LECTURE NO. 20

6.1 Power Series and Functions

Wright State University

What Is a Power Series?

- A Power Series is a series with the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

  where x is a variable and c_n are constants.
- The series above is centered at 0.
- In general, a power series may be centered at an arbitrary value a as follows

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

- A power series centered at 2:

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$$

Set $x-2=0$

$x=2 \leftarrow$ center

Convergence of a Power Series

- Given a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

- We want to know the x-values for which the series is convergent.
- The series is convergent if $x = a$, i.e., the center (We just add up a bunch of 0's).
- What about other values of x?
- Always use Ratio Test to find them.

$\sum_{n=0}^{\infty} a_n$

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

absolutely convergent if $L < 1$

divergent if $L > 1$

inconclusive if $L = 1$.

Example no. 1: use Ratio Test on a Power Series

$$\sum_{n=0}^{\infty} n!(x-1)^n$$

Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}^{\,(n+1)}\ \cancel{(x-1)^{n+1}}^{\,(x-1)}}{\cancel{n!}\ \cancel{(x-1)^n}}\right| = \lim_{n\to\infty}\left|(n+1)(x-1)\right| \begin{cases} = 0 & \text{if } x=1 \\ = \infty & \text{if } x\neq 1 \end{cases}$$

The power series $\sum_{n=0}^{\infty} n!(x-1)^n$ is only convergent at $x=1$,

(i.e. at the center.) (By Ratio Test!)

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{n!}$$

center: $x+2=0 \quad x=-2$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+2)^{n+1}}{(n+1)!}}{\frac{(x+2)^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x+2)^{n+1}}^{(x+2)}}{\cancel{(n+1)!}_{(n+1)}}\cdot\frac{\cancel{n!}}{\cancel{(x+2)^n}}\right| = \lim_{n\to\infty}\left|\frac{x+2}{\underbrace{n+1}_{\to\infty}}\right| = 0 < 1$$

By Ratio Test, $\sum_{n=0}^{\infty}\frac{(x+2)^n}{n!}$ is convergent for all values of x.

Interval of convergence: $(-\infty, \infty)$

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)3^n}$$

Centered at 0

Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+2)\,3^{n+1}}}{\frac{x^n}{(n+1)\,3^n}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}^{\,x}}{(n+2)\,\cancel{3^{n+1}}_{\,3}}\cdot\frac{(n+1)\,\cancel{3^n}}{\cancel{x^n}}\right|$$

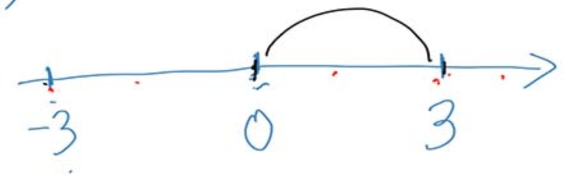
$$= \lim_{n\to\infty}\left|\frac{x}{3}\cdot\frac{n+1}{n+2}\right| = \frac{|x|}{3} < 1$$

$$\frac{n+1}{n+2} \to 1 \qquad \approx \left|\frac{x}{3}\right|$$

$$\frac{|x|}{3} < 1 \to |x| < 3$$

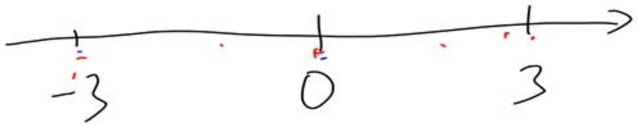
$$|x| < 3$$

Radius of Convergence



When $\frac{|x|}{3} < 1$, the power series will be convergent.

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)\,3^n}$$



$$\frac{|x|}{3} < 1 \Rightarrow |x| < 3$$

What happens if $x=3$ or -3 ?

$x=3$: $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{(n+1)\,3^n} = \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$ p-series $p=1$. divergent!

$x=-3$. $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(n+1)\,3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{(n+1)\,3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$: use AST (alternating series Test)

$\frac{1}{n+1}$ $\to 0$, decreasing. Convergent by AST

Radius of Convergence: $R = 3$

Interval of Convergence: $[-3, \; 3)$

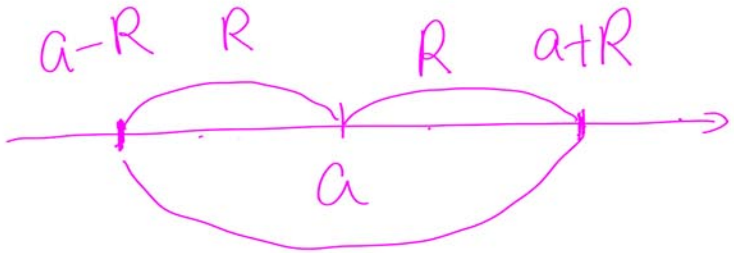
Summary on Convergence of a Power Series

- Given a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

- There are three possibilities concerning convergence:

  1) The series is only convergent at the center;
  2) The series is convergent for all real number x;
  3) There is a positive value R such that the series is convergent when $|x-a| < R$ and divergent when $|x-a| > R$; R is called Radius of Convergence.



$$\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$$

Centered at 0

$2x$

Ratio Test

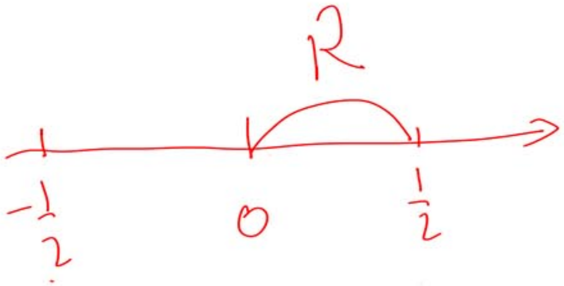
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2x)^{n+1}}{(n+1)^2}}{\frac{(2x)^n}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(2x)^{n+1}}}{(n+1)^2}\cdot\frac{n^2}{\cancel{(2x)^n}}\right| = \lim_{n\to\infty}\left|2x\cdot\frac{n^2}{(n+1)^2}\right|$$

1

$$= |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

$|x - \text{center}| <$ a number

Radius of convergence

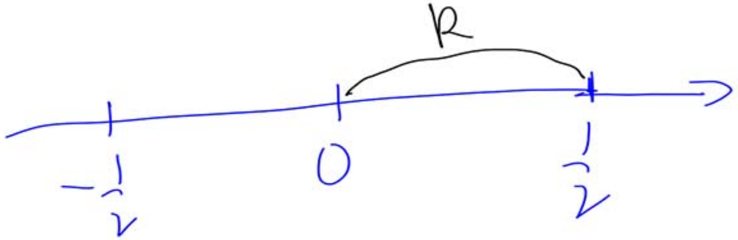


$$\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{x\to\infty}\frac{x^2}{(x+1)^2}$$

$$= \lim_{x\to\infty}\frac{2x}{2(x+1)} = \lim_{x\to\infty}\frac{2}{2}$$

$$= 1$$

$$\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$$



$x=\frac{1}{2}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(2\cdot\frac{1}{2})^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series $p=2$ so convergent!

$x=-\frac{1}{2}$. $\displaystyle\sum_{n=1}^{\infty} \frac{(2(-\frac{1}{2}))^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ alternating series convergent by AST

Interval of Convergence $[-\frac{1}{2}, \frac{1}{2}]$

Radius of Convergence $\frac{1}{2}$