LECTURE NO. 14

5.1 Sequences

Wright State University

Why do we study squences and series?

- How would you calculate e^2?
- Inside a calculator, e^x is represented as follows:

3! : 3 factorial

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$3! = 3 \cdot 2 \cdot 1$

- So $e^2 = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots$; the calculator will perform the computation up to a certain number of decimal places.
- Recall that $\int e^{t^2} dt$ is impossible to do!
- However, e^{t^2} can be written as an infinite sum of "good functions":

$$e^{t^2} = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \frac{t^8}{4!} + \cdots$$

What is a sequence?

- A sequence $\{a_n\}$ is a set of numbers listed in order:

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

- We can write a sequence by listing the numbers:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

- Or we can give a general formula: $a_n = 2^n, n \geq 0$; $1, 2, 4, 8, 16, 32, \cdots$
- We may also define a sequence by a recursive relation.
- The Famous Fibonacci Sequence:

$$f_1 = f_2 = 1, f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3$$

$f_3 = f_2 + f_1 = 2$

$f_4 = f_3 + f_2 = 3$

$f_5 = f_4 + f_3 = 5$

$\cdots$

Arithmetic Sequence and Geometric Sequence

- In an arithmetic sequence, the difference between every pair of consecutive terms is the same.
- An arithmetic sequence $\{a_n\}$: $2, 7, 12, 17, 22, 27, 32, \cdots$
- General Formula: $a_n = 2 + 5(n-1) = 5n - 3, n \geq 1.$
- In a geometric sequence, the ratio between every pair of consecutive terms is the same.
- A geometric sequence $\{b_n\}$: $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \frac{3}{16}, \cdots$ Common Ratio $= -\frac{1}{2}$
- General Formula: $b_n = 3 \cdot (-\frac{1}{2})^{n-1}, n \geq 1.$

Finding Explicit Formulas

- $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \cdots$

$n = 1, 2, 3, 4, 5, \cdots$

$(-1)^n$ or $(-1)^{n+1}$

$$a_n = (-1)^n \cdot \frac{n}{n+1}, \quad n \geq 1$$

- $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \cdots$

$n = 1, 2, 3, 4, 5, \cdots$

N: Geometric Sequence

D: Arithmetic Sequence

$$b_n = \frac{3^n}{4+3(n-1)} = \frac{3^n}{3n+1}, \quad n \geq 1$$

Convergence of a sequence

- A sequence is convergent if
$$\lim_{n\to\infty} a_n = \text{ a number};$$
otherwise, it is divergent.
- Three ways to find $\lim_{n\to\infty} a_n$:
  - Use Algebra: ex. $a_n = \cos(\frac{2}{3n+1})$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(\frac{2}{\underbrace{3n+1}_{\infty}}) = \cos(0) = 1$$
.
  - Use L'Hôpital's Rule
  - Use Squeeze Theorem

$$b_n = \frac{n^2+1}{2n+3}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2+1}{2n+3}$$

divide all terms by the highest power of n in $\underline{\underline{D}}$.

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n} + \frac{1}{n}}{\frac{2n}{n} + \frac{3}{n}} = \lim_{n\to\infty} \frac{\overset{\infty}{n} + \underset{\to 0}{\frac{1}{n}}}{2 + \underset{\to 0}{\frac{3}{n}}}$$

$$= \infty \quad \text{(divergent)}$$

Use LHR to decide if the sequence $a_n = \frac{2^n}{n^2}$ is convergent.

We need to find $\underset{n\to\infty}{\lim} \dfrac{2^n}{n^2} = \underset{x\to\infty}{\lim} \dfrac{2^x}{x^2}$ ← we can use LHR to this limit

$$\lim_{x\to\infty} \frac{2^x}{x^2} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2^x \ln 2}{2x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} = \infty$$

Therefore $\underset{n\to\infty}{\lim} \dfrac{2^n}{n^2} = \infty$, the sequence $\{a_n\}$ is divergent.

Use Squeeze Theorem to decide if $b_n = \frac{\sin n}{n}$ is convergent.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$n \to \infty$: $-\frac{1}{n} \to 0$, $\frac{\sin n}{n} \to 0$, $\frac{1}{n} \to 0$

By Squeeze Theorem

The sequence $b_n = \frac{\sin n}{n}$ is convergent to 0.

Other Properties of a Sequence

- A sequence $\{a_n\}$ is increasing if $a_n \leq a_{n+1}$ for all n.
- Example of an increasing sequence: $a_n = \frac{n}{n+1}, n \geq 1$
- A sequence $\{a_n\}$ is decreasing if $a_n \geq a_{n+1}$ for all n.
- Example of a decreasing sequence: $a_n = \frac{1}{n}, n \geq 1$
- A sequence $\{a_n\}$ is bounded if there are real numbers m and M such that $m \leq a_n \leq M$ for all n.
- Example of a bounded sequence: $a_n = (-1)^n \frac{2n}{2n+1}, n \geq 1$
- $$-1 \leq a_n \leq 1$$