LECTURE NO. 19

5.6 Ratio and Root Test

Wright State University

Ratio Test

- Ratio Test is used to determine the absolute convergence of a series.
- For any series

$$\sum_{n=1}^{\infty} a_n$$

Let

$$L=\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|.$$

- ▶ If L < 1, then the series is absolutely convergent.
- ▶ If L > 1, then the series is divergent.
- ▶ If L = 1, then Ratio Test is inconclusive.



An Example on Ratio Test

Rate Test
$$\sum_{n=1}^{\infty} \frac{2^n}{n!} \qquad n! \quad \text{factorial}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} = \lim_{n \to \infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \qquad (n+1)!$$

$$= \lim_{n \to \infty} \frac{2}{n!} \qquad n+1$$

$$= \lim_{n \to \infty} \frac{2}{n+1} = 0 < 1$$
Therefore, the Sectes is absolutely convergent by Ride Test.

Another Example on Ratio Test

Red Test
$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$$

$$\begin{vmatrix} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1}}{(n+1)^3} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n}$$

$$= \lim_{n\to\infty} \frac{3^n}{(n+1)^3} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} 3 \cdot \left(\frac{n}{n+1} \right)^3 = \lim_{n\to\infty} 3 \cdot \left(\frac{1+n}{n+1} \right)^3$$

$$= \lim_{n\to\infty} \frac{3^n}{(n+1)^3} \cdot \frac{3^n}{n^3} \cdot \frac{1+n^3}{n^3} \cdot \frac{3^n}{n^3}$$

$$= \lim_{n\to\infty} \frac{3^n}{(n+1)^3} \cdot \frac{3^n}{(n+1)^3} \cdot \frac{3^n}{n^3}$$

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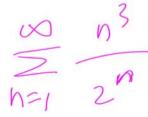
$$= \lim_{n\to\infty} \frac{3^n}{(n+1)^3} \cdot \frac{3^n}{($$

Some Remarks on Ratio Test

• Ratio Test fails if L = 1, for example,

$$\sum_{n=1}^{\infty} \frac{2n}{3n^2 + 1}$$

• When to use Ratio Test?



- 1) If the terms contain factorials, n!, (2n+1)!, etc, or
- 2) if the terms contain a mixture of exponential terms $(2^n, (-3)^n)$ and polynomial terms $(n^2, n^{\frac{3}{2}})$.

Summary on Series - 1 Two Important Classes

- Geometric Series and p-Series. We know exactly when they are convergent/divergent.
- Geometric Series

$$\sum_{n=1}^{\infty} a \cdot r^{n-1}$$

- 1) If |r| < 1, then the geometric series is convergent (to $\frac{a}{1-r}$).
- 2) If $|r| \ge 1$, then the geometric series is divergent.
- p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- 1) If p > 1, then the *p*-series is convergent.
- 2) If $p \le 1$, then the *p*-series is divergent.



Summary on Series - 2 the Testing Method

Given a series

$$\sum_{n=1}^{\infty} a_n$$

- First check if $\lim_{n\to\infty} a_n = 0$; if no, the series is divergent by Test for Divergence.
- Next check if it is an alternating series? If yes, then try Alternating Series Test.
- Then check if the series is a positive series; if yes, try either Comparison Test or Integral Test.
- Finally, we may try Ratio Test; especially in the following two cases.
 - 1) If the terms contain factorials, n!, (2n+1)!, etc, or
 - 2) if the terms contain a mixture of exponential terms $(2^n, (-3)^n)$ and polynomial terms $(n^2, n^{\frac{3}{2}})$.