Instructions:

- The exam is open book. Please finish the exam by your own.
- All the answers should be typed in so that your work will be fully evaluated.
- Please email your finished exam to Prof. Jianliang Qian (email: jqian@msu.edu) by **5PM, Friday, May 1st, 2020.**
- No late exam will be accepted.

1. (a) (5 points) For $A \in \mathbf{C}^{m\times n}$, show that

$$\|A\|_\infty \leq \sqrt{n}\|A\|_2.$$

(b) (5 points) Show that the inequality in part a) is sharp.

2. (10 points) Let $A \in C^{m \times n}$ with $m \geq n$. Show that A^*A is nonsingular if and only if A has full rank.

3. (10 points) Let $A \in C^{m \times m}$, and let $\mathbf{a}_j$ be its jth column. Prove the following inequality:

$$|\det(A)| \leq \prod_{j=1}^{m} \|\mathbf{a}_j\|_2.$$

4. (a) (5 points) State the Singular Value Decomposition (SVD) theorem for $A \in \mathbf{C}^{m\times n}$.

   (b) (5 points) Let $A \in \mathbf{C}^{m\times n}$. Set $\sigma = \|A\|_2$. Show that there are vectors $\mathbf{v} \in \mathbf{C}^n$ and $\mathbf{u} \in \mathbf{C}^m$ with $\|\mathbf{v}\|_2 = \|\mathbf{u}\|_2 = 1$ such that $A\mathbf{v} = \sigma\mathbf{u}$.

   (c) (5 points) Find an SVD of

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

5. (15 points) Let

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Use the QR factorization of A to solve the least-squares problem

$$\min \|A\mathbf{x} - \mathbf{b}\|_2,$$

where $\mathbf{b} = [0, 0, 3, 2]^T$ with T indicating transpose.

6. (15 points) Let $A \in R^{n\times n}$ be symmetric and positive definite and $n = j + k$. Partition A into the following 2 by 2 blocks:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is $j\times j$ and A_{22} is $k\times k$. Let R_{11} be the Cholesky factor of A_{11}: $A_{11} = R_{11}^T R_{11}$, where R_{11} is upper triangular with positive main-diagonal entries. Let $R_{12} = \left(R_{11}^{-1}\right)^T A_{12}$ and let $\tilde{A}_{22} = A_{22} - R_{12}^T R_{12}$.

(a) Prove that A_{11} is positive definite.

(b) Prove that

$$\tilde{A}_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

(c) Prove that $\tilde{A}_{22}$ is positive definite.

7. (10 points) Show that if $A \in \mathbf{R}^{m\times m}$ is symmetric and positive definite, then solving the linear system $A\mathbf{x} = \mathbf{b}$ amounts to computing

$$\mathbf{x} = \sum_{i=1}^{m} \frac{c_i}{\lambda_i}\mathbf{v}_i,$$

where λ_i are the eigenvalues of A and $\mathbf{v}_i$ are the corresponding eigenvectors, and c_i are some constants determined by $\mathbf{b}$ and $\mathbf{v}_i$.

8. (15 points) Let $A \in \mathbf{C}^{m\times n}, m \geq n$, with linearly independent columns:

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n].$$

Find eigenvalues and eigenvectors of the projection matrix

$$P = I - A(A^*A)^{-1}A^*.$$