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Physics 841 - Homework 8 **due Mon., Mar. 16, 2020**

Problems

1. **Potential of Charge and Conducting Sphere:** Consider a grounded conducting sphere of radius R (centered at the origin) in presence of a point charge q located outside of the sphere (at position $(0, 0, a)$ with $a > R$). (We discussed this setup in the class but skipped some details in the derivation.)

   (a) (5 pts) Write the potential as a sum of two terms: (i) the potential of the point charge and (ii) a general solution of the azimuthally symmetric Laplace equation (using Legendre polynomials).

   (b) (10 pts) Determine the unknown coefficients in term (ii) using suitable boundary conditions.

   (c) (10 pts) In this example, the term (ii) can be rewritten in a closed and suggestive form. Please perform this resummation and interpret your result in terms of (image) point charges.

   (d) (10 pts) Derive the surface charge density on the sphere in terms of q, R, a and $x = \cos\theta$. [Optional: discuss the limits $a \to R$ and $a \to \infty$.]

   (e) (10 pts) Calculate the induced charge.

   (f) (10 pts) Plot (or draw qualitatively) the following quantities in dependence of the distance from the center of the sphere (along a line from $(0, 0, 0)$ to $(0, 0, a)$): term (i), term (ii), the sum of (i) and (ii), the actual potential.

   (g) (5 pts) How is the potential outside of the sphere modified if the sphere is held at a fixed potential ? (Consult Jackson.)

2. **Green function:** Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

   (a) (10 pts) Write down the appropriate Green function $G(\vec{r}, \vec{r}\,')$.

   (b) (20 pts) If the potential on the plane $z = 0$ is specified to be $\phi = \phi_0$ inside a circle of radius R centered at the origin, and $\phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (s, φ, z). Why r?

   (c) (10 pts) Find the formula for $\phi(0, \varphi, z)$ along the axis of the circle ($s = 0$) by explicitly integrating the expression in (b).

Problem 1

(a)

(i) $\phi_q(\vec{r}) = \dfrac{q}{4\pi\epsilon_0 |\vec{r} - a\vec{e}_z|}$

(ii) $\phi_s(\vec{r}) = \dfrac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left(A_l r^l + B_l \dfrac{1}{r^{l+1}} \right) P_l(\cos\theta)$

$$\phi(\vec{r}) = \phi_q + \phi_s$$
$$= \frac{1}{4\pi\epsilon_0}\left[q\left(r^2 + a^2 - 2ra\cos\theta\right)^{-1/2} + \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta)\right]$$

(b) We want $\lim_{r\to\infty} \phi = \infty$, so $\boxed{A_l = 0 \ \forall l}$

The other boundary condition is $\phi|_{r=R} = 0$

We use:

An important expansion is that of the potential at **x** due to a unit point charge at **x′**:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \qquad (3.38)$$

So:

$$\phi = \frac{1}{4\pi\epsilon_0} \sum_{l=0} \left(q \frac{r_<^l}{r_>^{l+1}} + \frac{B_l}{r^{l+1}} \right) P_l \cos\theta$$

For $r = R$, $a > R \Rightarrow r_< = R$, $r_> = a$

$$\Rightarrow \phi|_{r=R} = \frac{1}{4\pi\epsilon_0} \sum_{l=0} \Bigl(\underbrace{q \frac{R^l}{a^{l+1}} + \frac{B_l}{R^{l+1}}}_{\stackrel{!}{=} 0} \Bigr) P_l \cos\theta \stackrel{!}{=} 0$$

$$\Rightarrow \boxed{B_l = -q \frac{R^{2l+1}}{a^{l+1}}}$$

(c)

We have:

$$\phi = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left(\frac{r^l}{a^{l+1}} - \frac{R}{a} \frac{(R^2/a)^l}{r^{l+1}} \right) P_l \cos\Theta$$

Using (3.38) backwards gives:

$$\boxed{\phi = \frac{q}{4\pi\epsilon_0} \left(\underbrace{\frac{1}{|\vec{r} - a\vec{e}_z|}}_{\text{actual charge}} - \underbrace{\frac{R/a}{|\vec{r} - \left(\frac{R^2}{a}\right)\vec{e}_z|}}_{\text{image charge } q' = -\frac{R}{a}q,\ \text{location } \frac{R^2}{a}\vec{e}_z} \right)}$$

(d) $\sigma = -\epsilon_0 \frac{\partial \phi}{\partial r}\Big|_{r=R}$

$$= -\frac{q}{4\pi} \frac{\partial}{\partial r} \left[(r^2 + a^2 - 2ar\cos\Theta)^{-1/2} - \frac{R}{a}\left(r^2 + \frac{R^4}{a^2} - 2\frac{R^2}{a} r\cos\Theta\right)^{-1/2} \right]\Bigg|_{r=R}$$

$$= \frac{q}{4\pi} \left[\frac{R - a\cos\Theta}{(R^2 + a^2 - 2aR\cos\Theta)^{3/2}} - \frac{R(R - R^2/a \cos\Theta)}{a\left(R^2 + \frac{R^4}{a^2} - 2\frac{R^3}{a}\cos\Theta\right)^{3/2}} \right]$$

= [simplify]

$$= \frac{q}{4\pi R a} \cdot \frac{\frac{R^2}{a^2} - 1}{\left(1 + \frac{R^2}{a^2} - \frac{2R}{a}\cos\Theta\right)^{3/2}}$$

(e)

$$\Rightarrow Q = \int_0^{2\pi}\int_0^{\pi} \sigma R^2 \sin\theta \, d\theta \, d\varphi$$

$$\overset{\tilde{R}:=R/a}{=} \frac{2\pi q \tilde{R}(\tilde{R}^2-1)}{4\pi} \int_0^{\pi} (1+\tilde{R}^2 - 2\tilde{R}\cos\theta)^{-3/2} \sin\theta \, d\theta$$

$$\boxed{= -q \, R/a}$$

(g) It'll simply be shifted by the same constant factor.

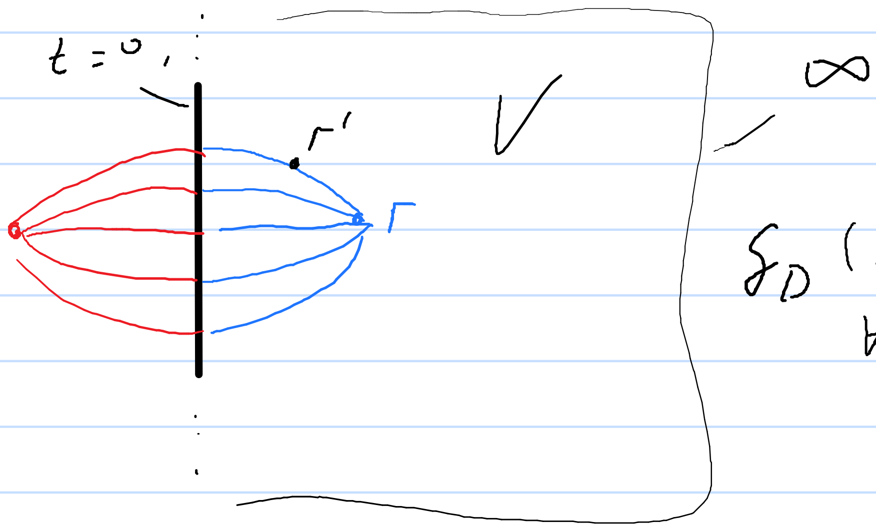
Problem 2

(a)

8.5.2 Calculation of Dirichlet Green Functions

We have seen that the Dirichlet Green function is the electrostatic potential of a unit-strength point charge in the presence of a grounded boundary. For planar and spherical boundaries, the method of images is sufficient to find $G_D(\mathbf{r}, \mathbf{r}')$. ~~In the sections to follow, we outline three other methods used~~

So here we can simply use the method of images:



$$G_D(\vec{r}, \vec{r}') = 0 \quad \forall \vec{r} \in \partial V, \ \vec{r}' \in V$$

Therefore the Green function is:

$$4\pi\varepsilon_0 \cdot G_D(\vec{r}, \vec{r}') = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{-1/2} - \left[(x-x')^2 + (y-y')^2 + (z + z')^2\right]^{-1/2}$$

depending on definition

mirror charge

(b)

This choice makes the last integral in (8.52) zero. Therefore, once we have solved for $G_D(\mathbf{r}, \mathbf{r}')$ and supplied the boundary data $\varphi_S(\mathbf{r}_S)$, the unique solution for the potential is

$$\varphi(\mathbf{r} \in V) = \int_V d^3r' \, \rho(\mathbf{r}') G_D(\mathbf{r}', \mathbf{r}) - \epsilon_0 \int_S dS' \, \varphi_S(\mathbf{r}') \frac{\partial G_D(\mathbf{r}', \mathbf{r})}{\partial n'}. \tag{8.54}$$

Here: $\rho(r) = 0$

$$\Rightarrow \quad \phi = -\varepsilon_0 \int_{\partial V} da' \, \phi_S \frac{\partial G_D}{\partial n'}$$

$$\phi = -\varepsilon_0 \int_{\partial V} da' \, \phi \cdot \frac{\partial g_D}{\partial n'}$$

∂V is a box with the plane $z=0$ as one sides. All other sides are at infinity and therefore don't contribute to the integral becaus $\phi = 0$ at infinity.

Also the potential is 0 outside the circle. What's left is:

$$\phi = -\varepsilon_0 \, \phi_0 \int_{-\infty}^{\infty}\int_0^R \int_0^{2\pi} \frac{\partial g_D}{\partial n} \, r \, d\varphi \, dr \, \delta(z) dz$$

here: $n = -z$ (pointing out)

$$\Rightarrow \quad \phi = + \varepsilon_0 \phi_0 \int_{-\infty}^{\infty}\int_0^R \int_0^{2\pi} \frac{\partial g_D}{\partial z} \, r \, d\varphi \, dr \, \delta(z) dz$$

plug in g_D in cylindrical: (r, φ, z) why would you call it s?

$$\phi = \frac{\phi_0}{4\pi} \int_{-\infty}^{\infty}\int_0^R \int_0^{2\pi} \frac{\partial}{\partial z'} \Big(\left[r^2 + r'^2 - 2rr'\cos(\varphi - \varphi') + (z-z')^2 \right]^{-\frac{1}{2}}$$
$$- \left[r^2 + r'^2 - 2rr'\cos(\varphi-\varphi') + (z+z')^2 \right]^{-1/2} \Big)$$
$$r' d\varphi' dr' \, \delta(z') \, dz$$

Derivative: $-(z \mp z') / [\cdots]^{3/2}$ and $z' = 0$

$$\Rightarrow \boxed{\phi = \frac{z \phi_0}{2\pi} \int_0^R \int_0^{2\pi} \frac{r' d\varphi' dr'}{(r^2 + r'^2 - 2rr'\cos\varphi' + z^2)^{3/2}}}$$

$\varphi' \to \varphi' + \varphi$ because of symmetry

(c) Evaluate the general solution for $r = 0$:

$$\phi\Big|_{r=0} = \frac{z\phi_0}{2\pi} \int_0^R \int_0^{2\pi} \frac{r' \, d\varphi' \, dr'}{(r'^2 + z^2)^{3/2}}$$

$$= z\phi_0 \int_0^R \frac{r' \, dr'}{(r'^2 + z^2)^{3/2}} \quad , \quad u := r'^2 + z^2$$

$$\Rightarrow du = 2r' \, dr'$$

$$\Rightarrow \boxed{\phi\Big|_{r=0} = \phi_0 \left(1 - \frac{z}{\sqrt{R^2 + z^2}} \right)}$$