LECTURE NO. 26

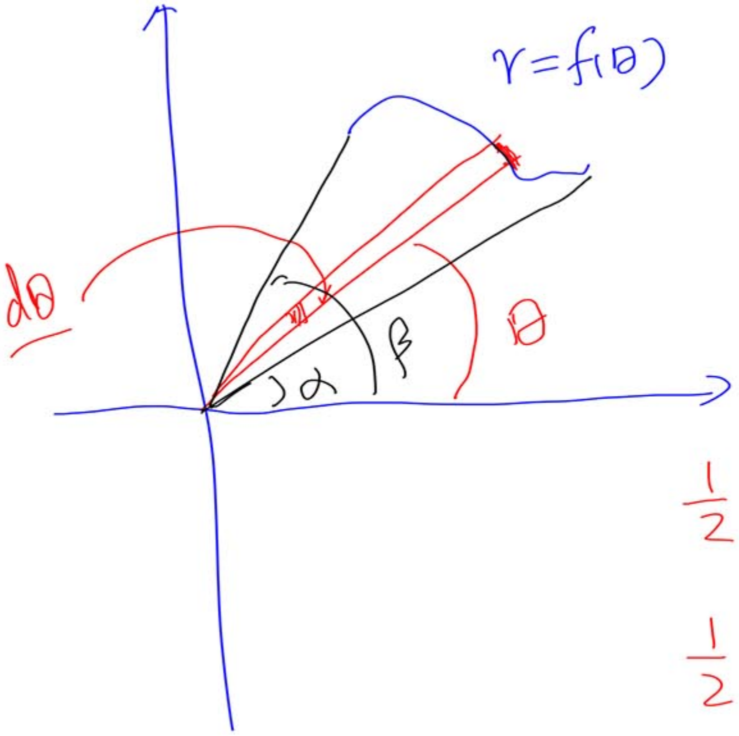
7.4 Area and Arc Length in Polar Coordinates

Wright State University

Area of Regions Bounded by Polar Curves

The area of the region bounded by $r = f(\theta), \alpha \leq \theta \leq \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta.$$



$\frac{1}{2} r^2 d\theta$

$\frac{1}{2}[f(\theta)]^2 d\theta$

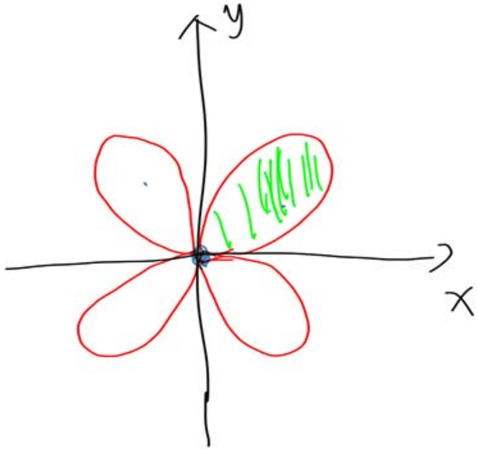
$\frac{1}{2} r^2 \theta$

$\pi r^2 \cdot \frac{\theta}{2\pi}$

$\parallel$

$\frac{1}{2} r^2 \theta$

Find the area of one petal of the flower defined by $r = 3\sin(2\theta)$.



$r = 3\sin(2\theta) = 0$

Two consecutive solutions

$\sin(2\theta) = 0$

$2\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$\theta = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \ldots$

$-\pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \pi$

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [3\sin(2\theta)]^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 9\sin^2 2\theta\, d\theta$$

$$\sin^2(2\theta) = \frac{1-\cos(4\theta)}{2}$$

$$\boxed{\begin{aligned} \sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\ \cos^2\theta &= \frac{1+\cos(2\theta)}{2} \end{aligned}}$$

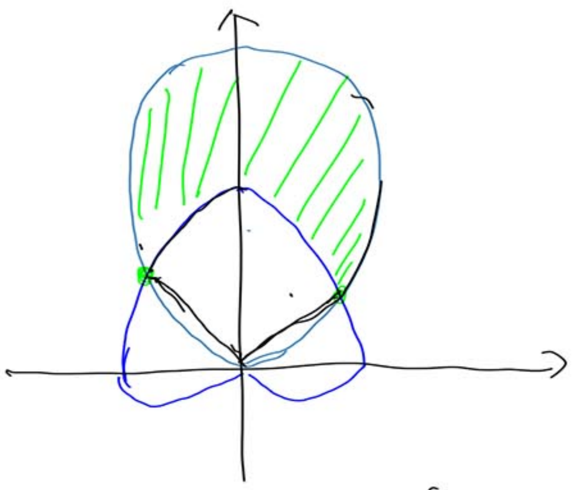
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 9 \cdot \frac{1-\cos(4\theta)}{2}\, d\theta = \frac{9}{4}\int_0^{\frac{\pi}{2}} 1-\cos(4\theta)\, d\theta$$

$$= \frac{9}{4}\left(\theta - \frac{\sin(4\theta)}{4}\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{9}{4}\left(\frac{\pi}{2}\right) - \frac{9}{4}(0) = \frac{9\pi}{8}$$

FINAL ANSWER

Find the area outside the "heart" $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.



set $2+2\sin\theta = 6\sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$

Area of shaded part $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (6\sin\theta)^2 \, d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2+2\sin\theta)^2 \, d\theta$

$(2+2\sin\theta)^2 = 4+8\sin\theta+4\sin^2\theta$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 36\sin^2\theta - 4 - 8\sin\theta - 4\sin^2\theta \, d\theta$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 32\sin^2\theta - 4 - 8\sin\theta \, d\theta$ $\quad \left(\sin^2\theta = \frac{1-\cos(2\theta)}{2}\right)$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 16(1-\cos(2\theta)) - 4 - 8\sin\theta \, d\theta$

$\frac{1}{2}\left(16\left(\theta - \frac{\sin 2\theta}{2}\right) - 4\theta + 8\cos\theta\right)\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$ (Book P666)

$= 4\pi$ (final answer)

Three forms of Arc Length Formulas

1) For $y = f(x), a \le x \le b$,

$$\text{Arc Length}= \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

2) For parametric curve $x = x(t), y = y(t), a \le t \le b$,

$$\text{Arc Length}= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

3) For polar curve $r = f(\theta), \alpha \le \theta \le \beta$,

$$\text{Arc Length}= \int_\alpha^\beta \sqrt{r^2 + [\frac{dr}{d\theta}]^2} d\theta$$

$r = f(\theta)$

$x = r\cos\theta$

$y = r\sin\theta$

- To get 3), treat polar curve as $x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$, and then use Formula 2).

Find the arc length of $r = 2 + 2\cos\theta$, $0 \le \theta \le \pi$.

$$\text{Arc length} = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{(2+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{4 + 8\cos\theta + \underbrace{4\cos^2\theta + 4\sin^2\theta}_{4}}\, d\theta$$

$$= \int_0^{\pi} \sqrt{8 + 8\cos\theta}\, d\theta = 2\int_0^{\pi} \sqrt{2+2\cos\theta}\, d\theta$$

How do we integrate $2\int_0^{\pi}\sqrt{2+2\cos\theta}\,d\theta$?

$$\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \qquad \cos^2\left(\frac{\theta}{2}\right) = \frac{\cos\theta+1}{2}$$

$$2\int_0^{\pi}\sqrt{2+2\left(2\cos^2\frac{\theta}{2}-1\right)}\,d\theta = 2\int_0^{\pi}\sqrt{\cancel{2}+4\cos^2\left(\frac{\theta}{2}\right)\cancel{-2}}\,d\theta$$

$$=2\int_0^{\pi}\sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\,d\theta = 2\int_0^{\pi} 2\cos\frac{\theta}{2}\,d\theta \qquad \left(\cos k\theta \xrightarrow{A.D} \frac{\sin k\theta}{k}\right.$$

$$=4\int_0^{\pi}\cos\left(\frac{\theta}{2}\right)d\theta = 4\cdot\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{1}{2}} = 8\sin\frac{\theta}{2}\Big|_0^{\pi}$$

$$=8\sin\frac{\pi}{2} - 8\sin 0$$

$$=8 \leftarrow \text{final answer.}$$