Numerical Linear Algebra
Homework 3

Alexander Hurnisch
hurnisc6@msu.edu

Problem 5.3

See my submission of Homework 2.

6.1. If P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

Theorem 6.1. *A projector P is orthogonal if and only if $P = P^*$.*

Algebraic Proof:

$$(I-2P)(I-2P)^* = (I-2P)(I^* - 2\overset{\overset{?}{=}P}{P^*}) = (I-2P)(I-2P)$$

$$= (\underbrace{I-P}_{\text{complement}}-P)^2 = \underbrace{(I-P)^2}_{\text{projector}} - \underbrace{(I-P)P - P(I-P)}_{P-P^2=P-P=0} + \underbrace{P^2}_{\text{projector}}$$

$$= I - P + P = I \quad \square$$

Geometric Interpretation:

$2P$ is <u>not</u> a projector (Proof: $(2P)^2 = 4P^2 = 4P \neq 2P$).

$2P$ is the operation of projecting a vector onto the subspace range(P) and then stretching it by a factor of 2 in all directions of that subspace.

$$\begin{pmatrix} \text{range}(2P) = \text{range}(P) \\ \text{range}(I-2P) = \text{range}(I-P) \end{pmatrix}$$

Therefore $I - 2P$ mirrors any vector around the subspace P projects onto.

Which simply means subtracting the range(P) components from the vector twice, flipping the signs in the orthogonal subspace of range(P) which is range($I-P$). Doing this again to the same vector therefore restores the original vector, hence $(I-2P)$ is unitary.

6.2. Let E be the $m \times m$ matrix that extracts the "even part" of an m-vector: $Ex = (x + Fx)/2$, where F is the $m \times m$ matrix that flips $(x_1, \ldots, x_m)^*$ to $(x_m, \ldots, x_1)^*$. Is E an orthogonal projector, an oblique projector, or not a projector at all? What are its entries?

E is an ortogonal projector.

Proof:

First: E is a projector:

Since flipping $(x_m, \ldots, x_1)^*$ restores $(x_1, \ldots, x_m)^*$

$\Rightarrow F^2 = I \Leftrightarrow F$ is involutory.

$$\Rightarrow E^2 = \frac{1}{4}(I + 2F + \overset{=I}{F^2}) = \frac{2I + 2F}{4} = \frac{I+F}{2} = E$$

Second: E is an orthogonal projector.

$\Leftrightarrow E$ is hermitian (Theorem 6.1)

The matrix represantation of F looks like this:

$$F = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \cdots & & & \cdots & 1 & 0 & 0 \\ & \cdots & & 1 & & \cdots & \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \Rightarrow F \text{ is hermitian}$$

$$\Rightarrow E^* = \left(\frac{I+F}{2}\right)^* = \frac{I + F^*}{2} = \frac{I+F}{2} = E \quad \square$$

$$E = \frac{1}{2}\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 1 & 0 \\ \cdots & & 2 & & \cdots \\ 0 & 1 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

6.3. Given $A \in \mathbb{C}^{m \times n}$ with $m \geq n$, show that A^*A is nonsingular if and only if A has full rank.

Let the SVD of A be $A = U\Sigma V^*$

Using

Theorem 5.1. *The rank of A is r, the number of nonzero singular values.*

Proof. The rank of a diagonal matrix is equal to the number of its nonzero entries, and in the decomposition $A = U\Sigma V^*$, U and V are of full rank. Therefore $\operatorname{rank}(A) = \operatorname{rank}(\Sigma) = r$. $\square$

it follows that A is of full rank if and only if Σ is of full rank.

$$A^*A = V\Sigma^*\Sigma V^* \quad \text{(Eigenvalue decomposition)}$$

(A^*A non singular $\Leftrightarrow$ A^*A has p distinct eigenvalues.)

not crucial for proof

$\Rightarrow$ A^*A is non-singular if and only if Σ is of full rank.

$\Rightarrow$ A^*A is non-singular if and only if A is of full rank. $\square$

Also: A^*A is a square matrix

$\Rightarrow$ A^*A non-singular $\Leftrightarrow$ A^*A is of full rank!

6.4. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Answer the following questions by hand calculation.

(a) What is the orthogonal projector P onto range(A), and what is the image under P of the vector $(1,2,3)^*$?

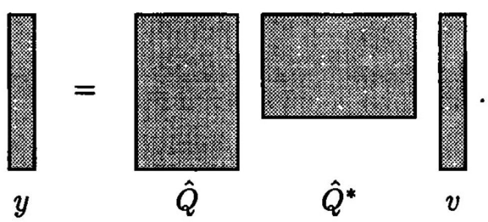
~~(b) Same questions for B.~~

(a) It's easy for A considering that:

p. 45, textbook

$$v \mapsto \sum_{i=1}^{n} (q_i q_i^*) v \tag{6.7}$$

is an orthogonal projector onto range($\hat{Q}$), and in matrix form, it may be written $y = \hat{Q}\hat{Q}^* v$:



Thus any product $\hat{Q}\hat{Q}^*$ is always a projector onto the column space of $\hat{Q}$, regardless of how $\hat{Q}$ was obtained, as long as its columns are orthonormal. Perhaps $\hat{Q}$ was obtained by dropping some columns and rows from a full factorization $v = QQ^*v$ of the identity,

The columns of A are othogonal already, we just need to normalize:

$$a_1 = (1, 0, 1)^* \Rightarrow \frac{a_1}{\|a_1\|_2} = \frac{1}{\sqrt{2}} a_1$$

a_2 is already normalized:

$$\Rightarrow P = \left(\frac{a_1}{\sqrt{2}}, a_2 \right) \cdot \begin{pmatrix} a_1^*/\sqrt{2} \\ a_2^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P \cdot (1, 2, 3)^* = (2, 2, 2)^*$$