Counting Techniques

- often necessary to determine the total # of outcomes in a sample space in order to determine probabilities associated with them

multiplication rule

- operation involving k steps

  # of ways of computing step 1 : n_1

  step 2 : n_2

$$\# = n_1 \cdot n_2 \cdot \ldots \cdot n_k$$

ex: new car; may choose from three different stereos ← $n_1 = 3$

four colors → $n_2 = 4$

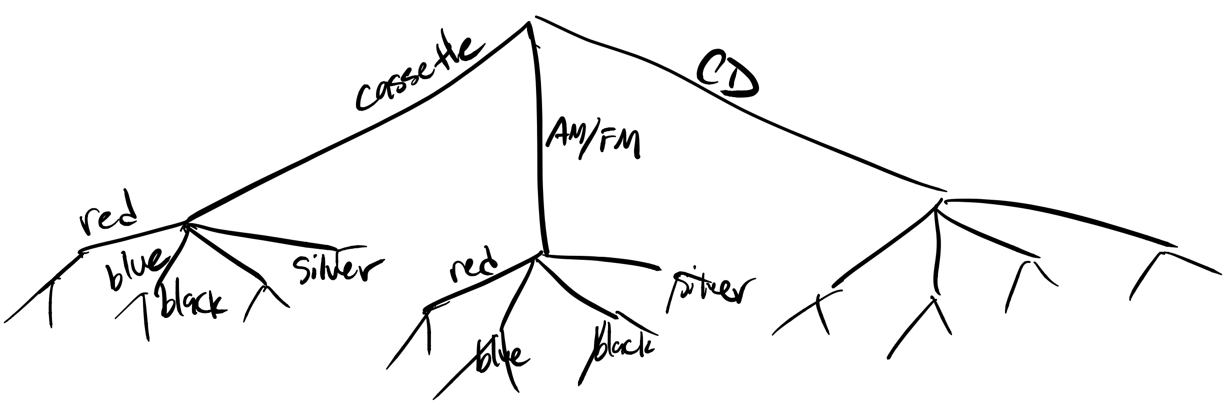
$n_3 = 2$ ← automatic or stick transmission

$n_4 = 2$ ← with or without A/C

∴ if no dependence between options (i.e., all red cars have manual transmission), then the number of possible cars is

$$n_1 \cdot n_2 \cdot n_3 \cdot n_4 = 3 \cdot 4 \cdot 2 \cdot 2 = \underbrace{48}_{\text{unique cars}}$$

∴ tree diagrams are useful here, can be modified to show dependence



sequences of elements

permutations: **ordered** sequences of elements

ex: $S\ \{1\ 2\ 3\}$

possible ordered sequences:

$\{123 \quad 321 \quad 132 \quad 213 \quad 231 \quad 312\}$

∴ we can see that there are six permutations of $\{1\,2\,3\}$

∴ the # gets big fast!

$\#$ of permutations $= n!$

∴ in this case: $n = 3$

∴ $\# = 3! = 3 \times 2 \times 1 = 6$

permutations of subsets

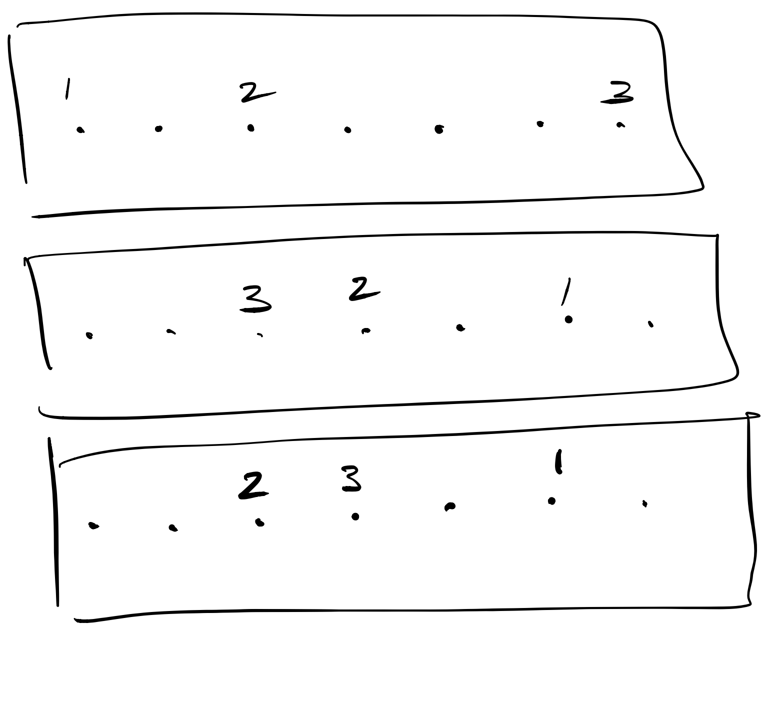
r elements selected from n total elements

$$P^n_r = \frac{n!}{(n-r)!}$$

notation!

ex: seven pilot holes in a wooden panel

- how many ways can you put three screws in those seven pilot holes if <u>order matters</u>?



$$P_3^7 = \frac{7!}{(7-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}} = 210$$

∴ we would have been here all day trying to figure this out by hand!

\# of permutations of $n = n_1 + n_2 + n_3 + \dots n_r$ objects

n_1 : one type
n_2 : different type etc.

$$\# = \frac{n!}{n_1! \cdot n_2! \cdot n_3! \dots n_r!}$$

ex: hospital operating room must schedule

$n_1 = 3$ knee surgeries

$n_2 = 2$ hip surgeries

∴ how many possible ordered sequences?

k h k h k

h h k k k

etc ..

$n = n_1 + n_2 = 5$

$$\# = \frac{5!}{2!\,3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2!\,\cancel{3!}}$$

$= 10$

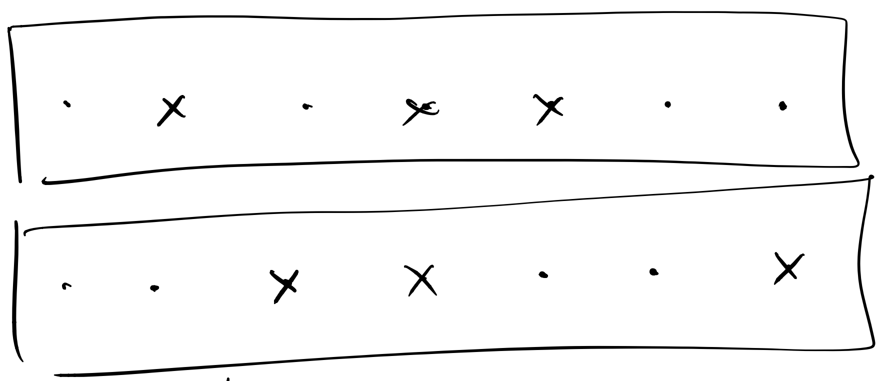
<u>Combinations</u>: like permutations, but <u>order is not important</u>

$$C^{n}_{r} = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

parenthetical notation!

this term in denominator can greatly reduce # vs. permutations

ex: pilot hole example



$$\binom{7}{3} = \frac{7!}{3!\,(7-3)!} = \frac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4!}}{\underbrace{\cancel{3!}}_{6}\;\cancel{4!}}$$

$$= 35 \ll 210\,!$$

- **probability** quantifies the likelihood or chance that an outcome in the sample space of a random experiment will occur

  - may range from 0 to 1 (or 100%)

- the way in which we compute probabilities depends on the nature of the experiment

Model of Equally-Likely Outcomes

- in a sample space of n outcomes, each outcome has the same probability of $\frac{1}{n}$

ex: fair six-sided die ← singular of dice!

- the probability of rolling each number is $\frac{1}{n}$

$$= \frac{1}{6} \cong 16.7\%$$

ex: let's say we have a sample space with four outcomes

$$S \{a \ b \ c \ d\}$$

.. let's say we know

$$P(a) = 0.1 \quad (\text{or } 10\%)$$

$$P(b) = 0.3 \quad (\text{or } 30\%)$$

$$P(c) = 0.5 \qquad \text{etc.}$$

$$P(d) = 0.1$$

define events $A \{a \ b\}$

$B \{b \ c \ d\}$

$C \{d\}$

-- for a discrete sample space:

$$\boxed{\text{the probability of an \underline{event} is the sum of the probabilities of the outcomes in that event}}$$

ex: $P(A) = P(a) + P(b) = 0.1 + 0.3 = 0.4$ or $\underline{40\%}$

$$P(B) = P(b) + P(c) + P(d)$$

$$= 0.3 + 0.5 + 0.1$$

$$= \underline{0.9 \text{ or } 90\%}$$

$$P(C) = P(d) = \underline{0.1 \text{ or } 10\%}$$

∴ this concept applies to <u>set operations</u> as well!

$$P(A') = P\{c\ d\} = 0.5 + 0.1 = 0.6$$

or 60%

$P(B') = 0.1$, $P(C') = 0.9$

(check!)

$$P(A \cap B) = P(b) = 0.3$$

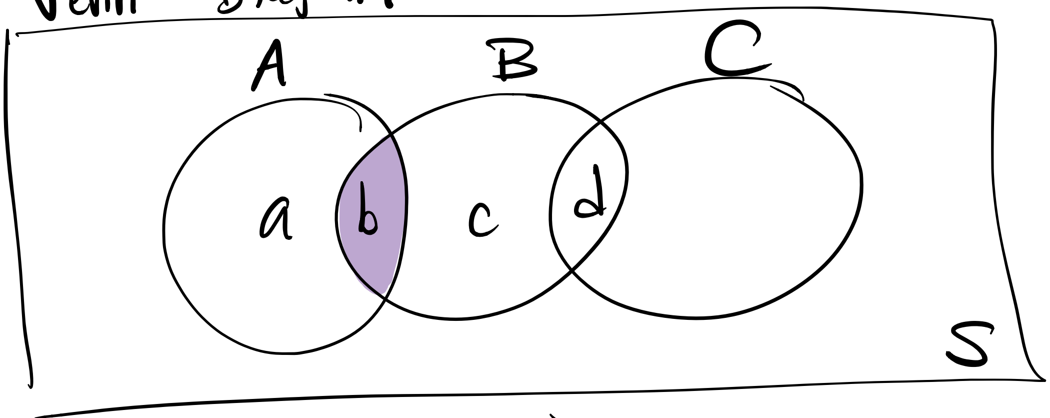
$$P(A \cup B) = 1 \text{ or } 100\%$$

check!

∴ verify w/ Venn diagram!

$$P(A \cap C) = P\{\overset{\text{null set}}{\emptyset}\} = 0$$

Venn Diagram



∴ shade $P(A \cap B)$