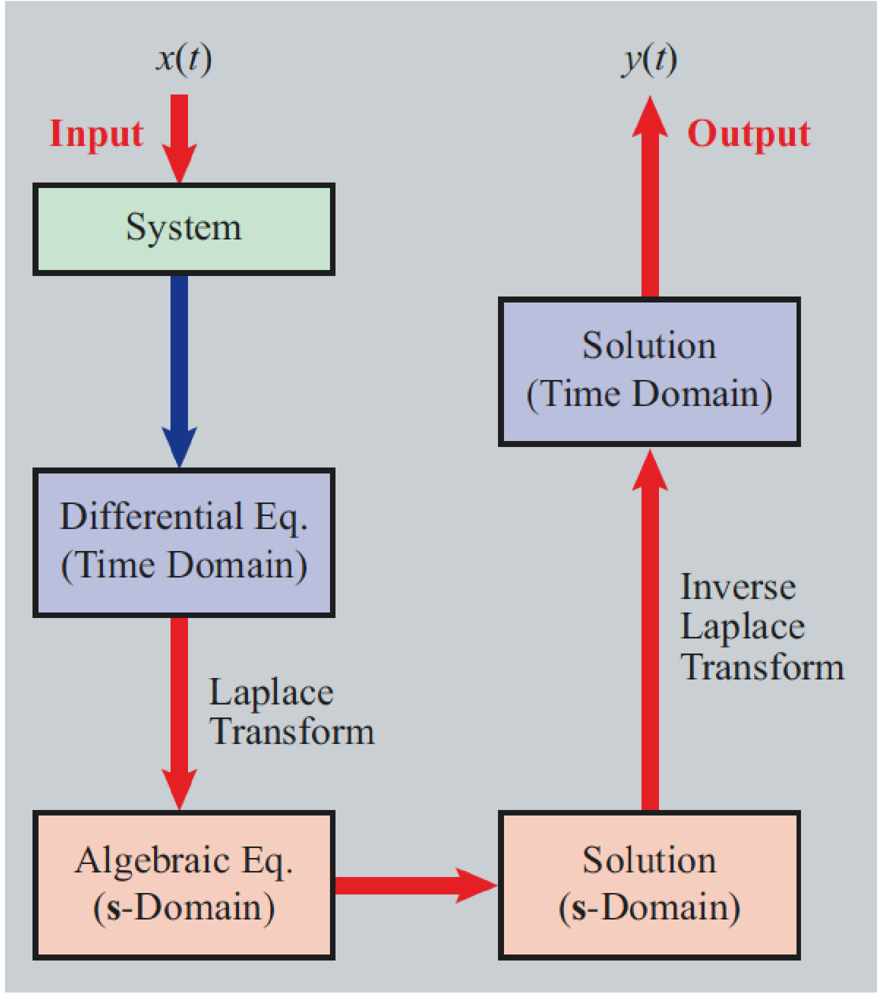


3. LAPLACE TRANSFORM

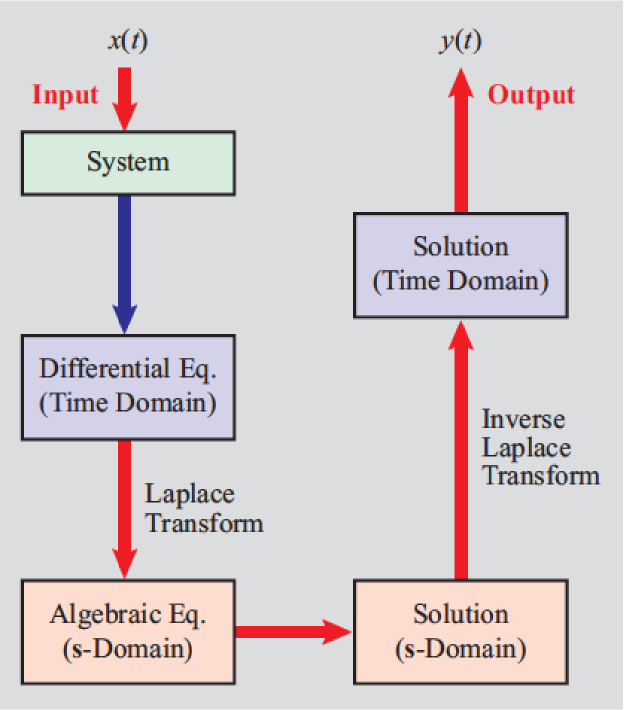
3 Laplace Transform

Contents

Objectives

Learn to:

- Compute the Laplace transform of a signal.
- Apply partial fraction expansion to compute the inverse Laplace transform.
- Perform convolution of two functions using the Laplace transform.
- Relate system stability to the poles of the transfer function.
- Interrelate the six different descriptions of LTI systems.



The beauty of the Laplace-transform technique is that it transforms a complicated differential equation into a straightforward algebraic equation. This chapter covers the A-to-Z of how to transform a differential equation from the time domain to the complex frequency s domain, solve it, and then inverse transform the solution to the time domain.

3-1 Definition of the (Unilateral) Laplace Transform

The symbol $\mathcal{L}[x(t)]$ is a shorthand notation for "the Laplace transform of function $x(t)$." Usually denoted $\mathbf{X}(\mathbf{s})$, the ***unilateral Laplace transform*** is defined as

$$\mathbf{X}(\mathbf{s}) = \mathcal{L}[x(t)] = \int_{0^-}^{\infty} x(t)\, e^{-\mathbf{s}t}\, dt, \tag{3.1}$$

where $\mathbf{s}$ is a complex variable—with a real part σ and an imaginary part ω—given by

$$\mathbf{s} = \sigma + j\omega. \tag{3.2}$$

When we apply the Laplace transform technique to physically realizable systems, we select the start time for the system operation as $t = 0^-$, making the unilateral transform perfectly suitable for handling causal signals with non-zero initial conditions.

3-1.1 Uniqueness Property

The uniqueness property of the Laplace transform states:

▶ A given $x(t)$ has a unique Laplace transform $\mathbf{X}(\mathbf{s})$, and vice versa. ◀

In symbolic form, the uniqueness property can be expressed by

$$x(t) \quad \longleftrightarrow \quad \mathbf{X}(\mathbf{s}) \tag{3.3a}$$

The **two-way arrow** is a shorthand notation for the combination of the two statements

$$\mathcal{L}[x(t)] = \mathbf{X}(\mathbf{s}) \qquad \text{and} \qquad \mathcal{L}^{-1}[\mathbf{X}(\mathbf{s})] = x(t). \tag{3.3b}$$

The first statement asserts that $\mathbf{X}(\mathbf{s})$ is the Laplace transform of $x(t)$, and the second statement asserts that $\mathcal{L}^{-1}[\mathbf{X}(\mathbf{s})]$, which is the ***inverse Laplace transform*** of $\mathbf{X}(\mathbf{s})$, is $x(t)$.

3-1.3 Inverse Laplace Transform

Equation (3.1) allows us to obtain Laplace transform $\mathbf{X}(\mathbf{s})$ corresponding to time function $x(t)$. The inverse process, denoted $\mathcal{L}^{-1}[\mathbf{X}(\mathbf{s})]$, allows us to perform an integration on $\mathbf{X}(\mathbf{s})$ to obtain $x(t)$:

$$x(t) = \mathcal{L}^{-1}[\mathbf{X}(\mathbf{s})] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathbf{X}(\mathbf{s})\, e^{\mathbf{s}t}\, d\mathbf{s}, \tag{3.5}$$

where $\sigma > \sigma_c$. The integration, which has to be performed in the two-dimensional complex plane, is rather cumbersome and to be avoided if an alternative approach is available for converting $\mathbf{X}(\mathbf{s})$ into $x(t)$.

Instead of applying Eq. (3.5), we can generate a table of Laplace transform pairs for all of the time functions commonly encountered in real systems and then use it as a look-up table to transform the **s**-domain solution to the time domain.

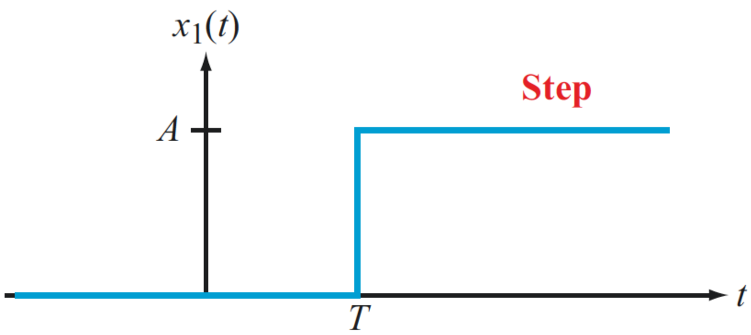
Example 3-1: Laplace Transforms of Singularity Functions

Determine the Laplace transforms of the signal waveforms displayed in Fig. 3-1.

Solution:

(a) The step function in Fig. 3-1(a) is given by

$$x_1(t) = A\,u(t-T).$$



$$\mathbf{X}_1(\mathbf{s}) = \int_{0^-}^{\infty} x_1(t)\, e^{-\mathbf{s}t}\, dt$$

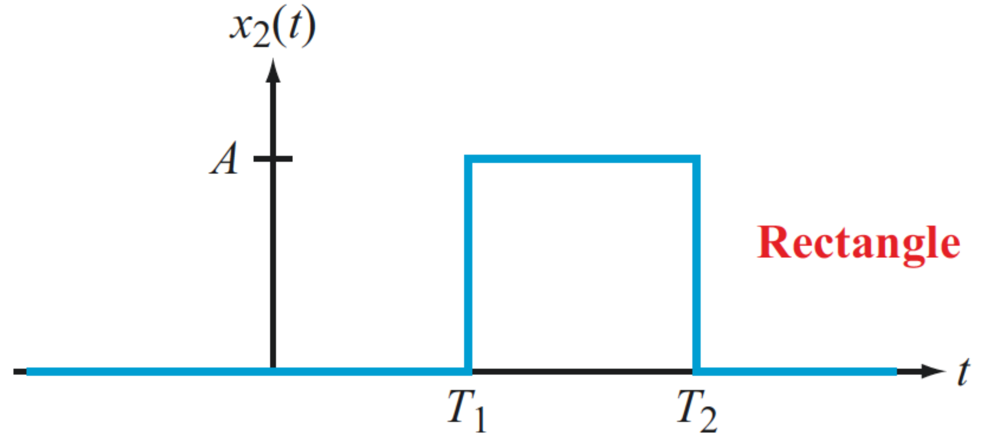
$$= \int_{0^-}^{\infty} A\,u(t-T)\, e^{-\mathbf{s}t}\, dt$$

$$= A \int_{T}^{\infty} e^{-\mathbf{s}t}\, dt$$

$$= -\frac{A}{\mathbf{s}}\, e^{-\mathbf{s}t}\bigg|_{T}^{\infty} = \frac{A}{\mathbf{s}}\, e^{-\mathbf{s}T}.$$

For the special case where $A = 1$ and $T = 0$ (i.e., the step occurs at $t = 0$), the transform pair becomes

$$u(t) \leftrightarrow \frac{1}{\mathbf{s}}. \qquad (3.6)$$



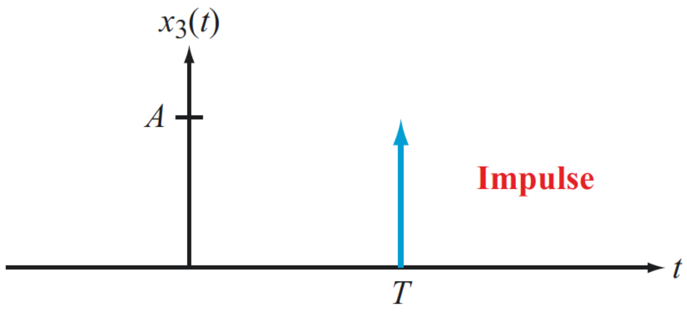
(b)

(b) The rectangle function in Fig. 3-1(b) can be constructed as the sum of two step functions:

$$x_2(t) = A[u(t - T_1) - u(t - T_2)],$$

and its Laplace transform is

$$\begin{aligned}
\mathbf{X}_2(\mathbf{s}) &= \int_{0^-}^{\infty} A[u(t - T_1) - u(t - T_2)]e^{-\mathbf{s}t}\, dt \\
&= A \int_{0^-}^{\infty} u(t - T_1)\, e^{-\mathbf{s}t}\, dt - A \int_{0^-}^{\infty} u(t - T_2)\, e^{-\mathbf{s}t}\, dt \\
&= \frac{A}{\mathbf{s}}[e^{-\mathbf{s}T_1} - e^{-\mathbf{s}T_2}].
\end{aligned}$$



(c)

(c) The impulse function in Fig. 3-1(c) is given by

$$x_3(t) = A\,\delta(t-T),$$

and the corresponding Laplace transform is

$$\mathbf{X}_3(\mathbf{s}) = \int_{0^-}^{\infty} A\,\delta(t-T)\,e^{-\mathbf{s}t}\,dt = Ae^{-\mathbf{s}T},$$

For the special case where $A = 1$ and $T = 0$, the Laplace transform pair simplifies to

$$\boxed{\delta(t) \;\leftrightarrow\; 1.} \qquad (3.7)$$

Example 3-2: Laplace Transform Pairs

Obtain the Laplace transforms of (a) $x_1(t) = e^{-at}\, u(t)$ and (b) $x_2(t) = [\cos(\omega_0 t)]\, u(t)$.

Solution:

(a) Application of Eq. (3.1) gives

$$\mathbf{X}_1(\mathbf{s}) = \int_{0^-}^{\infty} e^{-at}\, u(t)\, e^{-\mathbf{s}t}\, dt$$

$$= \left. \frac{e^{-(\mathbf{s}+a)t}}{-(\mathbf{s}+a)} \right|_0^{\infty} = \frac{1}{\mathbf{s}+a}\,.$$

Hence,

$$e^{-at}\, u(t) \leftrightarrow \frac{1}{\mathbf{s}+a}\,. \qquad (3.8)$$

(b) We start by expressing $\cos(\omega_0 t)$ in the form

$$\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}].$$

Next, we take advantage of Eq. (3.8):

$$\begin{aligned}
\mathbf{X}_2(\mathbf{s}) &= \mathcal{L}[\cos(\omega_0 t)\, u(t)] \\
&= \frac{1}{2}\,\mathcal{L}[e^{j\omega_0 t}\, u(t)] + \frac{1}{2}\,\mathcal{L}[e^{-j\omega_0 t}\, u(t)] \\
&= \frac{1}{2}\,\frac{1}{\mathbf{s}-j\omega_0} + \frac{1}{2}\,\frac{1}{\mathbf{s}+j\omega_0} \\
&= \frac{\mathbf{s}}{\mathbf{s}^2+\omega_0^2}\,.
\end{aligned}$$

Hence,

$$[\cos(\omega_0 t)]\, u(t) \leftrightarrow \frac{\mathbf{s}}{\mathbf{s}^2+\omega_0^2}\,.$$

Table 3-2: Examples of Laplace transform pairs. Note that $x(t) = 0$ for $t < 0^-$ and $T \geq 0$.

Laplace Transform Pairs			
	$x(t)$		$\mathbf{X}(\mathbf{s}) = \mathcal{L}[x(t)]$
1	$\delta(t)$	$\longleftrightarrow$	1
1a	$\delta(t-T)$	$\longleftrightarrow$	$e^{-T\mathbf{s}}$
2	$u(t)$	$\longleftrightarrow$	$\dfrac{1}{\mathbf{s}}$
2a	$u(t-T)$	$\longleftrightarrow$	$\dfrac{e^{-T\mathbf{s}}}{\mathbf{s}}$
3	$e^{-at}\, u(t)$	$\longleftrightarrow$	$\dfrac{1}{\mathbf{s}+a}$
3a	$e^{-a(t-T)}\, u(t-T)$	$\longleftrightarrow$	$\dfrac{e^{-T\mathbf{s}}}{\mathbf{s}+a}$
4	$t\, u(t)$	$\longleftrightarrow$	$\dfrac{1}{\mathbf{s}^2}$
4a	$(t-T)\, u(t-T)$	$\longleftrightarrow$	$\dfrac{e^{-T\mathbf{s}}}{\mathbf{s}^2}$
5	$t^2\, u(t)$	$\longleftrightarrow$	$\dfrac{2}{\mathbf{s}^3}$
6	$te^{-at}\, u(t)$	$\longleftrightarrow$	$\dfrac{1}{(\mathbf{s}+a)^2}$
7	$t^2e^{-at}\, u(t)$	$\longleftrightarrow$	$\dfrac{2}{(\mathbf{s}+a)^3}$
8	$t^{n-1}e^{-at}\, u(t)$	$\longleftrightarrow$	$\dfrac{(n-1)!}{(\mathbf{s}+a)^n}$
9	$\sin(\omega_0 t)\, u(t)$	$\longleftrightarrow$	$\dfrac{\omega_0}{\mathbf{s}^2+\omega_0^2}$
10	$\sin(\omega_0 t+\theta)\, u(t)$	$\longleftrightarrow$	$\dfrac{\mathbf{s}\sin\theta+\omega_0\cos\theta}{\mathbf{s}^2+\omega_0^2}$
11	$\cos(\omega_0 t)\, u(t)$	$\longleftrightarrow$	$\dfrac{\mathbf{s}}{\mathbf{s}^2+\omega_0^2}$
12	$\cos(\omega_0 t+\theta)\, u(t)$	$\longleftrightarrow$	$\dfrac{\mathbf{s}\cos\theta-\omega_0\sin\theta}{\mathbf{s}^2+\omega_0^2}$
13	$e^{-at}\sin(\omega_0 t)\, u(t)$	$\longleftrightarrow$	$\dfrac{\omega_0}{(\mathbf{s}+a)^2+\omega_0^2}$
14	$e^{-at}\cos(\omega_0 t)\, u(t)$	$\longleftrightarrow$	$\dfrac{\mathbf{s}+a}{(\mathbf{s}+a)^2+\omega_0^2}$
15	$2e^{-at}\cos(bt-\theta)\, u(t)$	$\longleftrightarrow$	$\dfrac{e^{j\theta}}{\mathbf{s}+a+jb}+\dfrac{e^{-j\theta}}{\mathbf{s}+a-jb}$
15a	$e^{-at}\cos(bt-\theta)\, u(t)$	$\longleftrightarrow$	$\dfrac{(\mathbf{s}+a)\cos\theta+b\sin\theta}{(\mathbf{s}+a)^2+b^2}$
16	$\dfrac{2t^{n-1}}{(n-1)!}e^{-at}\cos(bt-\theta)\, u(t)$	$\longleftrightarrow$	$\dfrac{e^{j\theta}}{(\mathbf{s}+a+jb)^n}+\dfrac{e^{-j\theta}}{(\mathbf{s}+a-jb)^n}$

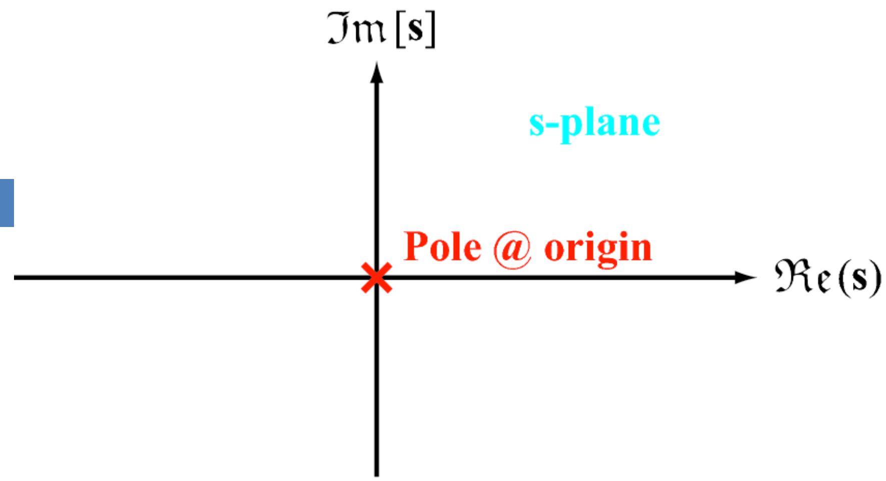
Poles & Zeros

In general, $\mathbf{X}(\mathbf{s})$ is a ***rational function*** of the form

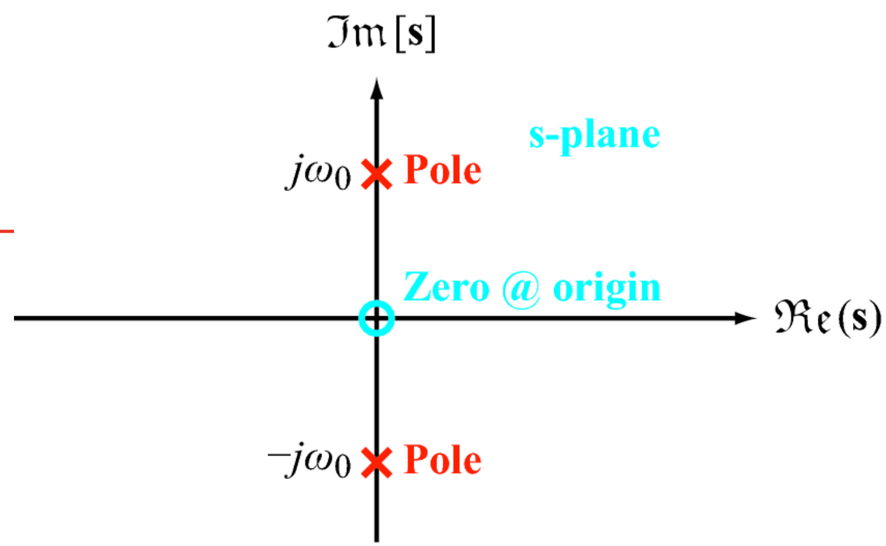
$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})},$$

▶ The ***zeros*** of $\mathbf{X}(\mathbf{s})$ are the values of $\mathbf{s}$ that render $\mathbf{N}(\mathbf{s}) = 0$, which also are called the ***roots*** of $\mathbf{N}(\mathbf{s})$. Similarly, the ***poles*** of $\mathbf{X}(\mathbf{s})$ are the roots of its denominator $\mathbf{D}(\mathbf{s})$. ◀

$$\mathbf{X}_4(\mathbf{s}) = \frac{\mathbf{s}}{\mathbf{s}^2 + \omega_0^2} = \frac{\mathbf{s}}{(\mathbf{s} - j\omega_0)(\mathbf{s} + j\omega_0)}$$



(a) $\mathbf{X}_1(\mathbf{s}) = 1/\mathbf{s}$ has a single pole



(b) $\mathbf{X}_4(\mathbf{s}) = \mathbf{s}/[(\mathbf{s} - j\omega_0)(\mathbf{s} + j\omega_0)]$

Table 3-1: Properties of the Laplace transform for causal functions; i.e., $x(t) = 0$ for $t < 0^-$.

Property	$x(t)$		$\mathbf{X}(\mathbf{s}) = \mathcal{L}[x(t)]$
1. Multiplication by constant	$K\ x(t)$	$\longleftrightarrow$	$K\ \mathbf{X}(\mathbf{s})$
2. Linearity	$K_1\ x_1(t) + K_2\ x_2(t)$	$\longleftrightarrow$	$K_1\ \mathbf{X}_1(\mathbf{s}) + K_2\ \mathbf{X}_2(\mathbf{s})$
3. Time scaling	$x(at), \quad a > 0$	$\longleftrightarrow$	$\frac{1}{a}\ \mathbf{X}\left(\frac{\mathbf{s}}{a}\right)$
4. Time shift	$x(t-T)\ u(t-T),\ T \geq 0$	$\longleftrightarrow$	$e^{-T\mathbf{s}}\ \mathbf{X}(\mathbf{s})$
5. Frequency shift	$e^{-\mathbf{a}t}\ x(t)$	$\longleftrightarrow$	$\mathbf{X}(\mathbf{s}+\mathbf{a})$
6. Time 1st derivative	$x' = \frac{dx}{dt}$	$\longleftrightarrow$	$\mathbf{s}\ \mathbf{X}(\mathbf{s}) - x(0^-)$
7. Time 2nd derivative	$x'' = \frac{d^2x}{dt^2}$	$\longleftrightarrow$	$\mathbf{s}^2\mathbf{X}(\mathbf{s}) - \mathbf{s}x(0^-) - x'(0^-)$
8. Time integral	$\int_{0^-}^{t} x(t')\ dt'$	$\longleftrightarrow$	$\frac{1}{\mathbf{s}}\ \mathbf{X}(\mathbf{s})$
9. Frequency derivative	$t\ x(t)$	$\longleftrightarrow$	$-\frac{d}{d\mathbf{s}}\ \mathbf{X}(\mathbf{s}) = -\mathbf{X}'(\mathbf{s})$
10. Frequency integral	$\frac{x(t)}{t}$	$\longleftrightarrow$	$\int_{\mathbf{s}}^{\infty} \mathbf{X}(\mathbf{s}')\ d\mathbf{s}'$
11. Initial value	$x(0^+)$	$=$	$\lim_{\mathbf{s}\to\infty} \mathbf{s}\ \mathbf{X}(\mathbf{s})$
12. Final value	$\lim_{t\to\infty} x(t) = x(\infty)$	$=$	$\lim_{\mathbf{s}\to 0} \mathbf{s}\ \mathbf{X}(\mathbf{s})$
13. Convolution	$x_1(t) * x_2(t)$	$\longleftrightarrow$	$\mathbf{X}_1(\mathbf{s})\ \mathbf{X}_2(\mathbf{s})$

Example 3-5: Applying the Frequency Differentiation Property

Given that

$$\mathbf{X}(\mathbf{s}) = \mathcal{L}[e^{-at}\, u(t)] = \frac{1}{\mathbf{s}+a}\ ,$$

apply Eq. (3.35) to obtain the Laplace transform of $te^{-at}\, u(t)$.

Solution:

$$\mathcal{L}[te^{-at}\, u(t)] = -\frac{d}{d\mathbf{s}}\,\mathbf{X}(\mathbf{s}) = -\frac{d}{d\mathbf{s}}\left[\frac{1}{\mathbf{s}+a}\right]$$
$$= \frac{1}{(\mathbf{s}+a)^2}\ .$$

Example 3-6: Laplace Transform

Obtain the Laplace transform of

$$x(t) = t^2 e^{-3t} \cos(4t)\, u(t).$$

Solution:
The given function is a product of three functions. We start with the cosine function which we will call $x_1(t)$:

$$x_1(t) = \cos(4t)\, u(t). \tag{3.39}$$

According to entry #11 in Table 3-2, the corresponding Laplace transform is

$$\mathbf{X}_1(\mathbf{s}) = \frac{\mathbf{s}}{\mathbf{s}^2 + 16}\,. \tag{3.40}$$

Next we define

$$x_2(t) = e^{-3t} \cos(4t)\, u(t) = e^{-3t}\, x_1(t), \tag{3.41}$$

and we apply the frequency shift property (entry #5 in Table 3-1) to obtain

$$\mathbf{X}_2(\mathbf{s}) = \mathbf{X}_1(\mathbf{s}+3) = \frac{\mathbf{s}+3}{(\mathbf{s}+3)^2 + 16}\,, \tag{3.42}$$

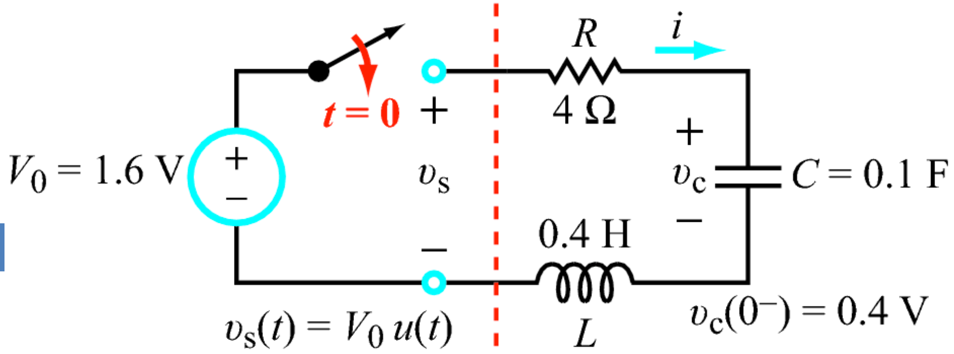
where we replaced $\mathbf{s}$ with $(\mathbf{s}+3)$ everywhere in the expression of Eq. (3.40). Finally, we define

$$x(t) = t^2\, x_2(t) = t^2 e^{-3t} \cos(4t)\, u(t), \tag{3.43}$$

and we apply the frequency derivative property (entry #9 in Table 3-1), twice:

$$\begin{aligned} \mathbf{X}(\mathbf{s}) = \mathbf{X}_2''(\mathbf{s}) &= \frac{d^2}{d\mathbf{s}^2}\left[\frac{\mathbf{s}+3}{(\mathbf{s}+3)^2+16}\right] \\ &= \frac{2(\mathbf{s}+3)[(\mathbf{s}+3)^2-48]}{[(\mathbf{s}+3)^2+16]^3}\,. \end{aligned} \tag{3.44}$$

Circuit Example



$$\upsilon_s(t) = V_o\, u(t)$$

For the loop, application of KVL for $t \geq 0^-$ gives

$$\upsilon_R + \upsilon_C + \upsilon_L = \upsilon_s,$$

where

$$\upsilon_R = R\, i(t),$$

$$\upsilon_C = \frac{1}{C}\int_{0^-}^{t} i(t')\, dt' + \upsilon_C(0^-),$$

$$\upsilon_L = L\,\frac{di(t)}{dt},$$

and

$$\upsilon_s = V_0\, u(t).$$

Substitution of these expressions in the KVL equation gives

$$R\, i(t) + \left[\frac{1}{C}\int_{0^-}^{t} i(t')\, dt' + \upsilon_C(0^-)\right] + L\,\frac{di(t)}{dt} = V_o\, u(t)$$

$$R\, i(t) \leftrightarrow R\, \mathbf{I}(\mathbf{s})$$

$$\frac{1}{C}\int_{0^-}^{t} i(t')\, dt' + \upsilon_C(0^-) \leftrightarrow \frac{1}{C}\left[\frac{\mathbf{I}(\mathbf{s})}{\mathbf{s}}\right] + \frac{\upsilon_C(0^-)}{\mathbf{s}}$$

(time integral property)

$$L\,\frac{di(t)}{dt} \leftrightarrow L[\mathbf{s}\,\mathbf{I}(\mathbf{s}) - i(0^-)]$$

(time derivative property)

$$V_o\, u(t) \leftrightarrow \frac{V_o}{\mathbf{s}}$$

(transform of step function).

$$R\, \mathbf{I}(\mathbf{s}) + \frac{\mathbf{I}(\mathbf{s})}{C\mathbf{s}} + \frac{\upsilon_C(0^-)}{\mathbf{s}} + L\mathbf{s}\, \mathbf{I}(\mathbf{s}) = \frac{V_o}{\mathbf{s}}$$

Solution of Circuit Example

$$R\,\mathbf{I}(s) + \frac{\mathbf{I}(s)}{Cs} + \frac{\upsilon_C(0^-)}{s} + Ls\,\mathbf{I}(s) = \frac{V_0}{s}$$

$$\begin{aligned}
\mathbf{I}(s) &= \frac{V_0 - \upsilon_C(0^-)}{L\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]} \\
&= \frac{1.6 - 0.4}{0.4\left(s^2 + \frac{4}{0.4}s + \frac{1}{0.4\times 0.1}\right)} \\
&= \frac{3}{s^2 + 10s + 25} = \frac{3}{(s+5)^2}\,.
\end{aligned}$$

According to entry #6 in Table 3-2, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^2}\right] = te^{-at}\,u(t).$$

Hence,

$$i(t) = 3te^{-5t}\,u(t).$$

Partial Fraction

Consider for example the expression

$$\mathbf{X}(\mathbf{s}) = \frac{4}{\mathbf{s}+2} + \frac{6}{(\mathbf{s}+5)^2} + \frac{8}{\mathbf{s}^2+4\mathbf{s}+5}. \tag{3.50}$$

The inverse transform, $x(t)$, is given by

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[\mathbf{X}(\mathbf{s})] \\ &= \mathcal{L}^{-1}\left[\frac{4}{\mathbf{s}+2}\right] + \mathcal{L}^{-1}\left[\frac{6}{(\mathbf{s}+5)^2}\right] \\ &\quad + \mathcal{L}^{-1}\left[\frac{8}{\mathbf{s}^2+4\mathbf{s}+5}\right]. \end{aligned} \tag{3.51}$$

By comparison with the entries in Table 3-2, we note the following:

(a) The first term in Eq. (3.51), $4/(\mathbf{s}+2)$, is functionally the same as entry #3 in Table 3-2 with $a = 2$. Hence,

$$\mathcal{L}^{-1}\left[\frac{4}{\mathbf{s}+2}\right] = 4e^{-2t}\ u(t). \tag{3.52a}$$

(b) The second term, $6/(\mathbf{s}+5)^2$, is functionally the same as entry #6 in Table 3-2 with $a = 5$. Hence,

$$\mathcal{L}^{-1}\left[\frac{6}{(\mathbf{s}+5)^2}\right] = 6te^{-5t}\ u(t). \tag{3.52b}$$

(c) The third term $8/(\mathbf{s}^2+4\mathbf{s}+5)$, is similar (but not identical) in form to entry #13 in Table 3-2. However, it can be rearranged to assume the proper form:

$$\frac{8}{\mathbf{s}^2+4\mathbf{s}+5} = \frac{8}{(\mathbf{s}+2)^2+1}.$$

Consequently,

$$\mathcal{L}^{-1}\left[\frac{8}{(\mathbf{s}+2)^2+1}\right] = 8e^{-2t}\sin t\ u(t). \tag{3.52c}$$

Combining the results represented by Eqs. (3.52a–c) gives

$$x(t) = [4e^{-2t} + 6te^{-5t} + 8e^{-2t}\sin t]\ u(t). \tag{3.53}$$

Solution is straightforward, as long as **X(s)** is a sum of partial fractions of the proper form for transformation to the time domain

Partial Fraction

Consider for example the expression

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The inverse transform, $x(t)$, is given by

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Combining the results represented by Eqs. (3.52a–c) gives

$$x(t) = [4e^{-2t} + 6te^{-5t} + 8e^{-2t}\sin t]\ u(t). \qquad (3.53)$$

Solution is straightforward, as long as **X**(**s**) is a sum of partial fractions of the proper form for transformation to the time domain

Proper Form

$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})} = \frac{b_m\mathbf{s}^m + b_{m-1}\mathbf{s}^{m-1} + \cdots + b_1\mathbf{s} + b_0}{a_n\mathbf{s}^n + a_{n-1}\mathbf{s}^{n-1} + \cdots + a_1\mathbf{s} + a_0}, \tag{3.54a}$$

where all of the a_i and b_j coefficients are real and the powers m and n are positive integers.

▶ The roots of $\mathbf{N}(\mathbf{s})$, namely the values of $\mathbf{s}$ at which $\mathbf{N}(\mathbf{s}) = 0$, are called the ***zeros*** of $\mathbf{X}(\mathbf{s})$, and we designate them $\{\mathbf{z}_i,\ i = 1, 2, \ldots, m\}$. A polynomial of order m has m roots. Similarly, the roots of $\mathbf{D}(\mathbf{s}) = 0$ are called the ***poles*** of $\mathbf{X}(\mathbf{s})$ and are designated $\{\mathbf{p}_i,\ i = 1, 2, \ldots, n\}$. ◀

Function $\mathbf{X}(\mathbf{s})$ can be expressed in terms of its poles and zeros as

$$\mathbf{X}(\mathbf{s}) = \mathbf{C}\,\frac{(\mathbf{s}-\mathbf{z}_1)(\mathbf{s}-\mathbf{z}_2)\ldots(\mathbf{s}-\mathbf{z}_m)}{(\mathbf{s}-\mathbf{p}_1)(\mathbf{s}-\mathbf{p}_2)\ldots(\mathbf{s}-\mathbf{p}_n)} = \mathbf{C}\,\frac{\prod_{i=1}^{m}(\mathbf{s}-\mathbf{z}_i)}{\prod_{i=1}^{n}(\mathbf{s}-\mathbf{p}_i)}, \tag{3.54b}$$

where $\mathbf{C}$ is a constant.

An important attribute of $\mathbf{X}(\mathbf{s})$ is the ***degree*** of its numerator, m, relative to that of its denominator, n.

$m<n$: strictly proper rational function

X(s) can be expanded into a partial fraction using standard recipes

$m=n$: proper rational function

1 preparatory division step is needed before **X**(s) can be expanded into a partial fraction

$m>n$: improper rational function

2 or more preparatory division steps are needed before **X**(s) can be expanded into a partial fraction

Case 1: $m = n$

Consider the function

$$\mathbf{X}_1(\mathbf{s}) = \frac{2\mathbf{s}^2 + 8\mathbf{s} + 6}{\mathbf{s}^2 + 2\mathbf{s} + 1}.$$

Since $m = n = 2$, this is a proper function, but not a strictly proper function. To convert it to the latter, we use the following steps:

Step 1: Factor out a constant equal to the ratio of the coefficient of the highest term in the numerator to the coefficient of the highest term in the denominator, which in the present case is $2/1 = 2$.

Step 2: Apply the ***division relationship***

$$\frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})} = 1 + \frac{\mathbf{N}(\mathbf{s}) - \mathbf{D}(\mathbf{s})}{\mathbf{D}(\mathbf{s})}. \tag{3.55}$$

For $\mathbf{X}_1(\mathbf{s})$, this two-step process leads to

$$\begin{aligned}
\mathbf{X}_1(\mathbf{s}) &= 2\left(\frac{\mathbf{s}^2 + 4\mathbf{s} + 3}{\mathbf{s}^2 + 2\mathbf{s} + 1}\right) && \textbf{(step 1)} \\
&= 2\left[1 + \frac{(\mathbf{s}^2 + 4\mathbf{s} + 3) - (\mathbf{s}^2 + 2\mathbf{s} + 1)}{(\mathbf{s}^2 + 2\mathbf{s} + 1)}\right] && \textbf{(step 2)} \\
&= 2\left[1 + \frac{2\mathbf{s} + 2}{\mathbf{s}^2 + 2\mathbf{s} + 1}\right] \\
&= 2 + \underbrace{\frac{4\mathbf{s} + 4}{\mathbf{s}^2 + 2\mathbf{s} + 1}}_{\text{strictly proper function}}.
\end{aligned}$$

Case 2: $m > n$

Function

$$\mathbf{X}_2(\mathbf{s}) = \frac{6\mathbf{s}^3 + 4\mathbf{s}^2 + 8\mathbf{s} + 6}{\mathbf{s}^2 + 2\mathbf{s} + 1}$$

is an improper rational function. To convert $\mathbf{X}_2(\mathbf{s})$ into a form in which the highest power terms are $\mathbf{s}^3$ in both the numerator and denominator, we factor out $6\mathbf{s}$:

$$\mathbf{X}_2(\mathbf{s}) = 6\mathbf{s}\left[\frac{\mathbf{s}^3 + (2/3)\mathbf{s}^2 + (8/6)\mathbf{s} + 1}{\mathbf{s}^3 + 2\mathbf{s}^2 + \mathbf{s}}\right].$$

Next, we apply the division relationship, which yields

$$\begin{aligned}
\mathbf{X}_2(\mathbf{s}) \\
&= 6\mathbf{s}\left[1 + \frac{[\mathbf{s}^3 + (2/3)\mathbf{s}^2 + (8/6)\mathbf{s} + 1] - [\mathbf{s}^3 + 2\mathbf{s}^2 + \mathbf{s}]}{\mathbf{s}^3 + 2\mathbf{s}^2 + \mathbf{s}}\right] \\
&= 6\mathbf{s}\left[1 + \frac{-(4/3)\mathbf{s}^2 + (2/6)\mathbf{s} + 1}{\mathbf{s}^3 + 2\mathbf{s}^2 + \mathbf{s}}\right] \\
&= 6\mathbf{s} + \underbrace{\frac{(-8\mathbf{s}^2 + 2\mathbf{s} + 6)}{\mathbf{s}^2 + 2\mathbf{s} + 1}}_{\text{proper function}}.
\end{aligned}$$

The second term of $\mathbf{X}_2(\mathbf{s})$ is a proper function. We can convert it into a strictly proper function by following the recipe in Case 1. The final outcome is

$$\mathbf{X}_2(\mathbf{s}) = (6\mathbf{s} - 8) + \underbrace{\frac{18\mathbf{s} + 14}{\mathbf{s}^2 + 2\mathbf{s} + 1}}_{\text{strictly proper function}}.$$

Distinct Real Poles

Given a proper rational function defined by

$$\mathbf{X}(s) = \frac{\mathbf{N}(s)}{\mathbf{D}(s)} = \frac{\mathbf{N}(s)}{(s-p_1)(s-p_2)\dots(s-p_n)}, \tag{3.61}$$

with distinct real poles p_1 to p_n, such that $p_i \neq p_j$ for all $i \neq j$, and $m < n$ (where m and n are the highest powers of s in $\mathbf{N}(s)$ and $\mathbf{D}(s)$, respectively), then $\mathbf{X}(s)$ can be expanded into the equivalent form:

$$\begin{aligned}\mathbf{X}(s) &= \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_n}{s-p_n} \\ &= \sum_{i=1}^{n} \frac{A_i}{s-p_i}\end{aligned} \tag{3.62}$$

with expansion coefficients A_1 to A_n given by

$$A_i = (s-p_i)\,\mathbf{X}(s)|_{s=p_i}, \quad i = 1, 2, \dots, n. \tag{3.63}$$

In view of entry #3 in Table 3-2, the inverse Laplace transform of Eq. (3.62) is obtained by replacing a with $-p$:

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}[\mathbf{X}(s)] \\ &= [A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t}]\, u(t).\end{aligned} \tag{3.64}$$

Example

$$\mathbf{X}(s) = \frac{s^2 - 4s + 3}{s(s+1)(s+3)}$$

$$\mathbf{X}(s) = \frac{A_1}{s} + \frac{A_2}{(s+1)} + \frac{A_3}{(s+3)}$$

$$A_1 = s\,\mathbf{X}(s)|_{s=0} = \left.\frac{s^2 - 4s + 3}{(s+1)(s+3)}\right|_{s=0} = 1,$$

$$A_2 = \frac{(-1)^2 + 4 + 3}{(-1)(-1+3)} = -4.$$

$$A_3 = (s+3)\,\mathbf{X}(s)|_{s=-3} = \left.\frac{s^2 - 4s + 3}{s(s+1)}\right|_{s=-3} = 4.$$

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}[\mathbf{X}(s)] \\ &= \mathcal{L}\left[\frac{1}{s} - \frac{4}{s+1} + \frac{4}{s+3}\right] \\ &= [1 - 4e^{-t} + 4e^{-3t}]\, u(t).\end{aligned}$$

Repeated Real Poles

Expansion coefficients B_1 to B_m are determined through a procedure that involves multiplication by $(\mathbf{s} - p)^m$, differentiation with respect to $\mathbf{s}$, and evaluation at $\mathbf{s} = p$:

$$B_j = \left\{ \frac{1}{(m-j)!} \frac{d^{m-j}}{d\mathbf{s}^{m-j}} [(\mathbf{s} - p)^m \mathbf{X}(\mathbf{s})] \right\} \Bigg|_{\mathbf{s}=p}, \quad j = 1, 2, \ldots, m. \tag{3.71}$$

For the m, $m-1$, and $m-2$ terms, Eq. (3.71) reduces to

$$B_m = (\mathbf{s} - p)^m \mathbf{X}(\mathbf{s})|_{\mathbf{s}=p}, \tag{3.72a}$$

$$B_{m-1} = \left\{ \frac{d}{d\mathbf{s}} [(\mathbf{s} - p)^m \mathbf{X}(\mathbf{s})] \right\} \Bigg|_{\mathbf{s}=p}, \tag{3.72b}$$

$$B_{m-2} = \left\{ \frac{1}{2!} \frac{d^2}{d\mathbf{s}^2} [(\mathbf{s} - p)^m \mathbf{X}(\mathbf{s})] \right\} \Bigg|_{\mathbf{s}=p}. \tag{3.72c}$$

Example 3-7: Repeated Poles

Determine the inverse Laplace transform of

$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})} = \frac{\mathbf{s}^2 + 3\mathbf{s} + 3}{\mathbf{s}^4 + 11\mathbf{s}^3 + 45\mathbf{s}^2 + 81\mathbf{s} + 54}$$

Numerical evaluation of **X**(**s**) reveals that **s**=-2 and **s**=-3 are roots of **D**(**s**).

Since $\mathbf{s} = -2$ is a root of $\mathbf{D}(\mathbf{s})$, we should be able to factor out $(\mathbf{s} + 2)$ from it. Long division gives

$$\begin{aligned}\mathbf{D}(\mathbf{s}) &= \mathbf{s}^4 + 11\mathbf{s}^3 + 45\mathbf{s}^2 + 81\mathbf{s} + 54 \\ &= (\mathbf{s} + 2)(\mathbf{s}^3 + 9\mathbf{s}^2 + 27\mathbf{s} + 27).\end{aligned}$$

Next, we factor out $(\mathbf{s} + 3)$:

$$\begin{aligned}\mathbf{D}(\mathbf{s}) &= (\mathbf{s} + 2)(\mathbf{s} + 3)(\mathbf{s}^2 + 6\mathbf{s} + 9) \\ &= (\mathbf{s} + 2)(\mathbf{s} + 3)^3.\end{aligned}$$

Hence, $\mathbf{X}(\mathbf{s})$ has a distinct real pole at $\mathbf{s} = -2$ and a triple repeated pole at $\mathbf{s} = -3$, and the given expression can be rewritten as

$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{s}^2 + 3\mathbf{s} + 3}{(\mathbf{s} + 2)(\mathbf{s} + 3)^3}\,.$$

Example 3-7 continued

$$\mathbf{X}(\mathrm{s}) = \frac{A}{\mathrm{s}+2} + \frac{B_1}{\mathrm{s}+3} + \frac{B_2}{(\mathrm{s}+3)^2} + \frac{B_3}{(\mathrm{s}+3)^3},$$

with

$$A = (\mathrm{s}+2)\,\mathbf{X}(\mathrm{s})|_{\mathrm{s}=-2} = \left.\frac{\mathrm{s}^2+3\mathrm{s}+3}{(\mathrm{s}+3)^3}\right|_{\mathrm{s}=-2} = 1,$$

$$B_3 = (\mathrm{s}+3)^3\,\mathbf{X}(\mathrm{s})|_{\mathrm{s}=-3} = \left.\frac{\mathrm{s}^2+3\mathrm{s}+3}{\mathrm{s}+2}\right|_{\mathrm{s}=-3} = -3,$$

$$B_2 = \left.\frac{d}{d\mathrm{s}}\left[(\mathrm{s}+3)^3\,\mathbf{X}(\mathrm{s})\right]\right|_{\mathrm{s}=-3} = 0,$$

and

$$B_1 = \left.\frac{1}{2}\,\frac{d^2}{d\mathrm{s}^2}\left[(\mathrm{s}+3)^3\,\mathbf{X}(\mathrm{s})\right]\right|_{\mathrm{s}=-3} = -1.$$

Hence,

$$\mathbf{X}(\mathrm{s}) = \frac{1}{\mathrm{s}+2} - \frac{1}{\mathrm{s}+3} - \frac{3}{(\mathrm{s}+3)^3},$$

$$\mathcal{L}^{-1}[\mathbf{X}(\mathrm{s})] = \left[e^{-2t} - e^{-3t} - \frac{3}{2}\,t^2 e^{-3t}\right] u(t).$$

3-5.4 Distinct Complex Poles

The Laplace transform of a certain system is given by

$$\mathbf{X}(\mathbf{s}) = \frac{4\mathbf{s}+1}{(\mathbf{s}+1)(\mathbf{s}^2+4\mathbf{s}+13)} .$$

$\mathbf{s}^2 + 4\mathbf{s} + 13 = 0$ gives

$$\mathbf{p}_1 = -2 + j3, \qquad \mathbf{p}_2 = -2 - j3.$$

$$\mathbf{X}(\mathbf{s}) = \frac{A}{\mathbf{s}+1} + \frac{\mathbf{B}_1}{\mathbf{s}+2-j3} + \frac{\mathbf{B}_2}{\mathbf{s}+2+j3}$$

$$A = (\mathbf{s}+1)\,\mathbf{X}(\mathbf{s})\big|_{\mathbf{s}=-1} = \left.\frac{4\mathbf{s}+1}{\mathbf{s}^2+4\mathbf{s}+13}\right|_{\mathbf{s}=-1} = -0.3,$$

$$\begin{aligned}\mathbf{B}_1 &= (\mathbf{s}+2-j3)\,\mathbf{X}(\mathbf{s})\big|_{\mathbf{s}=-2+j3}\\ &= \left.\frac{4\mathbf{s}+1}{(\mathbf{s}+1)(\mathbf{s}+2+j3)}\right|_{\mathbf{s}=-2+j3}\\ &= \frac{4(-2+j3)+1}{(-2+j3+1)(-2+j3+2+j3)}\\ &= \frac{-7+j12}{-18-j6} = 0.73e^{-j78.2^\circ},\end{aligned}$$

$$\begin{aligned}\mathbf{B}_2 &= (\mathbf{s}+2+j3)\,\mathbf{X}(\mathbf{s})\big|_{\mathbf{s}=-2-j3}\\ &= \left.\frac{4\mathbf{s}+1}{(\mathbf{s}+1)(\mathbf{s}+2-j3)}\right|_{\mathbf{s}=-2-j3} = 0.73e^{j78.2^\circ}.\end{aligned}$$

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}[\mathbf{X}(\mathbf{s})]\\ &= \mathcal{L}^{-1}\left(\frac{-0.3}{\mathbf{s}+1}\right) + \mathcal{L}^{-1}\left(\frac{0.73e^{-j78.2^\circ}}{\mathbf{s}+2-j3}\right)\\ &\quad + \mathcal{L}^{-1}\left(\frac{0.73e^{j78.2^\circ}}{\mathbf{s}+2+j3}\right)\\ &= [-0.3e^{-t} + 0.73e^{-j78.2^\circ}e^{-(2-j3)t}\\ &\quad + 0.73e^{j78.2^\circ}e^{-(2+j3)t}]\,u(t).\\ &= [-0.3e^{-t} + 1.46e^{-2t}\cos(3t - 78.2^\circ)]\,u(t).\end{aligned}$$

Table 3-3: Transform pairs for four types of poles.

Pole	$\mathbf{X(s)}$	$x(t)$
1. Distinct real	$\dfrac{A}{\mathbf{s}+a}$	$Ae^{-at}\,u(t)$
2. Repeated real	$\dfrac{A}{(\mathbf{s}+a)^n}$	$A\,\dfrac{t^{n-1}}{(n-1)!}\,e^{-at}\,u(t)$
3. Distinct complex	$\left[\dfrac{Ae^{j\theta}}{\mathbf{s}+a+jb}+\dfrac{Ae^{-j\theta}}{\mathbf{s}+a-jb}\right]$	$2Ae^{-at}\cos(bt-\theta)\,u(t)$
4. Repeated complex	$\left[\dfrac{Ae^{j\theta}}{(\mathbf{s}+a+jb)^n}+\dfrac{Ae^{-j\theta}}{(\mathbf{s}+a-jb)^n}\right]$	$\dfrac{2At^{n-1}}{(n-1)!}\,e^{-at}\cos(bt-\theta)\,u(t)$

Transfer Function

$$x(t) \leftrightarrow \mathbf{X(s)}$$
$$h(t) \leftrightarrow \mathbf{H(s)}$$
$$y(t) \leftrightarrow \mathbf{Y(s)}$$

According to Eq. (2.38), the system output $y(t)$ is convolution of $x(t)$ with $h(t)$,

$$y(t) = x(t) * h(t) = \int_{0^-}^{\infty} x(\tau)\, h(t-\tau)\, d\tau,$$

$$\mathcal{L}[x(t) * h(t)] = \int_{0^-}^{\infty} \left[\int_{0^-}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-\mathbf{s}t}\, dt$$
$$= \int_{0^-}^{\infty} x(\tau) \left[\int_{0^-}^{\infty} h(t-\tau)\, e^{-\mathbf{s}t}\, dt \right] d\tau,$$

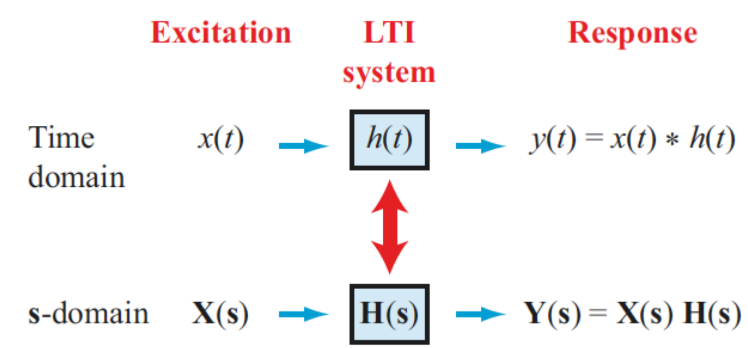
Because τ in the inner integral represents nothing more than a constant time shift, we can introduce the dummy variable $\mu = t - \tau$ and replace dt with $d\mu$,

$$\mathcal{L}[x(t) * h(t)] = \int_{0^-}^{\infty} x(\tau)\, e^{-\mathbf{s}\tau} \left[\int_{-\tau}^{\infty} h(\mu)\, e^{-\mathbf{s}\mu}\, d\mu \right] d\tau$$
$$= \int_{0^-}^{\infty} x(\tau)\, e^{-\mathbf{s}\tau}\, d\tau \int_{0^-}^{\infty} h(\mu)\, e^{-\mathbf{s}\mu}\, d\mu$$
$$= \mathbf{X(s)\, H(s)}. \qquad (3.87)$$

$$\boxed{y(t) = x(t) * h(t) \leftrightarrow \mathbf{Y(s)} = \mathbf{X(s)\, H(s)}.} \qquad (3.88)$$

► Convolution in the time domain corresponds to multiplication in the **s**-domain. ◄

In symbolic form:



Example 3-10: Transfer Function

The output response of a system excited by a unit step function at $t = 0$ is given by

$$y(t) = [2 + 12e^{-3t} - 6\cos 2t]\, u(t).$$

Determine: (a) the transfer function of the system and (b) its impulse response.

Solution:

(a) The Laplace transform of a unit step function is

$$\mathbf{X}(\mathbf{s}) = \frac{1}{\mathbf{s}}\,.$$

By using entries #2, 3, and 11 in Table 3-2, we obtain the Laplace transform of the output response

$$\mathbf{Y}(\mathbf{s}) = \frac{2}{\mathbf{s}} + \frac{12}{\mathbf{s}+3} - \frac{6\mathbf{s}}{\mathbf{s}^2+4}\,.$$

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{Y}(\mathbf{s})}{\mathbf{X}(\mathbf{s})} = 2 + \frac{12\mathbf{s}}{\mathbf{s}+3} - \frac{6\mathbf{s}^2}{\mathbf{s}^2+4}$$

Application of the division relationship to convert the function into proper form leads to:

$$\mathbf{H}(\mathbf{s}) = 2 + \left(12 - \frac{36}{\mathbf{s}+3}\right) + \left(-6 + \frac{24}{\mathbf{s}^2+4}\right)$$

$$= 8 - \frac{36}{\mathbf{s}+3} + \frac{24}{\mathbf{s}^2+4}\,.$$

The corresponding inverse transform is

$$h(t) = 8\delta(t) - 36e^{-3t}\, u(t) + 12\sin(2t)\, u(t).$$

Poles and System Stability

3-7.1 Strictly Proper Rational Function ($m < n$)

If $\mathbf{H}(\mathbf{s})$ has n poles, several combinations of distinct and repeated poles may exist.

Case 1: All n poles are distinct

Application of partial fraction expansion (Section 3-5.1) leads to

$$\mathbf{H}(\mathbf{s}) = \sum_{i=1}^{n} \frac{\mathbf{A}_i}{\mathbf{s} - \mathbf{p}_i}, \tag{3.90}$$

where $\mathbf{p}_1$ to $\mathbf{p}_n$ are real or complex poles and $\mathbf{A}_1$ to $\mathbf{A}_n$ are their corresponding residues. According to Section 3-5, the inverse Laplace transform of $\mathbf{H}(\mathbf{s})$ is

$$h(t) = [\mathbf{A}_1 e^{\mathbf{p}_1 t} + \mathbf{A}_2 e^{\mathbf{p}_2 t} + \cdots \mathbf{A}_n e^{\mathbf{p}_n t}]\, u(t).$$

In Section 2-6.4, we demonstrated that an LTI system with such an impulse response is BIBO stable if and only if all of the exponential coefficients have negative real parts. That is,

$$\Re\mathfrak{e}[\mathbf{p}_i] < 0, \qquad i = 1, 2, \ldots, n. \tag{3.92}$$

▶ The condition defined by Eq. (3.92) requires that $\mathbf{p}_1$ to $\mathbf{p}_n$ reside in the ***open left half-plane (OLHP)*** in the **s**-domain. The imaginary axis is not included in the OLHP; a system with poles along the imaginary axis is not BIBO stable. ◀

Case 2: **Some poles are repeated**

$m<n$

► A system whose transfer function $\mathbf{H}(\mathbf{s})$ is a strictly proper rational function is BIBO stable if and only if its distinct and repeated poles, whether real or complex, reside in the OLHP of the $\mathbf{s}$-domain, which excludes the imaginary axis. Furthermore, the locations of the zeros of $\mathbf{H}(\mathbf{s})$ have no bearing on the system's stability. ◄

Proper Rational Function ($m = n$)

► A system whose transfer function is a proper rational function obeys the same rules with regard to system stability as a strictly proper rational function. ◄

Improper Rational Function ($m > n$)

► A system whose transfer function is an improper rational function is not BIBO stable, regardless of the locations of its poles and zeros. ◄

Important Additional Stability Concepts (beyond Chapter 2 discussions in course text)

1. A LTIC system is s.t.b. <u>asymptotically stable</u> i.f.f. all the poles of its transfer function **H(s)** are in the OLHP. The poles may be simple or repeated.
2. A LTIC system is s.t.b. <u>unstable</u> i.f.f. either one or both of the following conditions exist:
   i. At least one pole of **H(s)** is in the RHP
   ii. There are repeated poles of **H(s)** on the imaginary axis
3. A LTIC system is <u>marginally stable</u> i.f.f. there are no poles of **H(s)** in the RHP and some unrepeated poles on the imaginary axis.

Example 3-12: Dangerous Consequences of Unstable Systems

A system has a transfer function $\mathbf{H(s)} = \mathbf{s} + 1$. Because $\mathbf{H(s)}$ is an improper rational function, the system is not BIBO stable. The undesirability of such a system can be demonstrated by examining the consequence of having high-frequency noise accompany an input signal. Suppose the input to the system is

$$x(t) = x_s(t) + x_n(t),$$

where $x_s(t)$ is the intended input signal and $x_n(t)$ is the unintended input noise, respectively. Furthermore, let

$$x_s(t) = 10 \sin(10^3 t)\; u(t)$$

and

$$x_n(t) = 10^{-2} \sin(10^7 t)\; u(t).$$

Note that the amplitude of the noise is 1000 times smaller than that of the signal, but the angular frequency of the signal is 10^3 rad/s, compared with 10^7 rad/s for the noise. Determine the ***signal-to-noise ratio*** [the ratio of the average power of $x_s(t)$ to that of $x_n(t)$] at the input and output of the system.

Example 3-12 Cont.

(a) **At input**

The signal-to-noise ratio is defined in terms of the average power carried by a waveform. Per Eq. (1.38), the average power of a sinusoidal signal of amplitude 10 is

$$P_s(\text{at input}) = \tfrac{1}{2}\,(10)^2 = 50.$$

Similarly, for the noise component, we have

$$P_n(\text{at input}) = \tfrac{1}{2}\,(10^{-2})^2 = 5 \times 10^{-5}.$$

The signal-to-noise ratio is

$$(S/N)_{\text{input}} = \frac{P_s}{P_n} = \frac{50}{5 \times 10^{-5}} = 10^6.$$

Example 3-12 Cont.

(b) **At output**

For $\mathbf{H}(\mathbf{s}) = \mathbf{s} + 1$, the output is

$$\mathbf{Y}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\,\mathbf{X}(\mathbf{s}) = \mathbf{s}\,\mathbf{X}(\mathbf{s}) + \mathbf{X}(\mathbf{s}).$$

From Table 3-1, the following two Laplace transform properties are of interest:

$$x(t) \quad \longleftrightarrow \quad \mathbf{X}(\mathbf{s})$$

$$\frac{dx}{dt} \quad \longleftrightarrow \quad \mathbf{s}\,\mathbf{X}(\mathbf{s}) - \mathbf{s}\,x(0^-).$$

In the present case, $x(t)$ consists of sinusoidal waveforms, so $x(0^-) = 0$. Application of the Laplace transform properties leads to

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[\mathbf{Y}(\mathbf{s})] \\ &= \mathcal{L}^{-1}[\mathbf{s}\,\mathbf{X}(\mathbf{s}) + \mathbf{X}(\mathbf{s})] \\ &= \frac{dx}{dt} + x(t) \\ &= \frac{d}{dt}\,[(x_s + x_n)] + (x_s + x_n) \\ &= \frac{d}{dt}[10\sin(10^3 t) + 10^{-2}\sin(10^7 t)] \\ &\quad + 10\sin(10^3 t) + 10^{-2}\sin(10^7 t) \\ &= \underbrace{[10^4\cos(10^3 t) + 10\sin(10^3 t)]}_{\text{output signal at } 10^3 \text{ rad/s}} \\ &\quad + \underbrace{[10^5\cos(10^7 t) + 10^{-2}\sin(10^7 t)]}_{\text{output noise at } 10^7 \text{ rad/s}}. \end{aligned}$$

The average powers associated with the signal and noise are

$$P_s(\text{@output}) = \tfrac{1}{2}\,(10^4)^2 + \tfrac{1}{2}\,(10)^2 \simeq 5 \times 10^7,$$

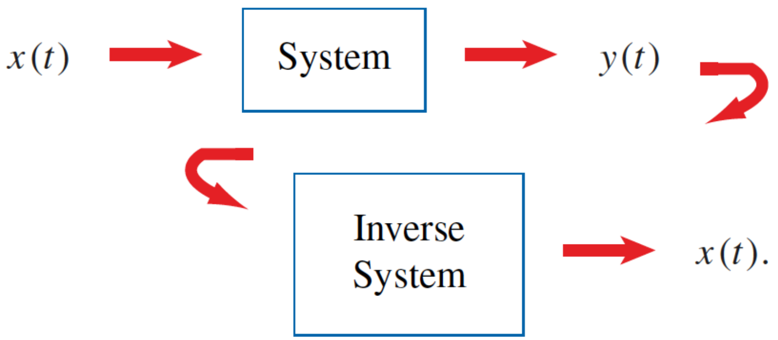
$$P_n(\text{@output}) = \tfrac{1}{2}\,(10^5)^2 + \tfrac{1}{2}\,(10^{-2})^2 \simeq 5 \times 10^9,$$

and the signal-to-noise ratio is

$$(S/N)_{\text{output}} = \frac{5 \times 10^7}{5 \times 10^9} = 10^{-2}.$$

Whereas the noise component was inconsequential at the input end of the system, propagation through the unstable system led to the undesirable consequence that at the output the noise power became two orders of magnitude greater than that of the signal power.

Invertible Systems



Example

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{Y}(\mathbf{s})}{\mathbf{X}(\mathbf{s})} = \frac{b_1\mathbf{s} + b_2}{\mathbf{s}^2 + a_1\mathbf{s} + a_2}$$

Transfer function of Inverse system:

$$\mathbf{G}(\mathbf{s}) = \frac{1}{\mathbf{H}(\mathbf{s})} = \frac{\mathbf{s}^2 + a_1\mathbf{s} + a_2}{b_1\mathbf{s} + b_2}$$

Inverse system is not stable because it is an improper rational function

► A BIBO stable and causal LTI system has a BIBO stable and causal inverse system if and only if all of its poles and zeros are in the open left half-plane, and they are equal in number (its transfer function is proper). Such a system is called a ***minimum phase*** system. ◄

Unilateral vs Bilateral Laplace

1. For causal signals, we can use unilateral Laplace transform to relate H(s) to the impulse response:

$$\mathbf{H}(\mathbf{s}) = \int_{0^-}^{\infty} h(\tau)\, e^{-\mathbf{s}\tau}\, d\tau$$

2. For everlasting signals, we should use the bilateral Laplace transform:

$$\mathbf{H}_{\mathrm{b}}(\mathbf{s}) = \int_{-\infty}^{\infty} h(\tau)\, e^{-\mathbf{s}\tau}\, d\tau$$

$$x(t) = A\cos(\omega t + \phi)$$

is an everlasting signal

$$\mathbf{H}_{\mathrm{b}}(\mathbf{s}) = \mathbf{H}(\mathbf{s}) \textit{ for causal systems.}$$

$$\cos\omega t \longrightarrow \boxed{h(t)} \longrightarrow y(t) = |\mathbf{H}(\omega)|\cos(\omega t + \theta),$$

where

$$\mathbf{H}(\omega) = |\mathbf{H}(\omega)|\underline{/\theta}$$

and $$\mathbf{H}(\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau.$$

Hence,

$$\mathbf{H}(\omega) = \mathbf{H}(\mathbf{s})\big|_{\mathbf{s}=j\omega} \qquad \left(\begin{array}{c}\text{sinusoidal input}\\ \text{causal system}\end{array}\right)$$

Example 3-14: Sinusoidal Signal

An LTI system with a transfer function

$$\mathbf{H}(\mathbf{s}) = 100/(\mathbf{s}^2 + 15\mathbf{s} + 600)$$

has an input signal

$$x(t) = 10\cos(20t + 30^\circ).$$

Determine the output response $y(t)$.

Solution: At $\omega = 20$ rad/s, we have

$$\mathbf{H}(\omega) = \mathbf{H}(\mathbf{s})|_{\mathbf{s}=j20} = \frac{100}{(j20)^2 + 15(j20) + 600}$$

$$= \frac{100}{200 + j300} = 0.28\underline{/-56.31^\circ}.$$

Application of Eq. (3.120) leads to

$$y(t) = A|\mathbf{H}(\omega)|\cos(\omega t + \theta + \phi)$$

$$= 10 \times 0.28\cos(20t - 56.31^\circ + 30^\circ)$$

$$= 2.8\cos(20t - 26.31^\circ).$$

3-10 Interrelating Different Descriptions of LTI Systems

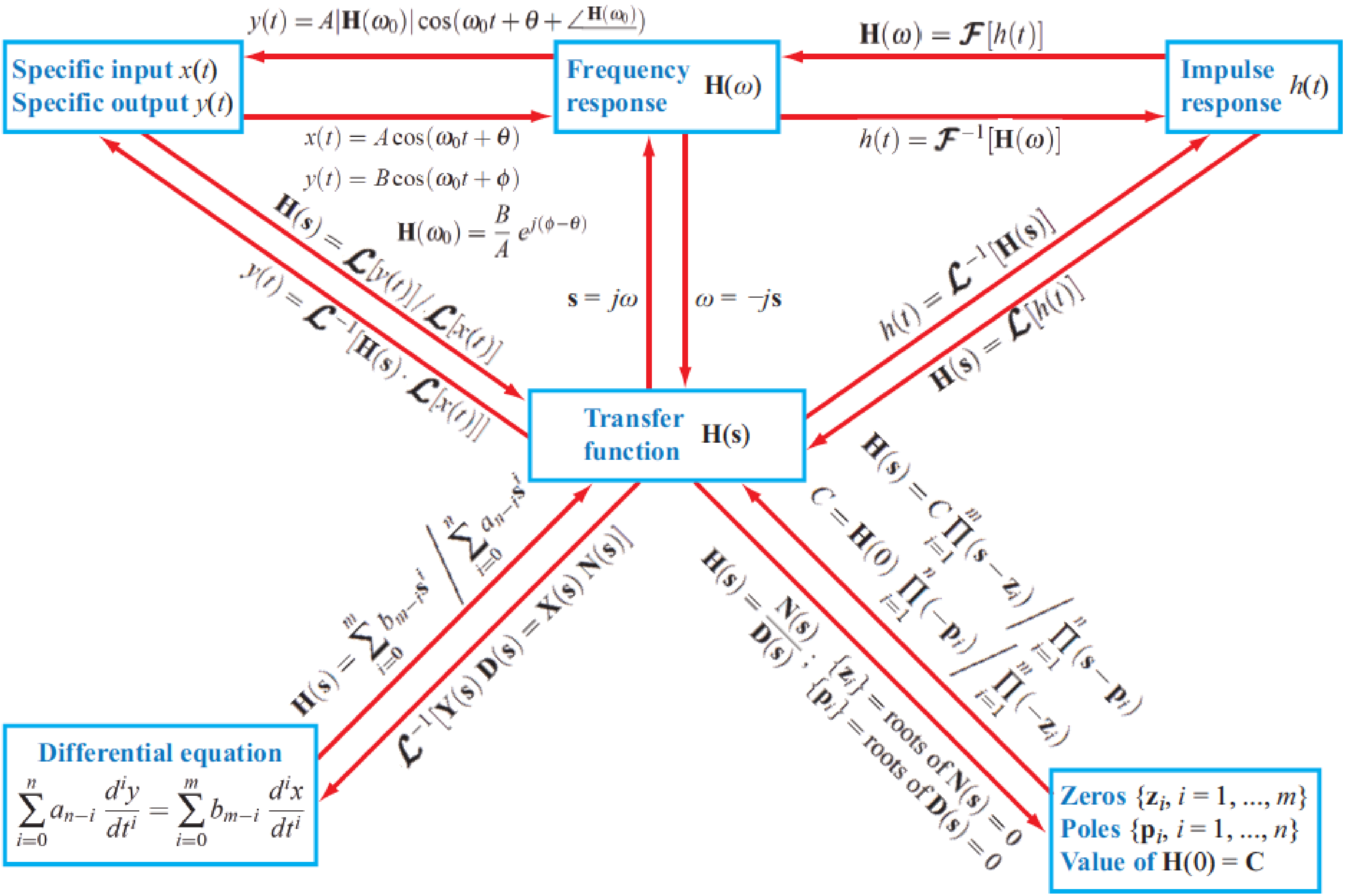
An LTI system can be described in six different ways, namely:

- Transfer function $\mathbf{H}(\mathbf{s})$
- Impulse response $h(t)$
- Poles $\{\mathbf{p}_i,\ i = 1, \ldots, n\}$ and zeros $\{\mathbf{z}_i,\ i = 1, \ldots, m\}$
- Specific input-output pair, $x(t)$ and $y(t)$, where

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

- Differential Equation (LCCDE)
- Frequency response $\mathbf{H}(\omega)$

Given any one of these descriptions, we can easily determine the other five (Fig. 3-5). Specifically, the transfer function $\mathbf{H}(\mathbf{s})$ can be related to the other descriptions as follows.



Specific input $x(t)$
Specific output $y(t)$
Frequency response $\mathbf{H}(\omega)$
Impulse response $h(t)$
Transfer function $\mathbf{H}(\mathbf{s})$
Differential equation
$\sum_{i=0}^{n} a_{n-i} \frac{d^i y}{dt^i} = \sum_{i=0}^{m} b_{m-i} \frac{d^i x}{dt^i}$
Zeros $\{\mathbf{z}_i, i = 1, ..., m\}$
Poles $\{\mathbf{p}_i, i = 1, ..., n\}$
Value of $\mathbf{H}(0) = \mathbf{C}$
$y(t) = A|\mathbf{H}(\omega_0)| \cos(\omega_0 t + \theta + \angle \mathbf{H}(\omega_0))$
$x(t) = A\cos(\omega_0 t + \theta)$
$y(t) = B\cos(\omega_0 t + \phi)$
$\mathbf{H}(\omega_0) = \frac{B}{A} e^{j(\phi - \theta)}$
$\mathbf{H}(\omega) = \mathcal{F}[h(t)]$
$h(t) = \mathcal{F}^{-1}[\mathbf{H}(\omega)]$
$\mathbf{s} = j\omega$
$\omega = -j\mathbf{s}$
$\mathbf{H}(\mathbf{s}) = \mathcal{L}[y(t)]/\mathcal{L}[x(t)]$
$y(t) = \mathcal{L}^{-1}[\mathbf{H}(\mathbf{s}) \cdot \mathcal{L}[x(t)]]$
$h(t) = \mathcal{L}^{-1}[\mathbf{H}(\mathbf{s})]$
$\mathbf{H}(\mathbf{s}) = \mathcal{L}[h(t)]$
$\mathbf{H}(\mathbf{s}) = \sum_{i=0}^{m} b_{m-i}\mathbf{s}^i \Big/ \sum_{i=0}^{n} a_{n-i}\mathbf{s}^i$
$\mathcal{L}^{-1}[\mathbf{Y}(\mathbf{s})\mathbf{D}(\mathbf{s}) = \mathbf{X}(\mathbf{s})\mathbf{N}(\mathbf{s})]$
$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})}$; $\{\mathbf{z}_i\}$ = roots of $\mathbf{N}(\mathbf{s}) = 0$
$\{\mathbf{p}_i\}$ = roots of $\mathbf{D}(\mathbf{s}) = 0$
$C = \mathbf{H}(0) \prod_{i=1}^{n}(-\mathbf{p}_i) \Big/ \prod_{i=1}^{m}(-\mathbf{z}_i)$
$\mathbf{H}(\mathbf{s}) = C\prod_{i=1}^{m}(\mathbf{s} - \mathbf{z}_i) \Big/ \prod_{i=1}^{n}(\mathbf{s} - \mathbf{p}_i)$

Example 3-15: Relate $h(t)$ to Other LTI System Descriptions

Given an LTI system with $h(t) = e^{-2t}\, u(t) + e^{-4t}\, u(t)$, determine the following: (a) $\mathbf{H}(\mathbf{s})$, (b) $\mathbf{H}(\omega)$, (c) LCCDE, (d) poles and zeros, and (e) the output response $y(t)$ due to an input $x(t) = \delta(t) + e^{-3t}\, u(t)$. Assume zero initial conditions.

Solution:

(a)

$$\mathbf{H}(\mathbf{s}) = \mathcal{L}[h(t)] = \mathcal{L}[e^{-2t}\, u(t) + e^{-4t}\, u(t)]. \qquad (3.130)$$

Application of entry #3 in Table 3-2 leads to

$$\begin{aligned}\mathbf{H}(\mathbf{s}) &= \frac{1}{\mathbf{s}+2} + \frac{1}{\mathbf{s}+4} \\ &= \frac{(\mathbf{s}+4)+(\mathbf{s}+2)}{(\mathbf{s}+2)(\mathbf{s}+4)} = \frac{2\mathbf{s}+6}{\mathbf{s}^2+6\mathbf{s}+8}\,. \end{aligned} \qquad (3.131)$$

(b)

$$\mathbf{H}(\omega) = \mathbf{H}(\mathrm{s})\big|_{\mathrm{s}=j\omega} = \frac{2(j\omega)+6}{(j\omega)^2 + j6\omega + 8} = \frac{6 + j2\omega}{(8-\omega^2) + j6\omega}$$

(c)

$$\mathbf{H}(\mathrm{s}) = \frac{\mathbf{Y}(\mathrm{s})}{\mathbf{X}(\mathrm{s})} = \frac{2\mathrm{s}+6}{\mathrm{s}^2+6\mathrm{s}+8}\,.$$

Cross multiplying gives

$$\mathrm{s}^2\,\mathbf{Y}(\mathrm{s}) + 6\mathrm{s}\,\mathbf{Y}(\mathrm{s}) + 8\mathbf{Y}(\mathrm{s}) = 2\mathrm{s}\,\mathbf{X}(\mathrm{s}) + 6\mathbf{X}(\mathrm{s}).$$

For a system with zero initial conditions, differentiation in the time domain corresponds to multiplication by s in the s-domain (property #6 in Table 3-1). Hence, the time-domain equivalent of the preceding equation is

$$\frac{d^2y}{dt^2} + 6\,\frac{dy}{dt} + 8y = 2\,\frac{dx}{dt} + 6x.$$

$$\mathbf{H}(\mathrm{s}) = \frac{1}{\mathrm{s}+2} + \frac{1}{\mathrm{s}+4}$$

$$= \frac{(\mathrm{s}+4)+(\mathrm{s}+2)}{(\mathrm{s}+2)(\mathrm{s}+4)} = \frac{2\mathrm{s}+6}{\mathrm{s}^2+6\mathrm{s}+8}$$

(d) From Eq. (3.131), we have

$$2\mathrm{s}+6 = 0 \;\rightarrow\; \text{zero } \{-3\}$$

and

$$\mathrm{s}^2 + 6\mathrm{s} + 8 = 0 \;\rightarrow\; \text{poles } \{-2, -4\}\,.$$

(e) For $x(t) = \delta(t) + e^{-3t}\,u(t)$, application of entries #1 and 3 in Table 3-2 leads to

$$\mathbf{X}(\mathrm{s}) = 1 + \frac{1}{\mathrm{s}+3} = \frac{\mathrm{s}+4}{\mathrm{s}+3}\,.$$

Output $\mathbf{Y}(\mathrm{s})$ is then given by

$$\mathbf{Y}(\mathrm{s}) = \mathbf{H}(\mathrm{s})\;\mathbf{X}(\mathrm{s})$$

$$= \frac{2(\mathrm{s}+3)}{\mathrm{s}^2+6\mathrm{s}+8} \cdot \frac{\mathrm{s}+4}{\mathrm{s}+3}$$

$$= \frac{2(\mathrm{s}+3)(\mathrm{s}+4)}{(\mathrm{s}+2)(\mathrm{s}+4)(\mathrm{s}+3)} = \frac{2}{\mathrm{s}+2}\,.$$

Hence,

$$y(t) = \mathcal{L}^{-1}\left(\frac{2}{\mathrm{s}+2}\right) = 2e^{-2t}\,u(t).$$

Example 3-16: Determine $\hat{\mathbf{H}}(\omega)$ from LCCDE

Given

$$\frac{d^2y}{dt^2} + 5\,\frac{dy}{dt} + 6y = \frac{d^2x}{dt^2} + 5\,\frac{dx}{dt} + 4x$$

for a system with zero initial conditions, determine the frequency response function $\mathbf{H}(\omega)$.

Solution:

Transferring the LCCDE to the **s**-domain gives

$$\mathbf{Y}(\mathbf{s})[\mathbf{s}^2 + 5\mathbf{s} + 6] = \mathbf{X}(\mathbf{s})[\mathbf{s}^2 + 5\mathbf{s} + 4],$$

from which we obtain the transfer function

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{Y}(\mathbf{s})}{\mathbf{X}(\mathbf{s})} = \frac{\mathbf{s}^2 + 5\mathbf{s} + 4}{\mathbf{s}^2 + 5\mathbf{s} + 6}\,.$$

The frequency response $\hat{\mathbf{H}}(\omega)$ is then given by

$$\begin{aligned}\mathbf{H}(\omega) &= \mathbf{H}(\mathbf{s})\big|_{\mathbf{s}=j\omega} \\ &= \frac{(j\omega)^2 + j5\omega + 4}{(j\omega)^2 + j5\omega + 6} = \frac{(4-\omega^2) + j5\omega}{(6-\omega^2) + j5\omega}\,.\end{aligned}$$

Example 3-17: Determine H(s) from its Poles and Zeros

Transfer functions $\mathbf{H}(\mathbf{s})$ has zero $\{+1\}$ and pole $\{-3\}$. Also, $\mathbf{H}(0) = -1$. Obtain an expression for $\mathbf{H}(\mathbf{s})$.

Solution:

$\mathbf{H}(\mathbf{s})$ has one zero at $\mathbf{s} = 1$ and one pole at $\mathbf{s} = -3$. Hence, for some constant $\mathbf{C}$, $\mathbf{H}(\mathbf{s})$ is given by

$$\mathbf{H}(\mathbf{s}) = \mathbf{C}\,\frac{\mathbf{s}-1}{\mathbf{s}+3}\,.$$

At $\mathbf{s} = 0$, $\mathbf{H}(0) = -1$. Hence,

$$-1 = \mathbf{C}\,\frac{(-1)}{3}\,,$$

so $\mathbf{C} = 3$, and then

$$\mathbf{H}(\mathbf{s}) = 3\,\frac{\mathbf{s}-1}{\mathbf{s}+3}\,.$$

LTI System Response Partitions

1. Zero-State/Zero-Input Partition

The total response is a sum of the following:

- **Zero-input response (ZIR)** = response of the system in the absence of an input excitation
- **Zero-state response (ZSR)** = response of the system when it is in zero state (that is, initial conditions are zero / no stored energy)

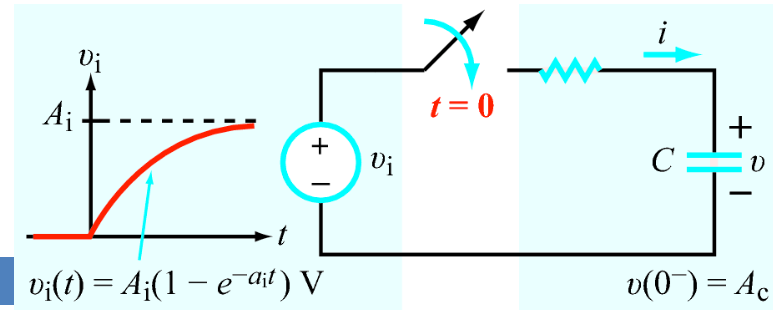
2. Natural/Forced Partition

- **Natural response** = all terms that vary with t according to the system's modes
- **Forced response** = all remaining terms,

3. Transient/Steady State Partition

- **Transient response** = all terms that decay to zero as $t \to \infty$, and a
- **Steady-state response** = all terms that remain after the demise of the transient response.

RC Circuit Example



Input Excitation **RC Circuit**

Initial condition

$$v(0^-) = A_c.$$

$$RC = \frac{1}{a_c}.$$

Transforming Eq. (3.134) to the **s**-domain entails

$$v(t) \rightarrow \mathbf{V}(\mathbf{s}),$$

$$\frac{dv}{dt} \rightarrow \mathbf{s}\,\mathbf{V}(\mathbf{s}) - v(0^-) = \mathbf{s}\,\mathbf{V}(\mathbf{s}) - A_c$$

(by property #6 in Table 3-1),

$$v_i(t) \rightarrow \mathbf{V}_i(\mathbf{s}),$$

Differential Equation

$$RC\,\frac{dv}{dt} + v = v_i(t)$$

which leads to

$$\underbrace{\left(\frac{\mathbf{s}}{a_c} + 1\right)\mathbf{V}(\mathbf{s})}_{\text{Zero state}} = \underbrace{\frac{A_c}{a_c}}_{\text{Initial condition}} + \underbrace{\mathbf{V}_i(\mathbf{s})}_{\text{Input excitation}}.$$

1. Zero-State/Zero-Input Partition

$$\underbrace{\left(\frac{s}{a_c}+1\right)}_{\text{Zero state}} \mathbf{V}(s) = \underbrace{\frac{A_c}{a_c}}_{\text{Initial condition}} + \underbrace{\mathbf{V}_i(s)}_{\text{Input excitation}}.$$

$$\underbrace{\mathbf{V}(s)}_{\text{Total response}} = \underbrace{\frac{A_c}{s+a_c}}_{\text{Zero-input response}} + \underbrace{\frac{a_c\,\mathbf{V}_i(s)}{s+a_c}}_{\text{Zero-state response}}.$$

Input: $$\upsilon_i(t) = A_i[1-e^{-a_i t}]\,u(t)\ \text{V}$$

with amplitude A_i and exponential coefficient a_i. corresponding s-domain expression is

$$\mathbf{V}_i(s) = \frac{A_i}{s} - \frac{A_i}{s+a_i} = \frac{A_i a_i}{s(s+a_i)}.$$

Inserting Eq. (3.139) into Eq. (3.137) leads to

$$\mathbf{V}(s) = \frac{A_c}{s+a_c} + \frac{A_i a_c a_i}{s(s+a_c)(s+a_i)}.$$

Partial fraction expansion leads to:

$$\mathbf{V}(s) = \frac{A_c}{s+a_c} + \left[\frac{A_i}{s} + \frac{A_i a_i}{(a_c-a_i)}\left(\frac{1}{s+a_c}\right) + \frac{A_i a_c}{(a_i-a_c)}\left(\frac{1}{s+a_i}\right)\right]$$

Time-domain solution:

$$\upsilon(t) = \underbrace{A_c e^{-a_c t}\,u(t)}_{\text{ZIR}} + \underbrace{\left[A_i + \left(\frac{A_i a_i}{a_c-a_i}\right)e^{-a_c t} + \left(\frac{A_i a_c}{a_i-a_c}\right)e^{-a_i t}\right] u(t).}_{\text{ZSR}} \quad (3.14$$

For $A_c = \upsilon(0^-) = 2$ V, $a_c = 3\ \text{s}^{-1}$, $A_i = 6$ V, and $a_i = 2\ \text{s}^{-}$ we have

$$\upsilon(t) = \underbrace{2e^{-3t}\,u(t)}_{\text{ZIR}} + \underbrace{[6+12e^{-3t}-18e^{-2t}]\,u(t)}_{\text{ZSR}}. \quad (3.14$$

2. Natural/Forced Partition

$$\underbrace{\mathbf{V}(\mathbf{s})}_{\text{Total response}} = \underbrace{\frac{A_c}{\mathbf{s}+a_c}}_{\text{Zero-input response}} + \underbrace{\frac{a_c\,\mathbf{V}_i(\mathbf{s})}{\mathbf{s}+a_c}}_{\text{Zero-state response}}.$$

$$\upsilon(t) = \underbrace{2e^{-3t}\,u(t)}_{\text{ZIR}} + \underbrace{[6+12e^{-3t}-18e^{-2t}]\,u(t)}_{\text{ZSR}}.$$

The function $e^{-a_c t} = e^{-3t}$ is called a ***mode*** of the system. It is independent of the input function because $a_c = 1/(RC)$ is characteristic of the RC circuit alone.

Grouping terms involving the system mode leads to:

$$\upsilon(t) = [2e^{-3t} + 12e^{-3t}]\,u(t) + [6-18e^{-2t}]\,u(t)$$

$$= \underbrace{14e^{-3t}\,u(t)}_{\text{Natural response}} + \underbrace{[6-18e^{-2t}]\,u(t)}_{\text{Forced response}}. \qquad (3$$

3. Transient/Steady State Partition

$$\underbrace{\mathbf{V}(\mathbf{s})}_{\text{Total response}} = \underbrace{\frac{A_c}{\mathbf{s} + a_c}}_{\text{Zero-input response}} + \underbrace{\frac{a_c \, \mathbf{V}_i(\mathbf{s})}{\mathbf{s} + a_c}}_{\text{Zero-state response}}.$$

$$\upsilon(t) = \underbrace{2e^{-3t}\, u(t)}_{\text{ZIR}} + \underbrace{[6 + 12e^{-3t} - 18e^{-2t}]\, u(t)}_{\text{ZSR}}.$$

Terms that decay to 0 as t approaches infinity constitute the transient response.

Partitioning the expression for $\upsilon(t)$ along those lines gives

$$\upsilon(t) = \underbrace{[14e^{-3t} - 18e^{-2t}]\, u(t)}_{\text{Transient response}} + \underbrace{6u(t)}_{\text{Steady-state response}}. \qquad (3.145)$$