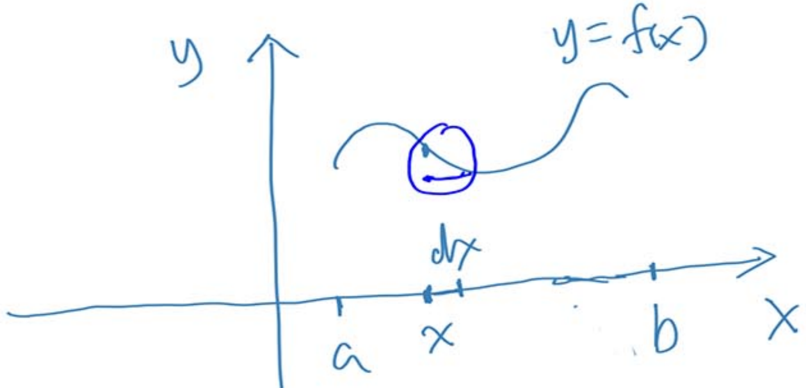
# LECTURE NO. 6

## 2.4 Arc Length of a Curve and Surface Area

Wright State University

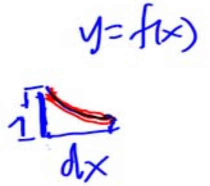



$x$ is a point in $[a, b]$

$dx$ is a tiny change in $x$

$$\text{Arc length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$? = dy$




$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)\, dx$$

$$\text{Red Piece} = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(dx)^2 + (f'(x)\, dx)^2}$$

$$= \sqrt{(dx)^2 (1 + f'(x)^2)}$$

$$= \sqrt{1 + [f'(x)]^2}\, dx$$

# Let $f(x) = 2x^{\frac{3}{2}}$. Find the arc length of $f(x)$ over $[0,1]$.

$$\text{Arc length} = \int_0^1 \sqrt{1+[f'(x)]^2}\,dx$$

$$f(x) = 2x^{\frac{3}{2}} \qquad f'(x) = 2\cdot\frac{3}{2}x^{\frac{1}{2}} = 3x^{\frac{1}{2}}$$

$$\text{Arc length} = \int_0^1 \sqrt{1+(3x^{\frac{1}{2}})^2}\,dx = \int_0^1 \sqrt{1+9x}\,dx$$

Substitution $\quad u = 1+9x \quad \frac{du}{dx} = 9 \quad dx = \frac{du}{9} \qquad x: 0 \longrightarrow 1$

$$u = 1+9x: 1 \longrightarrow 10$$

$$\int_1^{10} \sqrt{u}\,\frac{du}{9} = \frac{1}{9}\int_1^{10} u^{\frac{1}{2}}\,du = \frac{1}{9}\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{10}$$

$$= \frac{2}{27}\cdot 10^{\frac{3}{2}} - \frac{2}{27}$$

Final Answer

# Set up an integral for the arc length of $g(x) = \sin x$ over $[0, \pi]$.

Arc length $= \int_0^{\pi} \sqrt{1 + [g'(x)]^2}\, dx$

Since $g(x) = \sin x$, $\quad g'(x) = \cos x$

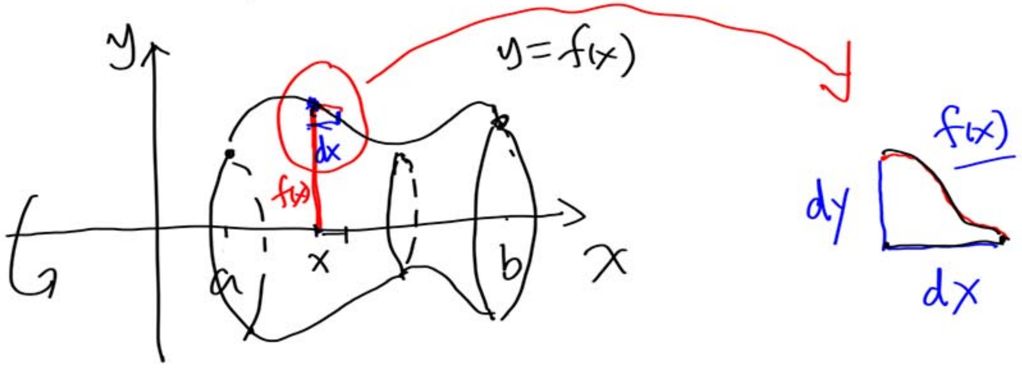
Arc length $= \underbrace{\int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx}$

This is the integral we need!

FINAL ANSWER

# Area of a surface of Revolution.

$y = f(x) \qquad a \le x \le b$




<u>Vase</u>

$\frac{dy}{dx} = f'(x) \qquad dy = f'(x)\,dx$

<u>Red Piece</u> $= \sqrt{(dx)^2+(dy)^2} = \sqrt{(dx)^2+(f'(x)\,dx)^2}$

$= \sqrt{1+f'(x)^2}\,dx$

After Rotation, the red piece yields "a cylinder"

height $= \sqrt{1+[f'(x)]^2}\,dx \qquad$ Radius $= f(x)$

$2\pi \cdot \text{Radius} \cdot \text{height} = 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

Area of A Surface of Revolution $= \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

Given $f(x) = \sqrt{x}$ on $[1, 4]$. Find the area of the surface generated by revolving $f(x)$ around $x$-axis.

$$\text{Surface Area} = \int_1^4 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$$

$$\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\text{Surface Area} = \int_1^4 2\pi \sqrt{x} \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\, dx = \int_1^4 2\pi \sqrt{x}\cdot\sqrt{1+\frac{1}{4x}}\, dx$$

$$= \int_1^4 2\pi \sqrt{x\cdot\left(1+\frac{1}{4x}\right)}\, dx = \int_1^4 2\pi \sqrt{x+\frac{1}{4}}\, dx$$

$x: 1 \to 4$

Substitution $u = x + \frac{1}{4}$ $\quad \frac{du}{dx} = 1 \quad dx = du \quad u: x+\frac{1}{4}.\ \frac{5}{4} \to \frac{17}{4}$

$$\int_{\frac{5}{4}}^{\frac{17}{4}} 2\pi \underset{\underset{u^{\frac{1}{2}}}{\parallel}}{\sqrt{u}}\, du = 2\pi \frac{2}{3} u^{\frac{3}{2}} \Big|_{\frac{5}{4}}^{\frac{17}{4}} = \frac{4}{3}\pi \left(\frac{17}{4}\right)^{\frac{3}{2}} - \frac{4}{3}\pi\left(\frac{5}{4}\right)^{\frac{3}{2}}$$

No Need to Simplify!

Set up an integral for the area of the surface generated by rotating $y = \sqrt[3]{x}$ with $1 \le x \le 8$ around $y$-axis.

$y = f(x) \quad a \le x \le b$

Surface Area Rotated around $x$-axis

$$\int_a^b 2\pi f(x) \sqrt{1+[f'(x)]^2}\, dx$$

Rotate around $y$, we can treat $x$ as a function of $y$

say $x = g(y) \quad c \le y \le d$

$$\text{Surface Area} = \int_c^d 2\pi\, g(y) \sqrt{1+[g'(y)]^2}\, dy$$

$y = \sqrt[3]{x}$ solve for $x$: $x = y^3$ since $1 \le x \le 8$, $y = \sqrt[3]{x}$, $\Rightarrow 1 \le y \le 2$

$\frac{dx}{dy} = 3y^2$

$$\text{Surface Area} = \int_1^2 2\pi\, y^3 \sqrt{1+(3y^2)^2}\, dy$$

(you may practice using substitution to solve this integral ;)

# List of Formulas

- The Arc length of $y = f(x)$ with $a \leq x \leq b$ is given by

$$\int_a^b \sqrt{1 + [f'(x)]^2} dx$$

- The Arc length of $x = g(y)$ with $c \leq y \leq d$ is given by

$$\int_c^d \sqrt{1 + [g'(y)]^2} dy$$

- Surface Area by rotating $y = f(x)$ with $a \leq x \leq b$ around $x$-axis is

$$\int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

- Surface Area by rotating $x = g(y)$ with $c \leq y \leq d$ by $y$-axis is $\int_c^d 2\pi g(y) \sqrt{1+[g'(y)]^2} dy$

$$\int^d 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy$$