# LECTURE NO. 11

3.4 Partial Fractions

Wright State University

# Integrals of Rational Functions

- Examples: $\int \frac{2x^2+3x-5}{x+2}dx$, $\int \frac{1}{x^2-3x+2}dx$, and etc
- A proper fraction if Degree of Numerator $<$ Degree of Denominator; otherwise it is improper.
- Use Long Division if the integrand (function to be integrated) is improper.
- If the integrand is a proper fraction, then we will use Partial Fraction.

# Use Long Division on $\int \frac{2x^2+3x-5}{x+2}\,dx$

$$
\begin{array}{r|l}
 & \;\;2x \quad -1 \\
\hline
x+2 & 2x^2+3x-5 \\
 & -\left(2x^2+4x\right) \\
\hline
 & \qquad -x-5 \\
 & \quad -\left(-x-2\right) \\
\hline
 & \qquad\quad -3
\end{array}
$$

← Quotient

↑ Remainder

$$\frac{2x^2+3x-5}{x+2} = \text{Quotient} + \frac{\text{Remainder}}{x+2} = 2x+1 + \frac{-3}{x+2}$$

$$\int \frac{2x^2+3x-5}{x+2}\,dx = \int 2x+1 - \frac{3}{x+2}\,dx$$

$$= x^2+x - \int \frac{3}{x+2}\,dx$$

Substitution $u = x+2$

$$= \boxed{x^2+x-3\ln|x+2|+C}$$

FINAL ANSWER

# What is Partial Fraction?

- Recall how we add/subtract two fractions:

$$\frac{1}{x-2} - \frac{1}{x-1} \to \frac{x-1}{(x-2)(x-1)} - \frac{x-2}{(x-2)(x-1)}$$
$$\to \frac{1}{(x-2)(x-1)} \to \frac{1}{x^2-3x+2}$$

- Partial Fraction is precisely the reverse of the operation above.
- $\int \frac{1}{x^2-3x+2}dx$ may be hard, but $\int \frac{1}{x-2} - \frac{1}{x-1}dx$ is easy!
- $\int \frac{1}{x-2} - \frac{1}{x-1}dx = \int \frac{1}{x-2}dx - \int \frac{1}{x-1}dx = \ln|x-2| - \ln|x-1| + C$

# Recall: Integral of Simple Rational Functions

- $\int \frac{2}{3x+5} dx \to u = 3x + 5$
- $\int \frac{1}{(2x+1)^3} dx \to u = 2x + 1$
- What about $\int \frac{4x+3}{x^2+9} dx$ ?
- Break it up: $\int \frac{4x}{x^2+9} dx + \int \frac{3}{x^2+9} dx$
- First part: $u = x^2 + 9$; Second Part: $\frac{1}{9} \int \frac{3}{(\frac{x}{3})^2+1} dx$, $u = \frac{x}{3}$.

$\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$

## $\int \frac{x-6}{x^2-4}\,dx$

$\frac{x-6}{x^2-4}$ partial fraction

$$\frac{x-6}{x^2-4} = \frac{x-6}{(x+2)(x-2)} = \frac{2}{x+2} + \frac{-1}{x-2}$$

Step 1. factor the D. $\frac{x-6}{(x+2)(x-2)}$

Step 2 write the decomposition. $(x+2)(x-2)\ \frac{x-6}{(x+2)(x-2)} = \left(\frac{A}{x+2} + \frac{B}{x-2}\right)(x-2)(x+2)$

Step 3 Find A & B

$$x-6 = A(x-2) + B(x+2)$$

$x=-2$: $-8 = A(-4)$ $A=2$

$x=2$ $-4 = 4B$ $B=-1$

$$\int \frac{x-6}{x^2-4}\,dx = \int \frac{2}{x+2} + \frac{-1}{x-2}\,dx$$

$$= \int \frac{2}{x+2}\,dx - \int \frac{1}{x-2}\,dx$$

$$= 2\ln|x+2| - \ln|x-2| + C$$

partial fraction

$$\frac{3x^2+7x-2}{x^3-x^2-2x} = \frac{3x^2+7x-2}{x(x^2-x-2)} = \frac{3x^2+7x-2}{x(x+1)(x-2)} = \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}\right)^{x(x+1)(x-2)}.$$

$$3x^2+7x-2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

$x=0$ : $-2 = -2A \quad A=1$

$x=-1$ : $-6 = B(-1)(-3) \quad B=-2$

$x=2 \quad 24 = C\,2\cdot(3) \quad C=4$

$$\int \frac{3x^2+7x-2}{x^3-x^2-2x}dx = \int \frac{1}{x} + \frac{-2}{x+1} + \frac{4}{x-2}\,dx$$

$$= \int \frac{1}{x}dx - 2\int \frac{1}{x+1}dx + 4\int \frac{1}{x-2}dx$$

$$= \ln|x| - 2\ln|x+1| + 4\ln|x-2| + C$$

FINAL ANSWER

partial fraction

$$\frac{x-2}{(2x-1)^2(x-1)} = \left(\frac{A}{(2x-1)^2} + \frac{B}{2x-1} + \frac{C}{x-1}\right)\cdot(2x-1)^2(x-1)$$

$$x-2 = A(x-1) + B(2x-1)(x-1) + C(2x-1)^2$$

$x=\frac{1}{2}$: $\frac{1}{2}-2 = (-\frac{1}{2})A \quad -\frac{3}{2} = -\frac{1}{2}A \quad A=3$

$x=1$: $-1 = C$

$x=0$. $-2 = -A+B+C$

$-2 = -3+B-1$

$B=2$

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{3}{(2x+1)^2} + \frac{2}{2x-1} + \frac{-1}{x-1}$$

$$\int \frac{3}{(2x-1)^2} + \frac{2}{2x-1} + \frac{-1}{x-1}\,dx$$

$u=2x-1$ $\quad$ $u=x-1$

$$-\frac{3}{2}\frac{1}{2x-1} + \ln|2x-1| - \ln|x-1| + C$$

final answer

$$\int \frac{3}{(2x-1)^2}\,dx$$

$u = 2x-1 \quad \frac{du}{dx} = 2 \quad dx = \frac{du}{2}$

$$\int \frac{3}{u^2}\,\frac{du}{2}$$

$u^n \to \frac{u^{n+1}}{n+1}$

$$\frac{3}{2}\int u^{-2}\,du$$

$$\frac{3}{2}\,\frac{u^{-1}}{-1} = -\frac{3}{2}\cdot\frac{1}{u}$$

$$= -\frac{3}{2}\,\frac{1}{2x-1}$$

$$\int \frac{2}{2x-1}\,dx$$

$u = 2x-1 \quad \frac{du}{dx} = 2 \quad dx = \frac{du}{2}$

$$\int \frac{2}{u}\,\frac{du}{2}$$

$$= \int \frac{1}{u}\,du = \ln|u|$$

$$= \ln|2x-1|$$

$$\frac{2x-3}{x^3+x} = \frac{2x-3}{x(x^2+1)} = \left(\frac{A}{x} + \frac{Bx+C}{x^2+1}\right)\cdot x(x^2+1)$$

quadratic irreducible factor!

$$2x-3 = A(x^2+1) + (Bx+C)\cdot x$$

$x=0$: $-3 = A$

$x=1$: $-1 = 2A + B + C$ $\quad -1 = -6 + B + C \quad B+C=5$

$x=-1$: $-5 = 2A + (-B+c)(-1)$ $\quad -5 = -6 + B - C \quad B-C = 1$

solve for B & C

$B=3$

$C=2$

$$\frac{2x-3}{x^3+x} = \frac{-3}{x} + \frac{3x+2}{x^2+1}$$

$$\int \frac{2x-3}{x^3+x}dx = \int \frac{-3}{x} + \frac{3x+2}{x^2+1}dx$$

$$\int \frac{2x-3}{x^3+x}\,dx = \int \frac{-3}{x} + \frac{3x+2}{x^2+1}\,dx$$

Break up the N!

$$= \int \frac{-3}{x}\,dx + \underbrace{\int \frac{3x}{x^2+1}\,dx}_{u=x^2+1} + \int \frac{2}{x^2+1}\,dx$$

$$= -3\ln|x| + \frac{3}{2}\ln|x^2+1| + 2\tan^{-1}x + C$$

FINAL ANSWER.

$u = x^2+1 \quad \frac{du}{dx} = 2x \quad dx = \frac{du}{2x}$

$$\int \frac{3x}{u}\,\frac{du}{2x} = \frac{3}{2}\int \frac{1}{u}\,du$$

$$= \frac{3}{2}\ln|u| = \frac{3}{2}\ln|x^2+1|$$

# Summary on Partial Fraction

- Distinct Linear Factors:

$$\frac{q(x)}{(x-r_1)(x-r_2)\cdots(x-r_k)} = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_k}{x-r_k}$$

- Repeated Linear Factors:

$$\frac{4x+3}{(x-2)^3(2x+1)^2} = \frac{A}{(x-2)^3} + \frac{B}{(x-2)^2} + \frac{C}{x-2} + \frac{D}{(2x+1)^2} + \frac{E}{2x+1}$$

- Irreducible Quadratic Factor:

$$\frac{3x+2}{(x^2+4)(x+3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+3}$$