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Physics 841 - Homework 6 **due Fri., Feb. 21, 2020**

Problems

1. **Practice with Legendre polynomials:** Consider a sphere of radius R (the center is at the origin) where there are no charges inside and outside and any possible charge is on the surface. The potential on the surface is $\phi|_S = V(\cos(\theta))$, θ is the inclination angle, as usual counted from the z-axis.

   (a) (20 pts) Using the method of separation of variables write down the general solution for the potential inside and outside the sphere as an expansion in Legendre polynomials. Relate the coefficients of the expansion to $\phi|_S$.

   (b) (15 pts) Find the solution for

   $$\phi|_S = V_0 \cos(3\theta).$$

   (c) (15 pts) Find the electric field at the point $(x = 0, y = 0, z = R/2)$ for the potential in (b).

2. **Summary of course topics (50 pts):** Please compile your *personal summary* of the course topics so far (special relativity and electrostatics up to and including cartesian multipole expansion). This should not just be a list of all possibly useful equations you can find. Your summary should reflect key concepts and the relations between them. Make sure you understand the content of the equations you assemble and how to apply them. (This assignment is intended to support your preparations for the exam. Please try to present a clear view of the topics, but you may want to avoid spending too much time just on perfecting the write-up.)

Problem 1

(a)

It's all in Jackson:

3.1 Laplace Equation in Spherical Coordinates

Jackson p. 84 ff.

In spherical coordinates (r, θ, ϕ), shown in Fig. 3.1, the Laplace equation can be written in the form:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0 \tag{3.1}$$

If a product form for the potential is assumed, then it can be written:

$$\Phi=\frac{U(r)}{r}P(\theta)Q(\phi) \tag{3.2}$$

When this is substituted into (3.1), there results the equation:

$$PQ\frac{d^2U}{dr^2}+\frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{d^2Q}{d\phi^2}=0$$

If we multiply by $r^2\sin^2\theta/UPQ$, we obtain:

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2}+\frac{1}{r^2\sin\theta P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]+\frac{1}{Q}\frac{d^2Q}{d\phi^2}=0 \tag{3.3}$$

The ϕ dependence of the equation has now been isolated in the last term. Consequently that term must be a constant which we call $(-m^2)$:

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2}=-m^2 \tag{3.4}$$

This has solutions

$$Q=e^{\pm im\phi} \tag{3.5}$$

In order that Q be single valued, m must be an integer if the full azimuthal range is allowed. By similar considerations we find separate equations for $P(\theta)$ and $U(r)$:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P=0 \tag{3.6}$$

$$\frac{d^2U}{dr^2}-\frac{l(l+1)}{r^2}U=0 \tag{3.7}$$

where $l(l+1)$ is another real constant.

From the form of the radial equation it is apparent that a single power of r (rather than a power series) will satisfy it. The solution is found to be:

$$U=Ar^{l+1}+Br^{-l} \tag{3.8}$$

but l is as yet undetermined.

3.2 Legendre Equation and Legendre Polynomials

The θ equation for $P(\theta)$ is customarily expressed in terms of $x=\cos\theta$, instead of θ itself. Then it takes the form:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P=0 \tag{3.9}$$

This equation is called the generalized Legendre equation, and its solutions are the associated Legendre functions. ~~Before considering (3.9) we will outline the~~

Special case: Legendre Polynomials

By manipulation of the power series solutions (3.11) and (3.14) it is possible to obtain a compact representation of the Legendre polynomials, known as *Rodrigues' formula:*

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \tag{3.16}$$

Since the Legendre polynomials form a complete set of orthogonal functions, any function $f(x)$ on the interval $-1\leq x\leq 1$ can be expanded in terms of them. The Legendre series representation is:

$$f(x)=\sum_{l=0}^{\infty} A_l P_l(x) \tag{3.23}$$

where

$$A_l=\frac{2l+1}{2}\int_{-1}^{1} f(x)P_l(x)\,dx \tag{3.24}$$

3.3 Boundary-Value Problems with Azimuthal Symmetry

From the form of the solution of the Laplace equation in spherical coordinates (3.2) it will be seen that, for a problem possessing azimuthal symmetry, $m=0$ in (3.5). This means that the general solution for such a problem is:

$$\Phi(r,\theta)=\sum_{l=0}^{\infty}[A_l r^l+B_l r^{-(l+1)}]P_l(\cos\theta) \tag{3.33}$$

The coefficients A_l and B_l can be determined from the boundary conditions. Suppose that the potential is specified to be $V(\theta)$ on the surface of a sphere of radius a, and it is required to find the potential inside the sphere. If there are no charges at the origin, the potential must be finite there. Consequently $B_l=0$ for all l. The coefficients A_l are found by evaluating (3.33) on the surface of the sphere:

$$\boxed{V(\theta)=\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)} \tag{3.34}$$

This is just a Legendre series of the form (3.23), so that the coefficients A_l are:

$$A_l=\frac{2l+1}{2a^l}\int_0^{\pi} V(\theta)P_l(\cos\theta)\sin\theta\,d\theta \tag{3.35}$$

To find the potential outside the sphere we merely replace $(r/a)^l$ by $(a/r)^{l+1}$. The resulting potential can be seen to be the same as (2.27), obtained by another means.

(b)

$$\phi|_S = V_0 \cos(3\theta)$$

radius R, and it is required to find the potential inside the sphere. If there are no charges at the origin, the potential must be finite there. Consequently $B_l = 0$ for all l. The coefficients A_l are found by evaluating (3.33) on the surface of the sphere:

$$\boxed{V(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)} \qquad (3.34)$$

This is just a Legendre series of the form (3.23), so that the coefficients A_l are:

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V(\theta) P_l(\cos\theta) \sin\theta \, d\theta \qquad P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l$$

So here: $A_l = \frac{2l+1}{2R} \int_0^{\pi} V_0 \cos(3\theta) P_l(\cos\theta) \sin\theta \, d\theta$

$x := \cos\theta \Rightarrow$

$dx = -\sin\theta \, d\theta$

and $\sin\theta = \sqrt{1-x^2}$

WolframAlpha

cos(3*arcos(x))

Alternate forms

$-\sin(3 \sin^{-1}(x))$

$\boxed{4x^3 - 3x}$

$x(2\cos(2\cos^{-1}(x)) - 1)$

So we have:

$$A_l = \frac{2l+1}{2R} \int_0^{\pi} V_0 \cos(3\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

$$A_l = +\frac{2l+1}{2R} V_0 \int_{-1}^{1} (4x^3 - 3x) \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l \, dx$$

$$= \frac{2l+1}{2^{l+1} R \, l!} V_0 \int_{-1}^{1} (4x^3 - 3x) \frac{d^l}{dx^l}(x^2-1)^l \, dx$$

$$A_l = \frac{2l+1}{2^{l+1} R^l l!} V_0 \underbrace{\int_{-1}^{1} (4x^3 - 3x) \frac{d^l}{dx^l}(x^2-1)^l \, dx}_{=:\, I}$$

$$(fg)' = f'g - fg' \Rightarrow fg + \int fg' = \int f'g$$

$$I = \ldots$$

I just realized that's complete overkill:

$$\phi|_S = V_0 \cos(3\theta) \overset{x=\cos\theta}{=} V_0(4x^3 - 3x)$$

$$= V_0\left(\frac{8}{5}P_3 - \frac{6}{10}P_1\right)$$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$\frac{8}{5}P_3 = 4x^3 - \frac{24}{10}x$$

$$3 - \frac{24}{10} = \boxed{\frac{6}{10}}$$

$$\Rightarrow A_1 = -\frac{6}{10}V_0$$

$$A_3 = \frac{8}{5}V_0$$

$A_l = 0$ otherwise

(b)

$\vec{E} = -\operatorname{grad}\phi$ Don't have any more time...

Summary of course topics

I actually do this all the time anyways. However, I do it a bit differently. I read through the textbook and mark important sections and make annotations. Additionally, I copy the most important sections into a file I call "Stuff to remember". This is mostly a collection of formulas and definitions which only works in conjunction with my textbook annotations. I'll attach it here and I hope that's all right and worth the full points. I don't want to waste too much time.