PHY 831: Statistical Mechanics
Homework 6

Due October 27th, 2020

1. Consider an ideal, non-relativistic three-dimensional Bose gas with spin zero, so that its number and pressure are given by

$$\begin{aligned} N &= \frac{V}{\ell_Q^3} G_{3/2}(e^{\beta\mu}) \\ P &= \frac{T}{\ell_Q^3} G_{5/2}(e^{\beta\mu}) \end{aligned}$$

   (a) Show that the isothermal compressibility κ_T and the adiabatic compressibility κ_S above the condensation temperature are given by

$$\kappa_T = \frac{V}{NT}\frac{G_{1/2}(\lambda)}{G_{3/2}(\lambda)},\ \kappa_S = \frac{3V}{5NT}\frac{G_{3/2}(\lambda)}{G_{5/2}(\lambda)},$$

   where

$$G_\nu(\lambda) = \frac{1}{\Gamma(\nu)}\int_0^\infty dx \frac{x^{\nu-1}}{\lambda^{-1}e^x - 1}$$

   are the Bose-Einstein functions and $\lambda = e^{\beta\mu}$. Note the relationship

$$\lambda\frac{dG_\nu(\lambda)}{d\lambda} = G_{\nu-1}(\lambda),$$

   which holds for $\nu > 1$ and can be found by directly taking the derivative and integrating by parts.

   (b) In the grand canonical ensemble, study the fluctuation in the number of particles N and discuss what happens to the number fluctuations as the system approaches the critical temperature.

2. Carry through the analysis of a Bose gas of ultra-relativistic particles (i.e. $\epsilon = pc$) with $\mu \neq 0$ and find the lower critical dimension for Bose-Einstein condensation for this gas. Also, find the critical temperature in dimensionalities in which condensation occurs. [The lower critical dimension is the highest dimension for which condensation does not occur. For example, in the non-relativistic gas the lower critical dimension is two.]

3. (Relies on material from class on Friday) Solid aluminum has a transverse speed of sound $c_{s,t} = 3.0 \times 10^5\,\mathrm{cm/s}$, a longitudinal speed of sound $c_{s,l} = 6.4 \times 10^5\,\mathrm{cm/s}$ and a density of $2.7\,\mathrm{g/cc}$. Each aluminum atom contributes three conduction electrons to the metal, while the rest of the electrons are bound to the ions.

(a) Calculate the transverse and longitudinal Debye temperatures $\Theta_{D,t}$ and $\Theta_{D,l}$ of the ion lattice.

(b) Determine the temperature at which the contribution to the heat capacity C_V from the phonons is equal to the contribution to C_V from the conduction electron (which you can assume to form a free gas inside the aluminum) . Assume the low-temperature limit for both heat capacities.

4. (Relies on material from class on Friday) Consider a solid which has a weird dispersion relation for sound, so that the frequency is related to the wave number by $\omega = ak^2$ and only longitudinal waves can be excited.

   (a) Find an expression for the phonon heat capacity.

   (b) Show that in the low-temperature limit the heat capacity goes as $C_V \propto T^\alpha$ and find the exponent α.

   (c) Show that in the high-temperature limit C_V goes to the result expected from the equipartition theorem.