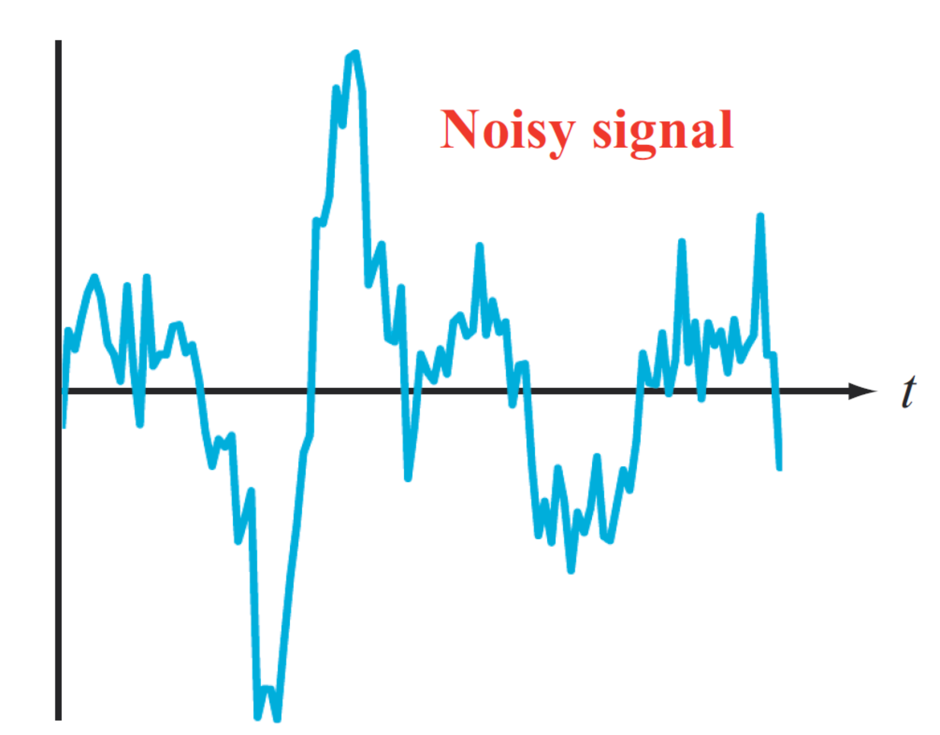
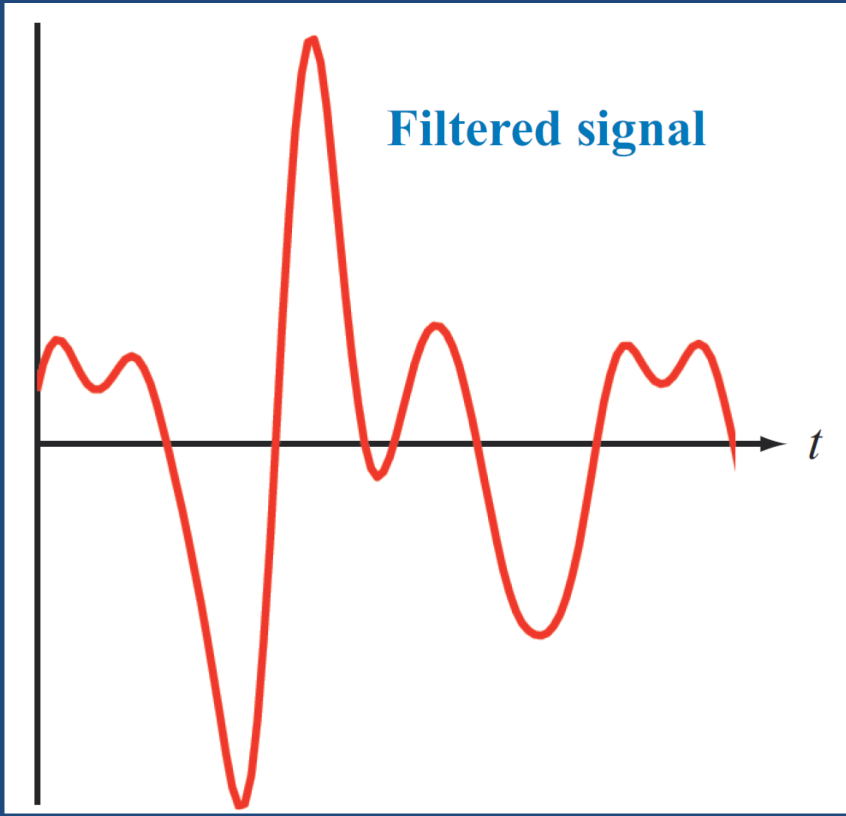




6. APPLICATIONS OF THE FOURIER TRANSFORM

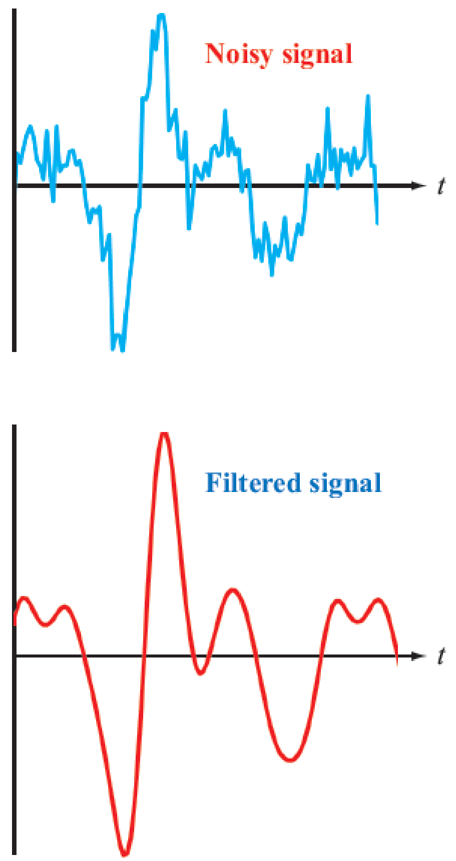
6 Applications of the Fourier Transform

Contents

Objectives

Learn to:

- Design lowpass, bandpass, highpass, bandreject, notch, comb, Butterworth, and resonator filters to remove noise or unwanted interference from signals.
- Compute the spectra of modulated signals.
- Compute the sampling rates necessary to avoid aliasing.



Noise filtering, ***modulation***, frequency division multiplexing, ***signal sampling***, and many related topics are among those treated in this chapter. These are examples of applications that rely on the properties of the ***Fourier transform*** introduced in Chapter 5.

Filtering and Signal Spectra

$$\mathbf{X}(\omega) \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ \mathbf{H}(\omega)\end{array}} \longrightarrow \mathbf{Y}(\omega)$$

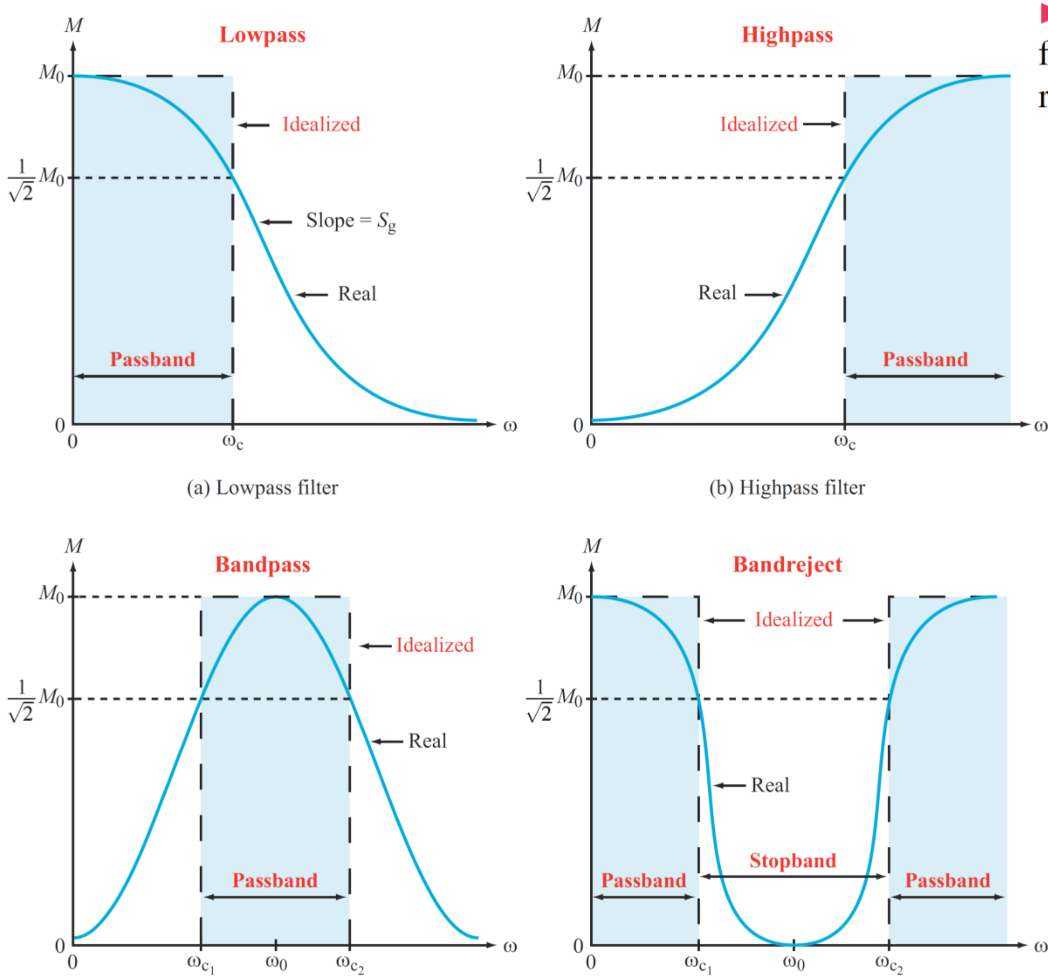
Filter

$$\mathbf{H}(\omega) = \frac{\mathbf{Y}(\omega)}{\mathbf{X}(\omega)}$$

$$\mathbf{H}(\omega) = M(\omega)\, e^{j\phi(\omega)}$$

$$M(\omega) = |\mathbf{H}(\omega)| \quad \text{and} \quad \phi(\omega) = \tan^{-1}\left\{\frac{\mathfrak{Im}[\mathbf{H}(\omega)]}{\mathfrak{Re}[\mathbf{H}(\omega)]}\right\}$$

Four Types of Filters



(a) Lowpass filter

(b) Highpass filter

(c) Bandpass filter

(d) Bandreject filter

▶ The corner frequency ω_c is defined as the angular frequency at which $M(\omega)$ is equal to $1/\sqrt{2}$ of the reference peak value:

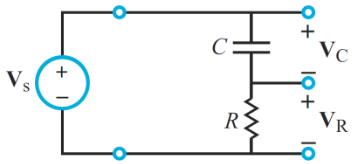
$$M(\omega_c) = \frac{M_0}{\sqrt{2}} = 0.707 M_0. \qquad (6.4)$$

The filter *passband is the range of ω over which the filter passes the input signal:*

- $0 \le \omega < \omega_c$, for lowpass filter,
- $\omega > \omega_c$, for highpass filter,
- $\omega_{c_1} < \omega < \omega_{c_2}$, for bandpass filter,
- $\omega < \omega_{c_1}$ and $\omega > \omega_{c_2}$, for bandreject filter. (6.6)

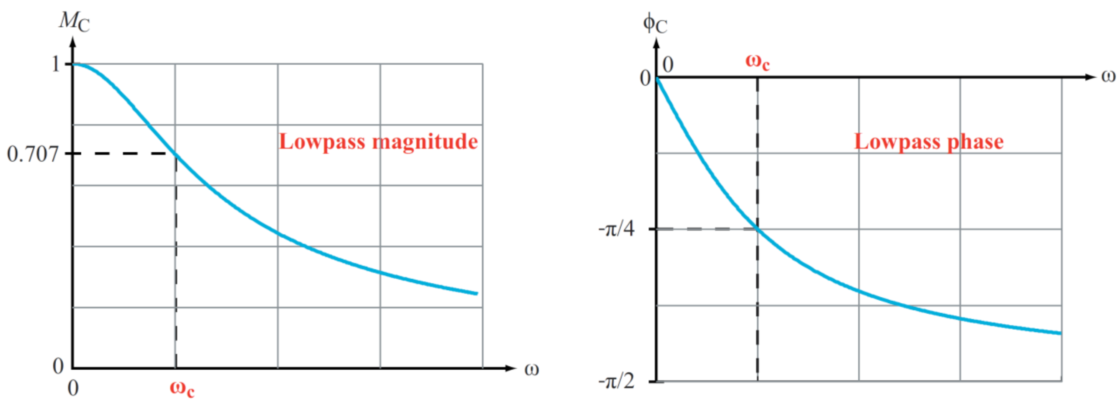
The ***stopband*** of the bandreject filter extends from ω_{c_1} to ω_{c_2}

RC Circuit Lowpass and Highpass Filters



(a) RC circuit

$$\mathbf{H}_C(\omega) = \frac{\mathbf{V}_C}{\mathbf{V}_s} = \frac{1}{1 + j\omega RC}$$

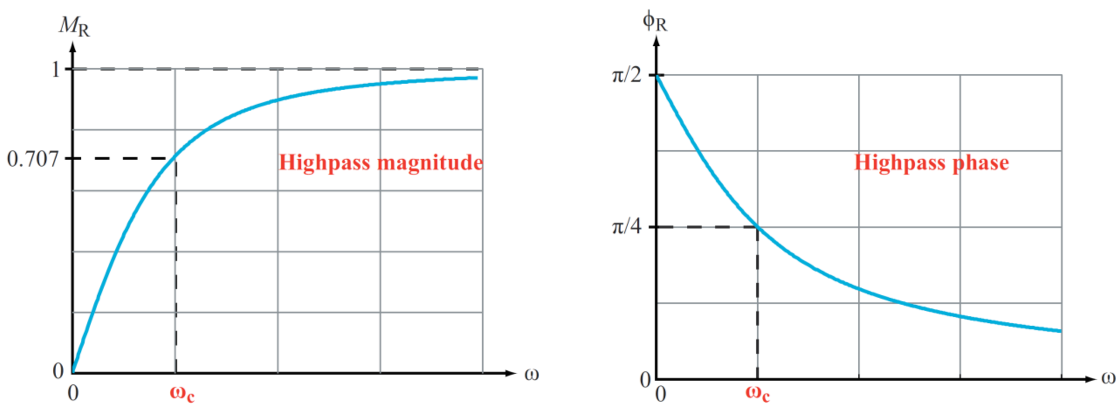


(b) Magnitude and phase angle of $\hat{\mathbf{H}}_C(w) = \mathbf{V}_C / \mathbf{V}_s$

$$M_C(\omega) = |\mathbf{H}_C(\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\phi_C(\omega) = -\tan^{-1}(\omega RC)$$

$$\mathbf{H}_R(\omega) = \frac{\mathbf{V}_R}{\mathbf{V}_s} = \frac{j\omega RC}{1 + j\omega RC}$$



(c) Magnitude and phase angle of $\hat{\mathbf{H}}_R(\omega) = \mathbf{V}_R / \mathbf{V}_s$

$$M_R(\omega) = |\mathbf{H}_R(\omega)| = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$

$$\phi_R(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega RC).$$

Decibels

M	M [dB]
10^N	$20N$ dB
10^3	60 dB
100	40 dB
10	20 dB
4	≈ 12 dB
2	≈ 6 dB
1	0 dB
0.5	≈ -6 dB
0.25	≈ -12 dB
0.1	-20 dB
10^{-N}	$-20N$ dB

$$M\ [\text{dB}] = 20 \log M = 20 \log |\mathbf{H}|$$

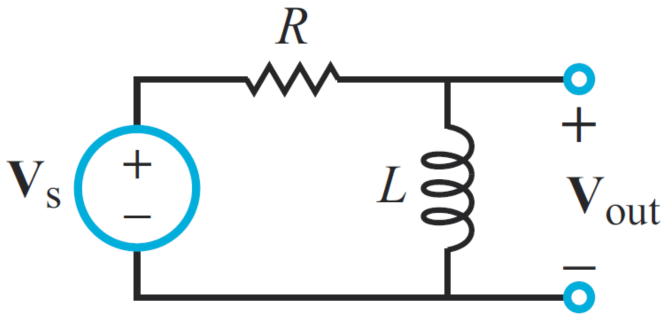
If $G = XY \rightarrow G\ [\text{dB}] = X\ [\text{dB}] + Y\ [\text{dB}]$

If $G = \dfrac{X}{Y} \rightarrow G\ [\text{dB}] = X\ [\text{dB}] - Y\ [\text{dB}]$.

Multiple Poles and Zeros: find frequency response of each pole and zero and add and subtract

Bode Plots for RL Circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{s}}} = \frac{j\omega L}{R + j\omega L} = \frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}$$



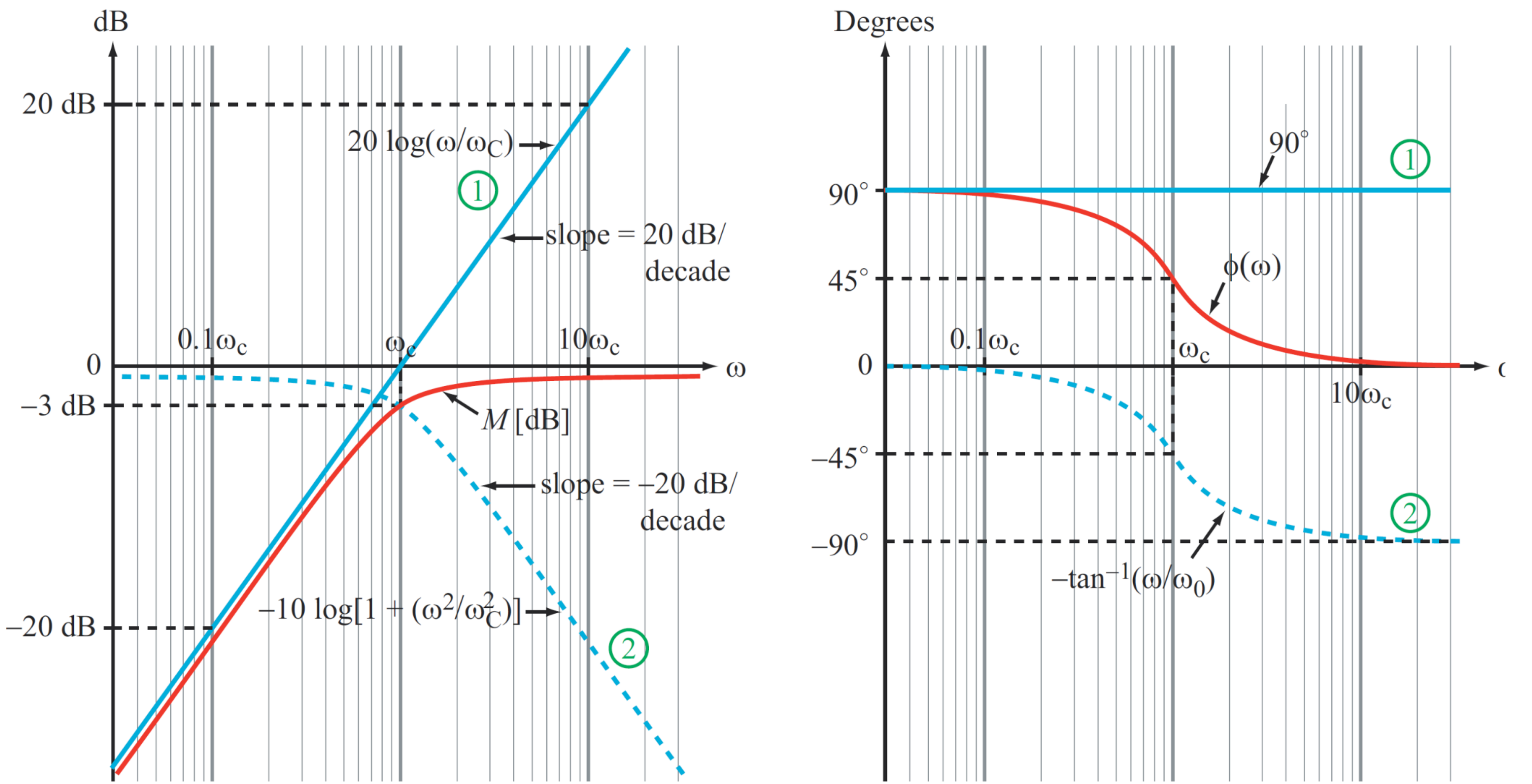
$$M = |\mathbf{H}(\omega)| = \frac{(\omega/\omega_c)}{|1 + j(\omega/\omega_c)|} = \frac{(\omega/\omega_c)}{\sqrt{1 + (\omega/\omega_c)^2}}$$

$$\phi(\omega) = \underbrace{90^\circ}_{1} - \underbrace{\tan^{-1}\left(\frac{\omega}{\omega_c}\right)}_{2}$$

$$\begin{aligned} M\ [\text{dB}] &= 20 \log M \\ &= 20 \log(\omega/\omega_c) - 20 \log[1 + (\omega/\omega_c)^2]^{1/2} \\ &= \underbrace{20 \log(\omega/\omega_c)}_{1} - \underbrace{10 \log[1 + (\omega/\omega_c)^2]}_{2}. \end{aligned}$$

Bode plots are on next slide

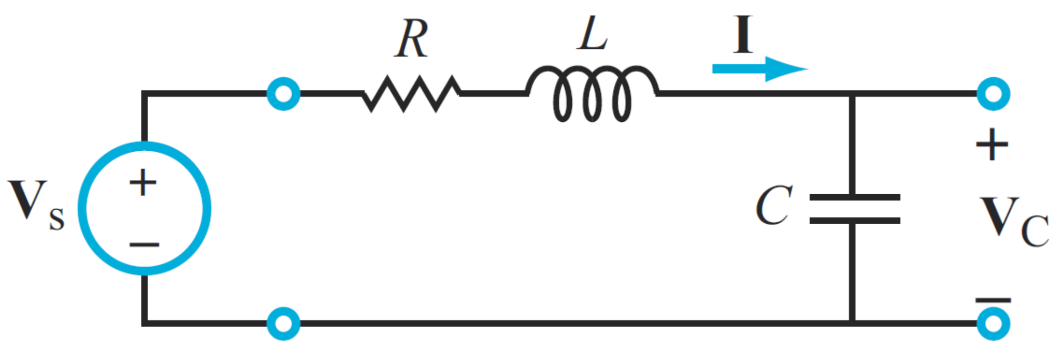
Bode Plots for RL Circuit



(b) Magnitude plot

(c) Phase plot

Bode Plots for Lowpass RLC Circuit

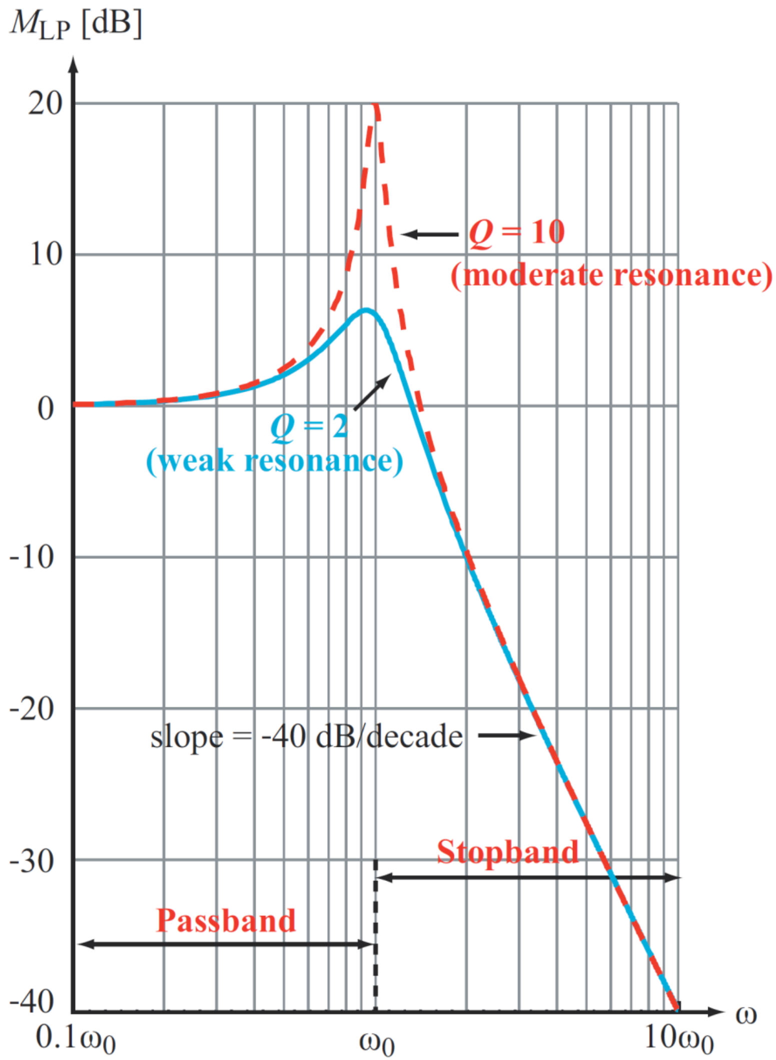


(a) $\mathbf{H}_{LP} = \mathbf{V}_C / \mathbf{V}_s$

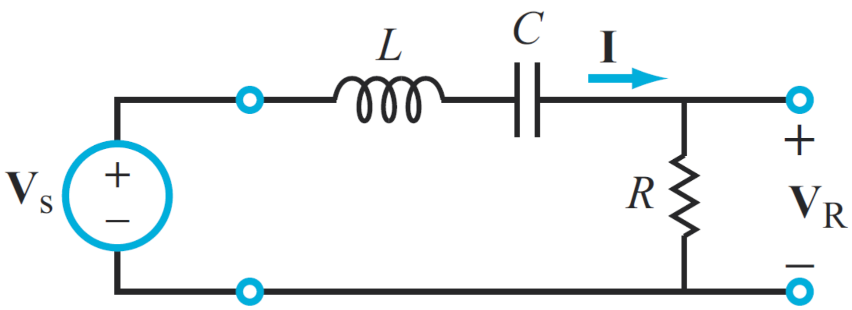
$$\mathbf{H}_{LP}(\omega) = \frac{\mathbf{V}_C}{\mathbf{V}_s} = \frac{(1/j\omega C)\mathbf{I}}{\mathbf{V}_s} = \frac{1}{(1-\omega^2 LC) + j\omega RC}$$

$$M_{LP}(\omega) = \frac{1}{[(1-\omega^2 LC)^2 + \omega^2 R^2 C^2]^{1/2}} \qquad \phi_{LP}(\omega) = -\tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right)$$

Bode Plots for Lowpass RLC Circuit



Bode Plots for Bandpass RLC Circuit

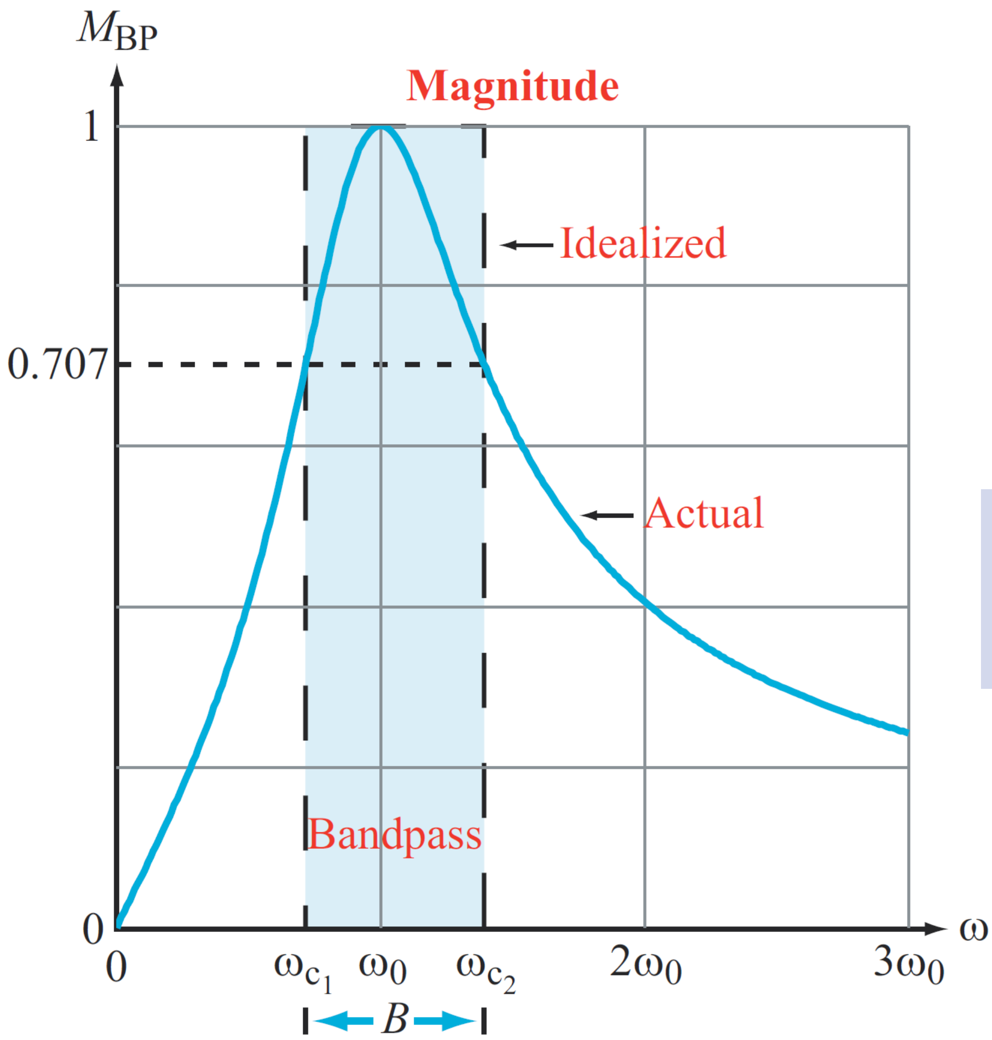


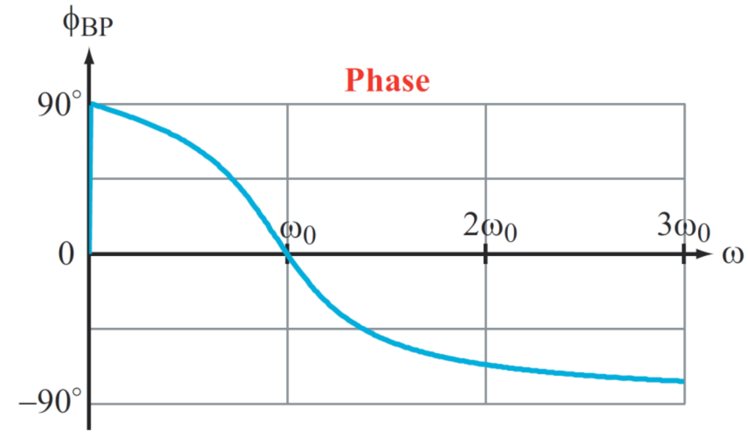
$$\mathbf{I} = \frac{\mathbf{V}_s}{R + j(\omega L - \frac{1}{\omega C})}$$

$$= \frac{j\omega C \mathbf{V}_s}{(1 - \omega^2 LC) + j\omega RC}$$

$$\mathbf{H}_{BP}(\omega) = \frac{\mathbf{V}_R}{\mathbf{V}_s} = \frac{R\mathbf{I}}{\mathbf{V}_s} = \frac{j\omega RC}{(1 - \omega^2 LC) + j\omega RC}$$

Bode Plots for Bandpass RLC Circuit





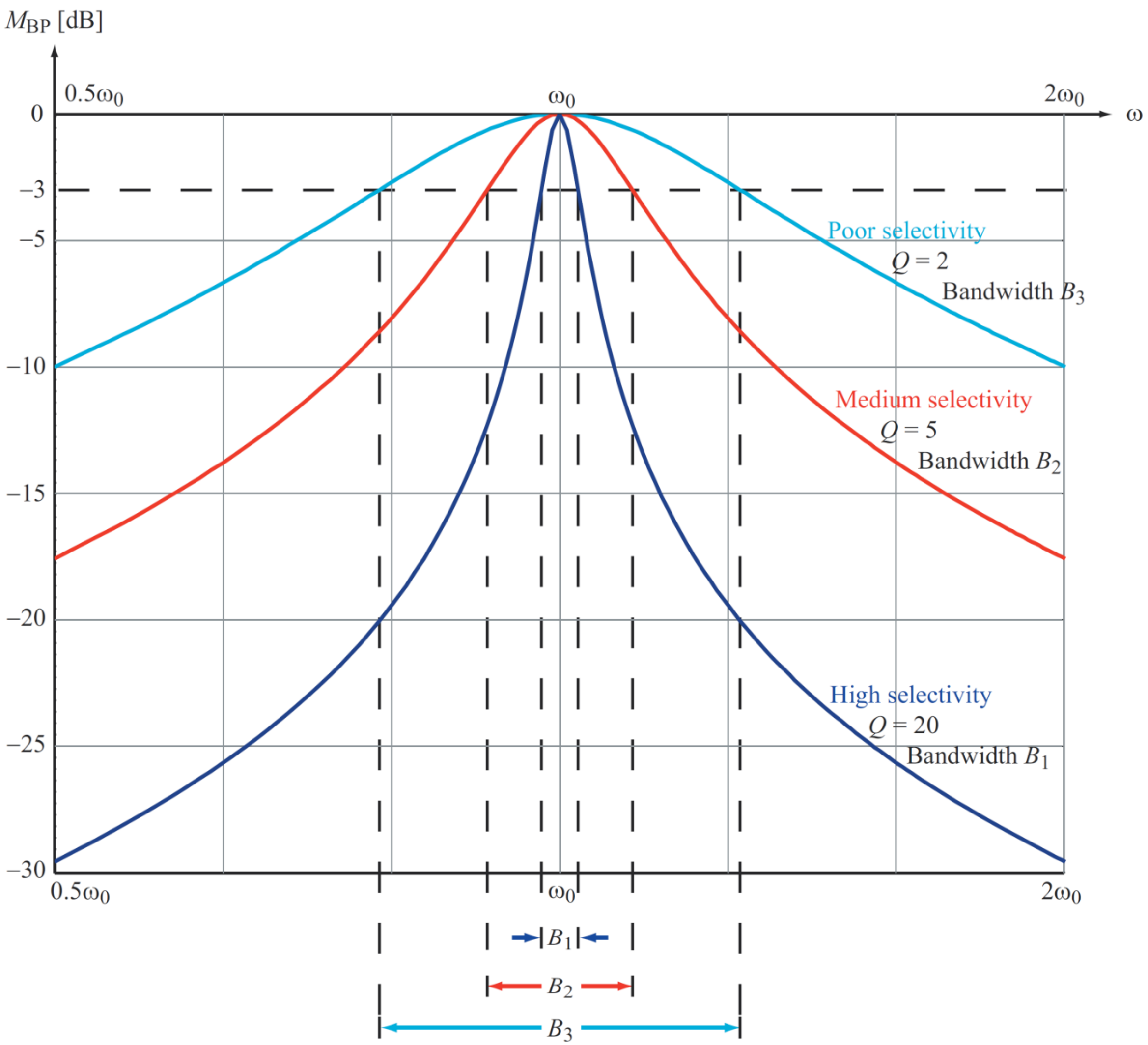
$$\omega_0 = \frac{1}{\sqrt{LC}} .$$

(resonant frequency)

$$B = \omega_{c_2} - \omega_{c_1} = \frac{R}{L}$$

$$Q = \frac{\omega_0}{B} .$$

RLC Circuit: Different Quality Factors



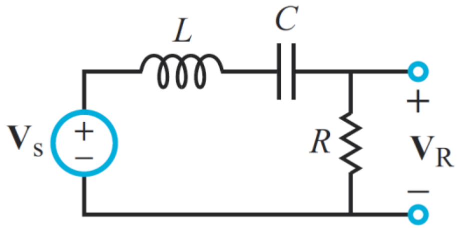
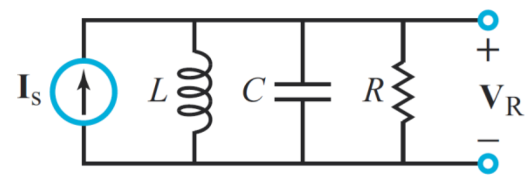
$$\omega_0 = \frac{1}{\sqrt{LC}} .$$

(resonant frequency)

$$B = \omega_{c_2} - \omega_{c_1} = \frac{R}{L}$$

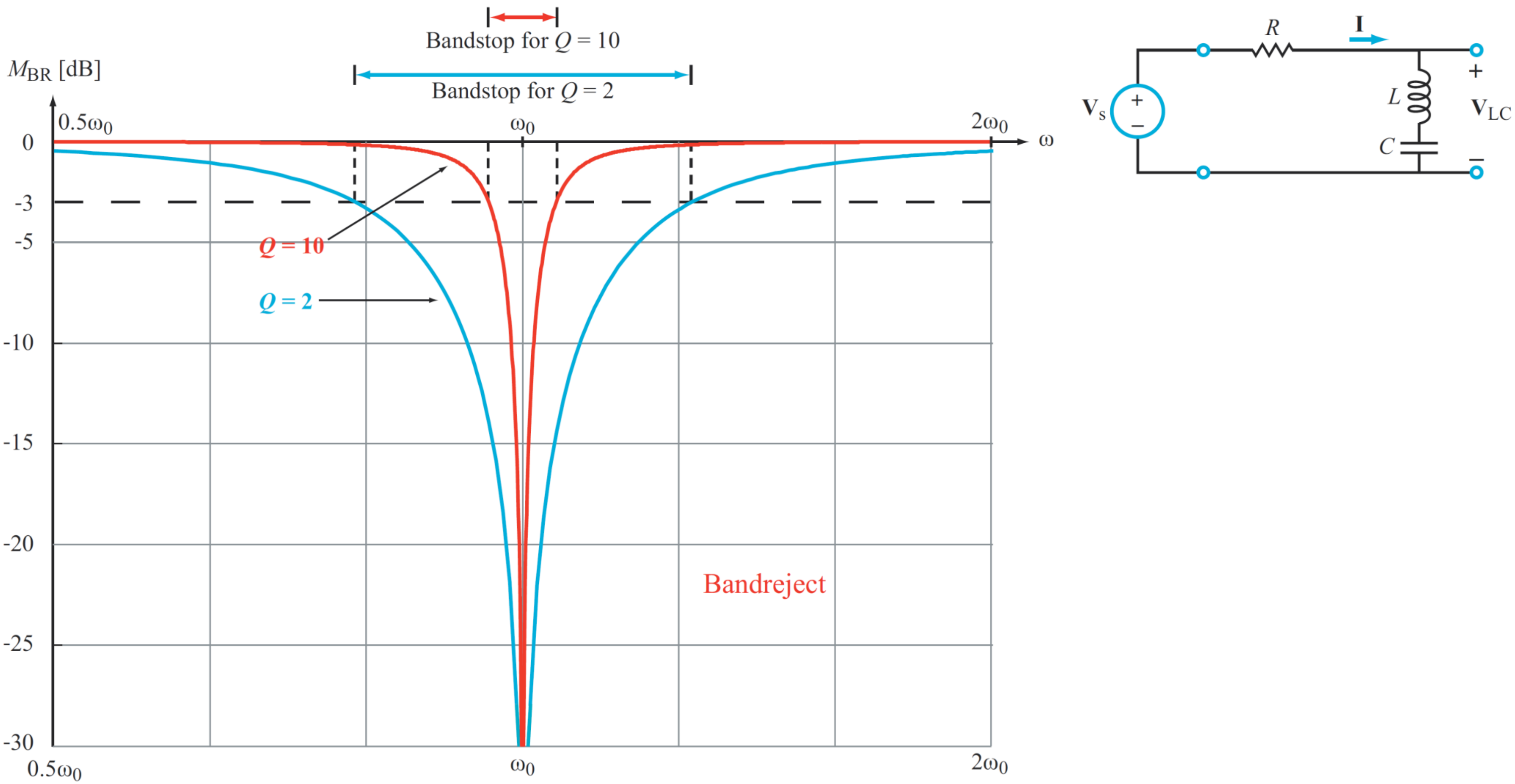
$$Q = \frac{\omega_0}{B} .$$

Series and Parallel RLC Circuits

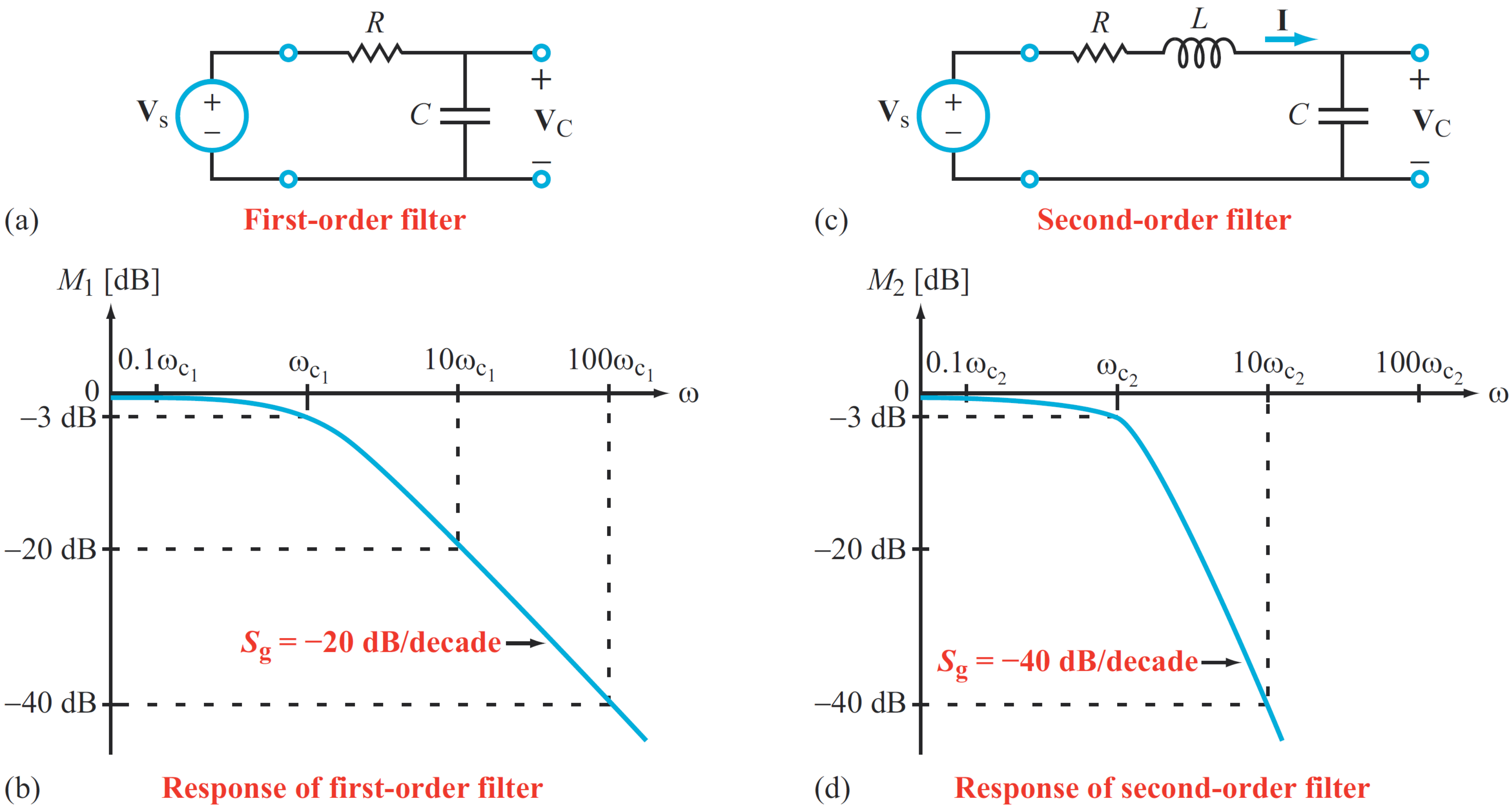
RLC circuit		
Transfer function	$\mathbf{H} = \dfrac{\mathbf{V}_\mathrm{R}}{\mathbf{V}_\mathrm{s}}$	$\mathbf{H} = \dfrac{\mathbf{V}_\mathrm{R}}{\mathbf{I}_\mathrm{s}}$
Resonant frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Bandwidth, B	$\dfrac{R}{L}$	$\dfrac{1}{RC}$
Quality factor, Q	$\dfrac{\omega_0}{B} = \dfrac{\omega_0 L}{R}$	$\dfrac{\omega_0}{B} = \dfrac{R}{\omega_0 L}$
Lower half-power frequency, ω_{c_1}	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$
Upper half-power frequency, ω_{c_2}	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$

Notes: (1) The expression for Q of the series RLC circuit is the inverse of that for Q of the parallel circuit. (2) For $Q \geq 10$, $\omega_{c_1} \simeq \omega_0 - \frac{B}{2}$, $\omega_{c_2} \simeq \omega_0 + \frac{B}{2}$.

Bandreject Filters: Q=2 and Q=10



1st Order and 2nd Order Lowpass



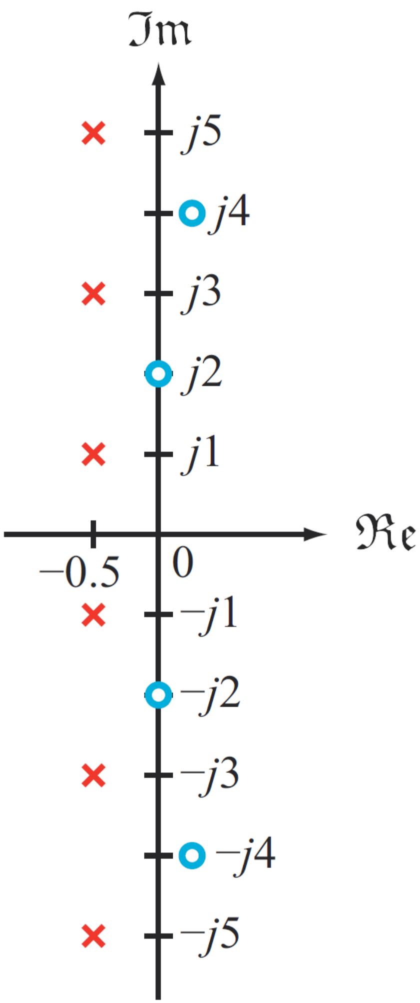
Poles and Zeros and Frequency Response

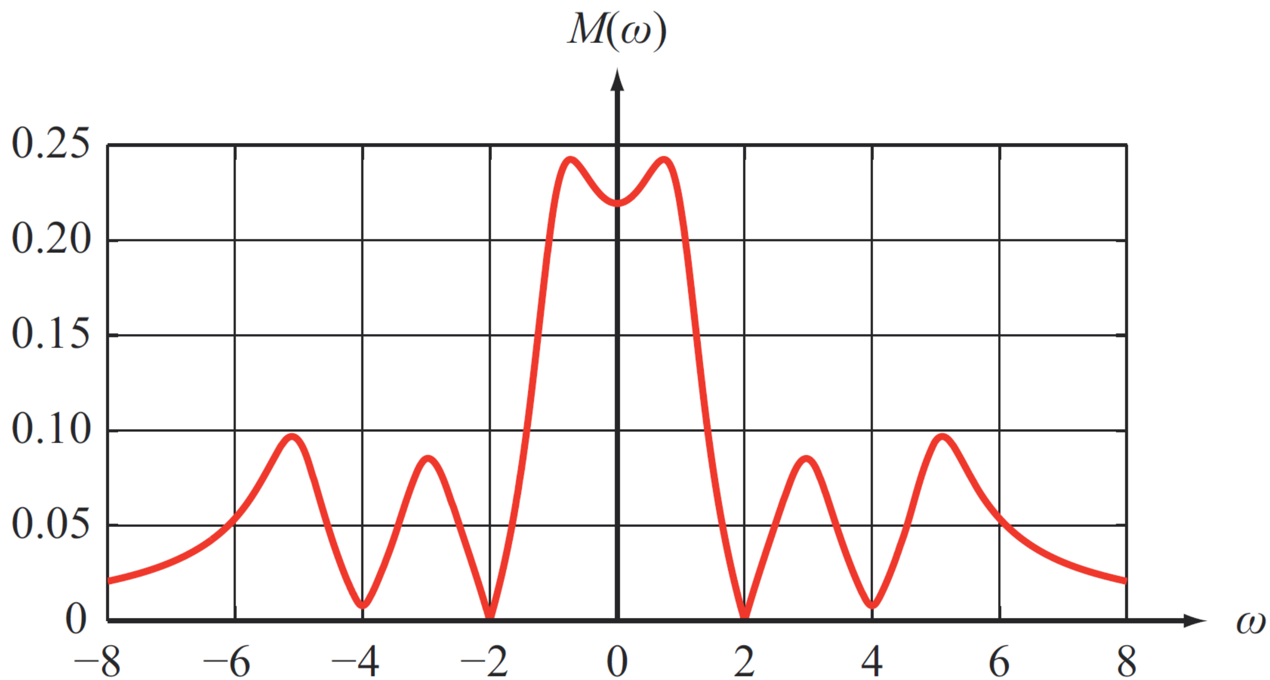
- The frequency response of a filter is governed—within a multiplicative scaling constant—by the locations of the poles and zeros of its transfer function $\mathbf{H}(\mathbf{s})$.

$$\mathbf{H}(\mathbf{s}) = \mathbf{C}\,\frac{\prod_{i=1}^{m}(\mathbf{s}-\mathbf{z}_i)}{\prod_{i=1}^{n}(\mathbf{s}-\mathbf{p}_i)} \quad \rightarrow \quad \mathbf{H}(\omega) = \mathbf{H}(\mathbf{s})\Big|_{\mathbf{s}=j\omega} = \mathbf{C}\,\frac{\prod_{i=1}^{m}(j\omega-\mathbf{z}_i)}{\prod_{i=1}^{n}(j\omega-\mathbf{p}_i)}$$

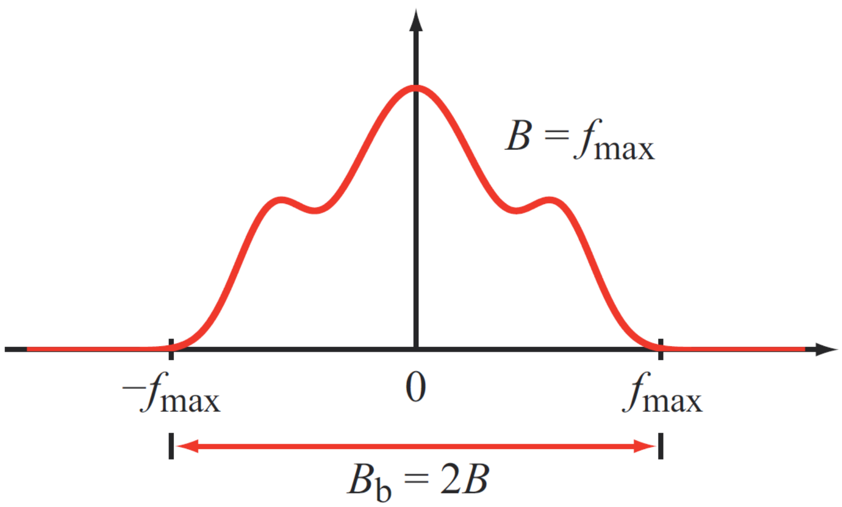
- If a system has zeros $\{a_i \pm jb_i\}$ and poles $\{c_i \pm jd_i\}$, the magnitude of its frequency response, $M(\omega)$, will have peaks at $\omega = \pm d_i$ if c_i is small, and dips at $\omega = \pm b_i$ if a_i is small.

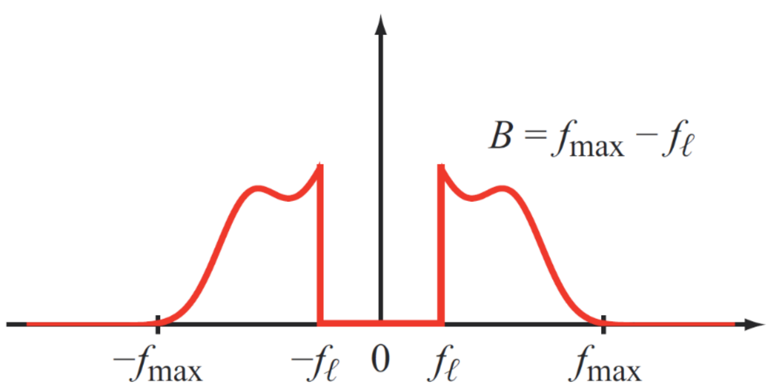
Example: Frequency Response from Locations of Poles and Zeros





Bandwidths of Signal Spectra





We will need these definitions for the next big topic: Amplitude modulation

Two Important Fourier Transforms

Modulation, AM radio and frequency-domain multiplexing are all just applications of the following Fourier transforms:

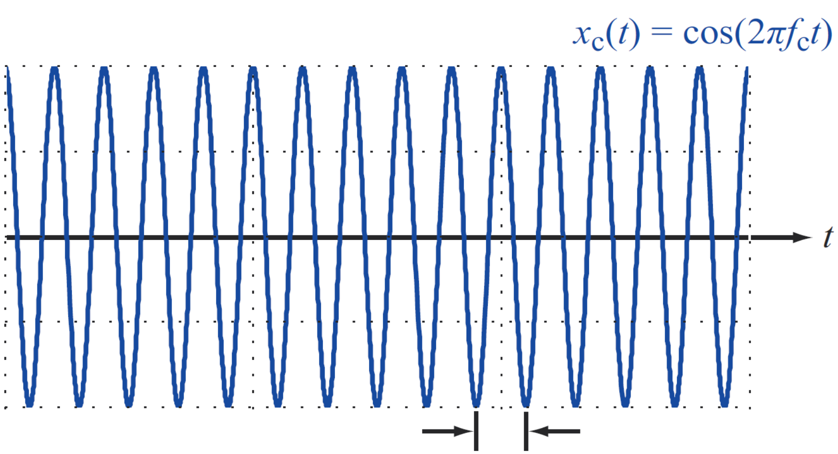
$$x_c(t) = A\cos(2\pi f_c t)$$

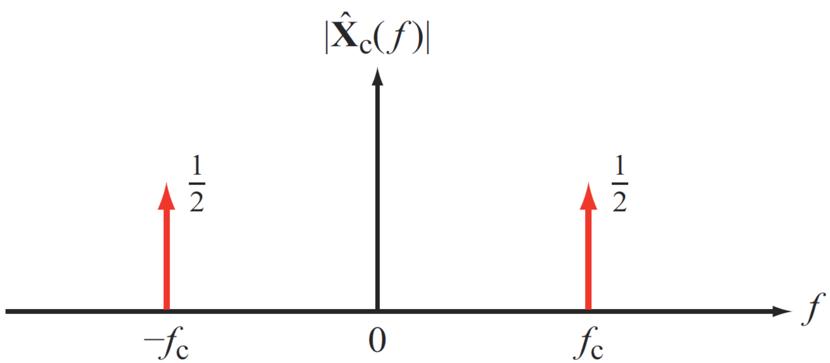
$$\mathbf{X}_c(f) = \frac{A}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

$$y_m(t) = x(t)\cos(2\pi f_c t)$$

$$\mathbf{Y}_m(f) = \frac{1}{2}[\mathbf{X}(f - f_c) + \mathbf{X}(f + f_c)].$$

(DSB modulation)





Double-Sideband Modulation and Demodulation (recovery of signal)

DSB Modulation of x(t): $y_m(t) = x(t)\cos(2\pi f_c t)$

DSB Demodulation of y(t): $y_d(t) = y_m(t)\cos(2\pi f_c t)$

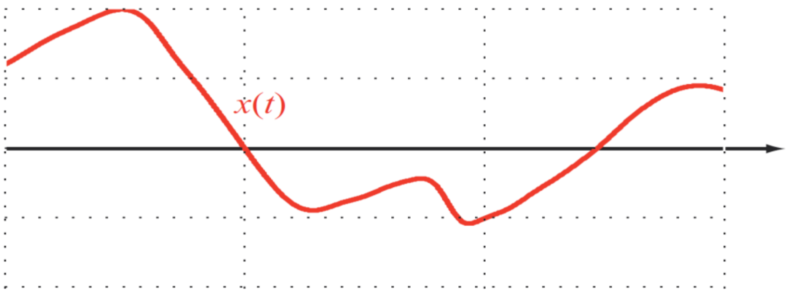
$$y_d(t) = x(t)\cos^2(2\pi f_c t) = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(4\pi f_c t).$$

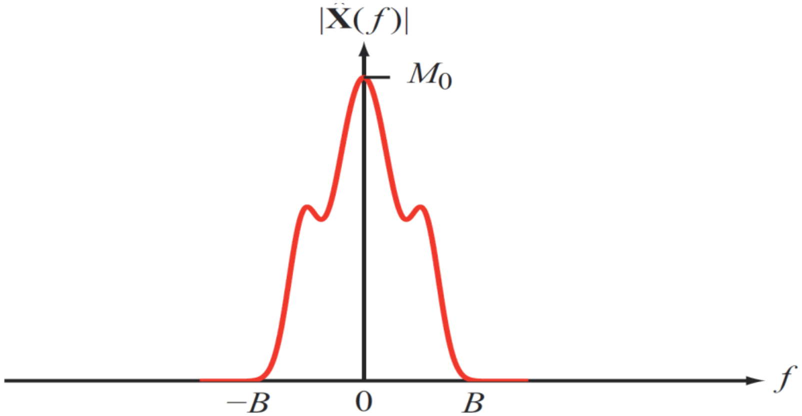
$$\mathbf{Y}_d(f) = \frac{1}{2}\mathbf{X}(f) + \frac{1}{4}\left[\mathbf{X}(f-2f_c) + \mathbf{X}(f+2f_c)\right]$$

<u>1st term</u>: Spectrum of original signal.

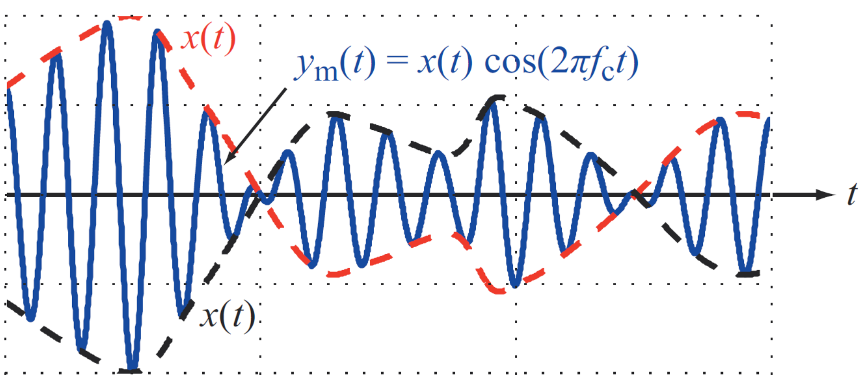
<u>2nd term</u>: Spectrum of original signal shifted up and down by twice the frequency of the carrier sinusoid.

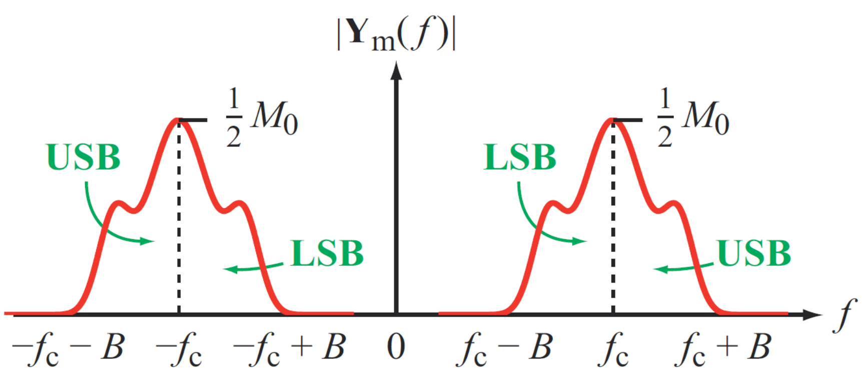
Double-Sideband Modulation



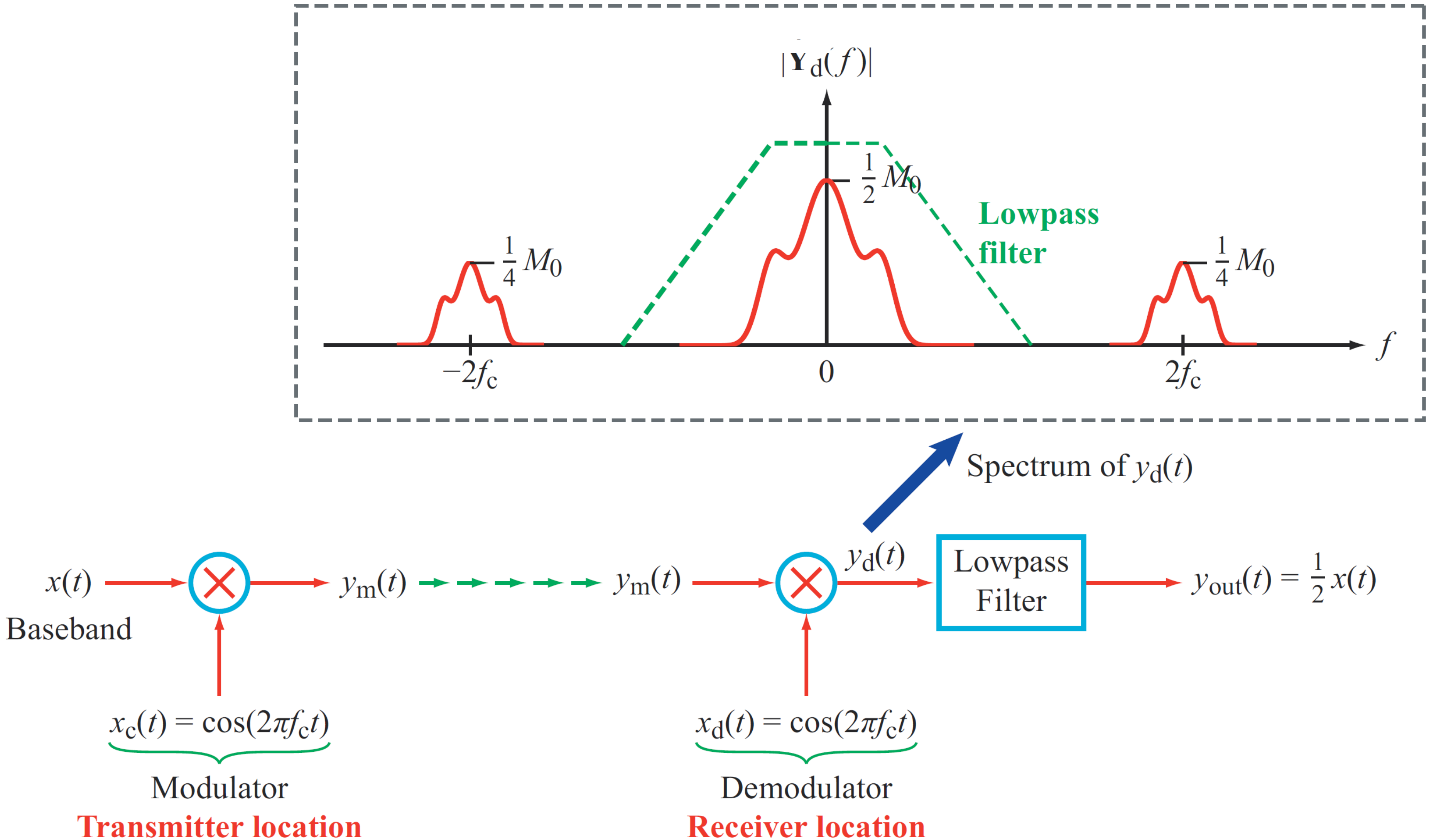


Multiplication of a signal by a sinusoid shifts its spectrum up and down by the frequency of the multiplying sinusoid:



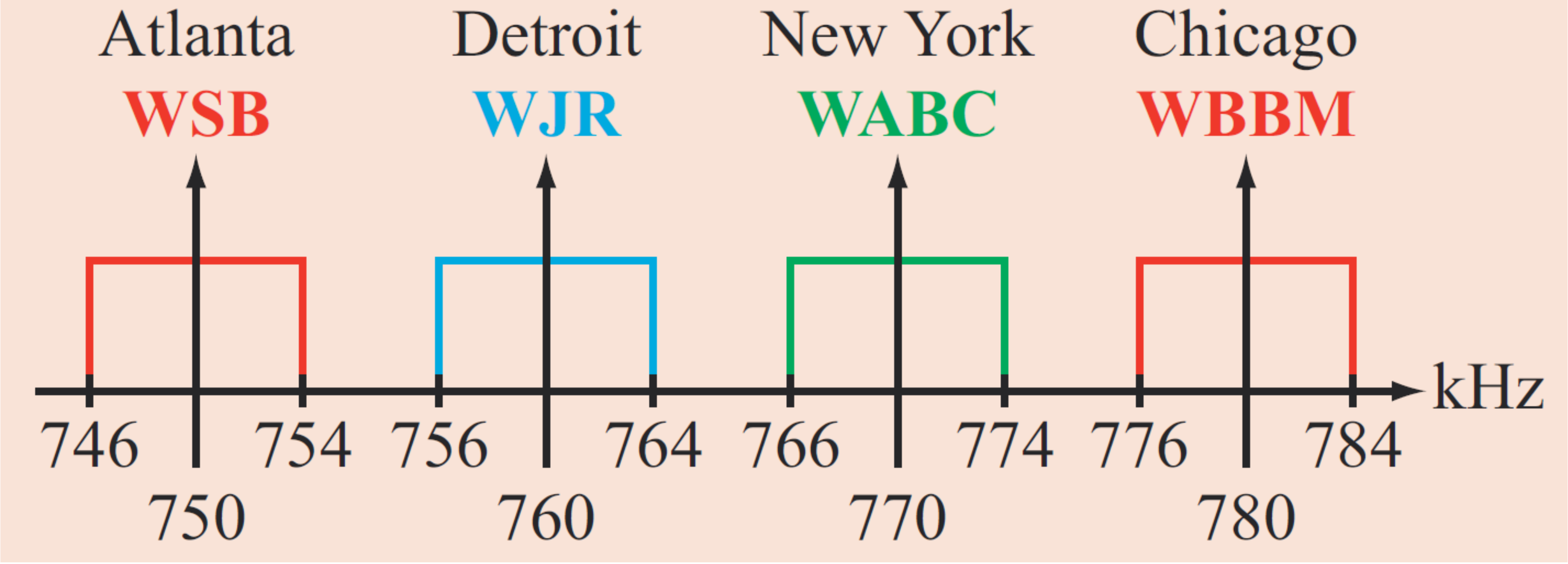


Double-Sideband Modulation and Demodulation (recovery of signal)



AM Radio Spectrum in the Midwest

Can transmit multiple radio signals in space using different carrier frequencies for each radio station.



Amplitude Modulation (AM Radio)

AM: Add a copy of the carrier to the DSB modulated signal.
Why? Can now use envelope detection to recover the signal.
This is much simpler than using DSB demodulation.

$$y_m(t) = [A + x(t)]\cos(2\pi f_c t) \quad = A\cos(2\pi f_c t) + x(t)\ \cos(2\pi f_c t)$$