Alexander Harnisch

Physics 841 - Homework 11 **due Fri., Apr. 3, 2020**

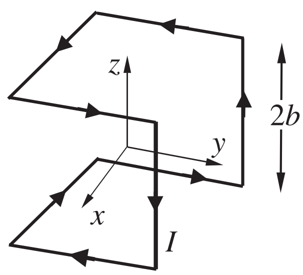
Problems

1. **Magnetic dipole moment:**
A current distribution produces the vector potential

$$\mathbf{A}(r, \theta, \varphi) = \hat{\varphi} \frac{\mu_0}{4\pi} \frac{A_0 \sin\theta}{r} \exp(-\lambda r).$$

   (a) (50 pts) Find the magnetic (dipole) moment of this current distribution. *Hint:* Find the current from the vector potential and then follow the definition of the magnetic moment. Avoid directly performing integrals, if possible.

2. **A current loop:**
A filamentary current loop traverses eight edges of a cube with side length $2b$ as shown in the figure. The origin is placed at the center of the cube.



   (a) (50 pts) Find the magnetic dipole moment $\mathbf{m}$ of this structure.

Problem 1

$$\mu_0 \vec{j} = \nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A})$$

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}} = 0$$

$$= \frac{\mu_0 A_0}{4\pi}\left(\frac{2\cos\Theta}{r^2}\vec{e}_r + \frac{\lambda\sin\Theta}{r}\vec{e}_\Theta\right)\exp(-\lambda r)$$

$$\Rightarrow \vec{j} = \frac{1}{\mu_0}\nabla\times\vec{B} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\overset{B}{\cancel{A}}_\varphi \sin\theta) - \frac{\partial \overset{B}{\cancel{A}}_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} = 0$$

$$+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \overset{B}{\cancel{A}}_r}{\partial\varphi} - \frac{\partial}{\partial r}(r\overset{B}{\cancel{A}}_\varphi)\right)\hat{\boldsymbol{\theta}} = 0$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r\overset{B}{\cancel{A}}_\theta) - \frac{\partial \overset{B}{\cancel{A}}_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$$

$$= \frac{A_0}{4\pi}\sin\Theta\left(\frac{2}{r^3} - \frac{\lambda^2}{r}\right)\exp(-\lambda r)\,\vec{e}_\varphi$$

With that we get the magnetic moment

$$\vec{m} = \frac{1}{2}\int d^3r\, \vec{r}\times\vec{j} = -\frac{A_0}{8\pi}\int d^3r\, \vec{e}_\Theta\, r\sin\Theta\left(\frac{2}{r^3} - \frac{\lambda^2}{r}\right)\exp(-\lambda r)$$

$$\vec{e}_\Theta = \cos\Theta\cos\varphi\,\vec{e}_x + \cos\Theta\sin\varphi\, e_y - z\sin\Theta\,\vec{e}_z$$

Since $\int_0^{2\pi}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}d\varphi = \vec{0}$ only the z-component will be non-zero:

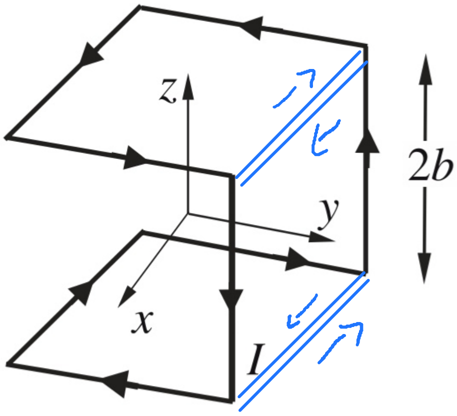
$$\vec{m} = \vec{e}_z\frac{A_0}{4}\overbrace{\int_0^\pi \sin^2\Theta\, d\Theta}^{\pi/2}\int_0^\infty r^3\exp(-\lambda r)\left(\frac{2}{r^3} - \frac{\lambda^2}{r}\right)dr$$

$$= \vec{e}_z\frac{\pi A_0}{8}\left(2\int_0^\infty \exp(-\lambda r)dr - \lambda^2\int_0^\infty r^2\exp(-\lambda r)\,dr\right)$$

$$= \vec{e}_z\frac{\pi A_0}{8}\left(2 - \lambda^2\frac{d^2}{d\lambda^2}\right)\overbrace{\int_0^\infty \exp(-\lambda r)dr}^{1/\lambda}$$

$$= \vec{e}_z\frac{\pi A_0}{8}\left(\frac{1}{\lambda} - \lambda^2 2\lambda^{-3}\right) = \boxed{0}$$

Problem 2



The problem can be simplified by adding the blue currents with net current zero on the right. Now we can simply use superposition of 3 square loops to find the over-all momentum.

$$\mathbf{m} = \frac{1}{2} I \oint_C \mathbf{r} \times d\boldsymbol{\ell}. \tag{11.19}$$

A corollary of Stokes' theorem[2] transforms (11.19) to

$$\mathbf{m} = \tfrac{1}{2} I \int_S d\mathbf{S}\, \nabla \cdot \mathbf{r} - \tfrac{1}{2} I \int_S dS_k \nabla r_k = I \int_S d\mathbf{S} \equiv I\mathbf{S}. \tag{11.20}$$

So the magnetic moment of the square loops is $I(2b)^2\vec{e}_s$, where the direction of $\vec{e}_s$ is given by the right-hand rule. The moment of the upper and lower loop cancels. So the remaining magnetic moment is:

$$\boxed{\vec{m} = 4 I b^2 \vec{e}_y}$$