A 25:

a) $\langle N_{on} \rangle = s + \alpha b$

b) Poisson $\left(p = \frac{\lambda^k e^{-\lambda}}{k!} \right)$

$$p_{on} = \frac{\langle N_{on} \rangle^{N_{on}} e^{-\langle N_{on} \rangle}}{N_{on}!} = \frac{(s+\alpha b)^{N_{on}} e^{-(s+\alpha b)}}{N_{on}!}$$

$$p_{off} = \frac{\langle N_{off} \rangle^{N_{off}} e^{-\langle N_{off} \rangle}}{N_{off}!} = \frac{b^{N_{off}} e^{-b}}{N_{off}!}$$

c) $\mathcal{L}(b,s) = p_{on}\, p_{off}$

$$= \frac{(s+\alpha b)^{N_{on}} e^{-(s+\alpha b)} b^{N_{off}} e^{-b}}{N_{on}!\, N_{off}!}$$

$$= \frac{(s+\alpha b)^{N_{on}} b^{N_{off}} e^{-(s+b(\alpha+1))}}{N_{on}!\, N_{off}!}$$

d) Nenner egal zum Maximieren

$\Rightarrow$ Maximiere $\tilde{\mathcal{L}} = N_{on}!\, N_{off}!\, \mathcal{L}$

$\tilde{\mathcal{L}}$ und $\ln(\tilde{\mathcal{L}})$ haben das Maximum an der gleichen Stelle

$\Rightarrow$ Maximiere $\ln(\tilde{\mathcal{L}})$

$$\ln(\tilde{\mathcal{L}}) = N_{on} \ln(s+\alpha b) + N_{off} \ln(b) - s - b(\alpha+1)$$

$$\frac{\partial \ln(\tilde{\mathcal{L}})}{\partial s} = \frac{N_{on}}{s+\alpha b} - 1 \stackrel{!}{=} 0$$

$$\Rightarrow N_{on} = \hat{s} + \alpha \hat{b} \qquad (1)$$

$$\frac{\partial \ln(\tilde{\mathcal{L}})}{\partial b} = \frac{N_{on}}{s+\alpha b}\alpha + \frac{N_{off}}{b} - \alpha - 1 \stackrel{!}{=} 0 \qquad (2)$$

(1) in (2)

$$\Rightarrow \frac{\hat{s}+\alpha\hat{b}}{\hat{s}+\alpha\hat{b}}\alpha + \frac{N_{off}}{\hat{b}} - \alpha - 1 = 0$$

$$\Leftrightarrow \frac{N_{off}}{\hat{b}} = 1 \quad \Leftrightarrow \boxed{\hat{b} = N_{off}}$$

$\hat{b}$ in (1)

$$\Rightarrow \boxed{\hat{s} = N_{on} - \alpha N_{off}}$$

e) $\sigma(\hat{\alpha}) = \left(\frac{d^2 F}{d\alpha^2}\Big|_{\hat{\alpha}} \right)^{-1/2}$

$$F = -\ln(\mathcal{L}) = -N_{on}\ln(s+\alpha b) - N_{off}\ln(b) + s + b(\alpha+1) + \ln(N_{on}!\,N_{off}!)$$

$$\frac{\partial F}{\partial s} = -\frac{N_{on}}{s+\alpha b} + 1$$

$$\frac{\partial F}{\partial b} = -\frac{N_{on}}{s+\alpha b}\alpha - \frac{N_{off}}{b} + \alpha + 1$$

$$\frac{\partial^2 F}{\partial s^2} = \frac{N_{on}}{(s+\alpha b)^2} \qquad \frac{\partial^2 F}{\partial s^2}\Big|_{s=\hat{s},\, b=\hat{b}} = \frac{1}{N_{on}}$$

$$\frac{\partial^2 F}{\partial s \partial b} = \frac{N_{on}}{(s+\alpha b)^2}\alpha = \frac{\partial^2 F}{\partial b \partial s} \qquad \frac{\partial^2 F}{\partial s \partial b}\Big|_{s=\hat{s},\, b=\hat{b}} = \frac{1}{N_{on}}\alpha$$

$$\frac{\partial^2 F}{\partial b^2} = \frac{N_{on}}{(s+\alpha b)^2}\alpha^2 + \frac{N_{off}}{b^2} \qquad \frac{\partial^2 F}{\partial b^2}\Big|_{s=\hat{s},\, b=\hat{b}} = \frac{1}{N_{on}}\alpha^2 + \frac{1}{N_{off}}$$

$$\Rightarrow \sigma^2 = \left(\frac{1}{N_{on}} \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 + \frac{N_{on}}{N_{off}} \end{pmatrix} \right)^{-1} = \frac{N_{on}}{\alpha^2 + \frac{N_{on}}{N_{off}} - \alpha^2} \begin{pmatrix} \alpha^2 + \frac{N_{on}}{N_{off}} & -\alpha \\ -\alpha & 1 \end{pmatrix}$$

$$= N_{off} \begin{pmatrix} \alpha^2 + \frac{N_{on}}{N_{off}} & -\alpha \\ -\alpha & 1 \end{pmatrix}$$

A 26

a) $E(\hat{\mu}) = E(\overline{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$

$= \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n} n\mu = \mu$

$\Rightarrow$ ist erwartungstreu

b) $Var(\overline{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} x_i\right)$

$= \frac{1}{n^2}\left[Var(x_1) + Var\left(\sum_{i=2}^{n} x_i\right) + 2\underbrace{Cov\left(X_1, \sum_{i=2}^{n} X_i\right)}_{=0,\ \text{unkorreliert}}\right]$

$= \ldots = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{1}{n^2} n\sigma^2$

$= \frac{\sigma^2}{n}$

c) $E(\hat{\sigma}^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right) = \frac{1}{n}\sum_{i=1}^{n} E\left((x_i-\mu)^2\right)$

$= \frac{1}{n}\sum_{i=1}^{n} Var(x_i) = \frac{1}{n} n\sigma^2 = \sigma^2$

$\Rightarrow S_0^2$ ist für die Varianz erwartungstreu

d) $E(\hat{\sigma}'^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{X})^2\right) = \frac{1}{n}\sum_{i=1}^{n} E\left((x_i-\overline{x})^2\right)$

$= \frac{1}{n}\sum_{i=1}^{n} E\left(((x_i-\mu) + (\mu-\overline{x}))^2\right)$

$= \frac{1}{n}\sum_{i=1}^{n} E\left((x_i-\mu)^2 + 2(x_i-\mu)(\mu-\overline{X}) + (\mu-\overline{X})^2\right)$

$= \frac{1}{n}\left(\sum_{i=1}^{n} E(x_i-\mu)^2 + E\left(\sum_{i=1}^{n}\left(2(x_i-\mu)(\mu-\overline{x}) + (\mu-\overline{X})^2\right)\right)\right)$

$= \frac{1}{n}\left(\sum_{i=1}^{n} E(x_i-\mu)^2 + E\left(2n(\overline{X}-\mu)(\mu-\overline{X}) + n(\mu-\overline{X})^2\right)\right)$

$= \frac{1}{n}\left(\sum_{i=1}^{n} E(x_i-\mu)^2 + E\left(-n(\mu-\overline{X})^2\right)\right)$

$= \frac{1}{n}\left(n\sigma^2 - n\frac{\sigma^2}{n}\right)$

$= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$

$\Rightarrow$ nicht erwartungstreu

$\Rightarrow$ Korrektur: $\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X})^2 \rightarrow \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2$

A27:

a) $\mathcal{L}(b) = \prod_{i=1}^{n} f(x_i|b) = \prod_{i=1}^{n} \frac{1}{b}\,\Theta(b-x_i)$ (untere Grenze ist fest auf Null)

$$= \frac{1}{b^n}\,\Theta(b-\max(X))$$

I $\Theta(b-\max(X)) \Rightarrow \mathcal{L}$ maximal für $b \geq \max(X)$

II $\frac{1}{b^n} \Rightarrow$ stetig fallend $\rightarrow$ b muss minimal sein

I & II $\Rightarrow$ Schätzer: $\hat{b} = \max(X)$

b) $E(\hat{b}) = E(\max(X))$

Wahrscheinlichkeitsdichte:

$$q(x) = n \cdot \underbrace{\left(\int_0^x \frac{1}{b}\,d\tilde{x}\right)^{n-1}}_{\text{Wsk, } n-1 \text{ mal einen Wert kleiner als } x \text{ zu ziehen}} \cdot \frac{1}{b}$$

$\rightarrow$ PDF, um genau den Wert x zu ziehen.

Das Maximum kann an n verschiedenen Stellen stehen.

$$q(x) = n\left(\frac{x}{b}\right)^{n-1}\frac{1}{b}$$

$$\Rightarrow E(\max(X)) = \int_0^b x\,q(x)\,dx = \int_0^b \frac{n}{b^n}\,x^n\,dx = \frac{n}{b^n}\;\frac{b^{n+1}}{n+1}$$

$$= \frac{n}{n+1}\,b$$

$\Rightarrow E(\hat{b}) = \frac{n}{n+1}\,b \neq b$

$\Rightarrow$ nicht erwartungstreu

$\Rightarrow$ Korrektur: $\hat{b}' = \frac{n+1}{n}\max(X)$

$$E(\hat{b}') = \frac{n+1}{n}\,E(\max(X)) = \frac{n+1}{n}\cdot\frac{n}{n+1}\,b = b$$