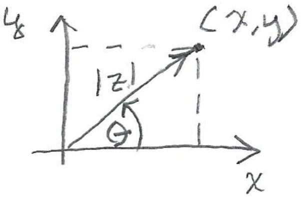
BACKGROUND REVIEW

Complex numbers: $z = x + jy = |z|e^{j\theta}$

Magnitude: $|z| = \sqrt{x^2+y^2}$

Angle: $\theta = \tan^{-1}\frac{y}{x}$



Complex conjugate: $z^* = x - jy$

Multiplication of two complex numbers:

$$zw = (x+jy)(a+jb) = (xa-yb) + j(ya+xb)$$

$$(3+j4)(1-j2) = (3+8) + j(4-6) = \begin{cases} 11 - j2 & \boxed{\text{Cartesian Form}} \\ \sqrt{11^2+2^2}\, e^{j\tan^{-1}\left(\frac{-2}{11}\right)} \\ = 11.18\, e^{-j10.3048^\circ} & \boxed{\text{Polar Form}} \end{cases}$$

Division of two complex numbers (EX):

$$\frac{3+j4}{1-j2} = \frac{(3+j4)(1+j2)}{(1-j2)(1+j2)} = \frac{(3-8)+j(4+6)}{5} = -1+j2$$

Euler's Identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

Vectors and Matrices

- ✓ definitions & properties*
- matrix algebra:
  - ✓ matrix addition/sub.
  - ✓ mult. of matrix by scalar
  - ✓ matrix multiplication
  - ✓ mult. of matrix by vector
  - ✓ matrix inversion

* definitions & properties

square	zero
identity	symmetric
diagonal	transpose

$AB \neq BA$ (in general)

$(A+B)C = AC + BC$

$AI = IA = A$

$\det(AB) = \det A \det B$

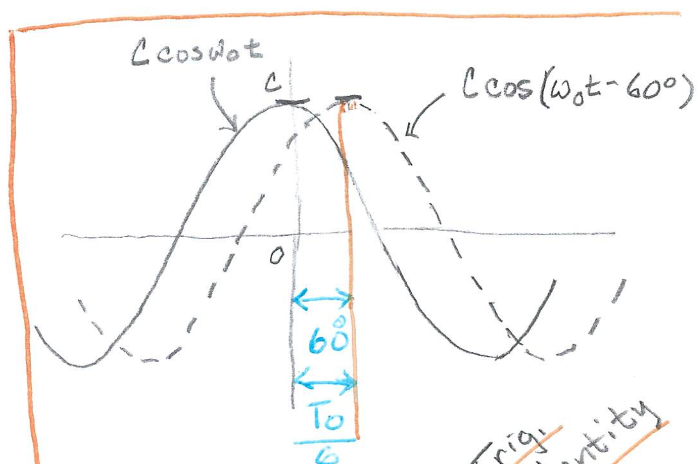
NB For matrix/vector multiplication, must be conformable

$\boxed{\text{Important Integration Formulas}}$

$$\int A\,dt = At + c, \text{ where } A \text{ is a constant}$$
$$\int e^{at}\,dt = \frac{1}{a}e^{at} + c$$
$$\int \cos(at)\,dt = \frac{1}{a}\sin(at) + c$$
$$\int \sin(at)\,dt = -\frac{1}{a}\cos(at) + c$$

$\boxed{\text{Important Differentiation Formulas}}$

$$\frac{d}{dt}A = 0, \text{ where } A \text{ is a constant}$$
$$\frac{d}{dt}At = A, \quad '' \quad '' \quad '' \quad '' \quad ''$$
$$\frac{d}{dt}e^{at} = a e^{at}$$
$$\frac{d}{dt}\cos(at) = -a\sin(at)$$
$$\frac{d}{dt}\sin(at) = a\cos(at)$$



$\boxed{\text{Sinusoids}}$

Adding

$$C\cos(\omega_0 t + \theta) = \underbrace{C\cos\theta}_{a}\cos\omega_0 t \underbrace{- C\sin\theta}_{b}\sin\omega_0 t \quad \text{Trig. Identity}$$
$$= a\cos\omega_0 t + b\sin\omega_0 t$$

$$a^2 + b^2 = C^2\cos^2\theta + C^2\sin^2\theta = C^2(\cancel{\cos^2\theta + \sin^2\theta}) = C^2 \Rightarrow C = \sqrt{a^2+b^2}$$

$$\frac{b}{a} = \frac{-C\sin\theta}{C\cos\theta} = -\tan\theta \Rightarrow \theta = \tan^{-1}\left(-\frac{b}{a}\right)$$

Sketching

$$C\cos(\omega_0 t + \theta), \quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} \qquad T_0 = \frac{1}{f_0} = \text{period}$$

$$\omega_0 T_0 = 2\pi \equiv 360° \Rightarrow \omega_0 = \frac{360°}{T_0}$$

✓ $\theta = 0$: $C\cos(\omega_0 t + \theta) = C$ @ $t = 0$

✓ $\theta \neq 0$: $C\cos(\omega_0 t + \theta) = C$ @ $\omega_0 t + \theta = 0$ or $t = -\frac{\theta}{\omega_0}$

✓ $\theta = -60°$

$$\Rightarrow t = -\frac{\theta}{\omega_0} = \frac{60°}{\omega_0} = \frac{60°}{360°}T_0 = \frac{T_0}{6}$$

$\boxed{\text{Solving Differential Equations using Laplace}}$

$$\frac{d^3y(t)}{dt^3} + \frac{d^2y(t)}{dt^2} - 8\frac{dy(t)}{dt} - 12y(t) = 3\frac{dx(t)}{dt} - x(t)$$

Laplace Transform

$\mathcal{L} \to$ $s^3Y(s) + s^2Y(s) - 8sY(s) - 12Y(s) = 3sX(s) - X(s)$

*(if i.c.s = 0) $(s^3 + s^2 - 8s - 12)Y(s) = (3s-1)X(s)$

repeated roots

$\boxed{\text{rational function}}$ $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{3s-1}{s^3+s^2-8s-12} = \dfrac{3s-1}{(s-3)(s+2)^2}$

$\boxed{\text{partial fraction expansion (PFE)}}$

$\deg Y(s) = m$

$\deg X(s) = n$

strictly proper functions $m < n$

proper " $m \le n$

improper " $m > n$

use long division on $H(s)$

$\to H(s) = H(s)_{poly} + H(s)_{s.p.}$

$H(s)$ EX is s.p. → use PFE

$$= \frac{A}{s-3} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

$$= \frac{8/25}{s-3} + \frac{7/5}{(s+2)^2} + \frac{-8/25}{s+2}$$

$\mathcal{L}^{-1}$ Inverse Laplace Transform

$$\mathcal{L}^{-1}(H(s)) = h(t)$$

$$= \frac{8}{25}e^{3t}u(t) + \frac{7}{5}te^{-2t}u(t) - \frac{8}{25}e^{-2t}u(t)$$

$$A = \left.\frac{3s-1}{(s+2)^2}\right|_{s=3} = \frac{8}{25}$$

$$B = \left.\frac{3s-1}{s-3}\right|_{s=-2} = \frac{7}{5}$$

$$C = \left.\frac{3(s-3)-(3s-1)}{(s-3)^2}\right|_{s=-2} = -\frac{8}{25}$$

✓ distinct roots

✓ repeated roots

✓ complex roots ...

Partial Fraction Expansion - Repeated Roots

found using "cover up" method

$$X(s)=\frac{1}{(s+1)(s+2)^4}=\frac{1}{s+1}+\frac{k_1}{s+2}+\frac{k_2}{(s+2)^2}+\frac{k_3}{(s+2)^3}-\frac{1}{(s+2)^4}$$

① Multiply both sides by s and let $s\to\infty$

$$0=1+k_1+0+0+0 \quad\Rightarrow\quad \boxed{k_1=-1}$$

$$\frac{1}{(s+1)(s+2)^4}=\frac{1}{s+1}-\frac{1}{s+2}+\frac{k_2}{(s+2)^2}+\frac{k_3}{(s+2)^3}-\frac{1}{(s+2)^4}$$

Choose s easy to evaluate but not a pole (in EX, not -1 or -2)

Setting $s=0$ and -3 on both sides yields

$$\frac{1}{16}=1-\frac{1}{2}+\frac{k_2}{4}+\frac{k_3}{8}-\frac{1}{16} \quad\Rightarrow\quad 4k_2+2k_3=-6$$

$$-\frac{1}{2}=-\frac{1}{2}+1+k_2-k_3-1 \quad\Rightarrow\quad k_2-k_3=0$$

$$\Downarrow$$

$$k_2=k_3=-1$$

$$X(s)=\frac{1}{s+1}-\frac{1}{s+2}-\frac{1}{(s+2)^2}-\frac{1}{(s+2)^3}-\frac{1}{(s+2)^4}$$

$$x(t)=\left[e^{-t}-\left(1+t+\frac{t^2}{2}+\frac{t^3}{6}\right)e^{-2t}\right]u(t)$$

Comment: Can tackle this problem in many ways.

② For $F(x)=\dfrac{a_0}{(x-\lambda)^r}+\dfrac{a_1}{(x-\lambda)^{r-1}}+\dots+\dfrac{a_{r-1}}{(x-\lambda)}$

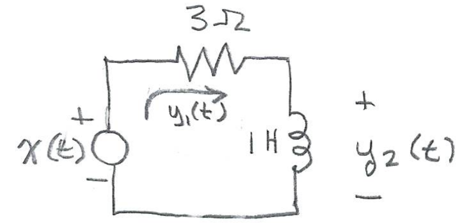
$$a_j=\frac{1}{j!}\frac{d^j}{dx^j}\left[(x-\lambda)^r F(x)\right]\Big|_{x=\lambda}$$

③ After determining two coefficients above $\left[\frac{1}{s+1}, -\frac{1}{(s+2)^4}\right]$ "cover up" method

clear fractions and equate coefficients of same degree of s and solve simultaneous equations

④ Use a different set of values for method ①
(although note $s=0$, $s=-3$ yield simple results)

EE 2010 - Analog Circuit Theory



Ex For the given circuit, find the differential equations relating outputs $y_1(t)$ and $y_2(t)$ to the input $x(t)$

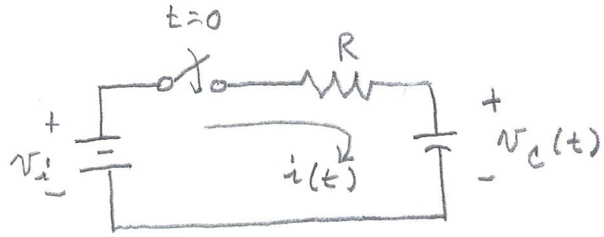
KVL (loop)

$$x(t) = 3y_1(t) + y_2(t) = 3y_1(t) + \frac{dy_1(t)}{dt} \quad ①$$

$$\frac{dx(t)}{dt} = 3\frac{dy_1(t)}{dt} + \frac{dy_2(t)}{dt}$$

$$= 3y_2(t) + \frac{dy_2(t)}{dt} \quad ②$$

For given $x(t)$, can you find $y_1(t)$? $y_2(t)$?



$R = 1\,k\Omega \quad C = 25\,\mu F \quad v_i = 10V$

$v_c(0^-) = 0V$

$\boxed{\text{FIND } v_c(t)}$

$$\tau = RC = 1000\left(\frac{25}{1000000}\right) = \frac{25}{1000} = .025$$

$$v_c(t) = v_i(t)\left(1 - e^{-t/\tau}\right)$$

NOTE: $\lim_{t\to\infty} v_c(t) = v_i$

$$\frac{1}{RC} = \frac{1000}{25} = 40$$

$t > 0$: $v_c(t) = 10\left(1 - e^{-40t}\right)$ volts

NOTE: ① Can also solve diff EQ w/ Laplace
② Can use impedance form / voltage divider

<u>Cramer's Rule</u> - convenient way to solve simultaneous linear equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$x_k = \frac{|D_k|}{|A|} \quad k = 1, \ldots, n$$

where $|D_k|$ obtained by replacing kth column of $|A|$ by $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

<u>EX</u>

$$2x_1 + x_2 + x_3 = 3$$
$$x_1 + 3x_2 - x_3 = 7$$
$$x_1 + x_2 + x_3 = 1$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}}_{y}$$

$$|A| = 6 - 1 + 1 - 3 + 2 - 1 = 4 \neq 0$$

$\Rightarrow \exists$ unique soln x_1, x_2, x_3

$$x_1 = \frac{1}{|A|} \begin{vmatrix} 3 & 1 & 1 \\ 7 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{8}{4} = 2$$

$$x_2 = \frac{1}{|A|} \begin{vmatrix} 2 & 3 & 1 \\ 1 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{4}{4} = 1$$

$$x_3 = \frac{1}{|A|} \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 1 & 1 \end{vmatrix} = -\frac{8}{4} = -2$$