A 12 a)

$$f_{Gauß} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$f_{2D} = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho}} \exp\left[-\frac{1}{2(1-\rho)^2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right)\right]$$

$$\Rightarrow f = \frac{1}{2\pi\sigma_x\sigma_{y|x}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right] \exp\left[-\frac{1}{2}\left(\frac{y-\mu_{y|x}}{\sigma_{y|x}}\right)^2\right]$$

$$= \frac{1}{2\pi\sigma_x\sigma_{y|x}} \exp\left[-\frac{1}{2}\left(\frac{x^2+\mu_x^2-2x\mu_x}{\sigma_x^2} + \frac{y^2+\mu_{y|x}^2-2y\mu_{y|x}}{\sigma_{y|x}^2}\right)\right]$$

$$= \frac{1}{2\pi\sigma_x\sigma_{y|x}} \exp\left[-\frac{1}{2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \frac{y^2+a^2+2abx+b^2x^2-2ya-2ybx}{\sigma_{y|x}^2}\right)\right]$$

$$= \frac{1}{\underbrace{2\pi\sigma_x\sigma_{y|x}}} \exp\left[-\frac{1}{2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-a}{\sigma_{y|x}}\right)^2 + \frac{2abx+b^2x^2-2ybx}{\sigma_{y|x}^2}\right)\right]$$

$$\Rightarrow \sigma_{y|x} = \sigma_y'\sqrt{1-\rho^2}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y'\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\underbrace{\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \frac{1}{(1-\rho^2)}\left(\frac{y-a}{\sigma_y'}\right)^2 + \frac{1}{(1-\rho^2)}\frac{2abx+b^2x^2-2ybx}{\sigma_y'^2}\right)}_{=:A}\right]$$

$$A = \frac{x^2+\mu_x^2-2x\mu_x}{\sigma_x^2} + \frac{1}{(1-\rho^2)}\left(\frac{y-a}{\sigma_y'}\right)^2 + \frac{1}{(1-\rho^2)}\frac{2abx+b^2x^2-2ybx}{\sigma_y'^2}$$

$$= \frac{1}{1-\rho^2}\left(\left(\frac{y-a}{\sigma_y'}\right)^2 + \frac{x^2+\mu_x^2-2x\mu_x}{\sigma_x^2} - \frac{\rho^2x^2\sigma_y'^2+\rho^2\mu_x^2\sigma_y'^2-2\rho^2\mu_x\sigma_y'^2x-2abx\sigma_x^2-b^2x^2\sigma_x^2+2ybx\sigma_x^2}{\sigma_x^2\sigma_y'^2}\right)$$

$$A = \frac{1}{1-\rho^2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \frac{y^2+a^2-2ya}{\sigma_y'^2}\right.$$

$$\left. - \frac{\rho^2 x^2 \sigma_y'^2 + \rho^2 \mu_x^2 \sigma_y'^2 - 2\rho^2 \mu_x \sigma_y'^2 x - 2abx\sigma_x^2 - b^2 x^2 \sigma_x^2 + 2ybx\sigma_x^2}{\sigma_x^2 \sigma_y'^2}\right)$$

$$=: \frac{1}{1-\rho^2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + B\right)$$

B01, 17g) $\mu_{Y|X} = \mu'_Y + \rho \frac{\sigma'_Y}{\sigma_X}(x-\mu_X)$

Bei uns: $\mu_{Y|X} = a + bx$

$$\Rightarrow a = \mu'_Y - \rho \frac{\sigma'_Y}{\sigma_X}\mu_X, \qquad b = \rho \frac{\sigma'_Y}{\sigma_X}$$

$$a^2 = \mu_Y'^2 + \rho^2 \frac{\sigma_Y'^2}{\sigma_X^2}\mu_X^2 - 2\mu'_Y \rho \frac{\sigma'_Y}{\sigma_X}\mu_X$$

$$\Rightarrow B = \frac{1}{\sigma_X^2 \sigma_Y'^2}\Big(y^2\sigma_X^2 + \mu_Y'^2\sigma_X^2 + \rho^2\sigma_Y'^2\mu_X^2 - 2\mu'_Y\rho\sigma'_Y\sigma_X\mu_X$$

$$- 2y\mu'_Y\sigma_X^2 + 2y\rho\sigma'_Y\sigma_X\mu_X - \rho^2 x^2 \sigma_Y'^2 - \rho^2\mu_X^2\sigma_Y'^2$$

$$+ 2\rho^2\mu_X\sigma_Y'^2 x + 2\mu'_Y\rho\sigma'_Y\sigma_X x - 2\rho^2\sigma_Y'^2\mu_X x$$

$$+ \rho^2\sigma_Y'^2 x^2 - 2y\rho\sigma'_Y\sigma_X x\Big)$$

$$= \left(\frac{y-\mu'_Y}{\sigma'_Y}\right)^2 + 0 + 0 + 0$$

$$- \frac{2\rho}{\sigma_X\sigma'_Y}\left(\mu'_Y\mu_X - y\mu_X - \mu'_Y x + xy\right)$$

$$= \left(\frac{y-\mu'_Y}{\sigma'_Y}\right)^2 - 2\rho\left(\frac{\mu_X - x}{\sigma_X}\right)\left(\frac{\mu'_Y - y}{\sigma'_Y}\right)$$

$$\Rightarrow f = \frac{1}{2\pi\sigma_X\sigma'_Y\sqrt{1-\rho}}\exp\left[-\frac{1}{2(1-\rho)^2}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu'_Y}{\sigma'_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu'_Y}{\sigma'_Y}\right)\right)\right]$$

$$\sigma_{y|x} = \sigma_y' \sqrt{1-\rho^2}$$

$$\rho = b \frac{\sigma_x}{\sigma_y'}$$

$$\mu_{y|x} = a + bx$$

$$\mu_{y|x} = \mu_y' + \rho \frac{\sigma_y'}{\sigma_x}(x - \mu_x)$$

$$\Rightarrow \mu_y' = a + bx - b(x - \mu_x) = a + b\mu_x = 3{,}1$$

$$\sigma_{y|x} = \sigma_y' \sqrt{1 - b^2 \frac{\sigma_x^2}{\sigma_y'^2}}$$

$$\Leftrightarrow \sigma_{y|x}^2 = \sigma_y'^2 \left(1 - b^2 \frac{\sigma_x^2}{\sigma_y'^2}\right) = \sigma_y'^2 - b^2 \sigma_x^2$$

$$\Rightarrow \sigma_y' = \sqrt{\sigma_{y|x}^2 + b^2 \sigma_x^2} = \frac{\sqrt{541}}{10}$$

$$\Rightarrow \rho = b\,\sigma_x \frac{1}{\sqrt{\sigma_{y|x}^2 + b^2 \sigma_x^2}} \simeq 0{,}90286$$

A 13

$g_1(x) = 0$

$\Rightarrow \vec{P}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$g_2(x) = -\frac{3}{4}x$

$\Rightarrow \vec{P}_2 = \frac{1}{\sqrt{16+9}} \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$g_3(x) = -\frac{5}{4}x$

$\Rightarrow \vec{P}_3 = \frac{1}{\sqrt{25+16}} \begin{pmatrix} 4 \\ -5 \end{pmatrix} = \frac{1}{\sqrt{41}} \begin{pmatrix} 4 \\ -5 \end{pmatrix}$

A 14

Population 0: $(2;2;1)$ $(2;3;2)$ $(2;1;2)$

$(1;2;0)$ $(3;2;0)$

Population 1: $(2{,}5;2{,}5;0)$ $(2{,}5;1{,}5;0)$ $(4;2;0)$

$(5{,}5;2{,}5;0)$ $(5{,}5;1{,}5;0)$

a) $\mu_0 = \frac{1}{5}\begin{pmatrix} 10 \\ 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$\mu_1 = \frac{1}{5}\begin{pmatrix} 20 \\ 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$

$S_w = \sum_j S_j$

$S_j = \sum_{i=0}^{4} (\vec{x}_i - \vec{\mu}_j)(\vec{x}_i - \vec{\mu}_j)^T$

$S_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}(0\ 0\ 0) + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}(0\ 1\ 1) + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}(0\ -1\ 1)$

$+ \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}(-1\ 0\ -1) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}(1\ 0\ -1)$

$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

$+ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$$S_1 = \begin{pmatrix} -1,5 \\ 0,5 \\ 0 \end{pmatrix} \begin{pmatrix} -1,5 & 0,5 & 0 \end{pmatrix} + \begin{pmatrix} -1,5 \\ -0,5 \\ 0 \end{pmatrix} \begin{pmatrix} -1,5 & -0,5 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1,5 \\ 0,5 \\ 0 \end{pmatrix} \begin{pmatrix} 1,5 & 0,5 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 1,5 \\ -0,5 \\ 0 \end{pmatrix} \begin{pmatrix} 1,5 & -0,5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2,25 & -0,75 & 0 \\ -0,75 & 0,25 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2,25 & 0,75 & 0 \\ 0,75 & 0,25 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 2,25 & 0,75 & 0 \\ 0,75 & 0,25 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2,25 & -0,75 & 0 \\ -0,75 & 0,25 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_W = S_0 + S_1$$

$$S_W = \begin{pmatrix} 11 & & \\ & 3 & \\ & & 4 \end{pmatrix} \quad \Bigg| \quad S_W^{-1} = \begin{pmatrix} \frac{1}{11} & & \\ & \frac{1}{3} & \\ & & \frac{1}{4} \end{pmatrix}$$

$$S_B = (\vec{\mu}_1 - \vec{\mu}_2)(\vec{\mu}_1 - \vec{\mu}_2)^T$$

$$= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

b)

$$S_W^{-1} S_B = \begin{pmatrix} \frac{4}{11} & 0 & -\frac{2}{11} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}$$

$$\det(S_W^{-1} S_B - \lambda \mathbb{1}) = -\lambda \begin{vmatrix} 4/11 - \lambda & -\frac{2}{11} \\ -\frac{1}{2} & 1/4 - \lambda \end{vmatrix}$$

$$= -\lambda \left[\left(\frac{4}{11} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \frac{1}{11} \right] \overset{!}{=} 0$$

$$-\lambda\left[\left(\frac{4}{11}-\lambda\right)\left(\frac{1}{4}-\lambda\right)-\frac{1}{11}\right]\overset{!}{=}0$$

$$\Rightarrow \lambda = 0$$

$$\vee \quad \frac{1}{11}-\frac{4}{11}\lambda-\frac{1}{4}\lambda+\lambda^2-\frac{1}{11}=0$$

$\Rightarrow \lambda = 0$ ist doppelte NS

$$\vee \quad \lambda=\frac{4}{11}+\frac{1}{4}=\frac{16+11}{44}=\frac{27}{44}$$

$\underline{\lambda = 0}$

$$\begin{pmatrix} \frac{4}{11} & 0 & -\frac{2}{11} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix} \quad 11\,\mathrm{I} + 8\,\underline{\mathrm{III}}$$

$$\begin{pmatrix} \frac{4}{11} & 0 & -\frac{2}{11} \\ 0 & 0 & 0 \\ 0 & 0 & -2+2 \end{pmatrix}$$

$$\Rightarrow \frac{4}{11}v_1-\frac{2}{11}v_3=0 \iff v_1=\frac{1}{2}v_3$$

$$\Rightarrow \text{EV:}\quad \vec{u}_1=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix} \qquad \vec{u}_2=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}$$

allgemein: $\left\{\vec{u}_{1,2}\in\mathbb{R}^3 \;\middle|\; \vec{u}_{1,2}=\begin{pmatrix}a\\b\\2a\end{pmatrix}\right\}$

$\lambda = \frac{27}{44}$

$$\begin{pmatrix} \frac{16-27}{44} & 0 & -\frac{8}{44} \\ 0 & -\frac{27}{44} & 0 \\ -\frac{22}{44} & 0 & \frac{11-27}{44} \end{pmatrix} = \frac{-1}{44} \begin{pmatrix} 11 & 0 & 8 \\ 0 & 27 & 0 \\ 22 & 0 & 16 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 0 & 8 \\ 0 & 27 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow v_2 = 0 \ , \ 11 v_1 + 8 v_3 = 0 \Leftrightarrow v_1 = -\frac{8}{11} v_3$$

$$\Rightarrow \left\{ \vec{u}_3 \in \mathbb{R}^3 \ \middle| \ \vec{u}_3 = \begin{pmatrix} -a \\ 0 \\ \frac{11}{8} a \end{pmatrix} \right\}$$

$$\vec{u}_3 = \frac{1}{\sqrt{185}} \begin{pmatrix} -8 \\ 0 \\ 11 \end{pmatrix}$$

c) Projektion $\vec{\lambda}$, bei der $D(\vec{\lambda}) = \frac{\vec{\lambda}^T S_B \vec{\lambda}}{\vec{\lambda}^T S_W \vec{\lambda}}$ extremal wird, erfüllt das Eigenwertproblem $S_W^{-1} S_B \vec{\lambda} = D \vec{\lambda}$

$$\Rightarrow \vec{\lambda} \in \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \}$$

$$D(\vec{u}_{1,2}) = \frac{(a \;\; b \;\; 2a)\begin{pmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ 2a \end{pmatrix}}{(a \;\; b \;\; 2a)\begin{pmatrix} 11 & & \\ & 3 & \\ & & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ 2a \end{pmatrix}}$$

$$= \frac{(4a-4a \quad 0 \quad -2a+2a)\begin{pmatrix} a \\ b \\ 2a \end{pmatrix}}{(11a \quad 3b \quad 8a)\begin{pmatrix} a \\ b \\ 2a \end{pmatrix}} = 0$$

$$D(\vec{u}_3) = \frac{(a \;\; 0 \;\; \tfrac{11}{8}a)\begin{pmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -a \\ 0 \\ 11/8\, a \end{pmatrix}}{(-a \;\; 0 \;\; \tfrac{11}{8}a)\begin{pmatrix} 11 & & \\ & 3 & \\ & & 4 \end{pmatrix}\begin{pmatrix} -a \\ 0 \\ 11/8\, a \end{pmatrix}}$$

$$= \frac{(-4a + \tfrac{11}{4}a \quad 0 \quad -2a+\tfrac{11}{8}a)\begin{pmatrix} -a \\ 0 \\ 11/8\, a \end{pmatrix}}{(-11a \quad 0 \quad \tfrac{11}{2}a)\begin{pmatrix} -a \\ 0 \\ 11/8\, a \end{pmatrix}}$$

$$= \frac{4 - \frac{11}{4} - \frac{11}{4} + \frac{121}{64}}{11 + \frac{121}{16}} = \frac{25}{1188} > 0$$

$\Rightarrow$ Die Projektion, die $D(\vec{\lambda})$ maximiert ist (normiert): $\vec{\lambda} = \frac{1}{\sqrt{185}}\begin{pmatrix} -8 \\ 0 \\ 11 \end{pmatrix}$

Verifizierung:

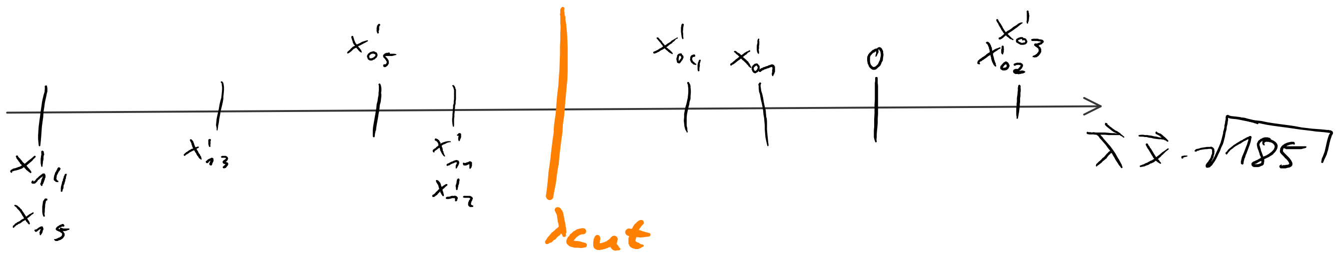
$$S_W^{-1}(\vec{\mu}_0 - \vec{\mu}_1) = \begin{pmatrix} \frac{1}{11} & & \\ & \frac{1}{3} & \\ & & \frac{1}{4} \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{11} \\ 0 \\ \frac{2}{8} \end{pmatrix} = \sqrt{185} \cdot \frac{2}{88}\,\vec{\lambda} \;\checkmark$$

d) $\vec{\lambda} = \frac{1}{\sqrt{185}} \begin{pmatrix} -8 \\ 0 \\ 11 \end{pmatrix}$

$x_i' = \vec{\lambda}^T \vec{x}_i$

$x_0' \in \left\{ \frac{-5}{\sqrt{185}} ; \frac{6}{\sqrt{185}} ; \frac{6}{\sqrt{185}} ; \frac{-8}{\sqrt{185}} ; \frac{-24}{\sqrt{185}} \right\}$

$x_1' \in \left\{ \frac{-20}{\sqrt{185}} ; \frac{-20}{\sqrt{185}} ; \frac{-32}{\sqrt{185}} ; \frac{-44}{\sqrt{185}} ; \frac{-44}{\sqrt{185}} \right\}$



e) λ_{cut} so gewählt, dass die Reinheit 100% ist, und die Effizienz noch sehr gut ist.

Reinheit $= 1$

Effizienz $= \frac{4}{5} = 0{,}8$

(Vorraussetzung: Pop. 0 sind die Signaldaten)

A 15

Metropolis - Hastings:

$$M_{i \to j} = \min\left(1, \frac{f(x_j)}{f(x_i)} \quad \frac{g(x_j|x_i)}{g(x_i|x_j)}\right)$$

$$\frac{g(x_j|x_i)}{g(x_i|x_j)} \longrightarrow 1 \Rightarrow \text{Metropolis-Hastings}$$

$$\longrightarrow \text{Metropolis}$$

g sei gaußförmig (B02, A7):

$$\frac{g(x|y)}{g(y|x)} = \frac{\sigma_y}{\sigma_x} \exp\left[\frac{1}{2}\left(u_x^2 - u_y^2\right)\right] = \frac{\sigma_y}{\sigma_x} \exp\left[\frac{1}{2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)\right]$$

$x_j = x_{i+1}, \quad \sigma_{x_j} = \sigma_{x_i}$

$$\Rightarrow \frac{g(x_j|x_i)}{g(x_i|x_j)} = \frac{\sigma_{x_j}}{\sigma_{x_j}} \exp\left[\frac{1}{2}\left(\left(\frac{x_j - \overset{=x_i}{x_{j-1}}}{\sigma_{x_j}}\right)^2 - \left(\frac{x_i - \overset{=x_j}{x_{i+1}}}{\sigma_{x_j}}\right)^2\right)\right]$$

$$= 1 \quad e^0 \quad = 1$$