PHY-841: Classical Electrodynamics / Midterm Exam 1 / February 21, 2020

SECRET STUDENT NUMBER: 150

Please READ all of the following before starting the exam:

- Do not write your name or student PID on any page of the exam. If you require extra paper, write the secret student number and the relevant problem number on the extra pages.
- You may use a simple calculator, but no phone, networking device, external notes, books, etc.
- All problems are in S.I. units. Please give your answers in terms of the given variables and use S.I. units, too.
- A complete answer usually includes a derivation of the result (unless stated otherwise). Show all work as neatly and logically as possible to maximize your credit. State clearly which equations were used. Circle or otherwise indicate your final answers.
- This exam has 4 problems for a total of 100 points. There are also some bonus points. You can additionally solve bonus points questions if time permits. Please make sure that you have all of the pages.
- **Good luck!**

$$
\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) &= \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}), \\
\vec{a} \cdot (\vec{b} \times \vec{c}) &= \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}), \\
(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}), \\
\vec{\nabla} \times (\vec{\nabla} \psi) &= 0, \\
\vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) &= 0, \\
\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}, \\
\vec{\nabla} \cdot (\psi \vec{a}) &= \vec{a} \cdot \vec{\nabla} \psi + \psi \vec{\nabla} \cdot \vec{a}, \\
\vec{\nabla} \times (\psi \vec{a}) &= \vec{\nabla} \psi \times \vec{a} + \psi \vec{\nabla} \times \vec{a}, \\
\vec{\nabla}(\vec{a} \cdot \vec{b}) &= (\vec{a} \cdot \vec{\nabla})\vec{b} + (\vec{b} \cdot \vec{\nabla})\vec{a} + \vec{a} \times (\vec{\nabla} \times \vec{b}) + \vec{b} \times (\vec{\nabla} \times \vec{a}), \\
\vec{\nabla} \cdot (\vec{a} \times \vec{b}) &= \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b}), \\
\vec{\nabla} \times (\vec{a} \times \vec{b}) &= \vec{a}(\vec{\nabla} \times \vec{b}) - \vec{b}(\vec{\nabla} \times \vec{a}) + (\vec{b} \cdot \vec{\nabla})\vec{a} - (\vec{a} \cdot \vec{\nabla})\vec{b}, \\
\vec{\nabla} \cdot \vec{r} &= 3, \\
\vec{\nabla} \times \vec{r} &= 0, \\
\vec{\nabla} \cdot \hat{r} &= 2/r, \\
\vec{\nabla} \times \hat{r} &= 0, \\
\vec{\nabla} r &= \hat{r}, \\
\vec{\nabla} \frac{1}{r} &= -\frac{\hat{r}}{r^2}, \\
\vec{\nabla} \cdot (\hat{r} f(r)) &= \frac{2}{r} f + \frac{df}{dr}, \\
(\vec{a} \cdot \vec{\nabla})\hat{r} &= \frac{1}{r}[\vec{a} - \hat{r}(\vec{a} \cdot \hat{r})] = \frac{\vec{a}_\perp}{r}, \\
\vec{\nabla}^2 \left(\frac{1}{r}\right) &= -4\pi \delta(\vec{r}), \\
\int_V d^3r \; \vec{\nabla} \cdot \vec{A} &= \int_S d\vec{S} \cdot \vec{A}, \\
\int_V d^3r \; \vec{\nabla} \psi &= \int_S \psi \; d\vec{S}, \\
\int_V d^3r \; \vec{\nabla} \times \vec{A} &= \int_S d\vec{S} \times \vec{A}, \\
\int_V d^3r \; (\phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi) &= \int_S \phi \; d\vec{S} \cdot \vec{\nabla} \psi, \\
\int_V d^3r \; (\phi \nabla^2 \psi - \psi \nabla^2 \phi) &= \int_S (\phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi) \cdot d\vec{S}, \\
\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S} &= \oint d\vec{\ell} \cdot \vec{A}, \\
\int_S d\vec{S} \times \vec{\nabla} \psi &= \oint_C d\vec{\ell} \; \psi.
\end{aligned}
$$

$$\begin{aligned}
\mathscr{L} &= -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \\
L &= \frac{1}{\gamma}(-mc^2 - qA_\mu u^\mu) \\
\frac{dp^\mu}{d\tau} &= qF^{\mu\nu}u_\nu \\
\partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0 \\
\partial_\mu J^\mu &= 0 \\
(g_{\mu\nu}) &= \mathrm{diag}(1,-1,-1,-1), \\
\vec{\beta} &= \vec{v}/c, \\
\gamma &= 1/\sqrt{1-\beta^2}, \\
x'^\mu &= \Lambda^\mu{}_\nu x^\nu \\
(\Lambda^\mu{}_\nu) &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
x^\mu &= (ct, x, y, z), \\
\partial^\mu &= ((1/c)\partial/\partial t, -\vec{\nabla}) \\
k^\mu &= (\omega/c, \vec{k}), \\
u^\mu &= (\gamma c, \gamma\vec{v}), \\
p^\mu &= (E/c, \vec{p}), \\
A^\mu &= (\phi/c, \vec{A}), \\
J^\mu &= (c\rho, \vec{j}), \\
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu, \\
\tilde{F}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}
\end{aligned}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho, \qquad \vec{\nabla}\cdot\vec{B} = 0,$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{j}, \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0,$$

$$\begin{aligned}
\int_{S(V)}\vec{E}\cdot d\vec{A} &= \frac{1}{\epsilon_0}\int_V \mathrm{d}^3r\ \rho \\
\frac{d\vec{p}}{dt} &= q(\vec{E} + \vec{v}\times\vec{B}) \\
\vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \\
\vec{B} &= \vec{\nabla}\times\vec{A} \\
p_i &= \int d^3\,r'\rho(\vec{r}\,')\,r_i' \\
Q_{ij} &= \int d^3r'\,\rho(\vec{r}\,')\,(3r_i'r_j' - \delta_{ij}r'^2) \\
\phi &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\mathrm{tot}}}{r} + \frac{p_i\hat{r}_i}{r^2} + \frac{1}{2!}Q_{ij}\frac{\hat{r}_i\hat{r}_j}{r^3} + \dots\right) \\
U &= Q_{\mathrm{tot}}\phi(\vec{r}) - p_iE_i(\vec{r}) - \frac{1}{6}Q_{ij}\partial_iE_j + \dots
\end{aligned}$$

Problem 1. Infinite wire with regions of different density [15 pts].

An infinite straight wire of vanishing cross section is positioned along the z-axis. It is charged with the following linear density ($\lambda > 0$, $a > 0$):

- $+\lambda$ for $z > a$,
- $-\lambda$ for $-a \leqslant z \leqslant a$,
- $+\lambda$ for $z < -a$.

Given that the electric field at an arbitrary point $\vec{r} = (\rho, \phi, z)$ from a charged segment of vanishing cross section and linear charge density λ positioned in $-a \leqslant z \leqslant a$ is

$$\begin{aligned}
\vec{E}(\vec{r}) &= E_\rho \hat{\rho} + E_z \hat{z}, \\
E_\rho &= \frac{\lambda}{4\pi\epsilon_0} \frac{1}{\rho} \left\{ \frac{z+a}{\sqrt{(z+a)^2+\rho^2}} - \frac{z-a}{\sqrt{(z-a)^2+\rho^2}} \right\}, \\
E_z &= \frac{\lambda}{4\pi\epsilon_0} \left\{ \frac{1}{\sqrt{(z-a)^2+\rho^2}} - \frac{1}{\sqrt{(z+a)^2+\rho^2}} \right\},
\end{aligned}$$

find the electric field from the whole wire at the point $\vec{r} = (\rho, \phi, z)$

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Problem 2. Electrodynamics, top-down approach [20 pts].

For real-world Electrodynamics where the photon is massless we narrowed down possible Lagrangian densities to

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu, \tag{1}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor and J^μ is the external current density.

1. [20 pts] Describe what main constraints, symmetries, etc. we imposed that lead to this particular form of the Lagrangian.

2. [**Bonus** 10 pts] Give examples, where possible, of terms that do not satisfy these constraints and thus had to be excluded.

Problem 3. Quadrupole moment tensor [30 pts].

A right circular cone of height h and base of radius R is positioned such that its symmetry axis coincides with the z-axis and the base is in the $x - y$ plane. The cone is charged with uniform charge density ρ_0. Find:

1. [10 pts] Total charge Q.
2. [20 pts] Quadrupole moment tensor Q_{ij}.
3. [**Bonus** 10 pts] Dipole moment $\vec{p}$.

Helpful hint:

$$\int_0^1 dt\, t^{m-1}(1-t)^{n-1} = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad \Gamma(n) = (n-1)!, \quad \Gamma(1) = 0! = 1.$$

Problem 4. Scattering [35 pts].

Consider an electron with mass m_e at rest. It is hit by an incoming photon with energy E (given) and spatial momentum $\vec{k}$. As a result of the collision, the photon is scattered by angle θ (given) and the electron acquires non-zero spatial momentum. Let k^μ (k'^μ) be the initial (final) 4-momentum of the photon, and p^μ (p'^μ) that of electron. Let the energy and spatial momentum of the photon after the collision be E' and $\vec{k}\,'$ and that of electron E'_e, $\vec{p}\,'_e$.

1. [5 pts] Draw a momentum diagram that represents the described process.

2. [5 pts] Write down all four 4-vectors k^μ, k'^μ, p^μ, p'^μ in terms of the quantities introduced in the problem.

3. [25 pts] The angle of scattering θ is the angle between $\vec{k}$ and $\vec{k}\,'$. Use the conservation of energy-momentum in the 4-vector form, as well as the following relation:

$$k^\mu - k'^\mu = p'^\mu - p^\mu$$

   to construct invariants and derive the ratio E'/E in terms of E, m_e, θ.

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