Calculus II

Lab 10 Solution

#1 $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ — ①

Since $\frac{1}{1+3x} = \frac{1}{1-(-3x)}$, replace x in ① by $-3x$,

we get

$$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3x)^n = \sum_{n=0}^{\infty} (-3)^n x^n$$

b) $$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3)^n x^n = 1 - 3x + 9x^2 - 27x^3 + \cdots$$

$$\underbrace{\int \frac{1}{1+3x}\,dx}_{\text{substitution}} = \int \sum_{n=0}^{\infty} (-3)^n x^n \,dx = \int 1 - 3x + 9x^2 - 27x^3 + \cdots \,dx$$

$$\frac{1}{3}\ln(1+3x) = C + \sum_{n=0}^{\infty} (-3)^n \cdot \frac{x^{n+1}}{n+1}$$

~~So~~ set $x=0$ we see that $C=0$

Therefore $\ln(1+3x) = 3 \cdot \sum_{n=0}^{\infty} (-3)^n \frac{x^{n+1}}{n+1}$

$$\left(\text{or } \sum_{n=0}^{\infty} (-1)^n \cdot 3^{n+1} \cdot \frac{x^{n+1}}{n+1} \right)$$

c) $$\frac{1}{1+3x} = \sum_{n=0}^{\infty} (-3)^n x^n = 1 - 3x + 9x^2 - 27x^3 + \cdots$$

Take derivatives on both sides, we get

$$-\frac{1}{(1+3x)^2} \cdot 3 = \sum_{n=0}^{\infty} (-3)^n n x^{n-1}$$

So $\frac{1}{(1+3x)^2} = -\frac{1}{3} \sum_{n=0}^{\infty} (-3)^n n x^{n-1}$ or $\sum_{n=0}^{\infty} (-3)^{n-1} n x^{n-1}$

#2 $f(x) = \sqrt[3]{7+x}$

$f(1) = 2$ $\quad f'(x) = \frac{1}{3}(7+x)^{-\frac{2}{3}}$ $\quad f'(1) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}(7+x)^{-\frac{5}{3}}$ $\quad f''(1) = -\frac{2}{9}\cdot\frac{1}{32} = -\frac{1}{144}$

$f^{(3)}(x) = \frac{10}{27}(7+x)^{-\frac{8}{3}}$ $\quad f^{(3)}(1) = \frac{10}{27}\cdot\frac{1}{256} = \frac{5}{3456}$

3rd-degree Taylor Polynomial centered at 1

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= 2 + \frac{1}{12}(x-1) + \frac{-1/144}{2!}(x-1)^2 + \frac{5/3456}{3!}(x-1)^3$$

#3 Maclaurin Series $\quad f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$

$f(x) = \sin 2x$ $\quad f(0) = 0$

$f'(x) = 2\cos 2x$ $\quad f'(0) = 2$

$f''(x) = -4\sin 2x$ $\quad f''(0) = 0$

$f'''(x) = -8\cos(2x)$ $\quad f^{(3)}(0) = -8$

From the Pattern, we get Maclaurin Series is

$$0 + 2x + 0 - \frac{8x^3}{3!} + 0 + \frac{32x^5}{5!} + 0 - \frac{128x^7}{7!} + \cdots$$

$$\cancel{\sum_{n=0}^{\infty} (-1)^{n+1} (2x)^{n}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$$

#4 $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(2n+1)}$

Ratio Test

$x-1$

$$\lim_{n\to\infty}\left|\frac{\dfrac{(x-1)^{n+1}}{3^{n+1}(2(n+1)+1)}}{\dfrac{(x-1)^n}{3^n(2n+1)}}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{\cancel{3^{n+1}}(2n+3)}\cdot\frac{\cancel{3^n}(2n+1)}{\cancel{(x-1)^n}}\right|$$

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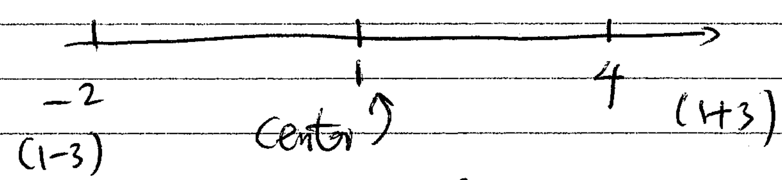
$\left(\text{as } n\to\infty \quad \lim_{n\to\infty}\frac{2n+1}{2n+3} = 1\right)$

$$= \left|\frac{x-1}{3}\right| < 1 \Rightarrow |x-1| < 3$$

So Radius of convergence is 3.



$x=-2$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n}{3^n(2n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ convergent by AST

$x=4$: $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{3^n(2n+1)} = \sum_{n=0}^{\infty}\frac{1}{2n+1}$ use Limit Comparison Test

compare with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}$.

divergent

Therefore interval of convergence

$= [-2, 4)$