LECTURE NO. 22

6.3 Taylor and Maclaurin Series

Wright State University

Two ways to Find Power Series Representation of a Function

1. Start from the series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

   and use algebra, differentiation, integration to find power series representations of other related functions.

2. Use Taylor Series (or Maclaurin Series) Formula to find the power series representation of a given function.

Taylor Series Formula

- Suppose that $f(x)$ is represented by a power series centered at a, that is

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

- Plug in a into x we get $c_0 = f(a)$.
- Take derivatives on both sides, we get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

- Set $x = a$ we get $c_1 = f'(a)$.
- Take derivatives again, we get

c_3 c_4

$$f''(x) = 2c_2 + 3 \cdot 2 \cdot (x-a) + 4 \cdot 3 \cdot (x-a)^2 + \cdots$$

- Set $x = a$, we get $c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}$

$2c_2 = f''(a)$

Taylor Series Formula -Continued.

- Recall from previous slide

$$f''(x) = 2c_2 + 3 \cdot 2 \cdot c_3(x-a) + 4 \cdot 3 \cdot c_4(x-a)^2 + \cdots$$

- Take derivatives again, we get

$$f^{(3)}(x) = 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot \overset{c_4}{\vee} (x-a) + \cdots$$

- Set $x = a$ we get $c_3 = \frac{f^{(3)}(a)}{6} = \frac{f^{(3)}(a)}{3!}$.
- Similarly, we can get $c_4 = \frac{f^{(4)}(a)}{4!}$
- So if a function $f(x)$ is represented by a power series centered at a, that is

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

- Then the coefficient c_n must equal $\frac{f^{(n)}(a)}{n!}$.

The formula for Taylor Series and Maclaurin Series

- Any given function can be represented by its Taylor Series centered at a as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

- If the center a happens to be 0, then the Taylor Series is also called Maclaurin Series.
- The formula for Maulaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Find the Maclaurin series for $f(x) = e^x$.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{4}(0)}{4!}x^4 + \cdots$$

$f(0) = e^0 = 1$

$f'(x) = e^x \quad f'(0) = 1$

$f''(x) = e^x \quad f''(0) = 1$

For any n, $f^{(n)}(0) = 1$

$$f(x) = e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (0! = 1,\ 1! = 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Find the Maclaurin Series for $\sin x$ and $\cos x$.

$$f(x) = \sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$f(0) = 0$,

$f'(x) = \cos x \quad f'(0) = 1$

$f''(x) = -\sin x \quad f''(0) = 0$

$f^{(3)}(x) = -\cos x \quad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x$

$$\sin x = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$(\sin x)' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)'$$

$$\cos x = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

Find the Taylor Series for $f(x) = \frac{1}{x}$ centered at 1

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \cdots$$

$$f(x) = \frac{1}{x} \qquad f(1) = 1$$

$$f'(x) = -x^{-2} \qquad f'(1) = -1$$

$$f''(x) = (-1)(-2)\,x^{-3} \qquad f''(1) = (-1)(-2) = (-1)^2 \cdot 2!$$

$$f^{(3)}(x) = (-1)(-2)(-3)\,x^{-4} \qquad f^{(3)}(1) = (-1)(-2)(-3) = (-1)^3\, 3!$$

$$f^{(4)}(x) = (-1)(-2)(-3)(-4)\,x^{-5} \qquad f^{(4)}(1) = (-1)(-2)(-3)(-4) = (-1)^4 \cdot 4!$$

$$f^{n}(1) = (-1)^n\, n!$$

$$f(x) = 1 - (x-1) + (-1)^2(x-1)^2 + (-1)^3(x-1)^3 + (-1)^4(x-1)^4 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

Find the third-degree Taylor Polynomial for $f(x) = \sqrt[3]{x}$ centered at 8.

$$f(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 + \frac{f^{(3)}(8)}{3!}(x-8)^3 + \boxed{\cdots}$$

Third-degree Taylor polynomial: $f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 + \frac{f^{(3)}(8)}{3!}(x-8)^3$

now we need to find $f(8)$, $f'(8)$, $f''(8)$, $f^{(3)}(8)$

$f(x) = x^{\frac{1}{3}}$ $\quad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ $\quad f'(8) = \frac{1}{3}\left(8^{-\frac{2}{3}}\right) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$ $\quad f''(8) = -\frac{2}{9}(8)^{-\frac{5}{3}} = -\frac{2}{9}\cdot\frac{1}{32} = -\frac{1}{144}$

$f^{(3)}(x) = \frac{10}{27}x^{-\frac{8}{3}}$ $\quad f^{(3)}(8) = \frac{10}{27}(8)^{-\frac{8}{3}} = \frac{10}{27}\cdot\frac{1}{256} = \frac{5}{27\cdot 128} = \frac{5}{3456}$

Third-degree Taylor Polynomial

$$2 + \frac{1}{12}(x-8) + \frac{-1/144}{2!}(x-8)^2 + \frac{5/3456}{3!}(x-8)^3$$

$\approx f(x) = \sqrt[3]{x}$

FINAL ANSWER.

List of known Maclaurin Series

- $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$
- $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$
- $$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
- $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

- $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

- $$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$