1.

Scheme A

$$-\frac{p}{h^2}(U_{i+1} - 2U_i + U_{i-1}) + qU_i = \lambda^h U_i, \quad 1 \le i \le N,$$

$$U_0 = 0, \quad U_{N+1} = U_{N-1}.$$

Let $\vec{u}$ be $(u_1, u_2, \dots, u_N)^T$

Then the eigenvalue problem in matrix form is:

$$A\vec{u} = \lambda^h \vec{u} \quad \text{with}$$

$$A = -\frac{p}{h^2}\begin{pmatrix} 1 & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ & 0 & & & \ddots \\ & & & 2 & -2 \end{pmatrix} + q I_N$$

Simplified $A\vec{u} = \tilde{\lambda}\vec{u} = \frac{h^2}{p}\lambda^h \vec{u}, \quad A \in \mathbb{R}^{N \times N}$:

$$A = \begin{pmatrix} 2+\tilde{q} & -1 & & & 0 \\ -1 & 2+\tilde{q} & -1 & & \\ & & \ddots & & \\ 0 & & -1 & 2+\tilde{q} & -1 \\ & & & -2 & 2+\tilde{q} \end{pmatrix}, \quad \tilde{q} = \frac{h^2}{p} q$$

$$\Rightarrow \lambda^h = \frac{p}{h^2}\tilde{\lambda}$$

Or even further, since all diagonal entries are the same. But I guess using that is not the point in the assignment, so I'll go with the unsimplified version that does not require scaling and shifting the eigenvalues! Same for Scheme B.

Scheme B

$$-\frac{p}{h^2}(U_{i+1}-2U_i+U_{i-1})+\frac{q}{6}(U_{i+1}+4U_i+U_{i-1})=\frac{\lambda^h}{6}(U_{i+1}+4U_i+U_{i-1}), 1\le i\le N,$$
$$U_0=0,\quad U_{N+1}=U_{N-1}. \tag{9}$$

$\Leftrightarrow \quad A\vec{u} = \tilde{\lambda}^h B \vec{u}$

with $A = -\frac{6p}{h^2}\begin{pmatrix} -2 & +1 & & 0 \\ +1 & -2 & +1 & \\ & & \ddots & \\ 0 & & +2 & -2 \end{pmatrix} + qB,$

$B = \begin{pmatrix} 4 & 1 & & 0 \\ 1 & 4 & 1 & \\ & & \ddots & \\ 0 & & 2 & 4 \end{pmatrix}$ so $A = (q - \frac{6p}{h^2})B - \frac{p}{h^2} I_N$

Or, again all in one matrix (which I guess is how we're supposed to do it):

$A\vec{u} = \lambda^h B\vec{u}$ with

$A = -\frac{p}{h^2}\begin{pmatrix} 1 & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ 0 & & & 2 & -2 \end{pmatrix} + q\cdot B$

$B = \frac{1}{6}\begin{pmatrix} 1 & 4 & 1 & & 0 \\ & 1 & 4 & 1 & \\ & & & \ddots & \\ 0 & & & 2 & 4 \end{pmatrix}$

2.

$$\lambda_l^h = \frac{pk_h(l)+qm_h(l)}{m_h(l)}, \qquad k_h(l) = 2h^{-2}(1-\cos((l-\frac{1}{2})h)),$$

$$\hat{\lambda}_l^h = pk_h(l)+q. \qquad m_h(l) = \frac{1}{3}(2+\cos((l-\frac{1}{2})h)).$$

$$\lambda_j = p(j-\frac{1}{2})^2+q, \quad j=1,2,\cdots.$$

Convergence of λ_l^h

$$\lim_{h\to 0}\lambda_l^h = \lim_{h\to 0}\frac{pk_h(l)+qm_h(l)}{m_h(l)} = \lim_{h\to 0} p\frac{k_h(l)}{m_h(l)} + q$$

$= q$

$$= p\lim_{h\to 0}\frac{2h^{-2}(1-\cos((l-\frac{1}{2})h))}{\frac{1}{3}(2+\cos((l-\frac{1}{2})h))} + q$$

$$= p\lim_{h\to 0}\frac{6-6\cos[(l-\frac{1}{2})h]}{h^2(2+\cos[(l-\frac{1}{2})h])} + q$$

Numerator $\to 0$

Denominator $\to 0$

$\Rightarrow$ use L'Hospital.

$$= p\lim_{h\to 0}\frac{6(l-\frac{1}{2})\sin[(l-\frac{1}{2})h]}{2h(2+\cos[(l-\frac{1}{2})h]) - h^2(l-\frac{1}{2})\sin[(l-\frac{1}{2})h]} + q$$

Again.

$$= p\lim_{h\to 0}\frac{6(l-\frac{1}{2})^2\cos[(l-\frac{1}{2})h]}{2(2+\cos[(l-\frac{1}{2})h]) - \underbrace{4h(l-\frac{1}{2})\sin[(l-\frac{1}{2})h]) - h^2(l-\frac{1}{2})^2\cos[(l-\frac{1}{2})h]}_{\to 0}} + q$$

$$= p(l-\frac{1}{2})^2\lim_{h\to 0}\frac{6\overbrace{\cos[(l-\frac{1}{2})h]}^{\to 1}}{2(2+\underbrace{\cos[(l-\frac{1}{2})h]}_{\to 1})} + q$$

$$= p(l-\frac{1}{2})^2\frac{6}{2(2+1)} + q = p(l-\frac{1}{2})^2 + q = \lambda_l$$

To show that is converges quadratically we can take a look at the Taylor expansion in dependence of h around $h=0$:

$$\frac{2h^{-2}(1-\cos((l-\frac{1}{2})h))}{\frac{1}{3}(2+\cos((l-\frac{1}{2})h))} = (l-\tfrac{1}{2})^2 + \tfrac{1}{192}(1-2l)^4 h^2 + O(h^4)$$

So for $h \to 0$ the leading order is $O(h^2)$, therefore λ_l^h converges quadratically to λ_l as $h \to 0$. $\square$

I realized the explicit limit was totally unnecessary, but well...

Convergence of $\hat{\lambda}_l^h$

$$\hat{\lambda}_l^h = pk_h(l) + q. = p\, 2h^{-2}(1-\cos((l-\frac{1}{2})h)) + q$$

$$\overset{\text{Taylor Series}}{=} p(l-\tfrac{1}{2})^2 + q - \tfrac{1}{12}p(l-\tfrac{1}{2})^4 h^2 + O(h^4)$$

$$\xrightarrow{h\to 0} p(l-\tfrac{1}{2})^2 + q = \lambda_l \text{ quadratically. } \square$$

Idea for 6.

Since we only need to find the smallest eigenvalue, I think the best approach is to use a modified unshifted inverse iteration:

We have: $A u = \lambda B u$

$$\Longleftrightarrow \quad u = \lambda A^{-1} B u$$

So do an inverse iteration by solving

$$A w_k = u_k \qquad \text{for } w$$

$$\text{where } u_k = B v_k \,, \quad \frac{v_k}{\|v_k\|_2} = 1$$

Then the modified Rayleigh coefficient (and our eigenvalue estimate) is:

$$A w_k = r (B w_k)$$

B is symmetric

$$\Rightarrow \quad r_k = \frac{w_k^T B^T A w_k}{w_k^T w_k} = \frac{w_k^T B A w_k}{w_k^T w_k}$$

Repeat with $v_{k+1} = \dfrac{w_k}{\|w_k\|_2}$ until convergence.