Special Relativity

Postulates

1. POSTULATE OF RELATIVITY
The laws of nature and the results of all experiments performed in a given frame of reference are independent of the translational motion of the system as a whole. More precisely, there exists a triply infinite set of equivalent Euclidean reference frames moving with constant velocities in rectilinear paths relative to one another in which all physical phenomena occur in an identical manner.

2. POSTULATE OF THE CONSTANCY OF THE SPEED OF LIGHT
The speed of light is independent of the motion of its source.

Lorentz Transformation

$$\left.\begin{aligned} x_0' &= \gamma(x_0-\beta x_1) \\ x_1' &= \gamma(x_1-\beta x_0) \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\}$$

$$\left.\begin{aligned} x_0 &= \gamma(x_0'+\beta x_1') \\ x_1 &= \gamma(x_1'+\beta x_0') \\ x_2 &= x_2' \\ x_3 &= x_3' \end{aligned}\right\}$$

$$\boldsymbol{\beta}=\frac{\mathbf{v}}{c}, \qquad \beta=|\boldsymbol{\beta}|$$

$$\gamma=(1-\beta^2)^{-1/2}$$

$$A = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \qquad A=\begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Addition of Velocities

are $u_i'=c\,dx_i'/dx_0'$ and $u_i=c\,dx_i/dx_0$. This means that the components of velocity transform according to

$$\left.\begin{aligned} u_\parallel &= \frac{u_\parallel'+v}{1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}} \\ \mathbf{u}_\perp &= \frac{\mathbf{u}_\perp'}{\gamma_v\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)} \end{aligned}\right\} \tag{11.31}$$

light by adding two velocities, even if each is very close to c. For the simple case of parallel velocities the addition law is

$$u=\frac{u'+v}{1+\frac{vu'}{c^2}} \tag{11.33}$$

Proper Time

Another useful concept is *proper time.* Consider a system, which for definiteness we will think of as a particle, moving with an instantaneous velocity $\mathbf{u}(t)$ relative to some inertial system K. In a time interval dt its position changes by $d\mathbf{x}=\mathbf{u}\,dt$. From (11.25) the square of the corresponding infinitesimal invariant interval ds is

$$ds^2 = c^2\,dt^2 - |d\mathbf{x}|^2 = c^2\,dt^2(1-\beta^2)$$

where here $\beta = u/c$. In the coordinate system K' where the system is instantaneously at rest the space-time increments are $dt' \equiv d\tau$, $d\mathbf{x}'=0$. Thus the invariant interval is $ds = c\,d\tau$. The increment of time $d\tau$ in the instantaneous rest frame of the system is thus a *Lorentz invariant quantity* that takes the form,

$$d\tau = dt\sqrt{1-\beta^2(t)} = \frac{dt}{\gamma(t)} \tag{11.26}$$

The time τ is called the *proper time of the particle or system.* It is the time as seen in the rest frame of the system. From (11.26) it follows that a certain proper time interval $\tau_2-\tau_1$ will be seen in the frame K as a time interval,

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2(\tau)}} = \int_{\tau_1}^{\tau_2} \gamma(\tau)\,d\tau \tag{11.27}$$

Equation (11.27) or (11.26) expresses the phenomenon known as *time dilatation.* A moving clock runs more slowly than a stationary clock. For equal time intervals in the clock's rest frame, the time intervals observed in the frame K are greater by a factor $\gamma > 1$. This paradoxical result is verified daily in

Relativistic Doppler Shift

relativistic Doppler shift. Consider a plane wave of frequency ω and wave vector $\mathbf{k}$ in the inertial frame K. In the moving frame K' this wave will have, in general, a different frequency ω' and wave vector $\mathbf{k}'$, but the phase of the wave is an invariant:

$$\phi = \omega t - \mathbf{k}\cdot\mathbf{x} = \omega' t' - \mathbf{k}'\cdot\mathbf{x}' \tag{11.28}$$

[Parenthetically we remark that because the equations of (11.16) are linear the plane wave in K with phase ϕ indeed remains a plane wave in frame K'.] Using (11.16) and the same arguments as we did in going from (11.7) to (11.8), we find that the frequency $\omega' = ck_0'$ and wave vector $\mathbf{k}'$ are given in terms of $\omega = ck_0$ and $\mathbf{k}$ by

$$\left.\begin{aligned} k_0' &= \gamma(k_0 - \boldsymbol{\beta}\cdot\mathbf{k}) \\ k_\parallel' &= \gamma(k_\parallel - \beta k_0) \\ \mathbf{k}_\perp' &= \mathbf{k}_\perp \end{aligned}\right\} \tag{11.29}$$

The Lorentz transformation of $(k_0, \mathbf{k})$ has exactly the same form as for $(x$

The frequency and wave number of any plane wave thus form a 4-vector.

For *light waves*, $|\mathbf{k}| = k_0$, $|\mathbf{k}'| = k_0'$. Then the results (11.29) can be expressed in the more familiar form of the Doppler shift formulas

$$\boxed{\cos\theta' = \frac{\cos\theta + \beta}{1+\beta\cos\theta} \qquad \begin{aligned} \omega' &= \gamma\omega(1-\beta\cos\theta) \\ \tan\theta' &= \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \end{aligned} \qquad (11.30)}$$

where θ and θ' are the angles of $\mathbf{k}$ and $\mathbf{k}'$ relative to the direction of $\mathbf{v}$. The inverse equations are obtained by interchanging primed and unprimed quantities and reversing the sign of β.

Relativistic Momentum and Energy

the momentum of a particle of mass m and velocity $\mathbf{u}$ is

$$\mathbf{p}=\gamma m\mathbf{u}=\frac{m\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}} \tag{11.46}$$

Thus the second, square-bracketed, term in (11.48) is absent. The total energy of a particle of mass m and velocity $\mathbf{u}$ is

$$E=\gamma mc^2=\frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} \tag{11.51}$$

$$E=\sqrt{c^2p^2+m^2c^4} \tag{11.55}$$

$$\mathbf{u}=\frac{c^2\mathbf{p}}{E} \tag{11.53}$$

Four Vectors

$$s^2=x_0{}^2-x_1{}^2-x_2{}^2-x_3{}^2 \tag{11.59}$$

We can therefore rephrase the kinematics of special relativity as the consideration of the group of all transformations that leave s^2 invariant. Technically, this group is called the *homogeneous Lorentz group*. It contains ordinary rotations as well as the Lorentz transformations of Section 11.3. ~~The group of transforma-~~

From the first postulate it follows that the mathematical equations expressing the laws of nature must be *covariant*, that is, invariant in form, under the transformations of the Lorentz group. ~~They must therefore be relations among~~

Tensors of rank k associated with the space-time point x are defined by their transformation properties under the transformation $x \to x'$. A *scalar* (tensor of rank zero) is a single quantity whose value is not changed by the transformation. The interval s^2 (11.59) is obviously a Lorentz scalar. For tensors of rank one, called *vectors*, two kinds must be distinguished. The first is called a *contravariant vector* A^α with four components A^0, A^1, A^2, A^3 that are transformed according to the rule,

$$A'^\alpha=\frac{\partial x'^\alpha}{\partial x^\beta}A^\beta \tag{11.61}$$

~~We will henceforth employ this *summation convention* for repeated indices~~. A *covariant vector* or tensor of rank one B_α is defined by the rule,

$$B'_\alpha=\frac{\partial x^\beta}{\partial x'^\alpha}B_\beta \tag{11.62}$$

Note that contravariant vectors have superscripts and covariant vectors have subscripts, corresponding to the presence of $\partial x'^\alpha/\partial x^\beta$ and its inverse in the rule of transformation. ~~It can be verified from (11.61) that if the law of transformation~~

A contravariant tensor of rank two $F^{\alpha\beta}$ consists of 16 quantities that transform according to

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \tag{11.63}$$

A covariant tensor of rank two $G_{\alpha\beta}$ transforms as

$$G'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} G_{\gamma\delta} \tag{11.64}$$

and the mixed second rank tensor $H^{\alpha}{}_{\beta}$ transforms as

$$H'^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} H^{\gamma}{}_{\delta} \tag{11.65}$$

$$(ds)^2 = g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} \tag{11.68}$$

$$g = \mathrm{diag}(1, -1, -1, -1)$$

$$x_{\alpha} = g_{\alpha\beta} x^{\beta} \tag{11.72}$$

$$x^{\alpha} = g^{\alpha\beta} x_{\beta} \tag{11.73}$$

$$\begin{aligned} F^{\cdot\cdot\alpha\cdot\cdot}_{\cdot\cdot} &= g^{\alpha\beta} F^{\cdot\cdot\cdot\cdot}_{\cdot\cdot\beta} \\ G^{\cdot\cdot\cdot}_{\cdot\cdot\cdot\alpha\cdot} &= g_{\alpha\beta} G^{\cdot\cdot\cdot\beta}_{\cdot\cdot\cdot\cdot} \end{aligned} \tag{11.74}$$

Unit convention in Jackson

velocity of light. In particle kinematics the symbols,

$$\left.\begin{matrix} p \\ E \\ m \\ \\ v \end{matrix}\right\} \quad \text{stand for} \quad \left\{\begin{matrix} cp \\ E \\ mc^2 \\ \\ \dfrac{v}{c} \end{matrix}\right.$$

Thus the connection between momentum and total energy is written as $E^2 = p^2 + m^2$, a particle's velocity is $\mathbf{v} = \mathbf{p}/E$, and so on. As energy units, the eV

Four velocity

$$U_0 \equiv \frac{dx_0}{d\tau} = \frac{dx_0}{dt}\frac{dt}{d\tau} = \gamma_u c \qquad \mathbf{U} \equiv \frac{d\mathbf{x}}{d\tau} = \frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} = \gamma_u \mathbf{u} \tag{11.36}$$

Comparison of (11.34) and (11.35) with the inverse of (11.22) suggests that *the four quantities* $(\gamma_u c, \gamma_u \mathbf{u})$ transform in the same way as $(x_0, \mathbf{x})$ and so form a 4-vector under Lorentz transformations. These four quantities are called the *time and space components of the 4-velocity* $(U_0, \mathbf{U})$.

4- momentum conservation and collisions

relativistic $\vec{p}$

$$p^\mu = m V^\mu = m(\gamma c, \gamma\vec{v}) = (E/c, \vec{p})$$

$$V_\mu V^\mu = \gamma^2(c^2 - \vec{v}^2) = \frac{c^2 - v^2}{1 - \frac{v^2}{c^2}}$$

$$= c^2 \frac{c^2 - v^2}{c^2 - v^2} = c^2$$

$$\Rightarrow p_\mu p^\mu = m^2c^2 = (m\gamma c)^2 - (m\gamma\vec{v})^2$$

With $M = \gamma m \Rightarrow p_\mu p^\mu = (Mc)^2 - (M\vec{v})^2 = (mc)^2$

$$c^2 p_\mu p^\mu = \underbrace{E}_{Mc^2 = \gamma mc^2}{}^2 - (pc)^2 = E_0^2 \quad , \quad E_0 = mc^2$$

$$E = \gamma E_0 = E_0 + T = mc^2 + T \quad , T = E_0(\gamma - 1)$$

Electro Statics

General Stuff

$$\mathbf{F}=q\mathbf{E} \tag{1.1}$$

where $\mathbf{F}$ is the force, $\mathbf{E}$ the electric field, and q the charge. In this equation it is assumed that the charge q is located at a point, and the force and the electric field are evaluated at that point.

Coulomb's law can be written down similarly. If $\mathbf{F}$ is the force on a point charge q_1, located at $\mathbf{x}_1$, due to another point charge q_2, located at $\mathbf{x}_2$, then Coulomb's law is

$$\mathbf{F}=kq_1q_2\frac{(\mathbf{x}_1-\mathbf{x}_2)}{|\mathbf{x}_1-\mathbf{x}_2|^3} \tag{1.2}$$

Note that q_1 and q_2 are algebraic quantities which can be positive or negative. The constant of proportionality k depends on the system of units used.

The electric field at the point $\mathbf{x}$ due to a point charge q_1 at the point $\mathbf{x}_1$ can be obtained directly:

$$\mathbf{E}(\mathbf{x})=kq_1\frac{(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3} \tag{1.3}$$

The experimentally observed linear superposition of forces due to many charges means that we may write the electric field at $\mathbf{x}$ due to a system of point charges q_i, located at $\mathbf{x}_i$, $i=1, 2, \ldots, n$, as the vector sum:

$$\mathbf{E}(\mathbf{x})=\sum_{i=1}^{n} q_i\frac{(\mathbf{x}-\mathbf{x}_i)}{|\mathbf{x}-\mathbf{x}_i|^3} \tag{1.4}$$

If the charges are so small and so numerous that they can be described by a charge density $\rho(\mathbf{x}')$ [if Δq is the charge in a small volume $\Delta x\,\Delta y\,\Delta z$ at the point $\mathbf{x}'$, then $\Delta q=\rho(\mathbf{x}')\,\Delta x\,\Delta y\,\Delta z$], the sum is replaced by an integral:

$$\mathbf{E}(\mathbf{x})=\int\rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,d^3x' \tag{1.5}$$

Gauss's Law

charge density $\rho(\mathbf{x})$, Gauss's law becomes:

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da=\cancel{4\pi}\int_V \rho(\mathbf{x})\,d^3x \;/\varepsilon_0 \text{ in SI} \tag{1.11}$$

where V is the volume enclosed by S.

Equation (1.11) is one of the basic equations of electrostatics. Note that it

using the divergence theorem. The *divergence theorem* states that for any well-behaved vector field $\mathbf{A}(\mathbf{x})$ defined within a volume V surrounded by the closed surface S the relation

$$\oint_S \mathbf{A}\cdot\mathbf{n}\,da=\int_V \nabla\cdot\mathbf{A}\,d^3x$$

holds between the volume integral of the divergence of $\mathbf{A}$ and the surface integral of the outwardly directed normal component of $\mathbf{A}$. The equation in fact can be used as the definition of the divergence (see *Stratton*, p. 4).

Now the divergence theorem allows us to write this as:

$$\int_V (\nabla\cdot\mathbf{E}-4\pi\rho)\,d^3x=0 \tag{1.12}$$

for an arbitrary volume V. We can, in the usual way, put the integrand equal to zero to obtain

Maxwell Eq. $$\nabla\cdot\mathbf{E}=\cancel{4\pi}\rho/\varepsilon_0 \text{ in SI} \tag{1.13}$$

which is the differential form of Gauss's law of electrostatics. This equation can

and $\operatorname{rot}\vec{E}=\vec{0}$ follows from Coulomb

Scalar Potential

quently we define the *scalar potential* $\Phi(\mathbf{x})$ by the equation:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi \tag{1.16}$$

Then (1.15) shows that the scalar potential is given in terms of the charge density by

$$\Phi(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \tag{1.17}$$

where the integration is over all charges in the universe, and Φ is arbitrary only to the extent that a constant can be added to the right side of (1.17).

Poisson & Laplace

$$\nabla^2\Phi = -4\pi\rho \tag{1.28}$$

This equation is called the *Poisson equation*. In regions of space where there is no charge density, the scalar potential satisfies the *Laplace equation*:

$$\nabla^2\Phi = 0 \tag{1.29}$$

Multipole Expansion (Cartesian)

In equations (4.4)–(4.6), q is the total charge, or monopole moment, $\mathbf{p}$ is the electric dipole moment:

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')\, d^3x' \tag{4.8}$$

and Q_{ij} is the traceless quadrupole moment tensor:

$$Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')\, d^3x' \tag{4.9}$$

We see that the lth multipole coefficients [$(2l+1)$ in number] are linear combinations of the corresponding multipoles expressed in rectangular coordinates. The expansion of $\Phi(\mathbf{x})$ in rectangular coordinates:

$$\Phi(\mathbf{x}) = \frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \cdots \tag{4.10}$$

For a dipole $\mathbf{p}$ along the z axis, the fields in (4.11) reduce to the familiar form:

$$\left.\begin{aligned} E_r &= \frac{2p\cos\theta}{r^3} \\ E_\theta &= \frac{p\sin\theta}{r^3} \\ E_\phi &= 0 \end{aligned}\right\} \tag{4.12}$$

These dipole fields can be written in vector form by recombining (4.12) or by directly operating with the gradient on the dipole term in (4.10). The result for the field at a point $\mathbf{x}$ due to a dipole $\mathbf{p}$ at the point $\mathbf{x}_0$ is:

$$\mathbf{E}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{|\mathbf{x}-\mathbf{x}_0|^3} \tag{4.13}$$

where $\mathbf{n}$ is a unit vector directed from $\mathbf{x}_0$ to $\mathbf{x}$.