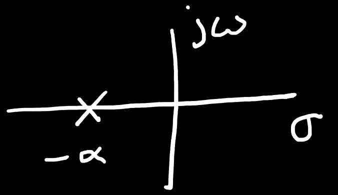
Frequency Response

If $H(s)$ has all its poles in the LHP, then the frequency response

$$H(j\omega) = H(s)\Big|_{s=j\omega}$$

EX $$H(s) = \frac{s^2+2s+2}{s^2+4s+13} = \frac{(s+1)^2+1^2}{(s+2)^2+3^2} \qquad H(j\omega) = \frac{-\omega^2+2j\omega+2}{-\omega^2+4j\omega+13} = \frac{(2-\omega^2)+j2\omega}{(13-\omega^2)+j4\omega}$$

CASE 1: Nonrepeated real poles



If $\alpha > 0$, $h(t) = e^{-\alpha t}u(t) \Rightarrow$ STABLE

If $\alpha = 0$, $h(t) = u(t) \Rightarrow$ MARGINALLY STABLE

In a marginally stable system, bounded i/ps may produce bounded or unbounded o/ps

EX Marginally stable system w/ $h(t) = u(t)$

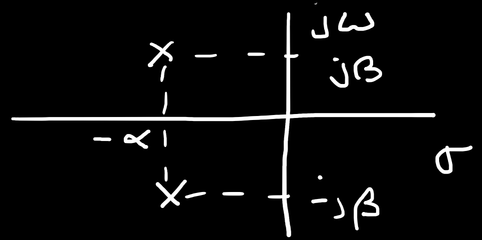
Input		Output	
(i) $x(t) = u(t)$	bounded	$y(t) = r(t)$	unbounded
(ii) $x(t) = \cos t\, u(t)$	bounded	$y(t) = \sin t\, u(t)$	

CASE 2: Repeated real poles

If $\alpha > 0$, $h(t) = te^{-\alpha t}u(t) \Rightarrow$ STABLE

If $\alpha < 0$, $h(t) = r(t) \Rightarrow$ UNSTABLE

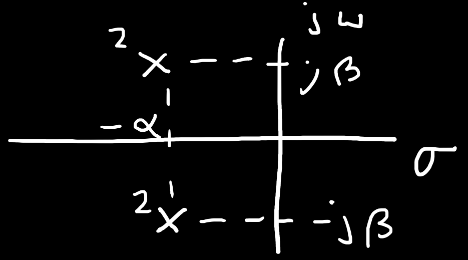
CASE 3: Nonrepeated quadratic poles



If $\alpha > 0$, $h(t) = e^{-\alpha t} \sin\beta t \, u(t) \Rightarrow$ STABLE

If $\alpha = 0$, $h(t) = \sin\beta t \, u(t) \Rightarrow$ MARGINALLY STABLE

CASE 4: Repeated quadratic poles



If $\alpha > 0$, $h(t) = [\sin\beta t - \beta t\cos\beta t]\, e^{-\alpha t} u(t) \Rightarrow$ STABLE

If $\alpha = 0$, $h(t) = [\sin\beta t - \beta t\cos\beta t]\, u(t) \Rightarrow$ UNSTABLE

✓ The position of the poles w.r.t. the σ-axis determines the frequency of oscillation of $h(t)$

✓ The position of the poles w.r.t. the $j\omega$-axis determines the damping factor

✓ For a stable system, $H(s)$ must satisfy

1) No poles in RHP

2) No repeated poles on the $j\omega$-axis

WHY? 3) Degree of $P(s) \neq$ Degree of $Q(s)$

If $Q(s)$ not in factored form, how do you determine stability?

1. Factor $Q(s)$ & find poles (MATLAB, computer)
2. If any of the coefficients of $Q(s)$ $a_0, a_1, \ldots, a_n$ are negative, $Q(s)$ has RHP roots
3. If any coefficients of $Q(s) = 0$, $Q(s)$ has roots in RHP or on $j\omega$-axis
4. If all coefficients of $Q(s)$ are positive, apply Routh-Hurwitz

Frequency Response

What if the i/p is a single freq., $x(t) = e^{j\omega t}$?

$$y(t) = x(t) * h(t) = e^{j\omega t} * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} h(\tau)\,d\tau$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\,d\tau \qquad \Rightarrow y(t) = e^{j\omega t} H(j\omega)$$

- ✓ observe: $y(t)$ contains $x(t) = e^{j\omega t}$, but weighted by $H(j\omega)$
- ✓ $H(j\omega)$: Response of the LTIC system due to input, $x(t) = e^{j\omega t}$
- ✓ $H(j\omega)$ can be evaluated at any value of ω
- ✓ $H(j\omega)$ is known as the <u>frequency</u> <u>response</u> of the system

← polar form $H(j\omega) = |H(j\omega)|\, e^{j\underline{\angle H(j\omega)}}$ a complex # - function of ω

<u>Input</u>		<u>Output</u>
$e^{j\omega t}$	$\Rightarrow$	$e^{j\omega t} H(j\omega)$
$\cos\omega t$	$\Rightarrow$	$Re[e^{j\omega t} H(j\omega)]$
$\cos\omega t$	$\Rightarrow$	$\lvert H(j\omega)\rvert \cos[\omega t + \angle H(j\omega)]$
$\cos(\omega t + \theta)$	$\Rightarrow$	$\lvert H(j\omega)\rvert \cos[\omega t + \theta + \angle H(j\omega)]$

Sinusoidal i/p of radian freq. ω, system response also sinusoid of same freq. ω

amplitude of o/p = $|H(j\omega)| \times$ i/p amplitude

phase " " = i/p phase shifted by $\angle H(j\omega)$

Ex $|H(j10)| = 3$, $\angle H(j10) = -30$

amplifies a sinusoid of freq $\omega = 10$ by factor of 3

delays " " " " " " $30°$

For $x(t) = 5\cos(10t + 50°)$, system response $= (3 \times 5)\cos(10t + 50 - 30)$

$= 15\cos(10t + 20°)$

$|H(j\omega)|$ = amplitude gain

Plot $|H(j\omega)|$ vs. ω = amplitude response or magnitude response

" $\angle H(j\omega)$ vs. ω = phase response

Magnitude and phase responses together $\Rightarrow$ frequency response

Caution: Frequency response valid only for stable systems

" " meaningless for unstable systems

Ex 4.23

Find the frequency response of a system whose t.f. is $H(s) = \frac{s+0.1}{s+5}$

$H(j\omega) = \frac{j\omega + 0.1}{j\omega + 5}$, $|H(j\omega)| = \frac{\sqrt{\omega^2 + 0.01}}{\sqrt{\omega^2 + 25}}$ and $\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$

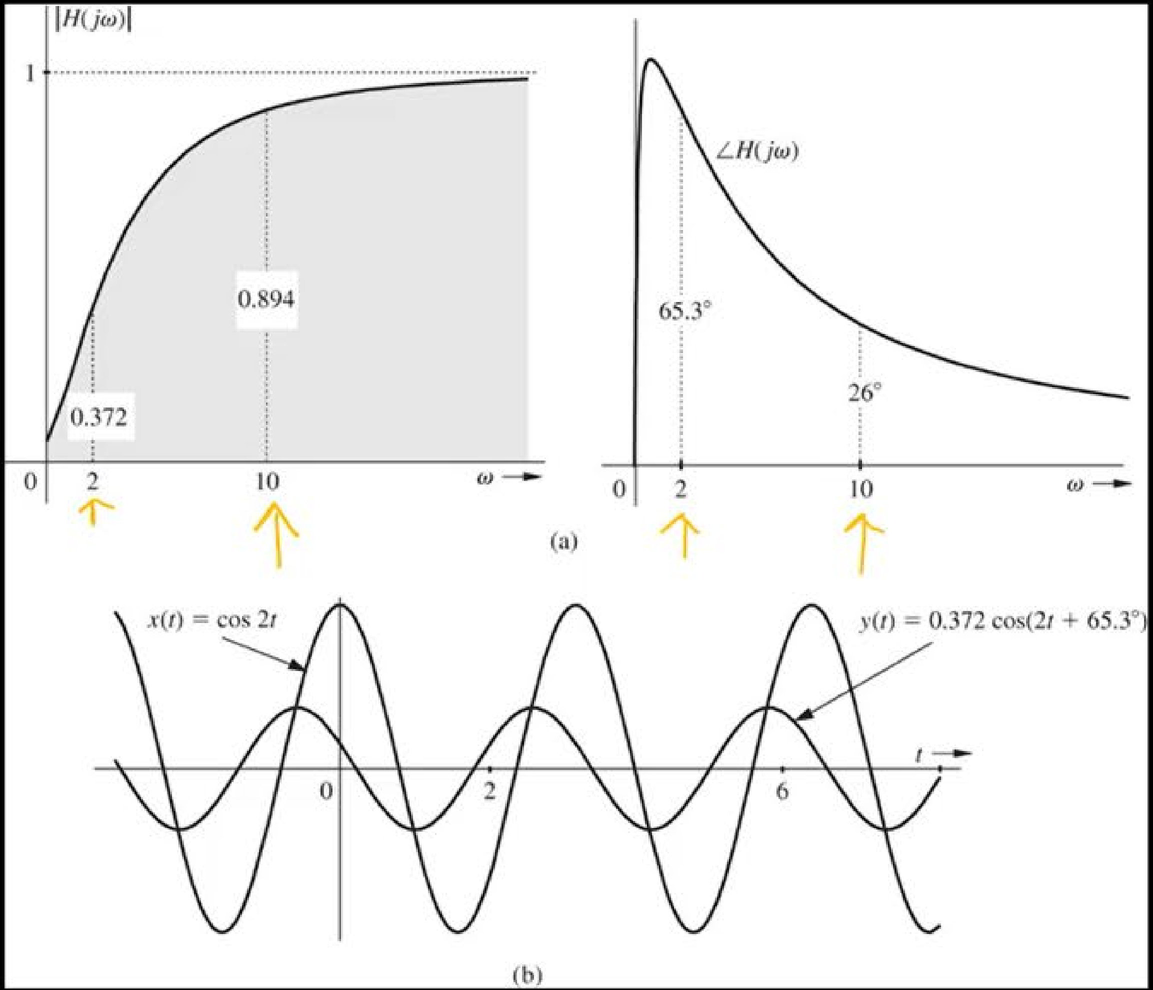
For input $x(t) = \cos 2t$, $\omega = 2$

$$|H(j2)| = \frac{\sqrt{(2)^2 + 0.01}}{\sqrt{(2)^2 + 25}} = 0.372$$

$$\angle H(j2) = \tan^{-1}\left(\frac{2}{0.1}\right) - \tan^{-1}\left(\frac{2}{5}\right)$$
$$= 87.1^\circ - 21.8^\circ = 65.3^\circ$$

$$\Rightarrow y(t) = 0.372 \cos(2t + 65.3^\circ)$$

Read directly off graph



For input $\cos(10t - 50^\circ)$, read directly from plots

$|H(j10)| = 0.894$ and $\angle H(j\omega) = 26^\circ$

$$\Rightarrow y(t) = 0.894 \cos(10t - 50^\circ + 26^\circ)$$
$$= 0.894 \cos(10t - 24^\circ)$$

Use MATLAB to generate plots following methods discussed on bottom of p. 415

Ex 4.28

a) Ideal delay of T seconds, w/ t.f. $H(s) = e^{-sT}$

$\Rightarrow H(j\omega) = e^{-j\omega T}$

$|H(j\omega)| = 1$

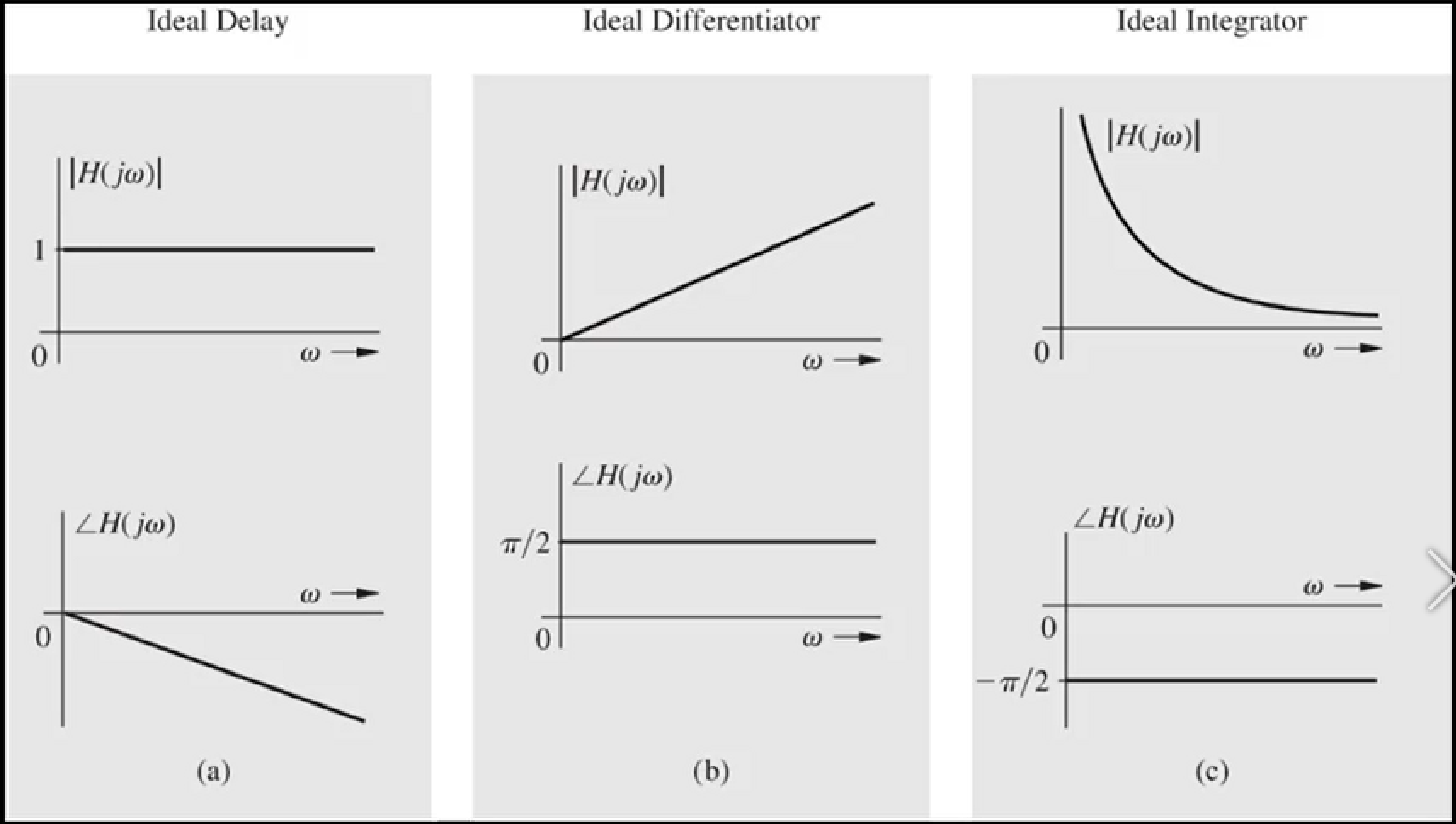
$\angle H(j\omega) = -\omega T$

b) Ideal differentiator, w/ t.f. $H(s) = s$

$\Rightarrow H(j\omega) = j\omega = e^{j\frac{\pi}{2}}$

$|H(j\omega)| = \omega$

$\angle H(j\omega) = \frac{\pi}{2}$



c) Ideal integrator, w/ t.f. $H(s) = \frac{1}{s}$

$\Rightarrow H(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega} = \frac{1}{\omega} e^{-j\frac{\pi}{2}} \Rightarrow |H(j\omega)| = \frac{1}{\omega}$ and $\angle H(j\omega) = -\frac{\pi}{2}$

Steady-State Response to Causal Sinusoidal Inputs

Consider $x(t) = e^{j\omega t}u(t)$ which starts at $t=0$; $X(s) = \frac{1}{s+j\omega}$

$$H(s) = \frac{P(s)}{Q(s)} \Longrightarrow Y(s) = X(s)\,H(s) = \frac{P(s)}{\underbrace{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}_{Q(s)}\;(s-j\omega)}$$

In PFE, let $k_1, k_2, \ldots k_n$ be coefficients corresponding to $Q(s)$ terms

Coefficient corresponding to last term $(s-j\omega)$ is

$$\left.\frac{P(s)}{Q(s)}\right|_{s=j\omega} = H(j\omega) \Longrightarrow Y(s) = \sum_{i=1}^{n} \frac{k_i}{s-\lambda_i} + \frac{H(j\omega)}{s-j\omega}$$

$$\Longrightarrow y(t) = \underbrace{\sum_{i=1}^{n} k_i e^{\lambda_i t}u(t)}_{\text{transient}} + \underbrace{H(j\omega)\,e^{j\omega t}u(t)}_{\text{steady state}}$$

$$y_{ss}(t) = |H(j\omega)| \cos\left[\omega t + \angle H(j\omega)\right] u(t)$$