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Physics 841 - Homework 12 **due Fri., Apr. 17, 2020**

Problems

1. **Non-relativistic particle in homogeneous magnetic field:** Consider the motion of a non-relativistic point particle in a static homogeneous magnetic field, ignoring radiation. Assume $\vec{B}$ is in the z direction, $\vec{B}(\vec{r}, t) = B_0\hat{z}$.

   (a) (20 pts) Starting from the Lorenz force, derive the trajectory $\vec{r}_0(t)$ of the particle and identify the non-relativistic cyclotron frequency. What is the change of kinetic energy of the particle with time ?

2. **Fields in a hollow cylinder:** Using cylindrical coordinates (ρ, φ, z), consider electric and magnetic fields

$$\vec{E} = \hat{\rho}\frac{c_1}{\rho}, \qquad \vec{B} = \hat{\varphi}\frac{c_2}{\rho},$$

   with constant c_1 and c_2 inside the volume bounded by $a \leq \rho \leq b$, i.e. inside an infinitely long cylinder with a hole.

   (a) (10 pts) Determine the Poynting vector $\vec{S}$ inside the volume.

   (b) (10 pts) Determine the total flux of energy in the fields through a cross-sectional surface with $a \leq \rho \leq b$.

   (c) (10 pts) Determine the energy per unit length $d\mathcal{E}/dz$ and the momentum per unit length $d\vec{p}/dz$ in the fields (for $a \leq \rho \leq b$).

3. **Force due to a plane wave:** An incident monochromatic plane wave described by a vector potential $\vec{A} = \vec{A}_0 \cos(\omega t - \vec{k}\vec{r})$ is completely absorbed by a sphere of radius R.

   (a) (10 pts) Find the electric and magnetic fields. (Take into account that for a plane wave $\vec{k}\vec{A} = 0$.)

   (b) (20 pts) Determine the Maxwell's stress tensor.

   (c) (20 pts) Find the force $\vec{F}$ exerted by the wave on the sphere averaged over the period $T = 2\pi/\omega$ using the result of part (b).

Problem 1

I'll assume the particle carries the electric charge q and mass m.

Then $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \overset{\vec{E}=\vec{0}}{=} q\vec{v} \times \vec{B} = m \frac{d}{dt}\vec{v} = m\ddot{\vec{r}}$

$$\Rightarrow \ddot{\vec{r}} = \underbrace{\frac{qB}{m}}_{=:\omega} \dot{\vec{r}} \times \vec{e}_z = \omega \dot{\vec{r}} \times \vec{e}_z = \omega \begin{pmatrix} v_y \\ -v_x \\ 0 \end{pmatrix} \quad (1)$$

This means the movement in z-direction is free. No matter what the initial v_x and v_y components are, as long as $v_x^2 + v_y^2 > 0$, the particle will follow a circular trajectory in the xy plane with angular frequency $\omega = qB/m$.

The kinetic energy will be constant [and oscillate between the x and y component of the movement (except for the uniform z component). no proof]

Proof First: the circular trajectory.

Since the magnetic flux is homogeneous and invariant under rotations around the z axis, we can freely choose the origin of the coordinate system and its azimuthal orientation.

So we choose a coordinate system parameterized in cylindrical coordinates, so that in that system the initial location of the particle is

$$\vec{r}_0 = R\,\vec{e}_x = R\,\vec{e}_r \quad , \quad \varphi_0 = 0 \quad , \quad R = \frac{mv}{qB}$$

and its initial velocity

$$\vec{v}_0 = -R\dot{\varphi}_0\,\vec{e}_\varphi + v_z\,\vec{e}_z$$

Again, as argued above, such a parameterization can always be found without loss of generality.

Now the solution to (1) is obvious:

$$\boxed{r(t) = \vec{r}_0 - R\,\omega\,t\,\vec{e}_\varphi + v_z t\,\vec{e}_z} \qquad (2)$$

We verify by differentiating twice:

$$\frac{d}{dt}\vec{r} = -R\,\omega\,\vec{e}_\varphi + v_z\,\vec{e}_z = \dot{\vec{r}}$$

$$\Rightarrow \frac{d^2}{dt^2}\vec{r} = -R\omega\,\dot{\vec{e}}_\varphi = -R\omega\frac{d}{dt}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} = -R\omega\,\dot{\varphi}\begin{pmatrix}-\cos\varphi\\ -\sin\varphi\\ 0\end{pmatrix}$$

$$= R\omega\dot{\varphi}\,\vec{e}_r$$

we identify $\omega = \dot{\varphi} = \dot{\varphi}_0$, then (Cyclotron freq)

$$\frac{d^2}{dt^2}\vec{r} = R\omega^2\vec{e}_r = -R\omega^2\,e_\varphi \times e_z = \omega\,\dot{\vec{r}} \times \vec{e}_z$$

$\vec{e}_z \times \vec{e}_z = \vec{0}$

Which shows that (2) solves (1) for the given boundary conditions.

Second part of the proof: The kinetic energy is constant.

$$T = \frac{m}{2}\dot{r}^2 = \frac{m}{2}\left(-R\omega\vec{e}_\varphi + v_z\vec{e}_z\right)^2$$

$$= \frac{m}{2}\left\{R^2\omega^2\left(\sin^2\varphi + \cos^2\varphi\right) + V_z^2\right\}$$

$$= \frac{m}{2}R^2\omega^2 + v_z^2$$

$$\Rightarrow \boxed{\frac{d}{dt}T = 0} \quad \square$$

Problem 2

(a)

$$\vec{S} = \mu_0^{-1}\, \vec{E} \times \vec{B} = \mu_0^{-1} \frac{c_1 c_2}{r^2}\, \vec{e}_r \times \vec{e}_\varphi = \frac{c_1 c_2}{\mu_0 r^2}\, \vec{e}_z$$

(b)

$$W_{ab} = \int_A \vec{S}\, d\vec{n} = 2\pi \frac{c_1 c_2}{\mu_0} \int_a^b r^{-2}\, r\, dr$$

$$= 2\pi \frac{c_1 c_2}{\mu_0} \int_a^b r^{-1}\, dr = 2\pi \frac{c_1 c_2}{\mu_0} \ln\left(\frac{b}{a}\right)$$

(c) (Zangwill 15.31) $u_{\mathrm{EM}} = \frac{1}{2}\epsilon_0 \left(\mathbf{E}\cdot\mathbf{E} + c^2 \mathbf{B}\cdot\mathbf{B}\right)$

Here: $u = \frac{\varepsilon_0}{2r^2}\left(c_1^2 + c^2 c_2^2\right) = \frac{1}{2r^2}\left(\varepsilon_0 c_1^2 + \frac{c_2^2}{\mu_0}\right)$

$$\Rightarrow \frac{d\mathcal{E}}{dz} = 2\pi \int_a^b u\, r\, dr = \pi \int_a^b \left(\varepsilon_0 \frac{c_1^2}{r} + \frac{c_2^2}{\mu_0 r}\right) dr$$

$$= \pi\left(\varepsilon_0 c_1^2 + \frac{c_2^2}{\mu_0}\right) \ln\left(\frac{b}{a}\right)$$

$$g = \frac{\vec{S}}{c^2} \Rightarrow \frac{d\vec{p}}{dz} = \frac{2\pi}{c^2}\, \vec{e}_z \int_a^b |\vec{S}|\, r\, dr$$

$$= \frac{2\pi}{c^2}\, \vec{e}_z \int_a^b \frac{c_1 c_2}{\mu_0 r}\, dr$$

$$= \frac{2\pi c_1 c_2}{c^2} \ln\left(\frac{b}{a}\right) \vec{e}_z$$

Problem 3

(a) Let the direction of the plane wave be the z-direction : $\vec{k} = k\,\vec{e}_z$.

And since $\vec{k}\,\vec{A} = 0$ and the symmetry of the problem (the sphere is invariant under rotations), we can choose the direction of $\vec{A}_0$ freely in the xy-plane <u>without loss of generality</u>.
So let's choose

$$\vec{A}_0 = A_0 \vec{e}_x \Rightarrow \vec{A} = A_0 \cos(\omega t - kz)\,\vec{e}_x$$

$$\Rightarrow \vec{E} = -\frac{\partial \vec{A}}{\partial t} = \omega A_0 \sin(\omega t - kz)\,\vec{e}_x$$

and
$$\vec{B} = \mathrm{rot}\,\vec{A} = -A_0 \frac{\partial}{\partial z} \cos(\omega t - kz)\,\vec{e}_y$$
$$= k A_0 \sin(\omega t - kz)\,\vec{e}_y$$

As expected with $\vec{S} \parallel \vec{k} \parallel \vec{e}_z = \vec{e}_x \times \vec{e}_y$.

(b) $$\mathbf{T} = \epsilon_0 \left[\mathbf{EE} + c^2\mathbf{BB} - \tfrac{1}{2}\mathbf{I}(E^2 + c^2B^2)\right].$$ (dyadic notation $\vec{V} \cdot \vec{V}^T = \vec{V}\vec{V}$)

$$\vec{E}\vec{E} = \underbrace{\omega^2 A_0^2 \sin^2(\omega t - kz)}_{=E^2}\, \vec{e}_x \cdot \vec{e}_x^{\,T}$$

$$\vec{B}\vec{B} = \underbrace{k^2 A_0^2 \sin^2(\omega t - kz)}_{=B^2}\, \vec{e}_y \cdot \vec{e}_y^{\,T}$$

$$\Rightarrow T = \frac{\epsilon_0}{2}\Big[(E^2 - c^2B^2)\,\vec{e}_x\vec{e}_x + (c^2B^2 - E^2)\,\vec{e}_y\vec{e}_y$$
$$+ (E^2 + c^2B^2)\,\vec{e}_z\vec{e}_z\Big]$$

using $c^2 k^2 = \omega^2$ we see that

$$c^2 B^2 - E^2 = 0 \quad \text{and therefore}$$

$$T = \frac{\varepsilon_0}{2}\left(E^2 + c^2 B^2\right)\vec{e}_z \vec{e}_z = \varepsilon_0 A_0^2 \omega^2 \sin^2(\omega t - kz)\vec{e}_z\vec{e}_z$$

$$\left(\Leftrightarrow T_{ij} = \varepsilon_0 A_0^2 \omega^2 \sin^2(\omega t - kz)\,\delta_{ij}\,\delta_{i3}\right)$$

(c)

Since $\nabla \cdot \mathbf{T}$ is a *vector* with components

$$(\nabla \cdot \mathbf{T})_j = \sum_i \frac{\partial}{\partial x_i} T_{ij}, \tag{15.47}$$

(15.44) takes the compact form

$$\mathbf{F}_{\text{mech}} = \frac{d\mathbf{P}_{\text{mech}}}{dt} = \int_V d^3r \left\{ -\frac{1}{c^2}\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot \mathbf{T} \right\}. \tag{15.48}$$

We note in passing that the $\partial \mathbf{S}/\partial t$ term in (15.48) precludes writing the total mechanical force as a surface integral as we did in electrostatics and magnetostatics. Time-harmonic fields are an important exception where the Poynting vector term disappears after averaging over one period of oscillation.

Applies here

We assume that the entire momentum is absorbed by the sphere.

$$\text{So} \quad \langle F_i \rangle_{T = \frac{2\pi}{\omega}} = \frac{1}{T}\int_0^T dt \int_V d^3r\, \nabla \cdot T$$

$$= \frac{1}{T}\int_0^T dt \int_{\partial V} da \sum_j T_{ij} n_j$$

So obviously $F_x = F_y = 0$ and $F_z = F$, $\vec{F} = F\vec{e}_z$

$$\langle F \rangle_T = \frac{1}{T}\int_0^T dt \int_{\partial V} d\vec{a}\; T_{zz}$$

$$\langle F \rangle_T = \frac{1}{T}\int_0^T dt \int_{\partial V} d\vec{n} \cdot \vec{e}_z \; T_{zz}$$

$$= \int_{\partial V} d\vec{n}\, \vec{e}_z \, \frac{1}{T}\int_0^T dt \; \varepsilon_0 A_0^2 \omega^2 \sin^2(\omega t - kz)$$

sphere absorbs the momentum!

$$= \varepsilon_0 A_0^2 \omega^2 R^2 \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_0^{\pi/2} d\theta \sin\theta \underbrace{\vec{e}_r \cdot \vec{e}_z}_{\cos\theta}}_{\frac{1}{2}} \underbrace{\int_0^T \frac{dt}{T} \sin^2(\omega t - k\cos\theta R)}_{\sin^2 \text{ over one full period } = \frac{1}{2}}$$

$$= \frac{\varepsilon_0 \pi}{2} A_0^2 \omega^2 R^2$$