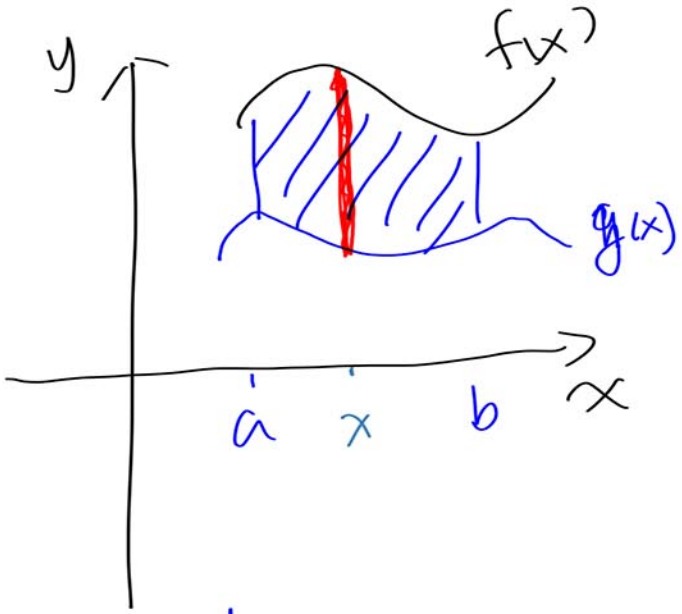
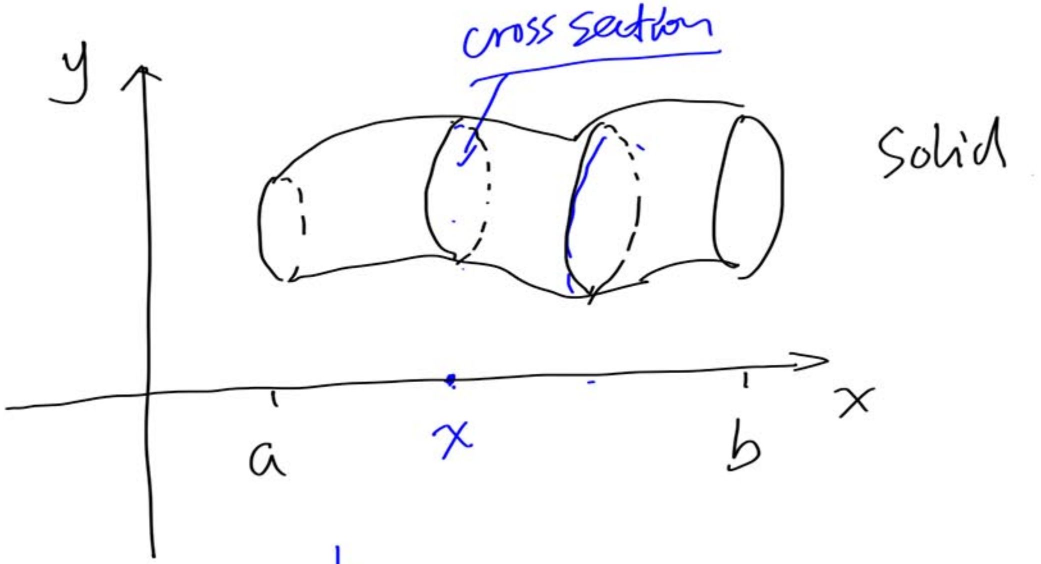
LECTURE NO. 4

2.2 Determining Volumes by Slicing

Wright State University

From Area to Volume

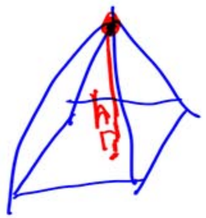




$$\text{Area} = \int_a^b \underbrace{f(x) - g(x)}_{\substack{\shortparallel \\ \text{length of cutting segment}}}\, dx$$

$$\underline{\text{Volume}} = \int_a^b \underline{\text{cross sectional area}}\, dx$$

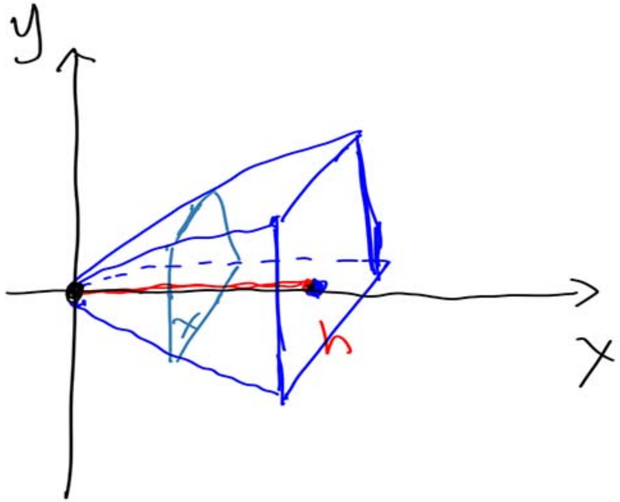
Volume of a Pyramid with Square Base



base : $b \times b$

height : h

$V = \frac{1}{3} b^2 h$



$$\text{Volume} = \int_0^h \text{cross sectional area } dx$$

C.S.A

Similarity : Small pyramid is similar to big pyramid

height of small pyramid $= x$

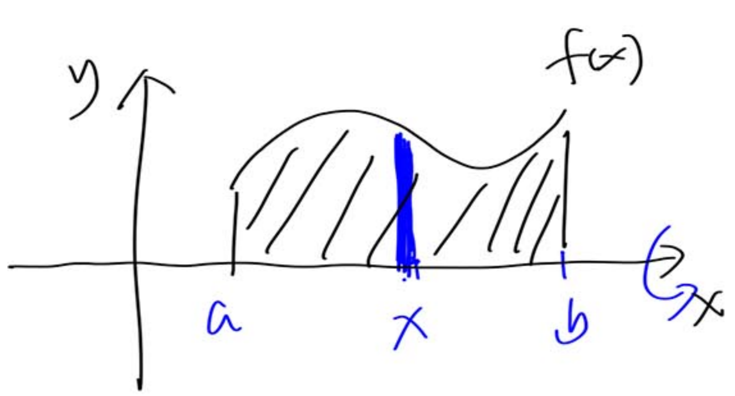
ratio between heights $= \frac{x}{h}$ = ratio of side lengths

$$\frac{x}{h} = \frac{\ell}{b} \qquad \ell = \frac{b}{h} x \qquad \text{C.S.A} = \frac{b^2}{h^2} x^2$$

$$V = \int_0^h \frac{b^2}{h^2} x^2 \, dx = \frac{b^2}{h^2} \frac{x^3}{3} \Big|_0^h = \frac{b^2}{h^2} \frac{h^3}{3}$$

$$= \frac{1}{3} b^2 h$$

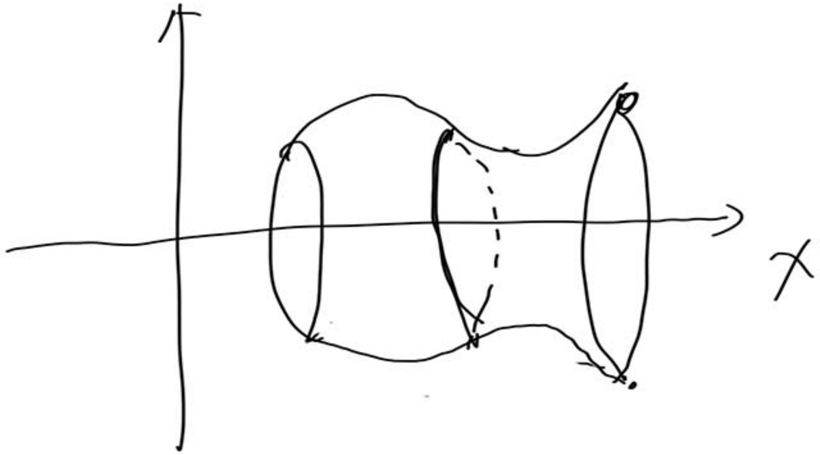
Solids of Revolution



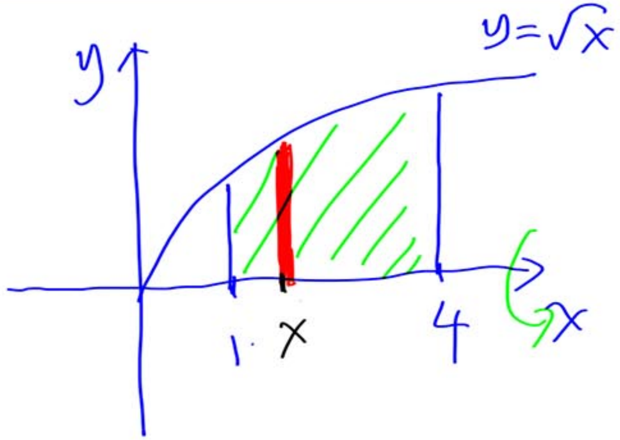
→ washer

$$\text{Volume} = \int_a^b \text{cross sectional area } dx$$

Disk



Find the volume of the solid generated by rotating the region between $f(x) = \sqrt{x}$ and the x-axis over the interval $[1, 4]$ around x-axis.



$$V = \int_1^4 \text{cross sectional area } dx$$

C.S.A.

cross section is a disk!

length of the cutting segment = Radius of the disk

$$\sqrt{x} = R$$

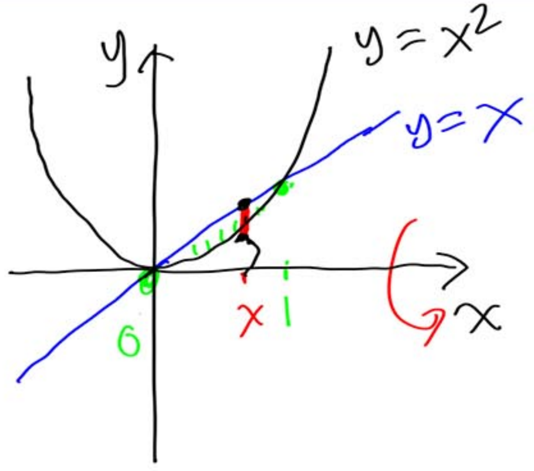
$$C.S.A = \pi R^2 = \pi (\sqrt{x})^2 = \pi x$$

Disk Method

$$V = \int_1^4 \pi x \, dx = \frac{\pi}{2} x^2 \Big|_1^4 = \frac{\pi}{2}(16-1) = \frac{15}{2}\pi$$

FINAL ANSWER.

Find the volume of the solid formed by revolving the region enclosed by $y = x$ and $y = x^2$ around x-axis.



$$\text{Volume} = \int_0^1 \underbrace{\text{cross sectional area}}_{\text{C.S.A.}}\, dx$$

washer

$$\text{C.S.A.} = \pi (OR)^2 - \pi (IR)^2$$

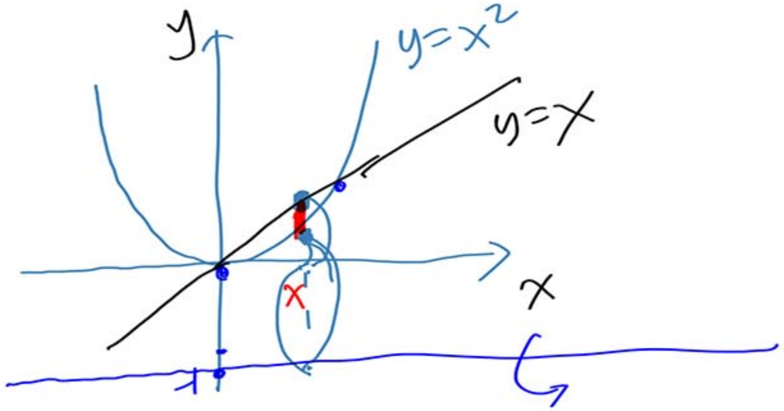
$IR = x^2 \qquad OR = x \qquad C.S.A = \pi x^2 - \pi (x^2)^2$

$$V = \int_0^1 \pi x^2 - \pi x^4 \, dx = \pi \int_0^1 x^2 - x^4 \, dx$$

$$V = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{2}{15}\pi$$

Washer Method

What if the same region is rotated around the line $y = -1$?



$$\text{Volume} = \int_0^1 \text{Cross sectional Area } dx$$

Washer Method $\quad \pi(OR)^2 - \pi(IR)^2$

$IR = x^2+1 \qquad OR = x+1 \qquad C.S.A = \pi(x+1)^2 - \pi(x^2+1)^2$

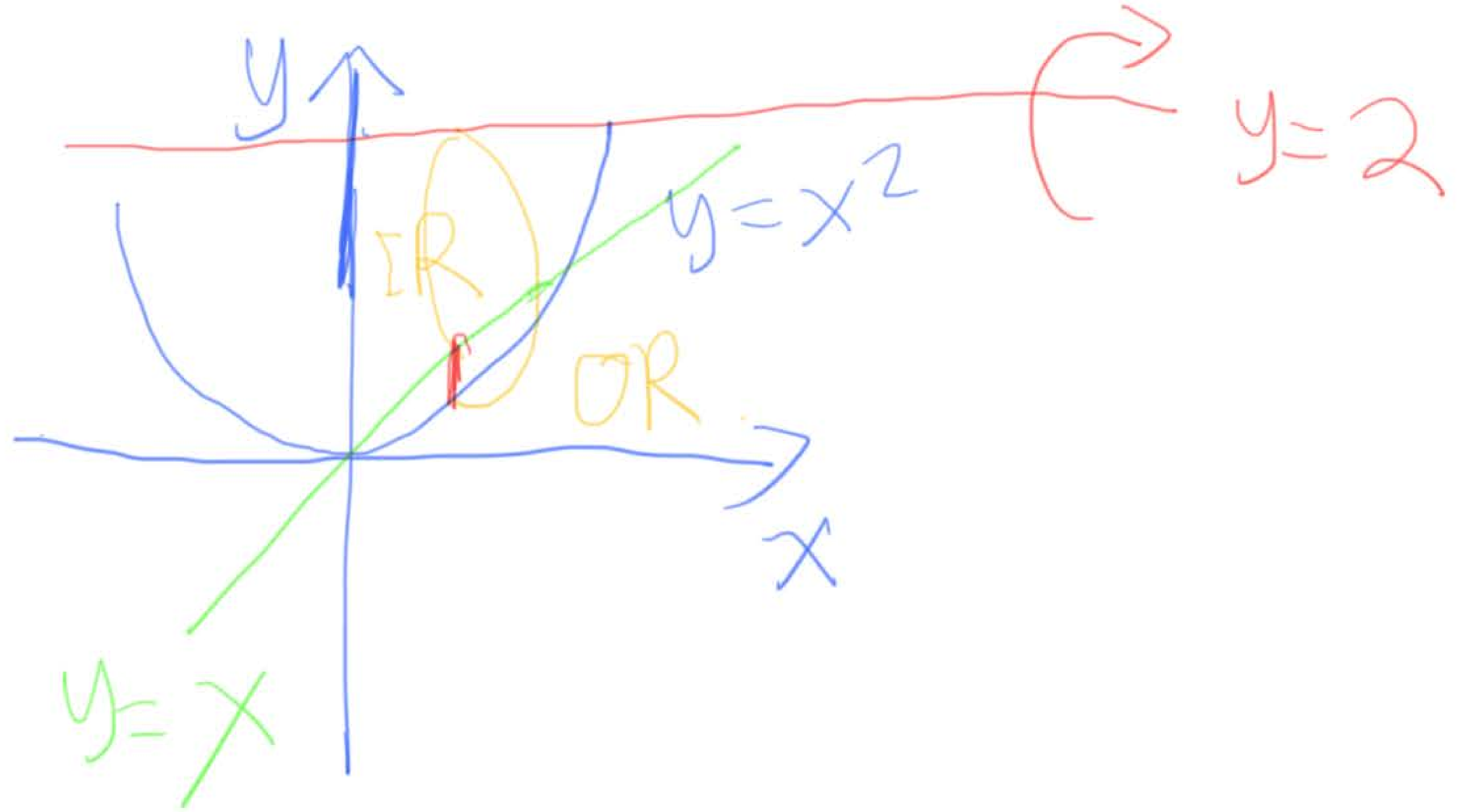
$$V = \int_0^1 \pi(x+1)^2 - \pi(x^2+1)^2 \, dx$$

$$V = \pi \int_0^1 x^2+2x+1 - (x^4+2x^2+1) \, dx$$

$$V = \pi \int_0^1 -x^2+2x-x^4 \, dx = \pi\left(-\frac{x^3}{3} + x^2 - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \frac{7}{15}\pi$$

What if the region is rotated around the line $y = 2$?



Inner Radius $= 2 - x$

Outer Radius $= 2 - x^2$

Area $= \pi(2-x^2)^2 - \pi(2-x)^2$

$$V = \int_0^1 \pi(2-x^2)^2 - \pi(2-x)^2\,dx$$

$$= \pi\int_0^1 (4-4x^2+x^4) - (4-4x+x^2)\,dx$$

$$= \pi\int_0^1 -5x^2+x^4+4x\,dx$$

$$= \pi\left(-\frac{5}{3}x^3+\frac{x^5}{5}+2x^2\right)\Big|_0^1$$

$$= \pi\left(-\frac{5}{3}+\frac{1}{5}+2\right) = \frac{8}{15}\pi$$

Final answer.