PHY 831: Statistical Mechanics
Homework 5

Due October 20th, 2020

1. Show that in the grand canonical ensemble, for a three-dimensional gas of spin S particles with single-particle energies $\epsilon = |\vec{p}|^2/2m$, the pressure can be written as
$$P = (2S+1)\int \frac{d^3p}{(2\pi\hbar)^3}\frac{|\vec{v}||\vec{p}|}{3} f_{\mp}(\epsilon(\vec{p})), \tag{1}$$
where $\vec{v} = \vec{p}/m$.

2. Find the density of single-particle states for particles trapped in three-dimensional parabolic potential. Assume the single particle energy levels are $\epsilon = \hbar\omega(m_x + m_y + m_z)$ where $m_i = 0, 1, 2, ..., \infty$, i.e. neglect the zero point energy of the quantum harmonic oscillator. How does this differ from the density states of a gas with energy momentum relation $\epsilon = |\vec{p}|c$ trapped in a box with side lengths L, where c is a constant?

3. Consider a gas of N non-relativistic electrons (spin = 1/2) confined to a two-dimensional area A with mass m in contact with a reservoir with temperature T and chemical potential μ.

   (a) Find the Fermi energy, ϵ_F of the system.

   (b) Calculate the two-dimensional "pressure" (i.e. $-\left(\frac{\partial F}{\partial V}\right)_{T,N}$) of the system when $T = 0$.

   (c) What is the heat capacity of the electrons at fixed N ($\left(\frac{\partial E}{\partial T}\right)_{N,V}$) when $T \ll \epsilon_F$, to first order in T?

   (d) What is the heat capacity of the electrons at fixed μ ($\left(\frac{\partial E}{\partial T}\right)_{\mu,V}$) when $T \ll \epsilon_F$, to first order in T?

4. (From last week:) Assume there are N random variables labeled by $i = \{1, ..., N\}$ that each obey the arbitrary normalized probability distribution $g(x)$, so that they have averages $\langle x_i^n \rangle = \int dx x^n g(x_i)$. Assume that $\langle x_i \rangle = 0$ and $\langle x_i^2 \rangle = \sigma^2$ for all i. Show that the distribution of the average of these random variables, $\bar{x} = \frac{1}{N}\sum_i x_i$, in the large N limit is given by
$$P(\bar{x}) = \frac{1}{\sqrt{2\pi\sigma^2/N}} e^{-\frac{N\bar{x}^2}{2\sigma^2}},$$
which is essentially the central limit theorem, which says that the probability distribution of the sum of a large number of random variables tends to a Gaussian (or normal) distribution. Therefore, it is maybe not so surprising that this

distribution shows up quite often in statistical mechanics. This also shows the standard deviation of $\bar{x}$ is $\propto 1/\sqrt{N}$, since we have $\int_{-\infty}^{\infty} dx x^2 \exp(-x^2/2\sigma^2) = \sqrt{2\pi}\sigma^{3/2}$, and the distribution of $\bar{x}$ goes to a delta function in the large-N limit [Hints: Use $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{ixy}$ and $\int_{-\infty}^{\infty} \exp(iay - by^2) = \sqrt{\pi/b} \exp(-a^2/4b)$.]