CMSE 823: Final Programming Project

Eigenvalue problems for Sturm-Liouville Equations

Due Date: 5PM, Thursday, April 23, 2020

---

1 Purpose

Eigenvalues are related to properties of structures, such as the buckling of beam columns and shells, the vibration of elastic bodies, and multigroup diffusion in nuclear reactors.

The goal of this **final** programming project is to solve the 1D Sturm-Liouville spectrum problems.

Besides performing the required calculations and producing the required plots, please include a writeup of no more than *5* pages in length. The writeup should *succinctly* address the relevant results of your computations and the algorithms used to generate these results.

No project report will be accepted after the due date.

2 Description of Problems

In order to illustrate the strength/weaknesses of a given eigenvalue solver, we will consider the following Sturm-Liouville(SL) spectrum problems:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = \lambda u, \quad 0 < x < \pi, \tag{1}$$

$$u(0) = 0, \quad u'(\pi) = 0. \tag{2}$$

It is known that such an SL problem has an infinite sequence of real eigenvalues:

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots \to \infty,$$

and an associated complete set of orthonormal eigenfunctions:

$$(u_j, u_k) = \int_0^\pi u_j(x)u_k(x)dx = \delta_{jk},$$

where δ_{jk} is the Dirac-delta function: δ_{jk}=1 when $j = k$, and δ_{jk}= 0 when $j \neq k$.

In the case of constant p and q, the analytical eigenfunctions are sinusoidal:

$$u_j(x) = \sqrt{\frac{2}{\pi}}\sin\left(j - \frac{1}{2}\right)x, \quad j = 1, 2, \cdots, \tag{3}$$

$$\lambda_j = p(j - \frac{1}{2})^2 + q, \quad j = 1, 2, \cdots. \tag{4}$$

When $j = 1$, λ_1 is the fundamental frequency, and u_1 is the associated normal mode. Usually we are interested in a few leading eigenvalues.

3 Discretization

We will concentrate on the constant-coefficient problem:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u=\lambda u, \quad 0<x<\pi, \tag{5}$$

$$u(0)=0, \quad u'(\pi)=0, \tag{6}$$

where p and q are constant functions. Then the above equation can be transformed into a matrix eigenvalue problems using the finite difference method.

The interval $[0,\pi]$ is partitioned into N subintervals, each interval of width $h=\frac{\pi}{N}$. Let $x_i=i*h$ denote grid points and let U_i denote $U(x_i)$.

We will study two different discretizations.

1. Scheme A:

$$-\frac{p}{h^2}(U_{i+1}-2U_i+U_{i-1})+qU_i=\lambda^h U_i, \quad 1\le i\le N, \tag{7}$$

$$U_0=0, \quad U_{N+1}=U_{N-1}. \tag{8}$$

2. Scheme B:

$$-\frac{p}{h^2}(U_{i+1}-2U_i+U_{i-1})+\frac{q}{6}(U_{i+1}+4U_i+U_{i-1})=\frac{\lambda^h}{6}(U_{i+1}+4U_i+U_{i-1}), 1\le i\le N,$$

$$U_0=0, \quad U_{N+1}=U_{N-1}. \tag{9}$$

Scheme A yields an usual eigenvalue problem, and Scheme B yields a generalized eigenvalue problem.

For the generalized eigenvalue system (9), the components of the l-th eigenvector are

$$U_j^l=\sqrt{\frac{2}{\pi}}\sin((l-\frac{1}{2})jh);$$

the associated eigenvalue λ_l^h is

$$\lambda_l^h=\frac{pk_h(l)+qm_h(l)}{m_h(l)},$$

where

$$k_h(l)=2h^{-2}(1-\cos((l-\frac{1}{2})h)),$$

$$m_h(l)=\frac{1}{3}(2+\cos((l-\frac{1}{2})h)).$$

For the usual eigenvalue system (7), the components of the l-th eigenvector remain unchanged, and the eigenvalue $\hat{\lambda}_l^h$ is

$$\hat{\lambda}_l^h=pk_h(l)+q.$$

4 Questions

1. Formulate the eigenvalue problems in terms of matrices.

2. Show that both λ_l^h and $\hat{\lambda}_l^h$ converge to λ_l in the same $O(h^2)$ accuracy.

3. For the eigenvalue problem (7), use the **inverse power method** to compute the smallest eigenvalue. Numerically demonstrate that the smallest eigenvalue converges to the true smallest eigenvalue in the second order by taking $p = 1$, $q = 5$, and $N = 2^n$, $n = 4, 5, 6, 7, 8, 9$. You may plot the convergence order as a curve of slope 2.

4. For the eigenvalue problem (7), use the **shifted power method** to compute the smallest eigenvalue. Numerically demonstrate that the smallest eigenvalue converges to the true smallest eigenvalue in the second order by taking $p = 1$, $q = 5$, and $N = 2^n$, $n = 4, 5, 6, 7, 8, 9$.

5. For the eigenvalue problem (7), write an efficient **QR iteration** code with deflation to compute all eigenvalues. Numerically demonstrate that the smallest eigenvalue converges to the true smallest eigenvalue in the second order by taking $p = 1$, $q = 5$, and $N = 2^n$, $n = 4, 5, 6, 7, 8, 9$.

6. For the generalized eigenvalue problem (9), design a method to compute the smallest eigenvalue. Numerically demonstrate that the smallest eigenvalue converges to the true smallest eigenvalue in the second order by taking $p = 1$, $q = 5$, and $N = 2^n$, $n = 4, 5, 6, 7, 8, 9$.

7. You may use the Matlab eigenvalue solver to calibrate your computations.

5 What to turn in

There are two parts that you need to turn in: the codes and the writeup.

5.1 The codes

You should write a short description (README file) of how to run the codes that you have written.

5.2 The writeup

Besides performing the required calculations and producing the required plots, please turn in a writeup of no more than *5* pages in length.

The writeup should *succinctly* address the relevant results of your computations and the algorithms used to generate these results.

5.3 Turn in

Please include all your codes and writeup in a tar ball so that the grader can easily manage those files. Please email your tar ball to the grader, He Lyu, at **lyuhe@msu.edu** by the due date. To indicate that it is a CMSE 823 project, please put "**823 Project**" in the subject line of the email.

5.4 No project report will be accepted after the due date.