Rules for Significant Figures:

125	3 sig fig	This number has a numeral in all three places.
.001	1 sig fig	The zeroes are considered "place holders" and only the thousandths place is significant.
.101	3 sig fig	Here the zero is measured as a zero and all three are significant.
500	1 sig fig	This one can be ambiguous. If it is written "as is" without knowing how it was measured then you can only take it to have 1 significant figure. But, if it is known how it was measured it could be up to 3 significant figures.

The Number of Significant Figures of an Answer Depending on the Mathematical Operation:

Add. & Subtr. (concerns the number of places)

$$\begin{array}{l} 125 \\ \underline{+12.8} \\ 137.8 \Rightarrow 138 \end{array} \qquad \begin{array}{l} 125 \\ \underline{-12.8} \\ 112.2 \Rightarrow 112 \end{array}$$

The answer can only be as accurate as the numeral with the least accuracy in the summation or difference.

Mult. & Div. (concerns the number of significant figures)

$$(571)(2.2) = 1256.2 \Rightarrow 1300$$
$$(571)/(2.2) = 259.5454 \Rightarrow 260$$

The answer will have as many significant figures as the numeral which has the least number of significant figures in the multiplication or division.

Mult. & Div. by a pure number (such as in a formula: $2\pi r$)

The pure number is looked at as having an infinite number of zeros, such as 2 = 2.0000000..., thus, it always has the largest numbers of significant figures. Only the variable in the equation that is measured in the experiment governs how many significant figures the answer will have.

Whenever someone uses a device that measures some quantity (such as a meter-stick, gram balance, or stopwatch), they should be aware of the fact that this device is mass-produced. A manufacturer will reproduce the appropriate scale (such as a meter-stick) to a certain degree of precision. Usually, the higher the precision of the instrument, the higher the cost in making that instrument. The manufacturer will state the degree of precision with these higher cost devices. Otherwise, if no degree of precision is stated, it can be taken that the instrument is accurate to within half of the smallest marked increment. In the case of a wooden meter-stick, the smallest increment is usually one millimeter. This results in an uncertainty that is just 0.05 percent of the entire length of the meter-stick. An uncertainty that is small when taking data in the beginning lab.

The material of the device is subject to environmental changes. Again, with the wooden meter-stick, temperature and humidity could change the length of the meter-stick. With this expansion or contraction of the material the increments will either become spaced farther apart or closer together than when it was manufactured. Fortunately, the expansion/contraction of wood over an extreme temperature difference is not much. Also, it is known that the device will read off by the same amount, either always too large, or too small. **This is a source of systematic uncertainty.**

The experimenters themselves incorporate uncertainty when reading the measuring device. With the wooden meter-stick a length is measured of an object. The length does not always fall exactly on an increment so the experimenter estimates between marks to better represent that length. This is usually done by visually breaking up the smallest increment into tenths and estimating. **This is a source of random uncertainty.**

Finally, an object that is to be measured may not be uniform in its shape. The surface area of a tabletop can be found by multiplying the length of the table by its width. This use of the formula presumes that the tabletop is a perfect rectangle. However, if someone was to measure the length of the tabletop at different positions (fig. 1) it can be found that each measurement is slightly off from another. This is due to the inconsistencies of the table as it was made, and further if the table had been banged up a bit through the years. **This is a source of random uncertainty.**

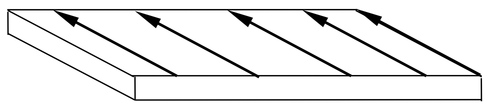

fig. 1

The more measurements taken, the better a representation of a measure of the length can be found. With this beginning laboratory, and the time allotted to take measurements, five measurements of any dimension should give us a reasonable representation of an average value for that measurement. The average value may not be the true value, but it is our best approximation. Uncertainty analysis allows a range about this average value such that one of those values within the range is most likely the true value.

Also, you will use the following formula for determining the standard deviation of several measurements of a dimension (or mass, or time, or velocity, etc.). More about this formula is shown in the example that follows.

$$\sigma = \sqrt{\frac{(\bar{x} - x_1)^2 + (\bar{x} - x_2)^2 + \cdots + (\bar{x} - x_n)^2}{n-1}}$$

To determine the total uncertainty (or absolute uncertainty) associated with the measurements of length and width the standard deviation and the systematic uncertainty of each measurement is combined into a total uncertainty value.

$$\delta = \sqrt{(\sigma)^2 + (sys.)^2}$$

The Greek letter δ (lower-case delta) represents the total uncertainty of the particular measurement.

If two measured quantities are being added or subtracted the total uncertainty in their answer is:

$$C = A + B \quad or \quad C = A - B$$
$$\delta_C = \sqrt{\delta_A^2 + \delta_B^2}$$

If two measured quantities are being multiplied or divided the fractional uncertainty in their answer is:

$$C = AB \quad or \quad C = A/B$$
$$\frac{\delta_C}{C} = \sqrt{\left(\frac{\delta_A}{A}\right)^2 + \left(\frac{\delta_B}{B}\right)^2}$$

(The percent uncertainty is merely the fractional uncertainty times 100.)

And the total uncertainty can then be determined:

$$\therefore \delta_C = C\sqrt{\left(\frac{\delta_A}{A}\right)^2 + \left(\frac{\delta_B}{B}\right)^2}$$

Finally, if a formula has a power involved in a measured quantity the fractional uncertainty can be determined (k is a constant value and not a measured value):

$$C = kAB^n$$
$$\frac{\delta_C}{C} = \sqrt{\left(\frac{\delta_A}{A}\right)^2 + \left(n\frac{\delta_B}{B}\right)^2}$$

Now for a sample set of data:
The length and width of a tabletop was measured 5 times each. The surface area of the tabletop is to be calculated and the associated uncertainties are also to be calculated.

length		
20.25	± .05	cm
20.23	± .05	cm
20.22	± .05	cm
20.25	± .05	cm
20.23	± .05	cm
20.24	±.05	cm

←avg.→

width		
15.98	± .05	cm
16.03	± .05	cm
15.98	± .05	cm
15.92	± .05	cm
15.96	± .05	cm
15.97	±.06	cm

Note that each of the measurements for length and for width has a systematic uncertainty of ±.05 centimeters. This value is determined by taking half of the smallest increment of the measuring device. Here, the device was a centimeter-scale ruler with millimeter markings (0.1 centimeter markings) as its smallest unit. The hundredths place was estimated.

The average of the 5 measurements in the length results in a value of 20.236 cm. However, following the rules of significant figures has the answer written as 20.24 cm. The same goes for the width. The average for it is 15.974 cm., and, following the rules of significant figures, the answer is written as 15.97 cm.

As you can see the measurements do vary a little bit caused by the random uncertainty associated with the object's inconsistent dimensions and our ability to estimate the measurement to the nearest hundredth of a centimeter. This tells you that uncertainty should be considered and a range of values should be determined for each of these measurements. To determine a reasonable amount of deviation that the numbers are showing use the following statistical analysis formula for determining the Standard Deviation.

$$\sigma_x = \sqrt{\frac{(\bar{x}-x_1)^2 + (\bar{x}-x_2)^2 + (\bar{x}-x_3)^2 + (\bar{x}-x_4)^2 + (\bar{x}-x_5)^2}{5-1}}$$

where $\bar{x}$ is the full average value of the measurement and $x_1,\ldots,x_5$ are each of the measurements made. The value of 5 – 1 comes from the general formula stated as n – 1. Since we have 5 measurements then n takes on that value.

For the length value the standard deviation is equal to 0.013416408 cm. as read off of a calculator display. Similarly, the width's standard deviation is found to be equal to 0.0397492138 cm.

To determine the total uncertainty (or absolute uncertainty) associated with the measurements of length and width the standard deviation and the systematic uncertainty of each measurement is combined into a total uncertainty value.

$$\delta = \sqrt{(\sigma)^2 + (sys.)^2}$$

The Greek letter δ (lower-case delta) represents the total uncertainty of the particular measurement. A subscript is usually included to designate what is being measured. In the case mentioned before total uncertainties in the length and width can now be determined.

$$\delta_L = \sqrt{(0.013416408)^2 + (0.05)^2} = 0.051768716$$
$$\delta_W = \sqrt{(0.0397492138)^2 + (0.05)^2} = 0.063874877$$

Uncertainties can at most have two significant figures. In addition, the number of decimal places of the uncertainties needs to match those of the value it is stated with. Take note here that the standard deviation in the length measurement is small compared to the systematic uncertainty. Thus, the total uncertainty of the length seems to be the same as the systematic uncertainty. This is not always the case, as you can see with the total uncertainty of the width. You, as experimenters, will find that the total uncertainty of a measurement can have a larger value than the systematic uncertainty due to the irregularities associated in measuring objects.

The surface area of the table and its total uncertainty in the surface area can now be determined. Multiply the full average values found for the length and width together first.

$$\text{Surface Area} = (\text{length}) \times (\text{width}) = (20.236\,\text{cm}) \times (15.974\,\text{cm}) = 323.249864\,\text{cm}^2$$

Using the rules of significant figures the recorded value of the Surface Area = 323.2 cm^2.

The fractional uncertainty of the Surface Area is found by:

$$\frac{\delta_A}{\mathrm{A}} = \sqrt{\left(\frac{\delta_{\mathrm{L}}}{\mathrm{L}}\right)^2 + \left(\frac{\delta_W}{W}\right)^2} = \sqrt{\left(\frac{0.051768716}{20.236}\right)^2 + \left(\frac{0.0638748777}{15.974}\right)^2} = 0.004747062$$

So, the total uncertainty in the Surface Area is found by multiplying through by the Surface Area value of 323.249864 cm^2.

$$\delta_A = (0.0047470052) \times (323.249864\ \mathrm{cm}^2) = 1.534468794\ \mathrm{cm}^2$$

If we were to look at the fractional uncertainties of length and width we would be able to tell which one contributes more uncertainty to the final value of area by which ever one has the largest value. The fractional uncertainty shows us the precision of the measurement. If we looked only at the total uncertainty of a measurement it would not give us a good idea of the precision of that measurement. However, the ratio of the total uncertainty to the average value will. [For instance, a measurement of 100.0 cm ± 0.1 cm is much more precise of a measurement than 1.0 cm ± 0.1 cm, even though they have the same total uncertainties. The first value's fractional uncertainty is equal to 0.001, while the second value's fractional uncertainty is equal to 0.1. The first value's fractional uncertainty is very small indicating a higher precision in its measurement.]

Again, the rules of significant figures come into play. Here we see that the average value for the Surface Area is only accurate to the tenths place. The uncertainty value cannot be more or less accurate than this value. The uncertainty can have at most two significant figures, (but could have only one), to match up with the same number of decimal places as the average value determined.

$$\text{Surface Area} = 323.2\,\mathrm{cm}^2 \pm 1.5\,\mathrm{cm}^2$$

ERROR ANALYSIS OF STRAIGHT LINE PLOTS

The following method will be used for determining the uncertainty of the slope and y-intercept of a straight line plot. This method is adapted from one proposed by John L. Safko (American Journal of Physics, 33 379 (1965). As you read the directions, follow the graph on the next page to fully understand the method.

1. Draw the best line, L_0, through the data points. Find the slope, K_0, of this line. Use graph points the line passes through, not data points. Find the y-intercept, A_0.
2. Draw lines L_1 and L_2 parallel to L_0 on both sides of L_0 such that they pass through the data points farthest away from L_0. Remember to take the error bars into account.
3. Draw two lines, y_1 and y_2, parallel to the y-axis and passing through the data points at largest and smallest x. Again, take uncertainty in the x-direction into account.
4. The four lines, L_1, L_2, y_1, and y_2, form a parallelogram with corners a, b, c, and d. Determine the slope, K_{max}, for the line $\overline{bc}$. Determine the slope, K_{min}, for the line $\overline{ad}$. 5. The uncertainty in the slope is defined as:

$$\frac{K_{max} - K_{min}}{2}$$

6. The uncertainty of the y-intercept can be obtained by extending the diagonals of the parallelogram to the y-axis and labeling the resulting points A_{max} and A_{min}.
7. The uncertainty in the y-intercept is defined as:

$$\frac{A_{max} - A_{min}}{2}$$

As an example, consider the graph on the following page.

- The graph has a title and the axes are labeled.
- Each data point is circled and error bars are drawn.
- Calculations are written clearly on the graph.
- Lines bc and ad do not have to be drawn in order to calculate the slope.
- Data points are not used to determine the slope of L_0.

Title of Graph