Problems

1. **Lorentz transformations:** Let $\Lambda = (\Lambda^\mu{}_\nu)$ be the 4x4 matrix of a Lorentz boost in the x direction and $g = (g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$ the 4x4 matrix of the metric tensor.

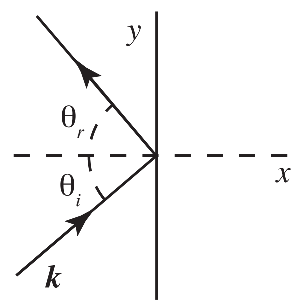
   (a) (5 pts) Show that the Lorentz transformation fulfills $\Lambda^T g \Lambda = g$ and consequently leaves the scalar product of two four-vectors invariant.

   (b) (5 pts) If x^μ is a contravariant vector, what kind of object is $\frac{\partial}{\partial x^\mu}$? Show by studying its transformation properties.

   (c) (10 pts) Show that two Lorentz boosts *both in the x direction* with rapidities ζ_1 and ζ_2 are equivalent to a single boost with rapidity ζ_3. Derive the value of ζ_3 in terms of ζ_1 and ζ_2.

2. **Doppler effect and aberration of light:**

   A light source emits light of frequency ω_S with a wave vector $\mathbf{k}$ in the xy plane, where $|\mathbf{k}| = \omega_S/c$. The light is reflected from a plane mirror parallel to the yz plane. The angle of incidence θ_i and the angle of reflection θ_r are defined with respect to the normal to the mirror, as shown in the figure. Now consider the entire device (both the source and the mirror) in motion with relativistic velocity $\beta = v/c$ in the positive x-direction, with respect to the laboratory. Predict the results of the measurements made in the laboratory for:



   (a) (20 pts) the frequencies of the incident and reflected waves (expressed in terms of ω_S, β, and the angle of incidence and reflection θ_S in the device frame),

   (b) (20 pts) the cosine of the angle of incidence (expressed in terms of $\cos\theta_S$ and β),[1]

   (c) (10 pts) the relation between angle of incidence and angle of reflection (both in the lab frame).

3. **Lorentz transformations for acceleration, Jackson, 11.5 (30 pts):** A coordinate system K' moves with a velocity $\mathbf{v}$ relative to another system K. In K' a particle has a velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$. Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\mathbf{v}$ are

$$\mathbf{a}_\parallel = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_\parallel, \quad \mathbf{a}_\perp = \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left(\mathbf{a}'_\perp + \frac{\mathbf{v}}{c^2} \times (\mathbf{a}' \times \mathbf{u}')\right). \tag{1}$$

[1] The change in direction of light between two different inertial frames is known as the aberration of light. It also occurs classically, but the relativistic formula gives more pronounced effects at large v/c.