LECTURE NO. 19

5.6 Ratio and ~~Root~~ Test

Wright State University

Ratio Test

- Ratio Test is used to determine the absolute convergence of a series.
- For any series

$$\sum_{n=1}^{\infty} a_n$$

- Let

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

  - If $L < 1$, then the series is absolutely convergent.
  - If $L > 1$, then the series is divergent.
  - If $L = 1$, then Ratio Test is inconclusive.

An Example on Ratio Test

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$n!$ factorial

Ratio Test

$$\frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{n}}{1\cdot 2\cdot 3\cdots \cancel{n}\cdot(n+1)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty}\frac{\overset{2}{\cancel{2^{n+1}}}}{\underset{n+1}{\cancel{(n+1)!}}}\cdot\frac{\cancel{n!}}{\cancel{2^n}}$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$$

Therefore, the series is absolutely convergent by Ratio Test.

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(n+1)^3}}{\frac{(-3)^n}{n^3}} \right| = \lim_{n\to\infty} \frac{\overset{3}{\cancel{3^{n+1}}}}{(n+1)^3} \cdot \frac{n^3}{\cancel{3^n}}$$

$$= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3}$$

LHR

algebra

$$= \lim_{n\to\infty} 3 \cdot \left(\frac{n}{n+1}\right)^3 = \lim_{n\to\infty} 3 \cdot \left(\frac{1}{1+\frac{1}{n}}\right)^3$$

$\frac{1}{n} \to 0$

$$= 3 > 1$$

The series $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$ is divergent.

Some Remarks on Ratio Test

- Ratio Test fails if $L = 1$, for example,

$$\sum_{n=1}^{\infty} \frac{2n}{3n^2+1}$$

$$\sum_{n=1}^{\infty} \frac{n^3}{2^n}$$

- When to use Ratio Test?

  1) If the terms contain factorials, $n!$, $(2n+1)!$, etc, or

  2) if the terms contain a mixture of exponential terms (2^n, $(-3)^n$) and polynomial terms (n^2, $n^{\frac{3}{2}}$).

- Geometric Series and p-Series. We know exactly when they are convergent/divergent.
- Geometric Series

$$\sum_{n=1}^{\infty} a \cdot r^{n-1}$$

1) If $|r| < 1$, then the geometric series is convergent (to $\frac{a}{1-r}$).
2) If $|r| \geq 1$, then the geometric series is divergent.

- p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

1) If $p > 1$, then the p-series is convergent.
2) If $p \leq 1$, then the p-series is divergent.

- Given a series

$$\sum_{n=1}^{\infty} a_n$$

- First check if $\lim_{n\to\infty} a_n = 0$; if no, the series is divergent by Test for Divergence.
- Next check if it is an alternating series? If yes, then try Alternating Series Test.
- Then check if the series is a positive series; if yes, try either Comparison Test or Integral Test.
- Finally, we may try Ratio Test; especially in the following two cases.
  1) If the terms contain factorials, $n!$, $(2n+1)!$, etc, or
  2) if the terms contain a mixture of exponential terms (2^n, $(-3)^n$) and polynomial terms (n^2, $n^{\frac{3}{2}}$).