Numerical Linear Algebra
Homework 1

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1.3. Generalizing Example 1.3, we say that a square or rectangular matrix R with entries r_{ij} is *upper-triangular* if $r_{ij} = 0$ for $i > j$. By considering what space is spanned by the first n columns of R and using (1.8), show that if R is a nonsingular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular. (The analogous result also holds for lower-triangular matrices.)

$$e_j = \sum_{i=1}^{m} z_{ij} a_i. \tag{1.8}$$

$$\Leftrightarrow \quad \vec{e}_j = A \vec{z}_j \quad , \quad Z = A^{-1}$$

$$\begin{bmatrix} \Bigg| & & \Bigg| \\ e_1 & \cdots & e_m \\ \Bigg| & & \Bigg| \end{bmatrix} = I = AZ = AA^{-1}$$

Here: $A = R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & 0 & \ddots & \vdots \\ & & & r_{nn} \end{bmatrix}$

Let $Z = R^{-1}$. We want to prove that Z is upper triangular.

Let $\vec{z}_i$, $1 \leq i \leq n$ be the columns of Z.

Since R is rectangular $\Rightarrow \det R = \prod_{i=1}^{n} r_{ii}$

$\Rightarrow$ all r_{ii} $1 \leq i \leq n$ are non-zero, otherwise Z would not exist.

$\Rightarrow \quad \vec{e}_1 = r_{11} \vec{z}_1 \quad \Rightarrow \quad \boxed{\begin{array}{l} \vec{z}_1 = r_{11}^{-1} \vec{e}_1 \qquad (1) \\ z_{i1} = 0, \ i > 1 \\ z_{1k} \neq 0, \ i = 1 \end{array}}$

We can continue with induction from here:

$$\vec{e}_{i+1} = \sum_{j=1}^{n} \vec{z}_j \, r_{j,i+1} = \sum_{j=1}^{i+1} \vec{z}_j \, r_{j,i+1}$$

Because $r_{j,i+1} = 0$ for $j > i+1$

$$= \sum_{j=1}^{i} \vec{z}_j \, r_{j,j+1} + \vec{z}_{i+1} \, r_{i+1,i+1}$$

$$\Rightarrow \vec{z}_{i+1} = \vec{r}^{\,-1}_{(i+1)(i+1)} \Big(\underbrace{\vec{e}_{i+1}}_{\substack{e_{i+1,k} \\ = \delta_{i+1,k}}} - \underbrace{\sum_{j=1}^{i} \vec{z}_j \, r_{j,j+1}}_{\substack{(1) \Rightarrow z_{jk} = 0 \\ \text{for } k > j \\ z_{jk} \neq 0 \\ \text{for } k \leq j}} \Big)$$

$$\Rightarrow z_{i+1,k} = 0 \text{ for } k > i+1$$
$$\neq 0 \text{ for } k \leq i+1$$

$\square$

1.4. Let $f_1, \dots, f_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, \dots, d_8$, there exists a set of coefficients $c_1, \dots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \qquad i = 1, \dots, 8.$$

(a) Show by appealing to the theorems of this lecture that $d_1, \dots, d_8$ determine $c_1, \dots, c_8$ uniquely.
(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, \dots, d_8$ to coefficients $c_1, \dots, c_8$. What is the i, j entry of A^{-1}?

It's not clearly stated, but I think it's mentioned somewhere in the book that the default field is $\mathbb{C}$, so:

$$f_i : [1, 8] \subset \mathbb{R} \to \mathbb{C}, \quad i = 1, \dots, 8$$

Let $F \in \mathbb{C}^{8 \times 8}$ be the Matrix

with $F_{ij} = f_j(i)$

$$\Rightarrow \sum_{j=1}^{8} c_j f_j(i) = d_i, \iff F\vec{c} = \vec{d}$$

with $\vec{c}, \vec{d} \in \mathbb{C}^8$

a) We know that $\vec{c}$ exist for <u>any</u> $\vec{d}$.
Therefore $\text{range}(F) = \mathbb{C}^8$.

$\iff$ which means F^{-1} exists (Theorem 1.3)

Therefore $\vec{c} = F^{-1}\vec{d}$ is unique. $\square$

Being strict: $\text{range}(F) = \mathbb{C}^8 \iff \vec{d}$ is unique.

b)

$$A\vec{d} = \vec{c} \quad \text{for } \underline{\text{any}} \ c \in \mathbb{C}^8$$

$$\Rightarrow A\vec{d} = A \underbrace{F\vec{c}}_{\vec{d} \text{ per definition}} = \vec{c} \quad \Rightarrow A = F^{-1}$$

Since $A^{-1} = F$: $\boxed{A^{-1}{}_{ij} = F_{ij} = f_j(i)}$

2.1. Show that if a matrix A is both triangular and unitary, then it is diagonal.

2.2. The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

(a) Prove this in the case $n = 2$ by an explicit computation of $\|x_1 + x_2\|^2$.

(b) Show that this computation also establishes the general case, by induction.

2.3. Let $A \in \mathbb{C}^{m\times m}$ be hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

(a) Prove that all eigenvalues of A are real.

(b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

2.1 If A is upper triangular, then A^{-1} is also upper tiangular. (Proof given by 1.3)
If A is lower triangular, then A^{-1} is also lower triangular (Proof analogous to 1.3)

$\Rightarrow$ Since A is unitary it must be diagonal. $\square$

2.2 a) For $n=2$: $x_i \in \mathbb{C}^m$

$$\sum_{i=1}^{n} \|\vec{x}_i\|^2 = \|\vec{x}_1\|^2 + \|\vec{x}_2\|^2$$

$$\left\|\sum_{i=1}^{n} \vec{x}_i\right\|^2 = \|\vec{x}_1 + \vec{x}_2\|^2 = \sum_{i=1}^{m} (x_{1,i} + x_{2,i})^2$$

$$= \sum_{i=1}^{m} (x_{1,i}^2 + x_{2,i}^2) + 2\sum_{i=1}^{m} x_{1i}\, x_{2i} = \|\vec{x}_1\|^2 + \|\vec{x}_2\|^2 + \underbrace{2x_2^* x_1}_{=0 \text{ because orthogonal set}}$$

$$= \sum_{i=1}^{n} \|\vec{x}_i\|^2 \quad \square$$

b)

$$\left\| \sum_{i=1}^{n+1} \vec{x_i} \right\|^2 = \left\| \underbrace{\sum_{i=1}^{n} \vec{x_i}}_{:= \vec{u}} + \vec{x}_{n+1} \right\|^2$$

$\vec{u}$ is a sum of vectors all perpendicular to $\vec{x}_{n+1}$ => $\vec{u}$ is perpendicular to $\vec{x}_{n+1}$: $\vec{u}^* \vec{x}_{n+1} = 0$

$$= \sum_{i=1}^{m} (u_i^2 + x_{n+1,i}) + 2 \sum_{i=1}^{m} u_i x_{n+1,i} = \|\vec{u}\|^2 + \|\vec{x}_{n+1}\| + 2 \overbrace{\vec{u}^* \vec{x}_{n+1}}$$

$$= \left\| \sum_{i=1}^{n} \vec{x}_i \right\|^2 + \|\vec{x}_{n+1}\|^2$$

Induction step assuming $\left\| \sum_{i=1}^{j} x_i \right\|^2 = \sum_{i=1}^{j} \|x_i\|^2$ for $1 \leq j \leq n$

$$\stackrel{\downarrow}{=} \sum_{i=1}^{n} \|\vec{x}_i\|^2 + \|\vec{x}_{n+1}\|^2 = \sum_{i=1}^{n+1} \|x_i\|^2 \quad \square$$

2.3. Let $A \in \mathbb{C}^{m \times m}$ be hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

(a) Prove that all eigenvalues of A are real.

(b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

a) That's a good one. Quantum Mechanics would have a serious problem if that wasn't true!

$$\lambda \|\vec{x}\|^2 = \lambda \vec{x}^* \vec{x} = \vec{x}^* (\lambda x) = \vec{x}^* A \vec{x} \overset{A \text{ is hermitian}}{=} \vec{x}^* A^* \vec{x} = (A\vec{x})^* \vec{x}$$
$$= \lambda^* \vec{x}^* \vec{x} = \lambda^* \|\vec{x}\|^2$$

$\vec{x}$ is an eigenvector $\Rightarrow \|\vec{x}\|^2 \neq 0$
$\Rightarrow \lambda^* = \lambda$
$\Leftrightarrow \lambda$ is real $\square$

b)

Let's say $A \vec{x}_i = \lambda_i \vec{x}_i$, $1 \leq i \leq n$

Then for $i \neq j$:

$$\lambda_j \vec{x}_i^* \vec{x}_j = \vec{x}_i^* (A \vec{x}_j) = (\vec{x}_i^* A) \vec{x}_j \overset{A^* = A}{=} (\vec{x}_i^* A^*) \vec{x}_j$$

$$= (A \vec{x}_i)^* \vec{x}_j = (\lambda_i \vec{x}_i)^* \vec{x}_j \overset{a)}{=} \lambda_i \vec{x}_i^* \vec{x}_j$$

It follows that $\vec{x}_i^* \vec{x}_j = 0$ because $\lambda_i \neq \lambda_j$ for $i \neq j$

$\square$

2.5. Let $S \in \mathbb{C}^{m\times m}$ be *skew-hermitian*, i.e., $S^* = -S$.

(a) Show by using Exercise 2.1 that the eigenvalues of S are pure imaginary. 2.3 not 2.1

(b) Show that $I - S$ is nonsingular.

(c) Show that the matrix $Q = (I-S)^{-1}(I+S)$, known as the *Cayley transform* of S, is unitary. (This is a matrix analogue of a linear fractional transformation $(1+s)/(1-s)$, which maps the left half of the complex s-plane conformally onto the unit disk.)

a) $S^* = -S = i^2 S \iff \frac{1}{i} S^* = iS \iff (iS)^* = iS$

$\Rightarrow iS$ is hermitian

2.3 $\Rightarrow iS$ has only real eigenvalues λ_i

$\Rightarrow$ all eigenvalues $i\lambda_i$ of iS are pure imaginary $\square$

b) If $I-S$ is non-singular, then $\text{null}(I-S) = \{\vec{0}\}$

So $(I-S)\vec{x} = 0$ can only be true for $\vec{x} = \vec{0}$.

Proof that $\vec{x} = \vec{0}$:

$(I-S)\vec{x} = 0 \iff \vec{x} = S\vec{x}$

$\iff S\vec{x} = \lambda\vec{x}, \ \lambda = 1$

We know from a) that $\lambda = 1$ is not possible $\Rightarrow \vec{x} = \vec{0}$ $\square$

Alternative without using a)

$$\vec{x}^*\vec{x} = (S\vec{x})^*\vec{x} = \vec{x}^* S^* \vec{x} \overset{S^* = -S}{=} -\vec{x}^* S \vec{x} = -\vec{x}^*\vec{x}$$

$\Rightarrow \vec{x} = \vec{0}$ $\square$

(c) Show that the matrix $Q = (I-S)^{-1}(I+S)$, known as the *Cayley transform* of S, is unitary. (This is a matrix analogue of a linear fractional transformation $(1+s)/(1-s)$, which maps the left half of the complex s-plane conformally onto the unit disk.)

c) $$Q^* = (I+S)^*\big(\underbrace{(I-S)}_{}{}^{-1}\big)^*$$

non-singular $\Rightarrow \big((I-S)^{-1}\big)^* = \big((I-S)^*\big)^{-1}$

$$= (I+S)^*\big((I-S)^*\big)^{-1}$$

notation used in textbook

$$\Rightarrow Q^*Q = (I+S)^*(I-S)^{-*}\cdot(I-S)^{-1}(I+S)$$

$$= (I+S^*)\big((I-S)(I-S)^*\big)^{-1}(I+S)$$

$S^* = -S$

$$= (I-S)\big((I-S)(I+S)\big)^{-1}(I+S)$$

$$= (I-S)\underbrace{(I-S^2)^{-1}\underbrace{(I+S)\cdot\underbrace{(I-S)(I-S)^{-1}}_{\cdot I}}_{=(I-S^2)}}_{=I}$$

$$= (I-S)(I-S)^{-1} = I \qquad \square$$

2.6. If $\vec{u}$ and $\vec{v}$ are m-vectors, the matrix $A = I + \vec{u}\vec{v}^*$ is known as a *rank-one perturbation of the identity*. a) Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\vec{u}\vec{v}^*$ for some scalar α, and give an expression for α.
b) For what $\vec{u}$ and $\vec{v}$ is A singular? If it is singular, what is null(A)?

a) If A is non-singular, then A^{-1} exists and vice versa. So we only have to show that $AA^{-1} = I$

$$AA^{-1} = (I + \vec{u}\vec{v}^*)(I + \alpha\vec{u}\vec{v}^*)$$

$$= I + \vec{u}\vec{v}^* + \alpha\vec{u}\vec{v}^* + \overbrace{\alpha\vec{u}\underbrace{\vec{v}^*\vec{u}}_{\text{scalar}}\vec{v}^*}^{= \alpha\vec{v}^*\vec{u}\,\vec{u}\vec{v}^*}$$

guess: $\boxed{\alpha = -\dfrac{1}{1+\vec{v}^*\vec{u}}}$

$$\Rightarrow AA^{-1} = I + \vec{u}\vec{v}^* - \frac{\vec{u}\vec{v}^*}{1+\vec{v}^*\vec{u}} - \frac{\vec{u}\vec{v}^* \cdot \vec{v}^*\vec{u}}{1+\vec{v}^*\vec{u}}$$

$$= I + \vec{u}\vec{v}^* - \frac{\vec{u}\vec{v}^*\cancel{(I+\vec{v}^*\vec{u})}}{\cancel{1+\vec{v}^*\vec{u}}} = I \quad \square$$

b) Again we use A non singular $\Leftrightarrow$ null $A = \{\vec{0}\}$

So if A is singular, there must exist at least one $\vec{x} \neq \vec{0}$ such that

$$A\vec{x} = \vec{x} + \vec{u}\vec{v}^*\vec{x} = \vec{0}$$

$$\Rightarrow \vec{x} = -\underbrace{\vec{v}^*\vec{x}}_{\text{scalar}}\,\vec{u}$$

gosh I prefer bra-ket notation, it's so much easier to see this...

$|x\rangle = -\langle v|x\rangle|u\rangle$

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That means: $\vec{x} = \beta \vec{u}$, $\beta \in \mathbb{C} \setminus \{0\}$

$$\Rightarrow A\vec{x} = \beta\vec{u} + \vec{u}(\vec{v}^* \beta\vec{u}) = \beta\vec{u}(1 + \vec{v}^*\vec{u}) = \vec{0}$$

$$\Rightarrow \boxed{\vec{v}^*\vec{u} = -1 \iff A \text{ is singular} \quad \square}$$

In that case:

$$\text{null}(A) = \{\beta\vec{u} \mid \beta \in \mathbb{C}\}$$