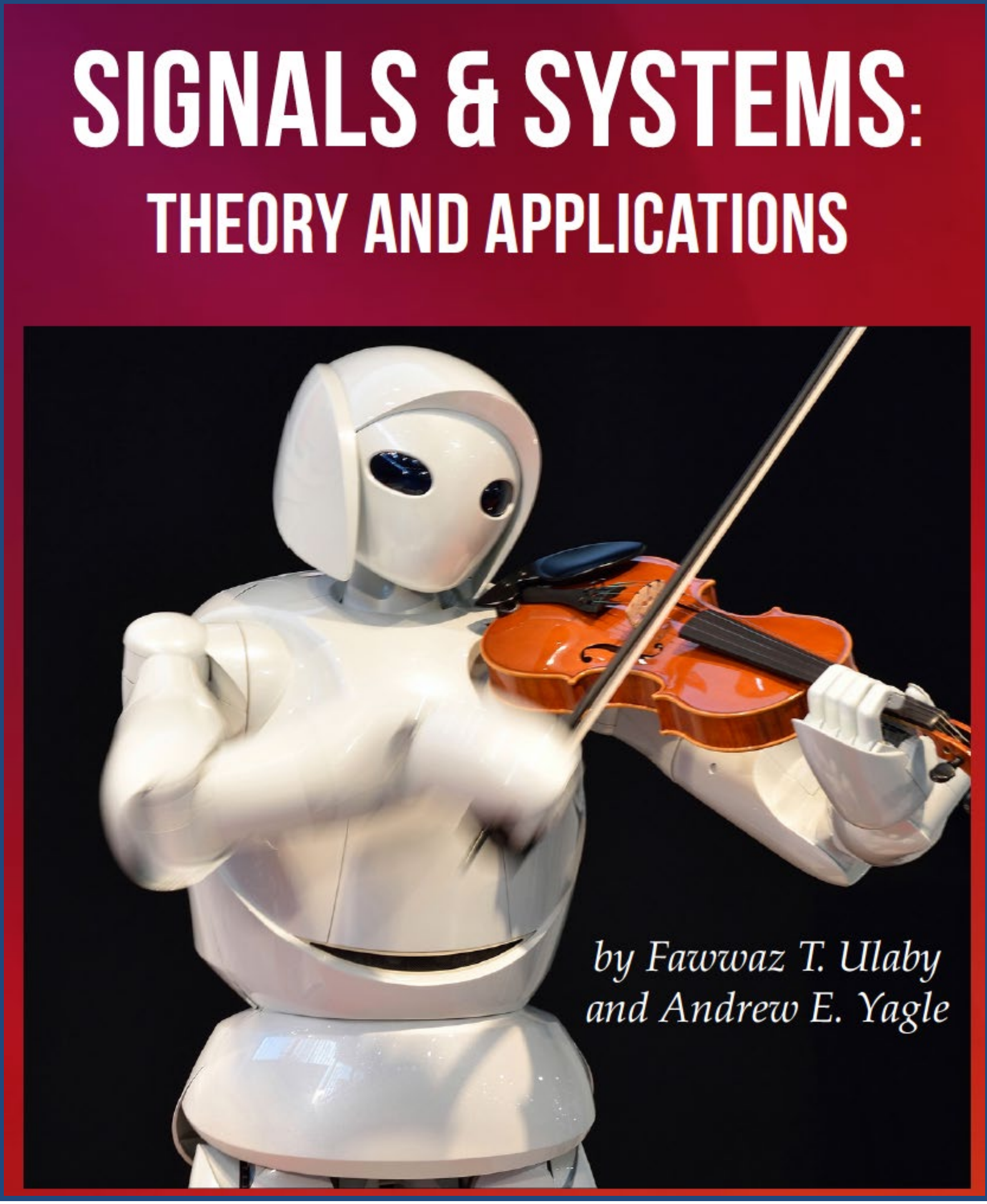
1. SIGNALS


SIGNALS & SYSTEMS:
THEORY AND APPLICATIONS
by Fawwaz T. Ulaby
and Andrew E. Yagle

1 Signals

Contents

Objectives

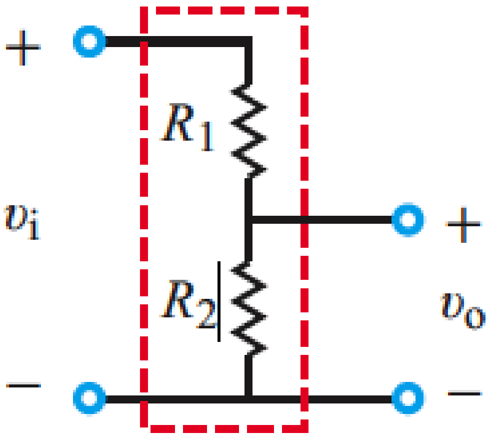
Learn to:

- Perform transformations on signals.
- Use step, ramp, pulse, and exponential waveforms to model simple signals.
- Model impulse functions.
- Calculate power and energy contents of signals.

Signals come in many forms: continuous, discrete, analog, digital, periodic, nonperiodic, with even or odd symmetry or no symmetry at all, and so on. Signals with special ***waveforms*** include ramps, exponentials, and impulses. This chapter introduces the vocabulary, the properties, and the ***transformations*** commonly associated with signals in preparation for exploring in future chapters ***how signals interact with systems***.

Overview

A system transforms input signals (excitations)
into output signals (responses)
to perform a certain operation.



A voltage divider is a simple system.

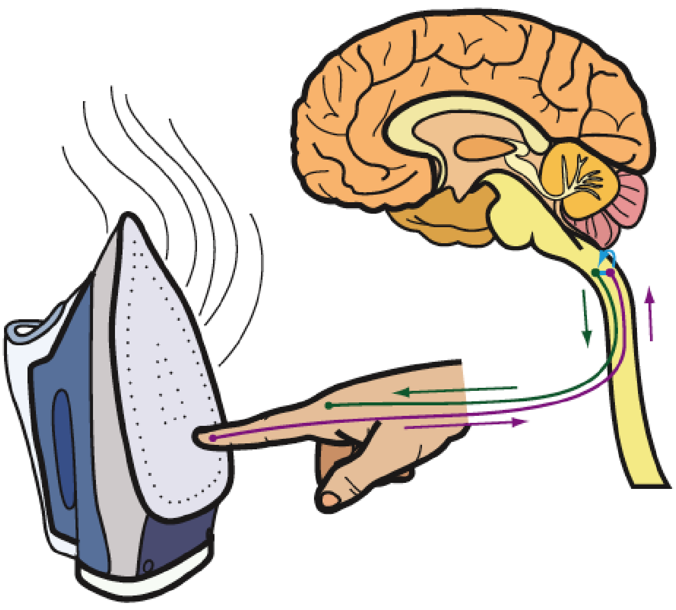
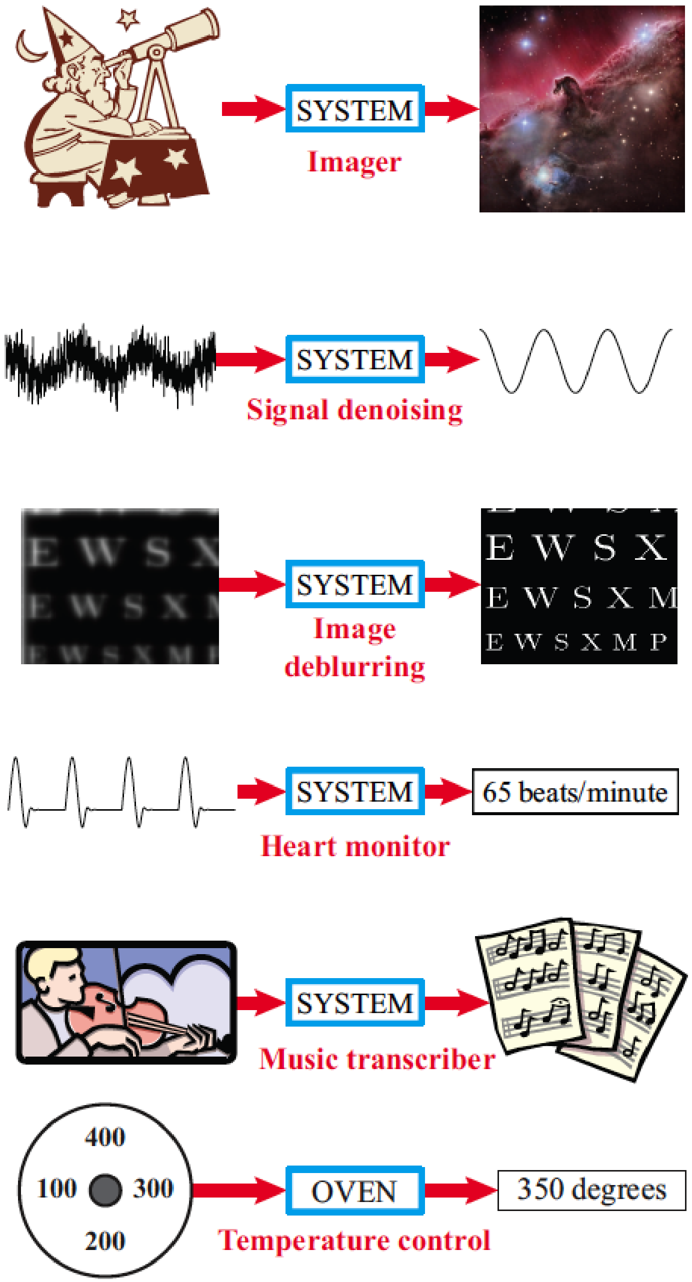


Figure 1-3: Finger-CNS-muscle communication.

Examples of Systems



Types of Signals

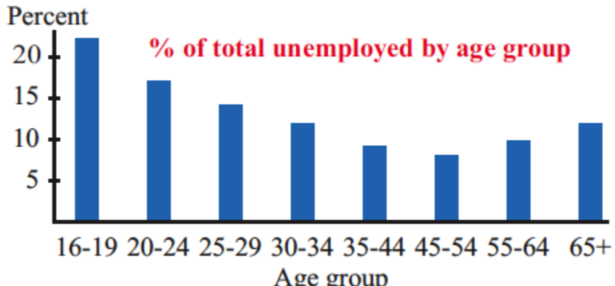
- Continuous vs Discrete
- Causal: $x(t) = 0$ for $t<0$
  vs Noncausal: $x(t) \neq 0$ for $t<0$
- Analog vs Digital

Acoustic pressure waveform

(a) Continuous-time signal

Brightness across discrete row of pixels

(b) Discrete-spatial signal

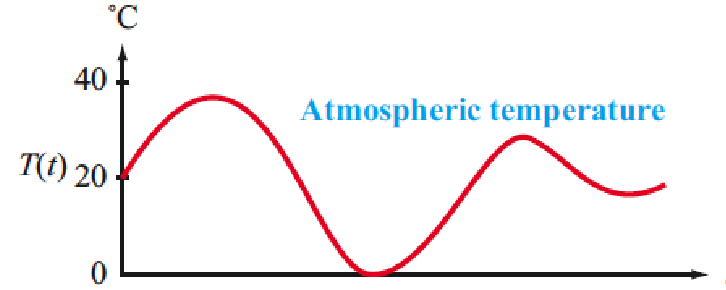


(c) Independent variable is age group

X-ray image

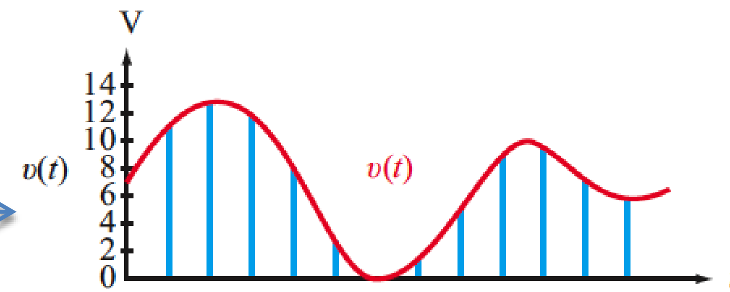
(d) 2-D spatial signal

Analog vs Digital



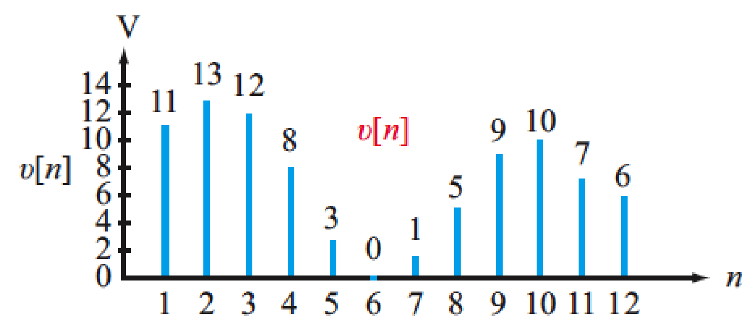
(a) Atmospheric temperature in ˚C

Sampling



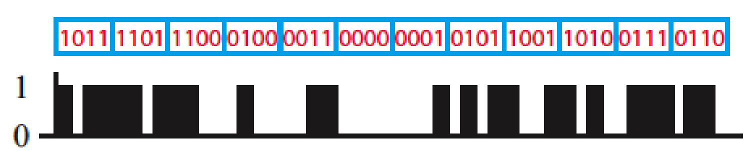
(b) Sensor voltage in volts

Discrete Signal



(c) Discrete version of (b)

Digital Signal



(d) Digital signal

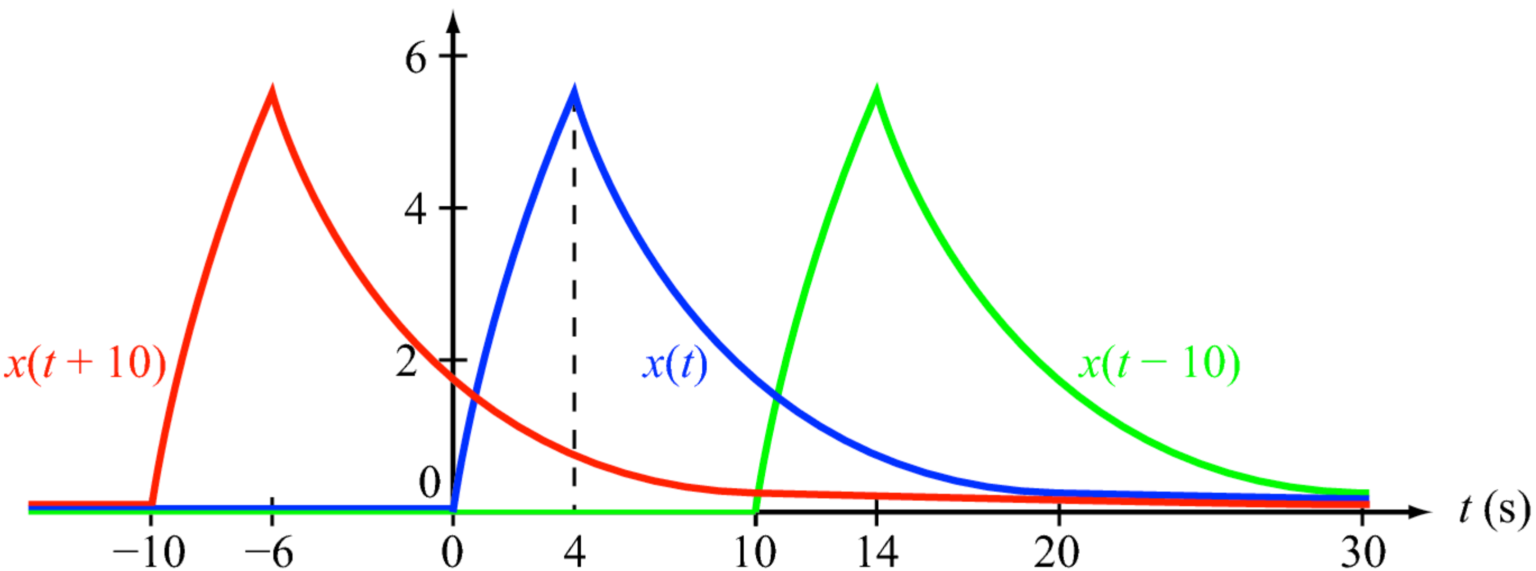
Signal Transformations

1-2.1 Time-Shift Transformation

If $x(t)$ is a continuous-time signal, a ***time-shifted*** version with delay T is given by

$$y(t) = x(t - T), \qquad (1.1)$$

wherein t is replaced with $(t - T)$ everywhere in the expression



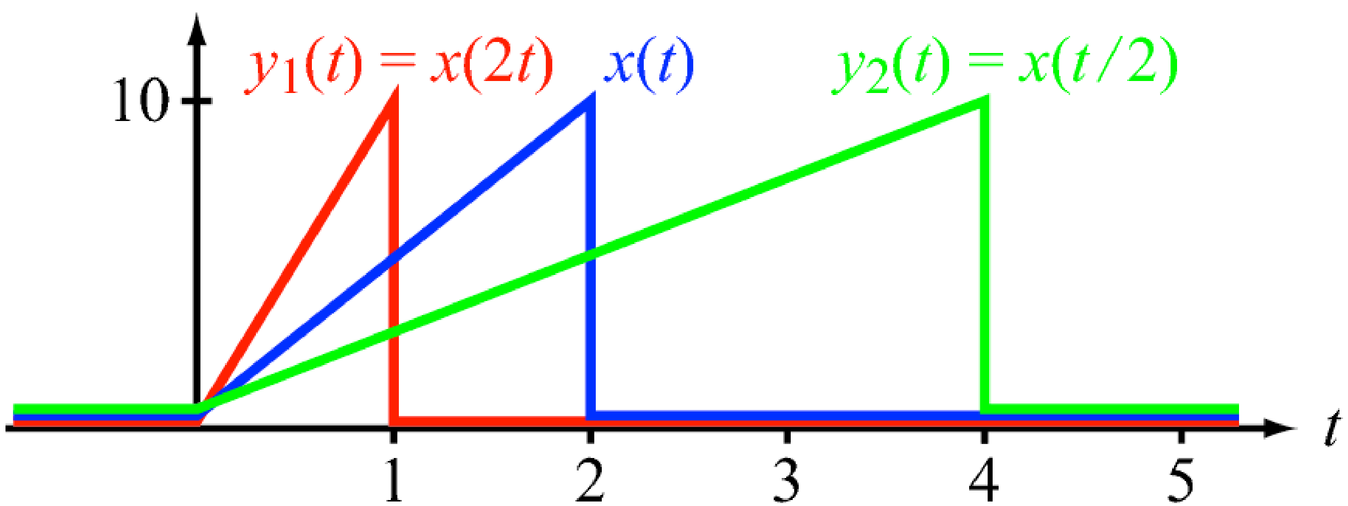
Signal Transformations

1-2.2 Time-Scaling Transformation

Mathematically, the *time-scaling transformation* can be expressed as

$$y(t) = x(at), \qquad (1.3)$$

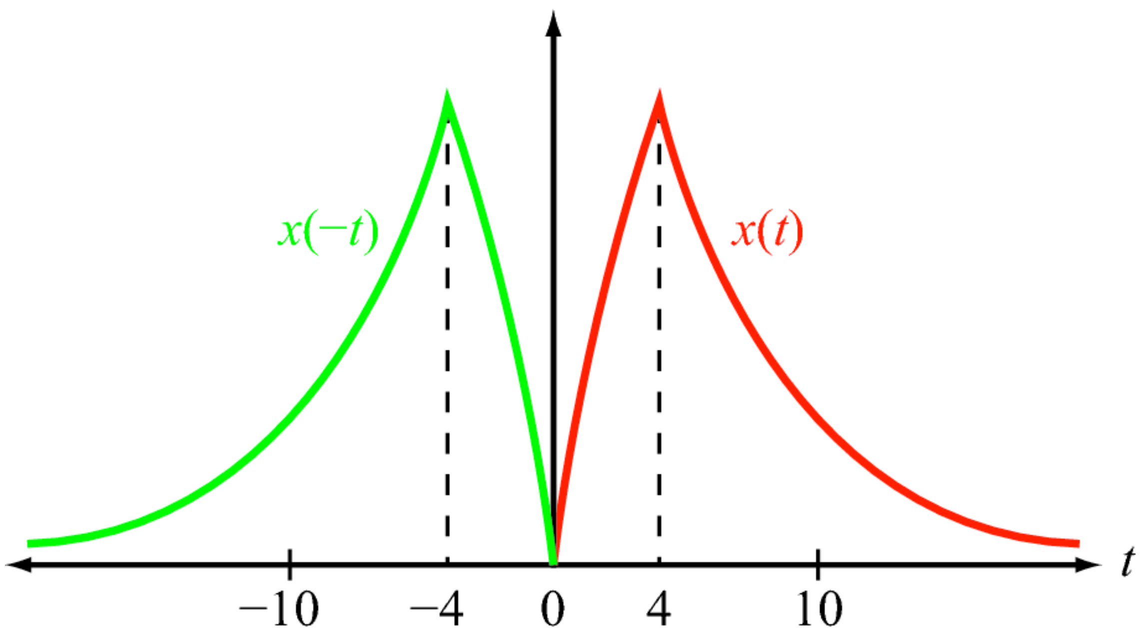
where a is a compression or expansion factor depending on whether its absolute value is larger or smaller than 1,



Signal Transformations

1-2.3 Time-Reversal Transformation

▶ Replacing t with $-t$ in $x(t)$ generates a signal $y(t)$ whose waveform is the mirror image of that of $x(t)$ with respect to the vertical axis. ◀



1-2.4 Combined Transformation

The three aforementioned transformations can be combined into a generalized transformation:

$$y(t) = x(at - b) = x\left(a\left(t - \frac{b}{a}\right)\right) = x(a(t - T)), \quad (1.5)$$

where $T = b/a$. We recognize T as the time shift and a as the compression/expansion factor.

Signal Transformation Procedure

The procedure for obtaining $y(t) = x(a(t - T))$ from $x(t)$ is as follows:

(1) Scale time by a:

- If $|a| < 1$, then $x(t)$ is expanded.
- If $|a| > 1$, then $x(t)$ is compressed.
- If $a < 0$, then $x(t)$ is also reflected.

This results in $z(t) = x(at)$.

(2) Time shift by T:

- If $T > 0$, then $x(t)$ shifts to the right.
- If $T < 0$, then $x(t)$ shifts to the left.

This results in $z(t - T) = x(a(t - T)) = y(t)$.

The procedure for obtaining $y(t) = x(at - b)$ from $x(t)$ reverses the order of time scaling and time shifting:

(1) Time shift by b.

(2) Time scale by a.

Example 1-1: Multiple Transformations

For signal $x(t)$ profiled in Fig. 1-10(a), generate the corresponding profile of $y(t) = x(-2t + 6)$.

Solution:

We start by recasting the expression for the dependent variable into the standard form given by Eq. (1.5),

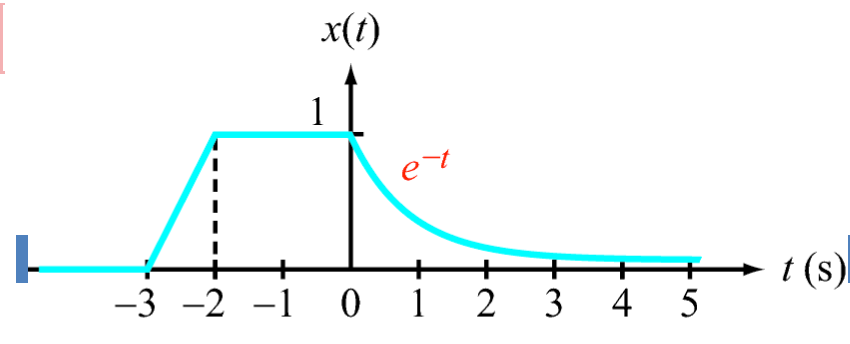
$$y(t) = x\left(-2\left(t - \frac{6}{2}\right)\right)$$
$$= x(-2(t-3)).$$

Reversal   Compression factor   Time-shift

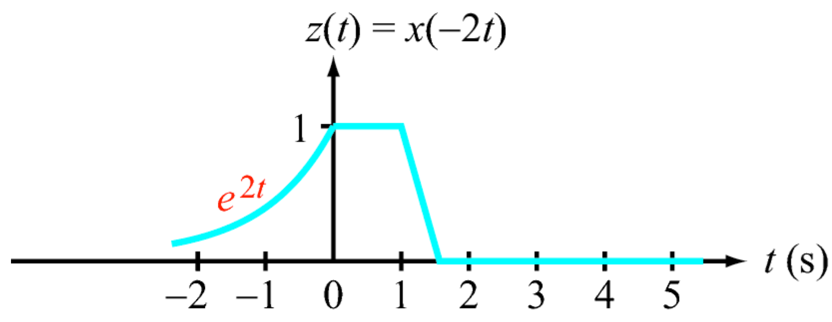
We need to apply the following transformations:

(1) Scale time by $-2t$: This causes the waveform to reflect around the vertical axis and then compresses time by a factor of 2. These steps can be performed in either order. The result, $z(t) = x(-2t)$, is shown in Fig. 1-10(b).

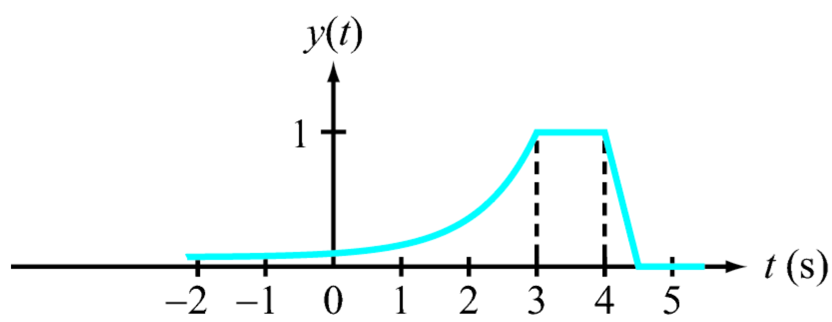
(2) Delay waveform $z(t)$ by 3 s: This shifts the waveform to the right by 3 s (because the sign of the time shift is negative). The result, $y(t) = z(t-3) = x(-2(t-3))$, is displayed in Fig. 1-10(c).



(a) $x(t)$



(b) $z(t)$



(c) $y(t)$

1-3.1 Even Symmetry

▶ A signal $x(t)$ exhibits ***even symmetry*** if its waveform is symmetrical with respect to the vertical axis. ◀

$$x(t) = x(-t) \qquad \textbf{(even symmetry)}.$$

1-3.2 Odd Symmetry

In contrast, the waveform in Fig. 1-11(c) has ***odd symmetry***.

▶ A signal exhibits odd symmetry if the shape of its waveform on the left-hand side of the vertical axis is the *inverted* mirror image of the waveform on the right-hand side. ◀

Equivalently,

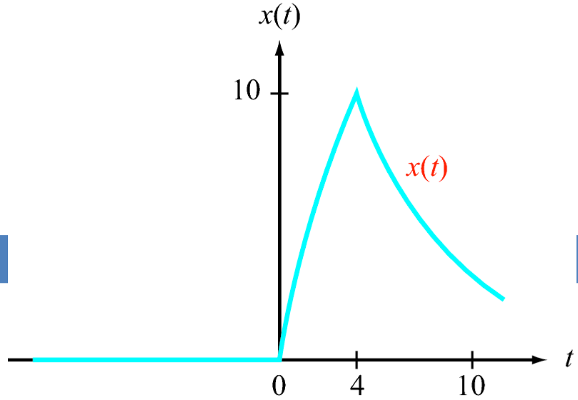
$$x(t) = -x(-t) \qquad \textbf{(odd symmetry)}. \tag{1.7}$$

Even/Odd Synthesis

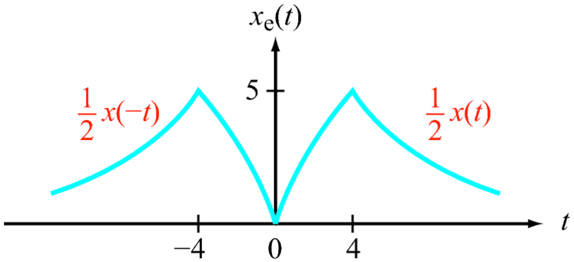
$$x(t) = x_e(t) + x_o(t),$$

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)],$$

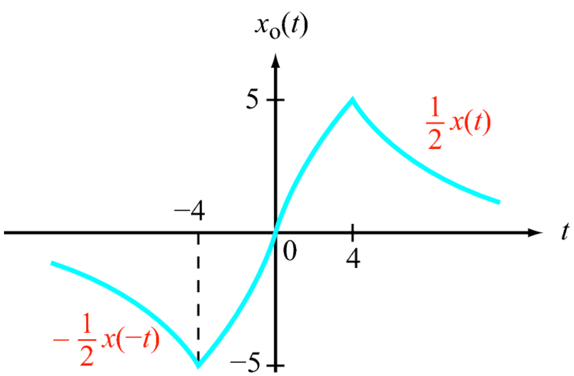
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)].$$



(a) $x(t)$



(b) $x_e(t)$



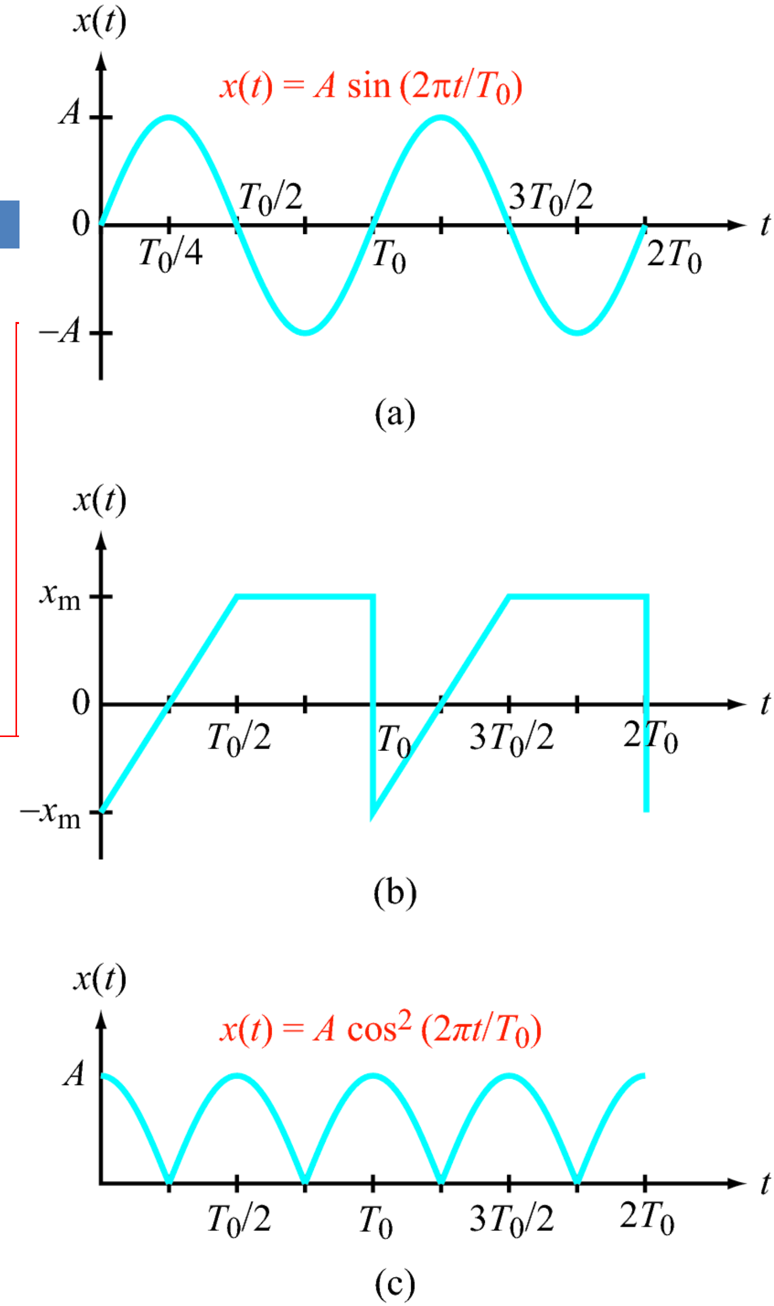
(c) $x_o(t)$

Periodic Signals

► A periodic signal $x(t)$ of period T_0 satisfies the ***periodicity*** property:

$$x(t) = x(t + nT_0) \tag{1.11}$$

for all integer values of n and all times t. ◄



(a)

(b)

(c)

Step Function

Ideal Step Function

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t > 0. \end{cases}$$

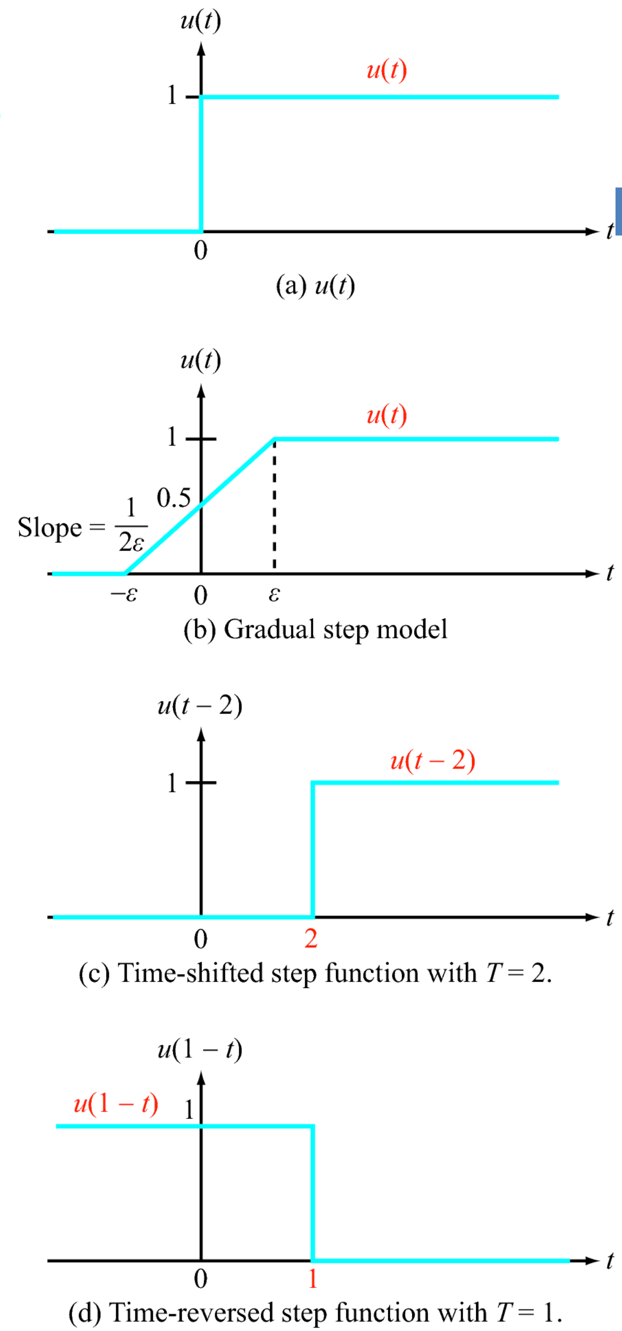
Realistic Step Function

$$u(t) = \lim_{\epsilon \to 0} \begin{cases} 0 & \text{for } t \le -\epsilon \\ \left[\frac{1}{2}\left(\frac{t}{\epsilon} + 1\right)\right] & \text{for } -\epsilon \le t \le \epsilon \\ 1 & \text{for } t \ge \epsilon, \end{cases}$$

Time-Shifted Step Function

$$u(T - t) = \begin{cases} 1 & \text{for } t < T, \\ 0 & \text{for } t > T. \end{cases}$$

Step Functions

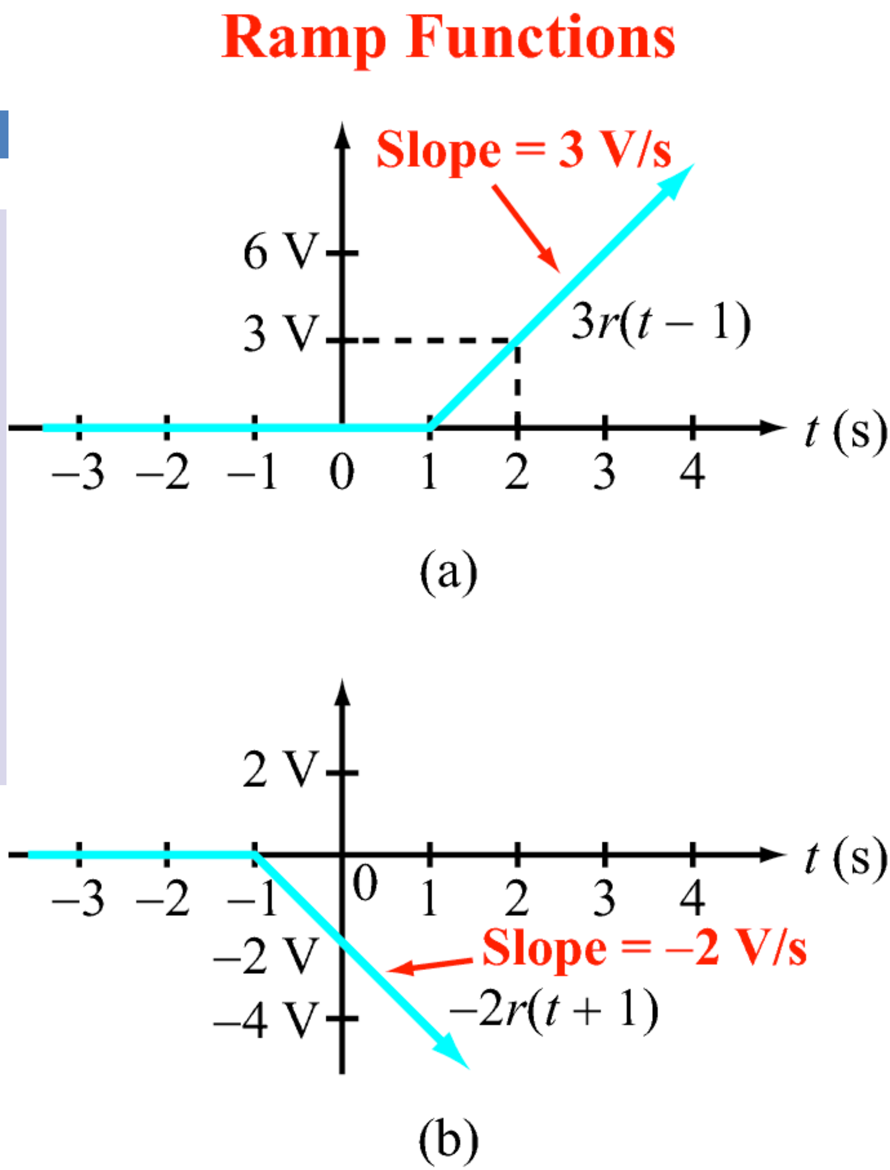


Ramp Function

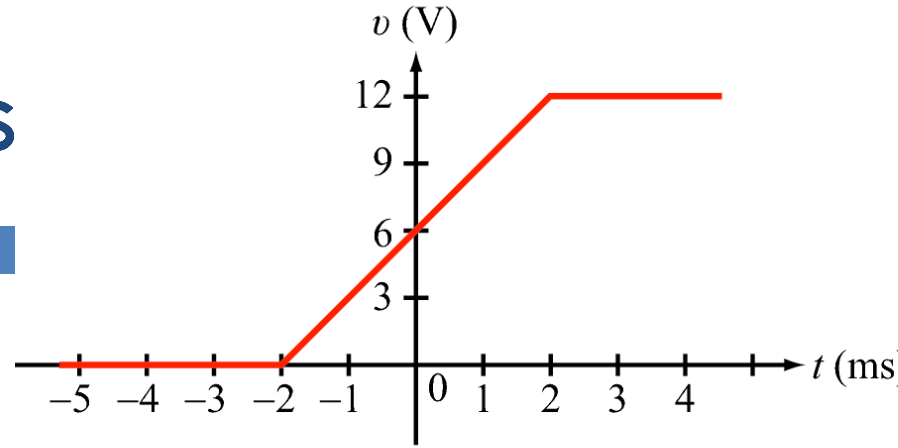
$$r(t) = \begin{cases} 0 & \text{for } t \le 0, \\ t & \text{for } t \ge 0, \end{cases}$$

and

$$r(t-T) = \begin{cases} 0 & \text{for } t \le T, \\ (t-T) & \text{for } t \ge T. \end{cases}$$

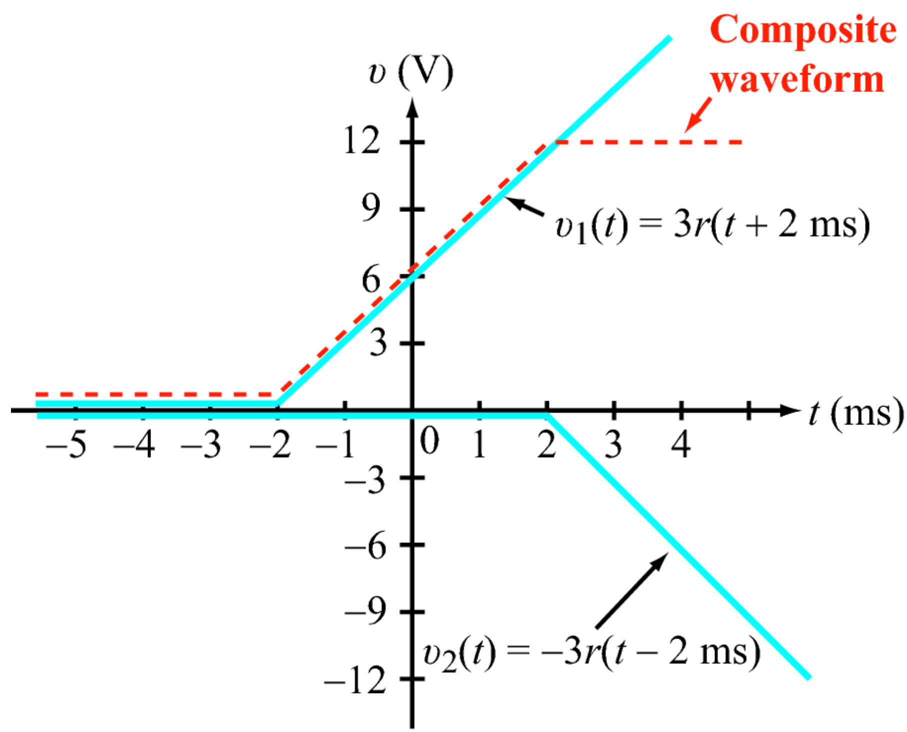


Waveform Synthesis



(a) Original function

$$\upsilon(t) = \upsilon_1(t) + \upsilon_2(t)$$
$$= 3r(t + 2\text{ ms}) - 3r(t - 2\text{ ms})$$



(b) As sum of two time-shifted ramp functions

Equivalently:

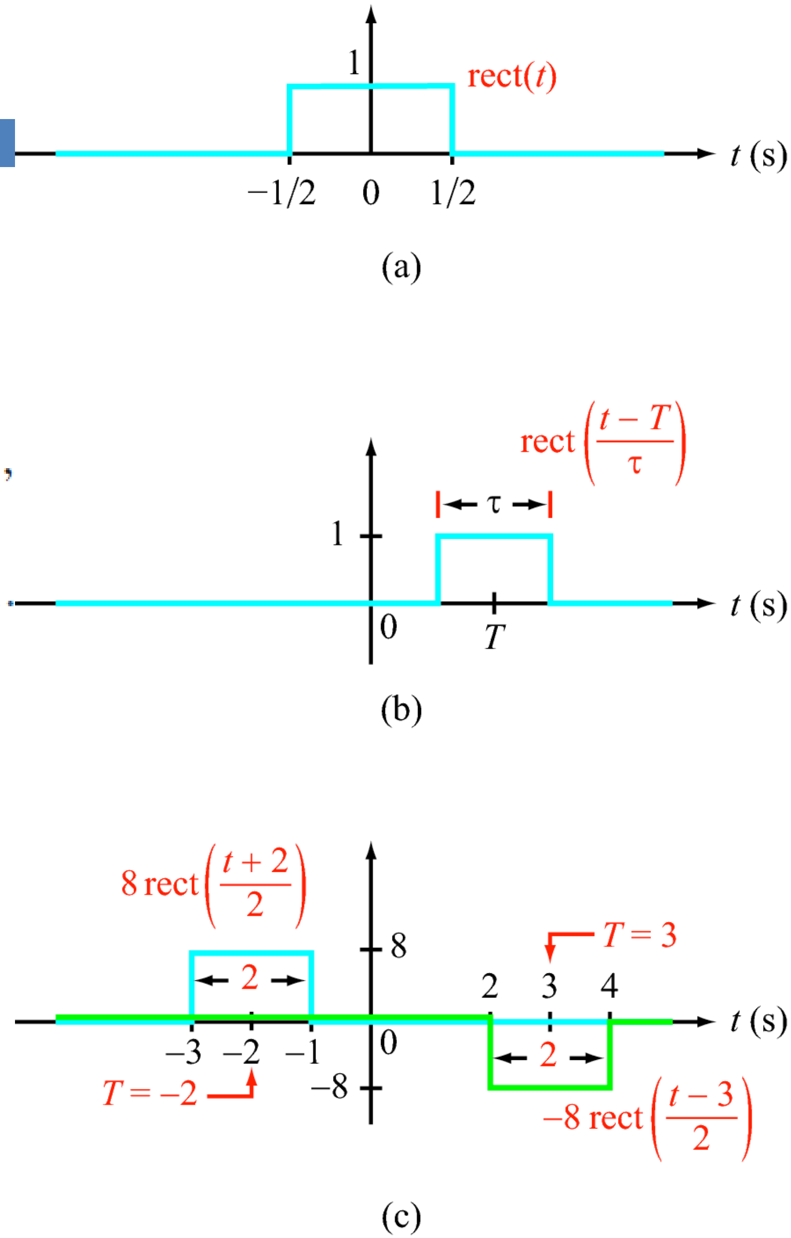
$$\upsilon(t) = 3(t + 2\text{ ms})\, u(t + 2\text{ ms})$$
$$- 3(t - 2\text{ ms})\, u(t - 2\text{ ms})$$

Rectangular Function (Pulse)

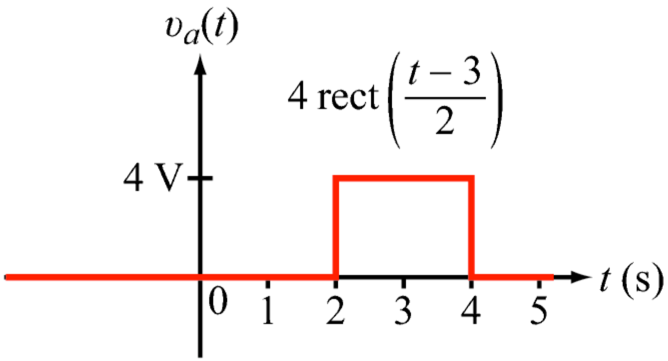
$$\mathrm{rect}\left(\frac{t-T}{\tau}\right) = \begin{cases} 0 & \text{for } t < (T-\tau/2), \\ 1 & \text{for } (T-\tau/2) < t < (T+\tau/2), \\ 0 & \text{for } t > (T+\tau/2). \end{cases}$$

(1.

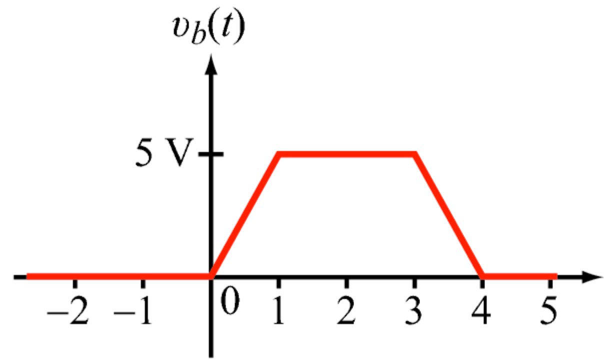
Rectangular Pulses



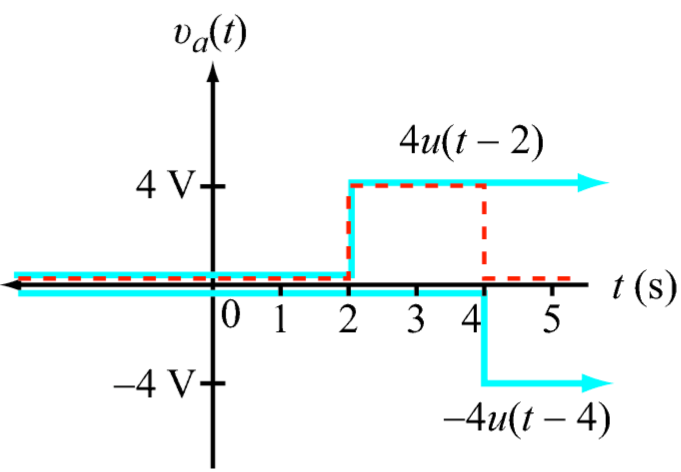
Waveform Synthesis



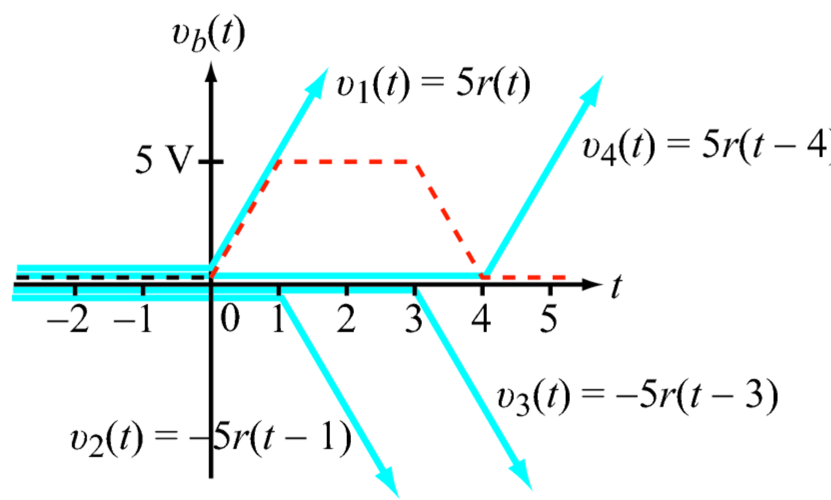
(a) Rectangular pulse



(b) Trapezoidal pulse



(c) $v_a(t) = 4u(t-2) - 4u(t-4)$



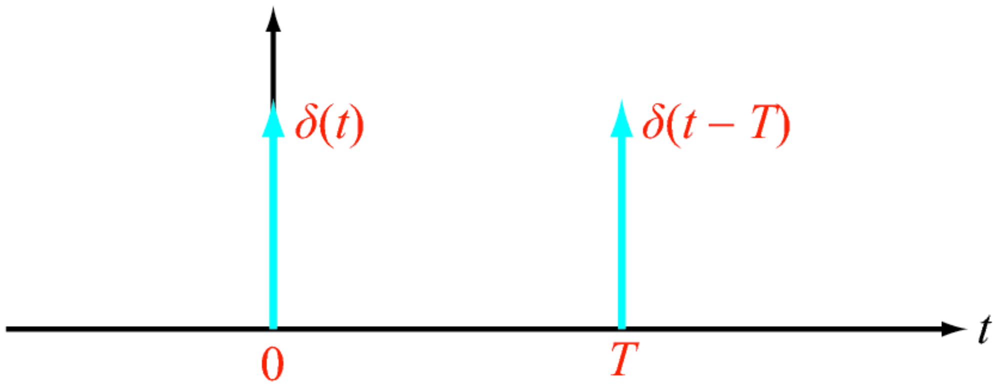
(d) $v_b(t) = v_1(t) + v_2(t) + v_3(t) + v_4(t)$

Impulse Function

$$\delta(t - T) = 0 \qquad \text{for } t \neq T$$

and

$$\int_{-\infty}^{\infty} \delta(t - T)\, dt = 1.$$

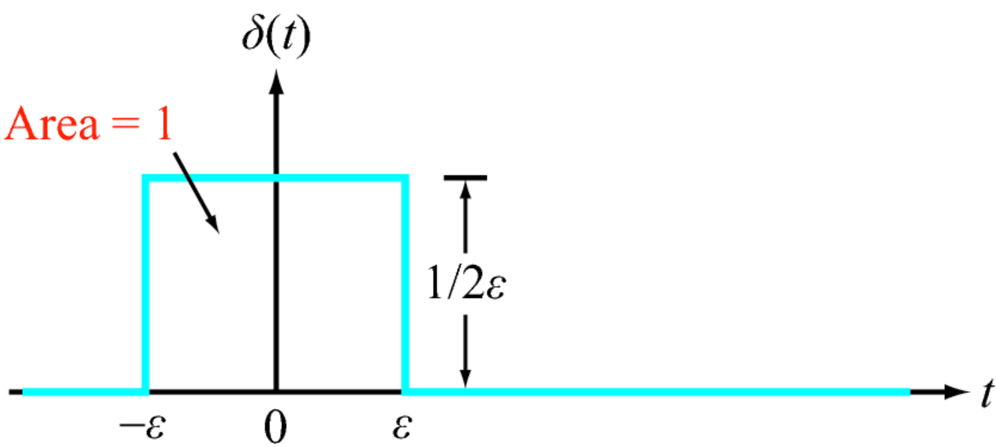


(a) $\delta(t)$ and $\delta(t - T)$

Relationship to u(t)

$$\frac{d}{dt}\,[u(t - T)] = \delta(t - T),$$

$$u(t - T) = \int_{-\infty}^{t} \delta(\tau - T)\, d\tau.$$



(b) Rectangle model for $\delta(t)$

Sampling Property of $\delta(t)$

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-T)\,dt = x(T).$$

(sampling property)

Example 1-6: Impulse Integral

Evaluate $\int_1^2 t^2\, \delta(2t-3)\, dt$.

Solution:

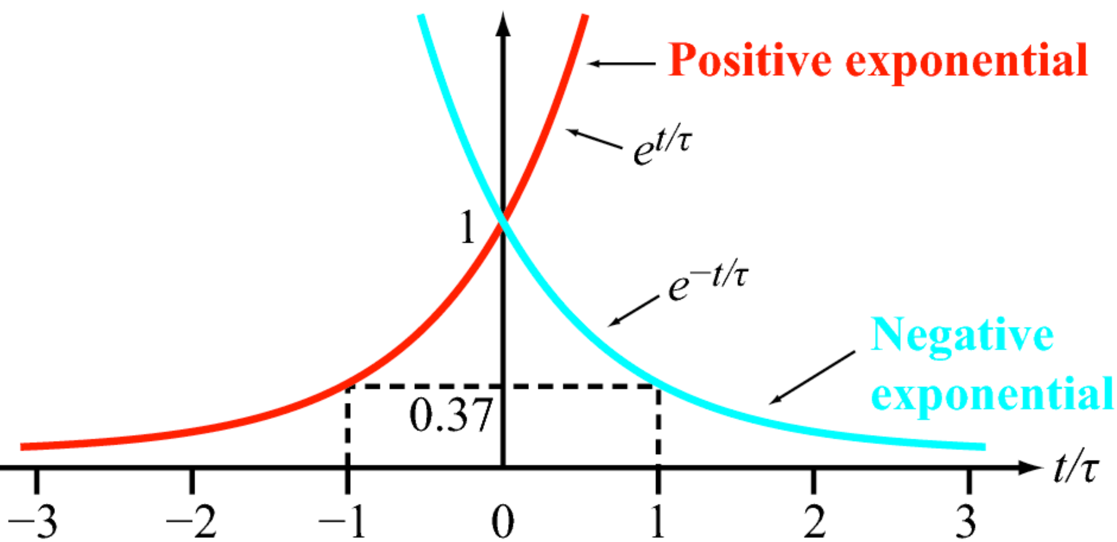
Using the time-scaling property, the impulse function can be expressed as

$$\delta(2t-3) = \delta\left(2\left(t-\frac{3}{2}\right)\right) = \frac{1}{2}\,\delta\left(t-\frac{3}{2}\right).$$

Hence,

$$\int_1^2 t^2\, \delta(2t-3)\, dt = \frac{1}{2}\int_1^2 t^2\, \delta\left(t-\frac{3}{2}\right) dt = \frac{1}{2}\left(\frac{3}{2}\right)^2 = \frac{9}{8}.$$

Exponential Waveform



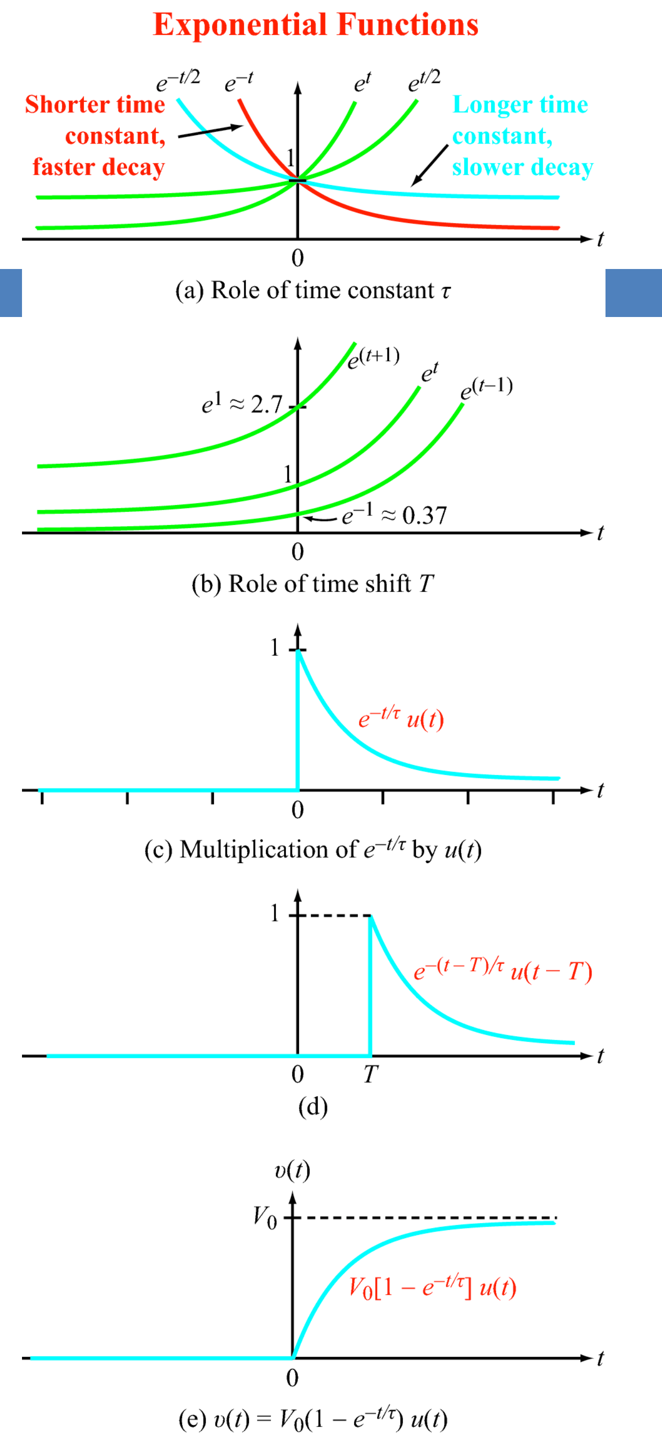
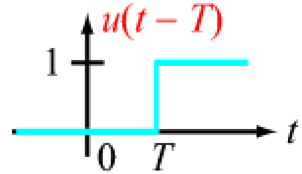
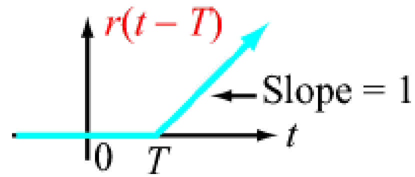
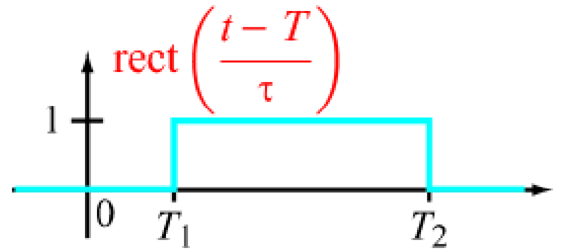
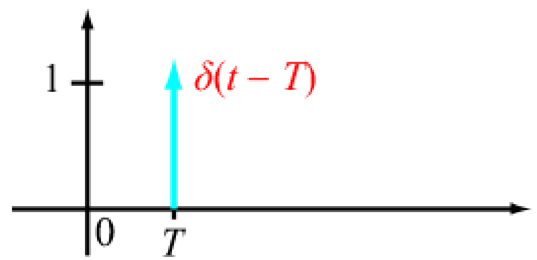
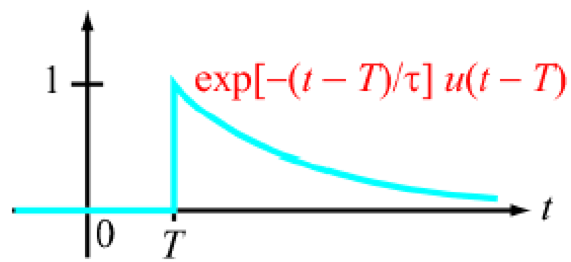


Table 1-2: Common nonperiodic functions.

Function	Expression	General Shape
Step	$u(t-T) = \begin{cases} 0 & \text{for } t < T \\ 1 & \text{for } t > T \end{cases}$	
Ramp	$r(t-T) = (t-T)\, u(t-T)$	
Rectangle	$\text{rect}\left(\frac{t-T}{\tau}\right) = u(t-T_1) - u(t-T_2)$ $T_1 = T - \frac{\tau}{2}; \quad T_2 = T + \frac{\tau}{2}$	
Impulse	$\delta(t-T)$	
Exponential	$\exp[-(t-T)/\tau]\, u(t-T)$	

Signal Power and Energy

For a signal x(t),

Average Power:

$$P_{\text{av}} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt.$$

Total Energy:

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt,$$

$$P_{\text{av}} = \lim_{T\to\infty} \frac{E}{T}$$

- P_{av} and E define three classes of signals:

(a) *Power signals*: P_{av} is finite and $E \to \infty$

(b) *Energy signals*: $P_{\text{av}} = 0$ and E is finite

(c) *Non-physical signals*: $P_{\text{av}} \to \infty$ and $E \to \infty$

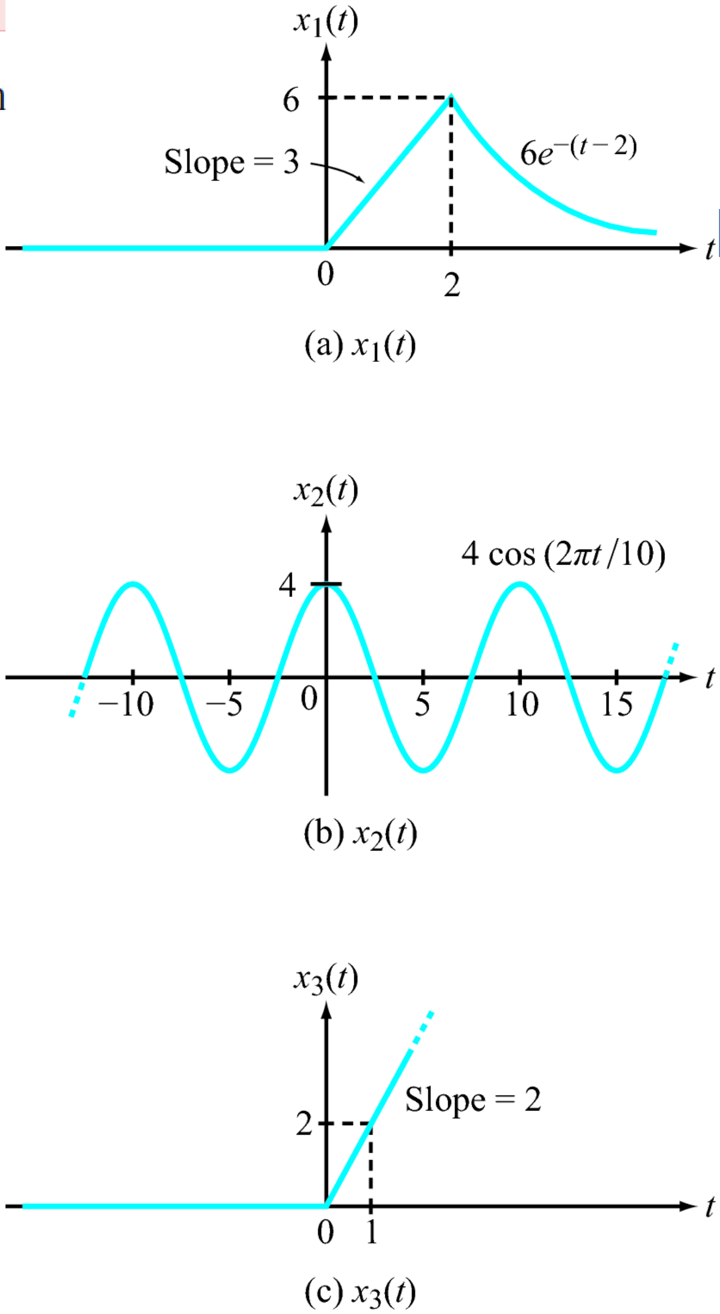
Example 1-7: Power and Energy

Evaluate P_{av} and E for each of the three signals displayed in Fig. 1-23.

Solution:

(a) Signal $x_1(t)$ is given by

$$x_1(t) = \begin{cases} 0 & \text{for } t \le 0, \\ 3t & \text{for } 0 \le t \le 2, \\ 6e^{-(t-2)} & \text{for } t \ge 2. \end{cases}$$



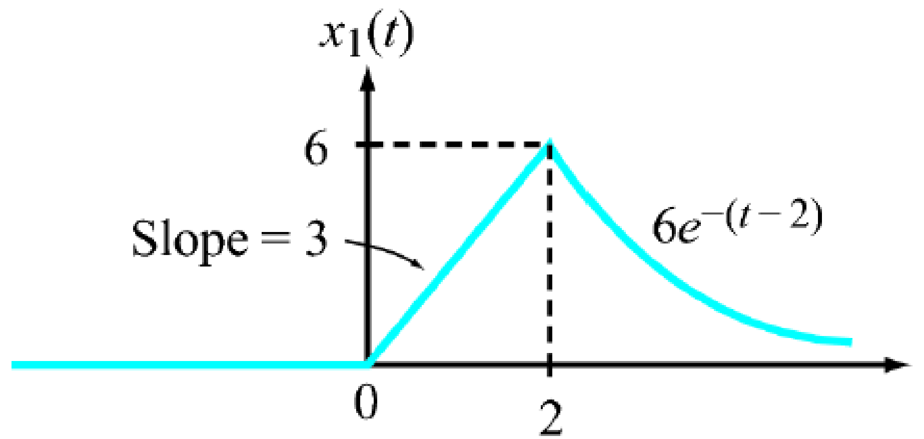
$$E_1 = \int_0^2 (3t)^2\, dt + \int_2^\infty [6e^{-(t-2)}]^2\, dt$$

$$= \int_0^2 9t^2\, dt + \int_2^\infty 36e^{-2(t-2)}\, dt$$

$$= \left.\frac{9t^3}{3}\right|_0^2 + 36e^4 \int_2^\infty e^{-2t}\, dt$$

$$= 24 + 36e^4 \left(\left.\frac{-e^{-2t}}{2}\right|_2^\infty \right)$$

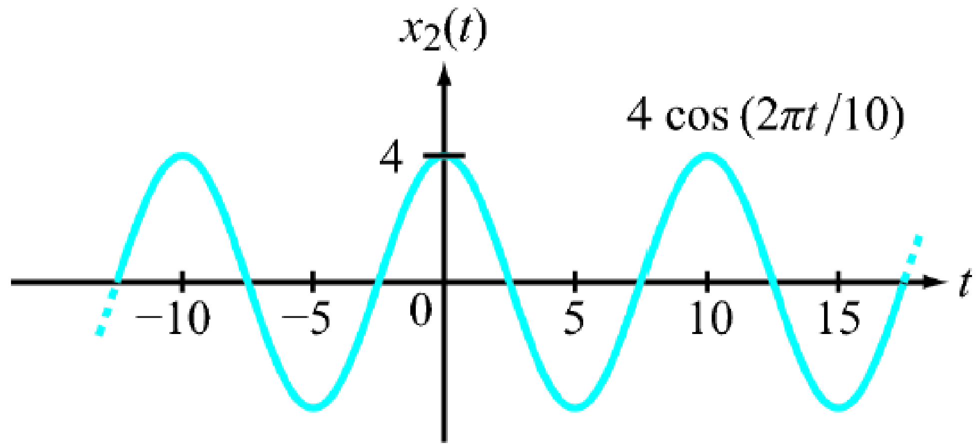
$$= 42.$$



(a) $x_1(t)$

$$P_{av} = \lim_{T\to\infty} \frac{E}{T}$$

Since E_1 is finite, it follows from Eq. (1.35) that $P_{av_1} = 0$.



(b) $x_2(t)$

$$x_2(t) = 4\cos\left(\frac{2\pi t}{10}\right).$$

From the argument of $\cos(2\pi t/10)$, the period is 10 s. Hence, application of Eq. (1.36) leads to

$$P_{\text{av}_2} = \frac{1}{10}\int_{-5}^{5}\left[4\cos\left(\frac{2\pi t}{10}\right)\right]^2 dt$$

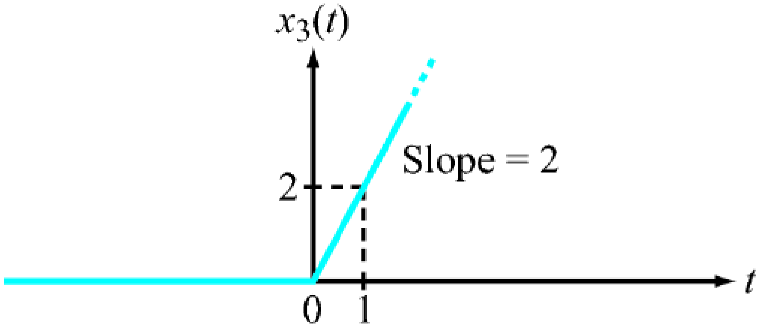
$$= \frac{1}{10}\int_{-5}^{5} 16\cos^2\left(\frac{2\pi t}{10}\right) dt$$

$$= 8.$$

Since P_{av_2} is finite, it follows that $E_2 \to \infty$.

(c) Signal $x_3(t)$ is given by

$$x_3(t) = 2r(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ 2t & \text{for } t \geq 0. \end{cases}$$



(c) $x_3(t)$

The time-averaged power associated with $x_3(t)$ is

$$P_{\text{av}_3} = \lim_{T\to\infty} \frac{1}{T} \int_0^{T/2} 4t^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \left[\left. \frac{4t^3}{3} \right|_0^{T/2} \right]$$

$$= \lim_{T\to\infty} \left[\frac{1}{T} \times \frac{4T^3}{24} \right]$$

$$= \lim_{T\to\infty} \left[\frac{T^2}{6} \right] \to \infty.$$

Moreover, $E_3 \to \infty$ as well.