Module # 10: **A Visit to Laplace Land** **Intro**

These notes are drawn from WIKIPEDIA and other sources. They are intended to offer a summary of topics to guide you in focused studies. You should augment this handout with notes taken in class, reading textbook(s), and working additional example problems.

Introduction

In *mathematics*, the **Laplace transform** is an *integral transform* named after its inventor *Pierre-Simon Laplace*. It transforms a function of a real variable t (often time) to a function of a *complex variable* s (*complex frequency*). The transform has many applications in science and engineering.

The Laplace transform is similar to the *Fourier transform*. While the Fourier transform of a function is a *complex function* of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable. The Laplace transform is usually restricted to transformation of functions of t with $t \geq 0$. Unlike the Fourier transform, the Laplace transform of a *distribution* is generally a *well-behaved* function. Techniques of complex variables can also be used to directly study Laplace transforms.

The Laplace transform is invertible on a large class of functions. The inverse Laplace transform takes a function of a complex variable s (often frequency) and yields a function of a real variable t (often time). Given a simple mathematical or functional description of an input or output to a *system*, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications.

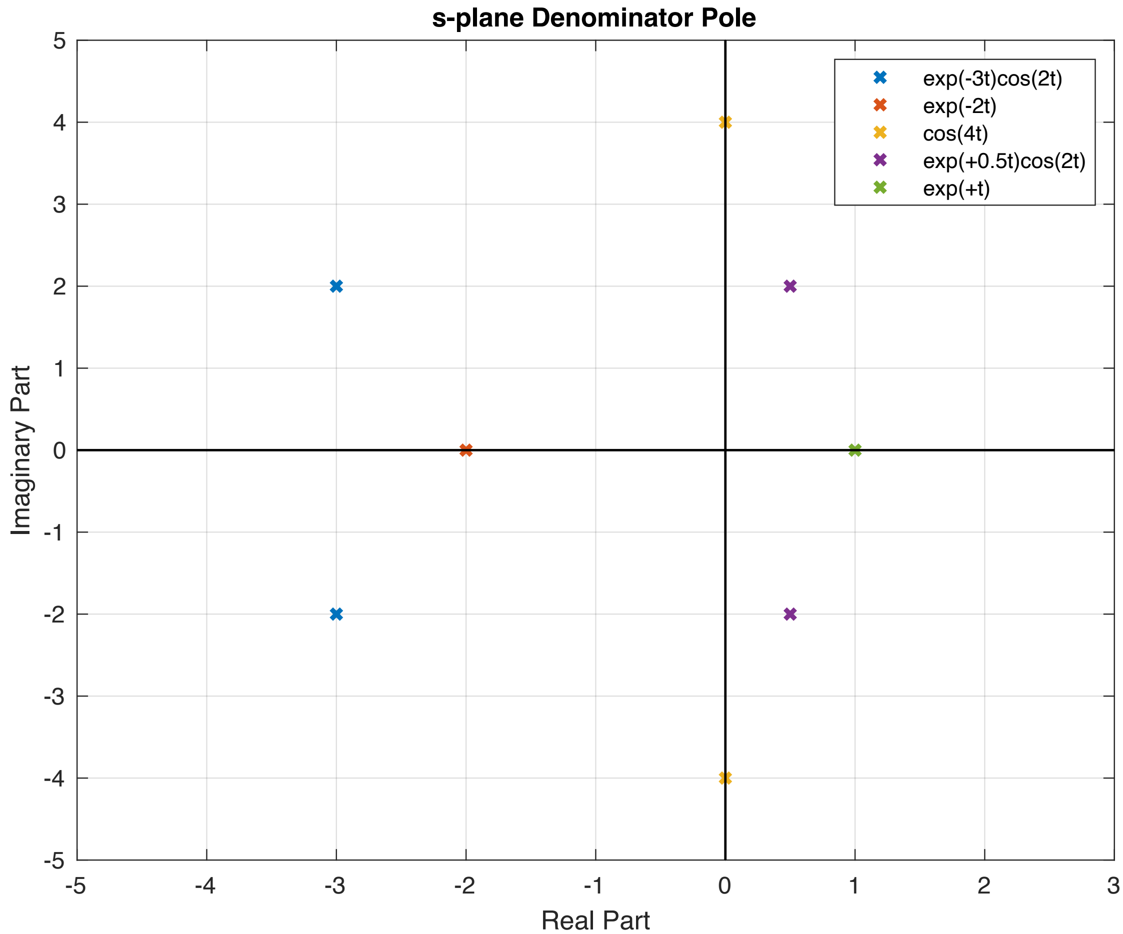
Formal Definition

The Laplace transform of a *function* $f(t)$, defined for all *real numbers* $t \geq 0$, is the function $F(s)$, which is a unilateral transform defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where s is a *complex number* frequency parameter: $s = \sigma + j\omega$, with real numbers σ and ω.

So what do functions look like in the s-domain? The functional form is determined by the **denominator polynomial**, and more precisely, the **zeros of the denominator polynomial**



Inverse Laplace Transform

The *inverse Laplace transform* is given by the following complex integral,

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi j} \lim_{T\to\infty} \oint_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\,ds$$

where γ is a real number so that the contour path of integration is in the region of convergence of $F(s)$.

In practice, it is typically more convenient to decompose a Laplace transform into known transforms of functions obtained from a table, and construct the inverse by inspection.

Properties and theorems

The Laplace transform has a number of properties that make it useful for analyzing linear *dynamical systems*. The most significant advantage is that *derivative/differentiation* becomes multiplication, and *integral/integration* becomes division, by s (similarly to *logarithms* changing multiplication of numbers to addition of their logarithms).

Because of this property, the Laplace variable s is also known as operator variable in the L domain: either derivative operator or integration operator. The transform turns *integral equations* and *differential equations* to *polynomial equations*, which are much easier to solve. Once solved, use of the inverse Laplace transform reverts to the original domain.

THE CENTRAL POINT:

The Laplace transformation from the *time domain* to the *frequency domain* transforms differential equations into algebraic equations and *convolution* into multiplication.

That's why we're using $\frac{1}{sC}$ instead of $\frac{1}{C}\int i(t)dt$ and Ls instead of $L\frac{di(t)}{dt}$ to characterize a dynamic system compactly and conveniently in terms of $H(s) = \frac{V_{out}(s)}{V_{in}(s)}$, or whatever input-output function is dictated by the problem. We can then use this convenient algebraic representation and the properties and functions below to **solve problems for dynamic systems that would otherwise be intractable.**

The following **table** is a list of properties of unilateral Laplace transform properties:

Properties of the unilateral Laplace transform

Operation	time-domain	s-domain
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
t-domain derivative	$f'(t)$	$sF(s) - f(0^+)$
Second derivative	$f^{(2)}(t)$	$s^2F(s) - sf(0^+) - f'(0^+)$
General derivative	$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+)$
t-domain integration	$\int_0^t f(\tau)\, d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$
s-domain derivative	$tf(t)$	$-F'(s)$
s-domain derivative	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
s-domain integration	$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)\, d\sigma$
Frequency shifting	$e^{at}f(t)$	$F(s-a)$
Time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Time scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Multiplication	$f(t)g(t)$	$\int F(\sigma)G(s-\sigma)\, d\sigma$
Convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\, d\tau$	$F(s) \cdot G(s)$
Complex conjugation	$f^*(t)$	$F^*(s^*)$
Cross-correlation	$f(t) * g(t)$	$F^*(-s^*) \cdot G(s)$
Periodic function	$f(t)$	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dt$
Initial value theorem	$f(0^+) = \lim_{s\to\infty} sF(s)$	
Final value theorem	$f(\infty) = \lim_{s\to 0} sF(s)$	

Table 1: Laplace transform properties

The following **table** contains selected Laplace transform pairs:

Laplace transform pairs

Function	**time-domain**	**s-domain**
	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
Unit impulse	$\delta(t)$	1
Delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$
Unit step	$u(t)$	$\frac{1}{s}$
Delayed unit step	$u(t-\tau)$	$\frac{1}{s}e^{-\tau s}$
Unit ramp	$t \cdot u(t)$	$\frac{1}{s^2}$
nth power	$t^n \cdot u(t)$	$\frac{n!}{s^{n+1}}$
nth root	$\sqrt[n]{t} \cdot u(t)$	
Exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$
Exponential approach	$(1-e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$
Sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$
Decaying sine	$e^{-\alpha t}\sin(\omega t) \cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
Decaying cosine	$e^{-\alpha t}\cos(\omega t) \cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Table 2: Laplace transform table

Utility in Differential Equations

All dynamic time-invariant linear systems are described by **constant-coefficient linear differential equations.** The utility Laplace transforms offers in solving these is transformational (pun). Let's just get to some examples.

In what follows, we use *engineering descriptions* (as opposed to mathematical descriptions).

Zero-Input Response

As implied, zero-input systems are described by differential equations of the form:

$$ay''(t) + by'(t) + cy(t) = 0$$

Using the time-domain derivative properties of Laplace transforms, we may write this as

$$a[s^2Y(s) - sy(0^+) - y'(0^+)] + b[sY(s) - y(0^+)] + cY(s) = 0$$

$$\underbrace{[as^2 + bs + c]}_{\text{characteristic equation}} \quad Y(s) = a[sy(0^+) + y'(0^+)] + b[y(0^+)]$$

$$Y(s) = \frac{a[sy(0^+) + y'(0^+)] + b[y(0^+)]}{[as^2 + bs + c]}$$

so that the solution, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ (which is easily obtainable) depends ONLY on the *initial conditions*. We will see that *initial conditions directly relate to energy stored in the dynamic elements,* that is, the **INITIAL STATES** of the system. Note, trivially, that if there is no energy stored, the solution is $y(t) = 0$. Note also, that the order of the characteristic equation is the *order of the system.*

Zero-State Response

Zero-state systems are implicitly driven by some input or forcing function $x(t)$. Such systems are described by differential equations of the form:

$$ay''(t) + by'(t) + cy(t) = dx'(t) + ex(t)$$

Using the time-domain derivative properties of Laplace transforms, we may write this as

$$\underbrace{[as^2 + bs + c]}_{\text{characteristic equation}} \quad Y(s) = \underbrace{[ds + e]}_{\text{numerator}} X(s)$$

$$\frac{Y(s)}{X(s)} = H(s) = \frac{ds + e}{as^2 + bs + c}$$

where $H(s)$ is the transfer function of the system, and where the roots of the characteristic equation are the *poles* of the system and where the roots of the numerator equation are the *zeroes* of the system.

Total Response

Here, we could use the properties of *linearity* to form the *total response as the sum of the zero-input response and the zero-state response*, but we will demonstrate this as follows. consider a system with both initial states and driven by some input or forcing function $x(t)$. Such systems are described by differential equations of the form:

$$ay''(t) + by'(t) + cy(t) = dx'(t) + ex(t)$$

Using the time-domain derivative properties of Laplace transforms, we may write this as

$$a[s^2Y(s) - sy(0^+) - y'(0^+)] + b[sY(s) - y(0^+)] + cY(s) = dsX(s) + eX(s)$$

$$\underbrace{[as^2 + bs + c]}_{\text{characteristic equation}} \quad Y(s) = a[sy(0^+) + y'(0^+)] + b[y(0^+)] + [ds + e]X(s)$$

$$Y(s) = \underbrace{\frac{a[sy(0^+) + y'(0^+)] + b[y(0^+)]}{[as^2 + bs + c]}}_{\text{zero-input}} + \underbrace{\frac{[ds + e]X(s)}{[as^2 + bs + c]}}_{\text{zero-state}}$$

$$Y(s) = \frac{a[sy(0^+) + y'(0^+)] + b[y(0^+)]}{[as^2 + bs + c]} + H(s)X(s)$$

where the transfer function

$$H(s) = \frac{ds + e}{as^2 + bs + c}$$

where the roots of the characteristic equation are the *poles* of the system and where the roots of the numerator of the transfer function are the *zeroes* of the system.

Applications to Steady-State Circuit Analysis

So what does this all mean to us? Thus far in the course, we have focused on the *steady-state response = zero-state response* of circuit systems. In doing so, we have used the steady-state (also zero-state) impedance models:

Steady-State Impedance Models

s-domain	**Steady State @ ω R/s**	**Steady State @ f Hz**
$Z_R = R$	$Z_R = R$	$Z_R = R$
$Z_C = \frac{1}{s \cdot C}$	$Z_C = \frac{1}{j\omega \cdot C}$	$Z_C = \frac{1}{j2\pi f \cdot C}$
$Z_L = s \cdot L$	$Z_L = j\omega \cdot L$	$Z_L = j2\pi f \cdot L$

Table 3: Steady-State Impedance Models for R, C, and L

The response of any time-invariant linear system (modeled by $h(t) \stackrel{\mathcal{L}}{\longleftrightarrow} H(s)$) to a stimulus signal $f(t)$ is the *convolution*

$$(f * h)(t) = \int_0^t f(\tau)h(t-\tau)\,d\tau \stackrel{\mathcal{L}}{\longleftrightarrow} F(s) \cdot H(s)$$

Pretty messy in time-domain, but just a *functional multiply* in s-domain. Moreover, we have thus far constrained our attention to *sinusoidal steady-state* (including DC, $f = 0$) - so we don't have to use the Laplace representation of input signals to find $F(s)$ (as in the table above). Instead, we employ the additional simplification where if the input is sinusoidal at a particular frequency, ω_0,

$$v_{in}(t) = A\cos(\omega_0 t + \theta)$$

the output is found as:

$$v_{out}(t) = A|H(\omega_0)|\cos(\omega_0 t + \theta + \angle H(\omega_0))$$

that is, the *amplitude* of the output is the *product* of the amplitude of the input signal and the magnitude of the transfer function, $A|H(\omega_0)|$ while the *phase* of the output is the *sum* of the phases of the input signal and the transfer function, $\theta + \angle H(\omega_0)$.

If the input is a DC voltage or current, i.e.,

$$v_{in}(t) = A$$

the output is found as:

$$v_{out}(t) = A|H(0)|\Re\left\{e^{(j\angle H(0))}\right\}$$

where for DC inputs, the "phase" $\angle H(0)$ will be 0 or π, that is, the systems can modify the output to be positive or negative.

Later, we'll consider:

Applications to Transient Responses

As we turn to *transient responses* **all analysis principles will remain valid**, only some models will change. in particular, as before, the response of any time-invariant linear system (modeled by $h(t) \overset{\mathcal{L}}{\longleftrightarrow} H(s)$) to a stimulus signal $f(t)$ is the *convolution*

$$(f * h)(t) = \int_0^t f(\tau)h(t-\tau)\,d\tau \overset{\mathcal{L}}{\longleftrightarrow} F(s) \cdot H(s)$$

Again, messy in time-domain, but just a *functional multiply* in s-domain. Here's where things change a bit.

Transient Response of Zero-State Systems

Systems are said to be in "zero-state" if none of the dynamic elements in the system (inductors and capacitors in our case) are storing energy. In this case, we proceed much the same as in the steady-state case using the same impedance models to find the transfer function $H(s)$. We then

Zero-State Impedance Models

t**-domain**	s**-domain**
$v(t) = i(t) \cdot R$	$V(s) = I(s) \cdot R$
$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau$	$V(s) = \frac{I(s)}{sC}$
$v(t) = L\frac{di(t)}{dt}$	$V(s) = Ls \cdot I(s)$

proceed to find $Out(s) = F(s) \cdot H(s)$ using the s-domain function transforms and properties listed above.

For example, to find the **step response** of a circuit, we would follow the

Node Voltage Procedure for Zero-State Systems:

1. Replace *all independent sources with symbolic representations*
2. Identify the essential ($\geq$ 3-element connections) nodes
3. Select a node as the reference node = the node at *ground potential* = 0 Volts
4. Identify and label the voltages at nodes that are readily deduced
5. Note the node-pairs linked by a *voltage source* and simplify accordingly
6. Assign voltage variables $v_a, v_b, \ldots$ to the remaining nodes with only one assignment for each linked node-pair, the other node in that pair assigned voltages such as "$v_1 - 20$" or "$v_4 + 3v_x$".
7. Employ s-domain impedance models: $Z_R(s) = R$, $Z_C(s) = \frac{1}{s \cdot C}$, and $Z_L(s) = s \cdot L$.
8. Apply $I_{\text{out}} = V_{\text{difference}}/Z$ for each branch leaving the node

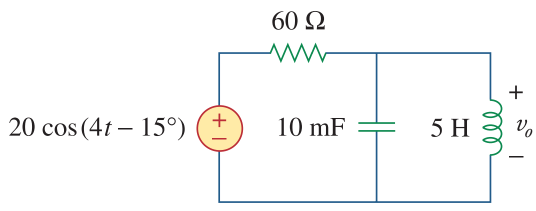
9. Enjoy the thrill of ending the consideration of each node with the powerful "$= 0$"
10. Add one additional equation for each dependent source specification if necessary

    Circuit analysis is now complete! *But you may be asked to:*

11. Find the transfer function: $H(s) = \frac{V_{out}(s)}{V_{in}(s)}$, or whatever input-output function is dictated by the problem.
12. Find $V_{out}(s) = V_{in}(s)H(s)$ for the specified function $V_{in}(s)$.
13. Find $v_{out}(t) = \mathcal{L}^{-1}\{V_{out}(s)\}$

The last step is easily done via MatLab's `ilaplace` command. An example:

Example 9.11:



Begin using a solver as before:

```
%%
%  Example 9.11
clear all
% Declare symbolic variables
syms Vin v0 s t
% Nodal analysis directly in solve()
[v0]=solve((v0-Vin)/60 + v0*0.01*s + v0/(5*s)== 0)
% Transfer function
H(s) = v0/Vin
```

which yields

```
H(s) = (5*s)/(3*s^2 + 5*s + 60)
```

Since the unit step $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$, we can find the response to an input of $3u(t)$ by finding $v_{out}(t) = \mathcal{L}^{-1}\{H(s)\frac{3}{s}\}$

```
% The response to 3u(t)
v(t) = ilaplace(H(s)*3/s)
```

which yields

```
v(t) = (2*695^(1/2)*exp(-(5*t)/6)*sin((695^(1/2)*t)/6))/139
```

How about a rectangular pulse? Since we can write a rectangular pulse of width 2 as $u(t) - u(t-2)$, we can find the response to a rectangular pulse with amplitude 3, i.e., $3u(t) - 3u(t-2)$ by using the properties above and finding $v_{out}(t) = \mathcal{L}^{-1}\left\{H(s)\left(\frac{3}{s} - \frac{3e^{-2s}}{s}\right)\right\}$

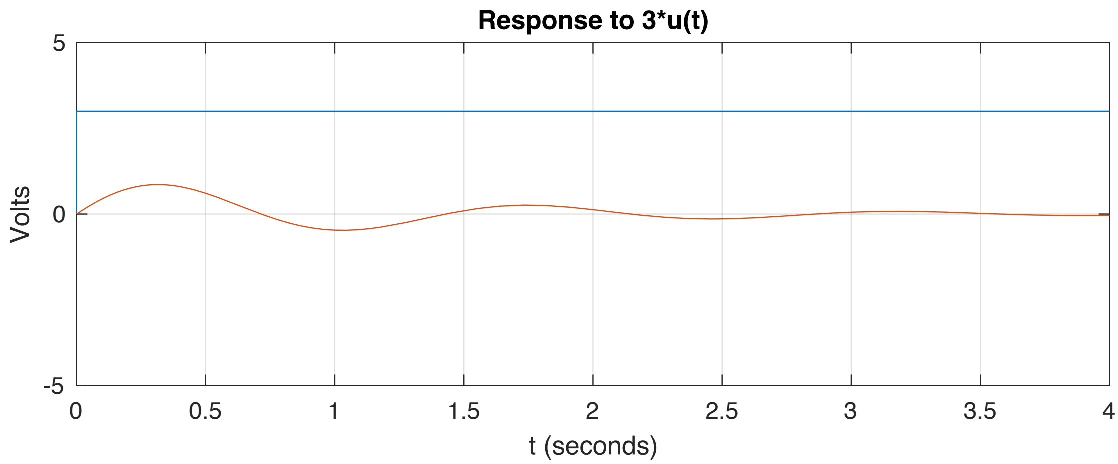
```
% The response to a rectangular pulse: 3u(t) - 3u(t-2)
v(t) = ilaplace(H(s)*(3/s-3/s*exp(-2*s)))
```

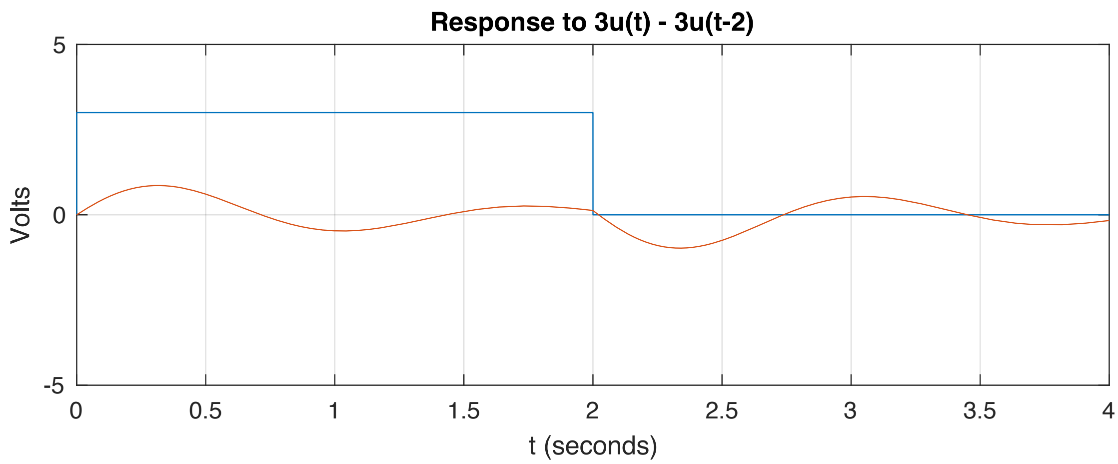
which yields

```
v(t) = (6*695^(1/2)*exp(-(5*t)/6)*sin((695^(1/2)*t)/6))/139 - (6*695^(1/2)*sin
    ((695^(1/2)*(t - 2))/6)*heaviside(t - 2)*exp(5/3 - (5*t)/6))/139
```

Which we could have deduced by inspection from the response to $3u(t)$ and the time-delay property.

To help visualize, I've included a plot of the excitation functions and the outputs below.





We could easily continue to find the response to any input we're interested in.

Transient Response of Non-Zero-State Systems

If one or more of the dynamic elements have store energy, things get a little trickier. First of all, we would use the initial-condition = initial-state models for the dynamic elements:

Total Response Impedance Models

t-domain	s-domain
$v(t) = i(t) \cdot R$	$V(s) = I(s) \cdot R$
$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v_C(t_0)$	$V(s) = \frac{I(s)}{sC} + \frac{v_C(t_0)}{s}$
$v(t) = L\frac{di(t)}{dt}$	$V(s) = Ls \cdot I(s) - L \cdot i_L(t_0)$

where $v(t_0)$ and $i(t_0)$ represent the initial voltage across the capacitor and initial current through the inductor at time t_0 – the initial states. These correspond to the dynamic element models that incorporate the initial-condition = initial-states as:

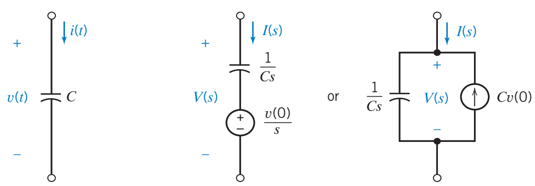


Figure 1: The Capacitor Model

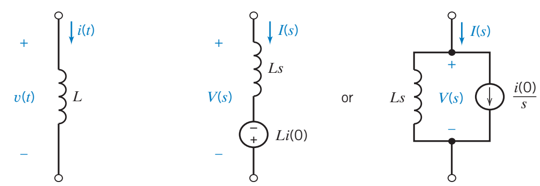


Figure 2: The Inductor Model

The good news is two-fold. First, in electrical engineering, initial states of elements are rarely of interest. Secondly, by *linearity* **the total response of a system is the sum of the zero-state response and the zero-input response**, the latter being the response due only to the stored-energy elements with no external inputs. We'll consider this type pf problem in the last portion of the course.