LECTURE NO. 21

6.2 Properties of Power Series

Wright State University

Representing Functions as Power Series

- Consider

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- This is a geometric series with common ratio $r = x$.
- Therefore, the series is convergent to $\frac{1}{1-x}$ when $|x| < 1$.
- So the function $\frac{1}{1-x}$ can be represented by a power series:

Function

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Power Series

- This can help us find power series representations of many other functions.

Construct other power series representations

- Recall that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- Find a power series representation for $\frac{1}{1-2x}$.

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

Here we plug $2x$ into x

Find a power series representation for $\frac{3x}{1+x^2}$

- First we find a power series representation for $\frac{1}{1+x^2}$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

- Next multiply by $3x$ on both sides:

$$(3x) \cdot \frac{1}{1+x^2} = \underbrace{\left(1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots\right)}_{\cdot 3x} = \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] \cdot 3x$$

$$\frac{3x}{1+x^2} = 3x + (-3x^3) + 3x^5 + (-3x^7) + \cdots = \sum_{n=0}^{\infty} \left((-1)^n x^{2n} \cdot 3x\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot 3 \cdot x^{2n+1}$$

Use differentiation to find a power series representation for $\frac{1}{(1+x)^2}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \cdots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= 1 - x + x^2 - x^3 + x^4 + \cdots$$

Recall that $\left(\frac{1}{1+x}\right)' = -\frac{1}{(1+x)^2}$ $\quad [(1+x)^{-1}]' = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$

$$\left(\frac{1}{1+x}\right)' = \left(1 - x + x^2 - x^3 + x^4 + \cdots\right)' = \left(\sum_{n=0}^{\infty} (-1)^n x^n\right)'$$

$$-\frac{1}{(1+x)^2} = 0 - 1 + 2x - 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n n x^{n-1} = \sum_{n=1}^{\infty} (-1)^n \cdot n x^{n-1}$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots = (-1) \sum_{n=1}^{\infty} (-1)^n n x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n x^{n-1}$$

Use integration to find a power series representation for $\ln(1+x)$

$$(\ln(1+x))' = \frac{1}{1+x}$$

antiderivative

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\int \frac{1}{1+x}\,dx = \int 1 - x + x^2 - x^3 + x^4 + \cdots \,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n \,dx$$

$$\ln(1+x) = c + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} \int (-1)^n x^n \,dx = c + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

set $x=0$ LHS $= \ln 1 = 0$ RHS $= c + 0$ $\Rightarrow c = 0$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$$

Use integration to find a power series representation for $\tan^{-1}(x)$

$$(\tan^{-1}x)' = \frac{1}{1+x^2}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

$$\int \frac{1}{1+x^2}\,dx = \int 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots \,dx$$

$$\tan^{-1}x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots$$

set $x=0$, $LHS = \tan^{-1}0 = 0$ $\quad RHS = C \Rightarrow C = 0$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^{2n-1}}{2n-1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Summary

- We started with

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

- We used it to find power series representations for other similar functions.
  - Algebra: ex. $\frac{3x}{1+x^2}$.
  - Differentiation: ex. $\frac{1}{(1+x)^2}$
  - Integration: ex. $\ln(1+x)$, $\tan^{-1} x$