LECTURE NO. 17

5.4 Comparison Test

Wright State University

Comparison Test

- ONLY Works for positive series.
- Need to compare a given series with a series that we know whether it is convergent/divergent.
- There are two classes of series we know exactly when they are convergent.

Geometric Series

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \cdots$$

① Convergent to $\frac{a}{1-r}$ if $|r| < 1$

② divergent.

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad (p > 0)$$

$$= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

① Convergent if $p > 1$.

② divergent if $p \leq 1$

The Idea of Simple Comparison Test

Small one divergent $\Rightarrow$ big one divergent

Big one convergent $\Rightarrow$ Small one convergent

- Given two positive series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ with } 0 \leq b_n \leq a_n.$$

- A positive series either adds up to a number (convergent) or to ∞ (divergent).
- If the smaller series adds up to ∞, then the big one must add up to ∞ (both divergent).
- If the big series adds up to a number, then the small one must add up to a number (both convergent).
- Inconclusive if small one adds up to a number (or if big one adds up to ∞).

Example No. 1 on Simple Comparison Test

- $$\sum_{n=1}^{\infty} \frac{1}{3^n + 5}$$

① Compare with the series $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

The new series is a Geometric Series with $r = \frac{1}{3}$ ∴ So it is convergent.

② $$\frac{1}{3^n+5} < \frac{1}{3^n}$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{3^n+5}$ is convergent by Simple Comparison Test.

Example No. 2 on Simple Comparison Test

- $$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

2, $\ln 2 < 2$ $\ln 3 < 3$, $\ln 4 < 4$, in general, we know $\ln n < n$

Now take the reciprocal: $\frac{1}{\ln n} > \frac{1}{n}$

Compare $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ with $\sum_{n=2}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent.

Therefore, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent by Simple Comparison Test.

- Simple Comparison Test does not work for

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Convergent

$$\frac{1}{n^2-1} > \frac{1}{n^2}$$

Limit Comparison Test

- Simple Comparison Test does not work for

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}$$

- Limit Comparison Test: Given two positive series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n$$

when n is big

$\frac{a_n}{b_n} \approx L$

or $a_n \approx L b_n$

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ i.e. a positive number,
- then the two series have the same convergent/divergent property.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$$

Geometric Series with $r = \frac{2}{3}$

(1) Compare with $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \frac{2}{3} + \frac{2^2}{3^2} + \frac{2^3}{3^3} + \cdots$: So it is convergent

(2) Find the limit: $\lim_{n\to\infty} \frac{\frac{2^n}{3^n-1}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{\cancel{2^n}}{3^n-1} \cdot \frac{3^n}{\cancel{2^n}} = \lim_{n\to\infty} \frac{3^n}{3^n-1} = \lim_{n\to\infty} \frac{\frac{3^n}{3^n}}{\frac{3^n}{3^n} - \frac{1}{3^n}}$

$= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{3^n}} = 1$ (a positive number!)

$\frac{1}{3^n} \to 0$

Therefore, $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$ is convergent by Limit Comparison Test.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{3n+5} \qquad n^{\frac{1}{2}}$$

① Compare with $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \frac{1}{3} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}}$ p-series with $p = \frac{1}{2}$ divergent

② Find $\lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{3n+5}}{\frac{\sqrt{n}}{3n}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{3n+5} \cdot \frac{3n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{3n}{3n+5}$

$= \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot \frac{3n}{3n+5} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{3}{3+\frac{5}{n}} = 1$

Therefore, $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{3n+5}$ is divergent by limit Comparison Test.

More on Limit Comparison Test

- In Limit Comparison Test, we calculate

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

- If L is a positive number, then $a_n \approx L \cdot b_n$ when n is big.
- What if $L = 0$?
- Then a_n is much smaller than b_n when n is big.

$a_n < b_n$

Big one convergent ⇒ Small one convergent.

- What if $L = \infty$?
- Then a_n is much bigger than b_n when n is big.

Small one divergent ⇒ Big one divergent.

$a_n > b_n$

Comparison Test vs Integral Test

- They both work for positive series. When to use which?
- Use Integral Test when the function is easy to integrate; for example

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^2(n)} \longrightarrow \int_2^{\infty} \frac{1}{x \ln^2 x}\, dx$$

$u = \ln x$ (substitution)

- We might want to use Comparison Test on this series:

$$\sum_{n=1}^{\infty} \frac{2n+5}{3n^3+n^2-n+7}$$

$$\sum_{n=1}^{\infty} \frac{2n+5}{3n^3+n^2-n+7}$$

(1) Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n^3} = \frac{2}{3}\sum_{n=1}^{\infty} \frac{n}{n^3} = \frac{2}{3}\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series $p=2$ convergent

(2) $$\lim_{n\to\infty} \frac{\frac{2n+5}{3n^3+n^2-n+7}}{\frac{2n}{3n^3}} = \lim_{n\to\infty} \frac{2n+5}{3n^3+n^2-n+7} \cdot \frac{3n^3}{2n} = 1$$

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{3n^3+n^2-n+7}$ is convergent by limit comparison Test.