Alexander Harnisch

Physics 841 - Homework 5 **due Fri., Feb. 14, 2020**

Problems

1. **Divergence theorem:** By using the divergence theorem evaluate the following integrals:

   (a) (5 pts)

   $$\oiint \vec{r}\,(\vec{a}\cdot\vec{n})\,dA, \tag{1}$$

   (b) (5 pts)

   $$\oiint (\vec{a}\cdot\vec{r})\,\vec{n}\,dA, \tag{2}$$

   where $\vec{a}$ is a constant vector and $\vec{n}$ is a vector normal to the surface element dA.

2. **Charged rod:** A uniformly charged rod with the total charge Q is placed along the z axis so that its ends are at $z = -a$ and $z = a$.

   (a) (15 pts) Find the potential $\phi(\vec{r})$ at an arbitrary point $\vec{r} = (x, y, z)$.

   (b) (15 pts) Find the electric field $\vec{E}(\vec{r})$.

   (c) (10 pts) Find the dipole and quadrupole moments of this system.

3. **Cartesian multipole moments of a charged ring:** A thin circular ring of radius R located in the xy-plane and centered at the z-axis has line charge density $+\lambda_0$ for $0 \leq \varphi < \pi$ and line charge density $-\lambda_0$ for $\pi \leq \varphi < 2\pi$, where φ is the azimuthal angle around the z-axis.

   (a) (15 pts) Calculate the components of the dipole moment of the ring.

   (b) (15 pts) Calculate the components of the quadrupole tensor of the ring. (Hint: exploit the symmetries of the tensor and the ring.)

   (c) (10 pts) Do your results depend on the choice of the origin of the coordinate frame?

   (d) (10 pts) Another point-like dipole $\vec{p}_2$ is located on the z-axis at a large distance $z \gg R$. Calculate the torque acting on this dipole.

Problem 1

using the divergence theorem. The *divergence theorem* states that for any well-behaved vector field $\mathbf{A}(\mathbf{x})$ defined within a volume V surrounded by the closed surface S the relation

$$\oint_S \mathbf{A}\cdot\mathbf{n}\,da = \int_V \nabla\cdot\mathbf{A}\,d^3x$$

holds between the volume integral of the divergence of $\mathbf{A}$ and the surface integral of the outwardly directed normal component of $\mathbf{A}$. The equation in fact can be used as the definition of the divergence (see *Stratton*, p. 4).

(a)

$$\oiint_{\partial V} \vec r\,(\vec a\cdot\vec n)\,dA = \sum_{i=1}^{3}\vec e_i \oiint_{\partial V} \underbrace{r_i\,\vec a}_{\vec A}\cdot\vec n\,dA$$

Divergence Theorem

$$= \sum_{i=1}^{3}\vec e_i\int_V \underbrace{\mathrm{div}(r_i\,\vec a)}_{=\partial_i r_i a_i = a_i}\,d^3r$$

$\int_V d^3r$

$$= \sum_{i=1}^{3}\vec e_i\int_V a_i\,d^3r = V\cdot\vec a$$

(b)

$$\int_V \nabla\cdot\mathbf{A}\,d^3x = \int_S \mathbf{A}\cdot\mathbf{n}\,da$$

$$\boxed{\int_V \nabla\psi\,d^3x = \int_S \psi\mathbf{n}\,da}$$

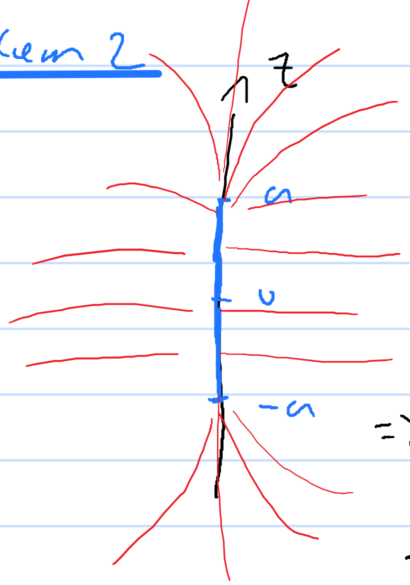
$$\int_V \nabla\times\mathbf{A}\,d^3x = \int_S \mathbf{n}\times\mathbf{A}\,da$$

$$\oiint_{\partial V}(\vec a\cdot\vec r)\,\vec n\,dA = \int_V \underbrace{\mathrm{grad}\Big(\sum_{i=1}^{3} a_i r_i\Big)}_{=\vec a}\,d^3r$$

$$= V\vec a$$

Problem 2

(a)



$$\Phi(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \qquad (1.17)$$

$$\rho(z) = \frac{Q}{2a}\,\Theta(x-a)\,\Theta(x+a)\,\delta(y)\,\delta(x)$$

$$\Rightarrow \phi(\vec{r}) = K \int_{-a}^{a} \frac{Q}{2a\,|\vec{r}-\vec{r}'|}\, d^3r'$$

$$= \frac{KQ}{2a} \int_{-a}^{a} \left(x^2+y^2+(z-z')^2\right)^{-1/2} dz$$

Indefinite integral

$$\int \frac{1}{\sqrt{x^2+y^2+(z-v)^2}}\, dv = \log\left(\sqrt{(v-z)^2+x^2+y^2}+v-z\right) + \text{constant}$$

(assuming a complex-valued logarithm)

$$= \frac{KQ}{2a} \log\left(\frac{\sqrt{(a-z)^2+x^2+y^2}+a-z}{\sqrt{(a+z)^2+x^2+y^2}-a-z}\right), \quad K^{-1} = 4\pi\epsilon_0 \text{ in SI}$$

(b) $E = -\mathrm{grad}\, \phi$

What would I possibly learn doing this by hand? It's just a waste of my time.

(c)

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')\, d^3x' \qquad (4.8)$$

$$\vec{p} = \int \vec{r}'\,\rho(\vec{r}')\, d^3\vec{r}' = \frac{Q}{2a}\int_{-a}^{a} z\,\vec{e}_z\, dz = \vec{0}$$

and Q_{ij} is the traceless quadrupole moment tensor:

$$Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')\, d^3x' \tag{4.9}$$

$$Q_{ij} = \frac{Q}{2a}\int_{-a}^{a} (3x_i'x_j' - r'^2\delta_{ij})\, dz \quad , \quad x = y = 0$$

$$\Rightarrow Q_{ij} = 0 \text{ for } i \neq j$$

$$Q_{11} = Q_{22} = \frac{Q}{2a}\int_{-a}^{a} -z^2\, dz = -\frac{Q}{6a}\left[z^3\right]_{-a}^{a}$$

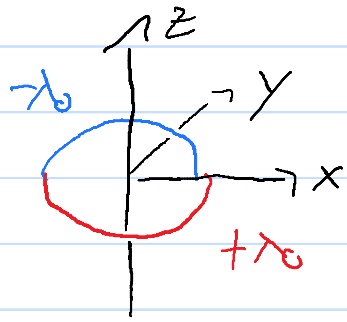
$$= -\frac{Q}{3}a^2$$

$$\Rightarrow Q_{33} = Tr(Q) - Q_{11} - Q_{22} = \frac{2}{3}Qa^2$$

$$\text{So } Q_{ij} = \frac{Q}{3}a^2\delta_{ij}\left(2\delta_{i3} - \delta_{i2} - \delta_{i1}\right)$$

Problem 3

(a)



$$\rho(r,\varphi,z) = \delta(z)\,\delta(R-r)\cdot\begin{cases} +\lambda_0 & , \; 0 \le \varphi < \pi \\ -\lambda_0 & , \; \pi \le \varphi < 2\pi \end{cases}$$

$$\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')\, d^3x' \qquad (4.8)$$

$$\vec{p} = \lambda_0 R^2 \left(\int_0^\pi \vec{e}_r \, d\varphi - \underbrace{\int_\pi^{2\pi} \vec{e}_r \, d\varphi}_{= \int_0^\pi -\vec{e}_r \, d\varphi} \right)$$

$$= \lambda_0 R^2 2 \int_0^\pi \vec{e}_r \, d\varphi$$

$$= \lambda_0 R^2 2 \int_0^\pi \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} d\varphi$$

$$= \lambda_0 R^2 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \lambda_0 R^2 4 \vec{e}_y$$

(b)

and Q_{ij} is the traceless quadrupole moment tensor:

$$Q_{ij} = \int (3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}')\, d^3x' \qquad (4.9)$$

$$Q_{ij} = \lambda_0 R^2 \left(\int_0^\pi 3x_ix_j - R^2\delta_{ij} \, d\varphi - \int_\pi^{2\pi} 3x_ix_j - R^2\delta_{ij} \, d\varphi \right)$$

$$\Rightarrow Q_{23} = Q_{32} = Q_{13} = Q_{31} = 0$$

$$Q_{33} = \lambda_0 R^4 \left(\underbrace{-\pi + \pi}_{\text{same}} \right) = 0$$

$$Q_{22} = \lambda_0 R^2 \left(\int_0^\pi 3\sin^2(\varphi) - R^2 \, d\varphi - \underbrace{\int_\pi^{2\pi} 3\sin^2(\varphi) - R^2 \, d\varphi}_{= 3\int_0^\pi \sin^2\varphi} \right)$$

cancels

$$= 0 \Rightarrow Q_{11} = 0$$

$\hookrightarrow$ same with $\sin^2\varphi \to \cos^2\varphi$, also $\mathrm{Tr}\,Q = 0$

Only $Q_{12} = Q_{21}$ left:

$$Q_{12} = Q_{21} = \lambda_0 R^2 \left(\underbrace{\int_0^{\pi} 3 \sin\varphi \cos\varphi \, d\varphi}_{0} - \underbrace{\int_{\pi}^{2\pi} 3 \sin\varphi \cos\varphi \, d\varphi}_{0} \right)$$

$$= 0$$

$$\boxed{\text{So: } \quad Q_{ij} = 0}$$

(c) $\vec{p}$ does not because it's the first non-zero term of the expansion.

Q does because $\vec{p}$ is non-zero.

(d) Field of the ring for large distance on z axis:

These dipole fields can be written in vector form by recombining (4.12) or by directly operating with the gradient on the dipole term in (4.10). The result for the field at a point $\mathbf{x}$ due to a dipole $\mathbf{p}$ at the point $\mathbf{x}_0$ is:

$$\mathbf{E}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{|\mathbf{x}-\mathbf{x}_0|^3} \tag{4.13}$$

where $\mathbf{n}$ is a unit vector directed from $\mathbf{x}_0$ to $\mathbf{x}$.

$\vec{p} = 4\lambda_0 R^2 \vec{e}_y$

$$\vec{E}(z\vec{e}_z) = \frac{3\vec{e}_z \underbrace{(\vec{p}\cdot\vec{e}_z)}_{0} - \vec{p}}{z^3} = -\frac{4\lambda_0 R^2}{z^3} \vec{e}_y$$

$$\Rightarrow \tau = \vec{E} \times \vec{p}_2 = -\frac{4\lambda_0 R^2}{z^3} \left(\vec{e}_y \times \vec{p}_2 \right)$$