Problems

1. **A strip with current:**
   A straight and infinitely long strip of width $2a$ carries a current $I$ which is uniformly distributed across the width of the strip. The strip is positioned in the $x = 0$ plane between $y = -a$ and $y = a$, the current is in the $z$ direction.

   (a) (25 pts) Find the magnetic field $\vec{B}$ at an arbitrary point $\vec{r} = (x, y, z)$.

   (b) (10 pts) To check your result consider the limiting case of large distances from the strip.

2. **A rotating sphere:**
   A sphere of radius $a$ carries a uniform surface-charge distribution $\sigma$. The sphere is rotated about a diameter with constant angular velocity $\omega$.

   (a) (35 pts) Find the vector potential $\vec{A}$ and the magnetic field $\vec{B}$ inside and outside the sphere.

3. **Current flow over a sphere:**
   A current $I$ starts at $z = -\infty$ and flows up the $z$-axis as a linear filament until it hits an origin-centered sphere of radius $R$. The current spreads uniformly over the surface of the sphere and flows up lines of longitude from the south pole to the north pole. The recombined current flows thereafter as a linear filament up the $z$-axis to $z = +\infty$.

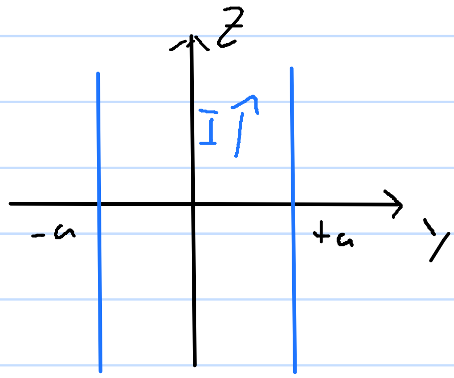
   (a) (5 pts) Find the current density on the sphere.

   (b) (20 pts) Use explicitly stated symmetry arguments and Ampere's law in integral form to find the magnetic field at every point in space.

   (c) (5 pts) Check that your solution satisfies the magnetic field matching conditions at the surface of the sphere.

# Problem 1

(a)




Biot Savart:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\vec{j}(\vec{r}) = \frac{I}{2a}\,\delta(x)\,\Theta(a-|y|)\,\vec{e}_z$$

$\vec{r}'$ on strip

$$\vec{j}\times(\vec{r}-\vec{r}') = \left[-(y-y')\vec{e}_x + x\,\vec{e}_y\right]\frac{I}{2a}$$

$$\text{So} \quad \vec{B}(\vec{r}) = \underbrace{\frac{\mu_0}{4\pi}\frac{I}{2a}\int_{-\infty}^{+\infty}\int_{-a}^{a}\frac{-(y-y')\vec{e}_x + x\vec{e}_y}{\left[x^2+(y-y')^2+(z-z')^2\right]^{3/2}}\,dy'\,dz'}_{\text{Mathematica}}$$

Mathematica

$$\boxed{\vec{B} = \frac{\mu_0 I}{4\pi a}\left[\left(\arctan\frac{a-y}{x} + \arctan\frac{a+y}{x}\right)\vec{e}_y + \left(\frac{1}{2}\ln\left(\frac{x^2+(y-a)^2}{x^2+(y+a)^2}\right)\right)\vec{e}_x\right]}$$

Can also be done by hand substituting $x^2+(y-y')^2$ and $z-z'$ and wasting a lot of time.

(b)

For large distances: $a \to 0$

so $\ln \frac{x^2+(y-a)^2}{x^2+(y+a)^2} \xrightarrow{a \to 0} 0$

and $\arctan \frac{a-y}{x} + \arctan \frac{a+y}{x} \xrightarrow{a \to 0} \arctan\left(\frac{-y}{x}\right) - \arctan\left(\frac{y}{x}\right)$
$= 0$

So $\lim\limits_{a \to 0} \vec{B}$ is case "$\frac{0}{0}$" where we can apply L'Hospital's rule

derivative of the denominator is simply $4\pi$

derivative of the numerator is:

$$\left[\frac{x}{x^2+(a-y)^2} + \frac{x}{x^2+(a+y)^2}\right]\vec{e}_y - \frac{1}{2}\left[\frac{2a-2y}{x^2+(y-a)^2} - \frac{2a+2y}{x^2+(y+a)^2}\right]\vec{e}_x$$

$$\Rightarrow \lim_{a \to 0} \vec{B} = \frac{\mu_0 I}{4\pi}\left(\frac{2x}{x^2+y^2}\vec{e}_y - \frac{2y}{x^2+y^2}\vec{e}_x\right)$$

$$= \frac{\mu_0 I}{4\pi} \cdot \frac{2}{r}\underbrace{\left(\cos\varphi\, \vec{e}_y - \sin\varphi\, \vec{e}_x\right)}_{\vec{e}_\varphi} = \boxed{\frac{\mu_0 I}{2\pi r}\vec{e}_\varphi}$$

polar coordinates
$r^2 = x^2+y^2$
$x = r\cos\varphi$
$y = r\sin\varphi$

As expected: The result for an infinite wire!

# Problem 2

$$\rho(\vec{r}) = \sigma\,\delta(|\vec{r}| - a)$$

$$\vec{j} = \rho\vec{v} = \rho\cdot(\vec{\omega}\times\vec{r}) = \sigma(\vec{\omega}\times\vec{r})\,\delta(|\vec{r}| - a)$$

Re-assembling the components (10.75) into a single vector gives the vector potential in the Coulomb gauge as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \qquad (10.76)$$

Zangwill

So here:

$$\vec{A}(\vec{r}) = \frac{\mu_0\sigma a^3}{4\pi}\,\vec{\omega}\times\underbrace{\int d\Omega'\,\frac{\vec{e}_{r'}}{|\vec{r}-\vec{r}'|}}$$

So: $\displaystyle\int d\Omega'\,\frac{\vec{e}_{r'}}{|\vec{r}-\vec{r}'|}$

$= \underbrace{I(r)}_{\text{just depends on } |\vec{r}|}\,\vec{e}_r$ $\Leftarrow$

We can choose the coordinate system for the integration freely. E.g.: $\vec{r}$ aligned along the z axis. Then it is obvious that the result must also point in that direction!

Taking the dot product on both sides with $\vec{e}_r$ yields:

$$I(r) = \int d\Omega'\,\frac{\cos\theta}{|\vec{r}-\vec{r}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}}\int d\Omega'\,P_1(\cos\theta)P_l(\cos\theta)$$

with $\theta = \sphericalangle(\vec{r},\vec{r}')$ and $r_< = \min(r,a)$, $r_> = \max(r,a)$

Since Legendre polynomials are orthogonal, only the $l=1$ term will be non-zero.

So : $I = \frac{4}{3}\pi \frac{r_<}{r_>^2}$

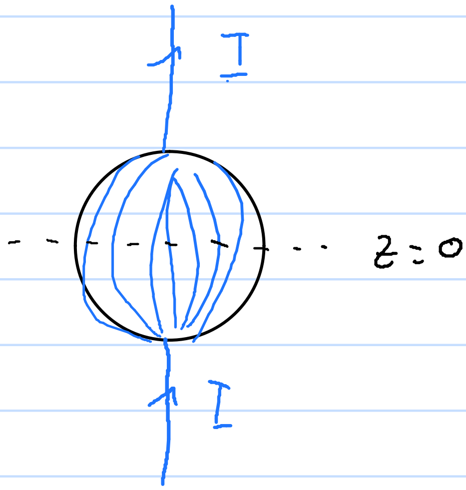
$$\Rightarrow \boxed{\begin{aligned}\vec{A}(\vec{r}) &= \frac{\mu_0 \sigma a^3}{3r} \frac{r_<}{r_>^2} (\vec{\omega} \times \vec{r}) \\ &= \begin{cases} \frac{\mu_0 \sigma a}{3} \vec{\omega} \times \vec{r}, & \text{inside} \\ \frac{\mu_0 \sigma a^4}{3r^3} \vec{\omega} \times \vec{r}, & \text{outside} \end{cases}\end{aligned}}$$

Then: $\vec{B} = \text{rot}\, A$

$$\boxed{\begin{aligned}&\text{Inside: } \vec{B} = \frac{\mu_0 \sigma a}{3} \text{rot}(\vec{\omega} \times \vec{r}) = \frac{2}{3} \mu_0 \sigma a \vec{\omega} \\ &\text{Outside: } \vec{B} = \frac{\mu_0 \sigma a^4}{3} \text{rot}\left(\frac{\vec{\omega} \times \vec{r}}{r^3}\right) \\ &\qquad = \frac{\mu_0 \sigma a^4}{3r^3} \left[3\vec{e}_r(\vec{\omega} \cdot \vec{e}_r) - \vec{\omega}\right]\end{aligned}}$$

## Problem 3

(a)




$$I = \int_C d\vec{s} \cdot \vec{K} \times \vec{e}_n$$

where $\vec{K}$ is the surface charge density and $C$ is any curve on the surface.

The intersections with the sphere and any z-plane are circles for which

$$I = \oint_C d\vec{s} \cdot \vec{K} \times \vec{e}_n$$

must hold. In that case:

because the current flows from $-z$ to $+z$.

$$d\vec{s} = R \sin\theta \, d\varphi \, \vec{e}_\varphi \quad \text{and} \quad \vec{K} = -K(\theta)\,\vec{e}_\theta$$

and $\vec{e}_n = \vec{e}_r$ on the sphere.

So: $I \overset{z=0 \text{ plane}}{=} 2\pi R \sin\theta \, K(\theta)$

$$\Rightarrow \boxed{\begin{aligned} \vec{K} &= -\frac{I}{2\pi R \sin\theta}\,\vec{e}_\theta \\ \vec{J} &= \vec{K}\,\delta(R-r) \end{aligned}}$$

(b) The problem has azimuthal symmetry.
So we can write $\vec{J}$ as superposition of a z-term and a cylindric-radial-term:

$$\vec{J} = j_\rho(\rho, z)\,\vec{e}_\rho + j_z(\rho, z)\,\vec{e}_z$$

For $z > R$ and $z < R$: $\vec{J} = I\,\vec{e}_z$

Maxwell says: $\mathrm{rot}\,\vec{B} = \mu_0 \vec{J}$ and we know from symmetry that $\partial_\varphi \vec{B} = 0$

$$\Rightarrow \quad \mathrm{rot}\,\vec{B} = -\partial_z B_\varphi \vec{e}_\rho + \rho^{-1}\partial_\rho(\rho B_\varphi)\vec{e}_z$$

$$\Rightarrow \quad \vec{B} = B_\varphi\,\vec{e}_\varphi$$ as one might expect.

For $|z| > R$ we therefore simply set the result of the infinit wire $\vec{B} = \frac{\mu_0 I}{2\pi\rho}\vec{e}_\varphi$

Inside the sphere:

$$\oint_c d\vec{r}\,\vec{B} = 2\pi\rho B_\varphi = 0 \Rightarrow B = 0$$

Outside the sphere:

$$\oint_c d\vec{r}\,\vec{B} = 2\pi\rho B_\varphi = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi\rho}$$

So all combined we have:

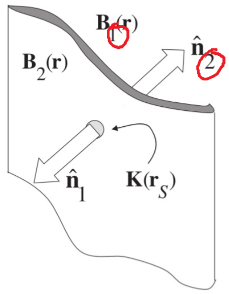
$$\boxed{\vec{B} = \begin{cases} 0, & \text{inside} \\ \frac{\mu_0 I}{2\pi\rho}\vec{e}_\varphi, & \text{outside} \end{cases}}$$

(c)

Subtracting the two equations in (10.36) gives $\mathbf{B}_1 - \mathbf{B}_2 = \mu_0 \mathbf{K}(\mathbf{r}_S) \times \hat{\mathbf{n}}_2$. Taking the dot product and cross product of this equation with $\hat{\mathbf{n}}_2$ produces the matching conditions

$$\hat{\mathbf{n}}_2 \cdot [\mathbf{B}_1 - \mathbf{B}_2] = 0 \tag{10.37}$$

$$\hat{\mathbf{n}}_2 \times [\mathbf{B}_1 - \mathbf{B}_2] = \mu_0 \mathbf{K}(\mathbf{r}_S). \tag{10.38}$$




why?!

**Figure 10.8:** A surface that carries an areal current density $\mathbf{K}(\mathbf{r}_S)$. The unit normal vectors $\hat{\mathbf{n}}_k$ points outward from region $k$.

On the sphere: $\hat{n}_2 = \vec{e}_r$, $\vec{B}_1 = B_{out} = \frac{\mu_0 I}{2\pi \sin\theta R} \vec{e}_\varphi$,

$\hat{n}_1 = -\vec{e}_r$, $B_2 = B_{in} = 0$

So: $\hat{n}_2 [\vec{B}_1 - \vec{B}_2] = \underbrace{\vec{e}_r \cdot \vec{e}_\varphi}_{\text{allways } 0} \cdot \frac{\mu_0 I}{2\pi R} = 0$ ✓

$$\hat{n}_2 \times [\vec{B}_1 - \vec{B}_2] = \vec{e}_r \times \frac{\mu_0 I}{2\pi R \sin\theta} \vec{e}_\varphi$$

$$= -\frac{\mu_0 I}{2\pi R \sin\theta} \vec{e}_\theta \overset{(a)}{=} \mu_0 \vec{K}$$ ✓