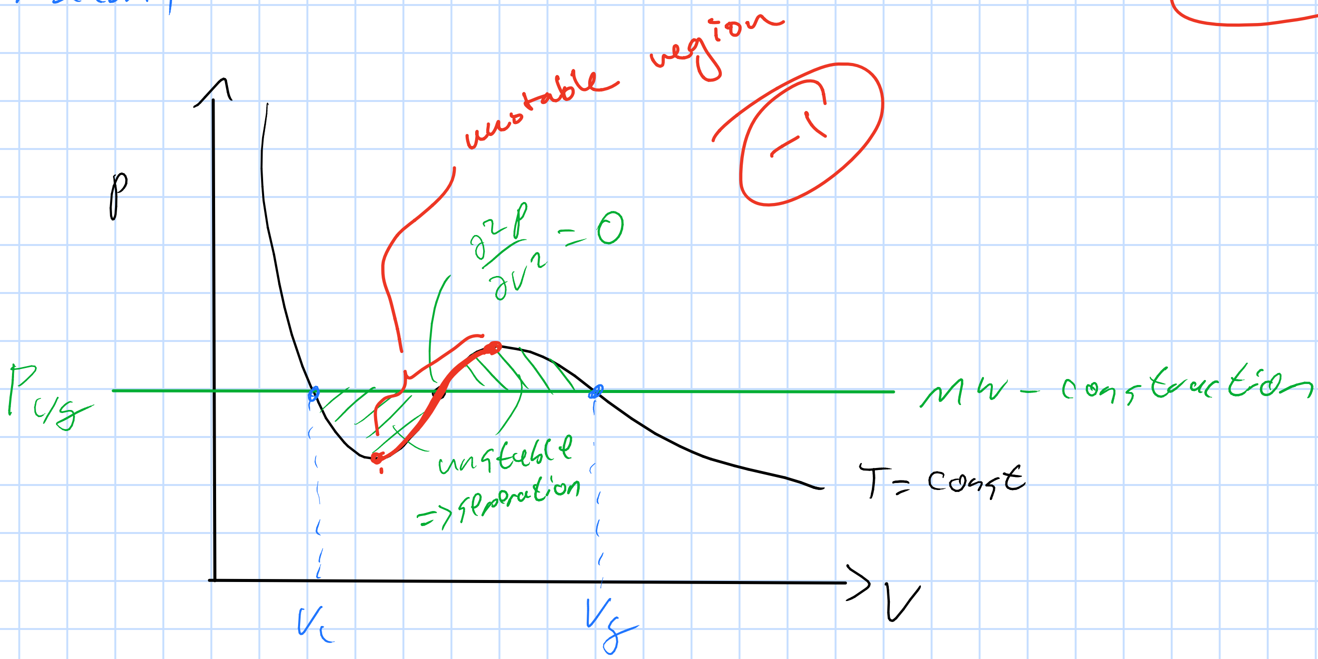
Problem 1



$$\int_{v_l}^{v_g} (p - p_{l/g})\, dv = 0$$

unstable between v_l and v_g (at $p_{l/g}$)

Problem 2

$$Z = 2^N \cosh^N\left(\beta \frac{qJ}{2} \langle\sigma\rangle + \beta h\right) = 2^N \cosh^N\left[\beta\mu(B+B')\right]$$

with $h = \mu B$, $B' = \frac{qJ}{2\mu}\langle\sigma\rangle$

Net magnetization is

$$M(B,T) = \mu N \langle\sigma\rangle = T\left(\frac{\partial \ln Z}{\partial B}\right)_T$$

$$\ln Z = N\ln 2 + N \ln\left[\cosh(\cdots)\right]$$

$$\Rightarrow \left(\frac{\partial \ln Z}{\partial B}\right)_T = N \frac{\partial}{\partial B} \ln\left[\cosh(\cdots)\right]$$

and $\langle\sigma\rangle = \frac{T}{\mu} \frac{\partial}{\partial B} \ln\left[\cosh(\cdots)\right]$

$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$

$\frac{\partial}{\partial x} \cosh x = \sinh x$

$$\Rightarrow \frac{\partial}{\partial x} \ln\left[\cosh(a \cdot x + y)\right] = a \cdot \tanh(ax + y)$$

So here: $\langle\sigma\rangle = \frac{T}{\mu} \frac{\partial}{\partial B} \ln\left[\cosh(a \cdot \beta)\right]$

with $a = \beta\mu = \frac{\mu}{T}$

$$\Rightarrow \langle\sigma\rangle = \underbrace{\frac{aT}{\mu}}_{=1} \tanh\left[\beta\mu(B+B')\right]$$

$$= \tanh\left[\beta\left(\frac{qJ}{2}\langle\sigma\rangle + h\right)\right]$$

Problem 3

$$F_z = \int \frac{d^3p}{(2\pi\hbar)^3} \underbrace{v_z \frac{p^2}{2m}}_{\text{single particle energy flux in } z\text{-direction}} f_1(\vec{p})$$

$$v_z = \frac{p\cos\theta}{m}$$

$$= (2\pi\hbar)^{-3} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\infty} dp\, p^2 \sin\theta \, \frac{p\cos\theta}{m} \, \frac{p^2}{2m} \cdot f_1$$

$$= (2\pi\hbar)^{-3} \, 2\pi \int_0^{\pi} d\theta \int_0^{\infty} dp \frac{p^5}{2m^2} \sin\theta\cos\theta \cdot \exp\left(-\beta \frac{p^2}{2m} + \beta\mu\right)(1 + a\cos\theta)$$

$$= \frac{2\pi}{(2\pi\hbar)^3} e^{\beta\mu} \underbrace{\int_0^{\pi} d\theta \sin\theta\cos\theta(1+a\cos\theta)}_{0 + \frac{2a}{3}\text{, given}} \underbrace{\int_0^{\infty} dp \frac{p^5}{2m^2} \exp\left(-\beta\frac{p^2}{2m}\right)}_{\text{some } \Gamma(\cdot)}$$

$$x := \beta \frac{p^2}{2m}$$

$$\Rightarrow dx = \frac{\beta p}{m} dp$$

$$= \frac{2\pi}{(2\pi\hbar)^3} e^{\beta\mu} \frac{2a}{3} \int_0^{\infty} \frac{m}{\beta p} dx \frac{p^{4}}{2m^2} \exp(-x) \qquad p^2 = \frac{2mx}{\beta}$$

$$= \frac{2\pi}{(2\pi\hbar)^3} e^{\beta\mu} \frac{2}{3} T^3 m \, 2a \underbrace{\int_0^{\infty} x^{3-1} \exp(-x)\, dx}_{\Gamma(2) = 2}$$

$$= \frac{8 \cdot 2\pi}{3(2\pi\hbar)^3} e^{\beta\mu} T^3 m = \boxed{\frac{2am}{3\pi^2\hbar^3} e^{\beta\mu} T^3}$$

Problem 4

(a) $$\frac{\partial e}{\partial t} = \frac{\partial}{\partial t}\frac{3}{2m}T = 0 = \frac{\partial e}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\alpha t = 0$$

$$\Rightarrow \frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} = 0 = -\frac{p}{\rho}\frac{\partial u}{\partial x}$$ ✓

(b) $$\frac{\partial u}{\partial t} = \alpha \quad , \frac{\partial u}{\partial x} = 0$$

$$p = \frac{T}{m}\rho_0 \exp(-\lambda x) \Rightarrow \frac{\partial p}{\partial x}\Big|_{t=0} = \frac{-T_0\rho_0}{m}\lambda \exp(-\lambda x)$$

So:
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha \overset{!}{=} \left(-\frac{1}{\rho}\frac{\partial p}{\partial x}\right)_{t=0} = \cancel{-}\cancel{(\rho_0 \exp(-\lambda x))}^{-1} \cdot \left(\cancel{+}\frac{T_0\cancel{\rho_0}}{m}\lambda\, \cancel{\exp(-\lambda x)}\right)$$

$$= \frac{T_0\lambda}{m} \Rightarrow \lambda = \frac{m\alpha}{T_0}$$

(c) $$\frac{\partial \rho}{\partial t} = \rho_0 \exp(-\lambda x) f'(t), \quad \frac{\partial \rho}{\partial x} = -\lambda\rho_0 \exp(-\lambda x) f(t), \quad \frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -u\frac{\partial \rho}{\partial x} \Leftrightarrow \cancel{\rho_0 \exp(-\lambda x)} f' = +\alpha t \lambda \cancel{\rho_0 \exp(-\lambda x)} f$$

So left with DE: $f' = \alpha t \lambda f$

$$\Rightarrow f = \frac{c}{2}\cdot \exp(\alpha\lambda t^2), \text{ some } c \in \mathbb{R}$$

$$\Rightarrow \rho(x,t) = \rho_0 \exp(-\lambda x)\frac{c}{2}\exp(\alpha\lambda t^2) \qquad \rho(x, t=0) \overset{!}{=} \rho_0 \exp(-\lambda x)$$

$$\Rightarrow f(0) \overset{!}{=} 1 \Rightarrow c = 2$$

So $$\rho(x,t) = \rho_0 \exp\left(\frac{m\alpha}{T_0}\left(\frac{\alpha t^2}{2} - x\right)\right)$$

-1 density profile just translating w/ velocity αt

Gets compressed over time at same location.
Falling water in gravity? No time left...