Numerical Linear Algebra
Homework 9

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24.1. For each of the following statements, prove that it is true or give an example to show it is false. Throughout, $A \in \mathbb{C}^{m\times m}$ unless otherwise indicated, and "ew" stands for eigenvalue. (This comes from the German "Eigenwert." The corresponding abbreviation for eigenvector is "ev," from "Eigenvektor.")

(a) If λ is an ew of A and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an ew of $A - \mu I$. ✓

(b) If A is real and λ is an ew of A, then so is $-\lambda$. ✗

(c) If A is real and λ is an ew of A, then so is $\bar{\lambda}$. ✗

(d) If λ is an ew of A and A is nonsingular, then λ^{-1} is an ew of A^{-1}. ✓

(e) If all the ew's of A are zero, then $A = 0$. ✗

(f) If A is hermitian and λ is an ew of A, then $|\lambda|$ is a singular value of A. ✓

(g) If A is diagonalizable and all its ew's are equal, then A is diagonal. ✓

(a) True. Proof:

λ is ew of $A \iff \det(A-\lambda I) = 0$

$\Rightarrow \det\big(A-\mu I - (\lambda-\mu)I\big) = \det(A-\lambda I) = 0$

$\iff \lambda-\mu$ is ew of $A-\mu I$

$\square$

(b) False. Proof:

Take the matrix $A = \lambda I$ with ew λ with algebraic multiplicity m. So here A is real and has ew λ, but not $-\lambda$.

$\square$

(c) False. Proof same as above with $\mathrm{im}(\lambda) \neq 0$.

$A = \lambda I$, $\lambda = a + bi$, $b \neq 0$, $a, b \in \mathbb{R}$

$\Rightarrow \lambda$ is ew but $\bar{\lambda}$ is not an ew.

$\square$

(d) True. Proof: $Ax = \lambda x \iff A\lambda^{-1}x = x$

$\iff \lambda^{-1}x \overset{\uparrow}{=} A^{-1}x$ $\square$

A non singular

Additionally we find the corresponding ev to be the same!

(e) False. Proof:

Take $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \neq 0$

$\det(A - \lambda I) = -(1-\lambda)(1+\lambda) + 1 = \lambda^2 \overset{!}{=} 0$

$\Rightarrow A$ has the only ew $\lambda = 0$ and $A \neq 0$. $\square$

(f) True. Proof in textbook:

Theorem 5.5. *If $A = A^*$, then the singular values of A are the absolute values of the eigenvalues of A.*

Follows from 5.4 ?!

Proof. As is well known (see Exercise 2.3), a hermitian matrix has a complete set of orthogonal eigenvectors, and all of the eigenvalues are real. An equivalent statement is that (5.1) holds with X equal to some unitary matrix Q and Λ a real diagonal matrix. But then we can write

$$A = Q\Lambda Q^* = Q|\Lambda|\operatorname{sign}(\Lambda)Q^*, \quad (5.2)$$

where $|\Lambda|$ and $\operatorname{sign}(\Lambda)$ denote the diagonal matrices whose entries are the numbers $|\lambda_j|$ and $\operatorname{sign}(\lambda_j)$, respectively. (We could equally well have put the factor $\operatorname{sign}(\Lambda)$ on the left of $|\Lambda|$ instead of the right.) Since $\operatorname{sign}(\Lambda)Q^*$ is unitary whenever Q is unitary, (5.2) is an SVD of A, with the singular values equal to the diagonal entries of $|\Lambda|$, $|\lambda_j|$. If desired, these numbers can be put into nonincreasing order by inserting suitable permutation matrices as factors in the left-hand unitary matrix of (5.2), Q, and the right-hand unitary matrix, $\operatorname{sign}(\Lambda)Q^*$. $\square$

(g) True. Proof: Let's go the other way:

A is diagonalizable and all ew's are λ

Let $B = \lambda I \overset{\downarrow}{=} X^{-1}AX$ with X non-singular

(similarity transformation)

$\Rightarrow A = X\lambda I X^{-1} = \lambda X X^{-1} = \lambda I = B$ $\square$

24.2. Here is *Gerschgorin's theorem*, which holds for any $m \times m$ matrix A, symmetric or nonsymmetric. *Every eigenvalue of A lies in at least one of the m circular disks in the complex plane with centers a_{ii} and radii $\sum_{j \neq i} |a_{ij}|$. Moreover, if n of these disks form a connected domain that is disjoint from the other $m - n$ disks, then there are precisely n eigenvalues of A within this domain.*

First Part

Second Part

(a) Prove the first part of Gerschgorin's theorem. (Hint: Let λ be any eigenvalue of A, and x a corresponding eigenvector with largest entry 1.)

(a) Let λ be any eigenvalue of A, and x a corresponding eigenvector x with largest entry $x_i = 1$ (which can always be obtained).

$(|x_j| \leq 1, j \neq i)$

So $$Ax = \lambda x \Rightarrow \sum_j a_{ij} x_j = \lambda \overset{1}{x_i} = \lambda$$

$$\Leftrightarrow \sum_{j \neq i} a_{ij} x_j + a_{ii} \overset{=1}{x_i} = \lambda$$

$$\Rightarrow |\lambda - a_{ii}| = \left|\sum_{j \neq i} a_{ij} x_j\right| \underset{\text{triangle inequality}}{\leq} \sum_{j \neq i} |a_{ij}||x_j| \underset{x_{j \neq i} \leq 1}{\leq} \sum_{j \neq i} |a_{ij}| \quad \square$$

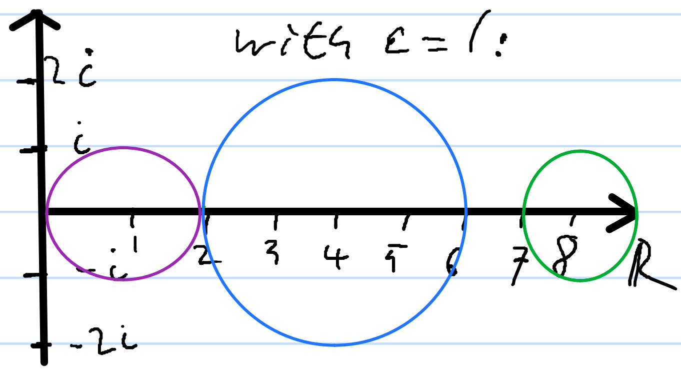
(c) Give estimates based on Gerschgorin's theorem for the eigenvalues of

$$A = \begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}, \qquad |\epsilon| < 1.$$

(d) Find a way to establish the tighter bound $|\lambda_3 - 1| \leq \epsilon^2$ on the smallest eigenvalue of A. (Hint: Consider diagonal similarity transformations.)

(c) The 3 disks are:

$D(8, 1)$, $D(4, 1+\varepsilon)$, $D(1, \varepsilon)$



Since none of them overlap, we know that each one of them contains exactly one ew.

Additionally, A is symmetric (hermitian), therefore all 3 ews are real (Theorem 24.7).

So we know: $\lambda_1 > \lambda_2 > \lambda_3$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$

with $|\lambda_1 - 8| \leq 1$

$|\lambda_2 - 4| \leq 1 + \varepsilon$

$|\lambda_3 - 1| \leq \varepsilon$

(d) Consider the similarity transformation
$A \to X^{-1} A X$ with $X = \mathrm{diag}(1, 1, \varepsilon^{-1})$:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\varepsilon \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 1 & 0 \\ 4 & 1 & \varepsilon \\ 0 & \varepsilon^2 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^{-1} \end{pmatrix} = \begin{pmatrix} 8 & 1 & 0 \\ 4 & 1 & 1 \\ 0 & \varepsilon^2 & 1 \end{pmatrix}$$

According to Theorem 24.3 B has the same eigenvalues as A. Therefore we can again apply Gerschgorin's Theorem as in part (c). This time we obtain

$$|\lambda_3 - 1| \leq \varepsilon^2 \qquad \square$$

25.1. (a) Let $A \in \mathbb{C}^{m \times m}$ be tridiagonal and hermitian, with all its sub- and superdiagonal entries nonzero. Prove that the eigenvalues of A are distinct. (Hint: Show that for any $\lambda \in \mathbb{C}$, $A - \lambda I$ has rank at least $m - 1$.)

(b) On the other hand, let A be upper-Hessenberg, with all its subdiagonal entries nonzero. Give an example that shows that the eigenvalues of A are not necessarily distinct.

(a) Let $B = A - \lambda I$, $\lambda \in \mathbb{C}$

$\Rightarrow$ B is also tridiagonal with all its sub- and superdiagonal entries nonzero.

Let's write B as

$$B = \begin{pmatrix} v^T & 0 \\ U & w \end{pmatrix}, \quad \text{with } v \text{ and } w \in \mathbb{C}^{m-1} \text{ and } U \in \mathbb{C}^{(m-1)\times(m-1)}$$

So U is upper-triangular and its diagonal entries are nonzero. Therefore, U is of full rank $m-1$. From that it directly follows, that $\operatorname{rank}(B) \geq m-1$ ($m-1$ if $\det B = 0$, m if $\det B \neq 0$).

If $\det(B) = 0 \Rightarrow \operatorname{rank}(B) = m-1$ and λ is an ev of A and the eigenspace is the nullspace of B.

$\dim(\operatorname{null}(B)) = m - \operatorname{rank}(B) = 1$ is the geometric multiplicity of λ.

We already know that A is hermitian.

Theorem 24.7. *A hermitian matrix is unitarily diagonalizable, and its eigenvalues are real.*

Theorem 24.5. *An $m \times m$ matrix A is nondefective if and only if it has an eigenvalue decomposition $A = X\Lambda X^{-1}$.*

$\Rightarrow$ Any hermitian matrix is nondefective.

So we have proven that all eigenvalues of A have geometric multiplicity 1 and A is nondefective.
Therefore, every eigenvalue has algebraic multiplicity 1 which means every eigenvalue is distinct. $\square$

(b)

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad p = (1-\lambda)^3$

Has $\lambda = 1$ with algebraic multiplicity 3.

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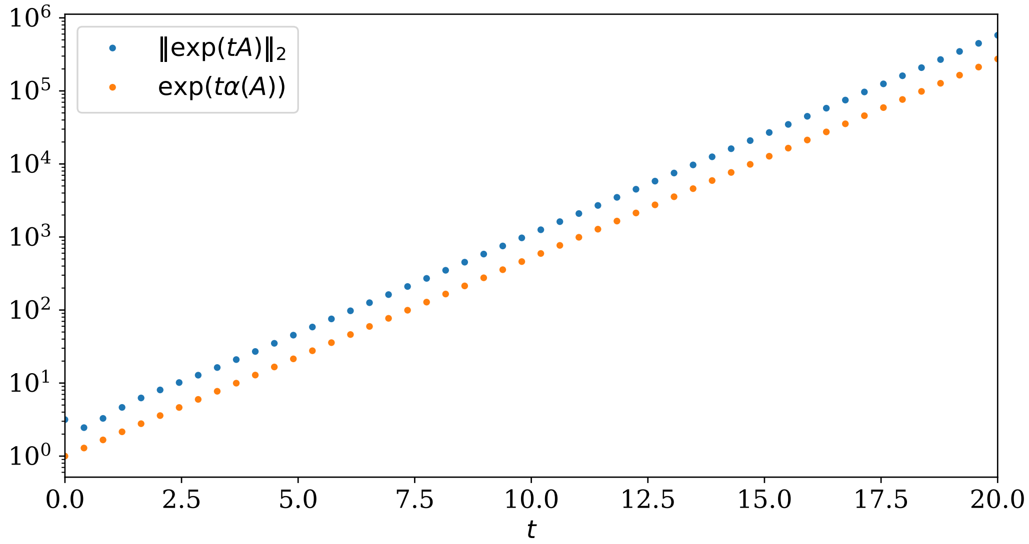
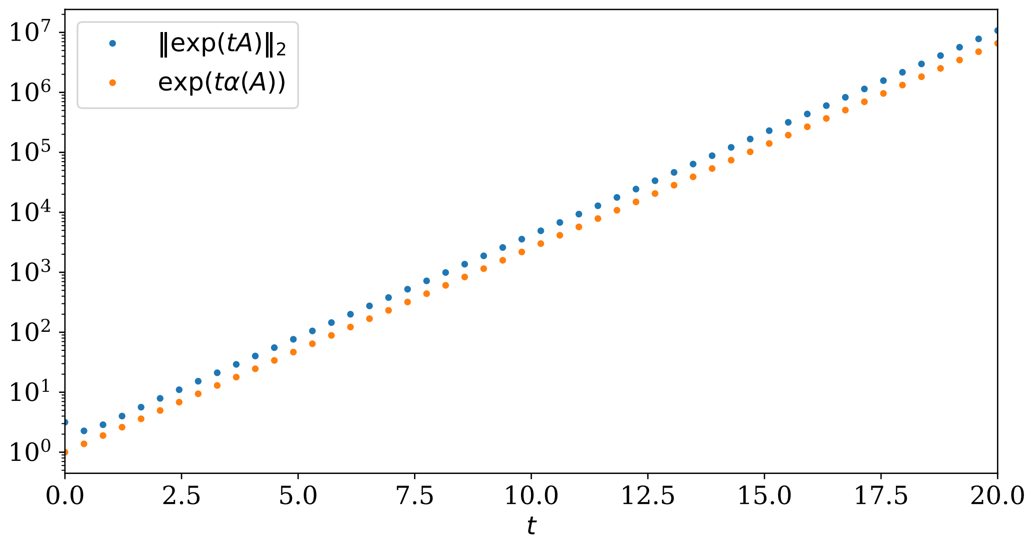
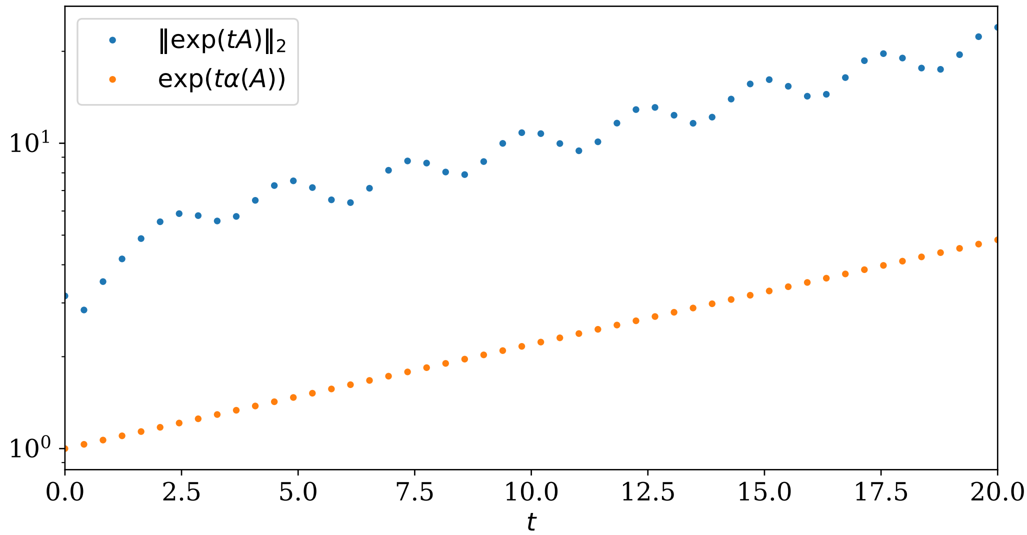
24.3

Here are some examples of the resulting plot. Most of the plots turn out to look something like this:

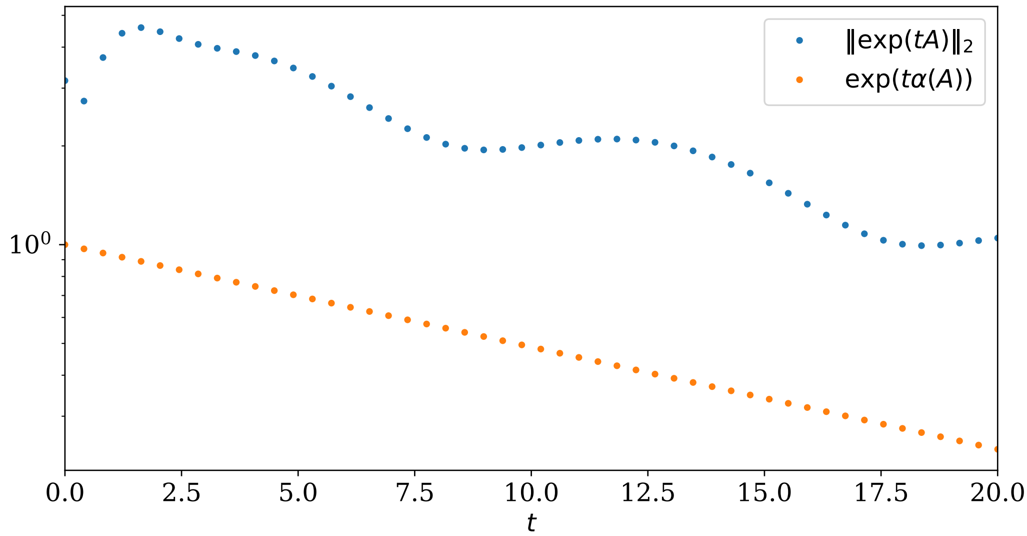




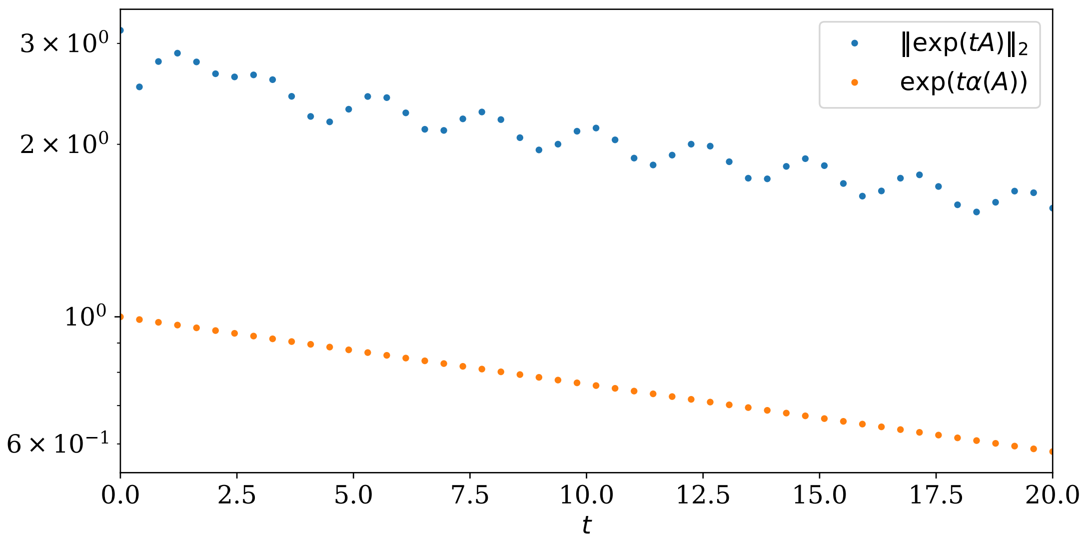
However, sometimes oscillations occur:



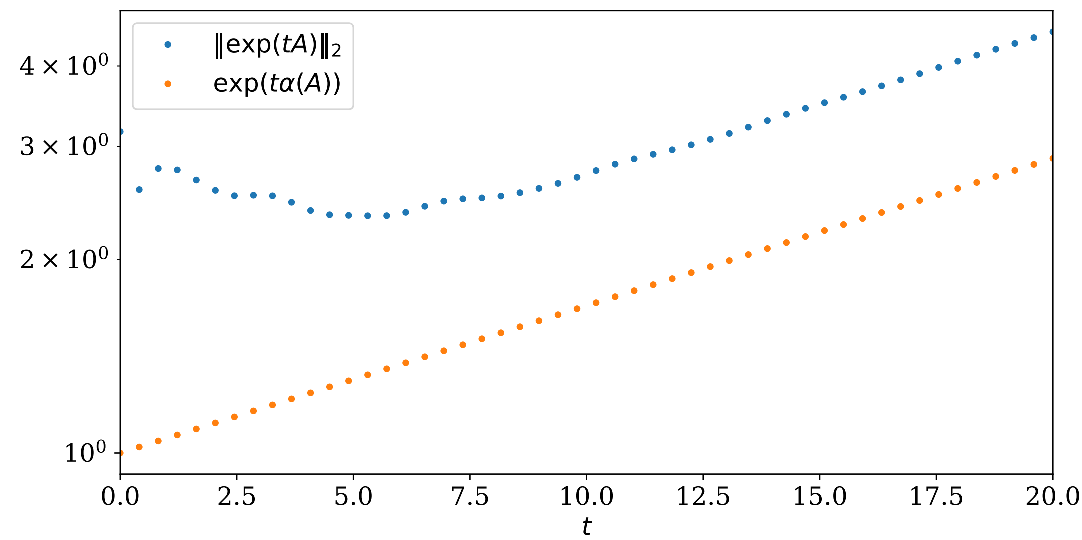
$\|\exp(tA)\|_2$
$\exp(t\alpha(A))$
10^1
10^0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t

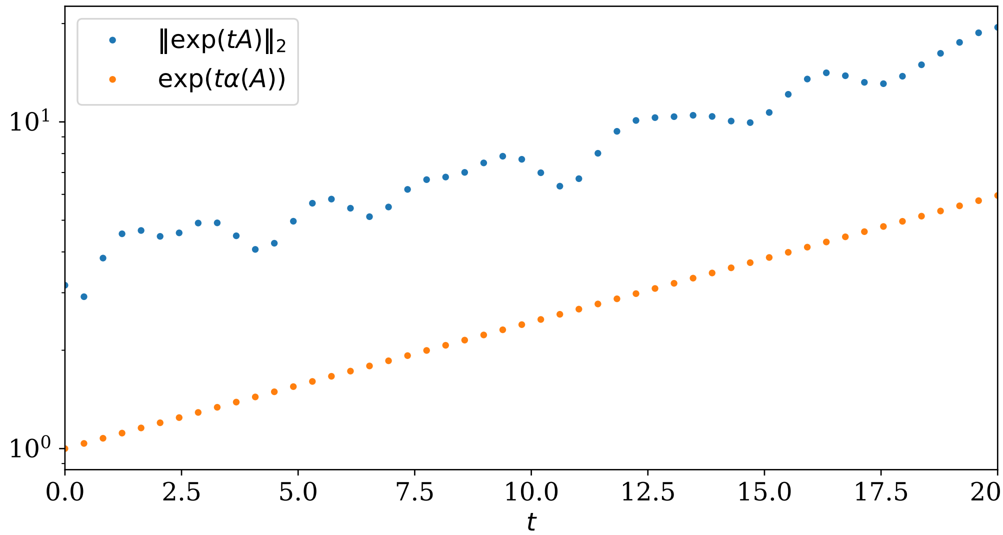


$\|\exp(tA)\|_2$
$\exp(t\alpha(A))$
10^0
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t



And there are also some where it seems to be something in between:





I've played around with this a lot and tried to figure out what leads to the different outcomes. Sadly I did not come up with anything conclusive. I think it most likely has something todo with the imaginary parts of the eigenvalues. Where some combination of more imaginary contribution leads to the oscillations. However, I did not find a conclusive rule.