Chapters 5 and 6: Continuous-Time Signal Analysis and Applications: The Fourier Transform

Perform design and analysis of systems and filters in the frequency domain

Focused early on continuous-time analysis of periodic signals using the Fourier Series

The Fourier Transform is applicable for any continuous-time signal

Advantages of the Laplace Transform over the Fourier Transform

- ✓ They are easier to calculate
- ✓ It is easier to include the effect of initial conditions in solving systems problems
- ✓ They exist for power signals as well as energy signals
- ✓ Insight into the frequency response of systems can be obtained from the complex frequency plane (s-plane)

Advantages of the Fourier Transform over the Laplace Transform

- ✓ They have the useful interpretation as a frequency spectrum
- ✓ They can be computed using fast digital computation techniques such as the FFT
- ✓ The inverse transform is a real integral

Fourier Transform

- The Fourier integral (transform) exists only for energy signals
- Periodic power signals have Fourier transforms that contain impulses at discrete frequencies
- To extend the Fourier transform to other power signals, a convergence factor $e^{-\sigma t}$ must be introduced to make the integral exist
- This changes the interpretation of the transform as a frequency spectrum

Fourier Transform

$$\boxed{\mathcal{F}[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$ [Analysis: Time-domain to Frequency-domain]

The Fourier Transform is complex with magnitude and phase: $X(\omega) = |X(\omega)|e^{\angle X(\omega)}$

The Amplitude (Magnitude) Spectrum

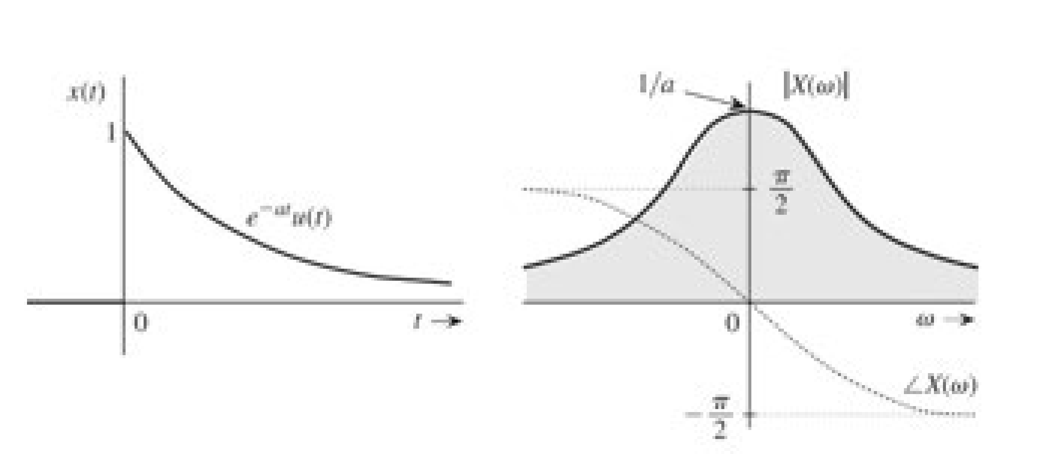
The Phase Spectrum

Inverse Fourier Transform

$$\boxed{\mathcal{F}^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega}$$ [Synthesis: Frequency- to Time-domain]

Example Find the Fourier Transform of an exponential $x(t) = e^{-at}u(t)$

$$\mathcal{F}[x(t)] = X(\omega) = \int_0^\infty x(t)e^{-j\omega t}dt = \int_0^\infty e^{-at}e^{-j\omega t}dt = \int_0^\infty e^{-(a+j\omega)t}dt = \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^\infty = \frac{1}{(a+j\omega)}$$



$$\boxed{\frac{1}{(a+j\omega)} \Leftrightarrow e^{-at}u(t)}$$

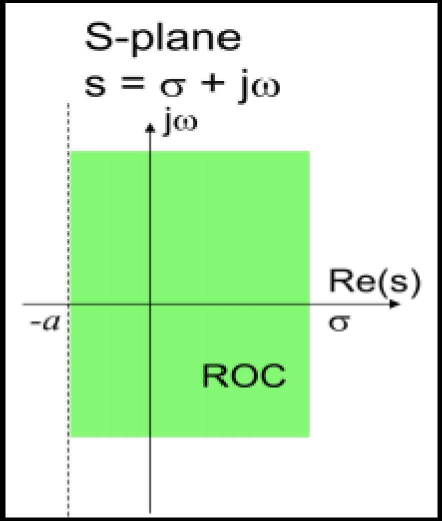
Magnitude: $|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$

Phase: $\angle X(\omega) = e^{-\tan^{-1}\left(\frac{\omega}{a}\right)}$

How does $X(\omega)$ relate to $X(s)$?

$$\mathcal{L}[x(t)] = X(s) = \int_0^\infty x(t)e^{-st}dt = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(a+s)t}dt$$

$$= \left.\frac{e^{-(a+s)t}}{-(a+s)}\right|_0^\infty = \frac{1}{(a+s)}, \text{if } Re(s) > -a$$



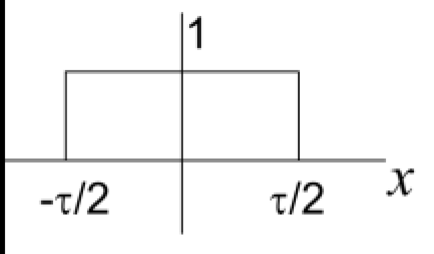
Therefore, $X(\omega) = X(s)|_{s=j\omega}$ if the ROC of $X(s)$ includes the $j\omega$-axis.

In this case, the Fourier Transform exists.

Useful Functions

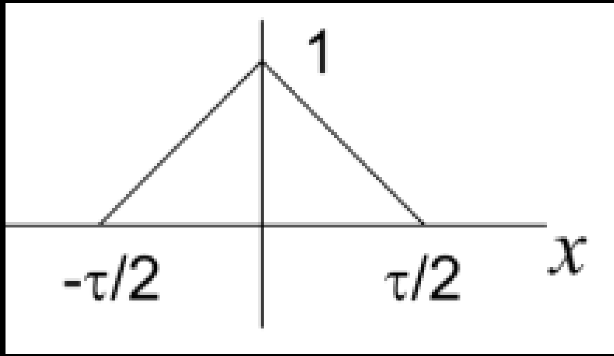
Gate Function

$$\mathrm{rect}\left(\frac{x}{\tau}\right) = \begin{cases} 0, & |x| > \tau/2 \\ \frac{1}{2}, & |x| = \tau/2 \\ 1, & |x| < \tau/2 \end{cases}$$



Triangle Function

$$\Delta\left(\frac{x}{\tau}\right) = \begin{cases} 0, & |x| \geq \tau/2 \\ 1 - 2\left|\frac{x}{\tau}\right|, & |x| < \tau/2 \end{cases}$$



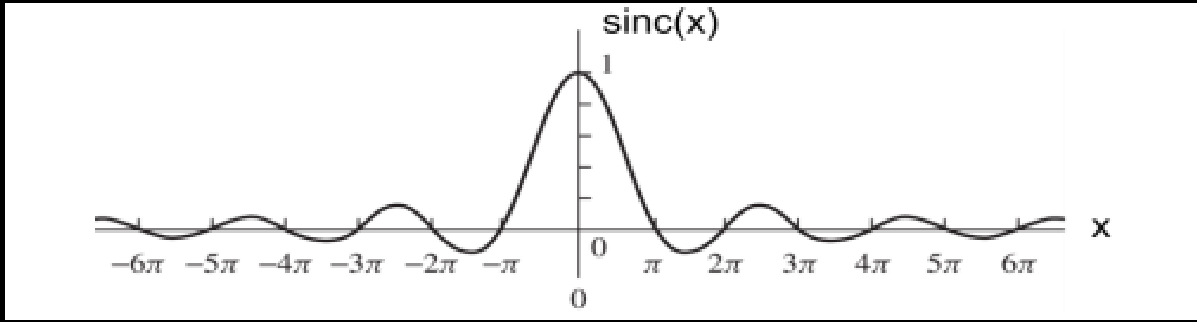
If width $\tau = 1$ in either the gate or triangle function, then it is called the unit gate function or the unit triangle function

Interpolation Function

$$\mathrm{sinc}(x) = \frac{\sin x}{x}$$

$$\mathrm{sinc}(x) = 0, \quad x = \pm k\pi$$

$$\mathrm{sinc}(x) = 1, \quad x = 0$$

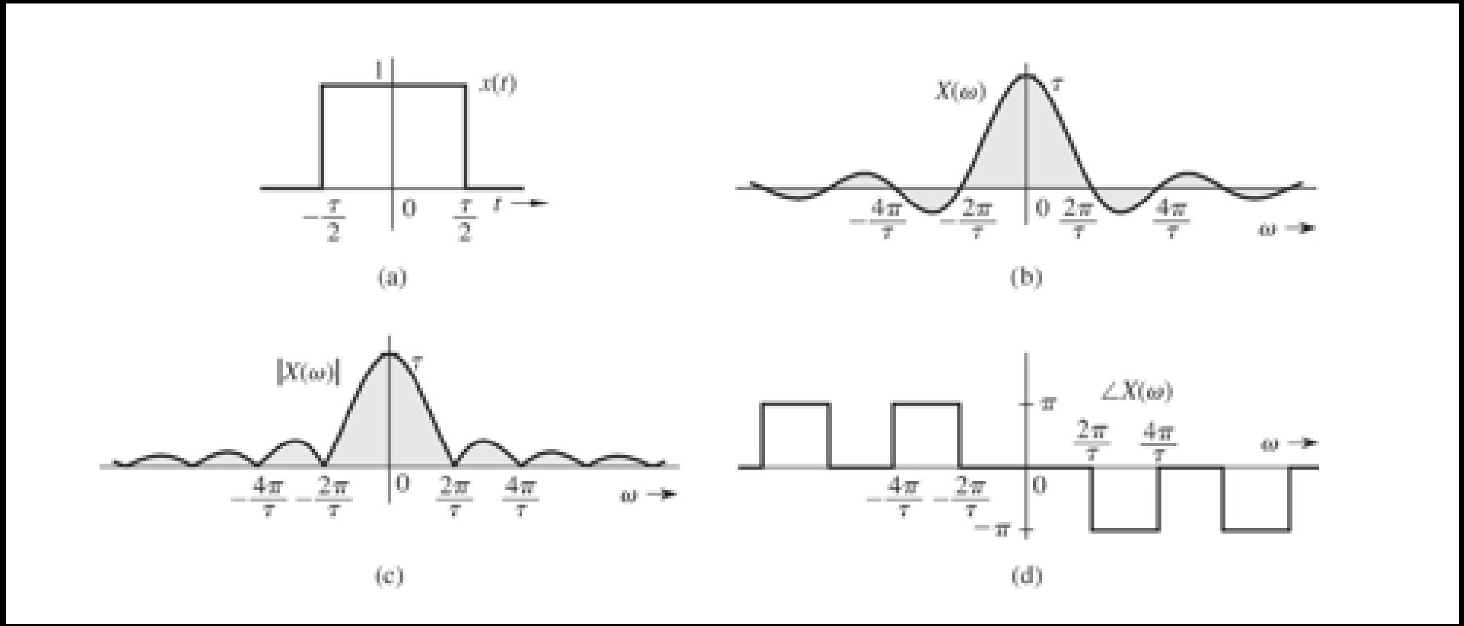


<u>Example</u> Fourier Transform of a Rectangular Pulse $x(t) = \text{rect}\left(\frac{t}{\tau}\right)$

$$X(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = -\frac{1}{j\omega}\left(e^{-\frac{j\omega\tau}{2}} - e^{\frac{j\omega\tau}{2}}\right) = 2\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$$

Fourier Transform Pair

$$\boxed{\text{rect}\left(\frac{t}{\tau}\right) \Leftrightarrow \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)}$$



<u>What is the bandwidth of the sinc spectrum?</u>

- The spectrum of a pulse extends from 0 to ∞
- However, much of the spectrum is concentrated within the first lobe ($\omega = 0$ to $2\pi/\tau$)
- Therefore, a rough estimate of the bandwidth of a rectangular pulse of width τ seconds is $2\pi/\tau$ rad/sec, or $1/\tau$ Hz

Existence of the Fourier Transform (Dirichlet Condition)

Sufficient Condition: $x(t)$ is absolutely integrable, $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

- The area under $|x(t)|$ should be finite
- $x(t)$ is single-valued with finite maxima and minima in any finite time interval
- $x(t)$ is piecewise continuous; that is, it has a finite number of discontinuities in any finite time interval

Observation 1

- The above conditions are not satisfied by many commonly occurring signals, such as $\cos(t), \sin(t)$ and $u(t)$. However, by allowing impulses in the frequency domain, we will define Fourier transforms for these signals also (more on this later…).

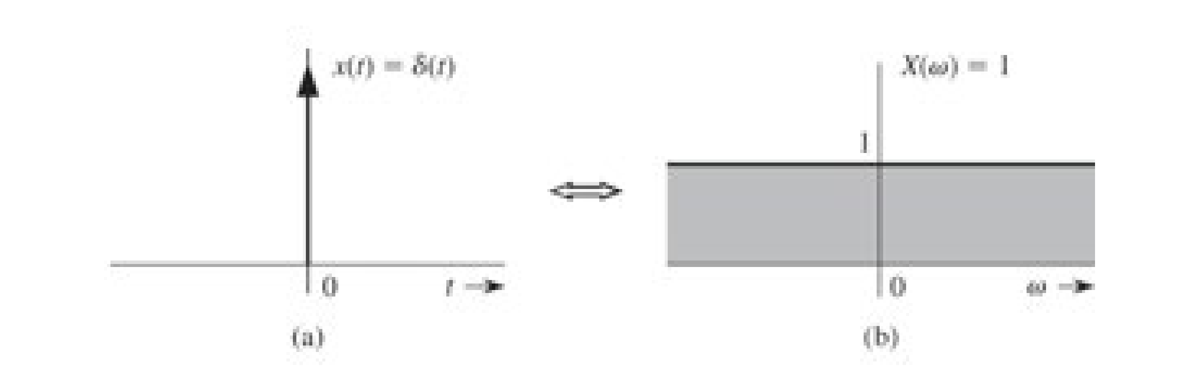
Observation 2

- Almost all practically occurring signals have a Fourier Transform!

Example Find the Fourier Transform of the Unit Impulse $\boxed{\delta(t) \Leftrightarrow 1}$

The Fourier Transform of the unit impulse can be calculated using the sampling property of the unit impulse from Chapter 1 : $x(t) = \delta(t);$ $X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt = 1; \quad \forall\, \omega$

Therefore, the Fourier Transform of the unit impulse has a constant contribution at all frequencies



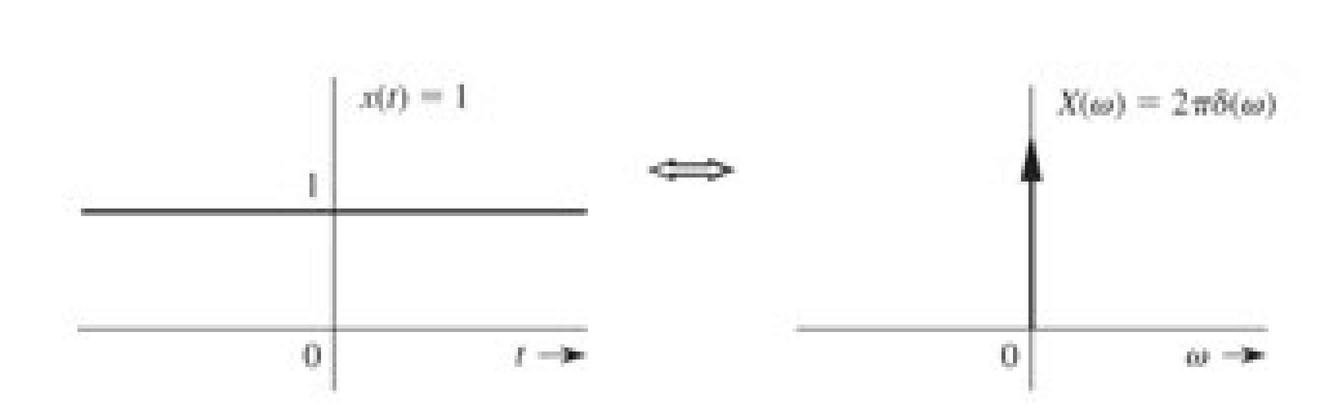
(a) (b)

Example Find the Inverse Fourier Transform of $\delta(\omega)$

$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$ (due to the sampling property of the impulse)

Fourier Transform Pairs:

$$\boxed{\frac{1}{2\pi} \Leftrightarrow \delta(\omega) \ \text{ or } \ 1 \Leftrightarrow 2\pi\delta(\omega)}$$



Related (Important) Example Find the Inverse Fourier Transform of $X(\omega) = \delta(\omega - \omega_0)$

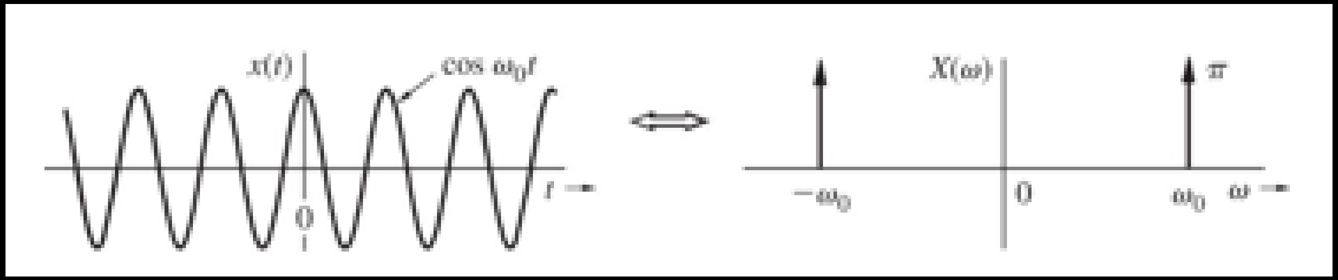
$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$ (due to sampling property of the impulse)

Fourier Transform Pairs: $\boxed{\frac{1}{2\pi} e^{\pm j\omega_0 t} \Leftrightarrow \delta(\omega \mp \omega_0) \text{ or } e^{\pm j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega \mp \omega_0)}$

Example Find the Fourier Transform of $x(t) = \cos(\omega_0 t)$

Using Euler's: $\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$; $\mathcal{F}[\cos(\omega_0 t)] = \frac{1}{2}\mathcal{F}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$

$\therefore \boxed{\cos(\omega_0 t) \Leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]}$



- Recall that a periodic signal does not satisfy the absolute integrability requirement
- Therefore, the Fourier integral does not exist and cannot be used directly to find its Fourier Transform
- The Fourier Transform of $A\cos(\omega_0 t)$ can be found by including impulses in the frequency domain (see above important example)

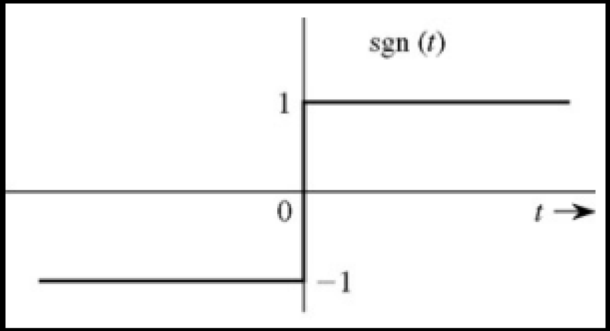
Example Find the Fourier Transform of a periodic signal (sum of many exponentials – Fourier Series)

$$x(t) = \sum_{n=-\infty}^{\infty} D_n\, e^{jn\omega_0 t}; \quad \omega_0 = 2\pi/T_0$$

Take the Fourier Transform of both sides and use the linearity property of the Fourier Transform

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n\, \delta(\omega - n\omega_0)$$

Example Find the Fourier Transform of a signum function



$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

$$= \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

$$\mathcal{F}[\mathrm{sgn}(t)] = \lim_{a\to 0}\left[\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right]$$

$$= \lim_{a\to 0}\left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \lim_{a\to 0}\left[\frac{-2j\omega}{a^2+\omega^2}\right] = \frac{2}{j\omega}$$

$$\boxed{\mathrm{sgn}(t) \Leftrightarrow \frac{2}{j\omega}}$$

Example Find the Fourier Transform of $u(t)$

Observe that $u(t) = \frac{1}{2}[1 + \mathrm{sgn}(t)]$

$$\mathcal{F}[u(t)] = \frac{1}{2}\mathcal{F}[1 + \mathrm{sgn}(t)] = \frac{1}{2}\left[2\pi\delta(\omega) + \frac{2}{j\omega}\right] = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\boxed{u(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}}$$

Example Find the Fourier Transform of an exponential $\boxed{e^{-at}u(t) \Leftrightarrow \frac{1}{a+j\omega}}$

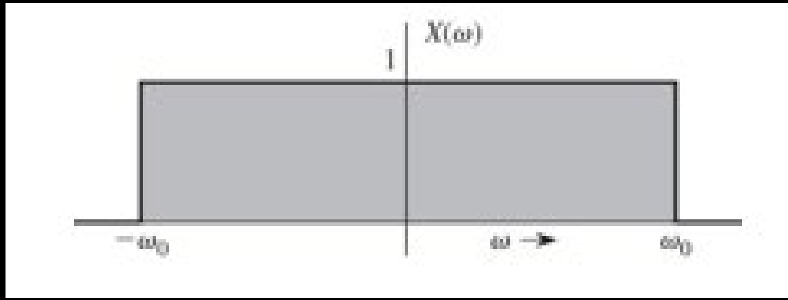
$$\int_0^\infty e^{-at}\, e^{-j\omega t} dt = \int_0^\infty e^{-(a+j\omega)t}\, dt = \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^\infty = \frac{1}{a+j\omega} = \frac{1}{\sqrt{a^2+\omega^2}} e^{-\tan^{-1}\left(\frac{\omega}{a}\right)}$$

Example Find the Inverse Fourier Transform of $X(\omega) = \mathrm{rect}\left(\frac{\omega}{2\omega_0}\right)$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} (1)e^{j\omega t} d\omega = \left.\frac{1}{2\pi(jt)} e^{j\omega t}\right|_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{2\pi(jt)}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) = \frac{\sin(\omega_0 t)}{\pi t}$$

$$= \frac{\omega_0}{\pi}\frac{\sin(\omega_0 t)}{\omega_0 t} = \frac{\omega_0}{\pi}\mathrm{sinc}(\omega_0 t)$$



$$\boxed{\frac{\omega_0}{\pi}\mathrm{sinc}(\omega_0 t) \Leftrightarrow \mathrm{rect}\left(\frac{\omega}{2\omega_0}\right)}$$

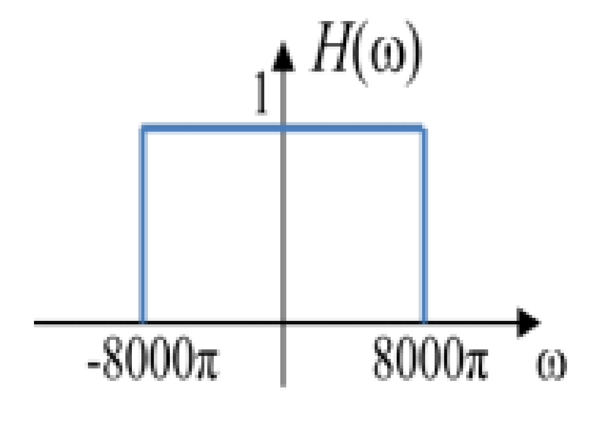
- Lowpass Filter: If $X(\omega)$ represents a filter spectrum, it passes all frequencies between 0 and ω_0
- This is known as an Anti-Aliasing Filter, which is very useful in the reconstruction of a sampled signal

Example Find the impulse response of an anti-aliasing filter $h(t)$ that can pass up to 4000 Hz $(\omega_0 = 2\pi(4000) = 8000\pi \text{ rad/sec})$. The desired Fourier spectrum is shown below.

$$h(t) = \mathcal{F}^{-1}[H(\omega)] = \frac{1}{2\pi}\int_{-8000\pi}^{8000\pi}(1)e^{j\omega t}d\omega = \frac{1}{2\pi(jt)}\left. e^{j\omega t}\right|_{-8000\pi}^{8000\pi}$$

$$= \frac{1}{2\pi(jt)}\left(e^{j8000\pi t} - e^{-j8000\pi t}\right) = \frac{\sin(8000\pi t)}{\pi t}$$

$$= 8000\,\frac{\sin(8000\pi t)}{8000\pi t} = 8000 \operatorname{sinc}(8000\pi t)$$



Connection Between the Fourier and Laplace Transforms

- If the ROC of $X(s)$ includes the $j\omega$-axis, then

$$\mathcal{F}[x(t)] = X(\omega) = \mathcal{L}[x(t)]\big|_{s=j\omega} = \left.\int_{-\infty}^{\infty} x(t)e^{-st}dt\right|_{s=j\omega} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Recall that the Fourier Transform of $u(t)$ is $\pi\delta(\omega) + \frac{1}{j\omega}$. The Laplace Transform of $u(t)$ is $1/s$ with ROC of $Re(s) > 0$. This ROC does not include the imaginary axis. Such is the case for $x(t)$ that are constant, exponentially growing, or oscillating with constant amplitude.

Properties of the Fourier Transform

Time Reversal (or Time Reflection)

$$\boxed{x(-t) \Leftrightarrow X(-\omega) \quad [X^*(\omega),\ \text{for real } x(t)]}$$

Proof: $\mathcal{F}[x(-t)] = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt = -\int_{\infty}^{-\infty} x(\tau)e^{j\omega\tau}d\tau = \int_{-\infty}^{\infty} x(\tau)e^{j\omega\tau}d\tau = X(-\omega)$

Let $\tau = -t;\ dt = -d\tau;\ Limits\!: t = \pm\infty \Rightarrow \tau = \mp\infty$

For real $x(t)$: $\mathcal{F}[x(-t)] = \int_{-\infty}^{\infty} x(\tau)e^{j\omega\tau}d\tau = \left(\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = X(-\omega)\right)^* = X^*(\omega)$

$X(-\omega) = X^*(\omega)$, i.e., $X(\omega) = X^*(-\omega)$

$\Rightarrow |X(\omega)| = |X(-\omega)|$; Magnitude is even

$\angle X(\omega) = -\angle X(-\omega)$; Phase is odd

Example Apply the reflection property to find the Fourier Transform of $x(t) = e^{-a|t|}$, a > 0

$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = x_1(t) + x_1(-t)$

$X_1(\omega) = \mathcal{F}[x_1(t)] = \mathcal{F}[e^{-at}u(t)] = \frac{1}{(a+j\omega)}$ (done previously)

$\mathcal{F}[x_1(-t)] = X_1(-\omega) = \frac{1}{(a-j\omega)}$ (according to time-reflection property)

$X(\omega) = X_1(\omega) + X_1(-\omega) = \frac{1}{(a+j\omega)} + \frac{1}{(a-j\omega)} = \frac{a+j\omega+a-j\omega}{a^2+\omega^2} = \frac{2a}{a^2+\omega^2}$

$$\boxed{e^{-a|t|} \Leftrightarrow \frac{2a}{a^2+\omega^2}}$$

Alternate Method   Use the Fourier integral to find the Fourier Transform of $e^{-a|t|}$, $a > 0$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{(a-j\omega)t} dt + \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{e^{(a-j\omega)t}}{a-j\omega}\Bigg|_{-\infty}^{0} + \frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\Bigg|_{0}^{\infty} = \frac{1}{(a+j\omega)} + \frac{1}{(a-j\omega)} = \frac{a+j\omega+a-j\omega}{a^2+\omega^2} = \boxed{\frac{2a}{a^2+\omega^2} \Leftrightarrow e^{-a|t|}}$$

Duality Property   If $x(t) \Leftrightarrow X(\omega)$, then $\mathcal{F}[X(t)] = 2\pi\, x(-\omega)$

Proof: $2\pi\, x(\mathrm{t}) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \Rightarrow \quad 2\pi\, x(-\mathrm{t}) = \int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$

Interchange symbols ω and t: $2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt = \mathcal{F}[X(t)]$

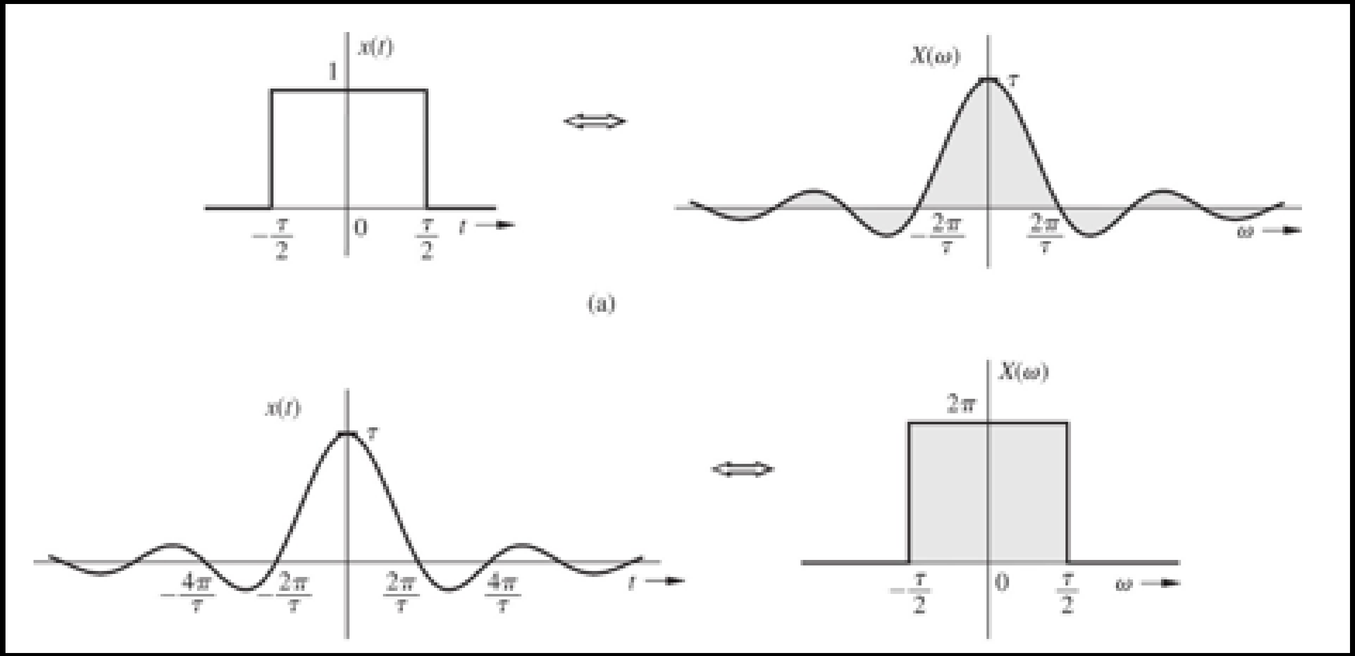
Example We proved earlier that $\underbrace{\mathrm{rect}\left(\frac{t}{\tau}\right)}_{x(t)} \Leftrightarrow \underbrace{\tau\, \mathrm{sinc}\left(\frac{\omega\tau}{2}\right)}_{X(\omega)}$

Using Duality we can find the Fourier Transform of a sinc function as follows:

$$\underbrace{\tau\, \mathrm{sinc}\left(\frac{t\tau}{2}\right)}_{X(t)} \Leftrightarrow \underbrace{2\pi\, \mathrm{rect}\left(\frac{\omega}{\tau}\right)}_{2\pi\, x(-\omega)}$$

The Fourier Transform of a time-domain sinc signal cannot be found using the Laplace Transform or the integral definition of the F.T. The duality property provides an indirect way to obtain it.

Time-Domain



Frequency-Domain

Linearity Property

Proof:

$$\mathcal{F}[ax(t)+by(t)] = \int_{-\infty}^{\infty}[ax(t)+by(t)]e^{-j\omega t}dt = \int_{-\infty}^{\infty} ax(t)\, e^{-j\omega t}dt + \int_{-\infty}^{\infty} by(t)\, e^{-j\omega t}dt$$

$$= a\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}dt + b\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}dt = aX(\omega) + bY(\omega)$$

$$\mathcal{F}[ax(t)+by(t)] = aX(\omega)+bY(\omega) \Longrightarrow \boxed{ax(t)+by(t) \Leftrightarrow aX(\omega)+bY(\omega)}$$

Scaling Property

Given $x(t) \Longleftrightarrow X(\omega)$ and $a \neq 0$, $\boxed{x(at) \Longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$

Proof:

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \quad \text{assume } a > 0, \text{c.o.v.: } \lambda = at;\ t = \frac{\lambda}{a};\ dt = (1/a)d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\left(\frac{\lambda}{a}\right)} \left(\frac{1}{a}\right) d\lambda = \left(\frac{1}{a}\right) \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\left(\frac{\omega}{a}\right)\lambda} d\lambda = \left(\frac{1}{a}\right) X\left(\frac{\omega}{a}\right)$$

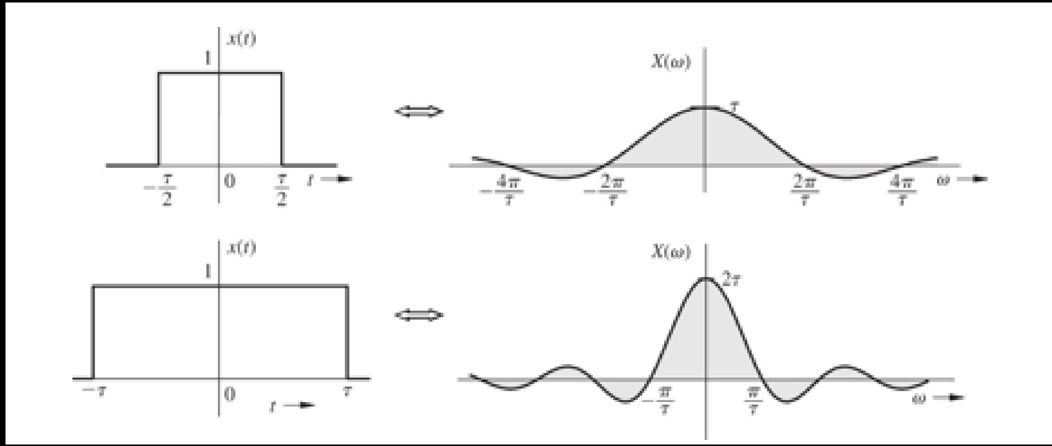
for $a < 0$, the negative value is offset by the change in the limits of integration, giving

$$x(at) \Longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Observations:

- $|a| > 1$: compression in time axis, expansion in frequency axis
- $|a| < 1$: expansion in time axis, compression in frequency axis
- Extent in time domain is inversely proportional to extent in frequency domain (bandwidth)

  $x(t)$ is wider $\longleftrightarrow$ spectrum is narrower
  $x(t)$ is narrower $\longleftrightarrow$ spectrum is wider



Time-Shifting (Delay) Property

Given $x(t) \Leftrightarrow X(\omega)$, $\boxed{x(t-t_0) \Leftrightarrow e^{-j\omega t_0}X(\omega)}$

Proof:

Let $t - t_0 = \tau$

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau+t_0)}d\tau = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}d\tau = e^{-j\omega t_0}X(\omega)$$

Similarly, $\boxed{x(t+t_0) \Leftrightarrow e^{j\omega t_0}X(\omega)}$

Observations:

- Delaying a signal by t_0 seconds does not change its amplitude spectrum
- The phase spectrum, however, is changed by $-\omega t_0$

Alternate Method for Finding the Fourier Transform of $\cos(\omega_0 t)$

- Proved earlier that, $\delta(t) \Leftrightarrow 1$
- Using the time-shifting property, $\delta(t-t_0) \Leftrightarrow e^{-j\omega t_0}$ and $\delta(t+t_0) \Leftrightarrow e^{j\omega t_0}$
- Also, $\delta(t+t_0) + \delta(t-t_0) \Leftrightarrow e^{j\omega t_0} + e^{-j\omega t_0} = 2\cos(\omega_0 t)$
- Using duality, $2\cos(t_0 t) \Leftrightarrow 2\pi[\delta(-\omega+t_0) + \delta(-\omega-t_0)] = 2\pi[\delta(\omega+t_0) + \delta(\omega-t_0)]$

Replacing t_0 by ω_0, we get the very useful result $\boxed{\cos(\omega_0 t) \Leftrightarrow \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]}$

Frequency Shift – Modulation Property

Given $x(t) \Leftrightarrow X(\omega)$, $\boxed{x(t)e^{-j\omega_0 t} \Leftrightarrow X(\omega + \omega_0)}$

Proof:

$$\mathcal{F}\left[x(t)e^{-j\omega_0 t}\right] = \int_{-\infty}^{\infty} x(t)e^{-j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega+\omega_0)t}dt = X(\omega + \omega_0)$$

Similarly, $\boxed{x(t)e^{j\omega_0 t} \Leftrightarrow X(\omega - \omega_0)}$

<u>Example</u> Find the Fourier Transform of $y(t) = x(t)\cos\omega_0 t$

Using Euler's write $x(t)\cos\omega_0 t = \frac{1}{2}x(t)\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

$$\mathcal{F}[x(t)\cos\omega_0 t] = \frac{1}{2}\mathcal{F}\left[x(t)e^{j\omega_0 t}\right] + \frac{1}{2}\mathcal{F}\left[x(t)e^{-j\omega_0 t}\right] = \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0)$$

$$\boxed{x(t)\cos\omega_0 t \Leftrightarrow \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0)}$$

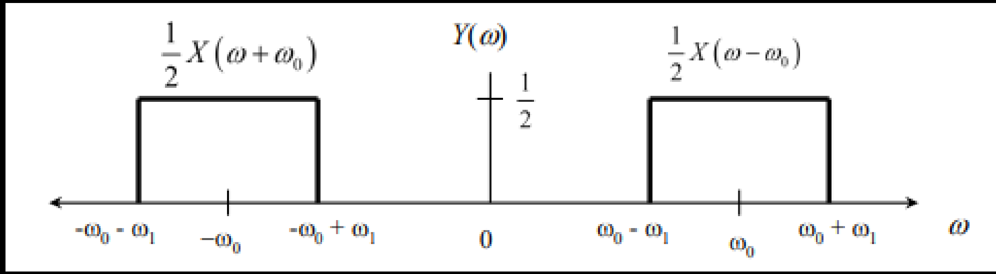
Amplitude Modulation:

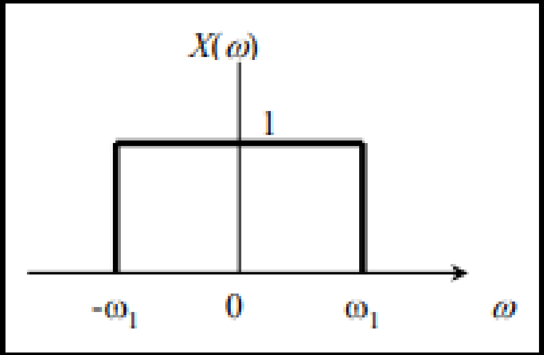
- $\cos\omega_0 t$ is the carrier and $x(t)$ is the modulating signal (message)
- $x(t)\cos\omega_0 t$ is the amplitude modulated signal

Example – Amplitude Modulation

$y(t) = x(t)\cos\omega_0 t$, where $x(t)$ is an ideal lowpass signal

Assume $\omega_1 \ll \omega_0$, i.e., the carrier frequency is much larger than the message bandwidth





Why Modulation?

- Modulation changes frequency content of a message from its baseband frequencies to higher frequencies making its transmission over the airwaves possible
- Music ($0 \le f \le 22\ KHz$) and speech ($100 \le f \le 5\ KHz$) are relatively low frequency signals requiring an antenna of length $\frac{3\times10^8}{4f}$ meters
- If $f = 30\ KHz \Rightarrow$ length of antenna is $2.5\ km \approx 1.5$ miles $\Rightarrow$ need to increase baseband frequencies

For Amplitude Modulation (AM): $f_c = 1000\ KHz \Rightarrow$ length of antenna ≈ 75 meters (rooftop)

For Frequency Modulation (FM): $f_c = 100\ MHz \Rightarrow$ length of antenna ≈ 75 cm

Convolution Property

Fourier Transform of Convolution: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$

Similar to Laplace, convolution in the time-domain corresponds to multiplication in the frequency domain

Time-Convolution: $\boxed{y(t) = x(t) * h(t) \Leftrightarrow Y(\omega) = X(\omega)H(\omega)}$

Proof:

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\left(H(\omega)e^{-j\omega\tau}\right)d\tau = H(\omega)\int_{-\infty}^{\infty} x(\tau)\,e^{-j\omega\tau}d\tau = H(\omega)X(\omega)$$

$t - \tau = \lambda$

$h(\lambda)e^{-j\omega(\lambda+\tau)}$

Frequency-Convolution: $\boxed{x_1(t)x_2(t) \Leftrightarrow \frac{1}{2\pi}X_1(\omega) * X_2(\omega)}$

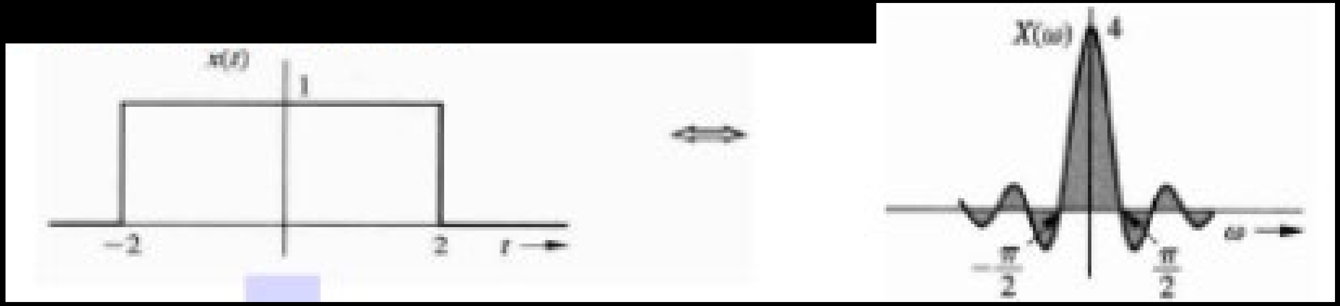
Example $x(t) * \delta(t) = \int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau = x(t)$ [using sampling property of $\delta(t)$]

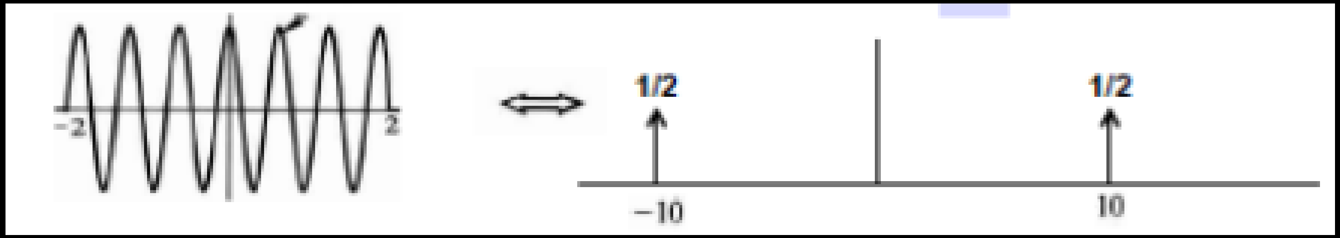
Example $x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} \delta(\tau-t_0)x(t-\tau)d\tau = x(t-t_0)$

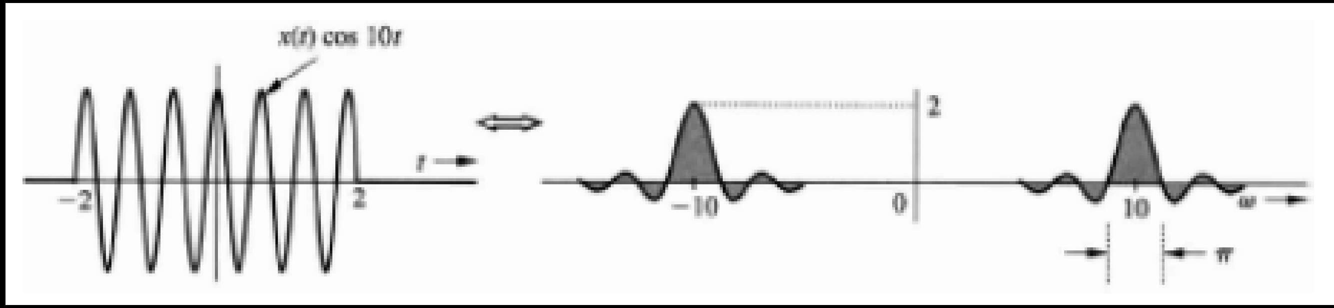
Example (Amplitude Modulation) $y(t) = x(t)\cos\omega_0 t$

$$Y(\omega) = \frac{1}{2\pi}X(\omega) * (\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]) = \frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$$

Example Find the spectrum of $x(t)\cos 10t$ where $x(t) = \text{rect}\left(\frac{t}{4}\right)$ using the convolution property







Signal Energy

Energy in the Time-Domain: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

Energy in the Frequency-Domain: $E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

Parseval's Theorem: Conservation of Energy

Energy in the Time-Domain = Energy in the Frequency-Domain

Proof:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) x^*(t) dt = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Example Find the energy of the signal $x(t) = e^{-at}u(t)$. Determine the frequency W rad/sec so that the energy contributed by the spectral components of all the frequencies below W is 95% of the signal energy E_x.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2at} dt = \frac{1}{2a}$$

This can be verified by Parseval's Theorem.
For this signal, $X(\omega) = \frac{1}{j\omega + a}$ and

$$E_x = \frac{1}{\pi}\int_0^{\infty} |X(\omega)|^2 d\omega = \frac{1}{\pi}\int_0^{\infty} \frac{1}{\omega^2 + a^2} d\omega$$
$$= \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\bigg|_0^{\infty} = \frac{1}{2a}$$

The band $\omega = 0$ to $\omega = W$ contains 95% of the signal energy; that is,

$$\frac{0.95}{2a} = \frac{1}{\pi}\int_0^{W} \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\bigg|_0^{W} = \frac{1}{\pi a}\tan^{-1}\frac{W}{a}$$

or $\frac{0.95\pi}{2} = \tan^{-1}\frac{W}{a} \implies W = 12.706a \text{ rad/sec}$

The spectral components from 0 (DC) to $12.706a$ rad/sec ($2.02a$ Hz) contribute 95% of the total signal energy; all the remaining spectral components in the band from $12.706a$ rad/sec to ∞ contribute only 5% of the signal energy

The Essential Bandwidth of a Signal

- The spectra of all practical signals extend to infinity
- However, because the energy of any practical signal is finite, the signal spectrum must approach $0\ as\ \omega \rightarrow \infty$
- Most of the signal energy is contained within a certain band of B Hz, and the energy contributed by the components beyond B Hz is negligible
- We can therefore suppress the signal spectrum beyond B Hz with little effect on the signal shape and energy
- The bandwidth B is called the essential bandwidth of the signal
- The criterion for selecting B depends on the error tolerance in a particular application
- In filter design half-power is often used, which is the frequency corresponding to 50% energy

Time-Differentiation Property

Given $x(t) \Longleftrightarrow X(\omega)$, $\boxed{\dfrac{d^n x(t)}{dt^n} \Longleftrightarrow (j\omega)^n X(\omega)}$

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \longrightarrow \frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(\omega)\, e^{j\omega t}d\omega \longrightarrow \boxed{\frac{dx(t)}{dt} \Longleftrightarrow (j\omega)X(\omega)} \quad \cdots$$

Example Use the time-differentiation property to find the Fourier transform of the triangle pulse $\Delta\left(\frac{t}{\tau}\right)$

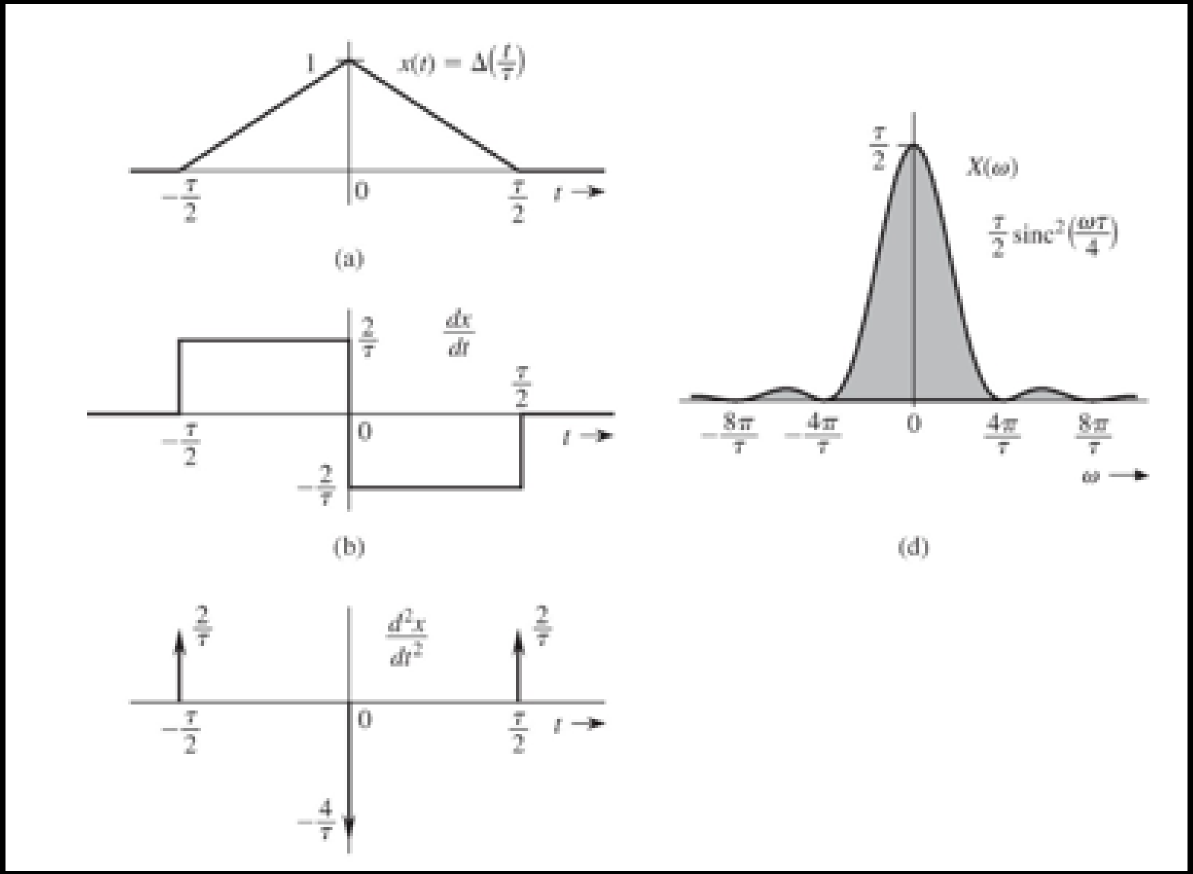
- $\dfrac{d^2x(t)}{dt^2} = \dfrac{2}{\tau}\left[\delta\left(t+\frac{\tau}{2}\right) - 2\delta(t) + \delta\left(t-\frac{\tau}{2}\right)\right]$
- From the time-differentiation property,

$$\frac{d^2x(t)}{dt^2} \Longleftrightarrow (j\omega)^2 X(\omega) = -\omega^2 X(\omega)$$

- From the time-shifting property, $\delta(t-t_0) \Longleftrightarrow e^{-j\omega t_0}$
- $\mathcal{F}\left[\dfrac{d^2x(t)}{dt^2}\right] = -\omega^2 X(\omega) = \dfrac{2}{\tau}\left[e^{j(\omega\tau/2)} - 2 + e^{-j(\omega\tau/2)}\right]$

$$= \frac{4}{\tau}\left[\cos\left(\frac{\omega\tau}{2}\right) - 1\right] = -\frac{8}{\tau}\sin^2\left(\frac{\omega\tau}{4}\right)$$

- $X(\omega) = \dfrac{8}{\omega^2\tau}\sin^2\left(\dfrac{\omega\tau}{4}\right) = \dfrac{\tau}{2}\left[\dfrac{\sin\left(\frac{\omega\tau}{4}\right)}{\left(\frac{\omega\tau}{4}\right)}\right]^2 = \dfrac{\tau}{2}\mathrm{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$



Time-Integration Property

Given $x(t) \Leftrightarrow X(\omega)$, $\boxed{\int_{-\infty}^{t} x(\tau)d\tau \Leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)}$

Proof: $u(t-\tau) = 1, \tau \leq t$ time-convolution property

$$\int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = x(t) * u(t) \Leftrightarrow X(\omega)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$
$$= \pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$$

Signal Transmission through LTIC Systems

- A linear system is characterized by its impulse response, $h(t) \Leftrightarrow H(\omega)$
- If $x(t)$ is the input to a linear system, then the output is given by

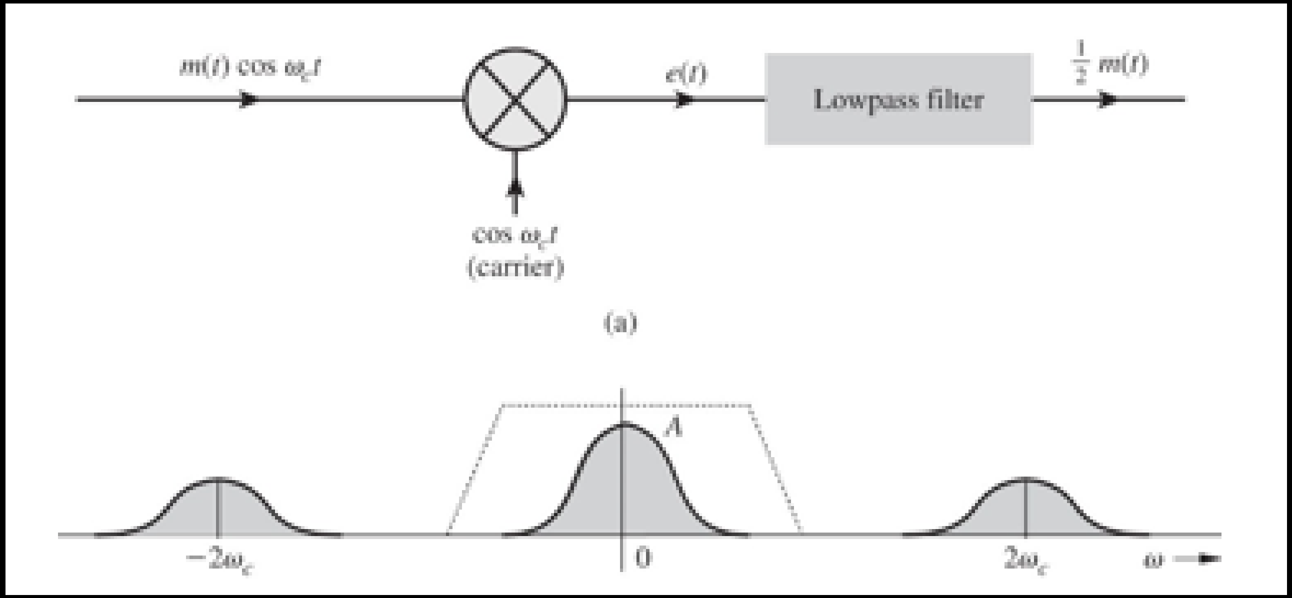
$$\boxed{y(t) = x(t) * h(t) \Leftrightarrow Y(\omega) = X(\omega)\,H(\omega)}$$

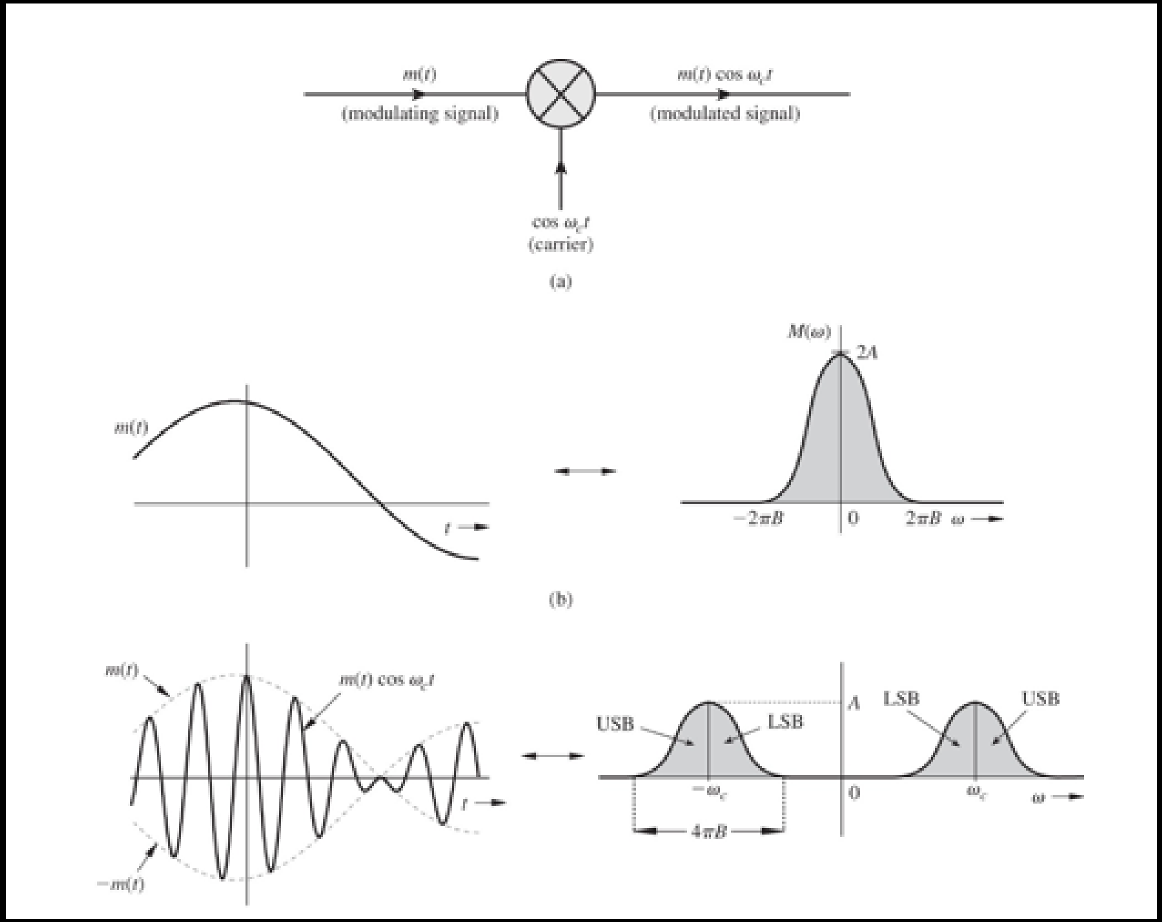
- This is only valid for stable systems!

Amplitude Modulation

Modulation Equation (at the Transmitter)

$$\phi_{AM}(t) = m(t)\cos\omega_c t$$





Demodulation Equation (at the Receiver)

$$\phi_{AM}(t) = m(t)\cos\omega_c t = m(t)\cos^2\omega_c t = \frac{m(t)}{2}[1+\cos 2\omega_c t] \Leftrightarrow \frac{1}{2}M(\omega) + \frac{1}{4}[M(\omega+2\omega_c) + M(\omega-2\omega_c)]$$