Analysis of Variance (ANOVA)

- recall two-sample hypothesis tests; what if we want to compare two concrete curing methods?
  - C.I.'s, paired t-test, hypothesis tests, etc..
- OK... what about <u>five</u> methods?
- ANOVA is great for comparing more than two <u>levels</u> of a <u>single factor experiment</u>

  levels ⇓ a/k/a <u>treatments</u>

  single factor experiment ⇓ this course!

Read Ch. 13-1 "Design of Experiments"

∴ in the case of five levels, sure we could test <u>every</u> pair; becomes cumbersome and inefficient <u>very</u> quickly.

ex: grocery bag manufacturer wants to improve tensile strength of paper grocery bags

∴ engineering thinks the hardwood concentration in the pulp is a contributing factor

∴ wants to test $a = 4$ levels

$\{5\% \quad 10\% \quad 15\% \quad 20\%\}$

(hardwood concentration)

∴ $n = 6$ test specimens or <u>replicates</u> of each level to be tested

∴ we have $N = an = 4 \cdot 6 = \underline{24}$ experiments

∴ tested in random order ∴ important!

∴ what if you did all the 5% samples first, then 10%, etc.. and later found out the humidity went up in the lab throughout the day?

∴ "confounding variable"

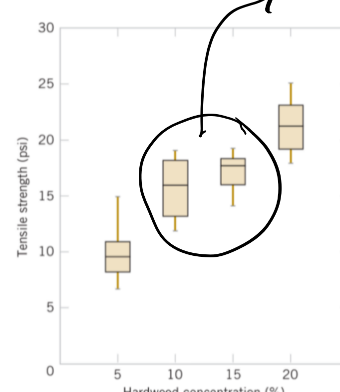
a/k/a "nuisance variable"

graphical representation of data:

TABLE • 13-1 Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations						Totals	Averages
	1	2	3	4	5	6		
5	7	8	15	11	9	10	60	10.00
10	12	17	13	18	19	15	94	15.67
15	14	18	19	17	16	18	102	17.00
20	19	25	22	23	18	20	127	21.17
							383	15.96

⇑ tensile strengths in psi



this pair doesn't seem to significantly differ; all other pairs look like it!

⇑ box & whisker plot

·· Ultimately comes down to a hypothesis test for the entire experiment :

$H_0 : \mu_1 = \mu_2 = \mu_3 \ldots = \mu_a$

(treatment means are all equal; no significant difference between treatments / levels of hardwood concentration)

$H_1 : \mu_1 \neq \mu_2$
$\mu_2 \neq \mu_3$ etc.. for at least one pair

·· We now just need a test statistic and rejection criteria that quantifies this whole pile of data

·· this example (hardwood concentration) uses a fixed effects model; treatments were specifically chosen {5% 10% 15% 20%}, conclusions cannot be extended to any other treatments [i.e., 12.5% etc..]

- otherwise: random effects or components of Variance model, in which treatments are a random sample from a larger pop. of many treatments (not in this class!)

- let $y_{i\cdot}$ (means total!) represent the total of the observations under the ith treatment; row total

$$y_{i\cdot} = \sum_{j=1}^{n} y_{ij}$$

- let $\bar{y}_{i\cdot}$ represent the mean of the observations under the ith treatment

$$\bar{y}_{i\cdot} = y_{i\cdot}/n$$

- $$y_{\cdot\cdot} = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij} \rightarrow \text{grand total}$$

$$\bar{y}_{\cdot\cdot} = y_{\cdot\cdot}/\underset{= an}{N} \rightarrow \text{grand average}$$

TABLE • 13-1 Tensile Strength of Paper (psi)

Hardwood Concentration (%)	Observations 1	2	3	4	5	6	Totals	Averages
5 $i=1$	y_{11} 7	y_{12} 8	y_{13} 15	11	9	10	$y_{1\cdot}$ 60	$\bar{y}_{1\cdot}$ 10.00
10 $i=2$	y_{21} 12	y_{22} 17	13	18	19	15	$y_{2\cdot}$ 94	$\bar{y}_{2\cdot}$ 15.67
15	14	18	19	17	16	18	$y_{3\cdot}$ 102	$\bar{y}_{3\cdot}$ 17.00
20	19	25	22	23	18	20	$y_{4\cdot}$ 127	$\bar{y}_{4\cdot}$ 21.17
							383	15.96
						y_{46} y_{an}	$y_{\cdot\cdot}$	$\bar{y}_{\cdot\cdot}$

The ANOVA Identity:

$$\boxed{SS_T = SS_{Tr} + SS_E}$$

SS_T : total sum of squares

$$\sum_i \sum_j \left(y_{ij} - \bar{y}_{\cdot\cdot}\right)^2$$

∴ measures total variability of data

∴ $an-1$ degrees of freedom

SS_{Tr} : treatment sum of squares

$$n \sum_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2$$

∴ measures variability due to differences between treatments

∴ $a-1$ d.o.f.

SS_E : error sum of squares

$$\sum_i \sum_j (y_{ij} - \bar{y}_{i\cdot})^2$$

∴ variability due to random error

∴ $a(n-1)$ d.o.f.

Computational formulae :

$$SS_T = \sum\sum y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$SS_{Tr} = \sum \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{N}$$

∴ by ANOVA identity :

$$\underbrace{SS_E = SS_T - SS_{Tr}}$$

∴ compute mean squares from sum-of-squares :

mean square of treatments :

$$MS_{Tr} = \frac{SS_{Tr}}{d.o.f.(Tr)} = \underbrace{\frac{SS_{Tr}}{a-1}}$$

mean square of error :

$$MS_E = \frac{SS_E}{d.o.f.(E)} = \underbrace{\frac{SS_E}{a(n-1)}}$$

test statistic for $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$

$$f_0 = \frac{MS_{Tr}}{MS_E}$$

critical value : $f_{\alpha, \underbrace{a-1}_{\text{num. degrees of freedom}}, \underbrace{a(n-1)}_{\text{den. degrees of freedom}}}$

reject H_0 if $f_0 > f_{critical}$

∴ ANOVA table summarizes these numbers:

Source	SS	d.o.f.	MS	f_0
Treatments	SS_{Tr}	$a-1$	MS_{Tr}	
Error	SS_E	$a(n-1)$	MS_E	
Total	SS_T	$an-1$	—	

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$$SS_T = \sum\sum y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$= 7^2 + 8^2 + 15^2 + \dots\dots - \frac{383^2}{4 \cdot 6}$$

$$= 512.96$$

d.o.f. : $an - 1$

$4 \cdot 6 - 1 = 23$

$$SS_{Tr} = \sum \frac{y_{i.}^2}{n} - \frac{y_{..}^2}{N}$$

$$= \frac{60^2 + 94^2 + 102^2 + 127^2}{6} - \frac{383^2}{24}$$

$$= 382.79$$

d.o.f. $= a - 1 = 3$

$$SS_E = SS_T - SS_{Tr} = 512.96 - 382.79$$

$$= 130.17$$

d.o.f. :
$a(n-1)$
$= 4(6-1) = 20$

$$MS_{Tr} = \frac{SS_{Tr}}{a-1} = \frac{382.79}{3} = 127.6$$

$$MS_E = \frac{SS_E}{a(n-1)} = \frac{130.17}{20} = 6.51$$

$$f_0 = \frac{MS_{Tr}}{MS_E} = \frac{127.6}{6.51} = 19.6$$

apparently!

$$f_{critical} = f_{.01,3,20} = 4.94$$

$$f_0 \gg f_{critical}$$

Strongly reject H_0 @ $\alpha = 0.01$

$\therefore$ treatment means of hardwood concentration are <u>not</u> equal! At least <u>one</u> pair of concentrations must differ! But which one?