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Physics 841 - Homework 1 **due Fri., Jan. 17, 2020**

Problems

1. **Space travel:** You are the head of an expedition to some planet X orbiting the closest star to us, Proxima Centauri. You know that the distance to X from Earth is 4.22 light-years and that the spaceship is going to travel with constant speed $v = \sqrt{0.9999}c$.

   (a) (5 pts) How long the travel is going to take from the perspective of an observer staying on Earth ?

   (b) (10 pts) You suspect that for an observer on the spaceship the clock is going to run differently. You want to know for sure, so you could take an adequate amount of supplies for the trip, but not more than necessary, since the storage space in the spaceship is limited. How long the travel is going to take from the perspective of an observer on the spaceship ?

2. **Alien rocket ship race:** As you safely arrive on planet X after the space trip and start studying everyday life of the alien population there, you attend a rocket ship race. There, among other things, you witness the following event. Rocket ships A and B move on the space track in the same direction on a straight line separated by distance $l = 2.26829 \cdot 10^8$ m with a speed $\frac{40}{41}c$. Ship A is leading. At the moment when ship A is at point a and ship B at point b, ship A breaks down and instantly slows down to speed $\frac{9}{41}c$, while ship B maintains its original speed. After a short moment ship B catches up with ship A and they collide. Thinking about this event you wonder about the following (for simplicity, take $c = 3 \cdot 10^8$ m/s):

   (a) (8 pts) In your reference frame, how much time passed from the moment of break-down to the moment the rockets collided ?

   (b) (11 pts) How fast was rocket ship B approaching in A's frame? How fast was A approaching in B's frame ?

   (c) (11 pts) You wonder if the crew of either ship could avoid the collision. How much time elapsed from the moment when ship A was at point a to the collision from the perspective of ship A ? How much time elapsed from the moment when ship B was at point b to the collision from the perspective of ship B ?

3. **Lorentz transformations:** As you explore the planet X and see that the laws of physics work there exactly in the same way as they do on Earth, you run into alien graduate students who are surprised by your great expertise and ask you to help them with the following problem. A reference frame K' is moving with respect to a frame K with velocity V along the x axis. The axes of K' are parallel to the axes of K. A clock at rest in K' at point (x'_0, y'_0, z'_0) shows time t'_0 at the moment when it passes by a clock which is at rest in the frame K at point (x_0, y_0, z_0) and shows time t_0.

   (a) (15 pts) Write down the Lorentz transformations that relate the space and time coordinates between the two frames.

Problem 1

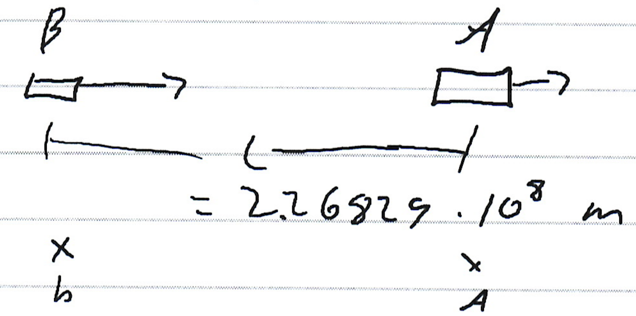
a) $t' = vd = 4.22 \frac{a}{c} \cdot \sqrt{0.9999}\, c \simeq 4.2198\ a$

b) $t = \frac{t'}{\gamma} = vd \cdot \underbrace{\sqrt{1-0.9999}}_{=0.01} \approx 0.042198\ a$

$\approx 15\,d\ 9\,h\ 36\,m$

Problem 2

$V_b = \frac{40}{41} c \qquad V_a = 9/41\, c$



$c = 3 \cdot 10^8 \frac{m}{s}$

a) $\Delta V = V_b - V_a = \frac{31}{41} c$

$\Rightarrow \Delta t = \frac{L}{\Delta V} = 2.26829 \cdot \cancel{10^8}\, \cancel{m} \cdot \frac{41}{31} \cdot \frac{1}{3} \cdot \cancel{10^{-8}} \frac{s}{\cancel{m}}$

$= 2.26829 \cdot \frac{41}{93}\ s$

$\approx 1\ s$

b)

Velocity addition : $u' = \frac{u - v}{1 - \frac{vu}{c^2}}$

So : $\Delta V_a = \frac{V_b - V_a}{1 - \frac{V_b V_a}{c^2}} = \frac{31/41}{1 - \frac{360}{1681}} c \approx 0.9621\ c$

$\Delta V_b = - \Delta V_a$

c) $\gamma_a = \left(1 - \left(\frac{9}{41}\right)^2\right)^{-1/2} = 1 + 1/40 \Rightarrow \Delta t_a = \gamma_a^{-1} \Delta t$

$\approx 0.9756\ s$

$\gamma_b = \left(1 - \left(\frac{40}{41}\right)^2\right)^{-1/2} = 4 + \frac{5}{9} \Rightarrow \Delta t_b = \gamma_b^{-1} \Delta t \simeq 0.2195\ s$

3. **Lorentz transformations:** As you explore the planet X and see that the laws of physics work there exactly in the same way as they do on Earth, you run into alien graduate students who are surprised by your great expertise and ask you to help them with the following problem. A reference frame K' is moving with respect to a frame K with velocity V along the x axis. The axes of K' are parallel to the axes of K. A clock at rest in K' at point (x'_0, y'_0, z'_0) shows time t'_0 at the moment when it passes by a clock which is at rest in the frame K at point (x_0, y_0, z_0) and shows time t_0. meaning they align

(a) (15 pts) Write down the Lorentz transformations that relate the space and time coordinates between the two frames.

100/100

Idea: First perform a translation into systems $\tilde{K}$ and $\tilde{K}'$, such that the origin (spacial and temporal) overline. Then the normal Lorentz transform applies between $\tilde{K}$ and $\tilde{K}'$. Then just translate back:

$\tilde{x} = x - x_0$; $\tilde{x}' = x' - x_0$

$\tilde{y} = y - y_0$

$\tilde{z} = z - z_0$

$\tilde{t} = t - t_0$; and so on

$$\Rightarrow \boxed{\begin{aligned} x' &= \gamma[(x-x_0) - \beta c(t-t_0)] + x_0' \\ y' &= y - y_0 + y_0' \\ z' &= z - z_0 + z_0' \\ t' &= \gamma[(t-t_0) - \beta/c\,(x-x_0)] + t_0' \end{aligned}}$$

4. **Boost in arbitrary direction:** Later on, you run into a mean alien professor who doubts that the Earth's civilization possesses enough knowledge of special relativity to make a trip from Earth to planet X on its own. He gives you the following problem as a test. Given a four vector $x^\mu = (1, 3, 2, 4)$ m (with $x^0 = ct$) in a frame K, consider how it is observed in a frame K' which moves with velocity $\vec{v}_0 = (0.1, 0.7, 0.3)c$ with respect to K.

   (a) (20 pts) Calculate the four vector x'^μ observed in the frame K'. First, derive the symbolic result for general $x^\mu = (x^0, \vec{x})$ and $\vec{v}_0$ from the formula for a boost along a coordinate axis. Then plug in the specific values above.

   (b) (10 pts) Determine the rapidity of the Lorentz boost from K to K'.

   (c) (10 pts) Is the four vector time-like, space-like or light-like ?

Problem 4

a)

x_1 axis. If the axes in K and K' remain parallel, but the velocity $\mathbf{v}$ of the frame K' in frame K is in an arbitrary direction, the generalization of (11.16) is

$$\left.\begin{aligned} x_0' &= \gamma(x_0 - \boldsymbol{\beta}\cdot\mathbf{x}) \\ \mathbf{x}' &= \mathbf{x} + \frac{(\gamma-1)}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x_0 \end{aligned}\right\} \tag{11.19}$$

Derivation:

The first equation here follows almost trivially from the first equation in (11.16). The second appears somewhat complicated, but is really only the sorting out of components of $\mathbf{x}$ and $\mathbf{x}'$ parallel and perpendicular to $\mathbf{v}$ for separate treatment in accord with (11.16).

The connection between β and γ given in (11.17) and the ranges $0 \leq \beta \leq 1$, $1 \leq \gamma \leq \infty$ allow the alternative parametrization,

$$\left.\begin{aligned} \beta &= \tanh\zeta \\ \gamma &= \cosh\zeta \\ \gamma\beta &= \sinh\zeta \end{aligned}\right\} \tag{11.20}$$

and so

where ζ is known as the *boost parameter* or *rapidity*. In terms of ζ the first two equations of (11.16) become

$$\begin{aligned} x_0' &= x_0\cosh\zeta - x_1\sinh\zeta \\ x_1' &= -x_0\sinh\zeta + x_1\cosh\zeta \end{aligned} \tag{11.21}$$

The structure of these equations is reminiscent of a rotation of coordinates, but with hyperbolic functions instead of circular, basically because of the relative negative sign between the space and time terms in (11.14) [see Section 11.7 and (11.95)].

$$\vec{\beta} = \frac{\vec{v}_0}{c} = (0.1,\ 0.7,\ 0.3)^T \quad , \quad x^\mu = (1, 3, 2, 4)^T \ \mathrm{m}$$

$$\Rightarrow s^2 = 28$$

$$\vec{\beta}^T \cdot \vec{x} = (0.1,\ 0.7,\ 0.3)\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \frac{29}{10}\ \mathrm{m}$$

$$\gamma = (1-\beta^2)^{-\frac{1}{2}} = \left(1 - \frac{59}{100}\right)^{-\frac{1}{2}} = \frac{10}{\sqrt{41}}$$

$$\boxed{x_0' = \gamma(x_0 - \vec{\beta}^T\vec{x}) = \frac{10}{\sqrt{41}}\left(1\,\mathrm{m} - \frac{29}{10}\,\mathrm{m}\right) = -\frac{19\sqrt{41}}{41}}$$

$$\vec{x}' = \vec{x} + \underbrace{\frac{\gamma - 1}{\beta^2}}_{\frac{\frac{10}{\sqrt{41}} - 1}{\frac{59}{100}}} \underbrace{(\vec{\beta}^T \vec{x})}_{\frac{29}{10} m} \vec{\beta} - \gamma \vec{\beta} x_0$$

$\vec{x} \to \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $\vec{\beta} \to (0.1, 0.7, 0.3)^T$, $\gamma \to \frac{10}{\sqrt{41}}$, $x_0 \to 1\,m$

$$\approx \left[\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + 2.761053211 \cdot \begin{pmatrix} 0.1 \\ 0.7 \\ 0.3 \end{pmatrix} - \frac{10}{\sqrt{41}} \begin{pmatrix} 0.1 \\ 0.7 \\ 0.3 \end{pmatrix} \right] m$$

$$\approx \begin{pmatrix} 3.119934959 \\ 2.839541914 \\ 4.359803678 \end{pmatrix} m$$

probably wrong // what's the point of this assignment?

$\Rightarrow s^2 \approx 27.9999$ ✓

check: close enough! (should be conserved)

b) $\cosh^{-1}\left(\frac{10}{\sqrt{41}}\right) \approx 1.015712827 = \zeta$

$\gamma \beta = \frac{10}{\sqrt{41}} \cdot \frac{\sqrt{59}}{10} \approx 1.1995342\,2$

$\Rightarrow \sinh^{-1}(\gamma\beta) \approx 1.015712827 = \zeta$ ✓

c) Since our metric is diag$(-1, +1, +1, +1)$:

$s^2 = -(ct)^2 + x^2 + y^2 + z^2$

$\Rightarrow s^2 = 28$, spacelike.