1. Assume that the Landau free energy is given by

$$F_L = \int d^3x \left[atm^2 + bm^6\right],$$

where $a$ and $b$ are constants and $t = (T - T_c)/T_c$. Find the critical exponent $\beta$.

Minimize wrt $m$:

$$\frac{\partial}{\partial m}\left[ atm^2 + bm^6 \right] \overset{!}{=} 0$$

$$\Leftrightarrow \quad 2at + 6bm^4 = 0$$

$$\Rightarrow \quad m = \left(-\frac{at}{3b}\right)^{1/4}$$

So $\beta = 1/4$

2. Consider a two-dimensional system that undergoes a phase transition and has a Landau-Ginzburg free energy near the critical point given by

$$F_L = \int d^2x \left[ t\vec{\psi}\cdot\vec{\psi} + u(\vec{\psi}\cdot\vec{\psi})^2 - 2u'\psi_x^2\psi_y^2 + g(\vec{\psi}\cdot\vec{\psi})^3 + \frac{\kappa}{2}(\nabla\psi_x)^2 + \frac{\kappa}{2}(\nabla\psi_y)^2 - \vec{h}\cdot\vec{\psi} \right]$$

where $\vec{\psi} = (\psi_x, \psi_y)$ is a two-component order parameter, $t = (T - T_c)/T_c$, $T_c$ is the critical temperature of the system, and $u$, $g$, and $\kappa$ are constants. $\vec{h}$ is an external ordering field.

(a) Find the most probable values of $\vec{\psi}$ when the system is uniform, $u = u' = 0$, $\vec{h} = 0$, and $g > 0$ for both $t > 0$ and $t < 0$.

With $\psi_x = \Psi\cos\Theta$, $\psi_y = \Psi\sin\Theta$:

$\frac{\partial F}{\partial \Theta} = 0 \Rightarrow$ minimum of $F$ can be reached for any $\Theta$

$$\frac{\partial F}{\partial \Psi} = 2At\Psi + 6g\Psi^5 = 0 \Rightarrow \Psi = \begin{cases} 0, & t>0 \\ (-t/3g)^{1/4}, & t<0 \end{cases}$$

(b) Find the critical exponent $\gamma_x$ for this model with $\partial\psi_x/\partial h_x \sim |t|^{-\gamma_x}$ when $u = u' > 0$ and $g = 0$.

$$F = A\left(t\psi_x^2 + t\psi_y^2 + u(\psi_x^4 + \cancel{2\psi_x^2\psi_y^2} + \psi_y^4) - \cancel{2u\psi_x^2\psi_y^2} - h_x\psi_x - h_y\psi_y\right)$$

$$\frac{\partial F}{\partial \psi_x} = 2t\psi_x + 4u\psi_x^3 - h_x = 0 \Rightarrow \psi_x = \begin{cases} 0, & t>0 \\ (-t/2u)^{1/2}, & t<0 \end{cases}, \quad h_x = 0$$

$$\frac{\partial h_x}{\partial \psi_x} = 2t + 12u\psi_x^2 = \begin{cases} 2t, & t>0 \\ 4|t|, & t<0 \end{cases}, \quad h_x = 0$$

$$\Rightarrow \gamma_x = \gamma_x' = 1$$

(c) What is the order of the phase transition if $u < 0$, $u' = 0$, and $g > 0$?

$$t\psi - 4|u|\psi^3 + 6|g|\psi^5 = 0$$

with $|t| \ll |u| \sim |g|$, $\psi = \pm\left(\frac{2|u|}{3|g|}\right)^{1/2}$

Order parameter is discontinous

$\Rightarrow$ 1-st order phase transition

(d) What kind of spontaneous symmetry breaking does this system exhibit if $u = u' > 0$ and $g = 0$?

$$F = \int d^2x \left[ t\psi_x^2 + t\psi_y^2 + u\psi_x^4 + u\psi_y^4 + \frac{K}{2}(\nabla\psi_x)^2 + \frac{K}{2}(\nabla\psi_y)^2 \right]$$

For $t<0$: $\psi_x = \psi_y = \pm\left(\frac{-t}{2u}\right)^{1/2}$ and the spontaneous symmetry breaking is discrete!

(e) What kind of spontaneous symmetry breaking does this system exhibit if $u' = 0$, $u > 0$, and $g > 0$?

From (a) we know the rotational symmetry is continous, so the spontaneous symmetry breaking is also continuous.

(f) Identify the Goldstone modes for this system when $u' = 0$ and find their contribution to the free energy. Use the finite Fourier transform

$$f(\vec{x}) = \frac{1}{\sqrt{A}} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} f_{\vec{q}}$$

and assume that

$$\int \frac{d^2x}{A} e^{i(\vec{q}+\vec{q}')\cdot\vec{x}} = \delta_{\vec{q},\vec{q}'}$$

where $A$ is the area of the system.

$$F_\theta = \frac{K}{2}\int d^2x \, (\nabla(\bar\psi\cos\theta))^2 + (\nabla(\bar\psi\sin\theta))^2$$

$$= \frac{K\bar\psi^2}{2}\int d^2x \, (\cos^2\theta + \sin^2\theta)(\nabla\theta)^2$$

Use FT: $\theta = A^{-1/2}\sum_{\vec{q}} \exp(i\vec{q}\vec{x})\,\theta_{\vec{q}}$

$$\nabla\theta = \frac{i}{\sqrt{A}}\sum_{\vec{q}} \vec{q}\,\exp(i\vec{q}\vec{x})\,\theta_{\vec{q}}$$

$$\Rightarrow F_\theta = -\frac{K\bar\psi^2}{2}\sum_{\vec{q}}\sum_{\vec{q}'} \vec{q}\cdot\vec{q}'\,\theta_{\vec{q}}\theta_{\vec{q}'}\underbrace{\int d^2x\, A^{-1}\exp(i(\vec{q}+\vec{q}')\cdot\vec{x})}_{\delta_{q,-q'}}$$

$$= \frac{K\bar\psi}{2}\sum_{\vec{q}} q^2\,\theta_{\vec{q}}\,\theta_{-\vec{q}} = \frac{K\bar\psi^2}{2}\sum_{\vec{q}} q^2|\theta_{\vec{q}}|^2$$

3. Assume the correlation function for a system with a scalar order parameter has the form

$$\Gamma(\vec{r}) = \langle m(\vec{r})m(0)\rangle - \langle m(0)\rangle^2 = Cr^{2-D-\eta}\exp(-r/\xi), \text{ with } \xi = \xi_0 t^{-\nu}$$

(a) Find the susceptibility in terms of $C$, $\xi$, $\eta$ and the dimensionality $D$. (You do not need to explicity calculate the angular part of the integral, i.e. just write your answer with $\Omega_D$ defined by $\int d^D r = \int d\Omega \int dr r^{D-1} \equiv \Omega_D \int dr r^{D-1}$)

$$\chi = \frac{V}{T}\int d^D x\, \Gamma(x) = \frac{CV}{T}\oint d\Omega_D \int_0^\infty dr\, r^{D-1} r^{2-D-\eta} \exp(-r/\xi)$$

$$= \frac{CV}{T}\Omega_D\, \xi^{2-\eta} \underbrace{\int_0^\infty dx\, x^{1-\eta} e^{-x}}_{\Gamma(2-\eta)}$$

$$= \frac{CV}{T}(1-\eta)!\,\Omega_D\, \xi^{2-\eta}$$

(b) Find the critical exponent $\gamma$ in terms of $\eta$ and $\nu$.

$\chi \propto |t|^{-\gamma}$ also $\chi \propto \xi^{2-\eta}$ an $\xi \propto |t|^{-\nu}$

$$\Rightarrow \gamma = \nu(2-\eta)$$

4. The impact of a gradient term on the liquid-gas phase boundary: Assume the free energy per unit length is given by $f(\rho, T) + \frac{\kappa}{2}(\partial_x \rho)^2$ where $\rho(x)$ is the local density (and assume a planar geometry). Here, the density distribution can be thought of as a Landau-Ginzburg field. The number of particles in is $N = \mathcal{A}\int_{x_g}^{x_l} dx \rho(x)$. Here $x_{l,g}$ are points sufficiently far into the liquid and gas phases that the gradient term goes to zero and $\mathcal{A}$ is the surface area between the two phases. Assume throughout the system is connected to an external particle reservoir with chemical potential $\mu_0$.

(a) Write down an integral expression for the contribution to the Grand Potential of this system between the points $x_l$ and $x_g$. Express it in terms of the local pressure $P(x)$ and local chemical potential $\mu(x)$. Note that the fundamental thermodynamic relation gives $-P(\rho(x), T) = f(\rho(x), T) - \mu(\rho(x), T)\rho(x)$.

$$\Omega = F - \mu_0 N = A \int_{x_g}^{x_l} dx \left( f(\rho, T) - \mu_0 \rho + \frac{\kappa}{2} (\nabla \rho)^2 \right)$$

$$= A \int_{x_g}^{x_l} dx \left( -P - (\mu_0 - \mu(\rho))\rho + \frac{\kappa}{2} (\nabla \rho)^2 \right)$$

(b) Find an integral expression for $\Delta\Omega$, the difference between this Grand Potential and the Grand Potential between $x_l$ and $x_g$ when $\kappa = 0$.

$$\Omega_{\kappa=0} = A \int_{x_g}^{x_l} dx \left( -P_{\kappa=0}(x) - (\mu_0 - \mu) \rho_{\kappa=0} \right)$$

step function $\qquad \mu_l = \mu_g = \mu_0$

$$\Rightarrow \Delta\Omega = A \int_{x_g}^{x_l} dx \left( P_0 - P - (\mu_0 - \mu)\rho + \frac{\kappa}{2} (\nabla \rho)^2 \right)$$

(c) Transform the integral over $x$ to an integral density from $\rho_l$ to $\rho_g$ and vary $\Delta\Omega$ wrt the function $h(\rho) = \partial_x \rho$ to find a condition that minimizes the Grand Potential over the liquid-gas transition region (i.e. think about minimizing the integrand wrt $h$). Plug this back into the expression for $\Delta\Omega$ to find

$$\Delta\Omega = \sqrt{2\kappa}\mathcal{A} \int_{\rho_g}^{\rho_l} d\rho \sqrt{P(\rho_g) - P(\rho) + (\mu(\rho) - \mu(\rho_g))\rho} \equiv \mathcal{A}\sigma$$

where $\sigma$ is the surface tension.

$$\Delta\Omega = A \int_{\rho_g}^{\rho_l} d\rho \left( \frac{P_0 - P - (\mu_0 - \mu)\rho}{d\rho/dx} + \frac{\kappa}{2} \frac{\partial \rho}{\partial x} \right)$$

$$\delta \Delta\Omega = A \int_{\rho_g}^{\rho_l} d\rho \left( \underbrace{- \frac{P_0 - P - (\mu_0 - \mu)\rho}{h^2} + \frac{\kappa}{2}}_{=0} \right) \delta h = 0$$

$$\Rightarrow h = \frac{\partial \rho}{\partial \rho} = \left( \frac{2}{\kappa} \left( P_0 - P - (\mu_0 - \mu) \rho \right) \right)^{1/2}$$

$$\Rightarrow \Delta\Omega = A \int_{\rho_g}^{\rho_l} d\rho ( h + h )$$

$$= A \sqrt{2\kappa} \int_{\rho_g}^{\rho_l} d\rho \left( P_0 - P(\rho) - (\mu_0 - \mu(\rho)) \rho \right)^{1/2}$$