PHY 831: Statistical Mechanics
Homework 2

Due September 21st, 2020

1. Show that
$$\left(\frac{\partial E}{\partial N}\right)_{T,V} = \mu - T\left(\frac{\partial \mu}{\partial T}\right)_{N,V}.$$

2. Prove the relationship
$$C_P = C_V + TV\frac{\alpha_P^2}{\kappa_T}.$$
Since the isothermal compressibility is always greater than zero for a thermodynamically stable gas, this implies the heat capacity at constant pressure is always greater than the heat capacity at constant volume.

3. Consider N spin-1/2 particles on a lattice (so that the particles are distinguishable) in a state with $N/2 + n$ up spins. The Hamiltonian for this system is $H = -\sum_j \sigma_j B$, where $\sigma_j = \pm 1$. This is a simple model for a paramagnetic system.

   (a) Show that the total number of such microstates is
   $$\Omega(n) = \frac{N!}{(N/2+n)!(N/2-n)!}.$$
   (I just want you to go through what we did in lecture here.)

   (b) If the total energy of the system is unspecified, the probability of a a particular value of n (which is proportional to the magnetization of the system) is $p(n) = \Omega(n)/2^N$ since there are 2^N possible states of the system. Show that for $N \gg n$, we have
   $$p(n) \approx \sqrt{\frac{2}{\pi N}} e^{-2n^2/N}$$
   (hint: use Sterling's formula including factors of $\ln(2\pi N)$.)

   (c) Verify that $p(n)$ is normalized.

   (d) Use $p(n)$ to calculate $\langle n^2 \rangle$ and $\langle n^4 \rangle$.

   (e) Assume that there are two paramagnets, each with N spins, in contact with a total energy of zero. What is the root-mean-square value of n for one of the systems?

4. Consider two identical particles that cannot occupy the same single-particle state (i.e. fermions), in a 3-level system with single-particle energies 0, ϵ, and 2ϵ.

(a) Find the canonical partition function Z_N.

(b) Calculate the average energy. Write down the $T = 0$ and $T = \infty$ limits of the average energy.

(c) Calculate the entropy of the system. Write down the $T = 0$ and $T = \infty$ limits of the average entropy.

(d) Repeat parts (a)-(c), but now assuming that the particles are indistinguishable but can occupy the same state.

(e) Repeat parts (a)-(c), but assume the particles are distinguishable and can occupy the same state.