PHY 831: Statistical Mechanics
Homework 9

Due December 11th, 2020

1. Assume that the Landau free energy is given by

$$F_L = \int d^3x \left[atm^2 + bm^6\right],$$

where a and b are constants and $t = (T - T_c)/T_c$. Find the critical exponent β.

2. Consider a two-dimensional system that undergoes a phase transition and has a Landau-Ginzburg free energy near the critical point given by

$$F_L = \int d^2x \left[t\vec{\psi}\cdot\vec{\psi} + u(\vec{\psi}\cdot\vec{\psi})^2 - 2u'\psi_x^2\psi_y^2 + g(\vec{\psi}\cdot\vec{\psi})^3 + \frac{\kappa}{2}(\nabla\psi_x)^2 + \frac{\kappa}{2}(\nabla\psi_y)^2 - \vec{h}\cdot\vec{\psi}\right]$$

where $\vec{\psi} = (\psi_x, \psi_y)$ is a two-component order parameter, $t = (T - T_c)/T_c$, T_c is the critical temperature of the system, and u, g, and κ are constants. $\vec{h}$ is an external ordering field.

   (a) Find the most probable values of $\vec{\psi}$ when the system is uniform, $u = u' = 0$, $\vec{h} = 0$, and $g > 0$ for both $t > 0$ and $t < 0$.

   (b) Find the critical exponent γ_x for this model with $\partial\psi_x/\partial h_x \sim |t|^{-\gamma_x}$ when $u = u' > 0$ and $g = 0$.

   (c) What is the order of the phase transition if $u < 0$, $u' = 0$, and $g > 0$?

   (d) What kind of spontaneous symmetry breaking does this system exhibit if $u = u' > 0$ and $g = 0$?

   (e) What kind of spontaneous symmetry breaking does this system exhibit if $u' = 0$, $u > 0$, and $g > 0$?

   (f) Identify the Goldstone modes for this system when $u' = 0$ and find their contribution to the free energy. Use the finite Fourier transform

$$f(\vec{x}) = \frac{1}{\sqrt{A}}\sum_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} f_{\vec{q}}$$

   and assume that

$$\int \frac{d^2x}{A} e^{i(\vec{q}+\vec{q}')\cdot\vec{x}} = \delta_{\vec{q},\vec{q}'}$$

   where A is the area of the system.

3. Assume the correlation function for a system with a scalar order parameter has the form

$$\Gamma(\vec{r}) = \langle m(\vec{r})m(0)\rangle - \langle m(0)\rangle^2 = Cr^{2-D-\eta}\exp(-r/\xi), \text{ with } \xi = \xi_0 t^{-\nu}$$

(a) Find the susceptibility in terms of C, ξ, η and the dimensionality D. (You do not need to explicity calculate the angular part of the integral, i.e. just write your answer with Ω_D defined by $\int d^D r = \int d\Omega \int dr r^{D-1} \equiv \Omega_D \int dr r^{D-1}$)

(b) Find the critical exponent γ in terms of η and ν.

4. The impact of a gradient term on the liquid-gas phase boundary: Assume the free energy per unit length is given by $f(\rho, T) + \frac{\kappa}{2}(\partial_x \rho)^2$ where $\rho(x)$ is the local density (and assume a planar geometry). Here, the density distribution can be thought of as a Landau-Ginzburg field. The number of particles in is $N = \mathcal{A} \int_{x_g}^{x_l} dx \rho(x)$. Here $x_{l,g}$ are points sufficiently far into the liquid and gas phases that the gradient term goes to zero and $\mathcal{A}$ is the surface area between the two phases. Assume throughout the system is connected to an external particle reservoir with chemical potential μ_0.

   (a) Write down an integral expression for the contribution to the Grand Potential of this system between the points x_l and x_g. Express it in terms of the local pressure $P(x)$ and local chemical potential $\mu(x)$. Note that the fundamental thermodynamic relation gives $-P(\rho(x), T) = f(\rho(x), T) - \mu(\rho(x), T)\rho(x)$.

   (b) Find an integral expression for $\Delta\Omega$, the difference between this Grand Potential and the Grand Potential between x_l and x_g when $\kappa = 0$.

   (c) Transform the integral over x to an integral density from ρ_l to ρ_g and vary $\Delta\Omega$ wrt the function $h(\rho) = \partial_x \rho$ to find a condition that minimizes the Grand Potential over the liquid-gas transition region (i.e. think about minimizing the integrand wrt h). Plug this back into the expression for $\Delta\Omega$ to find

$$\Delta\Omega = \sqrt{2\kappa}\mathcal{A} \int_{\rho_g}^{\rho_l} d\rho \sqrt{P(\rho_g) - P(\rho) + (\mu(\rho) - \mu(\rho_g))\rho} \equiv \mathcal{A}\sigma$$

   where σ is the surface tension.