Confidence Intervals

- consider some random variable X
  - we know the point estimator for population mean μ is

    $$\hat{\mu} = \overline{X}$$

  - so we can take a sample and calculate some point estimate, $\bar{x}$

    ↑ lower-case, actual #!

- how "good" is the point estimate?
  - in other words, how "reasonable" is this "reasonable value" for the unknown μ?

- if $\bar{x} = 1{,}000$, is μ likely to be between

  900 and 1100?
  990 and 1010?
  0 and a bajillion?

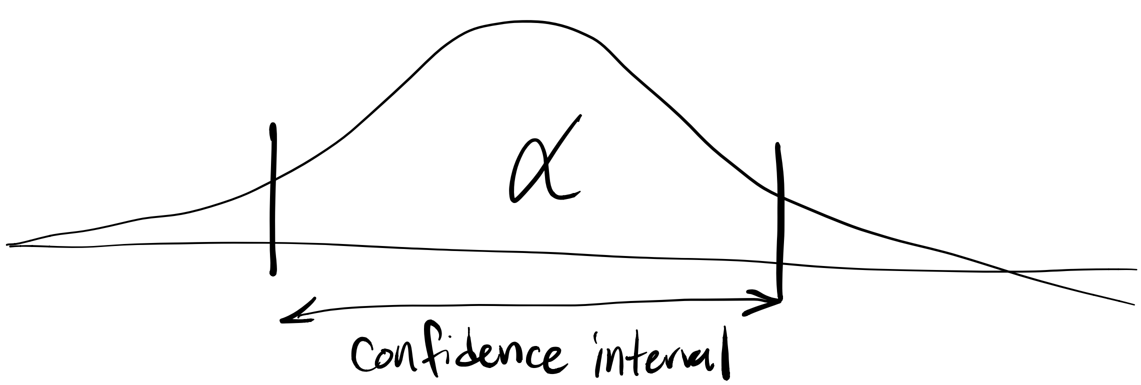
∴ we need an interval estimate that uses sample data to come up with a "likely range" for μ

Confidence Interval on Mean, Known σ^2

← population variance

∴ we know from the Central Limit Theorem that $\bar{x}$ is normally distributed w/ mean μ, if n is sufficiently large

∴ so let's come up with an interval that corresponds to a probability under the standard normal distribution



.. how do we compute a confidence interval for an unknown μ, based on a probability α? How do we select α?

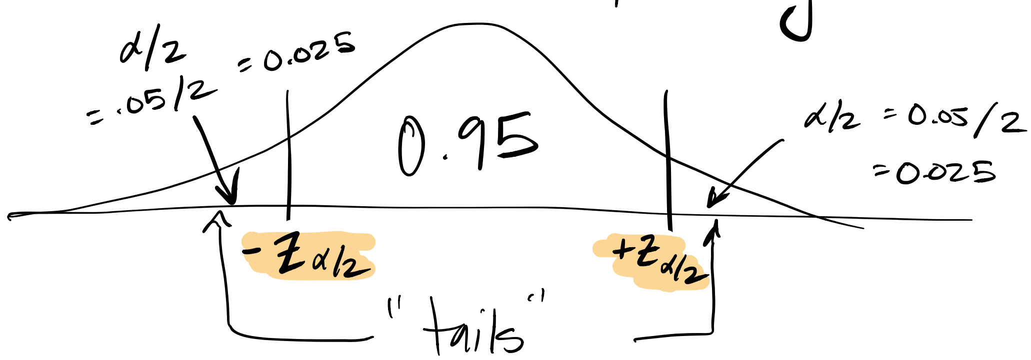
.. define level of confidence:

$$\text{conf.} = (1-\alpha) \times 100\%$$

.. for example, if we want to be 95% confident that the interval contains μ

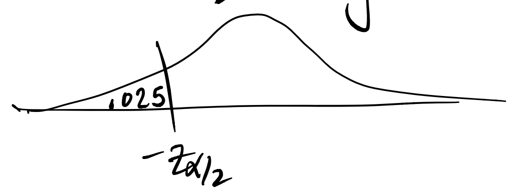
then $\alpha = 0.05$

.. we can come up with z-values that correspond to this probability



∴ Can we determine these z-values? Yes!

∴ z-table is cumulative; easy one is $-z_{\alpha/2}$

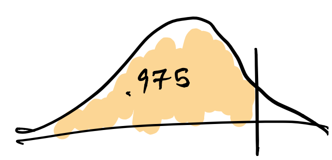


$$P(z < -1.96) = 0.024998$$

close enough! ∴ $-z_{.025} = -1.96$

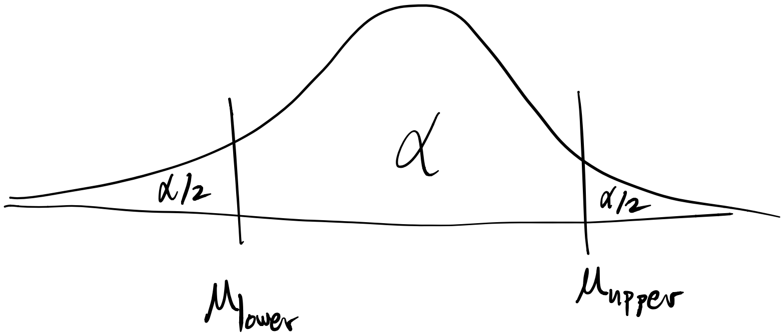
∴ standard normal is a symmetrical probability distribution (Gaussian family)

∴ $+z_{0.025} = +1.96$



∴ we want values of x corresponding to these values of z; then, we'll have an interval that corresponds to our sample data.

∴ simply solve the z-formula for x!



$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

$$\therefore \quad \mu_{upper} = \bar{x} + \frac{+z_{\alpha/2}\,\sigma}{\sqrt{n}}$$

$$\text{and} \quad \mu_{lower} = \bar{x} + \frac{-z_{\alpha/2}\,\sigma}{\sqrt{n}}$$

finally, a $(1-\alpha) \times 100\%$ Confidence interval on μ, given σ, is:

$$\boxed{\bar{x} - \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}}$$

shorthand notation:

$$\boxed{\mu : \bar{x} \pm \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}}$$

note: final answer is always in interval form, with unit.

example: $n = 10$ impact measurement tests performed on steel bars

results: $\bar{x} = 64.46$ J

joules

∴ known population standard deviation

$\sigma = 1$ J

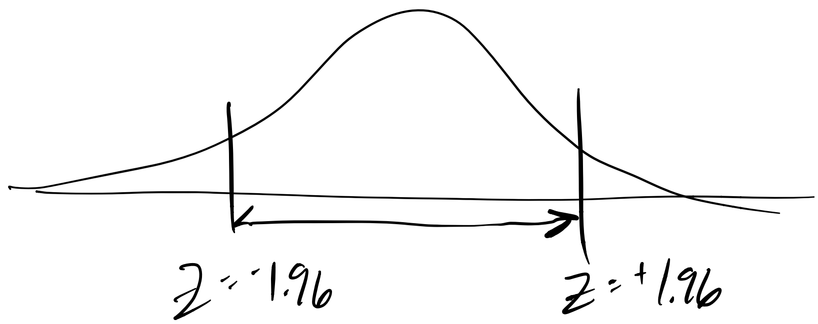
[if we don't know μ, how do we know σ^2?!
(more later)]

∴ Write a 95% confidence interval on μ.

common abbreviation!

95% C.I. ⟶ $\alpha = 0.05$

$$\therefore Z_{\alpha/2} = Z_{.05/2} = Z_{.025} = 1.96$$ ✓



$$\mu : \bar{x} \pm \frac{z_{\alpha/2}\sigma}{\sqrt{n}}$$

$$64.46 \pm \frac{1.96 \cdot 1}{\sqrt{10}}$$

$$64.46 - \frac{1.96 \cdot 1}{\sqrt{10}} < \mu < 64.46 + \frac{1.96 \cdot 1}{\sqrt{10}}$$

$$63.84 < \mu < 65.08 \quad (J)$$

∴ useful info !!!!

high n and low σ

∴ this is a nice, narrow range; good for problem solving interpretation: it is tempting to conclude that there is a 95% probability that the true mean μ is within the range 63.84 to 65.08 J.

FALSE.

∴ We will <u>never</u> know μ. Ever. EVER

∴ instead, we say we are 95% <u>confident</u>.

real interpretation: if an infinite # of samples is taken from the population, each of size n, <u>95% of them will correctly contain μ!</u>

Fig. 8-1 p. 255 (6th)

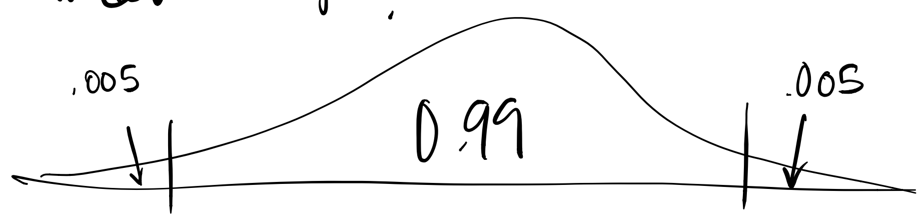
∴ how about a 99% C.I.?

∴ need $z_{\alpha/2} = z_{.01/2} = z_{.005} = 2.58$

$$\mu: \quad \bar{x} \pm \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}} \longrightarrow 64.46 \pm \frac{2.58 \cdot 1}{\sqrt{10}}$$

$$63.64 < \mu < 65.28 \quad (J)$$

∴ wider range!



∴ **but** we're more confident!

∴ **but** wider range is **worse** for problem solving!!!

$\alpha = 0.05$ is a good compromise and most widely used!

∴ What if you **need** a narrower interval @ some confidence level?

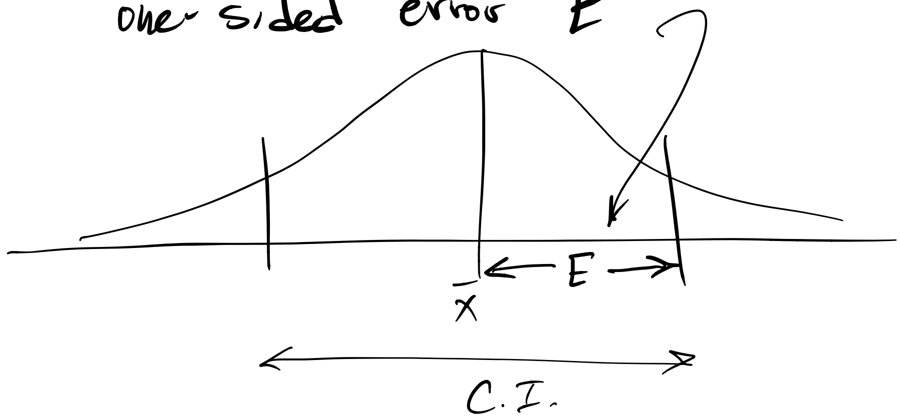
∴ What's the only thing left we can change?

∴ increase sample size

$$\bar{x} \pm \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}$$

$\uparrow$

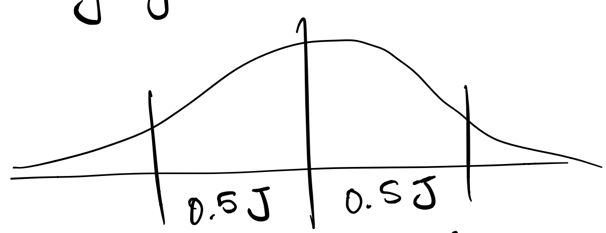
∴ sample size needed for a desired "one-sided error E"



$$n \geq \left(\frac{z_{\alpha/2}\,\sigma}{E}\right)^2 \quad \text{(round up)}$$

∴ impact measurement test example:

∴ say you want interval of ± 0.5 J



then $E = 0.5$

$$n = \left(\frac{z_{.025}\cdot 1}{E}\right)^2 = \left(\frac{1.96\cdot 1}{0.5}\right)^2 = 15.37$$

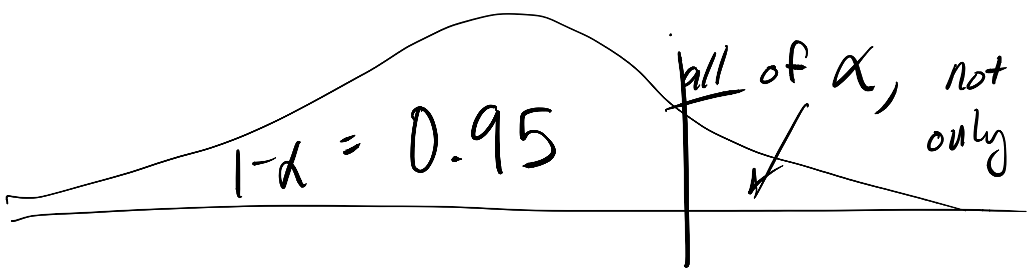
∴ use $n = 16$ tests

Confidence Bounds

∴ for some engineering problems, don't need a two-sided interval

upper one-sided $(1-\alpha) \times 100\%$ confidence bound:
use upper value of C.I., substitue

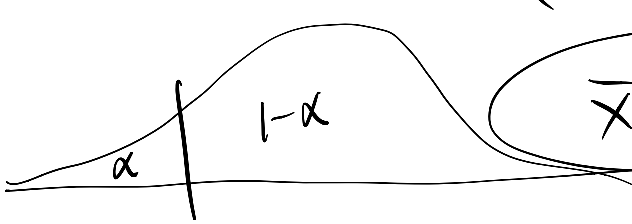
α for $\alpha/2$!!!



all of α, not $\alpha/2$, only one tail!

$$\therefore \; \mu < \bar{x} + Z_\alpha \sigma/\sqrt{n}$$

lower one-sided $(1-\alpha) \times 100\%$ confidence bound:



$$\bar{x} - Z_\alpha \sigma/\sqrt{n}$$

∴ formula sheet will only have two-sided C.I.'s!

ex: impact measurement, write lower one-sided 95% confidence bound

∴ need $Z_\alpha = Z_{.05} \cong 1.64$

∴ $\bar{x} - Z_\alpha \sigma/\sqrt{n} < \mu$

$64.46 - 1.64 \cdot 1/\sqrt{10} < \mu$

$63.94 < \mu$ (J)