Homework 2

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3.1

Prove that if W is an arbitrary nonsingular matrix, the function $\|\cdot\|_W$ defined by (3.3) is a vector norm.

Solution

Check the three conditions, since $\|\cdot\|$ is a vector norm, it satisfies those 3 conditions in (3.1). Then for $\|\cdot\|_W$,

- $\|x\|_W = \|Wx\| \geq 0$, and $\|x\|_W = \|Wx\| = 0$ only if $Wx = 0$, which holds only if $x = 0$ since W is non-singular.
- For all vectors x and y, $\|x+y\|_W = \|W(x+y)\| = \|Wx + Wy\| \leq \|Wx\| + \|Wy\| = \|x\|_W + \|y\|_W$.
- For all scalars $\alpha \in \mathbb{C}$, $\|\alpha x\|_W = \|W\alpha x\| = \|\alpha(Wx)\| = |\alpha|\|Wx\| = |\alpha|\|x\|_W$.

Then $\|x\|_W = \|Wx\|$ is a vector norm.

3.2

Let $\|\cdot\|_W$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the spectral radius of A, i.e., the largest absolute value $|\lambda|$ of an eigenvalue λ of A.

Solution

Let λ be the eigenvalue for A with largest absolute value, i.e., $|\lambda| = \rho(A)$, let $x_0 \neq 0$ be an eigenvector corresponding to λ, then $Ax_0 = \lambda x_0$.

$$\|Ax_0\| = \|\lambda x_0\| = |\lambda|\|x_0\| = \rho(A)\|x_0\|,$$

which gives

$$\rho(A) = \frac{\|Ax_0\|}{\|x_0\|} \leq \sup_{x\in\mathbb{C}^n,\, x\neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|.$$

Here the last equality is by the fact that $\|\cdot\|$ is an induced norm. So we have $\rho(A) \leq \|A\|$.

3.3

Vector and matrix p-norms are related by various inequalities, often involving the dimensions of m or n. For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general m,n) for which equality is achieved. In this problem, x is an m-vector and A is an $m \times n$ matrix.
(a) $\|x\|_\infty \leq \|x\|_2$
(b) $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$
(c) $\|A\|_\infty \leq \sqrt{n}\|A\|_2$
(d) $\|A\|_2 \leq \sqrt{m}\|A\|_\infty$

Solution

(a) Assume $x \in \mathbb{R}^m$, let x_0 be the entry in x with largest absolute value, i.e., $|x_0| = \|x\|_\infty$. Then,

$$\|x\|_2 = (\sum_{i=1}^{m} |x_i|^2)^{1/2} \geq (|x_0|^2)^{1/2} = \|x\|_\infty.$$

Example: $x = (1, 0, 0, \cdots, 0)^T$.
(b) Assume $x \in \mathbb{R}^m$, then

$$\|x\|_2 = \sqrt{\sum_{i=1}^{m} |x_i|^2} \leq \sqrt{\sum_{i=1}^{m} \|x\|_\infty^2} = \sqrt{m\|x\|_\infty^2} = \sqrt{m}\|x\|_\infty.$$

Example: $x = (1, 1, \cdots, 1)^T$.
(c) Use the inequalities in (a) and (b), we get

$$\|A\|_\infty = \sup_{x\in\mathbb{C}^n,\, x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \sup_{x\in\mathbb{C}^n,\, x\neq 0} \frac{\|Ax\|_2}{\frac{1}{\sqrt{n}}\|x\|_2} = \sqrt{n} \sup_{x\in\mathbb{C}^n,\, x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{n}\|A\|_2.$$

Example: $A^{m\times n} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, then $\|A\|_\infty = n,\ \|A\|_2 = \sqrt{n}$.

(d) Again, use the inequalities in (a) and (b), we get

$$\|A\|_2 = \sup_{x\in\mathbb{C}^n,\, x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \sup_{x\in\mathbb{C}^n,\, x\neq 0} \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{m} \sup_{x\in\mathbb{C}^n,\, x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n}\|A\|_\infty.$$

Example: $A^{m\times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$, then $\|A\|_\infty = 1,\ \|A\|_2 = \sqrt{m}$.

4.1

Determine the SVDs of the following matrices (by hand calculation):

(a)

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Solution

(a) Let $A = U\Sigma V^*$ be the SVD decomposition, then $A^*A = V\Sigma^*\Sigma V^*$.

$$A^*A = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$$

Then the eigenvalues of A^*A are $\lambda_1 = 9,\ \lambda_2 = 4$, thus $\sigma_1 = 3,\ \sigma_2 = 2$. It is easy to see that the corresponding eigenvectors are $(1,0)^T$ and $(0,1)^T$, which forms a orthonormal basis in $\mathbb{C}^2$. We have

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then we can also obtain U,

$$U = AV\Sigma^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(c) Let $A = U\Sigma V^*$ be the SVD decomposition, then $A^*A = V\Sigma^*\Sigma V^*$.

$$A^*A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}.$$

So $\sigma_1 = 2,\ \sigma_2 = 0$, then

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ \Sigma^*\Sigma = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = V \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} V^*,$$

which further gives

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Again, from $AV = U\Sigma$ we can see that

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4.4

False, if two matrices A and B have same singular values, they may not be unitarily equivalent.
Counter example :

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

A and B have same singular values, but they are not unitarily equivalent, since B is Hermitian, then QBQ^* is also Hermitian, but A is not.

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