PHY 831: Statistical Mechanics
Exam 3

November 23rd, 2020

Possibly useful information:

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} \text{ (for } a > 0)$$

$$\Gamma(n) = (n-1)! = \int_0^\infty dx x^{n-1} e^{-x}$$

$$\ln N! \approx N \ln N - N \text{ (for } N \gg 1)$$

$$\zeta(m) = \sum_{n=1}^{\infty} n^{-m} = \frac{1}{\Gamma(m)} \int_0^\infty dx \frac{x^{m-1}}{e^x - 1}$$

$$\zeta(1) = \infty \quad \zeta(2) = \tfrac{\pi^2}{6} \quad \zeta(3) = \frac{\pi^4}{90}$$

$$\int_0^\pi d\theta \sin\theta = 2, \quad \int_0^\pi d\theta \sin\theta\cos\theta = 0, \quad \int_0^\pi d\theta \sin\theta \cos^2\theta = \frac{2}{3}$$

1. (4 points) Approximately draw P vs. v for a van der Waals gas for a single temperature below the critical temperature. Label the unstable region and approximately draw the Maxwell construction for this isotherm.

2. (4 points) Consider the mean-field approximation to the Ising model

$$H_{MF} = -\frac{qJ}{2}\langle\sigma\rangle \sum_{i=1}^{N} \sigma_i - h \sum_{i=1}^{N} \sigma_i$$

   where q is a factor of order unity and h is the scaled external magnetic field. This system has the partition function

$$Z = \sum_{\sigma_1} \ldots \sum_{\sigma_N} e^{-\beta H_{MF}(\{\sigma_i\})} = 2^N \cosh^N(\beta \frac{qJ}{2}\langle\sigma\rangle + \beta h)$$

   Find the self-consistency equation for $\langle\sigma\rangle$.

3. (8 points) Assuming a three-dimensional system and given the single-particle distribution function

$$f_1(\vec{x}, \vec{p}, t_0) = \exp\left(-\beta\frac{\vec{p}^2}{2m} + \beta\mu\right)(1 + a\cos\theta), \tag{1}$$

   where the angle θ is defined by $\vec{p} = p(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, a is a constant with a magnitude less than one, μ is the chemical potential, calculate the energy flux in the z-direction.

4. Assume that you have a one-dimensional gas that can be modeled using Euler's equations of hydrodynamics. This gas has a temperature that is constant in time and space, i.e. $T(x,t) = T_0$, a velocity that is constant in space but changes linearly with time, i.e. $u(x,t) = \alpha t$, and an initial density $\rho(x, t = 0) = \rho_0 \exp(-\lambda x)$. Here λ, α, ρ_0, and T_0 are constants. Assume the gas is ideal so that the pressure is give by $P = \rho T/m$ and the internal energy per mass is given by $\epsilon = \frac{3}{2m}T$ (m is the mass of a particle). In one dimension, Euler's equations (assuming there are no external forces) are

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} &= -\rho\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} \\
\frac{\partial \epsilon}{\partial t} + u\frac{\partial \epsilon}{\partial x} &= -\frac{P}{\rho}\frac{\partial u}{\partial x}
\end{aligned} \tag{2}$$

(a) (2 points) Show that the energy evolution equation is satisfied by the assumed temperature and velocity profiles.

(b) (2 points) Find the value of λ that is required for the fluid velocity evolution equation to be satisfied at time zero.

(c) (4 points) Use the density evolution equation and the results from part (b) to find the density at all times and positions by assuming it has the form $\rho(x,t) = \rho(x, t = 0)f(t)$. Briefly describe what the solution means physically.