Problem 1 Alexander Harnisch (harnisch6)

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(a)
$$dF = -S\,dT - p\,dV + \mu\,dN$$
$$= \left(\frac{\partial F}{\partial T}\right)_{V,N} dT + \left(\frac{\partial F}{\partial V}\right)_{T,N} dV + \left(\frac{\partial F}{\partial N}\right)_{T,V} dN$$

$$\left(\frac{\partial}{\partial V}\left(\frac{\partial F}{\partial T}\right)_{V,N}\right)_{T,N} = \left(\frac{\partial}{\partial T}\left(\frac{\partial F}{\partial V}\right)_{T,N}\right)_{V,N}$$
$$= -\left(\frac{\partial S}{\partial V}\right)_{T,N} = -\left(\frac{\partial p}{\partial T}\right)_{V,N}$$
$$\Rightarrow \left(\frac{\partial S}{\partial V}\right)_{T,N} = \left(\frac{\partial p}{\partial T}\right)_{V,N} \quad \square$$

(b)
$$dE = T\,dS - p\,dV + \mu\,dN = T\,dS - \frac{NT}{V}\,dV + \mu\,dN$$
$$\Rightarrow dS = \frac{dE}{T} + \frac{N}{V}\,dV - \frac{\mu}{T}\,dN$$

$$\left(\frac{\partial P}{\partial T}\right)_{V,N} = \frac{N}{V} \overset{(a)}{=} \left(\frac{\partial S}{\partial V}\right)_{T,N}$$ (because $P = \frac{NT}{V}$)

$$c_s^2 = -\frac{V^2}{Nm}\left(\frac{\partial P}{\partial V}\right)_{S,N} = +\frac{V^2}{Nm}\left(\frac{\partial^2 E}{\partial V^2}\right)_{S,N}$$

$$\Rightarrow S = N\ln(V) + \text{const}(T)$$ Not helpful...

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_{N,V} = \frac{3}{2}N$$

$P = \frac{NT}{V} \Rightarrow \left(\frac{\partial P}{\partial V}\right)_{T,N} = -\frac{NT}{V^2}$ and $\left(\frac{\partial P}{\partial T}\right)_{N,V} = \frac{N}{V}$

Oh! :
$$c_s^2 = -\frac{V^2}{Nm}\left(\frac{\partial P}{\partial V}\right)_{S,N} = -\frac{V^2}{Nm}\left[\left(\frac{\partial P}{\partial V}\right)_{T,N} - \left(\frac{\partial P}{\partial T}\right)_{V,N}\left(\frac{\partial T}{\partial V}\right)_{V,N}\right]$$

$$= -\frac{V^2}{Nm}\left[-\frac{NT}{V^2} - \left(\frac{\partial P}{\partial T}\right)_{V,N}\left(\frac{\partial T}{\partial S}\right)_{V,N}\left(\frac{\partial S}{\partial V}\right)_{T,N}\right] = \frac{V^2}{Nm}\left[\frac{NT}{V^2} + \left(\frac{\partial P}{\partial T}\right)^2_{V,N}\underbrace{\left(\frac{\partial T}{\partial S}\right)_{V,N}}_{\frac{T}{C_V} = \frac{2}{3N}T}\right]$$

same (a)

$$= \frac{V^2}{Nm}\left[\frac{NT}{V^2} + \frac{N^2}{V^2}\frac{2T}{3N}\right]$$
$$= \frac{T}{m} + \frac{2}{3m}T = \frac{T}{m}\left(1 + \frac{2}{3}\right) = \frac{5}{3}\frac{T}{m}$$

Problem 2

$$\varepsilon = |\vec{p}|c =: pc \quad , \quad H = \sum_i^N p_i c$$

(a)

$N=1 \Rightarrow H = \varepsilon$

and $$Z_1 = \frac{1}{h^1 \cdot 1!} \int_{-\infty}^{\infty}\int_0^L e^{-\beta H(x,p)}\, dx\, dp$$

$p = |\vec{p}| \Rightarrow -\infty \to 0$

$$= \frac{L}{h} \int_0^\infty \exp(-\beta c p)\, dp + \int_{-\infty}^{0} e^{+\beta p c}\, dp$$

$$= \frac{L}{h} \frac{2}{-\beta c} \left[ \exp(-\beta c p) \right]_{p=0}^{p=\infty}$$

$$= \frac{L}{h} \frac{1}{-\beta c} \left[ 0 - 1 \right]$$

$$= \frac{L}{\beta c h}$$

-1

(b) No interaction, so $Z_N = Z_1^N$
(and all particles are indistinguishable)

$$\Rightarrow Z_N = \left(\frac{L}{\beta c h}\right)^N \frac{1}{N!}$$

-1

(c)

$$S = \frac{\partial}{\partial T} T \ln Z_N \quad (\because k_B = 1)$$

$$= \frac{\partial}{\partial T} T \ln\left(\left(\frac{TL}{ch}\right)^N\right) = \frac{\partial}{\partial T} T N \ln \frac{TL}{ch}$$

$$= N \ln\left(\frac{TL}{ch}\right) + NT \frac{L}{ch} \frac{ch}{TL} = N\left(\ln \frac{TL}{ch} + 1\right)$$

Problem 3

$$dE = TdS - PdV \quad (+\mu dN) \quad \swarrow \text{constant here}$$

with $P = \frac{E}{V}\left(\frac{V}{V_0}\right)^{\lambda}\left(\frac{E}{E_0}\right)^{\lambda}$

$T = E\left(\frac{E}{E_0}\right)^{\lambda}$

$$\Rightarrow \quad dS = \frac{1}{T}dE + \frac{P}{T}dV$$

$$= E^{-1}\left(\frac{E}{E_0}\right)^{-\lambda} dE + \frac{1}{V}\left(\frac{V}{V_0}\right)^{\lambda} dV$$

$$\Rightarrow \quad S(E_1, V_1) = \int_{E_0}^{E_1}\left(\frac{\partial S}{\partial E}\right)_V dE + \int_{V_0}^{V_1}\left(\frac{\partial S}{\partial V}\right)_E dV$$

$$= E_0^{\lambda}\int_{E_0}^{E_1} E^{-\lambda-1}\, dE + V_0^{-\lambda}\int_{V_0}^{V_1} V^{\lambda-1}\, dV$$

$$= E_0^{\lambda}\left[-\frac{1}{\lambda}E^{-\lambda}\right]_{E_0}^{E_1} + V_0^{-\lambda}\left[\frac{1}{\lambda}V^{\lambda}\right]_{V_0}^{V_1}$$

$$= \frac{1}{\lambda}\left[\frac{V_1^{\lambda} - V_0^{\lambda}}{V_0^{\lambda}} - E_0^{\lambda}\left(E_1^{-\lambda} - E_0^{-\lambda}\right)\right]$$

$$= \frac{1}{\lambda}\left[\left(\frac{V_1}{V_0}\right)^{\lambda} - 1 - \left(\frac{E_0}{E_1}\right)^{\lambda} + 1\right]$$

$$= \frac{1}{\lambda}\left[\left(\frac{V_1}{V_0}\right)^{\lambda} - \left(\frac{E_0}{E_1}\right)^{\lambda}\right]$$

So $$S(E, V) = \frac{1}{\lambda}\left[\left(\frac{V}{V_0}\right)^{\lambda} - \left(\frac{E_0}{E}\right)^{\lambda}\right]$$

and $$S(E_0, V_0) = \frac{1}{\lambda}[1 - 1] = 0 \quad \checkmark$$