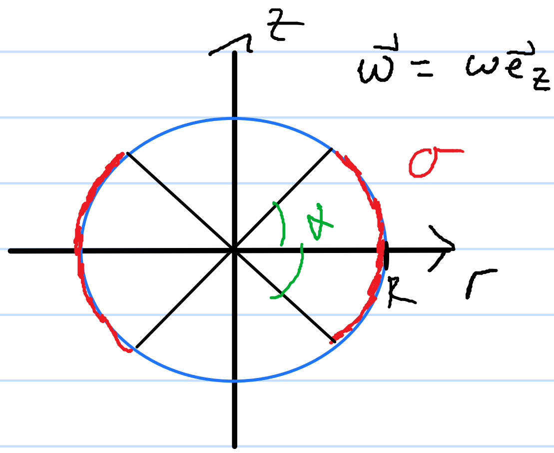
2. **A rotating sphere:** A sphere of radius R carries a uniform surface-charge distribution σ in the region of $\theta \in [\pi/2 - \alpha, \pi/2 + \alpha]$, α is a fixed parameter. The rest of the sphere has no charge. The sphere is rotated about the z axis with constant angular velocity ω.

   (a) (20 pts) Find the magnetic dipole moment $\vec{m}$.

   (b) (10 pts) Demonstrate that your solution is correct by considering two limits: $\alpha \to \pi/2$, $\alpha \to 0$. In the second limit assume that the total current is held fixed.

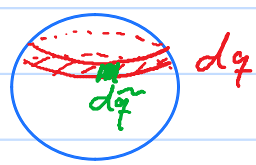
Problem 2

(a)



I think it is the easiest to approach the problem by seeing the total magnetic moment as superposition of the magnetic moments of infinitesimal rings.

The charge on such a ring is:



$$d\tilde{q} = \sigma\, dS = \sigma R^2 \sin\theta \, d\theta \, d\varphi \, .$$

in total: $dq = \int_0^{2\pi} d\tilde{q} = 2\pi \, d\tilde{q}$

So the current is:

$$d\vec{I} = \frac{\omega}{2\pi} dq \, \vec{e}_\varphi = \omega \sigma R^2 \sin\theta \, d\theta \, \vec{e}_\varphi = dI \, \vec{e}_\varphi \, .$$

Then the moment induced by such an infinitesimal ring is:

$$d\vec{m} = \frac{1}{2} dI \oint_{ring} \vec{r} \times d\vec{l}$$

(Zangwill 11.19, 11.20)

$$= dI \left(\pi R^2 \sin^2\theta\right) \vec{e}_z$$

$$= \pi \omega \sigma R^4 \sin^3\theta \, d\theta \, \vec{e}_z$$

With $d\vec{m} = \pi \omega \sigma R^4 \sin^3\theta \, d\theta \, \vec{e}_z$

we can now find the total moment by superposition:

$$\vec{m} = \int d\vec{m} = \pi \omega \sigma R^4 \vec{e}_z \underbrace{\int_{\pi/2-\alpha}^{\pi/2+\alpha} \sin^3\theta \, d\theta}_{\frac{1}{6}(9\sin(\alpha)+\sin(3\alpha))}$$

$$\boxed{= \pi \sigma R^4 \frac{1}{6} \left[9\sin(\alpha) + \sin(3\alpha)\right] \underbrace{\omega \vec{e}_z}_{\vec{\omega}}}$$

(b) With $\sigma = \frac{Q}{4\pi R^2}$ (Total charge for $\alpha = \frac{\pi}{2}$, NOT total charge for different α.)

$$\vec{m} = \frac{QR^2}{4} \cdot \frac{1}{6} \left[9\sin(\alpha) + \sin(3\alpha)\right] \vec{\omega}$$

$\alpha \to \pi/2$ Using the above form this becomes very easy now. It's just a spherical shell with uniform surface charge density $\sigma = Q/4\pi R^2$:

$$\vec{m} \xrightarrow{\alpha \to \pi/2} \frac{QR^2}{4} \frac{1}{6} \left[9 - 1\right] \vec{\omega} = \frac{QR^2}{3} \vec{\omega} \quad ✓$$

$$= \frac{4\pi R^4}{3} \vec{\omega}$$

Which is the correct result for a full spherical shell!

$\alpha \to 0$ (current held fix) Obviously here σ is not constant, if we want the current to be constant.

We expect the result for a current loop:

$$\vec{m}_{loop} = I A \vec{e}_z = I \pi R^2 \vec{e}_z$$ (follows from Zangwill 11.20)

Using $d\vec{I} = \frac{\omega}{2\pi} dq \, \vec{e}_\varphi = \omega \sigma R^2 \sin\Theta d\Theta \, \vec{e}_\varphi$ from (a)

the total current I passing through any longitude on the sphere's surface is:

$$I = \omega \sigma R^2 \int_{\pi/2-\alpha}^{\pi/2+\alpha} \sin\Theta d\Theta = 2\omega\sigma R^2 \sin(\alpha)$$

Therefore, if we want I to be constant:

$$\sigma(\alpha) = I \left(2\omega R^2 \sin(\alpha)\right)^{-1}.$$

Using the endresult from (a), we can now take the limit:

$$\lim_{\alpha\to 0} \vec{m}\Big|_{I \text{ constant}} = \lim_{\alpha \to 0} \pi \sigma(\alpha) R^4 \frac{1}{6}\left[9\sin(\alpha) + \sin(3\alpha)\right] \omega \vec{e}_z$$

$$= \frac{\pi \frac{R^4}{6} \omega \vec{e}_z}{2\omega R^2} \lim_{\alpha\to 0} \frac{9\sin(\alpha) + \sin(3\alpha)}{\sin(\alpha)}$$

applying L'Hospital

$$= \frac{\pi R^2}{12} I \vec{e}_z \frac{9\cos(0) + 3\cdot\cos(3\cdot 0)}{\cos(0)}$$

$$= \pi R^2 I \vec{e}_z \quad \checkmark$$

As expected: The result for a current loop!

Just wanted to add that I worked a total of 25 hours on this. Please keep that in mind when designing the subject exam.