4. (From last week:) Assume there are N random variables labeled by $i = \{1, ..., N\}$ that each obey the arbitrary normalized probability distribution $g(x)$, so that they have averages $\langle x_i^n \rangle = \int dx x^n g(x_i)$. Assume that $\langle x_i \rangle = 0$ and $\langle x_i^2 \rangle = \sigma^2$ for all i. Show that the distribution of the average of these random variables, $\bar{x} = \frac{1}{N}\sum_i x_i$, in the large N limit is given by

$$P(\bar{x}) = \frac{1}{\sqrt{2\pi\sigma^2/N}} e^{-\frac{N\bar{x}^2}{2\sigma^2}},$$

which is essentially the central limit theorem, which says that the probability distribution of the sum of a large number of random variables tends to a Gaussian (or normal) distribution. Therefore, it is maybe not so surprising that this distribution shows up quite often in statistical mechanics. This also shows the standard deviation of $\bar{x}$ is $\propto 1/\sqrt{N}$, since we have $\int_{-\infty}^{\infty} dx x^2 \exp(-x^2/2\sigma^2) = \sqrt{2\pi}\sigma^{3/2}$, and the distribution of $\bar{x}$ goes to a delta function in the large-N limit [Hints: Use $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy e^{ixy}$ and $\int_{-\infty}^{\infty} \exp(iay - by^2) = \sqrt{\pi/b}\exp(-a^2/4b)$.]

The distribution of the means is given by

$$p(\bar{x}) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \, g(x_i) \, \delta\left(\bar{x} - \frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

Using the hint $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp(ixy)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp(i\bar{x}y) \prod_{i=1}^{N} \underbrace{\int_{-\infty}^{\infty} dx_i \, g(x_i) \exp\left(-i\frac{x_i y}{N}\right)}_{=: I}$$

expand exp

$$I = \int_{-\infty}^{\infty} dx \, g(x) \exp\left(-i\frac{xy}{N}\right) = \sum_{n=0}^{\infty} \left(-\frac{iy}{n! N}\right)^n \langle x^n \rangle$$

$$= 1 - \frac{y^2}{2N^2}\langle x^2 \rangle + O\left(\frac{y^3}{N^3}\right)$$

$\to 0$ as $N \to \infty$

$$\Rightarrow \ln I \approx -\frac{y^2}{2N^2}\langle x^2 \rangle$$

And with that:

$$P(\bar{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp(i\bar{x}y) \prod_{i=1}^{N} I = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp(i\bar{x}y) \, I^N$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp(i\bar{x}y + N\ln I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp\left(i\bar{x}y - \frac{y^2}{2N}\langle x^2 \rangle\right)$$

as $N \to \infty$

$$= \frac{1}{2\pi}\left(\frac{2\pi N}{\langle x^2 \rangle}\right)^{1/2} \exp\left(-\frac{N\bar{x}^2}{2\langle x^2 \rangle}\right)$$

using the other hint

Which is the Gaussian PDF with

$\mu = 0$ and $\sigma = \sqrt{\frac{\langle x^2 \rangle}{N}}$.

1. Show that in the grand canonical ensemble, for a three-dimensional gas of spin S particles with single-particle energies $\epsilon = |\vec{p}|^2/2m$, the pressure can be written as

$$P = (2S+1)\int \frac{d^3p}{(2\pi\hbar)^3}\frac{|\vec{v}||\vec{p}|}{3} f_{\mp}(\epsilon(\vec{p})), \tag{1}$$

where $\vec{v} = \vec{p}/m$.

Using (280) from lecture notes:

$$P = \frac{T}{V}\ln Z = \mp\frac{T}{V}\int d\epsilon g(\epsilon)\ln\left[1 \mp \exp(-\beta(\epsilon-\mu))\right] \tag{280}$$

$$\frac{PV}{T} = \mp\int_{\varepsilon_{min}}^{\varepsilon_{max}} g(\varepsilon)\ln(1 \mp \exp(-\beta(\varepsilon-\mu)))\,d\varepsilon$$

here we have $g(\varepsilon) = \underbrace{(2S+1)}_{\text{spins}}\left(\frac{L}{2\pi\hbar}\right)^3 \int d^3p\, \delta(\varepsilon - \varepsilon_\circ(p))$

$$= \mp\left(\frac{L}{2\pi\hbar}\right)^3(2S+1)\int d^3p \ln(1 \mp \exp(-\beta(\varepsilon-\mu)))$$

$$= \mp\left(\frac{L}{2\pi\hbar}\right)^3(2S+1)\oint d\Omega(p)\int_0^\infty dp\, p^2 \ln\left(1 \mp \exp\left(-\beta\left(\frac{p^2}{2m}-\mu\right)\right)\right)$$

Integrat ...

$$= \frac{V}{(2\pi\hbar)^3}(2S+1)\oint d\Omega(p)\int_0^\infty dp \frac{p^4}{3Tm} f_{\mp}(\varepsilon(p))$$

$$= \frac{V}{(2\pi\hbar)^3 T}(2S+1)\int d^3p \frac{p^2}{3m} f_{\mp}(\varepsilon(p))$$

$$\Rightarrow \quad P = \frac{(2S+1)}{(2\pi\hbar)^3}\int d^3p \frac{pv}{3} f_{\mp}(\varepsilon(p))$$

2. Find the density of single-particle states for particles trapped in three-dimensional parabolic potential. Assume the single particle energy levels are $\epsilon = \hbar\omega(m_x + m_y + m_z)$ where $m_i = 0, 1, 2, ..., \infty$, i.e. neglect the zero point energy of the quantum harmonic oscillator. How does this differ from the density states of a gas with energy momentum relation $\epsilon = |\vec{p}|c$ trapped in a box with side lengths L, where c is a constant?

The number of states with single particle energy $\leq \varepsilon$:

$$\sigma(\varepsilon) = \sum_{m_x, m_y, m_z = 0}^{\infty} \Theta(\varepsilon - \hbar\omega(m_x + m_y + m_z))$$

for $T \gg \hbar\omega$: $\sum_{m}^{\infty} \to \int_0^\infty$

$$\Rightarrow \sigma(\varepsilon) = \iiint_0^\infty dm_x\, dm_y\, dm_z\, \Theta\left(\frac{\varepsilon}{\hbar\omega} - m_x - m_y - m_z\right)$$

$$= \int_0^{\varepsilon/\hbar\omega} dm_x \int_0^{\varepsilon/\hbar\omega - m_x} dm_y \int_0^{\varepsilon/\hbar\omega - m_x - m_y} dm_z$$

$$= \int_0^{\varepsilon/\hbar\omega} dm_x \int_0^{\varepsilon/\hbar\omega - m_x} dm_y \left(\frac{\varepsilon}{\hbar\omega} - m_x - m_y\right)$$

$$= \int_0^{\frac{\varepsilon}{\hbar\omega}} dm_x \left(\left(\frac{\varepsilon}{\hbar\omega} - m_x\right)^2 - \frac{1}{2}\left(\frac{\varepsilon}{\hbar\omega} - m_x\right)^2 \right)$$

$$= \frac{1}{2}\int_0^{\frac{\varepsilon}{\hbar\omega}} dm_x \left(\frac{\varepsilon}{\hbar\omega} - m_x\right)^2 = \frac{1}{2}\int_0^{\frac{\varepsilon}{\hbar\omega}} du\, u^2 = \frac{1}{6}\frac{\varepsilon^3}{(\hbar\omega)^3}$$

$$\Rightarrow g(\varepsilon) = \frac{d\sigma}{d\varepsilon} = \frac{1}{2}\frac{\varepsilon^2}{(\hbar\omega)^3} \propto \varepsilon^2$$

For a gas in a box with $\varepsilon = pc$:

$$g(\varepsilon) = \left(\frac{L}{2\pi\hbar}\right)^3 (2s+1) \underbrace{4\pi}_{\text{isotropic } \vec{p}} p^2 \left(\frac{\partial\varepsilon}{\partial p}\right)^{-1}$$

$$= (2s+1)\frac{L^3\varepsilon^2}{2\pi^2\hbar^3 c^3} \propto \varepsilon^2 \quad \text{same as harmonic oscillators!}$$

3. Consider a gas of N non-relativistic electrons (spin = 1/2) confined to a two-dimensional area A with mass m in contact with a reservoir with temperature T and chemical potential μ.

(a) Find the Fermi energy, ϵ_F of the system.

(b) Calculate the two-dimensional "pressure" (i.e. $-\left(\frac{\partial F}{\partial V}\right)_{T,N}$) of the system when $T = 0$.

(c) What is the heat capacity of the electrons at fixed N ($\left(\frac{\partial E}{\partial T}\right)_{N,V}$) when $T \ll \epsilon_F$, to first order in T?

(d) What is the heat capacity of the electrons at fixed μ ($\left(\frac{\partial E}{\partial T}\right)_{\mu,V}$) when $T \ll \epsilon_F$, to first order in T?

$$\varepsilon = \frac{p^2}{2m}$$

(a) $$g(\varepsilon)\,d\varepsilon = g(p)\,dp \Rightarrow g(\varepsilon) = \frac{A}{(2\pi\hbar)^2}(2s+1)\,2\pi p\underbrace{\left(\frac{d\varepsilon}{dp}\right)^{-1}}_{=\frac{m}{p}}$$

$$= \frac{mA}{\pi\hbar^2}$$

With that: $$N = \int_0^{\varepsilon_F} g(\varepsilon)\,d\varepsilon = \frac{mA}{\pi\hbar^2}\varepsilon_F \Rightarrow \varepsilon_F = \frac{\pi\hbar^2}{mA}N$$

$F = E - TS = E$ for $T = 0$

(b) $$P = -\left(\frac{\partial F}{\partial A}\right)_{T,N} = -\left(\frac{\partial E}{\partial A}\right)_{T=0,N}$$

$$E = \int_0^{\varepsilon_F} \varepsilon\, g(\varepsilon)\,d\varepsilon = \frac{mA}{2\pi\hbar^2}\varepsilon_F^2 = \frac{\pi\hbar^2}{2mA}N^2$$

$$\Rightarrow P = \frac{\pi\hbar^2}{2mA^2}N^2 = \frac{\varepsilon_F N}{2A}$$

(c)

- First, we can calculate the correction to the energy, where we would have $E = I$ with $\phi(\epsilon) = \epsilon g(\epsilon)$ so that

$$E = \int_0^{\epsilon_F} d\epsilon\,\epsilon g(\epsilon) + g(\epsilon_F)\frac{\pi^2 T^2}{6} \tag{346}$$

$$\Rightarrow C_V = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \left(\frac{\partial E}{\partial T}\right)_{\varepsilon_F} = \frac{\pi^2 T}{3} + \frac{mA}{\pi\hbar^2} = \frac{\pi m A T}{3\hbar^2}$$

(d) Generally: Most of the time, we will just consider the first T dependent correction so that

$$I \approx \int_0^{\mu} d\epsilon\,\phi(\epsilon) + \phi'(\mu)\frac{\pi^2 T^2}{6} \tag{340}$$

So here: $$E = \int_0^{\mu} \varepsilon\, g(\varepsilon)\,d\varepsilon + \frac{\partial}{\partial\varepsilon}\left(\varepsilon g(\varepsilon)\right)\Big|_{\varepsilon=\mu}\frac{\pi^2 T^2}{6}$$

$$= \frac{mA}{2\pi\hbar^2}\mu^2 + \frac{\pi m A}{6\hbar^2}T^2 \Rightarrow \left(\frac{\partial E}{\partial T}\right)_{V,\mu} = \frac{\pi m A}{3\hbar^2}T$$