Problems

1. **Spherical cavity:** Consider a sphere of radius R that has a hollow spherical cavity of radius b inside it. The center of the big sphere is at the origin, the center of the cavity is at $\vec{a}$. The volume of the big sphere (excluding the cavity) is uniformly charged with a charge density ρ_0.

   (a) (25 pts) Derive the electric field (both magnitude and direction) at an arbitrary point inside the cavity. (Find a compact expression in terms of the given parameters.)

2. **Hydrogen atom, Jackson 1.5 (25 pts):** The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\varepsilon_0} \frac{e^{-\alpha r}}{r} \left(1 + \frac{\alpha r}{2}\right) \tag{1}$$

   where q is the magnitude of electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

3. **Field of a thin disc:** An infinitely thin round disk of radius R has its symmetry axis on the z-axis. It is uniformly charged with total charge q.

   (a) (10 pts) Write an expression for the charge density of the disk $\rho(\vec{r})$ using appropriate coordinate variables.

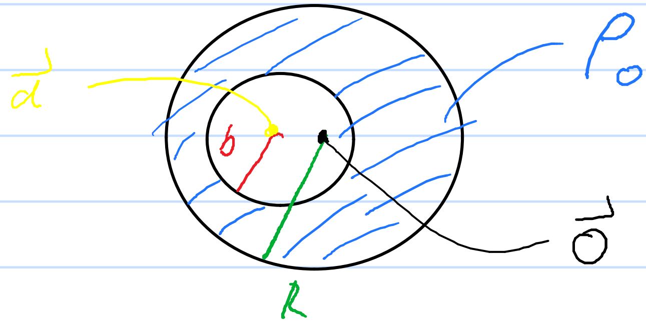
   (b) (5 pts) Determine the cartesian surface density $\sigma(x, y)$ (from your expression for $\rho(\vec{r})$).

   (c) (10 pts) Calculate by direct integration the electric field $\vec{E}(\vec{r})$ at an arbitrary point on the z-axis (from your expression for $\rho(\vec{r}\,')$).

   (d) (5 pts) Find the limits of the field for $z \gg R$ and for $z \ll R$ and explain the results.

4. **Equipotential surface (20 pts):** Two opposite point charges q_1 and $-q_2$ are positioned distance d apart. (Here q_1 and q_2 are unequal positive numbers.) Show that the equipotential surfaces in this system include a sphere of finite radius. Find the location of the center of the sphere and its radius. What is the value of the potential on the surface of this sphere ? (We use such a normalization that the value of the potential at infinity is zero.)

Problem 1



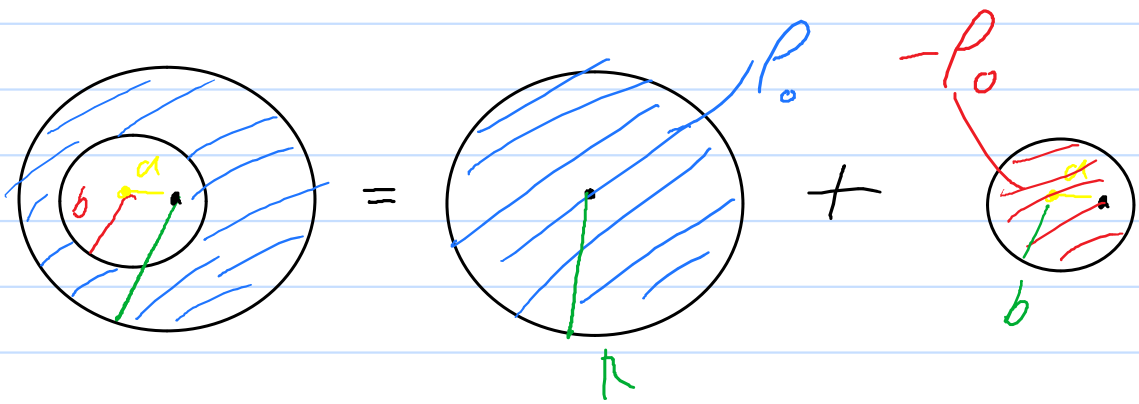
We can allways rotate the system in a way so that $\vec{d} = a\vec{e}_x$. Then we can find the electric field as the superposition of two fields of tw charged spheres on the x-axis:

Generally the E-field inside a charged sphere is:

$$\vec{E}_s = \frac{\rho_s}{3\varepsilon_0} r\vec{e}_r = \frac{\rho_s}{3\varepsilon_0}\vec{r}$$

(derived from Gauss: $4\pi\varepsilon_0 r^2 E = \frac{4}{3}\pi r^3 \rho$)

So here we have:



$$\boxed{\vec{E} = \frac{\rho_0}{3\varepsilon_0}\vec{a}} \quad = \frac{\rho_0}{3\varepsilon_0}\vec{r} + -\frac{\rho_0}{3\varepsilon_0}(\vec{r} - \vec{d})$$

Problem 2

$$\Phi = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

Generally: $\rho = -\varepsilon_0 \nabla^2 \phi$

One would expect the result to be a superposition of the electron's and the proton's charge distribution.

$\phi \xrightarrow{r \to 0} \infty$ so for $r=0$ (the protons position) we have to look at the expension:

$$\phi = \frac{q}{4\pi\varepsilon_0} \underbrace{\sum_{n=0}^{\infty} \frac{r^n}{n!}}_{=e^x} \left(1+\frac{\alpha r}{2}\right) r^{-1}$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}+\frac{\alpha}{2}+\dots\right) \overset{r\approx 0}{\approx} \frac{q}{4\pi\varepsilon_0 r}$$

Which is the potential of a positive point charge.

$\Rightarrow \rho_p = \delta^3(r)\, q$

For $r > 0$:

$$\rho_{e^-} = -\varepsilon_0 \nabla^2 = -\frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} e^{-\alpha r}\left(r^{-1}+\frac{\alpha}{2}\right)$$

$$= \frac{q}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r} e^{-\alpha r}\left(1+\alpha r+\frac{\alpha^2 r^2}{2}\right) = -\frac{q\alpha^3}{8\pi} e^{-\alpha r}$$

So in total: $\boxed{\rho(r) = \rho_p + \rho_{e^-} = \delta^3(r)\, q - \frac{q\alpha^3}{8\pi} e^{-\alpha r}}$

Problem 3

(a) In cylindrical coordinates:

$$\rho(r,\varphi,z) = \rho(r,z) = \frac{q}{\pi R^2}\,\Theta(R-r)\,\delta(z)$$

(b) $$\sigma_{z=0}(x,y) = \frac{q}{\pi R^2}\,\Theta(R^2-x^2-y^2)\,,\quad \rho(x,y,z) = \sigma_{z=0}(x,y)\,\delta(z)$$

(c)

$$\mathbf{E}(\mathbf{x}) = -\nabla\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d^3x' \cdot \varepsilon_0^{-1} \qquad (1.15)$$

$$\varepsilon_0\cdot\vec{E}(z)\Big|_{x=y=0} = -\nabla\int_0^{\infty}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{\rho(z,r=0)}{|z\vec{e}_z - r'\vec{e}_r + z'\vec{e}_z|}\,dz'\,d\varphi\,r'\,dr'$$

$$= -\nabla\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(z)\,dz'\,\frac{q}{\pi R^2}\,\Theta(R^2-r'^2)\,\frac{1}{\left|z\vec{e}_z - \begin{pmatrix}x'\\y'\\z'\end{pmatrix}\right|}\,dx'\,dy'$$

$$= -\frac{q}{\pi R^2}\nabla\int_0^R\frac{r}{\sqrt{r^2+z^2}}\,dr\cdot 2\pi$$

$$= -\frac{2q}{R^2}\frac{\partial}{\partial z}\left[\sqrt{r^2+z^2}\right]_{r=0}^{r=R} = -\frac{2q}{R^2}\frac{\partial}{\partial z}\left[\sqrt{R^2+z^2} - z\right]\vec{e}_z$$

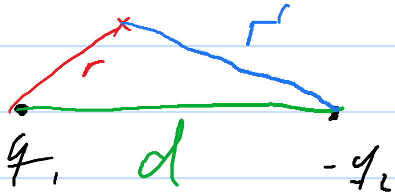
$$\boxed{= \frac{2q}{R^2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)\vec{e}_z}$$

(d) For $z \gg R$: $\frac{R}{z} \approx 0 \Rightarrow \vec{E} \approx \frac{2q}{\varepsilon_0 R^2}\left(1-\frac{z}{\left(z^2\left(1+\frac{R^2}{z^2}\right)\right)^{1/2}}\right)\vec{e}_z = \vec{0}$, $\quad = \frac{z}{z} = 1$

For $R \gg z$: $\frac{z}{R} \approx 0$

$$\Rightarrow \vec{E} \approx \frac{2q}{\varepsilon_0 R^2}\left(1-\frac{z}{R}\right) \approx \frac{2q}{\varepsilon_0 R^2}$$

Problem 4



$$\vec{r}' = \vec{r} - \vec{d} = \vec{r} - d\vec{e}_x$$

$$\vec{d} = d\vec{e}_x$$

Let's choose the origin at the location of q_1.
Then we have:

$$\phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r} - \frac{q_2}{\underbrace{|\vec{r} - d\vec{e}_x|}_{r'}}\right) + C$$

Because of symmetry, the equi-potential sphere's origin must be on the x-axis.

Since we want $\phi \xrightarrow{r\to\infty} 0 \Rightarrow C = 0$.

Let's assume $\phi|_{\underbrace{\partial S}_{\text{sphere surface}}} = 0$ also to find the sphere,

we can proof that this is correct afterwards.

So we just have to find the points on the x-axis where $\phi = 0$:

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{|x|} - \frac{q_2}{|x-d|}\right) = 0$$

$$\Leftrightarrow \frac{q_1}{q_2} = \frac{|x|}{|x-d|} \Rightarrow \frac{q_1^2}{q_2^2} = \frac{x^2}{(x-d)^2}$$

$$\frac{q_1^2}{q_2^2} = \frac{x^2}{(x-d)^2}$$

$$\Rightarrow x_\pm = \frac{q_1 d}{q_1 \pm q_2}$$

Fair to assume because the system is conserved under CP

Assume $q_1 > q_2$, then:

$$2r = x_+ - x_- = \frac{q_1 d}{q_1 + q_2} - \frac{q_1 d}{q_1 - q_2} = q_1 d \left(\frac{2 q_2}{q_2^2 - q_1^2} \right)$$

$$= 2 \frac{q_1 q_2}{q_2^2 - q_1^2} d \Rightarrow \boxed{r = \frac{q_1 q_2}{q_2^2 - q_1^2} d}$$

And the center is at:

$$x_+ - r = \frac{q_1 d}{q_1 + q_2} - \frac{q_1 q_2}{q_2^2 - q_1^2} d = \underbrace{d q_1 \left(\frac{1}{q_1 + q_2} - \frac{q_2}{q_2^2 - q_1^2} \right)}_{=: \Delta x}$$

For $q_2 > q_1$ it's all the same just reflected.

Now we just have to proof $\phi|_{\partial S} = 0$ everywhere on the sphere.

For that we shift the origin to the center of the sphere: $\vec{r} \to \vec{r} - \Delta x \vec{e}_x$

$$\tilde{\phi} = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1}{|\vec{r} - \Delta x \vec{e}_x|} - \frac{q_2}{|\vec{r} - (\Delta x - d) \vec{e}_x|} \right)$$

$$\tilde{\phi}|_{|\vec{r}|=r} = 0$$