1. Consider an ideal, non-relativistic three-dimensional Bose gas with spin zero, so that its number and pressure are given by

$$N = \frac{V}{\ell_Q^3} G_{3/2}(e^{\beta\mu})$$
$$P = \frac{T}{\ell_Q^3} G_{5/2}(e^{\beta\mu})$$

(a) Show that the isothermal compressibility κ_T and the adiabatic compressibility κ_S above the condensation temperature are given by

$$\kappa_T = \frac{V}{NT}\frac{G_{1/2}(\lambda)}{G_{3/2}(\lambda)}, \quad \kappa_S = \frac{3V}{5NT}\frac{G_{3/2}(\lambda)}{G_{5/2}(\lambda)},$$

where

$$G_\nu(\lambda) = \frac{1}{\Gamma(\nu)}\int_0^\infty dx \frac{x^{\nu-1}}{\lambda^{-1}e^x - 1}$$

are the Bose-Einstein functions and $\lambda = e^{\beta\mu}$. Note the relationship

$$\lambda \frac{dG_\nu(\lambda)}{d\lambda} = G_{\nu-1}(\lambda),$$

which holds for $\nu > 1$ and can be found by directly taking the derivative and integrating by parts.

$$\kappa_{T/S} = -\frac{1}{V}\left(\frac{\partial P}{\partial V}\right)^{-1}_{N,T/S}$$

Let's start with

$$\left(\frac{\partial P}{\partial V}\right)_{N,T} = \frac{T}{\ell_Q^3}\frac{\partial g_{5/2}}{\partial \lambda}\left(\frac{\partial \lambda}{\partial V}\right)_{N,T} = \frac{T}{\ell_Q^3}\frac{g_{3/2}}{\lambda}\left(\frac{\partial V}{\partial \lambda}\right)^{-1}_{N,T}$$

$$\left(\frac{\partial V}{\partial \lambda}\right)_{N,T} = -\frac{\ell_Q^3 N}{g_{3/2}}\frac{\partial g_{3/2}}{\partial \lambda} = -\ell_Q^3 N \frac{g_{1/2}}{\lambda g_{3/2}^2}$$

$$= -V\frac{g_{1/2}}{\lambda g_{3/2}}$$

$$\Rightarrow \left(\frac{\partial P}{\partial V}\right)_{N,T} = -\frac{1}{V}\frac{\cancel{\lambda} g_{3/2}}{g_{1/2}}\frac{T}{\ell_Q^3}\frac{g_{3/2}}{\cancel{\lambda}} = -\frac{TN}{V^2}\frac{g_{3/2}}{g_{1/2}}$$

And with that: $\kappa_T = \frac{V}{NT}\frac{g_{1/2}}{g_{3/2}}$

For κ_S: Keeping S constant should be the same as keeping $\lambda = e^{\beta\mu}$ constant.

Check: $S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu} = V\left(\frac{\partial P}{\partial T}\right)_{V,\mu}$

$=$ some function of λ

$$\text{So} \quad \left(\frac{\partial P}{\partial V}\right)_{S,N} = \left(\frac{\partial P}{\partial V}\right)_{\lambda,N} = \frac{5}{2}\,\frac{g_{5/2}}{\ell_Q^3}\left(\frac{\partial T}{\partial V}\right)_{\lambda,N}$$

$$\left(\frac{\partial V}{\partial T}\right)_{\lambda,N} = -\frac{3N\ell_Q^3}{2T\,g_{3/2}} = -\frac{3}{2}\frac{V}{T}$$

$$= -\frac{5}{3}\frac{V}{T}\frac{g_{5/2}}{\ell_Q^3} = -\frac{5NT}{3V^2}\frac{g_{5/2}}{g_{3/2}}$$

$$\text{And thus} : \kappa_S = -\frac{1}{V}\left(\frac{\partial P}{\partial V}\right)^{-1}_{N,S} = \frac{3V}{5NT}\frac{g_{3/2}}{g_{5/2}}$$

(b) In the grand canonical ensemble, study the fluctuation in the number of particles N and discuss what happens to the number fluctuations as the system approaches the critical temperature.

$$\mathrm{Var}(N) = \langle (N-\langle N\rangle)^2\rangle = \left(\frac{\partial^2 \ln Z}{\partial \alpha^2}\right)_{\beta,V}$$

$$= T\left(\frac{\partial N}{\partial \mu}\right)_{T,V} = T\left(\frac{\partial N}{\partial \lambda}\right)_{T,V}\left(\frac{\partial \lambda}{\partial \mu}\right)_{T,V}$$

$$= \frac{TV}{\ell_Q^3}\frac{\partial g_{3/2}}{\partial \lambda}\beta\lambda = \frac{V}{\ell_Q^3}g_{1/2}$$

Critical temperature: $\mu \to 0^- \Rightarrow \lambda \to 1$

and $\lim_{\lambda\to 1} g_{1/2}(\lambda) = \infty$.

So the fluctuations are singular when condensation occurs.

2. Carry through the analysis of a Bose gas of ultra-relativistic particles (i.e. $\epsilon = pc$) with $\mu \neq 0$ and find the lower critical dimension for Bose-Einstein condensation for this gas. Also, find the critical temperature in dimensionalities in which condensation occurs. [The lower critical dimension is the highest dimension for which condensation does not occur. For example, in the non-relativistic gas the lower critical dimension is two.]

$$\varepsilon = pc \quad \Rightarrow \quad \frac{\partial \varepsilon}{\partial p} = c$$

$$\langle N \rangle = \int_0^\infty \frac{g(\varepsilon)\, d\varepsilon}{\exp(\beta(\varepsilon-\mu)) - 1} = \int_0^\infty \frac{g(\varepsilon)\, d\varepsilon}{e^{\beta\varepsilon}\lambda^{-1} - 1}$$

$$g(\varepsilon) = \left(\frac{L}{2\pi\hbar}\right)^D \underbrace{S_D}_{\int d^D\Omega} \left(\frac{\varepsilon}{c}\right)^{D-1} c^{-1}$$

All the constants

$$\Rightarrow \quad N = C \cdot T^D \int_0^\infty \frac{x^{D-1}}{e^x/\lambda - 1}\, dx \quad , \quad x = \frac{\varepsilon}{T}$$

Use geometric sum expansion

$$= C T^D \sum_{i=0}^\infty \int_0^\infty x^{D-1} \exp(-(i+1)x)\, \lambda^{i+1}\, dx$$

$$u := (i+1)x \quad \Rightarrow \quad du = (i+1)\, dx$$

$$= T^D \sum_{i=0}^\infty (i+1)^{-D} \lambda^{i+1} \underbrace{\int_0^\infty u^{D-1} e^{-u}\, du}_{\Gamma(D)}$$

$$= T^D \Gamma(D) \sum_{i=1}^\infty i^{-D} \lambda^{D}$$

shift it

Again: Maximal for $\lambda \to 1$

then: $N \propto \sum_{i=1}^\infty i^{-D} = \zeta(D)$ for $D > 1$

diverges otherwise

So condensation for $D \geq 2$ and the lower critical is $D = 1$.

$$N = N_0 + N_{\varepsilon > 0} \quad , \quad N(\lambda=1, T_c) = \left(\frac{L}{2\pi\hbar}\right)^D \left(\frac{T}{c}\right)^D S_D\, \Gamma(D)\, \zeta(D)$$

$$\Rightarrow \quad T_c = \frac{2\pi\hbar c}{L} \left(\frac{N}{\Gamma(D)\zeta(D) S_D}\right)^{1/D}$$

3. (Relies on material from class on Friday) Solid aluminum has a transverse speed of sound $c_{s,t} = 3.0 \times 10^5\,\mathrm{cm/s}$, a longitudinal speed of sound $c_{s,l} = 6.4 \times 10^5\,\mathrm{cm/s}$ and a density of $2.7\,\mathrm{g/cc}$. Each aluminum atom contributes three conduction electrons to the metal, while the rest of the electrons are bound to the ions.

(a) Calculate the transverse and longitudinal Debye temperatures $\Theta_{D,t}$ and $\Theta_{D,l}$ of the ion lattice.

$$\omega_D = c_s \left(\frac{6\pi^2 N}{V} \right)^{1/3}$$

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$$\Theta = \frac{\hbar}{k_B} \omega_D = \frac{\hbar}{k_B} c_s (6\pi^2 n)^{1/3}$$

particle density

$$\Rightarrow \Theta_{D,t} \approx 350\ \mathrm{K}, \quad \Theta_{D,L} \approx 747\ \mathrm{K}$$

(b) Determine the temperature at which the contribution to the heat capacity C_V from the phonons is equal to the contribution to C_V from the conduction electron (which you can assume to form a free gas inside the aluminum) . Assume the low-temperature limit for both heat capacities.

$$C_V = \frac{12\pi^4}{5} N \left(\frac{T}{\Theta_D} \right)^3 \tag{432}$$

2 modes

$$\text{Here:}\quad C_{V,P} = \frac{4}{5}\pi^4 N \left[\left(\frac{T}{\Theta_{D,L}} \right)^3 + 2 \left(\frac{T}{\Theta_{D,t}} \right)^3 \right]$$

$$C_{V,e} = \frac{\pi^2}{2} 3N \frac{k_B T}{\varepsilon_F}$$

T_F

3 conduction electrons

$$\varepsilon_F = \frac{\hbar^2}{2m_e} (3\pi^2 n)^{3/2}$$

for 3d electron gas

$$\Rightarrow C_{V,p} \overset{!}{=} C_{V,e} \iff \frac{3}{2} \frac{T}{T_F} = \frac{4\pi^2}{5} \left(\Theta_{D,L}^{-3} + 2\Theta_{D,t}^{-3} \right) T^3$$

$$\Rightarrow T = \sqrt{\frac{15}{8\pi^2}}\, T_F^{-1/2} \left(\Theta_{D,L}^{-3} + 2\Theta_{D,t}^{-3} \right)^{-1/2}$$

$$\approx 5.4\ \mathrm{K}$$

4. (Relies on material from class on Friday) Consider a solid which has a weird dispersion relation for sound, so that the frequency is related to the wave number by $\omega = ak^2$ and only longitudinal waves can be excited.

(a) Find an expression for the phonon heat capacity.

$$g(\omega)\,d\omega = g(k)\,dk$$

$$\Rightarrow \quad g(\omega) = g(k)\left(\frac{d\omega}{dk}\right)^{-1}$$

$$= V\frac{\sqrt{\omega}}{4\pi^2 a^{3/2}}$$

$$g(k) = \frac{L^3}{(2\pi)^3}\cdot \underset{\overset{\downarrow}{4\pi,\,D=3}}{S_D}\, k^{\overset{D-1}{2}}$$

$$\frac{d\omega}{dk} = 2ak, \quad k = \sqrt{\frac{\omega}{a}}$$

$$\int_0^{\omega_D} d\omega\, g(\omega) \overset{!}{=} \overset{\text{3 modes}}{3}N$$

$$= \frac{V}{4\pi^2 a^{3/2}}\int_0^{\omega_D}\sqrt{\omega}\,d\omega = \frac{2}{3}\frac{V\omega_D^{3/2}}{4\pi^2 a^{3/2}}$$

$$\Rightarrow \quad \omega_D = a\left(18\pi^2\frac{N}{V}\right)^{2/3}$$

With that: $$E = \int_0^{\omega_D} d\omega\, g(\omega)\, f_-(\hbar\omega)\,\underbrace{\hbar\omega}_{\varepsilon}$$

$$= \frac{\hbar V}{4\pi^2 a^{3/2}}\int_0^{\omega_D}\frac{\omega^{3/2}\,d\omega}{\exp(\beta\hbar\omega)-1}$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V} = \frac{\hbar V}{4\pi^2 a^{3/2}}\int_0^{\omega_D}\frac{\hbar\omega^{5/2}\exp(\beta\hbar\omega)}{T^2\left(\exp(\beta\hbar\omega)-1\right)^2}\,d\omega$$

$x := \beta\hbar\omega$

$$= \frac{V}{4\pi^2}\left(\frac{T}{\hbar a}\right)^{3/2}\int_0^{\frac{\Theta_D}{T}}\frac{x^{5/2}e^x}{(e^x-1)^2}\,dx$$

$$= 3N\,\frac{3}{2}\left(\frac{T}{\Theta_D}\right)^{3/2}\int_0^{\frac{\Theta_D}{T}}\frac{x^{5/2}e^x}{(e^x-1)^2}\,dx = 3N\,D\!\left(\frac{T}{\Theta_D}\right)$$

(b) Show that in the low-temperature limit the heat capacity goes as $C_V \propto T^\alpha$ and find the exponent α.

low T: $\frac{T}{\Theta_D} \ll 1$, $\frac{\Theta_D}{T} \to \infty$

$$\Rightarrow D\left(\frac{T}{\Theta_D}\right) = \frac{3}{2}\left(\frac{T}{\Theta_D}\right)^{3/2} \underbrace{\int_0^\infty \cdots dx}_{\text{converges to some constant}}$$

$$\Rightarrow C_V \propto T^{3/2}$$

(c) Show that in the high-temperature limit C_V goes to the result expected from the equipartition theorem.

High T: $T \gg \Theta_D$, $\frac{\Theta_D}{T} \to 0 \Rightarrow e^x \approx 1+x$

$$\Rightarrow C_V = 3N\, D\left(\frac{T}{\Theta_D}\right) \approx 3N \frac{3}{2}\left(\frac{T}{\Theta_D}\right)^{3/2} \int_0^{\Theta_D/T} \frac{x^{5/2}(1+x)}{x^2}\, dx$$

$$\underset{x \approx 0}{\approx} 3N\frac{3}{2}\left(\frac{T}{\Theta_D}\right)^{3/2} \int_0^{\Theta_D/T} \frac{x^{5/2}}{x^2}\, dx$$

$$= 3N\, \frac{3}{2}\left(\frac{T}{\Theta_D}\right)^{3/2} \frac{2}{3}\left(\frac{\Theta_D}{T}\right)^{3/2}$$

$$= 3N$$