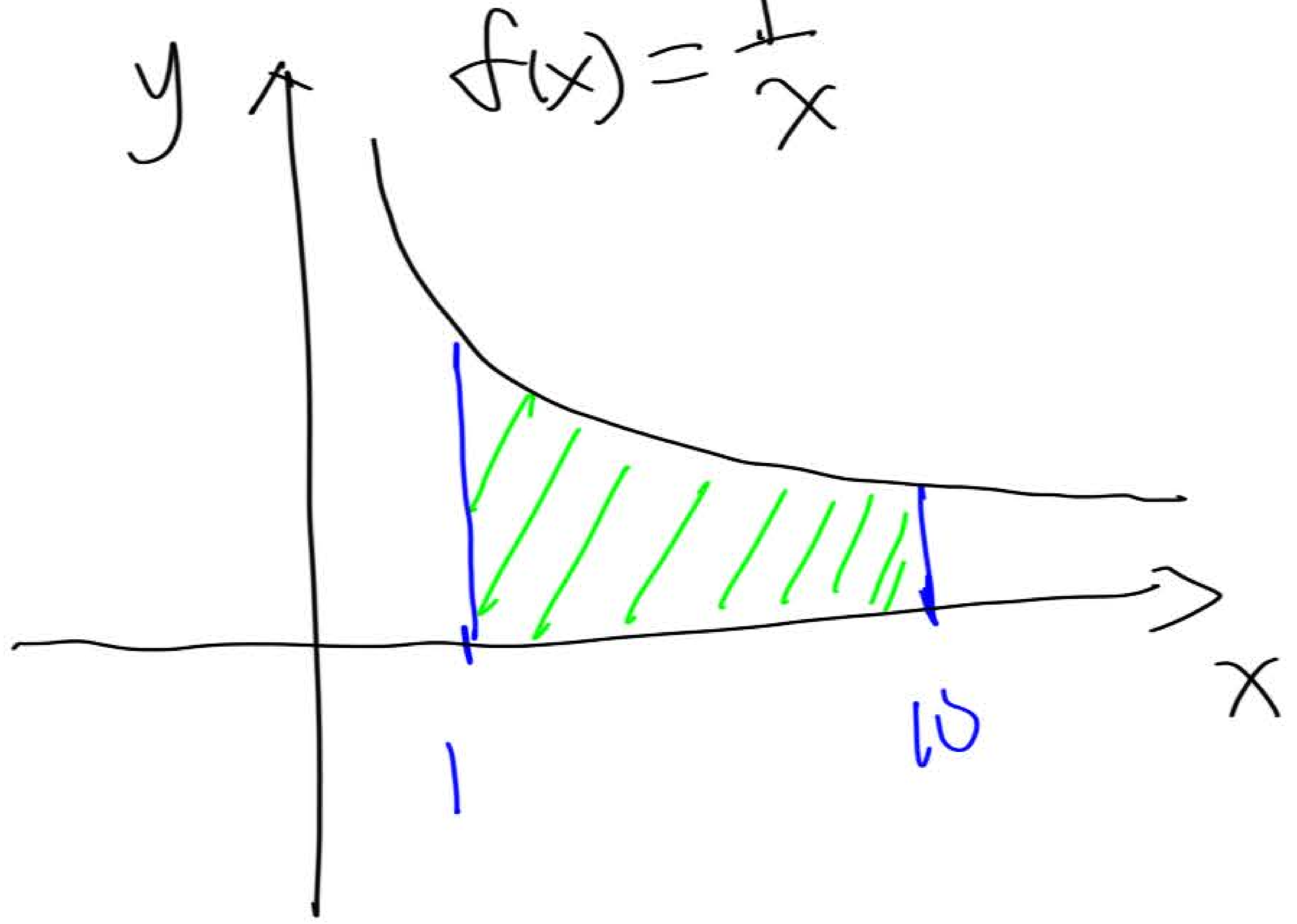
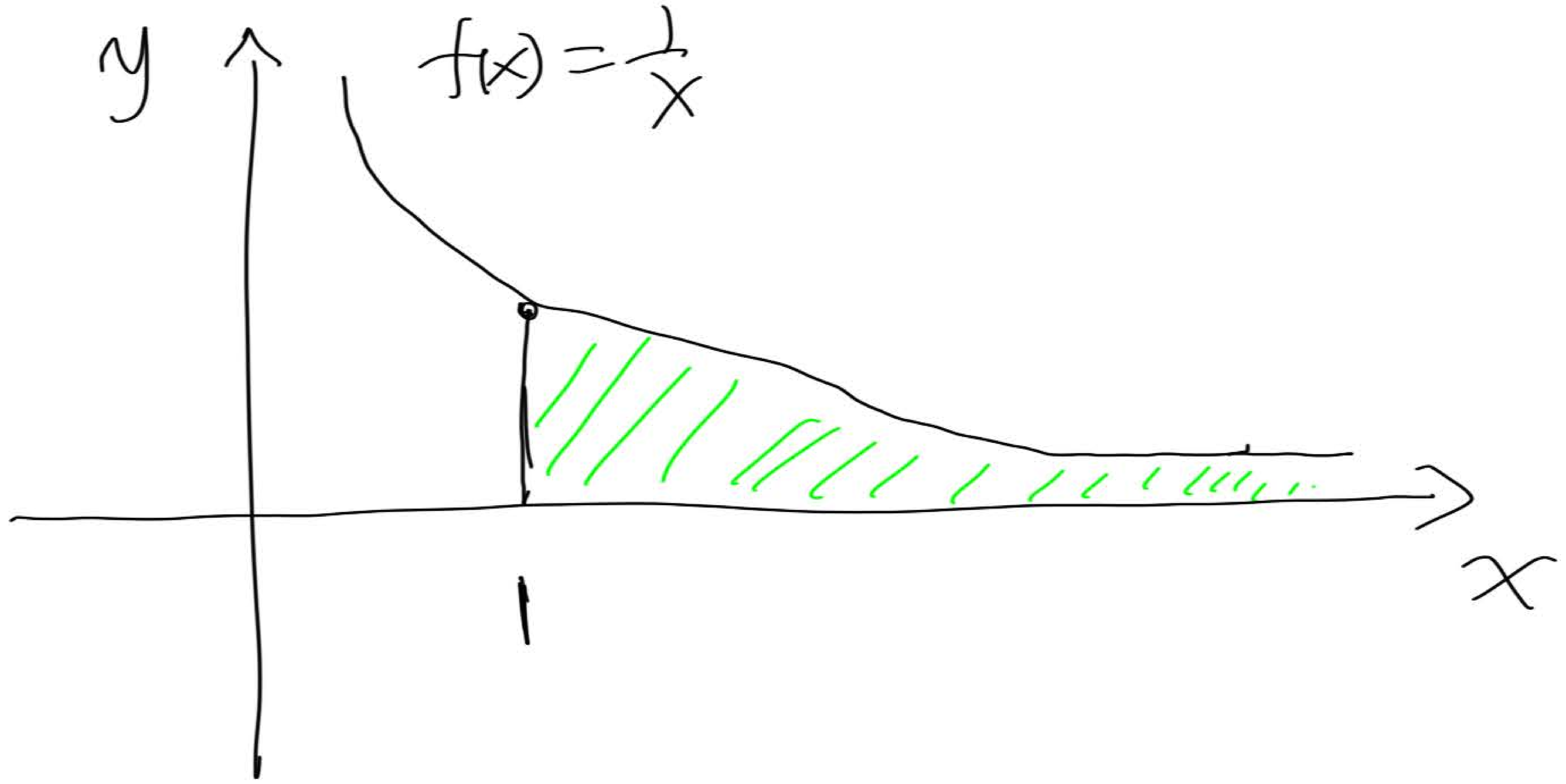
LECTURE NO. 13

3.7 Improper Integrals

Wright State University

Proper Integral vs Improper Integral





$$\int_1^{10} \frac{1}{x}\,dx = \text{Area of shaded region}$$

↓

a proper integral!

$$\int_1^{\infty} \frac{1}{x}\,dx = \text{Area of an unbounded region}$$

↓

An Improper integral

finite area (convergent!)

infinite area (divergent!)

$\int_1^\infty \frac{1}{x} dx$

↳ we need an antiderivative!

$$\ln|x| \Big|_1^\infty = \lim_{x\to\infty} \ln|x| - \underbrace{\ln 1}_{0} = \infty$$

This improper integral ($\int_1^\infty \frac{1}{x} dx$) is divergent!

final answer.

$\int_1^\infty \frac{1}{x^2} dx$

$$\int_1^\infty \frac{1}{x^2}dx = \int_1^\infty x^{-2}dx = \frac{x^{-1}}{-1}\Big|_1^\infty = -\frac{1}{x}\Big|_1^\infty$$

$$= \lim_{x\to\infty}\left(-\frac{1}{x}\right) - \left(-\frac{1}{1}\right) = 0-(-1) = 1$$

This improper integral $\left(\int_1^\infty \frac{1}{x^2}dx\right)$ is convergent to 1.

final answer

$\int_0^\infty xe^x\,dx$

$\int xe^x\,dx$

IBP. $\int u\,dv = uv - \int v\,du$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x + C$

$\int_0^\infty xe^x\,dx$

$= (xe^x - e^x)\Big|_0^\infty$

$$= \lim_{x\to\infty} (\underbrace{xe^x}_{\infty} - \underbrace{e^x}_{\infty}) - \underbrace{(0\cdot e^0 - e^0)}_{-1}$$

$= \lim_{x\to\infty} (x-1)e^x + 1$

$= \infty$

This improper integral is divergent!

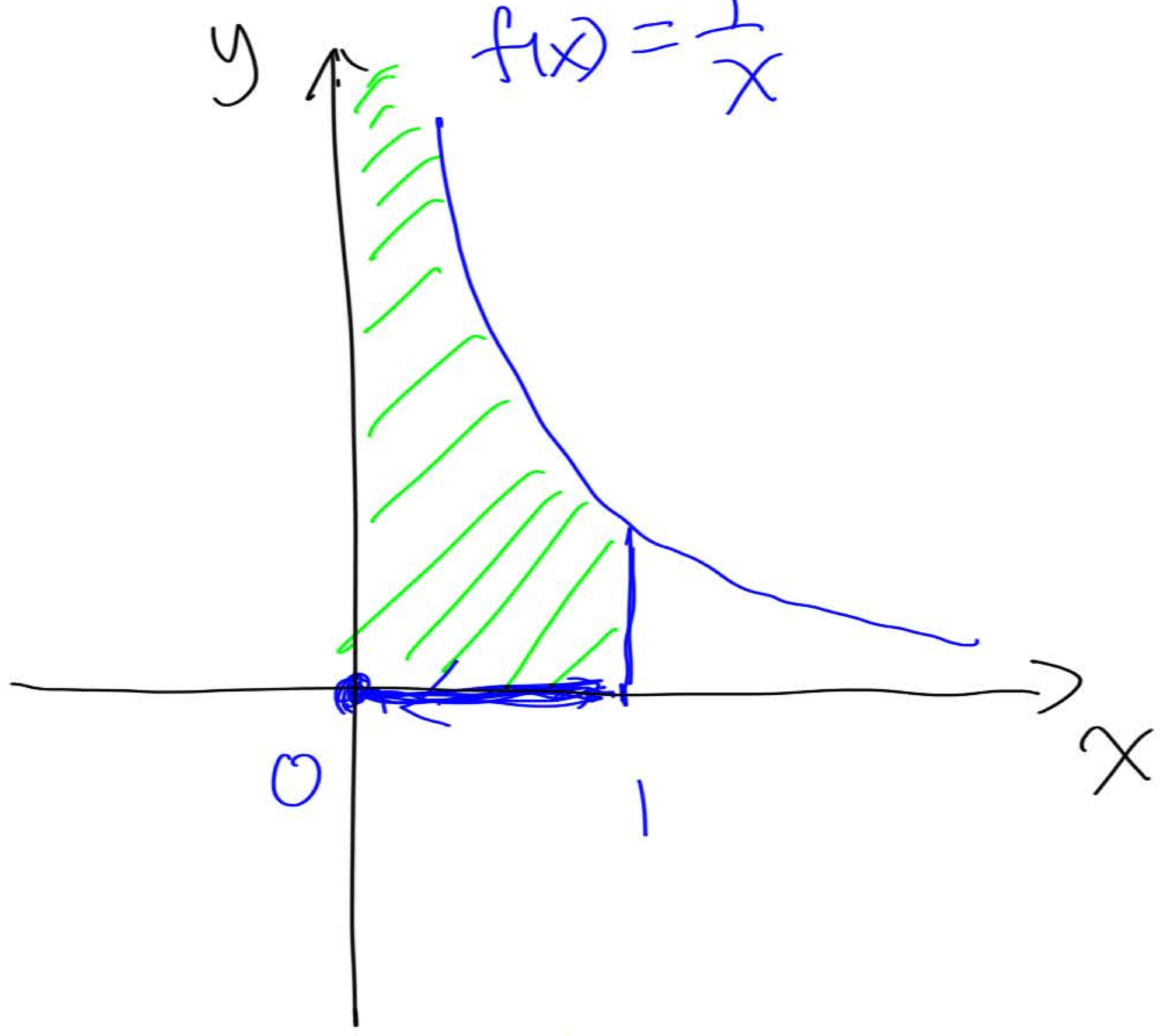
$\int_1^\infty \frac{1}{x^p}dx$

- It is divergent if $p = 1$; while it is convergent if $p = 2$.
- Try $p = 0.9$: $\int_1^\infty \frac{1}{x^{0.9}}dx$; and try $p = 1.1$: $\int_1^\infty \frac{1}{x^{1.1}}dx$
- $\int_1^\infty \frac{1}{x^{0.9}}dx$ is divergent; while $\int_1^\infty \frac{1}{x^{1.1}}dx$ is convergent.
- In general, we have
  - If $p > 1$, $\int_1^\infty \frac{1}{x^p}dx$ is convergent.
  - If $p \leq 1$, $\int_1^\infty \frac{1}{x^p}dx$ is divergent.

p - integral

$$\int_1^\infty \frac{1}{x^p} dx$$

Improper Integral of Second Type: $\int_0^1 \frac{1}{x}dx$

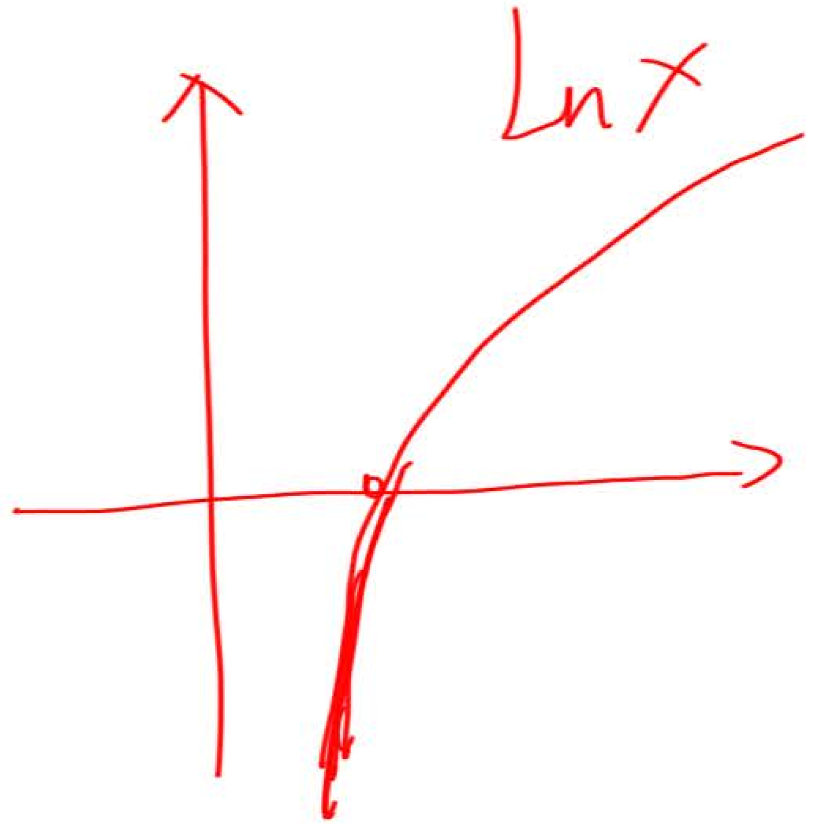


y-axis is a vertical asymptote!

So $\int_0^1 \frac{1}{x}dx$ corresponds to an unbounded region!

So it is an improper integral

$$\int_0^1 \frac{1}{x}dx = \ln|x|\Big|_0^1 = \ln 1 - \underbrace{\lim_{x\to 0^+} \ln|x|}_{-\infty} = \infty$$



The improper integral ($\int_0^1 \frac{1}{x}dx$) is divergent!

$\int_0^4 \frac{1}{\sqrt{4-x}}dx$

Why is this integral improper?

Because $x=4$ is a vertical asymptote!

$\int_0^4 \frac{1}{\sqrt{4-x}}dx$ $\quad u=4-x \quad \frac{du}{dx}=-1 \quad dx=-du$

$x: 0 \to 4.$

$u=4-x: 4 \to 0$

$u^n \xrightarrow[\text{derivative}]{\text{anti-}} \frac{u^{n+1}}{n+1}$

$$\int_4^0 \frac{1}{\sqrt{u}}(-du) = \int_0^4 \frac{1}{\sqrt{u}}du = \int_0^4 u^{-\frac{1}{2}}du$$

$$= 2u^{\frac{1}{2}}\Big|_0^4 = 4-0 = 4$$

This improper integral is convergent to 4.

$\int_3^5 \frac{2}{(x-3)^4}dx$

This is an improper integral because $x=3$ is a vertical asymptote!

$x: 3 \to 5$

$u = x-3 \qquad \frac{du}{dx} = 1 \qquad du = dx \qquad u = x-3: 0 \to 2$

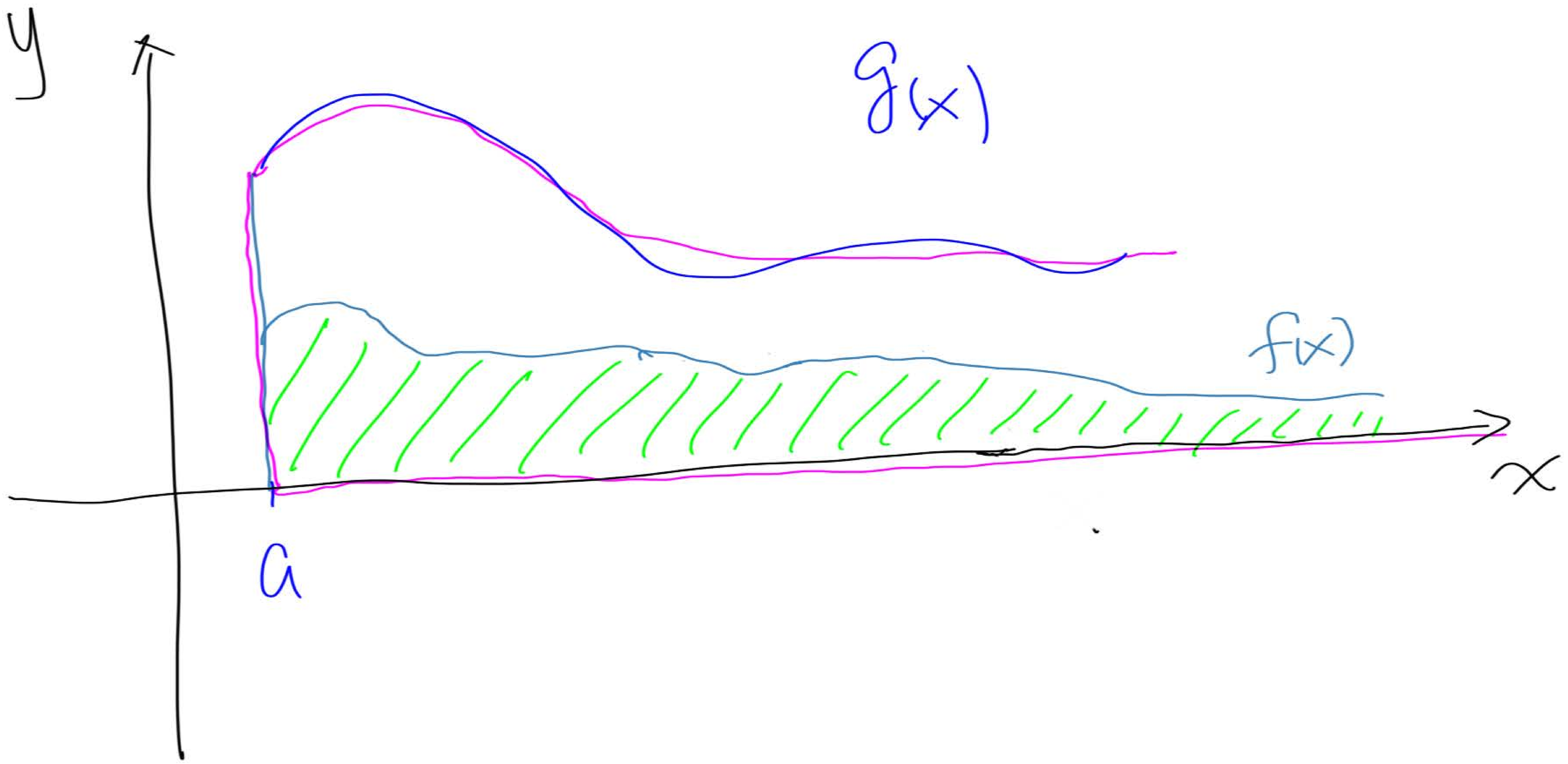
$$\int_0^2 \frac{2}{u^4}\,du = 2\int_0^2 u^{-4}\,du = 2\left.\frac{u^{-3}}{-3}\right|_0^2 \qquad \infty$$

$$= -\frac{2}{3}\left.\frac{1}{u^3}\right|_0^2 = -\frac{2}{3}\cdot\frac{1}{8} - \left(-\frac{2}{3}\cdot\frac{1}{0}\right)$$

This improper integral is divergent!

Comparison Test for Improper Integrals

- Given that $0 \leq f(x) \leq g(x)$, we want to compare $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$.



$\int_a^\infty f(x)\,dx$

-
- ▸ If $\int_a^\infty g(x)dx$ is convergent, then $\int_a^\infty f(x)dx$ is convergent.
  ▸ If $\int_a^\infty f(x)dx$ is divergent, then $\int_a^\infty g(x)dx$ is divergent.

Use Comparison Test to show $\int_1^{\infty} \frac{2+\sin x}{x^2} dx$ is convergent.

$$0 < \frac{2+\sin x}{x^2} \leq \frac{2+1}{x^2} = \frac{3}{x^2}$$

$$\int_1^{\infty} \frac{3}{x^2} dx = 3 \int_1^{\infty} \frac{1}{x^2} dx \longrightarrow \text{Convergent!}$$

Convergent!

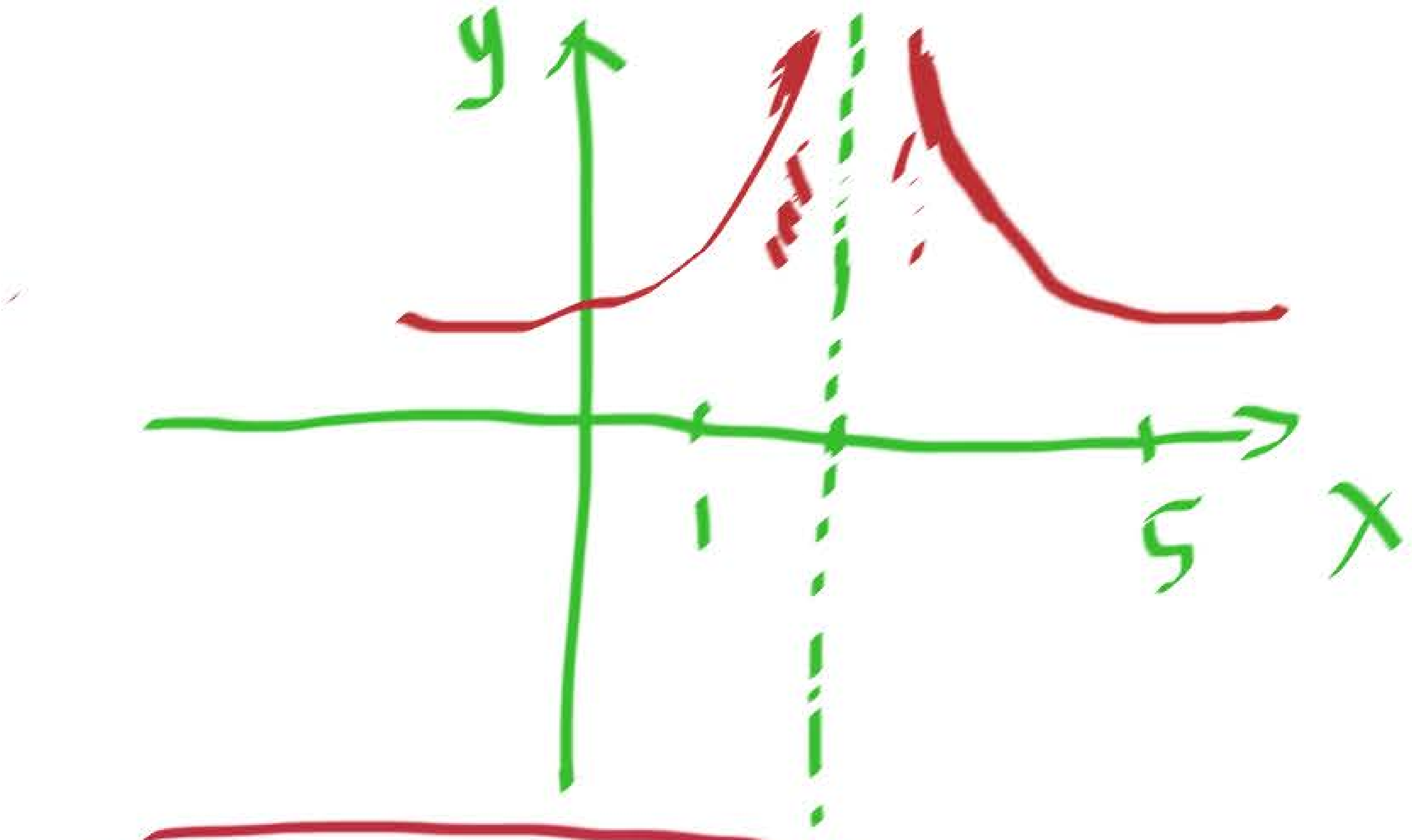
By Comparison Test, $\int_1^{\infty} \frac{2+\sin x}{x^2} dx$ is convergent!

$$\int_1^5 \frac{1}{(x-2)^2}\,dx \xrightarrow{\text{Break it up}} \int_1^2 \frac{1}{(x-2)^2}\,dx + \int_2^5 \frac{1}{(x-2)^2}\,dx$$

This is improper because $x=2$ is a V.A.

(The original integral is convergent if both new ones are convergent! Otherwise it is divergent.)

can be solved the same way!



$u = x-2 \quad du = dx$

$u: -1 \to 0$

$$\int_{-1}^{0} \frac{1}{u^2}\,du = \int_{-1}^{0} u^{-2}\,du$$

$$\frac{u^{-1}}{-1}\Big|_{-1}^{0} = -\frac{1}{u}\Big|_{-1}^{0}$$

$$\lim_{u\to 0^-}\left(-\frac{1}{u}\right) - 1$$

$$= \infty$$

Final answer. It is divergent!