Problems

1. **Potential of Charge and Conducting Sphere:** Consider a grounded conducting sphere of radius R (centered at the origin) in presence of a point charge q located outside of the sphere (at position $(0, 0, a)$ with $a > R$). (We discussed this setup in the class but skipped some details in the derivation.)

   (a) (5 pts) Write the potential as a sum of two terms: (i) the potential of the point charge and (ii) a general solution of the azimuthally symmetric Laplace equation (using Legendre polynomials).

   (b) (10 pts) Determine the unknown coefficients in term (ii) using suitable boundary conditions.

   (c) (10 pts) In this example, the term (ii) can be rewritten in a closed and suggestive form. Please perform this resummation and interpret your result in terms of (image) point charges.

   (d) (10 pts) Derive the surface charge density on the sphere in terms of q, R, a and $x = \cos\theta$. [Optional: discuss the limits $a \to R$ and $a \to \infty$.]

   (e) (10 pts) Calculate the induced charge.

   (f) (10 pts) Plot (or draw qualitatively) the following quantities in dependence of the distance from the center of the sphere (along a line from $(0, 0, 0)$ to $(0, 0, a)$): term (i), term (ii), the sum of (i) and (ii), the actual potential.

   (g) (5 pts) How is the potential outside of the sphere modified if the sphere is held at a fixed potential ? (Consult Jackson.)

2. **Green function:** Consider a potential problem in the half-space defined by $z \geq 0$, with Dirichlet boundary conditions on the plane $z = 0$ (and at infinity).

   (a) (10 pts) Write down the appropriate Green function $G(\vec{r}, \vec{r}\,')$.

   (b) (20 pts) If the potential on the plane $z = 0$ is specified to be $\phi = \phi_0$ inside a circle of radius R centered at the origin, and $\phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (s, φ, z).

   (c) (10 pts) Find the formula for $\phi(0, \varphi, z)$ along the axis of the circle ($s = 0$) by explicitly integrating the expression in (b).