PHY 831: Statistical Mechanics
Exam 2

October 30th, 2020

Possibly useful information:

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \text{ (for } a > 0)$$

$$\Gamma(n) = (n-1)! = \int_0^{\infty} dx x^{n-1} e^{-x}$$

$$\ln N! \approx N \ln N - N \text{ (for } N \gg 1)$$

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m} = \frac{1}{\Gamma(m)} \int_0^{\infty} dx \frac{x^{m-1}}{e^x - 1}$$

$$\zeta(1) = \infty \quad \zeta(2) = \frac{\pi^2}{6} \quad \zeta(3) = \frac{\pi^4}{90}$$

1. (5 pts) Consider an ensemble with fixed energy and number, but in which the volume is allowed to fluctuate. The ensemble average of the volume is constrained to be $\langle V \rangle$. Find the probability of microstate i of the system, p_i, and define the partition function for this ensemble, Z_H, by maximizing the Gibbs entropy.

2. (10 pts) For free bosons in a D-dimensional box with an energy-momentum relation $\epsilon = ap^s$, where a and s are positive constants, what is the dimension at which Bose-Einstein condensation begins to occur at low temperatures, in terms of D and s?

3. (10 points) Consider a one-dimensional solid of length L at temperature T containing N nuclei in a chain. Each nucleus contributes one spin-half conduction electron (so the rest of the electrons can be neglected). Model excitations of the lattice using a one-dimensional version of the Debye model, so that the density of states in frequency space is given by $g(\omega) = L/(2\pi c_s)$, where c_s is the sound speed, $k = \omega/c_s$ is the wavenumber, and the energy of a phonon is given by $\epsilon = \hbar\omega$. Since motion is only possible in the x-direction, the waves can have only one polarization. Treat the electrons as a free, non-relativistic gas confined to move in one-dimension.

   (a) What is the electron Fermi energy of this system?
   (b) What is the electron contribution to the energy of the system at zero temperature?
   (c) What is the Debye frequency for the lattice?
   (d) What is the phonon contribution to the energy for T small compared to the Debye temperature?