1. Find the second virial coefficient for a three-dimensional, non-relativistic, non-interacting Fermi gas. [Hint: remember for an ideal Bose gas we have $PV/T = V v_Q^{-1} g_{5/2}(e^{\beta\mu})$.]

From the lecture notes:

$$\frac{Pv}{T} = \sum_{m=1}^{\infty} a_m(T) \left(\frac{v_Q}{v}\right)^{m-1} \tag{520}$$

where $v = V/N$ is the volume per particle and we expect v_Q/v to be a reasonable expansion parameter. Here, $v_Q = \ell_Q^3 = (2\pi\hbar^2/mT)^{3/2}$. Therefore, at low-density/high-temperature we expect to only need the first few terms in the expansion.

- We can then expand the log of the partition function in powers of λ, giving

$$\log Z = \frac{V}{v_Q} \sum_{m=1}^{\infty} b_m(T,V)\lambda^m \tag{526}$$

with the first few coefficients of the expansion given by

$$\begin{aligned} b_1(T,V) &= \frac{Q_1}{V} \\ b_2(T,V) &= \frac{1}{2v_Q V}\left(Q_2 - Q_1^2\right) \\ b_3(T,V) &= \frac{1}{6v_Q^2 V}\left(Q_3 - 3Q_1Q_2 + 2Q_1^3\right). \end{aligned} \tag{527}$$

- For a quantum gas, we can define a configuration integral in a similar fashion (although it is often tricky to write down the canonical partition function of such a gas respecting the symmetry of the wave function)

$$Q_N = N! v_Q^N Z_N \tag{524}$$

so that we can combine to find a relationship between the coefficients

$$\sum_{m=1}^{\infty} a_m \left(\sum_{n=1}^{\infty} n b_n(T)\lambda^n\right)^{m-1} = \frac{\sum_{m=1}^{\infty} b_m(T)\lambda^m}{\sum_{m=1}^{\infty} m b_m(T)\lambda^m} \tag{530}$$

We can then move everything to the left hand side of this equation, collect coefficients of powers of λ, and then set the coefficients equal to zero to find the coefficients of the virial expansion. The first few of these are

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= -b_2 \\ a_3 &= 4b_2 - 2b_3 \\ &\dots \end{aligned} \tag{531}$$

- Clearly, there are no impacts from the symmetry of the wave function for the one particle partition function so that the classical result is the same as the quantum result. For the classical case it is easy to see $Q_1 = V$ so that $a_1 = b_1 = 1$ in general.

$$\Rightarrow \boxed{a_1 = b_1 = 1}$$

and we just need to find $a_2 = -b_2$

$$\ln Z = \frac{V}{V_Q} \sum_{m=1}^{\infty} b_m \lambda^m = \frac{V}{V_Q} F_{5/2}(\lambda)$$

for Fermi gas using (292) and (303)

$$F_{5/2} = \frac{1}{\Gamma(\frac{5}{2})} \int_0^\infty \frac{dx\, x^{3/2}}{\lambda^{-1}e^x + 1} = \frac{\lambda}{\Gamma(\frac{5}{2})} \int_0^\infty \frac{dx\, e^{-x} x^{3/2}}{1 + \lambda e^{-x}}$$

expand to 2nd order

$$= \frac{\lambda}{\Gamma(\frac{5}{2})} \underbrace{\int_0^\infty dx\, e^{-x} x^{5/2 - 1}}_{\Gamma(5/2)} - \frac{\lambda^2}{\Gamma(\frac{5}{2})} \underbrace{\int_0^\infty dx\, e^{-2x} x^{3/2}}_{\frac{1}{2^{5/2}}\Gamma(\frac{5}{2})} + O(\lambda^2)$$

$$= \lambda - \frac{\lambda^2}{2^{5/2}} + O(\lambda^2)$$

$$\Rightarrow \boxed{a_2 = -b_2 = \frac{1}{2^{5/2}}}$$

2. Given a system with the two-particle (or density-density) correlation function

$$g(R) = 1 + Ae^{-R/\ell}, \tag{1}$$

where A, and ℓ are constants, find the number fluctuations of the system,

$$\frac{\langle \Delta N^2 \rangle}{\langle N \rangle} = \frac{\langle N^2 - \langle N \rangle^2 \rangle}{\langle N \rangle}.$$

What is $\langle \Delta N^2 \rangle / N$ for a Boltzmann gas? What is $\langle \Delta N^2 \rangle / N$ for a non-interacting Fermi gas? [Calculate these directly from the grand canonical distribution function] Qualitatively, what does this tell us about spatial correlations in a Fermi gas?

$$\underbrace{1 + \frac{\langle N \rangle}{V} \int d^3R [g(\vec{R}) - 1] = \frac{\langle \Delta N^2 \rangle}{\langle N \rangle}} = \frac{T}{N} \left(\frac{\partial N}{\partial \mu} \right)_{T,V} = nT\kappa_T \tag{561}$$

Here:

$$\frac{\langle \Delta N^2 \rangle}{\langle N \rangle} = 1 + \underbrace{4\pi}_{S_3} \frac{\langle N \rangle}{V} A \int_0^\infty dR \underbrace{R^2}_{|J| \text{ in 3d spherical}} \exp\left(-\frac{R}{\ell}\right)$$

$$= 1 + 8\pi A \ell^3 \frac{\langle N \rangle}{V}$$

- We can also derive the fluctuations of number in the grand canonical ensemble as

$$\partial_\alpha^2 \ln Z(\beta, \alpha) = \langle \Delta N^2_{GC} \rangle = T \left(\frac{\partial N}{\partial \mu} \right)_{T,V} \qquad \text{Notes p. 4}$$

Boltzmann: $\langle \Delta N^2 \rangle = \partial_\alpha^2 \ln Z = \frac{\partial^2}{\partial \alpha^2} \int_0^\infty d\varepsilon\, g(\varepsilon) \exp(-\beta\varepsilon + \alpha)$

$$= \int_0^\infty d\varepsilon\, g(\varepsilon) \exp(-\beta\varepsilon + \alpha) = \langle N \rangle$$

$$\Rightarrow \frac{\langle \Delta N^2 \rangle}{\langle N \rangle} = 1 \qquad \text{uncorrelated!}$$

Fermi: $\langle \Delta N^2 \rangle = \partial_\alpha^2 \ln Z = \frac{\partial^2}{\partial \alpha^2} \int_0^\infty d\varepsilon\, g(\varepsilon) \ln(1 + \exp(-\beta\varepsilon + \alpha))$

$$= \int_0^\infty d\varepsilon\, g(\varepsilon) f_+(\varepsilon) [1 - f_+(\varepsilon)], \qquad \langle N \rangle = \int_0^\infty d\varepsilon\, f_+(\varepsilon)$$

$$\Rightarrow \frac{\langle \Delta N \rangle^2}{\langle N \rangle} = \frac{\int_0^\infty d\varepsilon\, g(\varepsilon) f_+(\varepsilon)[1 - f_+(\varepsilon)]}{\int_0^\infty d\varepsilon\, f_+(\varepsilon)} < 1 \qquad \text{anti correlated}$$

3. Consider a van Der Waals gas with equation of state

$$P = \frac{T}{v-b} - \frac{a}{v^2},$$

where a and b are constants and $v = V/N$. We want to think about the properties of the liquid-gas phase transition in the low temperature limit when the gas phase behaves like an ideal gas and the specific volume of the high-density phase approaches the density b so that we can consider only the first order correction in T. The first four parts of the problem derive some general properties of the van der Waals gas and the liquid-gas phase transition it undergoes, while the last three consider the properties of this transition in the low temperature limit.

(a) Derive the Maxwell relation

$$\left(\frac{\partial P}{\partial T}\right)_{N,V} = \left(\frac{\partial S}{\partial V}\right)_{T,N}$$

which also implies

$$T\left(\frac{\partial P}{\partial T}\right)_{N,V} = P + \left(\frac{\partial E}{\partial V}\right)_{N,T}.$$

$$-\left(\frac{\partial}{\partial T}\left(\frac{\partial F}{\partial V}\right)_{N,T}\right)_{N,V} = \left(\frac{\partial P}{\partial T}\right)_{N,V} = -\left(\frac{\partial}{\partial V}\left(\frac{\partial F}{\partial T}\right)_{N,V}\right)_{N,T} = \left(\frac{\partial S}{\partial V}\right)_{N,T}$$

$$\left(\frac{\partial E}{\partial V}\right)_{N,T} = \left(\frac{\partial F}{\partial V}\right)_{N,T} + T\left(\frac{\partial S}{\partial V}\right)_{N,T}$$

$$= -P + T\left(\frac{\partial P}{\partial T}\right)_{N,T}$$

$$\Rightarrow P + \left(\frac{\partial E}{\partial V}\right)_{N,T} = T\left(\frac{\partial P}{\partial T}\right)_{N,T}$$

(b) Find the difference in energies per particle at fixed temperature between specific volumes v_1 and v_2

$$e(T, v_1) - e(T, v_2),$$

using the result from part (a).

$$e(T, v_1) - e(T, v_2) = \int_{v_2}^{v_1} dv \left(\frac{\partial E}{\partial V}\right)_{N,T}$$

$$= \int_{v_2}^{v_1} dv \left(T\left(\frac{\partial P}{\partial T}\right)_{N,V} - P\right) = a\int_{v_2}^{v_1} \frac{dv}{v^2} = \frac{a}{v_2} - \frac{a}{v_1}$$

(c) Find the difference in entropy per particle at fixed temperature between specific volumes v_1 and v_2

$$s(T, v_1) - s(T, v_2),$$

using the result from part (a).

$$s(T, v_1) - s(T, v_2) = \int_{v_2}^{v_1} dv \left(\frac{\partial S}{\partial V}\right)_{T,N}$$

$$= \int_{v_1}^{v_2} dv \left(\frac{\partial P}{\partial T}\right)_{N,V} = \int_{v_2}^{v_1} \frac{dv}{v-b} = \ln\left(\frac{v_1-b}{v_2-b}\right)$$

(d) Find the conditions for liquid-gas phase coexistence. Stated another way, find the pressure of the both phases in terms of T, the specific volume in the gas phase, v_g, and the specific volume in the liquid phase, v_l.

$$\int_{V_g}^{V_l} P_g \, dv \overset{!}{=} \int_{V_g}^{V_l} P \, dv$$

$$\Leftrightarrow \quad P_g(V_l - V_g) = T \int_{V_g}^{V_l} \frac{dv}{v-b} - a \int_{V_g}^{V_l} \frac{dv}{v^2}$$

$$= T \ln\left(\frac{V_l - b}{V_g - b}\right) + \frac{a}{V_l} - \frac{a}{V_g}$$

$$\frac{V_g}{V_g}\frac{a}{V_l(V_l-V_g)} - \frac{V_l}{V_l}\frac{a}{V_g(V_l-V_g)}$$

$$\frac{a(\cancel{V_g - V_l})}{-V_g V_l(\cancel{V_g - V_l})}$$

$$\Rightarrow \quad P_l = P_g = \frac{T}{V_l - V_g} \ln\left(\frac{V_l - b}{V_g - b}\right) - \frac{a}{V_g V_l}$$

(e) At low temperature at the liquid-gas phase transition, the gas phase is at very low density and the liquid phase is nearly at the maximum density, $1/b$. Stated in terms of the specific volumes, $v_g \gg v_l \sim b$ and $v_g \gg \sqrt{a}$. In this limit, pressure equality of the two phases gives the liquid specific volume to first order in T as $v_l = b + Tb^2/a$. In this limit, show that the specific volume of the gas phase is approximately given by

$$v_g \approx \frac{Tb^2}{a} \exp\left(\frac{a}{Tb}\right)$$

Should be ideal gas:

$$P_g = \frac{T}{V_g - b} - \frac{a}{V_g^2} \approx \frac{T}{V_g}$$

Phase coexistance:

$$\frac{T}{\cancel{V_g}} \approx - \frac{T}{\cancel{V_g}} \ln\left(\frac{V_l - b}{V_g}\right) - \frac{a}{\cancel{V_g} V_l}$$

using $V_l = b + T\frac{b^2}{a} + O(b^3)$

$$\Rightarrow \quad 1 \approx -\ln\left(\frac{T\frac{b^2}{a}}{V_g}\right) - \frac{a}{Tb}$$

$$\text{Low } T: \ \frac{a}{Tb} \gg 1 \quad \Rightarrow \quad V_g \approx T\frac{b^2}{a}\exp\left(\frac{a}{Tb}\right)$$

(f) What is the latent heat across the phase transition at low temperature?

$$L = NT\left(s(T, V_g) - s(T, V_l)\right) = NT \ln\frac{V_g - b}{V_l - b}$$

$$\approx NT \ln\frac{V_g a}{Tb^2} = NT\frac{a}{Tb} = N\frac{a}{b}$$

(g) What is dP/dT along the phase co-existence line in the low temperature limit?

$$\left(\frac{\partial P}{\partial T}\right)_{PCL} = \frac{L}{NT(V_g - V_l)} \approx \frac{a}{bT} V_g^{-1} = \frac{a^2}{b^3 T} \exp\left(-\frac{a}{bT}\right)$$

4. Show that in the mean field approximation the magnetic susceptibility of the Ising model is given by

$$\chi = \left(\frac{\partial M}{\partial B}\right)_T = N\mu_B^2 \frac{1 - \langle\sigma\rangle^2}{T - (1 - \langle\sigma\rangle^2)T_c} \tag{2}$$

Ising: $M = \mu N \langle\sigma\rangle$

$$\Rightarrow \chi = \left(\frac{\partial M}{\partial B}\right)_T = \mu N \frac{\partial \langle\sigma\rangle}{\partial B}$$

and we know:

$$\langle\sigma\rangle = \tanh\left[\beta\left(\mu B + \underbrace{\frac{qJ}{2}}_{= T_c}\langle\sigma\rangle\right)\right]. \tag{589}$$

$$\Rightarrow \frac{\partial}{\partial B}\langle\sigma\rangle \underset{\text{Mathematica}}{=} \mu \frac{1 - \langle\sigma\rangle^2}{T - T_c(1 - \langle\sigma\rangle^2)}$$

$$\Rightarrow \chi = N\mu^2 \frac{1 - \langle\sigma\rangle^2}{T - T_c(1 - \langle\sigma\rangle^2)}$$

5. Consider the Ising ferromagnet in zero field, in the case where the spin can take three values $\sigma = -1, 0, 1$.

(a) Find the equation for the mean field free energy.

$$Z = \Big(\underset{\sigma=0}{1} + 2\cosh(\beta\mu B' + \beta\mu \overset{=0}{B})\Big)^N, \quad B' = \frac{qJ}{2}\frac{\langle\sigma\rangle}{\mu}$$

$$\Rightarrow F = -\frac{1}{\beta}\ln Z = -NT\left(1 + 2\cosh\left(\beta\frac{qJ}{2}\langle\sigma\rangle\right)\right)$$

(b) Find an implicit equation for the mean field magnetization.

$$\langle\sigma\rangle = \frac{T}{\mu N}\frac{\partial \ln Z}{\partial B}\bigg|_{B=0}$$

$$= \frac{T}{\mu}\left[\frac{\partial}{\partial B}\ln\left(1 + 2\cosh\left(\beta\frac{qJ}{2}\langle\sigma\rangle + \beta\mu B\right)\right)\right]_{B=0}$$

$$\Rightarrow \langle\sigma\rangle = \frac{\sinh\left(\frac{\beta qJ}{2}\langle\sigma\rangle\right)}{1/2 + \cosh\left(\frac{\beta qJ}{2}\langle\sigma\rangle\right)} = f(\langle\sigma\rangle)$$

(c) Find the critical temperature, is it lower or higher than the the $\sigma = \pm 1$ case?

Same tangent: $$\frac{\partial\langle\sigma\rangle}{\partial\langle\sigma\rangle} = 1 \overset{!}{=} \frac{\partial f}{\partial\langle\sigma\rangle} = \frac{\beta qJ\left(2 + \cosh\frac{\beta qJ}{2}\langle\sigma\rangle\right)}{\left(1 + 2\cosh\left(\frac{\beta qJ}{2}\langle\sigma\rangle\right)\right)^2}$$

$$\Rightarrow \beta qJ\left(2 + \cosh\frac{\beta qJ}{2}\langle\sigma\rangle\right) \overset{!}{=} \left(1 + 2\cosh\left(\frac{\beta qJ}{2}\langle\sigma\rangle\right)\right)^2$$

at $\langle\sigma\rangle = 0$: $\beta_c qJ\, 3 = 9$

$$\Rightarrow T_c = \frac{qJ}{3} < \frac{qJ}{2}$$

$\sigma = \pm 1$ case

So it is smaller by a factor of $3/2$!