PHY 831: Statistical Mechanics
Homework 1

Due September 14th, 2020

1. Starting with
$$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$$
show that the entropy can be derived from the Helmholtz free energy, defined as $F \equiv E - TS$, to be
$$S = -\left(\frac{\partial F}{\partial T}\right)_{N,V}$$

2. Consider a system in which the entropy $S(N, V, E)$ is an extensive quantity.

   (a) Show that
$$S = \left(\frac{\partial S}{\partial N}\right)_{V,E} N + \left(\frac{\partial S}{\partial V}\right)_{N,E} V + \left(\frac{\partial S}{\partial E}\right)_{N,V} E$$

   (b) Show that this in turn results in
$$E = TS - PV + \mu N$$

3. Derive the Maxwell relations
$$\left(\frac{\partial T}{\partial \mu}\right)_{N,P} = -\left(\frac{\partial N}{\partial S}\right)_{T,P}$$
and
$$\left(\frac{\partial P}{\partial T}\right)_{S,N} = \left(\frac{\partial S}{\partial V}\right)_{P,N}$$

4. A substance has the following properties:

   (i) At constant temperature T_0 the work done by it expanding from V_0 to V is
$$W = T_0 \ln \frac{V}{V_0} \tag{1}$$

   (ii) The entropy of the substance is give by
$$S = \frac{V}{V_0}\left(\frac{T}{T_0}\right)^a \tag{2}$$

   where V_0, T_0, and a are fixed constants.

(a) Calculate the Helmholtz free energy (relative to the Helmholtz free energy at (V_0, T_0)).

(b) Find the equation of state.

(c) Find the work done by an arbitrary expansion at an arbitrary constant temperature.

5. Consider a Carnot cycle where the working substance is an ideal gas with the equation of state $PV = NT$, energy $E = NT/(\gamma - 1)$, and entropy given by

$$S = N \ln \left[\left(\frac{E}{E_0} \right)^{\frac{1}{\gamma-1}} \frac{V}{V_0} \right] + N\mathcal{S}(N), \tag{3}$$

where γ is a constant. The cycle operates between temperatures T_1 and T_2 ($T_1 > T_2$) and decompresses at T_1 from volume V_a to volume V_b.

(a) Explicitly calculate the work done and heat gained or lost in each step of the cycle.

(b) Explicitly show this cycle has an efficiency

$$\eta = 1 - \frac{T_2}{T_1}$$