1. Consider the escape of molecules of a Maxwell-Boltzmann gas through an opening of area A in the walls of a vessel of volume V. Show that the escaping molecules have a mean kinetic energy of $2k_B T$, where T is the quasistatic temperature of the gas (i.e. assuming that the energy lost to the escaping molecules is small compared to the total energy of the system so the rest of the gas can be considered to be in equilibrium).

$$F_{n,z} = \int \frac{d^3p}{(2\pi\hbar)^3} v_z f(p) \Theta(v_z) \qquad (488)$$

with $f(p) = \exp\left(-\beta\left(\frac{p^2}{2m} - \mu\right)\right)$ here

And the energy flux will be

$$F_{\varepsilon,z} = \int \frac{d^3p}{(2\pi\hbar)^3} v_z f(p) \Theta(v_z) \cdot \underbrace{\frac{p^2}{2m}}_{\varepsilon}$$

With that, the mean energy of escaping particles is simply:

$$\left\langle \frac{p^2}{2m} \right\rangle_{escaping} = \frac{F_{\varepsilon,z} \cdot \cancel{A}}{F_{n,z} \cdot \cancel{A}} = \frac{\overbrace{\cancel{2\pi}}^{\text{azimuth}} \int_0^{\pi/2} d\theta \sin\theta \int_0^\infty dp \frac{p^5}{2m^2} \cancel{\cos\theta} \exp\left(-\beta\frac{p^2}{2m} \cancel{+\beta\mu}\right)}{\cancel{2\pi} \int_0^{\pi/2} d\theta \sin\theta \int_0^\infty dp\, p^3 \cancel{/m}\, \cancel{\cos\theta} \exp\left(-\beta\frac{p^2}{2m} \cancel{+\beta\mu}\right)}$$

$$= \frac{1}{2m} \frac{\int_0^\infty dp\, p^5 \exp\left(-\beta\frac{p^2}{2m}\right)}{\int_0^\infty dp\, p^3 \exp\left(-\beta\frac{p^2}{2m}\right)}$$

$$\Gamma(n) = (n-1)! = \int_0^\infty dx x^{n-1} e^{-x}$$

$$x = \beta\frac{p^2}{2m}$$

$$= \frac{1}{2m} \frac{\int_0^\infty (Tx\,2m)^2 e^{-x} dx}{\int_0^\infty Tx\,2m\, e^{-x} dx}$$

$$= T \frac{\Gamma(3)}{\Gamma(2)} = T\, 2! = 2T$$

2. In class we discussed the form of the Boltzmann collision term for the Maxwell-Boltzmann distribution. For Bosons/Fermions (upper sign/lower sign), the collision term of the Boltzmann equation takes the form

$$C[f] = \int \frac{d^3p_2}{(2\pi\hbar)^3} \int \frac{d^3p_1'}{(2\pi\hbar)^3} \int \frac{d^3p_2'}{(2\pi\hbar)^3} r(p_1\, p_2 \to p_1'\, p_2') \times \left[f(\vec{p}_1) f(\vec{p}_2)(1 \pm f(\vec{p}_1'))(1 \pm f(\vec{p}_2')) - f(\vec{p}_1') f(\vec{p}_2')(1 \pm f(\vec{p}_1))(1 \pm f(\vec{p}_2)) \right],$$

where f is the single-particle distribution function and all distribution functions are evaluated at the same point in space. The $1+f$ terms account for stimulated emission for Bosons. For Fermions the $1-f$ term accounts for final state blocking. If there is a high probability that there is a particle already in the final state of the reaction, then the rate at which the process proceeds is reduced since only one Fermion can be in a single-particle state. Show this collision integral goes to zero when f takes it's equilibrium form, $f_\mp(\vec{p}) = (\exp[\beta(p^2/2m - \mu)] \mp 1)^{-1}$.

We have:

$$\underbrace{\left[\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m}\cdot\nabla_{\vec{x}} + \vec{F}_{ext}\cdot\nabla_{\vec{p}}\right]}_{d/dt} f_1(\vec{x},\vec{p},t) = -\int dz_2 \vec{F}_{2,1}\cdot(\nabla_{\vec{p}_1} - \nabla_{\vec{p}_2}) f_2(z_1,z_2) \qquad (467)$$

$$\approx C[f_1], \qquad (468)$$

which becomes

$$\left\{\frac{\partial}{\partial t} + \frac{p_1}{m}\frac{\partial}{\partial x}\right\} f_\mp(p) = C[f_\mp]$$

here, because no external force and f does not depend on the direction of $\vec{p}$.

So showing that $C[f_\mp] = 0$ is equivalent to showing that

$$\left\{\frac{\partial}{\partial t} + \frac{p_1}{m}\frac{\partial}{\partial x}\right\} f_\mp(p) = 0$$

which is obviously the case, because

$$\frac{\partial}{\partial t} f_\mp = \frac{\partial}{\partial x} f_\mp = 0.$$

I realized we are probably supposed to show it explicitly (as Lake does in his official solution, so see next page)

Explicitly: $u := \exp(\beta(\varepsilon - \mu))$

$$\Rightarrow \quad f_\pm = (u \pm 1)^{-1}$$

$$1 + f_- = \frac{u-1}{u-1} + \frac{1}{u-1} = u f_-$$

$$1 - f_+ = \frac{u+1}{u+1} - \frac{1}{u+1} = u f_+$$

With that:

$$\begin{aligned} C[f] \;=\; & \int \frac{d^3 p_2}{(2\pi\hbar)^3} \int \frac{d^3 p_1'}{(2\pi\hbar)^3} \int \frac{d^3 p_2'}{(2\pi\hbar)^3} r(p_1\, p_2 \to p_1'\, p_2') \\ & \times \Big[f(\vec{p}_1) f(\vec{p}_2)(1 \pm f(\vec{p}_1'))(1 \pm f(\vec{p}_2')) - \underbrace{f(\vec{p}_1') f(\vec{p}_2')(1 \pm f(\vec{p}_1))(1 \pm f(\vec{p}_2))}_{\frac{u_1 u_2}{u_1' u_2'} f_\pm(p_1) f_\pm(p_2)(1 \mp f_\pm(p_1'))(1 \mp f_\pm(p_2'))} \Big] \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad C[f] \;=\; & \int \frac{d^3 p_2}{(2\pi\hbar)^3} \int \frac{d^3 p_1'}{(2\pi\hbar)^3} \int \frac{d^3 p_2'}{(2\pi\hbar)^3} r(p_1\, p_2 \to p_1'\, p_2') \\ & \times f(\vec{p}_1) f(\vec{p}_2)(1 \pm f(\vec{p}_1'))(1 \pm f(\vec{p}_2')) \underbrace{\left(\frac{u_1 u_2}{u_1' u_2'} - 1 \right)}_{=0} \end{aligned}$$

$$= 0$$

because $\dfrac{u_1 u_2}{u_1' u_2'} = \exp(\beta(\underbrace{\varepsilon_1 - \varepsilon_2 - \varepsilon_1' - \varepsilon_2'}_{=0,\ \text{energy conserved}})) = 1$

3. For an initial one particle distribution function of the form

$$f_1(\vec{x}, \vec{p}, t=0) = A \exp\left(-\frac{\vec{x}^2}{2L^2} - \frac{\vec{p}^2}{2mT_0}\right), \tag{1}$$

where A is a constant, find an expression for $f_1(\vec{x}, \vec{p}, t > 0)$ if the distribution evolves according to the collisionless Boltzmann equation. Show that this has the form of a Maxwell-Boltzmann distribution with a time dependent temperature $T(t)$, which implies f_1 is also a solution of the *collisional* Boltzmann equation.

- In the absence of interactions, i.e. $C[f_1] = 0$, the single-particle distribution is constant along single-particle trajectories in the phase space. In this case, the collisionless Boltzmann equation is solved by

$$f_1\Big(\vec{x}_0 + \underbrace{\int_{t_0}^{t} dt' \vec{v}(t')}_{t\frac{\vec{p}_0}{m}}, \vec{p}_0 + \cancel{\int_{t_0}^{t} dt' \vec{F}(t')}, t\Big) = f_1(\vec{x}_0, \vec{p}_0, t_0) \tag{469}$$

Which leaves: $f_1(\vec{x}_0 + t\frac{\vec{p}_0}{m}, \vec{p}_0, t) = f_1(\vec{x}_0, \vec{p}_0, 0)$

$$\Rightarrow f(\vec{x}, \vec{p}, t) = A \exp\left(-\frac{(\vec{x} - \frac{t\vec{p}}{m})^2}{2L^2} - \frac{p^2}{2mT_0}\right)$$

$$= A \exp\left(-\frac{x^2}{2L^2}\right) \exp\Big[-\frac{p^2}{2m}\underbrace{\left(T_0^{-1} + \frac{t^2}{L^2 m}\right)}_{T^{-1}(t)} + \langle\vec{p}, \vec{x}\rangle \frac{t}{mL^2}\Big]$$

$$= A \exp\left(-\frac{x^2}{2L^2}\right) \exp\Big[-\frac{1}{2mT(t)}\left(\vec{p} - \frac{tT(t)}{2L^2}\vec{x}\right)^2 + \frac{t^2 T(t)}{8mL^4} x^2\Big]$$

$$\boxed{= A(t) \exp\left\{-\frac{\left(\vec{p} - \frac{tT(t)}{2L^2}\vec{x}\right)^2}{2mT(t)}\right]}$$

MB form with $T(t)$!

$$\text{with } A(t) = A \exp\left[-\frac{x^2}{2L^2}\left(1 - \frac{t^2 T(t)}{4mL^2}\right)\right]$$

4. Consider a uniform fluid with background density ρ_0 and background velocity $u_0 = 0$. Assume that the disturbance is adiabatic (so that you can neglect the energy conservation equation of Euler's equations) and small (so that you only need to keeps terms to first order in $\delta\rho$ with $\rho = \rho_0 + \delta\rho$).

(a) Show that you can find solutions to the momentum and mass conservation equations of the form

$$\delta\rho(\vec{x}, t) = A\cos\left(\vec{k}\cdot\vec{x} - \omega(k)t + \phi\right).$$

These small disturbances correspond to sound waves in the fluid.

$\frac{\partial}{\partial t}\rho + \mathrm{div}(\rho\vec{u}) = 0$, $\frac{\partial}{\partial t}\vec{u}\rho + \mathrm{div}(\rho\vec{u}\times\vec{u}) = -\mathrm{grad}\, P$

expand to first order:

$\vec{u} = \delta\vec{u}$, $\rho = \rho_0 + \delta\rho$, $P = P_0 + \delta P = P_0 + \left(\frac{\partial P}{\partial\rho}\right)_S \delta\rho$

$\Rightarrow \frac{\partial}{\partial t}\delta\rho + \rho_0\,\mathrm{div}\,\delta\vec{u} = 0$

$\rho_0 \frac{\partial}{\partial t}\delta\vec{u} + 2\rho_0\,\underbrace{\mathrm{div}(\vec{u}\times\delta\vec{u})}_{=0} = -\mathrm{grad}\,\delta P = -\left(\frac{\partial P}{\partial\rho}\right)_S \mathrm{grad}\,\delta\rho$

$\frac{\partial}{\partial t}$

$\Rightarrow \frac{\partial}{\partial t}\delta\vec{u} + \left(\frac{\partial P}{\partial\rho}\right)_S \frac{\mathrm{grad}\,\delta\rho}{\rho_0} = 0$

$\Rightarrow \frac{\partial^2}{\partial t^2}\delta\rho + \rho_0\,\mathrm{div}\left(\frac{\partial}{\partial t}\delta\vec{u}\right) = 0$

$= \frac{\partial^2}{\partial t^2}\delta\rho - \rho\,\mathrm{div}\left(\left(\frac{\partial P}{\partial\rho}\right)_S \frac{\mathrm{div}\,\delta\rho}{\rho_0}\right)$

$= \left[\frac{\partial^2}{\partial t^2} - \left(\frac{\partial P}{\partial\rho}\right)_S \nabla^2\right]\delta\rho = 0$

Wave equation!

Assume solution $\delta\rho = A_0\cos(\vec{k}\vec{x} - \omega(k)t + \varphi_0)$

$\Rightarrow \left[-\omega^2 + k^2\left(\frac{\partial P}{\partial\rho}\right)_S\right]\delta\rho = 0$

(b) Write down $\omega(k)$ and identify the sound speed $\partial\omega/\partial k$.

We had:

$$\left[-\omega^2 + k^2\left(\frac{\partial P}{\partial \rho}\right)_s\right]\delta\rho = 0$$

$$\Rightarrow \quad \omega = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}\, k \quad \Rightarrow \quad \frac{\partial \omega}{\partial k} = c_s = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}$$

(c) Under what conditions is ω imaginary? What is the sign of the adiabatic compressibility in these conditions? (The adiabatic compressibility is given in terms of specific quantities by

$$\kappa_s^{-1} = \rho\left(\frac{\partial P}{\partial \rho}\right)_s$$

Imaginary if $\left(\frac{\partial P}{\partial \rho}\right)_s < 0$

so then: $\kappa_s = \Big(\underbrace{\rho}_{\geq 0}\ \underbrace{\left(\frac{\partial P}{\partial \rho}\right)_s}_{<0}\Big)^{-1} < 0$ negative!