Tests on Difference in Mean, Unknown Variance

∴ if $\underbrace{n \geq 30}$, substitute s_1^2 for σ_1^2

s_2^2 for σ_2^2,

per sample

use z-distribution formula

∴ if $n < 30$, two possible scenarios:

1.) case #1: population variances are assumed equal

$$\therefore \sigma_1^2 = \sigma_2^2 = \sigma^2$$

2.) case #2: pop. variances are assumed unequal

$$\therefore \sigma_1^2 \neq \sigma_2^2$$

∴ use different approaches!

Case #1 : $\sigma_1^2 = \sigma_2^2 = \sigma^2$

∴ we wish to test

hypothesized difference in means, often zero

$$H_0: \mu_1 - \mu_2 = \Delta_0$$

$$H_1: \mu_1 - \mu_2 \neq \Delta_0$$

∴ consider the pooled estimator of σ^2:

$$\boxed{S_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$$

∴ weighted sample variance, which is an estimator for population variance

- test statistic is then

$$T_0 = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

∴ rejection criteria for fixed-α tests:

critical values are $\pm t_{\alpha/2,\, \underbrace{n_1+n_2-2}_{\text{d.o.f.}}}$

ex: chemical process; two different catalysts available

∴ how do they affect yield?

catalyst #1: currently in use

catalyst #2: cheaper, should be adopted <u>if</u> it can be shown that there is no significant effect on yield!

test $H_0: \mu_1 - \mu_2 = 0$

$[\text{OR } \mu_1 = \mu_2]$

vs. $H_1: \mu_1 - \mu_2 \neq 0$

$\boxed{\text{using p-value approach}}$

∴ test data:

$\bar{x}_1 = 92.255\%$

$s_1 = 2.39\ \%$

$n_1 = 8$

these look pretty close!

(we've been fooled before!)

$\bar{x}_2 = 92.733\%$

$s_2 = 2.98\%$

$n_2 = 8$

∴ assume equal pop. variances

$$\therefore S_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} \Rightarrow S_p = 2.70$$

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

$$= -0.35$$

∴ let's look for this test statistic on t-table @ $\underbrace{n_1 + n_2 - 2}_{8+8-2 = 14}$ d.o.f.

$t_{0.40, 14} = 0.258$

$t_{0.25, 14} = 0.692$

∴ because it's a two-sided H_1!

$$0.25 < \frac{p\text{-value}}{2} < 0.40$$

∴ $\boxed{0.50 < p\text{-value} < 0.80}$

∴ no way we can reject H_0 at any reasonable α !!! Blows away $\alpha = 0.05$!

$\boxed{\text{fail to reject } H_0}$ ∴ data suggests the two catalysts do not give unequal yields!

[use cheaper one!]

Case # 2 : $\sigma_1^2 \neq \sigma_2^2$

(Variances assumed unequal)

∴ an <u>exact</u> t-value is unavailable

∴ an <u>approximate</u> one is

$$\boxed{T_0^* = \frac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}}$$

∴ the <u>catch</u> is that we need to adjust our degrees of freedom

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

round down.!!!!!

pro tip : compute $\frac{s_1^2}{n_1}$ and $\frac{s_2^2}{n_2}$ first

Critical values for fixed-α tests: $\pm t_{\alpha/2, v}$

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example: arsenic in drinking water measured in ten Phoenix suburbs (presumably municipal water supply) vs. ten desert communities (presumably well water)

test results:

(suburbs)

$\bar{x}_1 = 12.5$ parts per billion

$s_1 = 7.63$ ppb

(desert)

$\bar{x}_2 = 27.5$ ppb

$s_2 = 15.3$ ppb

pretty big difference in means, and std. deviations!

test $H_0: \mu_1 - \mu_2 = 0$

$H_1: \mu_1 - \mu_2 \neq 0$

using fixed-α approach @ $\alpha = 0.05$

$$t_0^* = \frac{12.5 - 27.5 - 0}{\sqrt{5.822 + 23.41}}$$

$\leftarrow \Delta_0$

$\frac{s_1^2}{n_1} = 5.822$

$\frac{s_2^2}{n_2} = 23.41$

$$= -2.774$$

pretty far out!

[almost three std. dev.]

$$V = \frac{(5.822 + 23.41)^2}{\frac{5.822^2}{9} + \frac{23.41^2}{9}} = 13.22$$

round down to 13 d.o.f.

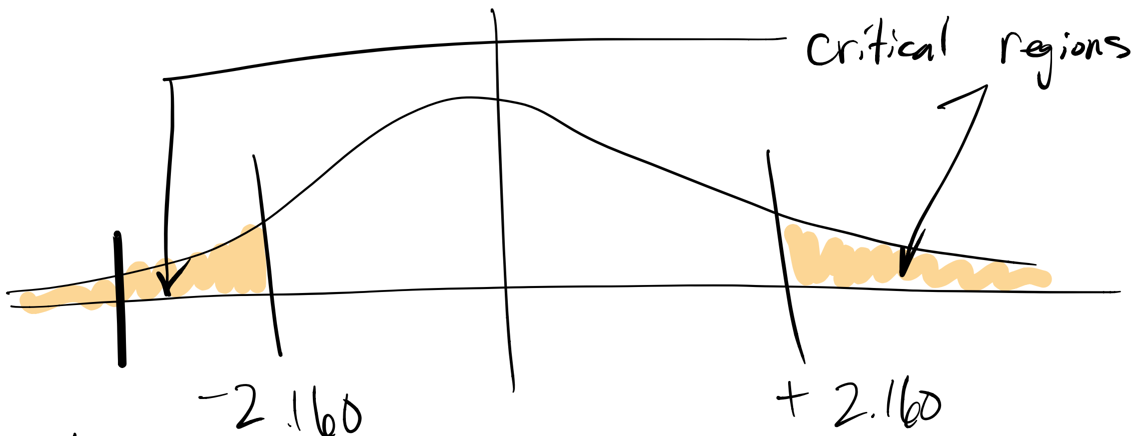
$\because n_1 + n_2 - 2 = 18$ d.o.f.

$\therefore$ adjustment to 13 d.o.f. is significant!

Critical values : $\pm t_{\alpha/2, v}$

$$\pm t_{.025, 13}$$

$$= \pm 2.160$$



$t_0^* = -2.774$

$$t_0^* < -t_{\alpha/2, v}$$

reject H_0

∴ data suggests Ar level is higher in desert communities

[P-value = 0.016 (software)]

∴ we can write confidence intervals on the difference in means for the two cases;

Case #1 $[\sigma_1^2 = \sigma_2^2 = \sigma^2]$

$$\mu_1 - \mu_2 : \quad \bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2, n_1+n_2-2}\, S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

chemical yield example:

$\bar{x}_1 = 92.255\%$

$S_1 = 2.39$

$n_1 = 8$

$\bar{x}_2 = 92.733$

$S_2 = 2.98$

$n_2 = 8$

$S_p = 2.70$

.. write 95% C.I. on the difference in mean chemical yields between the two catalyst:

$$t_{.025, n_1+n_2-2} = t_{.025,14} = 2.145$$

$$\mu_1 - \mu_2 : \quad 92.255 - 92.733 \pm 2.145 \cdot 2.70\sqrt{\frac{1}{8}+\frac{1}{8}}$$

$$-3.373 < \mu_1 - \mu_2 < 2.418 \quad [\% \text{ yield}]$$

.. note: this C.I. contains zero

∴ we would fail to reject

$$H_0 : \mu_1 - \mu_2 = \Delta_0 = 0$$

[in other words, Δ_0 is inside C.I.
∴ fail to reject]

Case #2 : $\sigma_1^2 \neq \sigma_2^2$

adjusted d.o.f. from nasty formula

$$\mu_1 - \mu_2 : \bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2, \nu} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

∴ Ar in drinking water example :

$t_{.025, 13} = 2.160$

$$\mu_1 - \mu_2 : 12.5 - 27.5 + 2.160\sqrt{5.822 + 23.41}$$

$$-26.68 < \mu_1 - \mu_2 < -3.32 \text{ (ppb)}$$

C.I. does not contain zero ;

∴ reject H_0 [C.I. does not contain Δ_0]

∴ data suggests $\mu_2 > \mu_1$ due to negative C.I. for $\mu_1 - \mu_2$