Michigan State University

CMSE 823 – Numerical Linear Algebra **Spring 2020**

Homework 5

Alexander Harnisch

1.

We re-use the implementations from the previous homework, you can find them again in the file *qr.py*.

2.

For solving the over and well-determined systems we can use Algorithm 11.2. For under-determined systems we simply transpose the system which effectively converts it into an over-determined system. Everything is pretty much the same.

So instead of $Ax = b$ we solve $x^{\mathrm{T}} A^{\mathrm{T}} = b^{\mathrm{T}}$ and perform the reduced QR decomposition on A^{T} instead. Now we have the solution:

$$x^{\mathrm{T}} = b^{\mathrm{T}} R^{-1} Q^{\mathrm{T}} \tag{1}$$

or equivalently

$$x = QR^{-\mathrm{T}} b = Qy \tag{2}$$

where we find y by forward substitution, since

$$R^{\mathrm{T}} y = b \tag{3}$$

with R^{T} being lower triangular.

I don't know how much of this is going to be discussed in class, I don't have time to wait for it, it doesn't fit my schedule. I also could not find this explicitly in the textbook.

You can find my implementation of the forward and backward substitution in *solve.py*. Note that for small n it is actually faster to calculate the inner product of x and r_j (or l_j in the forward case) in each step when x has been initialized as a zero vector. I showed both ways to implement it by implementing forward and backward subsection in those two different ways.

(a) (b) and (c)

To get the results for the given matrices run *hw05_02.py*. As you can see, the results are correct using all different methods for obtaining the QR decomposition. The output should look as follows:

```
Part (a)
--------
Solving ax=b with a =
[[1 2 3]
 [4 5 6]
 [7 8 7]
```

```
 [4 2 3]
 [4 2 2]]
and b =
[ 6. 15. 22.  9.  8.]

Classical Gram-Schmidt: x =
[1. 1. 1.]

Modified Gram-Schmidt: x =
[1. 1. 1.]

Householder Method: x =
[1. 1. 1.]

Part (b)
--------
Solving ax=b with a =
[[0.7     0.70711]
 [0.70001 0.70711]]
and b =
[1.40711 1.40712]

Classical Gram-Schmidt: x =
[0.99999352 1.00000641]

Modified Gram-Schmidt: x =
[0.99999352 1.00000641]

Householder Method: x =
[1. 1.]

Part (c)
--------
Solving ax=b with a =
[[1 2 3]
 [4 2 9]]
and b =
[ 6 15]

Classical Gram-Schmidt: x =
[0.42857143 0.85714286 1.28571429]

Modified Gram-Schmidt: x =
[0.42857143 0.85714286 1.28571429]

Householder Method: x =
[0.42857143 0.85714286 1.28571429]
```

3.

(a) and (b)

See the following handwritten page.

It's more convenient to start with 3. (b):

Originally we have the objective function

$$\|b - Ax\|_2^2 = (b - Ax)^*(b - Ax)$$

Which according to Theorem 11.1 is minimal if and only if

$$A^*Ax = A^*b.$$

We now add the L_2 normalization term to get the new objective function:

$$\|b - Ax\|^2 + \alpha^2\|x\|^2$$

$$= (b - Ax)^*(b - Ax) + \alpha^2 x^* x$$

$$= (\tilde{b} - \tilde{A}x)^*(\tilde{b} - \tilde{A}x)$$

with $\tilde{A} = \begin{pmatrix} A \\ \alpha I \end{pmatrix}$, $\tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}$

3. (a) Applying Theorem 11.1 to the new objective function immediately yields the new normal equations

$$\tilde{A}^* A x = \tilde{A}^* \tilde{b} = A^* b$$

$$\Leftrightarrow \quad (A^*A + \alpha^2 I)x = A^*b \qquad \square$$

(c)

You can find the code in the file *hilbert.py*.

(d)

Figure 1 shows the distance between the true and recovered solution with $n = 10$, $n = 20$ and $n = 30$ for different values of α between 10^{-15} and 1. With $\alpha = 0$ we get

$$\begin{aligned} n = 10: \quad & \|x_{\text{True}} - x_{\text{Solved}}\| = 0.002035339815873455, \\ n = 20: \quad & \|x_{\text{True}} - x_{\text{Solved}}\| = 292.0636304336893, \\ n = 30: \quad & \|x_{\text{True}} - x_{\text{Solved}}\| = 790.038304403896\,. \end{aligned} \tag{4}$$

These poor solutions without regularization are a direct result of the ill-conditioned nature of the Hilbert matrices.

(e)

As evident by Figure 1, it seems like the best choice for alpha here is

$$\alpha \approx 10^{-10}\,. \tag{5}$$

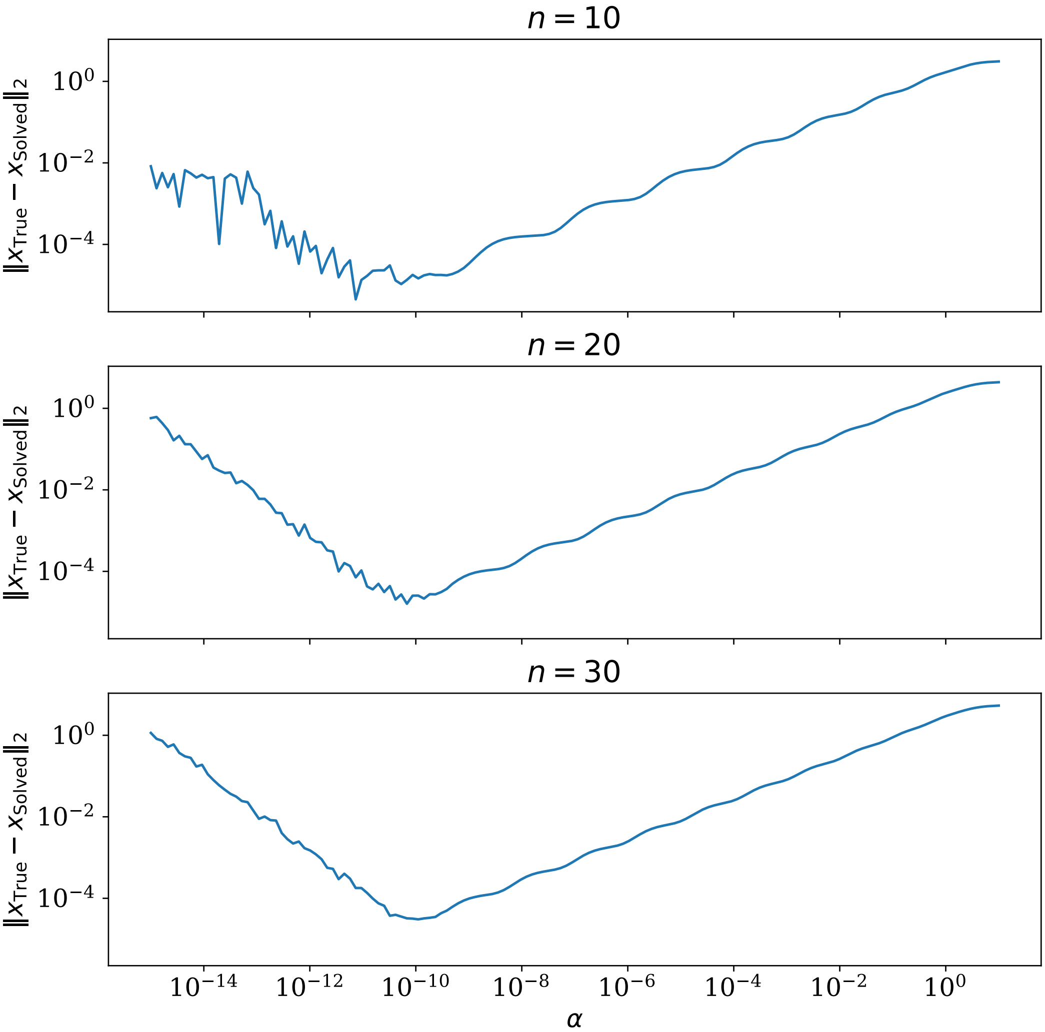

Figure 1: 2-norm between the true and recovered solution with $n = 10$, $n = 20$ and $n = 30$ for different values of α between 10^{-15} and 1.