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Physics 841 - Homework 2 **due Fri., Jan. 24, 2020**

Problems

1. **Lorentz transformations:** Let $\Lambda = (\Lambda^\mu{}_\nu)$ be the 4x4 matrix of a Lorentz boost in the x direction and $g = (g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ the $4x4$ matrix of the metric tensor.

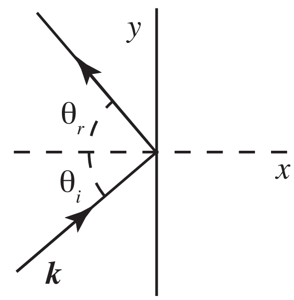
   (a) (5 pts) Show that the Lorentz transformation fulfills $\Lambda^T g \Lambda = g$ and consequently leaves the scalar product of two four-vectors invariant.

   (b) (5 pts) If x^μ is a contravariant vector, what kind of object is $\frac{\partial}{\partial x^\mu}$? Show by studying its transformation properties.

   (c) (10 pts) Show that two Lorentz boosts *both in the x direction* with rapidities ζ_1 and ζ_2 are equivalent to a single boost with rapidity ζ_3. Derive the value of ζ_3 in terms of ζ_1 and ζ_2.

2. **Doppler effect and aberration of light:**

   A light source emits light of frequency ω_S with a wave vector $\mathbf{k}$ in the xy plane, where $|\mathbf{k}| = \omega_S/c$. The light is reflected from a plane mirror parallel to the yz plane. The angle of incidence θ_i and the angle of reflection θ_r are defined with respect to the normal to the mirror, as shown in the figure. Now consider the entire device (both the source and the mirror) in motion with relativistic velocity $\beta = v/c$ in the positive x-direction, with respect to the laboratory. Predict the results of the measurements made in the laboratory for:



   (a) (20 pts) the frequencies of the incident and reflected waves (expressed in terms of ω_S, β, and the angle of incidence and reflection θ_S in the device frame),

   (b) (20 pts) the cosine of the angle of incidence (expressed in terms of $\cos\theta_S$ and β),[1]

   (c) (10 pts) the relation between angle of incidence and angle of reflection (both in the lab frame).

3. **Lorentz transformations for acceleration, Jackson, 11.5 (30 pts):** A coordinate system K' moves with a velocity $\mathbf{v}$ relative to another system K. In K' a particle has a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$. Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\mathbf{v}$ are

$$\mathbf{a}_\parallel = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3} \mathbf{a}'_\parallel, \quad \mathbf{a}_\perp = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_\perp + \frac{\mathbf{v}}{c^2} \times (\mathbf{a}' \times \mathbf{u}')\right). \tag{1}$$

[1] The change in direction of light between two different inertial frames is known as the aberration of light. It also occurs classically, but the relativistic formula gives more pronounced effects at large v/c.

Problem 1

(a)
$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda^T$$

$$\Lambda^T g \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ \gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\gamma^2\beta^2 - \gamma^2 = \gamma^2(\beta^2 - 1)$
$= -1$

$\gamma^2 - \gamma^2\beta^2 = 1$

$$\Rightarrow s^2 = x^T g x = x^T \Lambda^T g \Lambda x = x'^T g x' \quad \square$$

Jackson p. 535–536

(b)

Consider now the partial derivative operators with respect to x^α and x_α. The transformation properties of these operators can be established directly by using the rules of implicit differentiation. For example, we have

$$\frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial}{\partial x^\beta}$$

We will henceforth employ this *summation convention* for repeated indices. A *covariant vector* or tensor of rank one B_α is defined by the rule,

$$B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B_\beta \qquad (11.62)$$

p. 534

Comparison with (11.62) shows that *differentiation with respect to a contravariant component* of the coordinate vector transforms as the component of a *covariant vector operator.* From (11.72) it follows that differentiation with respect to a covariant component gives a contravariant vector operator. We therefore employ the notation,

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\nabla\right)$$
$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \nabla\right) \qquad (11.76)$$

(c)

$$A = \begin{pmatrix} \cosh\zeta & -\sinh\zeta & 0 & 0 \\ -\sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A(\zeta)$$

$$\Rightarrow \quad A_1 := A(\zeta_1), \quad A_2 := A(\zeta_2)$$

$$\Rightarrow \quad A_3 = A_2 A_1$$

$$= \begin{pmatrix} \cosh\zeta_1\cosh\zeta_2 + \sinh\zeta_1\sinh\zeta_2 & -\cosh\zeta_2\sinh\zeta_1 - \sinh\zeta_2\cosh\zeta_1 & 0 & 0 \\ -\sinh\zeta_2\cdot\cosh\zeta_1 - \cosh\zeta_2\sinh\zeta_1 & \sinh\zeta_2\sinh\zeta_1 + \cosh\zeta_2\cosh\zeta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{!}{=} \quad A(\zeta_3)$$

hyperbolic identities

$$\Rightarrow \quad \cosh\zeta_3 = \cosh\zeta_1\cosh\zeta_2 + \sinh\zeta_1\sinh\zeta_2 = \cosh(\zeta_1 + \zeta_2)$$

and $\sinh\zeta_3 = \sinh\zeta_2\cdot\cosh\zeta_1 + \cosh\zeta_2\sinh\zeta_1 = \sinh(\zeta_1 + \zeta_2)$

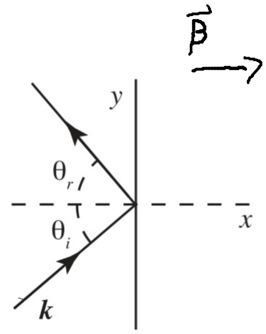
$$\Rightarrow \quad \boxed{\zeta_3 = \zeta_1 + \zeta_2}$$, surprise surprise! □

It's just concatination of rotational matrices.

Problem 2

2. **Doppler effect and aberration of light:**

A light source emits light of frequency ω_S with a wave vector $\mathbf{k}$ in the xy plane, where $|\mathbf{k}| = \omega_S/c$. The light is reflected from a plane mirror parallel to the yz plane. The angle of incidence θ_i and the angle of reflection θ_r are defined with respect to the normal to the mirror, as shown in the figure. Now consider the entire device (both the source and the mirror) in motion with relativistic velocity $\beta = v/c$ in the positive x-direction, with respect to the laboratory. Predict the results of the measurements made in the laboratory for:



(a) (20 pts) the frequencies of the incident and reflected waves (expressed in terms of ω_S, β, and the angle of incidence and reflection θ_S in the device frame), $\theta_S = \theta_i = \theta_r$ in device frame

(b) (20 pts) the cosine of the angle of incidence (expressed in terms of $\cos\theta_S$ and β),[1]

(c) (10 pts) the relation between angle of incidence and angle of reflection (both in the lab frame).

From Jackson p. 521–522!

relativistic Doppler shift. Consider a plane wave of frequency ω and wave vector $\mathbf{k}$ in the inertial frame K. In the moving frame K' this wave will have, in general, a different frequency ω' and wave vector $\mathbf{k}'$, but the phase of the wave is an invariant:

$$\phi = \omega t - \mathbf{k}\cdot\mathbf{x} = \omega' t' - \mathbf{k}'\cdot\mathbf{x}' \tag{11.28}$$

Lorentz transform

[Parenthetically we remark that because the equations of (11.16) are linear the plane wave in K with phase ϕ indeed remains a plane wave in frame K'.] Using (11.16) and the same arguments as we did in going from (11.7) to (11.8), we find that the frequency $\omega' = ck_0'$ and wave vector $\mathbf{k}'$ are given in terms of $\omega = ck_0$ and $\mathbf{k}$ by

Galilean Doppler shift

$$\left.\begin{aligned} k_0' &= \gamma(k_0 - \boldsymbol{\beta}\cdot\mathbf{k}) \\ k_\parallel' &= \gamma(k_\parallel - \beta k_0) \\ \mathbf{k}_\perp' &= \mathbf{k}_\perp \end{aligned}\right\} \tag{11.29}$$

Derivation

The Lorentz transformation of $(k_0, \mathbf{k})$ has exactly the same form as for $(x_0, \mathbf{x})$. *The frequency and wave number of any plane wave thus form a 4-vector.* The invariance (11.28) of the phase is the invariance of the "scalar product" of two 4-vectors (11.24). This correspondence is, in fact, an alternate path from (11.28) to the transformation law (11.29).

For light waves, $|\mathbf{k}| = k_0$, $|\mathbf{k}'| = k_0'$. Then the results (11.29) can be expressed in the more familiar form of the Doppler shift formulas

$\Rightarrow$

$$\begin{aligned} \omega' &= \gamma\omega(1 - \beta\cos\theta) \\ \tan\theta' &= \frac{\sin\theta}{\gamma(\cos\theta - \beta)} \end{aligned} \tag{11.30}$$

where θ and θ' are the angles of $\mathbf{k}$ and $\mathbf{k}'$ relative to the direction of $\mathbf{v}$. The inverse equations are obtained by interchanging primed and unprimed quantities and reversing the sign of β.

Here:

$$k_{y,i} = k_{y,r}, \quad k_{z,i} = k_{z,r}, \quad k_{x,i} = -k_{x,r}.$$

We just need to sub in the angles into the inner product:

$$\vec{\beta}\,\vec{k}_i = \beta k \cos\theta_s,$$

$$\vec{\beta}\,\vec{k}_r = \beta k \cos(\pi - \theta_s) = -\beta k \cos\theta_s$$

So, using (11.30) we see:

(a) $$\omega_i = \gamma\omega_s(1 - \beta\cos\theta_s)$$

In Lab frame $$\omega_r = \gamma\omega_s(1 + \beta\cos\theta_s)$$

and

(b) $$\tan\theta_i = \frac{\sin\theta_s}{\gamma(\cos\theta_s - \beta)}$$

in Lab frame

$$\Rightarrow \cos\theta_i = \frac{\cos\theta_s + \beta}{1 + \beta\cos\theta_s}$$ Derivation in lecture

and $$\cos\theta_r = \frac{\cos(\pi - \theta_s) + \beta}{1 - \beta\cos(\pi - \theta_s)} = \frac{\beta - \cos\theta_s}{1 + \beta\cos\theta_s}$$

Yeah I don't know what expression is asked for.

$$\frac{\cos\theta_i}{\cos\theta_r}\ ?$$

Problem 3

3. **Lorentz transformations for acceleration, Jackson, 11.5 (30 pts):** A coordinate system K' moves with a velocity $\mathbf{v}$ relative to another system K. In K' a particle has a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$. Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\mathbf{v}$ are

$$\mathbf{a}_\parallel = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_\parallel, \quad \mathbf{a}_\perp = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left(\mathbf{a}'_\perp + \frac{\mathbf{v}}{c^2}\times(\mathbf{a}'\times\mathbf{u}')\right). \tag{1}$$

The choice of coordinate axes is arbitrary, let's therefore choose $\frac{\vec{a}_\parallel}{|\vec{a}_\parallel|} = \frac{\vec{v}_\parallel}{|\vec{v}_\parallel|} =: \vec{e}_x$

$$\Rightarrow \frac{\vec{a}_\perp}{|\vec{a}_\perp|} = \frac{\vec{v}_\perp}{|\vec{v}_\perp|} = \vec{e}_y$$

for simplicity. We can generalize afterwards.

From Jackson p. 550-551:

The velocity components in each frame are $u_i' = c\,dx_i'/dx_0'$ and $u_i = c\,dx_i/dx_0$. This means that the components of velocity transform according to

$$\left.\begin{aligned} u_\parallel &= \frac{u'_\parallel + v}{1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}} \\ \mathbf{u}_\perp &= \frac{\mathbf{u}'_\perp}{\gamma_v\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)} \end{aligned}\right\} \tag{11.31}$$

$$= \frac{\vec{\beta}\cdot\vec{u}'}{c} \overset{\text{here}}{=} \frac{\beta u_x}{c}$$

For acceleration we go from there:

$$\vec{a} = c\frac{\partial \vec{u}}{\partial x^0}, \quad \vec{a}' = c\frac{\partial \vec{u}'}{\partial x'^0}$$ (prime and 0, not ten)

Also: $dx_0 = \gamma_v(dx_0' + \beta\, dx_1')$, Jackson p. 550

$$\Rightarrow \frac{dx^0}{dx'^0} = \gamma(1+\beta u_x'/c)$$

In components:

$$a_x = c\frac{du_x}{dx^0} = c\,\underbrace{\frac{dx'^0}{dx^0}}_{=\left(\frac{dx^0}{dx'^0}\right)^{-1} = \left(\gamma(1+\beta u'_x/c)\right)^{-1}}\frac{du_x}{dx'^0}$$

$$= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx'^0}\frac{u'_x + c\beta}{1+\beta u'_x/c}$$

Note: $\beta u'_x = \vec{\beta}\cdot\vec{u}'$

$$= \frac{c}{\gamma(1+\beta u'_x/c)}(1+\beta u'_x/c)^{-2}\left[(1+\beta u'_x/c)(a'_x/c) - (u'_x + c\beta)(\beta a'_x/c^2)\right]$$

$$= \frac{(1-\beta^2)a'_x}{\gamma\left(1+\beta\frac{u'_x}{c}\right)^3} = \frac{a'_x}{\gamma^3\left(1+\beta\frac{u'_x}{c}\right)^3} = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{vu'_x}{c^2}\right)^3}a'_x$$

using $\vec{a}_\parallel = a'_x\vec{e}_x$, $\vec{a}'_\parallel = \vec{a}'_\parallel = a'_\parallel\vec{e}_x$ and $\vec{v}\cdot\vec{u}' = vu'_x$

we can generalize to:

$$\mathbf{a}_\parallel = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_\parallel = \frac{\vec{a}'_\parallel}{\gamma^3\left(1+\vec{\beta}\cdot\vec{u}'/c\right)^3}$$

For a_y we get:

$$a_y = c\frac{du_y}{dx^0} = c\frac{dx'^0}{dx^0}\frac{du_y}{dx'^0}$$

$$= \frac{c}{\gamma(1+\beta u_x'/c)}\frac{d}{dx'^0}\frac{u_y'}{\gamma(1+\beta u_x'/c)}$$

$$= \frac{c}{\gamma^2(1+\beta\frac{u_x'}{c})}\frac{(1+\beta\frac{u_x'}{c})\frac{a_y'}{c} - u_y'\frac{\beta a_x'}{c^2}}{(1+\beta\frac{u_x'}{c})^2}$$

$$= \frac{a_y' + \frac{\beta}{c}(u_x' a_y' - u_y' a_x')}{\gamma^2(1+\beta\frac{u_x'}{c})^3}$$

Using $\vec{\beta}\cdot\begin{cases}\vec{u} = \beta u_x \\ \vec{v} = \beta v_x \\ \vec{a} = \beta a_x\end{cases}$ (sum with primes)

and $\vec{a}_\perp^{(')} = a_y^{(')}\vec{e}_y$ we generalize to:

$$\vec{a}_\perp = \frac{\vec{a}_\perp + \frac{\vec{a}'(\vec{\beta}\cdot\vec{u}') - \vec{u}'(\vec{\beta}\cdot\vec{a}')}{c}}{\gamma^2(1+\frac{\vec{\beta}\cdot\vec{u}}{c})^3}$$

Using $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{A}\cdot\vec{C}) - \vec{C}\cdot(\vec{A}\cdot\vec{B})$

we yield $\rightarrow$

$$\vec{a}_\perp = \frac{\vec{a}'_\perp + \frac{\vec{\beta}}{c} \times (\vec{a}' \times \vec{u}')}{\gamma^2 \left(1 + \frac{\vec{\beta} \cdot \vec{u}'}{c}\right)^3}$$

$$= \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v} \cdot \mathbf{u}'}{c^2}\right)^3} \left(\mathbf{a}'_\perp + \frac{\mathbf{v}}{c^2} \times (\mathbf{a}' \times \mathbf{u}')\right) \quad \square$$