Normal Distribution

∴ "bell curve!"

a/k/a Gaussian distribution

∴ De Moivre : Central Limit Theorem
1733, 100 years before Gauss

Central Limit Theorem: when a random experiment is replicated, as $n \uparrow$, the distribution of the mean value becomes normal and $\bar{x} \rightarrow \mu$, regardless of the distribution of the variable X.

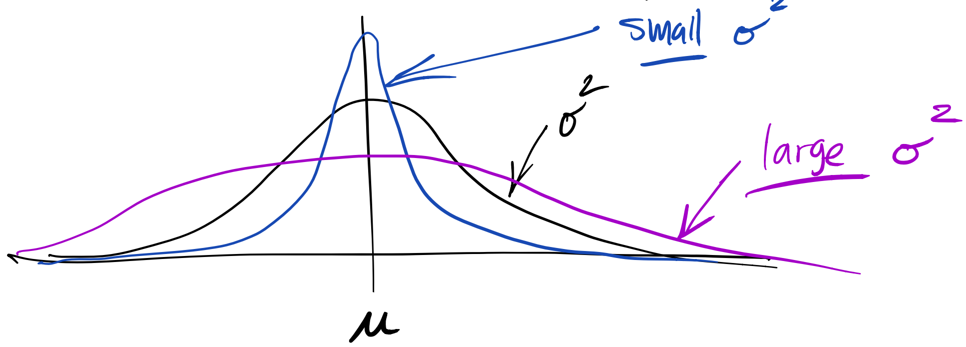
∴ this is huge!!!!!

.. probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

where $\mu = E(X) \rightarrow$ determines center

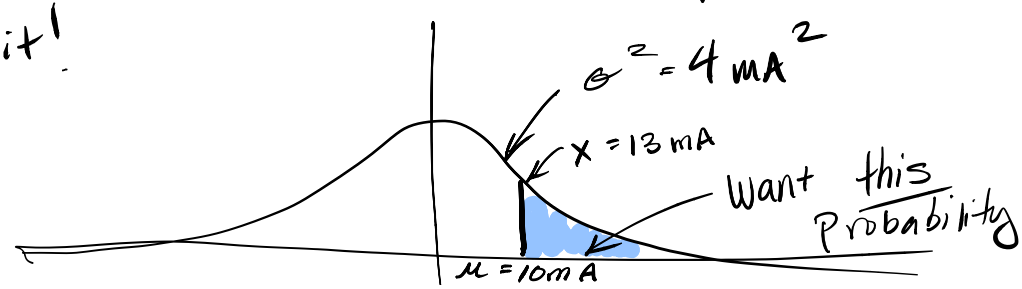
$\sigma^2 = V(X) \rightarrow$ determines width



ex: current measurements in a wire have known population mean $\mu = 10$ mA and variance $\sigma^2 = 4$ mA^2, normally distributed.

.. what is the probability that a measured current will have a value $>$ 13 mA

draw it!



- we could attempt to <u>integrate</u> under $f(x)$ above $x = 13$ mA

- no closed-form expression for $f(x)$ and it's <u>terrible anyway</u> !!!...

- we will determine this probability from a <u>table</u> of values

- can we expect a table for every μ and σ^2 possible? No.

  - introduce concept of <u>standardization</u>.

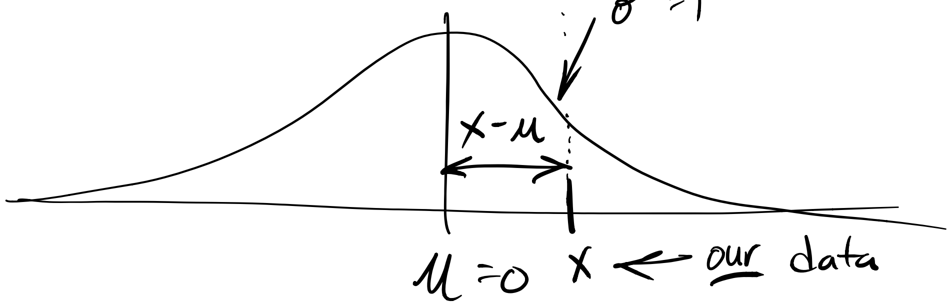
- take <u>our</u> μ and σ^2 and put them in terms of some "standard" values

$$\mu = 0$$
$$\sigma^2 = 1$$

- now we compute <u>Z-value</u>.

$$Z = \frac{x-\mu}{\sigma}$$

- what is the z-value?



- what is the difference $x-\mu$?

$$x - \mu = Z\sigma$$

∴ Z is the number of standard deviations we are away from the mean

- say you get your math test back and it's a 68
  - how "bad" did you do?
  - if the class average (mean) is a 54 with standard deviation of 7, you are <u>two</u> std. dev. above the mean → great!

.. Z tells us where we are relative to μ and σ^2

.. is a Z-table available? Yes.

.. Table III in Appendix

.. cumulative distribution

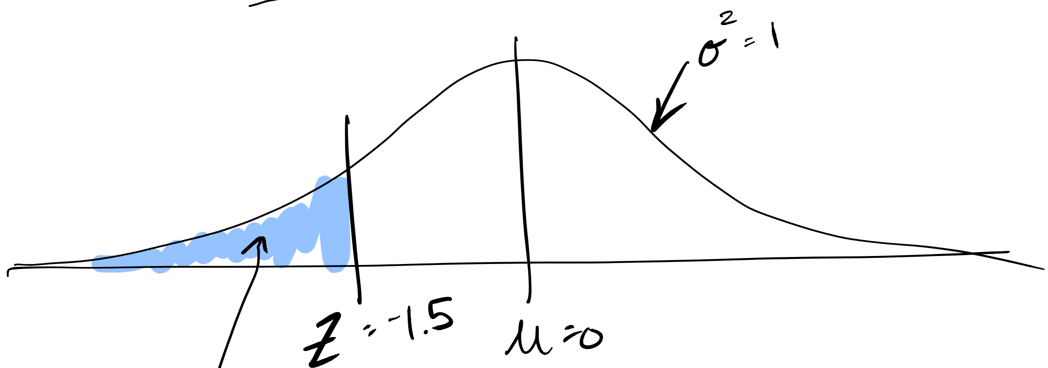


table gives you this;

$$P(Z \leq z)$$

$\rightarrow$ area to the left

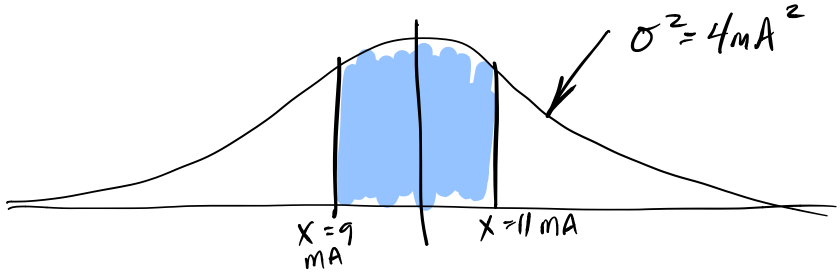
.. back to current in a wire

$\mu = 10$ mA

$\sigma^2 = 4$ mA^2

∴ What is the probability that a current measurement is between 9 and 11 mA ?

∴ first, I like to draw original probability:



$X - \mu = Z\sigma$

∴ gotta standardize

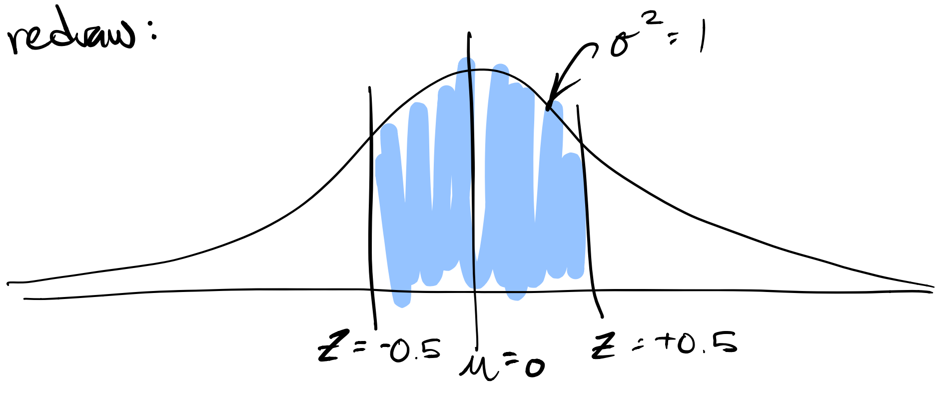
$$Z\Big|_{9mA} = \frac{X-\mu}{\sigma} = \frac{9-10}{\sqrt{4}} = -0.5$$

∴ 9 mA is half a standard deviation below μ

$$Z\Big|_{11mA} = \frac{11-10}{\sqrt{4}} = +0.5$$

∴ half a std. dev. above μ

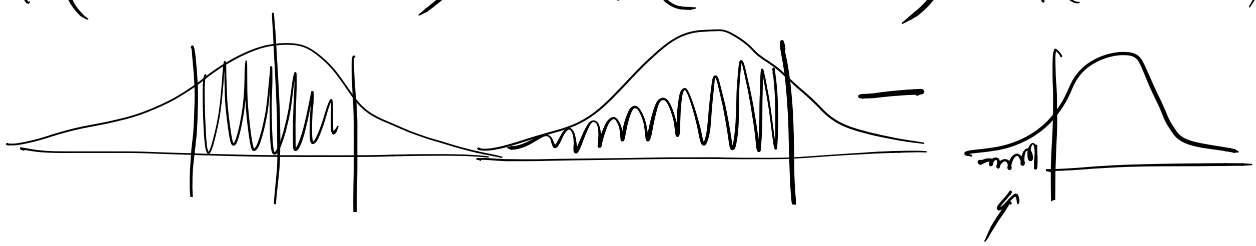
redraw:



- table doesn't give us this directly
- how do we get it in terms of cumulative probabilities?

cumulative

$$P(-0.5 < Z < +0.5) = P(Z < +0.5) - P(Z < -0.5)$$



$$P(Z < +0.5) = 0.691462$$

$$P(Z < -0.5) = 0.308538$$

$$P(-0.5 < Z < +0.5) = 0.691462 - 0.308538$$

$$= 0.382924 \text{ or } 38.29\%$$

$$= P(9 < X < 11) \text{ mA}$$

fun facts :

within <u>one</u> std. dev.

$P(-1 < Z < +1) = 0.6827$

$P(-2 < Z < +2) = 0.9545$

$P(-3 < Z < +3) = 0.9973$

$\pm 3\sigma \longrightarrow$ "six sigma"

Exponential Distribution

∴ consider a Poisson Process

∴ in a <u>discrete</u> Poisson process, X represents # of "flaws" in an interval

∴ we can look at the <u>same</u> problem a different way and call X the <u>"distance" between flaws</u>

→ continuous random variable!

- all about how problem is defined!

Probability density function:

$$f(x) = \lambda e^{-\lambda x}$$

... where λ : avg. # of flaws in an interval

$\therefore$ $E(x) = \mu = \frac{1}{\lambda}$

$$V(x) = \sigma^2 = \frac{1}{\lambda^2}$$

<u>Cumulative</u> distribution function:

$$F(x) = P(X \leq x) = 1 - e^{-\lambda x}$$

ex: Computer log-ons to large network modelled as a Poisson process w/ known mean of 25 logons/hour

- what is the probability of <u>no</u> log-ons for the next six minutes?

$\lambda = 25 \text{ logons}/\text{hour}$ ← already in correct form of "flaws per interval"

∴ x needs to be in the form "distance between flaws"

$$x = \frac{6 \text{ } \cancel{\text{minutes}}}{\text{event}} \cdot \frac{1 \text{ hour}}{60 \text{ } \cancel{\text{minutes}}} = 0.1 \text{ hours}/\text{event}$$

∴ what we're really trying to find is the probability that the time to the next logon exceeds 0.1 hours

$$P(x > 0.1) = \underbrace{1 - P(x < 0.1)}_{\text{compliment of event}} = 1 - F(0.1)$$

$$= 1 - (1 - e^{-\lambda x})$$

$$= 1 - (1 - e^{-25 \cdot 0.1})$$

$$= 0.082 \text{ or } 8.2\%$$

∴ if this had been a discrete Poisson problem:

known: average # of logons per hour is 25

X is now the # of logons in an interval

∴ pick interval length of 6 minutes

∴ now we simply find the probability of exactly zero logons in this interval

$= f(0)$

$$\lambda = 25 \frac{\text{logons}}{\cancel{\text{hour}}} \cdot \frac{1 \ \cancel{\text{hour}}}{10 \cdot \underbrace{6 \text{ minutes}}_{\text{interval}}} = 2.5 \frac{\text{logons}}{\underbrace{6 \text{ mins.}}_{\text{interval}}}$$

∴ back to discrete Poisson formula:

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!},$$

$$f(0) = \frac{e^{-2.5} \cdot 2.5^0}{\cancel{0!}} \quad 2.5^0 \rightarrow 1, \quad 0! \rightarrow 1$$

$f(0) = 0.082$ or 8.2%