11.1. Suppose the $m \times n$ matrix A has the form

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},$$

where A_1 is a nonsingular matrix of dimension $n \times n$ and A_2 is an arbitrary matrix of dimension $(m-n) \times n$. Prove that $\|A^+\|_2 \le \|A_1^{-1}\|_2$.

When $m = n$ then $A = A_1$ and $A^+ = A_1^{-1} = A^{-1}$.

So we can assume $m > n$.

Let $A = QR = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$ be the reduced QR decomposition of A. Then:

$$A^+ = (A^* A)^{-1} A^* = (R^* Q^* Q R)^{-1} R^* Q^*$$
$$= R^{-1} Q^* = A_1^{-1} Q_1 Q^*$$

$$\Rightarrow \|A^+\|_2 \le \|A_1^{-1}\|_2 \|Q_1 Q^*\|_2$$

So we need to prove that $\|Q_1 Q^*\|_2 \le 1$

Recall:

These processes of multiplication by a unitary matrix or its adjoint preserve geometric structure in the Euclidean sense, because inner products are preserved. That is, for unitary Q,

$$(Qx)^*(Qy) = x^* y, \qquad (2.9)$$

as is readily verified by (2.4). The invariance of inner products means that angles between vectors are preserved, and so are their lengths:

$$\|Qx\| = \|x\|. \qquad (2.10)$$

In the real case, multiplication by an orthogonal matrix Q corresponds to a rigid rotation (if $\det Q = 1$) or reflection (if $\det Q = -1$) of the vector space.