LECTURE NO. 12

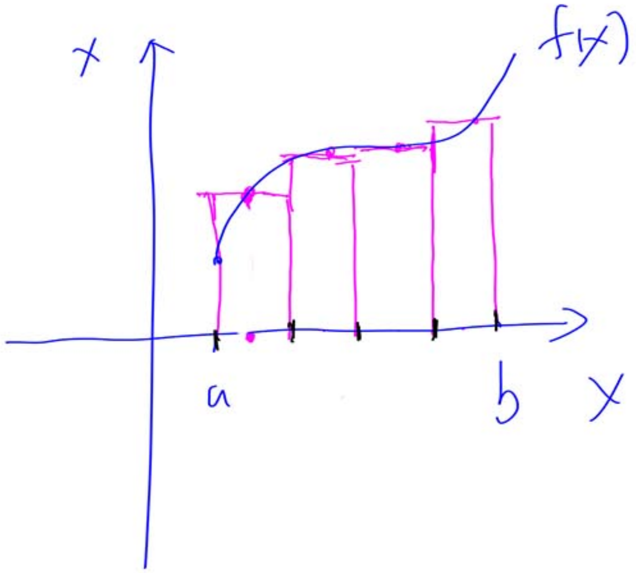
3.6 Numerical Integration

Wright State University

$\int_1^2 e^{x^2} dx$

$\int_2^4 \frac{\sin x}{x} dx$

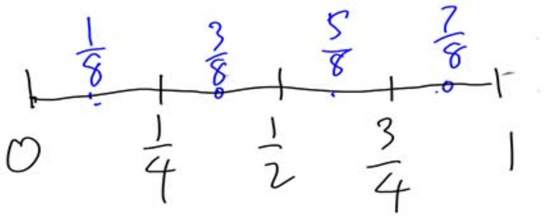
Recall the MidPoint Riemann Sum



$$\int_a^b f(x)\,dx$$

$\approx$ Sum of the areas of these rectangles.

Use Midpoint Riemann Sum to estimate $\int_0^1 x^2 dx$ using four subintervals.



4 subintervals $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

Midpoint Riemann Sum

$$= \left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{3}{8}\right) + f\left(\tfrac{5}{8}\right) + f\left(\tfrac{7}{8}\right) \right] \cdot \frac{1}{4}$$

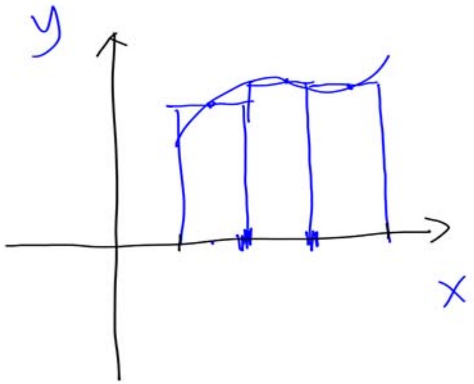
$$= \left(\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64} \right) \cdot \frac{1}{4}$$

$$= \frac{84}{64} \cdot \frac{1}{4} = \frac{21}{64} \approx 0.328125$$

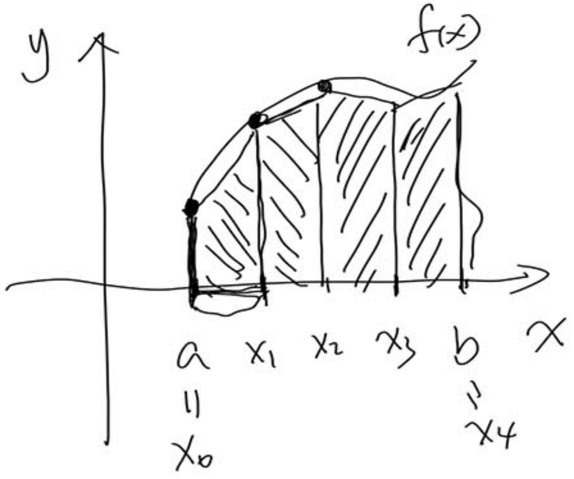
Midpoint

$$\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} = 0.3333\ldots$$

The Trapezoidal Rule



midpoint



$$\Delta x = \frac{b-a}{4}$$

$$\int_a^b f(x)\,dx \approx \frac{(f(x_0)+f(x_1))\cdot\Delta x}{2} + \frac{(f(x_1)+f(x_2))\Delta x}{2} + \cdots + \frac{(f(x_3)+f(x_4))\Delta x}{2}$$

Trapezoidal Rule

$$= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$$

Use Trapezoidal Rule with $n = 6$ to estimate $\int_1^4 \frac{1}{x} dx$.

$\int_a^b f(x)\,dx$ using n-subintervals $\quad \Delta x = \frac{b-a}{n}$

Trapezoidal Rule: $\frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$

$1 \quad \frac{3}{2} \quad 2 \quad \frac{5}{2} \quad 3 \quad \frac{7}{2} \quad 4$

$\Delta x = \frac{4-1}{6} = \frac{1}{2}$ $\qquad f(x) = \frac{1}{x}$

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Trapezoidal Rule: $\frac{\Delta x}{2}\left(f(1) + 2f(\frac{3}{2}) + 2f(2) + 2f(\frac{5}{2}) + 2f(3) + 2f(\frac{7}{2}) + f(4)\right)$

$\frac{1}{4}\left(1 + 2\cdot\frac{2}{3} + 2\cdot\frac{1}{2} + 2\cdot\frac{2}{5} + 2\cdot\frac{1}{3} + 2\cdot\frac{2}{7} + \frac{1}{4}\right)$ $\quad$ 1.41 55

$\frac{1}{4}\left(4 + \frac{4}{5} + \frac{4}{7} + \frac{1}{4}\right) = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{16} = \frac{560 + 112 + 80 + 35}{560} = \frac{787}{560}$

- In Trapezoidal Rule, we use a linear function on each subinterval to estimate $f(x)$ and the number of subintervals n can be any positive integer (even or odd).

- in Simpson's Rule, the number of subintervals n must be even; and we use a quadratic function to estimate $f(x)$ on two consecutive subintervals.

Trapezoidal Rule: $\frac{\Delta x}{2}\left(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right)$

- Simpson's Rule: (n is even) $\frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{n-1})+f(x_n)\right)$

$n=4$ $\frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\right)$

$n=6$ $\frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)\right)$

Formulas for Trapezoidal Rule and Simpson Rule to estimate $\int_a^b f(x)dx$

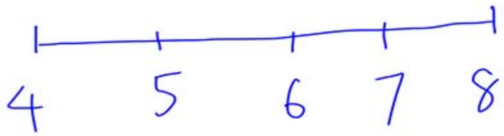
- n is the total number of subintervals and $\Delta x = \frac{b-a}{n}$ is the length of each subinterval.
- Trapezoidal Rule

$$\frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

- Simpson's Rule (n must be even):

$$\frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Use Simpson's Rule with $n = 4$ to estimate $\int_4^8 \sqrt{x}\,dx$.



$$\Delta x = \frac{8-4}{4} = \frac{4}{4} = 1$$

Simpson's Rule: $\frac{\Delta x}{3}\left(f(4) + 4f(5) + 2f(6) + 4f(7) + f(8)\right)$

$$f(x) = \sqrt{x} \qquad \Delta x = 1$$

$$\frac{1}{3}\left(\sqrt{4} + 4\sqrt{5} + 2\sqrt{6} + 4\sqrt{7} + \sqrt{8}\right)$$

$$\frac{1}{3}\left(2 + 4\sqrt{5} + 2\sqrt{6} + 4\sqrt{7} + 2\sqrt{2}\right)$$