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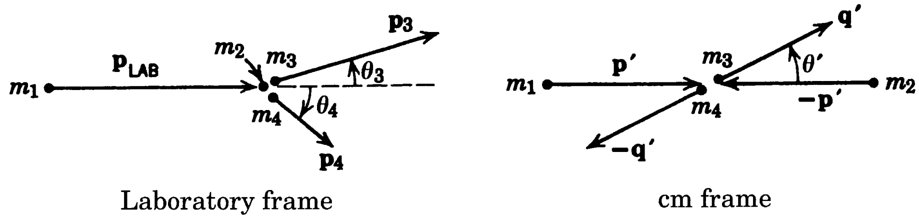
Physics 841 - Homework 3 **due Fri., Jan. 31, 2020**

Problems

1. **Collision with a particle at rest, Jackson, 11.23:** In a collision process a particle of mass m_2, at rest in the laboratory, is struck by a particle of mass m_1, momentum $\mathbf{p}_{LAB}$ and total energy E_{LAB}. In the collision the two initial particles are transformed into two others of mass m_3 and m_4. The configurations of the momentum vectors in the center of momentum (cm) frame (traditionally called the center-of-mass frame) and the laboratory frame are shown in the figure.



   (a) (10 pts) Use invariant scalar products to show that the total energy W in the cm frame has its square given by

$$W^2 = m_1^2 + m_2^2 + 2m_2E_{LAB} \tag{1}$$

   and that the cms 3-momentum $\mathbf{p}'$ is

$$\mathbf{p}' = \frac{m_2 \mathbf{p}_{LAB}}{W} \tag{2}$$

   (b) (8 pts) Show that the Lorentz transformation parameters $\boldsymbol{\beta}_{cm}$ and γ_{cm} describing the velocity of the cm frame in the laboratory are

$$\boldsymbol{\beta}_{cm} = \frac{\mathbf{p}_{LAB}}{m_2 + E_{LAB}}, \qquad \gamma_{cm} = \frac{m_2 + E_{LAB}}{W} \tag{3}$$

   (c) (8 pts) Show that the results of parts (a) and (b) reduce in the nonrelativistic limit to the familiar expressions,

$$W \simeq m_1 + m_2 + \left(\frac{m_2}{m_1 + m_2}\right)\frac{p_{LAB}^2}{2m_1} \tag{4}$$

$$\mathbf{p}' \simeq \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{p}_{LAB}, \qquad \boldsymbol{\beta}_{cm} \simeq \frac{\mathbf{p}_{LAB}}{m_1 + m_2} \tag{5}$$

2. **Converting photons to electron and positron:** Consider two photons with different energies that annihilate (in the vacuum) and produce an electron-positron pair. (I.e. a reaction with two photons – in, and electron and positron – out.)

   (a) (8 pts) For what ranges of initial photon energies and angles between their directions of propagation can this reaction take place? (In other words, give a relation, perhaps an inequality, that may contain photon energies, the angle, electron mass, speed of light, etc.)

Problem 1

Clearly this problem uses the convention:

velocity of light. In particle kinematics the symbols,

$$\left.\begin{matrix} p \\ E \\ m \\ v \end{matrix}\right\} \quad \text{stand for} \quad \left\{\begin{matrix} cp \\ E \\ mc^2 \\ \frac{v}{c} \end{matrix}\right.$$

Thus the connection between momentum and total energy is written as $E^2 = p^2 + m^2$, a particle's velocity is $\mathbf{v} = \mathbf{p}/E$, and so on. As energy units, the eV

4 Vector   3 Vector

(a)

$$\underbrace{\begin{aligned} \vec{P}_1 &= (E_{Lab}, \vec{p}_{Lab})^T \\ \vec{P}_2 &= (m_2, \vec{0})^T \end{aligned}}_{\text{In Lab frame}} \quad \begin{aligned} &\longrightarrow \\ &\longrightarrow \end{aligned} \quad \underbrace{\begin{aligned} \vec{P}_1' &= (E_1', \vec{p}')^T \\ \vec{P}_2' &= (E_2', -\vec{p}')^T \end{aligned}}_{\text{in CM frame}}$$

$$\vec{P}_1' + \vec{P}_2' = (E_1' + E_2', \vec{0})^T$$

$$\Rightarrow \underbrace{(\vec{P}_1' + \vec{P}_2')^2}_{\text{scalar}} = (E_1' + E_2')^2 = W^2$$

This is a Lorentz scalar!

$$\Rightarrow W^2 = (\vec{P}_1' + \vec{P}_2')^2 = (\vec{P}_1 + \vec{P}_2)^2$$

$$= \vec{P}_1^2 + \vec{P}_2^2 + 2\vec{P}_1 \cdot \vec{P}_2$$

$$= (m_1^2 + m_2^2 + 2 m_2 E_{Lab}) \cdot c^2$$ in our convention

$$(\vec{P}_1 \cdot \vec{P}_2)^2 = (\vec{P}_1' + \vec{P}_2')^2 \quad \text{idea to find } p'$$

Lorentz scalar

$$(\vec{P}_1 \cdot \vec{P}_2)^2 = (m_2 E_{Lab})^2 = m_2^2 (p_{Lab}^2 + m_1^2) = m_2^2 p_{Lab}^2 + m_1^2 m_2^2$$

$$= (\vec{P}_1' \cdot \vec{P}_2')^2 = (E_1' E_2' + p'^2)^2 = E_1'^2 E_2'^2 + 2 E_1' E_2' p'^2 + p'^4$$

$$= (p'^2 + m_1^2)(p'^2 + m_2^2) + 2 E_1' E_2' p'^2 + p'^4$$

$$= 2p'^4 + (m_1^2 + m_2^2) p'^2 + 2 E_1' E_2' p'^2 + m_1^2 m_2^2$$

$$= p'^2 (2p'^2 + m_1^2 + m_2^2 + 2 E_1' E_2') + m_1^2 m_2^2$$

$$= p'^2 (E_1'^2 + 2 E_1' E_2' + E_2'^2) + m_1^2 m_2^2 = p'^2 W^2 + m_1^2 m_2^2$$

$$\Rightarrow m_2^2 p_{Lab}^2 = p'^2 W^2 \Rightarrow p' = \frac{m_2}{W} p_{Lab}$$

$\vec{p}_{Lab}$ and $\vec{p}'$ both point in the boost direction.

$$\Rightarrow \vec{p}' = \frac{m_2 \vec{p}_{Lab}}{W} \quad \square$$

(b)

$$p' = \gamma_{cm}(p_{Lab} - \beta_{cm} E_{Lab})$$
$$-p' = \gamma_{cm}(0 - \beta_{cm} m_2)$$

add them up $\Rightarrow 0 = p_{Lab} - \beta_{cm}(E_{Lab} + m_2)$

$$\Rightarrow \vec{\beta}_{cm} = \frac{\vec{p}_{Lab}}{m_2 + E_{Lab}}$$

$$\gamma_{cm} = (1 - \beta_{cm}^2)^{-1/2} = \frac{m_2 + E_{Lab}}{((m_2 + E_{Lab})^2 - p_{Lab}^2)^{1/2}}$$

$$= \frac{m_2 + E_{Lab}}{(m_2^2 + 2m_2 E_{Lab} + E_{Lab}^2 - p_{Lab}^2)^{1/2}} = \frac{m_2 + E_{Lab}}{W}$$

(c) Non-relativistically:

$$E_{Lab} \approx m_1 + \frac{p_{Lab}^2}{2m_1}$$

$$\Rightarrow W^2 \approx m_1^2 + m_2^2 + 2m_2\left(m_1 + \frac{p_{Lab}^2}{2m_1}\right)$$

$$= (m_1 + m_2)^2 + \frac{m_2}{m_1} p_{Lab}^2$$

$$= (m_1 + m_2)^2 \left(1 + \frac{m_2}{(m_1+m_2)^2} \frac{p_{Lab}^2}{m_1}\right)$$

$$\Rightarrow W = (m_1 + m_2)\left(1 + \frac{m_2}{(m_1+m_2)^2} \frac{p_{Lab}^2}{m_1}\right)^{1/2}$$

I don't see how that leads to the given result.

(b) (6 pts) Consider now a head-on collision (the angle is π radians) and the photons of the same energy. Calculate the numerical value of the minimal photon energy required for the reaction to take place. Express the answer in SI units (Joules).

3. **Field tensor:** Consider the electromagnetic field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ in the conventions of our course (SI units, $(+,-,-,-)$ metric tensor).

   (a) (15 pts) Starting from the definition in terms of the potential A^μ, *derive* the matrix $(F^{\mu\nu})$ in terms of the components of the electric and magnetic fields, E_x, E_y, E_z, B_x, B_y, B_z.

   (b) (15 pts) Consider a Lorentz boost with relativistic velocity β in the positive x direction. *Derive* the components of the electric and magnetic fields in the moving frame in terms of the corresponding quantities in the original frame by transforming $F^{\mu\nu}$.

   (c) (10 pts) *Derive* the transformation properties of the electric and magnetic fields under parity (space inversion) and time reversal.

4. **To $\vec{B}\vec{E}$ or not to $\vec{B}\vec{E}$:** In a reference frame K there are a constant electric $\vec{E}$ and a magnetic $\vec{B}$ fields such that $\vec{E}\perp\vec{B}$.

   (a) (10 pts) With what velocity a reference frame K' should be moving with respect to K so that in K' there is only electric or only magnetic field ? *Derive* the value of the corresponding field in the K' frame as function of the original fields.

   (b) (10 pts) Does the solution always exist ? Is it unique ?

Problem 2

This problem discusses the Breit-Wheeler process
$\gamma\gamma' \to e^+ e^-$

(a) The total energy of the incoming photons must be at least the total rest mass energy of the two particles.

And the interaction must conserve energy and momentum.

In other words: 4-momentum conservation:

$$(P_\gamma + P_{\gamma'})^2 = (P_{e^+} + P_{e^-})^2 \quad \text{0 for photons}$$

$$\Rightarrow \underbrace{\left(\frac{E_{\gamma 0}}{c}\right)^2}_{0} + 2\left(E_\gamma E_{\gamma'}/c^2 - \vec{p}_\gamma \vec{p}_{\gamma'}\right) + \left(\frac{E_{\gamma' 0}}{c}\right)^2$$

$$= 2m_e^2c^2 + 2\left(E_{e^+}E_{e^-}/c^2 - \vec{p}_{e^+}\vec{p}_{e^-}\right)$$

$$\Leftrightarrow E_\gamma E_{\gamma'}/c^2 - |p_\gamma||p_{\gamma'}|\cos\Theta = m_e^2c^2 + E_{e^+}E_{e^-}/c^2 - \vec{p}_{e^+}\vec{p}_{e^-}$$

Since we just want to know the minimal required conditions, we can look at

$p_{e^-} = p_{e^+} = 0 \Rightarrow E_{e^+} = E_{e^-} = E_{e0} = m_e c^2 \Rightarrow$

Also for photons: $E = cp$

$$\Rightarrow \boxed{E_\gamma E_{\gamma'}(1-\cos\Theta) \geq m_e^2c^4 + m_e^2c^4 = 2E_{e0}^2}$$

(b) Using the result from (a):

$$E_\gamma E_{\gamma'} (1 - \cos\theta) \geq 2 m_e^2 c^4$$

Head-on collision: $\theta = \pi$ and $E_\gamma = E_{\gamma'}$

$\Rightarrow \quad E_\gamma^2 \geq m_e^2 c^4$

$\Rightarrow$ minimal photon energy:

$$E_\gamma \geq m_e c^2 \approx 8.187105065 \cdot 10^{-14} \, J$$

$$\approx 511 \, keV \quad \checkmark$$

What we would expect!

That requires a gamma ray laser to test in the lab.

Problem 3

(a) $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, $A = (\phi, \vec{A})$

$\Rightarrow F^{\alpha\alpha} = 0$, $F^{\alpha\beta} = -F^{\beta\alpha}$

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\nabla\right)$$
$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right)$$

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \underbrace{\frac{\partial}{\partial x_0} A^1}_{= \frac{1}{c}\frac{\partial A_x}{\partial t}} - \underbrace{\frac{\partial}{\partial x_1} \phi}_{= -\partial_x \phi}$$

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \qquad (11.134)$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$= \frac{1}{c}\frac{\partial A_x}{\partial t} + \partial_x \phi = -E_x = -F^{10}$$

$F^{02} = -E_y$, $F^{03} = -E_z$ analogous

So we have: $F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & ? & ? \\ E_y & ? & 0 & ? \\ E_z & ? & ? & 0 \end{pmatrix}$

Comment: might be E_i/c depending on the convention.

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\frac{\partial}{\partial x} A_y + \frac{\partial}{\partial y} A_x = -B_z$$
$$F^{13} = \partial^1 A^3 - \partial^3 A^1 = -\frac{\partial}{\partial x} A_z + \frac{\partial}{\partial z} A_x = B_y$$
$$F^{23} = \partial^2 A^3 - \partial^3 A^2 = -\frac{\partial}{\partial y} A_z + \frac{\partial}{\partial z} A_y = -B_x$$

So: $F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$

(b)

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad F' = \Lambda F \Lambda \iff F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}$$

$$= \begin{pmatrix} 0 & -E_x' & -E_y' & -E_z' \\ E_x' & 0 & -B_z' & B_y' \\ E_y' & B_z' & 0 & -B_x' \\ E_z' & -B_y' & B_x' & 0 \end{pmatrix}$$

With: $F'^{01} = \Lambda^0{}_0 F^{01} \Lambda_1{}^1 + \Lambda^0{}_1 F^{10} \Lambda_1{}^1$

$$\iff -E_x' = -\gamma^2 E_x - \gamma^2\beta^2 E_x$$

$$E_x' = (\gamma^2 + \gamma^2\beta^2) E_x = E_x$$

$$E_y' = \gamma(E_y - \beta B_z)$$

$$E_z' = \gamma(E_z + \beta B_y)$$

$$B_x' = B_x$$

$$B_y' = \gamma(B_y + \beta E_z)$$

$$B_z' = \gamma(B_z - \beta E_y)$$

(c) $$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \qquad (1)$$

Under T we get $\frac{d\vec{r}}{dt} = \vec{a} \xrightarrow{t\to -t} \vec{a}$ $\Rightarrow \left(\frac{d\vec{r}}{dt}\right)' = q\left(\vec{E}' - \vec{v}\times\vec{B}'\right)$

$\vec{v} \xrightarrow{t\to -t} -\vec{v}$

we want (1) to be invariant $\Rightarrow$ $\boxed{\begin{array}{l} t\to -t: \\ \vec{B} \to -\vec{B} \\ \vec{E} \to \vec{E} \end{array}}$

Under P we get the opposite:

$\vec{a} \xrightarrow{\vec{r}\to -\vec{r}} -\vec{a}$

$\vec{v} \to -\vec{v}$

$\Rightarrow -\left(\frac{d\vec{r}}{dt}\right)' = q\left(\vec{E}' - v\times\vec{B}'\right)$

$\Rightarrow$ $\boxed{\begin{array}{l} \vec{r}\to -\vec{r}: \\ \vec{B} \to \vec{B} \\ \vec{E} \to -\vec{E} \end{array}}$

Problem 4

(a) Let's choose $\vec{E} := E\vec{e}_y$ and the boost direction to be $\vec{e}_x$. This makes things easier while still being general as long as $\vec{B} \perp \vec{E}$, so we can point anywhere in the xz-plane.

so: $\vec{B} := B(\cos\varphi\, \vec{e}_x + \sin\varphi\, \vec{e}_z)$

Using that:

$$E'_y = \gamma(E - \beta B_z)$$
$$= \gamma E - \gamma\beta B \sin\varphi$$

$$E'_z = \gamma(E_z + \beta B_y) = 0$$

$$E'_x = E_x = 0$$

$$B'_x = B_x = B\cos\varphi$$

$$B'_y = \gamma(B_y + \beta E_z) = 0$$

$$B'_z = \gamma(B_z - \beta E_y)$$
$$= \gamma\beta \sin\varphi - \gamma\beta E$$

So first, if we want $E' = 0$:

$$E = \beta B \sin\varphi \Rightarrow \boxed{v = \frac{cE}{B\sin\varphi}}, \quad \varphi \neq 0$$

B can point anywhere except parallel to the boost direction.

If we want B' to be 0 we do the same thing the other way around:

$$\text{Let } \vec{B} := B\vec{e}_y, \quad \vec{E} := E(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_z)$$

Then we set:

$$E_x' = E_x = E\cos\phi \quad ; \quad B_x' = B_x = 0$$

$$E_z' = \gamma(E_z + \beta B_y) = \gamma\beta B \quad ; \quad E_y' = \gamma(E_y - \beta B_z) = 0$$

$$B_z' = \gamma(B_z - \beta E_y) = 0$$

$$B_y' = \gamma(B_y + \beta E_z) = \gamma(B + \beta E\sin\phi) \doteq 0$$

$$\Rightarrow \boxed{V = -\frac{Bc}{E\sin\phi}}$$

(b) Both: no.

It is not unique as evident by (a).

V depends on E and B and its direction.

It also does not always exist, e.g. if

$\frac{E}{B\sin\varphi} > 1$ then $V > c$ which is impossible.

(Or $\frac{B}{E\sin\phi} > 1$ in the other case)