Numerical Linear Algebra
Homework 2

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3.1. Prove that if W is an arbitrary nonsingular matrix, the function $\|\cdot\|_W$ defined by (3.3) is a vector norm.

$$\|x\|_W = \|Wx\|. \tag{3.3}$$

We need to show that (3.1) holds:

$$\begin{aligned} &(1)\ \|x\| \ge 0, \text{ and } \|x\| = 0 \text{ only if } x = 0,\\ &(2)\ \|x+y\| \le \|x\| + \|y\|, \\ &(3)\ \|\alpha x\| = |\alpha|\,\|x\|. \end{aligned} \tag{3.1}$$

(1) $\|\vec{x}\| \geq 0 \quad \forall x \Rightarrow \|W\vec{x}\| \geq 0$

$\|W\vec{x}\|$ is only 0 if $Wx = \vec{0}$, since W is nonsingular $W\vec{x} = \vec{0}$ only if $\vec{x} = \vec{0}$. Therefore $\underbrace{\|W\vec{x}\|}_{=\|\vec{x}\|_W} = 0$ only if $\vec{x} = \vec{0}$.

(2)

$$\|\vec{x}+\vec{y}\|_W = \|W(\vec{x}+\vec{y})\| = \|W\vec{x} + W\vec{y}\|$$

$$\leq \|W\vec{x}\| + \|W\vec{y}\| = \|\vec{x}\|_W + \|\vec{y}\|_W$$

(3)

$$\|\alpha\vec{x}\|_W = \|W(\alpha\vec{x})\| = |\alpha|\,\|W\vec{x}\| = |\alpha|\,\|\vec{x}\|_W$$

$\square$

3.2. Let $\|\cdot\|$ denote any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(A) \leq \|A\|$, where $\rho(A)$ is the *spectral radius* of A, i.e., the largest absolute value $|\lambda|$ of an eigenvalue λ of A.

Let $\vec{v}$ be the igenvector corresponding to λ:

$$A\vec{v} = \lambda \vec{v}$$

$$\Rightarrow \quad |\lambda| = \frac{\|A\vec{v}\|}{\|\vec{v}\|} \leq \sup_{\vec{x} \in \mathbb{C}^m \setminus \{0\}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \|A\| \quad \square$$

3.3. Vector and matrix p-norms are related by various inequalities, often involving the dimensions m or n. For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general m, n) for which equality is achieved. In this problem x is an m-vector and A is an $m \times n$ matrix.

(a) $\|x\|_\infty \leq \|x\|_2$,

(b) $\|x\|_2 \leq \sqrt{m}\,\|x\|_\infty$,

(c) $\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$,

(d) $\|A\|_2 \leq \sqrt{m}\,\|A\|_\infty$.

(a) That's just obvious. Say the maximum entry of $\vec{x}$ is x_i, then:

$$\|\vec{x}\|_\infty = |x_i| \leq \left(\sum_{j=1}^{m} |x_j|^2\right)^{1/2} = \|\vec{x}\|_2$$

$\square$

Equality for any multiple of a unit vector. $\alpha \vec{e_i}$

(b) Again, say x_i is the maximum entry of $\vec{x}$.

Then:

$$\|\vec{x}\|_2 = \left(\sum_{j=1}^{m} |x_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{m} |x_i|^2\right)^{1/2}$$

$$= \sqrt{m}\,|x_i| = \sqrt{m}\,\|\vec{x}\|_\infty$$

$\square$

(c) Follows from (a) and (b):

$$\frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty} \leqslant \frac{\|A\vec{x}\|_2}{1/\sqrt{n}\,\|\vec{x}\|_2} = \sqrt{n}\,\frac{\|A\vec{x}\|}{\|\vec{x}\|_2}, \quad \forall \vec{x} \in \mathbb{C}^n \setminus \{\vec{0}\}$$

$$\Leftrightarrow \quad \sup_{\vec{x}\in\mathbb{C}^n\setminus\{0\}} \frac{\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty} \leqslant \sup_{\vec{x}\in\mathbb{C}^n\setminus\{0\}} \sqrt{n}\,\frac{\|A\vec{x}\|}{\|\vec{x}\|_2}$$

$$\Leftrightarrow \quad \|A\|_\infty \leqslant \sqrt{n}\,\|A\|_2 \quad \square$$

(d) Same:

$$\frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2} \leqslant \frac{\sqrt{m}\,\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty}$$

$$\Leftrightarrow \sup_{\vec{x}\in\mathbb{C}^n\setminus\{0\}} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2} \leqslant \sup_{\vec{x}\in\mathbb{C}^n\setminus\{0\}} \frac{\sqrt{m}\,\|A\vec{x}\|_\infty}{\|\vec{x}\|_\infty}$$

$$\Leftrightarrow \quad \|A\|_2 \leqslant \sqrt{m}\,\|A\|_\infty \quad \square$$

4.1. Determine SVDs of the following matrices (by hand calculation):

(a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$A = U \Sigma V^*$$

(a)

$AA^* = A^2 = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$ => Eigenvalues are 9 and 4, eigenvectors are $\vec{e}_1, \vec{e}_2$

Theorem 5.4

$\Rightarrow \Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow U\Sigma = \Sigma \Rightarrow V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $AA^* = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ => only eigenvalue is 4 with eigenvector $\vec{e}_1$

$\Rightarrow \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow U\Sigma = \Sigma \Rightarrow V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

4.4. Two matrices $A, B \in \mathbb{C}^{m\times m}$ are *unitarily equivalent* if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m\times m}$. Is it true or false that A and B are unitarily equivalent if and only if they have the same singular values?

It is true. Tchnically the problem does not ask for a proof, but here you go:

Let $A = U_A \Sigma_A V_A^*$
$B = U_B \Sigma_B V_B^*$

Then $A = Q(U_B \Sigma_B V_B^*) Q^* = \underbrace{(QU_B) \Sigma_B (QV_B)^*}_{\text{also an SVD.}}$

The singular values are unique (proof in textbook) $\Rightarrow \Sigma_B = \Sigma_A$ $\square$

5.3. Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

(a) Determine, on paper, a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V.

(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.

(c) What are the 1-, 2-, ∞-, and Frobenius norms of A?

(d) Find A^{-1} not directly, but via the SVD.

(e) Find the eigenvalues λ_1, λ_2 of A.

(f) Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.

(g) What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

(a) Again, use Theorem 5.4 :

$$AA^* = AA^T = \begin{pmatrix} 125 & 75 \\ 75 & 125 \end{pmatrix}$$

$$\Rightarrow \quad (125-\lambda)^2 \overset{!}{=} 75^2$$

$$\Rightarrow \quad \left.\begin{matrix} \lambda_1 = 200 \\ \lambda_2 = 50 \end{matrix}\right\} \Rightarrow \begin{matrix} \sigma_1 = 10\sqrt{2} \\ \sigma_2 = 5\sqrt{2} \end{matrix} \Rightarrow \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$AA^T U = U\Sigma^2$$

$$\Leftrightarrow \quad \begin{pmatrix} 125 & 75 \\ 75 & 125 \end{pmatrix} U = U \begin{pmatrix} 200 & 0 \\ 0 & 50 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} U = U \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow \text{ choose } U \propto \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 2 \\ 8 & -2 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 8 & -2 \end{pmatrix} \checkmark$$

normalize: $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\Rightarrow V = A^T U \Sigma^{-1} = \begin{pmatrix} -2 & -10 \\ 11 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/10\sqrt{2} & 0 \\ 0 & 1/5\sqrt{2} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$= \begin{pmatrix} -12 & 8 \\ 16 & 6 \end{pmatrix} \begin{pmatrix} 1/10 & 0 \\ 0 & 1/5 \end{pmatrix} \cdot \frac{1}{2}$$

$$= \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \cdot \frac{1}{5}$$

(b) From (a):

$\sigma_1 = 10\sqrt{2}$

$\sigma_2 = 5\sqrt{2}$

$$A = \overset{\text{Left}}{U}\, \Sigma\, \overset{\text{Right}}{V^T}$$

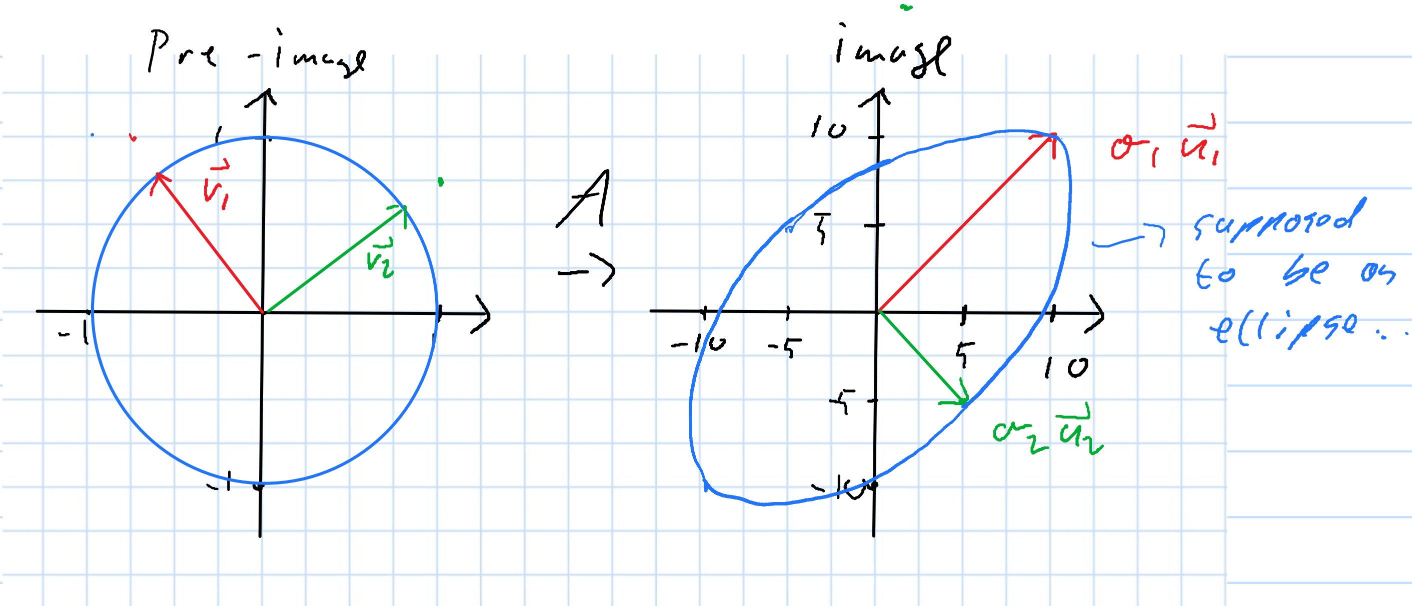
Left singular vectors:

$$\underbrace{2^{-1/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{=\vec{u}_1}, \quad 2^{-1/2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{u}_2$$

Right singular vectors:

$$\underbrace{5^{-1}\begin{pmatrix} -3 \\ 4 \end{pmatrix}}_{=\vec{v}_1}, \quad 5^{-1}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \vec{v}_2$$

Note that they are all normalized, like we want them to be!



(c) **Theorem 5.3.** $\|A\|_2 = \sigma_1$ *and* $\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$.

Proof. The first result was already established in the proof of Theorem 4.1: since $A = U\Sigma V^*$ with unitary U and V, $\|A\|_2 = \|\Sigma\|_2 = \max\{|\sigma_j|\} = \sigma_1$, by Theorem 3.1. For the second, note that by Theorem 3.1 and the remark following, the Frobenius norm is invariant under unitary multiplication, so $\|A\|_F = \|\Sigma\|_F$, and by (3.16), this is given by the stated formula. ☐

So $\|A\|_2 = 10\sqrt{2}$, $\|A\|_F = \sqrt{250} = 5\sqrt{10}$

Example 3.3. The 1-Norm of a Matrix. If A is any $m \times n$ matrix, then $\|A\|_1$ is equal to the "maximum column sum" of A. We explain and derive

So $\|A\|_1 = 16$

Example 3.4. The ∞-Norm of a Matrix. By much the same argument, it can be shown that the ∞-norm of an $m \times n$ matrix is equal to the "maximum row sum,"

$$\|A\|_\infty = \max_{1\le i\le m} \|a_i^*\|_1, \tag{3.10}$$

where a_i^* denotes the ith row of A. ☐

So $\|A\|_\infty = 15$

(d)

$$A^{-1} = (U\Sigma V^T)^{-1} = V\Sigma^{-1}U^T = \frac{1}{10}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1/10 & 0 \\ 0 & 1/5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= 100^{-1}\begin{pmatrix} 5 & -11 \\ 10 & -2 \end{pmatrix}$$

(e) $\lambda^2 - 3\lambda + 100 \stackrel{!}{=} 0 \Rightarrow \lambda_\pm = \dfrac{3 \pm \sqrt{391}\, i}{2}$

(f) The first part is always true.

But well: $\det A = -10 + 110 = 100$

$\lambda_+ \cdot \lambda_- = \frac{1}{4}(9 + 391) = 100$ ✓

$$\sigma_1 \sigma_2 = \sqrt{2}^2 \, 5 \cdot 10 = 100$$ ✓

(g) Generally in 2D: $A = \pi \underbrace{a\, b}_{\text{semi-axes}}$

We know that $\sigma_1 = a$, $\sigma_2 = b$

$$\Rightarrow A = \pi \sigma_1 \sigma_2 = 100\pi$$