2

Projective Measurement

2.1 The Stern–Gerlach Experiment

In 1922, Otto Stern and Walther Gerlach published what has come to be seen as a paradigmatic experiment on quantum measurement (Gerlach and Stern, 1922). It revealed the existence of the spin degree of freedom, although it was not recognized at the time. The Stern–Gerlach (SG) experiment sent a beam of hot silver atoms through a region with spatially varying magnetic field; see Fig. 2.1. The silver atoms formed a film on a collecting surface (and were originally detected when the smoke from Otto Stern's cigar turned the film black!). The beam of silver atoms was split apart into two regions that we now interpret as "spin up" and "spin down." The silver atom has one valence electron, and the valence electron's spin is responsible for this effect.

We identify the two important degrees of freedom as the quantum spin variable, and the transverse position of the atom after it has traversed some distance. The interaction between the spin and the diverging magnetic field comes from the magnetic dipole moment of the electron $\boldsymbol{\mu}$. In a constant field, the magnetic moment experiences a torque that orients it with the magnetic field, whereas in a gradient field, the moment experiences a force, the direction of which depends on the orientation of the moment. More formally, the Hamiltonian is of the form

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}, \tag{2.1}$$

with a magnetic field $\mathbf{B}$. Classically, we expect a force in the (say) z-direction,

$$F_z = \mu_z \frac{\partial B_z}{\partial z}, \tag{2.2}$$

when the magnetic field gradient is also in the z-direction. Here the subscripts indicate the component of the vector.

The SG experiment resulted in two distinct spots on the far screen, rather than a continuous blob, as would be classically expected from a random distribution of magnetic moment components. The experiment was very challenging, requiring an

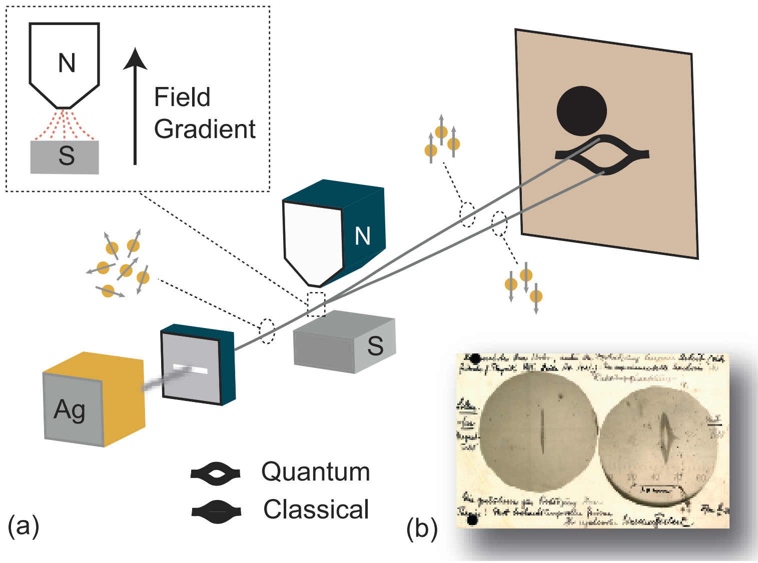

Figure 2.1 (a) In the original Stern–Gerlach experiment, a beam of silver atoms is generated in an oven and is collimated as it traverses through a magnetic field gradient. The quantized spin of the silver atoms' electrons interacts with the field, resulting in a force which causes the atoms to separate into spin-up and spin-down species, as recorded on a piece of photographic paper downstream. Classically, one would expect a random distribution to be peaked at the center. (b) Stern and Gerlach sent their results on a postcard to Niels Bohr, reproduced here with permission from the Niels Bohr Archive.

alignment of the slits and pole pieces to an inaccuracy of no more than 0.01 mm to generate the deviation of 0.2 mm finally observed by Stern and Gerlach (Friedrich and Herschbach, 1998). While the team sought to prove that the orbital angular momentum is quantized, as put forward by Bohr, additional analysis led them to the conclusion that in this case they were dealing with an underlying quantum degree of freedom (spin) where the magnetic moment takes on two values, $\pm\mu_z$. Indeed, if we rotate the SG device so the field gradient is in another direction (say x), then the two spots of silver atoms also appear, but now split in the x-direction. Technically, the value of μ_z can be related to the Bohr magneton, $\mu_B = e\hbar/(2m_e)$, where m_e is the electron mass, e the electron charge, and $\hbar$ the reduced Planck's constant, $\hbar = h/2\pi$. Planck's constant is given by $h = 6.62607015 \times 10^{-34}$ J s and is the characteristic physical constant of quantum physics, having the dimensions of an action – energy multiplied by time. The relation is $\mu_z = -g_s\mu_B/2$, where the electron g-factor $g_s \approx 2.002319$ comes from relativistic corrections (Odom et al., 2006). More generally, we can relate the magnetic moment of the electron to its spin operator $\mathbf{S}$ as $\boldsymbol{\mu}_s = -g_s\mu_B\mathbf{S}/\hbar$.

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Quantum mechanically, the spin has a quantum description with a state vector $|\psi\rangle$, and for such a two-state system, it can be represented in its own basis. We call it the $|+\rangle, |-\rangle$ basis when the magnetic field gradient is oriented in the z-direction, so any allowed quantum state has the form $|\psi\rangle = a|+\rangle + b|-\rangle$, where a, b are complex coefficients. We now consider the case of a single silver atom prepared in this state. Sending it through the SG apparatus will result in the atom arriving at the developing plate in either the high (+) or the low (−) position. Which one? This is one of the most puzzling facts in quantum physics:

Repeated experiments with identically prepared initial conditions can give rise to different final results.

One definition of insanity is doing the same thing over and over again and expecting different results. Quantum mechanics produces different results from the same initial conditions all the time, which may explain why so many practitioners of the subject are crazy. While it may seem like science loses all power to describe and predict in such a situation, we can adopt a less exacting standard than perfect prediction and be content with statistical predictions. That is, you do not try to determine where the atom goes every time; rather you are content to simply predict the probability of the result. Quantum mechanics is this kind of theory – it only gives us the odds of events happening. In the context of the SG device, quantum theory says that the probability P of the atom landing in the high (+) or low (−) position is given by

$$P_+ = |\langle +|\psi\rangle|^2 = |a|^2, \qquad P_- = |\langle -|\psi\rangle|^2 = |b|^2. \tag{2.3}$$

Here $|\ldots|^2$ indicates the squared magnitude of the complex numbers. We have adopted the Dirac notation, where bras $\langle\phi|$ can be combined with kets $|\psi\rangle$ to produce bra-kets, $\langle\phi|\psi\rangle$, a complex number. This notation is wide-spread and we encourage the unfamiliar reader to consult any elementary text in quantum mechanics for more details. This complex number may also be viewed geometrically as the *projection* of a vector in Hilbert space $|\psi\rangle$ onto a dual vector $\langle\phi|$, as we will discuss in more detail later.

Similarly, if we were to rotate the SG device in the x-direction, we would define the basis vectors in this direction as $|+x\rangle, |-x\rangle$, which are related to $|+\rangle, |-\rangle$ by a unitary transformation. The same state discussed earlier can be equivalently written as $|\psi\rangle = c|+x\rangle + d|-x\rangle$, where c, d are complex coefficients related to a, b by the unitary transformation, which involves the relative orientation of the rotated SG device to the original SG device. With this new representation of the quantum state, the probability assignments that correspond correctly to the experimental data are also of the form (2.3), but with the new coefficients c, d taking the place of the old coefficients a, b.

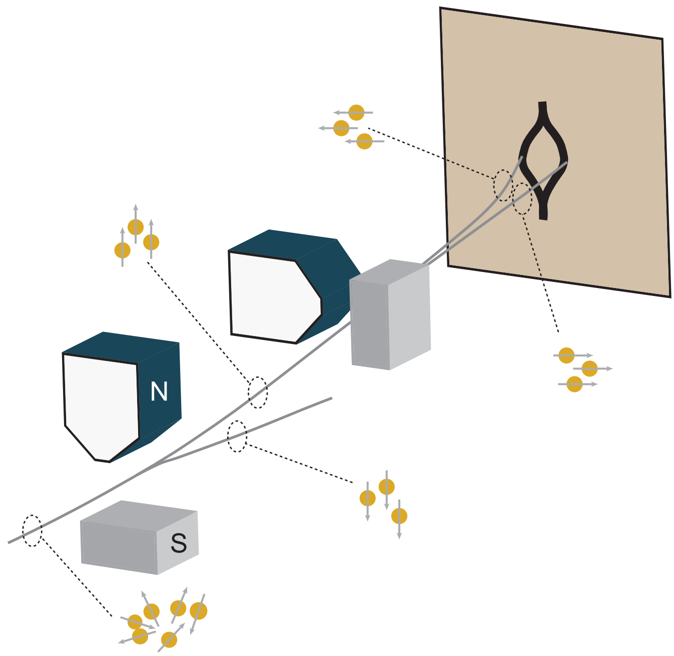

Figure 2.2 In a concatenated sequence of Stern–Gerlach-type experiments, each set of magnets induces a measurement along its respective field gradient. A random beam is first separated into components that align or anti-align vertically. If one of these components is measured by a magnet orthogonal to it, it is then decomposed into a new set of eigenstates that are oriented horizontally.

A crucial part of quantum formalism is to predict what happens to the quantum state post-measurement. In this case, the state of the silver atom is now found in either the high (+) or the low (−) position, and we say the spin state is *collapsed* to either the state $|+\rangle$ or $|-\rangle$, defined by whatever position we place our SG device in. This is nothing more than readjusting our state assignment to reflect what we know to be true. If the electron is found to have spin $|+\rangle$, then we must assign the state to now be $|+\rangle$ (see Fig. 2.2).

Sequences of SG Devices

We note in the previous section of this chapter that the measurement process is crucial to the probability assignments. Instead of detecting the position of the atom, suppose we consider a sequence of two back-to-back SG devices. If the second device has reversed the direction of the magnetic field gradient with a longer extent and we carefully arrange the spacing between the two devices together with the final detection screen, then tracking the dynamics of first the $|+\rangle$ state and then the

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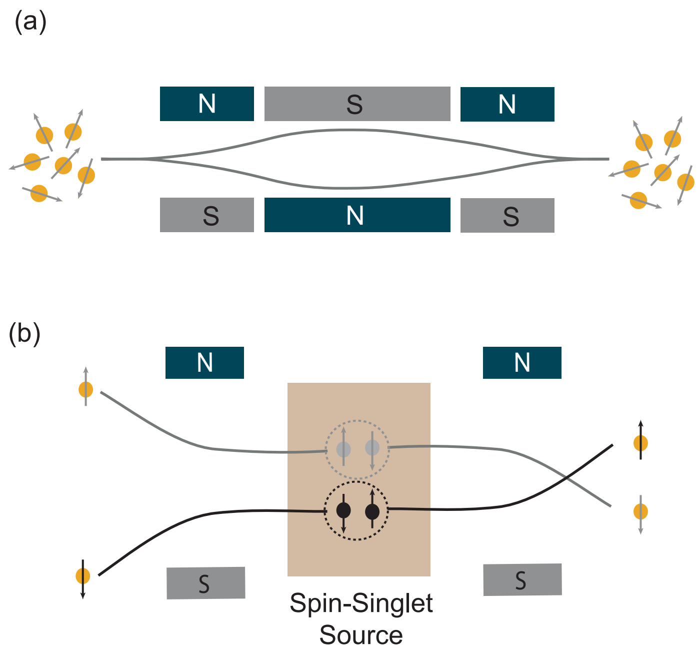

Figure 2.3 (a) Feynman proposed an arrangement of magnets where a beam of spins is separated and then recombined, demonstrating how entanglement can be manipulated before an irreversible projection onto the measurement screen takes place. (b) If a spin-singlet state is produced where the spin species are always anti-correlated, this property is detected when subjecting each component of the beam to a Stern–Gerlach-type measurement. This arrangement is central to many Bell-test experiments.

down $|-\rangle$ state reveals that the up (down) push of the first magnetic field gradient is compensated by the down (up) push of the second magnetic field gradient. The net result of this is nothing! Consequently, detection of the final position of the silver atom does not give any information to us about the spin state of the electron, so no spin measurement occurs. A more clever design described by Feynman (see Fig. 2.3(a)) consists of three SG magnets, two identical ones bookending a central magnet with twice the spatial extent but opposite polarity. The net result of this device is not only to split the beams and to bring them back together, but also to let them continue on their way (Feynman et al., 2011).

More generally, we can consider an arbitrary sequence of SG devices, where we can make a measurement of the spin state at each stage of the experiment by inserting a *beam block* after the SG device. This has the effect of removing the spin state we no longer wish to consider from the experiment, and then we can direct the remaining part of the beam into the next SG device. The next SG device can be oriented at some other angle relative to the first SG device. In this way, we can

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make a whole series of measurements, or we can also use the first measurement to prepare a known state to be analyzed by the second SG device and detection process.

Let us consider, for example, a general initial state, followed by a z-oriented SG device with the $|-\rangle$ state blocked, followed by an x-oriented SG device, followed by a detection screen. The first device will result in finding the atom passing the beam block with probability $|a|^2$, and in producing the state $|+\rangle$, as discussed in the previous section. The second device will find the atom in the position on the final screen corresponding to either state $|+x\rangle$ or state $|-x\rangle$. Using our probability assignment rule (2.3), we find the results:

$$P_{+x} = |\langle +x|+\rangle|^2, \qquad P_{-x} = |\langle -x|+\rangle|^2. \tag{2.4}$$

The experiment shows these probabilities are both $1/2$, so both outcomes are equiprobable. By rotating the SG devices in different directions, the coefficients determining the probabilities can be mapped out. For example, the x basis states are expressible in terms of the z basis states as

$$|\pm x\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle), \tag{2.5}$$

which are consistent with the previously noted probability assignments but contain more information, such as the fact that while the SG-x device will always find an $|+x\rangle$ deflected in the $+x$ direction on the screen, a SG-z device will find it deflected either up or down, half the time. From this kind of reasoning, the entire Hilbert space structure of quantum mechanics of spin-1/2 particles can be built up.

2.2 Measurements on Multiple Systems

Quantum mechanics becomes even more interesting when we consider the physics of multiple systems. Here we consider a variation on the SG device, where we have two SG devices, but also two particles. Suppose the two particles are both spin-1/2 for simplicity but emerge from a common source, which creates *entanglement* between them, a quantum mechanical correlation that we will now describe. Suppose the creation of this pair of particles conserved angular momentum. We would then expect that if one particle has a magnetic moment pointing up, the other should be pointing down. However, we have no preference as to which should be which. So, we write down a quantum state for both possibilities, $|+\rangle \otimes |-\rangle$, and $|-\rangle \otimes |+\rangle$. Here the $\otimes$ symbol indicates a direct product of two quantum states belonging in different Hilbert spaces. The first register indicates the first particle, while the second register indicates the second particle. Quantum mechanically, we are allowed

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to have coherence between these two states, so any combined quantum state of the form

$$|\psi\rangle = a|+\rangle \otimes |-\rangle + b|-\rangle \otimes |+\rangle \tag{2.6}$$

is possible. In the equiprobable case, we have that $|a|^2 = |b|^2 = 1/2$, but a and b may have a relative phase that is physically important. Let us take for simplicity the so-called *singlet* state,

$$|\psi_S\rangle = (1/\sqrt{2})(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle), \tag{2.7}$$

and suppose this is the state prepared before the experiment begins. The singlet turns out to have total spin 0, so it is a natural product of a decay process that conserves angular momentum from a spinless initial state.

We now consider two SG-z devices on either side of the source, so that by momentum conservation, if one spin enters the first SG device, the second enters the second device (see Fig. 2.3(b)). The question we are considering is where the spins land on the two different screens. One spin-1/2 particle must be registered in each for a successful detection. As in the single-particle case, the magnetic field gradient exerts a force on the particle, so if each particle can be registered high ($+$) or low ($-$) (corresponding to spin up or down), then there are in general four possibilities: $++, +-, -+, --$. What we find from experiments is that for the singlet state, results $++$ and $--$ never occur, and we only see $+-$ or $-+$. That is, whenever we see one SG device register "high," we know *for sure* that the other SG device will register "low," but we cannot say in advance which is which. But we can say that, by projecting on the bras $\langle -| \otimes \langle +|$ or $|\langle +| \otimes \langle -|$, half the time it will be one way, and the other half of the time it will be the other.

One of the amazing things about quantum mechanics is that we can keep this exact same source with the same singlet state, rotate the SG devices to be SG-x devices, and ask what happens. The quantum calculation is straightforward. We start with the singlet state (2.7), using the conversion from x into z basis states (2.5), to find, after some algebra,

$$|\psi_S\rangle = (1/\sqrt{2})(|-x\rangle \otimes |+x\rangle - |+x\rangle \otimes |-x\rangle). \tag{2.8}$$

This is the same singlet state as before! The only difference is an overall sign that does not affect the probability assignments in any basis. It turns out the singlet state has the remarkable property that no matter what basis it is expressed in (among the infinite number of choices), it remains invariant, up to an overall phase. The physical consequence of this is that no matter how the SG devices are turned, the same conclusions we first drew remain true: the spin orientation will always remain perfectly anti-correlated, but with no way of predicting which device will get the

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"high" register and which will get the "low" one. This type of experiment is behind the reasoning of Bell's inequality, discussed in the first chapter.

2.3 Mathematics of Projective Measurement

Now that we have physically motivated some of the basic phenomena of quantum measurement, we will give a more complete mathematical treatment to present the general structure of the theory more abstractly. We have been describing the simplest kind of measurement – the projective measurement. This corresponds to an idealized limit where perfect knowledge about the quantum system is gained by some means. It can be direct interaction, auxiliary system interaction, or inference about the system in some other way. The name comes from the description in terms of Hermitian (self-adjoint) *projection operators* $\hat{\Pi}_j$, where j indexes the number of possible outcomes that measurement can take. In the example of the SG device in the previous section, the single-particle case corresponds to two possible outcomes, whereas the two-particle case corresponds to four possible outcomes. A projection operator has the property that

$$\hat{\Pi}_i \hat{\Pi}_j = \hat{\Pi}_i \delta_{ij}. \tag{2.9}$$

Here, δ_{ij} is the Kronecker delta function – it takes a value of 1 if $i = j$, and is 0 otherwise. The projector set also obeys a completeness relation,

$$\sum_i \hat{\Pi}_i = \mathbf{1}, \tag{2.10}$$

where $\mathbf{1}$ indicates the identity operator. Here we consider the simplest case where the number of different detector outcomes equals the dimension of each projector $\hat{\Pi}_i$. The operators are defined by their action on quantum states. We let $|\psi\rangle$ be a Dirac "ket" residing in a Hilbert space, as mentioned in Section 2.1. This quantum state represents the quantum mechanical description of the system of interest. Suppose now a measurement is made, where a number of outcomes are possible. The probability of outcome j, P_j, is generally given by

$$P_j = \langle\psi|\hat{\Pi}_j^\dagger \hat{\Pi}_j|\psi\rangle = \langle\psi|\hat{\Pi}_j|\psi\rangle, \tag{2.11}$$

where we have made use of the fact that the projection operators are self-adjoint. We see immediately that the completeness relation (2.10) of the projectors implies that the probability distribution (2.11) is normalized ($\sum_j P_j = 1$), provided the state is properly normalized. It is convenient to write the projectors in terms of an outer product of some states $\{|j\rangle\}$ of the system Hilbert space with themselves as a rank-one operator,

$$\hat{\Pi}_j = |j\rangle\langle j|, \tag{2.12}$$

where $\langle j|$ is a "bra" dual-representation of the same ket. The projection property (2.9) implies that the state set $\{|j\rangle\}$ is orthonormal and consequently forms a basis in the system Hilbert space. If we express any state $|\psi\rangle$ in this basis, it takes the form

$$|\psi\rangle = \sum_j c_j |j\rangle, \tag{2.13}$$

where the complex coefficients c_j can also be expressed as $\langle j|\psi\rangle$, by the orthonormality of the state set $\{|j\rangle\}$.

We can understand why the operators $\hat{\Pi}_j$ are called projection operators by considering their action on the general ket $|\psi\rangle$,

$$\hat{\Pi}_j|\psi\rangle = |j\rangle\langle j|\psi\rangle = \langle j|\psi\rangle\, |j\rangle. \tag{2.14}$$

We notice that the operator takes whatever state we begin with and collapses it to the ket $|j\rangle$, the associated eigenstate of the projection operator. Furthermore, the state is no longer normalized, but has a complex coefficient, $\langle j|\psi\rangle$, which is the inner product of the state with the 'ket j, which is the projection of the state onto the measurement eigenstate. Its absolute square corresponds to the probability of the result j, as in Eq. (2.11). That is,

$$P_j = |c_j|^2 = |\langle j|\psi\rangle|^2. \tag{2.15}$$

How do we know which projection operators to use? The basis of measurement is specified by the settings of the measurement apparatus. The dimension of the system Hilbert space is set by the number of possible outcomes of a measurement on the system (assuming no degeneracies and a maximally informative type of measurement, a topic which we will return to later). Physically, this is set by the type of coupling arranged between the system of interest and the measuring device. That is, the basis of the coupling sets the basis of the projectors. We will see this more explicitly in the next section.

In the examples in the previous sections, the projection operators corresponding to a $+z$ measurement or a $-z$ measurement of the SG-z device correspond to

$$\hat{\Pi}_{+z} = |+\rangle\langle +|, \quad \hat{\Pi}_{-z} = |-\rangle\langle -|, \tag{2.16}$$

whereas the projection operators for a $+x$ measurement or a $-x$ measurement of the SG-x device correspond to

$$\hat{\Pi}_{+x} = |+x\rangle\langle +x|, \quad \hat{\Pi}_{-x} = |-x\rangle\langle -x|, \tag{2.17}$$

and so on. In the multiparty case we have parties A and B, so our sample space is expanded to four events with two indices, i,j. We can generalize the preceding definition and define the two-party projectors $\hat{\Pi}_{ij} = |i\rangle_{A\,A}\langle i| \otimes |j\rangle_{B\,B}\langle j|$, where $i,j = +,-$. This analysis can be extended to an arbitrarily large number of systems of varying dimension.

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Projection on Eigenstates of an Observable

Very often, we can associate the projectors of the measurements with eigenstates of a Hermitian operator, corresponding to the quantity that is being measured. This operator is typically called an "observable" $\hat{\mathcal{O}}$ in quantum mechanics and can be represented as

$$\hat{\mathcal{O}} = \sum_j \lambda_j \hat{\Pi}_j, \tag{2.18}$$

where λ_j are the eigenvalues of the operator, and the set $|j\rangle$ are the eigenvectors of the operator. The Hermitian nature of the operator implies that the eigenvalues are real, and the eigenvectors form a complete and orthonormal set of states as anticipated on the Hilbert space of the system. The state collapse discussed in the previous section implies that any postmeasurement state is an eigenstate of the measured observable associated with the eigenvalue λ_j. Exercise 2.5.2 proves that the set $\{\lambda_j\}$ are the eigenvalues of $\hat{\mathcal{O}}$.

The *expectation value* of the observable is the average value, taken over the set of all allowed outcomes, in the initial state $|\psi\rangle$,

$$\langle\hat{\mathcal{O}}\rangle = \sum_j P_j \lambda_j = \langle\psi|\hat{\mathcal{O}}|\psi\rangle. \tag{2.19}$$

Here, the expectation value may be interpreted as a weighted average of the eigenvalues of the measurement with respect to the probabilities P_j given in Eq. (2.11). Indeed, in individual realizations of the measurement process of the observable corresponding to $\mathcal{O}$ on identically prepared states, each outcome j occurs, associated with the system eigenvalue λ_j. Statistically, by averaging over this process, we obtain the expectation value. Similarly, higher-order moments of $\mathcal{O}$ may also be considered in order to capture the fluctuations of the process,

$$\langle\hat{\mathcal{O}}^n\rangle = \sum_j P_j \lambda_j^n, \tag{2.20}$$

which you will prove in Exercise 2.5.3.

A measuring device can resolve the physical values of the eigenvalues, such as a Stern–Gerlach device being sensitive to the value of the magnetic moment of the atom. As such, the measurement device resolves the system by its observable eigenvalues, sorting the system accordingly. As such, the postcollapse state corresponds to the eigenstate of the observable that is realized.

We now give some simple illustrations of these general properties for spin-1/2 systems. The observables corresponding to the spin components are written as $\hat{S}_i = (\hbar/2)\hat{\sigma}_i$, where $i = x, y, z$, and $\hat{\sigma}_i$ are the Pauli operators, named in honor of the Austrian physicist Wolfgang Pauli. The total spin has value of $\hat{\mathbf{S}}\cdot\hat{\mathbf{S}} = \sum_i \hat{S}_i^2 = \hbar^2/4$ and is fixed. If we suppose the spin-1/2 particle is in a uniform magnetic field $\mathbf{B}$, then the Hamiltonian is of the form (2.1). Aligning the z-direction along the magnetic field direction, this takes the simplified form $\hat{H} = -\mu_z B\hat{\sigma}_z$. Consequently, the eigenvectors of the Hamiltonian operator are given by the $|+\rangle, |-\rangle$ kets of the $\hat{\sigma}_z$ eigenkets, and the energy eigenvalues are given by $E_\pm = \pm\mu_z B$.

Other common example of operators in continuous variable systems are the position and momentum of a particle. Their expectation values in a quantum state $|\psi\rangle$ are given by $\langle\psi|\hat{\mathbf{x}}|\psi\rangle$, $\langle\psi|\hat{\mathbf{p}}|\psi\rangle$ and variances $\Delta x^2 = \langle\psi|\hat{\mathbf{x}}^2|\psi\rangle - \langle\psi|\hat{\mathbf{x}}|\psi\rangle^2$, and $\Delta p^2 = \langle\psi|\hat{\mathbf{p}}^2|\psi\rangle - \langle\psi|\hat{\mathbf{p}}|\psi\rangle^2$. These satisfy the well-known Heisenberg uncertainty principle, $\Delta x \Delta p \geq \hbar/2$.

Projection on Subspaces of an Observable

In the preceding section, we focused on the simplest case of projection on eigenstates of a Hermitian operator, and we implicitly assumed that all eigenvalues were distinct, the most common situation. As we mentioned in the previous section, the detector physically measures the system by the response of the system according to its observable eigenvalues. Suppose now there is a degeneracy in those eigenvalues, for example we let $\lambda_1 = \lambda_2 \neq \lambda_3, \ldots, \lambda_N$. Then, the observable can be written as

$$\hat{\mathcal{O}} = \lambda_1(\hat{\Pi}_1 + \hat{\Pi}_2) + \sum_{j=3}^{N} \lambda_j \hat{\Pi}_j. \tag{2.21}$$

Suppose the detector shows a result indicating the system takes on the eigenvalue λ_1, but gives no way to further resolve any information within the $\{|1\rangle, |2\rangle\}$ subspace. In that case the appropriate projection operator for the system is neither $\hat{\Pi}_1$ nor $\hat{\Pi}_2$, but rather $\hat{\Pi} = \hat{\Pi}_1 + \hat{\Pi}_2$. The operator $\hat{\Pi}$ is also a projection operator, as you will prove in Exercise (2.5.7).

It is now of great interest to investigate the postcollapse state of the system, given by

$$\hat{\Pi}|\psi\rangle = (\hat{\Pi}_1 + \hat{\Pi}_2)|\psi\rangle. \tag{2.22}$$

Taking the decomposition of the quantum state given in Eq. (2.13), we find the result

$$\hat{\Pi}|\psi\rangle = \sum_j c_j \hat{\Pi}|j\rangle = c_1|1\rangle + c_2|2\rangle. \tag{2.23}$$

Consequently, the physical (renormalized) state is given by

$$|\phi\rangle = \frac{c_1|1\rangle + c_2|2\rangle}{|c_1|^2 + |c_2|^2}. \tag{2.24}$$

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That is, the state does *not* collapse into any particular eigenstate of the measurement operator. Rather it falls into a superposition of the kets in the subspace defined by the degeneracy in the eigenvalue of the measured observable.

Let us consider an example of a rigid rotor of fixed length and moment of inertia, tethered at one end. It is prepared with a quantum wavefunction $\psi(\theta, \phi) = \langle\theta, \phi|\psi\rangle$, where θ, ϕ are the polar and azimuthal angles. We consider a measuring device that is able to measure the total angular momentum $\mathbf{L} \cdot \mathbf{L}$, but not any component of angular momentum. We will now find the event probability and postmeasurement state, given that the detector returns a value of $l = 1$, where l is the orbital angular momentum quantum number. That is, the eigenvalue of the Hermitian operator $\hat{L}^2 = \hat{\mathbf{L}} \cdot \hat{\mathbf{L}}$ is given by $\hbar^2 l(l+1)$, where $l = 1$ is the result.

We recall from quantum treatments of angular momentum that, in such a situation, several different values of the azimuthal quantum number m_l are permitted, where these are the eigenvalues of the z component of the angular momentum, $\hat{L}_z$. In this case, m_l can take the values $\{-1, 0, 1\}$. Denoting the eigenkets of $\hat{L}^2, \hat{L}_z$ as $|l, m\rangle$, we define the three projectors $\hat{\Pi}_{1,1} = |1,1\rangle\langle 1,1|$, $\hat{\Pi}_{1,0} = |1,0\rangle\langle 1,0|$, $\hat{\Pi}_{1,-1} = |1,-1\rangle\langle 1,-1|$. Drawing on the treatment earlier in this section, the correct projection operator for the $l = 1$ output of a detector that only detects total angular momentum is then given by

$$\hat{\Pi}_{l=1} = \hat{\Pi}_{1,1} + \hat{\Pi}_{1,0} + \hat{\Pi}_{1,-1}. \tag{2.25}$$

To find the probability of this outcome occurring (of all the possible values, $l = 0, 1, 2, \ldots$), we can use our result (2.11), so the probability $P_{l=1}$ is given by

$$P_{l=1} = \langle\psi|\hat{\Pi}_{l=1}|\psi\rangle. \tag{2.26}$$

This result can be further simplified by inserting complete sets of states $\int d\cos(\theta) d\phi\, |\theta, \phi\rangle\langle\theta, \phi| = \mathbf{1}$ and expressing the result in terms of integrals of the wavefunction and the spherical harmonics, $\langle\theta, \phi|l, m\rangle = Y_l^m(\theta, \phi)$:

$$\begin{aligned} Y_1^{-1}(\theta, \phi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{-i\phi} \sin\theta, \quad Y_1^1(\theta, \phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{i\phi} \sin\theta, \\ Y_1^0(\theta, \phi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta. \end{aligned} \tag{2.27}$$

The full expression is left to the reader. We can move on to finding the (unnormalized) postcollapse state as

$$|\phi\rangle = c_{-1}|1, -1\rangle + c_0|1, 0\rangle + c_1|1, 1\rangle, \tag{2.28}$$

where we define the coefficients using the extension of the result (2.23) to a three-state system,

$$c_m = \int d\cos(\theta)d\phi\, Y_1^m(\theta,\phi)^*\psi(\theta,\phi), \tag{2.29}$$

where $m = -1, 0, 1$. As we saw previously, the postcollapse state indeed has a definite value of l but resides in a quantum superposition of the three possible m values consistent with the measured l value.

2.4 Continuous Variables

While the previous discussion is focused on systems with a finite Hilbert space, it is possible to generalize the properties discussed there to continuous variable systems. Suppose we wish to measure a continuous variable like position or momentum. We define the eigenbasis of such an observable as $\{|z\rangle\}$, with associated projectors $\hat{\Pi}_z = |z\rangle\langle z|$. The continuous observable $\hat{\mathcal{O}}_c$ can then be written as

$$\hat{\mathcal{O}}_c = \int dz|z\rangle\langle z|\,\mathcal{O}(z), \tag{2.30}$$

where $\mathcal{O}(z)$ are the continuous eigenvalues of the operator. We must integrate over the index, rather than sum. We can handle the generalizations of Eqs. (2.9, 2.10) by replacing sums by integrals and Kronecker delta functions by Dirac delta functions. Consequently, we have

$$\hat{\Pi}_z\hat{\Pi}_w = \hat{\Pi}_z\delta(z-w), \tag{2.31}$$

and

$$\int dz\hat{\Pi}_z = \mathbf{1}. \tag{2.32}$$

Here the integrals are taken over the suitable range for the system of interest. If we consider the examples of position or momentum measurement, and write the projectors as outer products of position or momentum eigenstates, then the preceding properties can be understood as being identical to the statement that the inner product of these states with themselves yields a Dirac delta function (mathematically this is a distribution, strictly speaking).

2.5 Discussion of the Cardinal Properties of Projective Measurement

We conclude this chapter with a brief discussion of the cardinal properties of projective measurements that we have learned.

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(i) The projective nature of the measurement gives maximum distinguishability of the different eigenvalues of the observable being measured. It is also maximally disturbing. No matter which initial state you start with, the projection operator collapses them all to the eigenstate of the observable, $\hat{\Pi}_j|\psi\rangle \propto |j\rangle$, unless $|\psi\rangle$ has no component along $|j\rangle$, in which that measurement outcome will not appear.

(ii) The irreversible nature of projective measurement can be seen both mathematically as well as physically. The aforementioned projection property is a many-to-one mapping of quantum states to quantum states: Every possible state that is not orthogonal to $|j\rangle$ is reduced to that single state. Such an operation has no inverse in the mathematical sense. Physically, if we try to time-reverse the measurement process operationally, we run into problems. Consider the setup of the SG-z device and run it backwards. Suppose we put in the spin state $a|+\rangle + b|-\rangle$ and end in the state $|+\rangle$, corresponding to the atom deflecting up and being measured there. Now let us try and go backwards. If we time-reverse the momentum of the atom in a spin-up state, we have a left-moving, spin-down atom (since spin is an angular momentum, its time-reverse flips sign, like momentum). Now the atom will track the same course backwards, but the end of its course (corresponding to the beginning of the time-forward process) will not recover the initial situation. Even after again time-reversing at this point to try and recover it, we find the spin state is simply $|+\rangle$ and not the coherent superposition we began in.

(iii) Its instantaneous nature seems to hinge on what happens at the screen (or beam block). If at any point in time before detection of the atom's position, the process does indeed appear to be time-reversal invariant. If we describe both the orbital and spin degrees of freedom, then before detection, we have the state

$$|\Psi\rangle = a|+\rangle \otimes |\text{deflect up}\rangle + b|-\rangle \otimes |\text{deflect down}\rangle. \tag{2.33}$$

If we instead take this state and time-reverse it, we will then reverse the direction of the particle's momentum and spin direction via the time-reversal operator $\hat{\Theta}$, an anti-unitary operator, such that both momentum kets $\hat{\Theta}|p\rangle = |-p\rangle$ and spin direction kets $\hat{\Theta}|S_j\rangle = |-S_j\rangle$ are inverted. In this situation, the original state is recovered after running the process an equal amount of time and again time-reversing! Hence the collapse of the wavepacket appears to happen instantly upon reaching the detecting screen or beam block, resulting in the reduced state. This reasoning supports Heisenberg's argument mentioned in Chapter 1, that the detection of a particle at one place after a scattering event immediately suppresses the amplitude to find the particle anywhere else, no matter how far apart these possible locations are.

In the chapters to come we will see how all of these properties can be broken.

Exercises

Exercise 2.5.1 Consider a Stern–Gerlach apparatus with a fringing field of 100 gauss per cm. What length of the magnet is required to have a spacing of 1 mm on the screen placed 40 cm from the device? The oven temperature is approximately 250 degrees centigrade (use the typical velocity from this oven for a silver atom). First solve the general problem, and then put in specific values.

Exercise 2.5.2 Using Property (2.12) of the projection operators and the orthonormality of the set $|j\rangle$, prove λ_j are the eigenvalues of operator $\hat{\mathcal{O}}$.

Exercise 2.5.3 Show that the formula (2.20) is correct, given the properties (2.9) of projection operators.

Exercise 2.5.4 From the two properties of projective measurement operators (2.9, 2.10), show that the projection operators may be written as the outer product of a complete set of orthonormal basis states with themselves.

Exercise 2.5.5 Express the projection operators for $\pm x$ spin-1/2 measurements (2.17) in the z-basis.

Exercise 2.5.6 Consider a spin-1/2 system and a general SG-θ device that can project the spin onto an arbitrary angle. Recall that if we rotate a spin-1/2 system about a direction described by the unit vector $\hat{n}$ characterized by the polar and azimuthal angles, β and α respectively, we can describe the "up" spinor in the z-basis as

$$\chi = \begin{pmatrix} \cos(\beta/2)e^{-i\alpha/2} \\ \sin(\beta/2)e^{+i\alpha/2} \end{pmatrix}. \tag{2.34}$$

Construct the projection operators for such a device, expressed as matrices in the z-basis. If a spin is prepared in the $S_z = +\hbar/2$ eigenstate, what is the probability of finding it in the $S_\theta = +\hbar/2$ state after the SG-θ device?

Exercise 2.5.7 Prove the operator $\hat{\Pi} = \hat{\Pi}_1 + \hat{\Pi}_2$ is also a projection operator using Properties (2.9).

Exercise 2.5.8 Consider a three-level system prepared in the normalized state

$$|\psi\rangle = a|1\rangle + b|2\rangle + c|3\rangle, \tag{2.35}$$

where a, b, c are complex coefficients and kets $|1\rangle, |2\rangle, |3\rangle$ are an orthonormal basis.

Consider the operators

$$\hat{\mathcal{O}}_1 = \frac{1}{3}(\mathbf{1} + |1\rangle\langle 2| + |2\rangle\langle 1| + i|3\rangle\langle 2| - i|2\rangle\langle 3| - i|1\rangle\langle 3| + i|3\rangle\langle 1|),$$
$$\hat{\mathcal{O}}_2 = \frac{1}{2}(\mathbf{1} - |2\rangle\langle 2| + i|1\rangle\langle 3| - i|3\rangle\langle 1|),$$

$$\hat{\mathcal{O}}_3 = \frac{1}{6}(\mathbf{1} + 3|2\rangle\langle 2| - 2|1\rangle\langle 2| - 2|2\rangle\langle 1| + i|3\rangle\langle 1| - i|1\rangle$$
$$\langle 3| - 2i|3\rangle\langle 2| + 2i|2\rangle\langle 3|).$$

Here **1** is the identity operator.

(a) Show that the three operators form a complete set of projectors on the three-level system by verifying the fundamental properties of projection operators.

(b) If a measurement corresponding to those projection operators is implemented, what is the probability of finding result 2 (corresponding to $\hat{\mathcal{O}}_2$) for state $|\psi\rangle$. What does the state collapse to in this case?

Exercise 2.5.9 Recall the example given in the text of a measuring device that detected total angular momentum, but not any component of the angular momentum. Now consider a detector that can only detect the z component of the orbital angular momentum, but not its total angular momentum. Suppose such a detector returned a value of $m = 0$. What values of l are consistent with this result? Write down the correct projection operator for this result, and work out the angular dependence of the postcollapse state.

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