Homework Assignments: HW # 5

CMSE 823: Numerical Linear Algebra

1. Given an arbitrary matrix $A \in C^{m,n}$, you have constructed the QR decomposition by using the following three different procedures:

   (a) the classical Gram-Schmidt method;

   (b) the modified Gram-Schmidt method;

   (c) the Householder transform based method.

   Each method should return Q and R matrices in a suitable format.

   In this project we are going to use these three factorizations to solve least-squares problems.

2. Solve the following three systems by the three different QR decomposition methods.

   (a) Let

   $$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{pmatrix}$$

   Solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is taken such that the true solution is $\mathbf{x} = (1, 1, \cdots, 1)^T$.

   (b) Let

   $$A = \begin{pmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{pmatrix}$$

   Solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is taken such that the true solution is $\mathbf{x} = (1, 1)^T$.

   (c) Let

   $$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 9 \end{pmatrix}$$

   Solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (6, 15)^T$. Notice that this is an under-determined system.

3. Numerical discretization for inverse problems usually results in ill-conditioned linear systems. To solve such linear systems we use the Tikhonov regularization method. Then we may apply the QR method to solve the regularized system.

   Consider the following over-determined linear system:

$$A\mathbf{x} = \mathbf{b}, \tag{0.1}$$

   where $A \in R^{m\times n}$, $\mathbf{x} \in R^n$, and $\mathbf{b} \in R^m$.

   The Tikhonov regularization can be formulated as a minimization problem:

$$\min_{\mathbf{x}\in R^n} \|\mathbf{b} - A\mathbf{x}\|_2^2 + \alpha^2 \|\mathbf{x}\|_2^2$$

   where $\alpha \geq 0$ is a non-negative number.

   Questions:

   (a) prove that the necessary condition to minimize the above function is

$$(A^T A + \alpha^2\ I)\mathbf{x} = A^T\mathbf{b};$$

   (b) show that the above system can be rewritten as

$$\begin{pmatrix} A \\ \alpha I \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix};$$

   (c) apply the QR method with the above regularization to solve the Hilbert system $A\mathbf{x} = \mathbf{b}$, where A is the n-th order Hilbert matrix (use the matlab command **hilb** to check it out), $\mathbf{b}$ is taken such that the true solution is $\mathbf{x} = (1, 1, \cdots, 1)^T$;

   (d) to observe the performance of the above method, we may apply the QR method to solve the regularized system by taking

      i. $n = 10$, and $\alpha = 0, 0.00000001, 0.00001, 0.001, 0.1$, respectively;

      ii. $n = 20$, and $\alpha = 0, 0.00000001, 0.00001, 0.001, 0.1$, respectively;

      iii. $n = 30$, and $\alpha = 0, 0.00000001, 0.00001, 0.001, 0.1$, respectively.

   (e) can you choose an α which you think is ideal so that you get the best solution in the sense that it is closest to the true solution in the 2-norm sense?

4. What to turn in: you should write a short description (README file) of how to run the codes that you have written and email a tar ball of all the files to He Lyu at **lyuhe@msu.edu**. To make your email indicate that it is a CMSE project, please put "**823 Homework**" in the subject line.

Due date: Thursday, Feb. 13, 2020.