Problem 1

25/25

$S = -\sum_i p_i \ln p_i$ constraints: $\sum_i p_i = 1$ (1)

and $\langle V \rangle = \sum_i p_i V_i$ (2)

$\Rightarrow$ Lagrange function:

$$\mathcal{L} = -\sum_i p_i \ln p_i + \lambda_1 \left(1 - \sum_i p_i\right) + \lambda_2 \left(\langle V \rangle - \sum_i p_i V_i\right)$$

$$= \lambda_1 + \lambda_2 \langle V \rangle - \sum_i p_i \left(\ln p_i + \lambda_1 + \lambda_2 V_i\right)$$

Maximize it

$0 \overset{!}{=} \frac{\partial \mathcal{L}}{\partial \lambda_1} = (1)$, $0 \overset{!}{=} \frac{\partial \mathcal{L}}{\partial \lambda_2} = (2)$

some constant

$$0 \overset{!}{=} \frac{\partial \mathcal{L}}{\partial p_i} = -\ln p_i - 1 - \lambda_1 - \lambda_2 V_i \Rightarrow p_i = C \exp(-1 - \lambda_1 - \lambda_2 V_i)$$

(1) $$1 \overset{!}{=} \sum_i p_i = \underbrace{C \cdot \exp(-\lambda_1 - 1)}_{Z_H^{-1}} \sum_i \exp(-\lambda_2 V_i)$$

$$\Rightarrow Z_H = \sum_i \exp(-\lambda_2 V_i)$$

$$\underline{\underline{p_i = Z_H^{-1} \exp(-\lambda_2 V_i)}}$$

$\Rightarrow \ln p_i = -\lambda_2 V_i - \ln Z_H$

\# microstates

As usual, we can use the entropy $S = \ln \Omega$ to find the Lagrange multiplier.

(Just like $\beta = \frac{1}{T}$ for E and $\gamma = \frac{1}{\mu}$ for N)

Here: $\langle V \rangle \overset{(2)}{=} \frac{1}{Z_H} \sum_i V_i \exp(-\lambda_2 V_i) = -\frac{\partial \ln Z_H}{\partial \lambda_2}$

$$S = -\sum_i p_i \ln p_i = \sum_i p_i (\ln Z_H + \lambda_2 V_i) = \ln Z_H + \lambda_2 \langle V \rangle$$

and thus $\lambda_2 = \left(\frac{\partial S}{\partial V}\right)_{E,N} = \frac{P}{T}$ (Not a very useful ensemble in most situations)

Problem 2

$$p = \hbar k \quad , \quad k = \frac{n\pi}{L}$$

$$\varepsilon = a p^s \quad \Rightarrow \quad \frac{\partial \varepsilon}{\partial p} = s a p^{s-1}$$

$$g(\varepsilon)\, d\varepsilon = g_p(p)\, dp = \frac{d\varphi_p(p)}{dp}\, dp$$

$$\Rightarrow g(\varepsilon) = \left(\frac{L}{2\pi\hbar}\right)^D \left(\oint d\Omega_D\right) p^{D-1} \left(\frac{\partial \varepsilon}{\partial p}\right)^{-1}$$

$$= \left(\frac{L}{2\pi\hbar}\right)^D \Omega_D\, p^{D-1} \cdot \frac{p^{1-s}}{sa}$$

$$= \left(\frac{L}{2\pi\hbar}\right)^D \Omega_D \frac{p^{D-s}}{sa}$$

$$p^{D-s} = \left(\frac{\varepsilon}{a}\right)^{\frac{1}{s}(D-s)} \quad \varepsilon^{D/s-1} \quad a^{\frac{D-s}{s}} = a^{D/s-1}$$

$$= \left(\frac{L}{2\pi\hbar}\right)^D \Omega_D \frac{\varepsilon^{D/s-1}}{s a^{D/s}} =: C_D \cdot \varepsilon^{D/s-1}$$

$$\frac{\partial C_D}{\partial \varepsilon} = 0$$

So N at $\mu = 0$:

$$N = \int_0^\infty d\varepsilon\, g(\varepsilon) f_-(\varepsilon) = C_D \int_0^\infty \frac{\varepsilon^{D/s-1}\, d\varepsilon}{\exp(\beta\varepsilon) - 1} \quad , \quad x := \beta\varepsilon$$

$$= C_D\, T^{D/s} \int_0^\infty \frac{x^{D/s-1}}{e^x - 1}\, dx$$

$$= C_D\, T^{D/s}\, \Gamma(D/s)\, \zeta\left(\frac{D}{s}\right)$$

Finite only for $D/s > 1$, since the ground state does not contribute, it has to be finite when BE condensation occurs.

Thus: $\boxed{D > S \text{ for BE condensation!}}$

Problem 3

(a) free Electrons $\varepsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$

only $k = \frac{\pi}{L} n$, $n \in \mathbb{Z}$ allowed

$\Rightarrow$ For large L: $g_p(p) = \frac{L}{2\pi\hbar} \cdot \underset{\text{spin}}{2} \cdot 2$ $\leftarrow \oint \Omega_D = 2$ in 1D

$$g(\varepsilon)\, d\varepsilon = g_p(p)\, dp$$

$$\Rightarrow g(\varepsilon) = g_p(p) \left(\frac{d\varepsilon}{dp}\right)^{-1}$$

$$\frac{d\varepsilon}{dp} = \frac{p}{m}$$

$$p = \sqrt{2m\varepsilon}$$

$$= \frac{mL}{\pi\hbar} p^{-1}$$

$$= \frac{Lm}{\pi\hbar} (2m\varepsilon)^{-1/2} = \sqrt{\frac{m}{2}} \frac{L}{\pi\hbar} \varepsilon^{-1/2}$$

One free electron per nucleus, so N total:

$$N \overset{!}{=} \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon) = \frac{L}{\pi\hbar}\sqrt{\frac{m}{2}} \int_0^{\varepsilon_F} \varepsilon^{-1/2}\, d\varepsilon$$

$$= \frac{L}{\pi\hbar}\sqrt{\frac{m}{2}}\, 2\sqrt{\varepsilon_F}$$

$$\Rightarrow \boxed{\varepsilon_F = \frac{N^2\pi^2\hbar^2}{\cancel{8}\, L^2 m}}$$

(b) $E = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)\, d\varepsilon = \sqrt{\frac{m}{2}} \frac{L}{\pi\hbar} \int_0^{\varepsilon_F} \varepsilon^{1/2}\, d\varepsilon$

$$= \sqrt{\frac{m}{2}} \frac{L}{\pi\hbar} \frac{2}{3} \varepsilon_F^{3/2}$$

$$= \sqrt{\frac{m}{2}} \frac{L}{\pi\hbar} \frac{2}{3} \frac{N^3 \pi^3 \hbar^3}{2^{3/2} L^3 m^{3/2}}$$

$$= \frac{\pi^2\hbar^2}{L^2} \frac{2}{m\, 4 \cdot 3} N^3$$

$$= \frac{\pi^2\hbar^2}{6L^2} \frac{N^3}{m}$$

N nuclei and only 1 mode (Longitudinal)

(c)

$$N \overset{!}{=} \int_0^{\omega_D} g(\omega)\, d\omega = \int_0^{\omega_D} \frac{L}{2\pi c_s}\, d\omega = \frac{L}{2\pi c_s}\,\omega_D$$

$$\Rightarrow \boxed{\omega_D = \frac{2\pi N}{L}\, c_s}$$

(d)

$x := \beta\hbar\omega$

$\frac{d\omega}{dx} = \frac{1}{\beta\hbar}$

$$E = \int_0^{\omega_D} \varepsilon\, g(\omega)\, f(\omega)\, d\omega = \hbar \int_0^{\omega_D} \frac{\omega\, g(\omega)\, d\omega}{\exp(\beta\hbar\omega) - 1}$$

$$= \frac{\hbar L}{2\pi c_s} \int_0^{\omega_D} \frac{\omega\, d\omega}{\exp(\beta\hbar\omega) - 1} = \frac{\cancel{\hbar} L}{2\pi c_s} \int_0^{\beta\hbar\omega_D = \Theta} \frac{\frac{x}{\cancel{\hbar}\beta}\, \frac{dx}{\beta\hbar}}{\exp(x) - 1}$$

$$= \frac{L T^2}{2\pi\hbar c_s} \int_0^{\Theta/T} \frac{x\, dx}{\exp(x) - 1}$$

for $T \ll \Theta$:

$$\approx \frac{L T^2}{2\pi\hbar c_s} \underbrace{\int_0^{\infty} \frac{x\, dx}{\exp(x) - 1}}_{\zeta(2) = \frac{\pi^2}{6}} = \boxed{\frac{L\pi}{12\hbar}\, \frac{T^2}{c_s}}$$