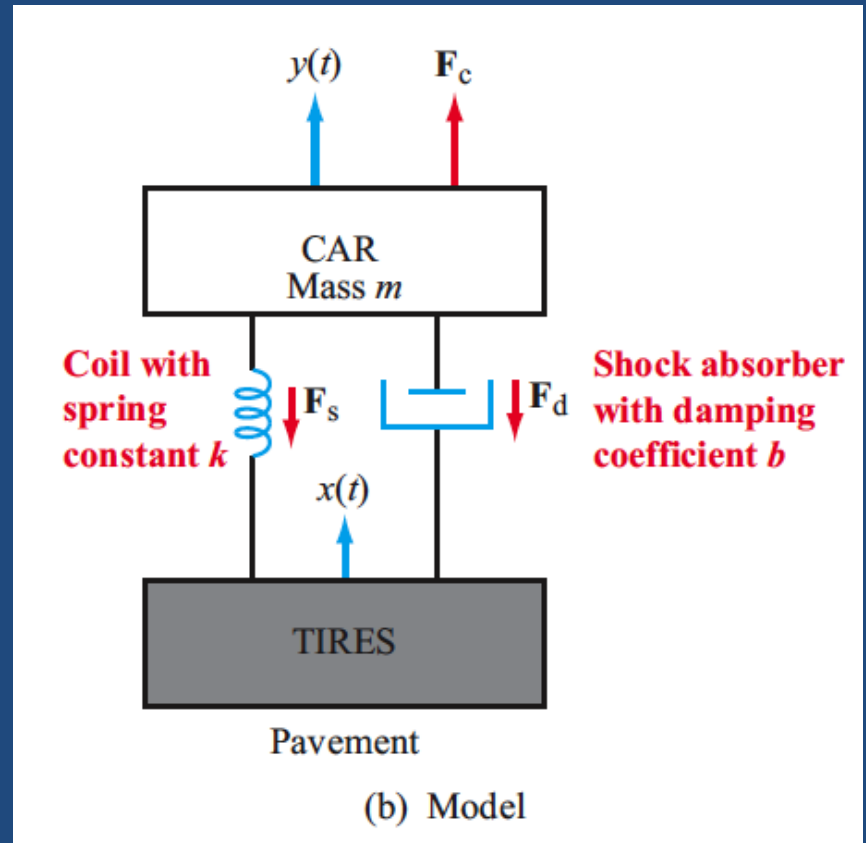


(a) Damping system



## 2. LTI SYSTEMS

# Linear Time-Invariant Systems

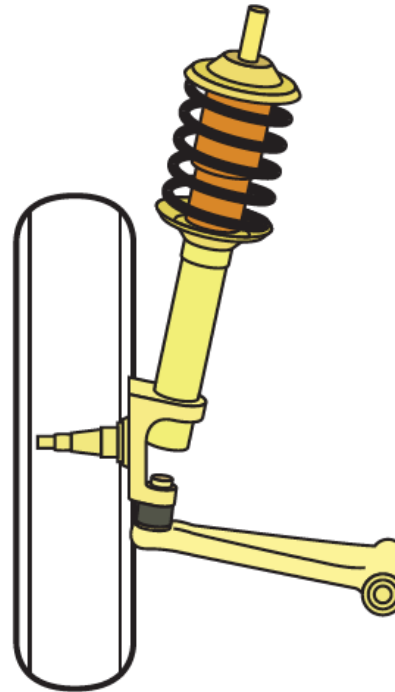
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## Objectives

Learn to:

- Describe the properties of LTI systems.
- Determine the impulse and step responses of LTI systems.
- Perform convolution of two functions.
- Determine causality and stability of LTI systems.
- Determine the overdamped, underdamped, and critically damped responses of second-order systems.
- Determine a car's response to various pavement profiles.

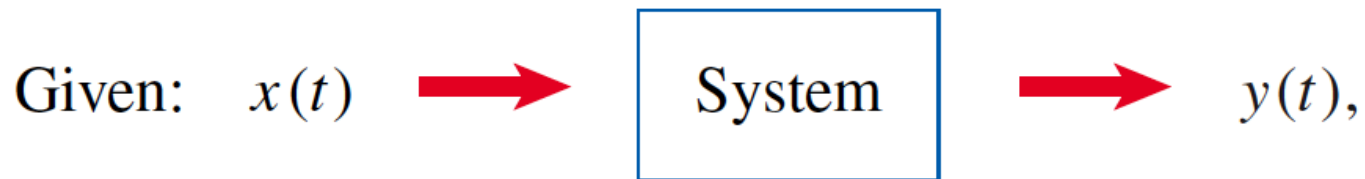


By modeling a car suspension system in terms of a differential equation, we can determine *the response* of the car's body to any pavement profile it is made to drive over. The same approach can be used to compute the response of any *linear system* to any *input excitation*. This chapter provides the language, the mathematical models, and the tools for characterizing linear, time-invariant systems.

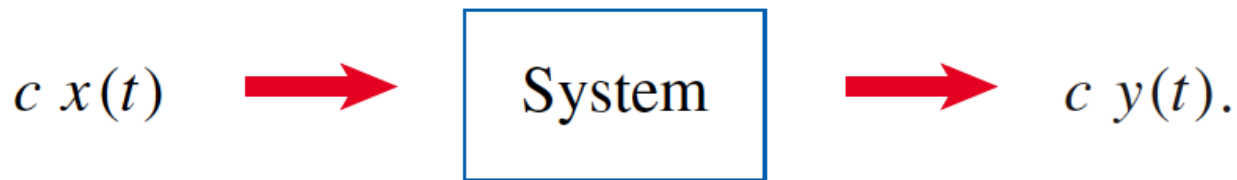
# LTI Systems

- A system is LTI if it is linear and time invariant.
- A system is linear if it has the scaling + additivity properties.

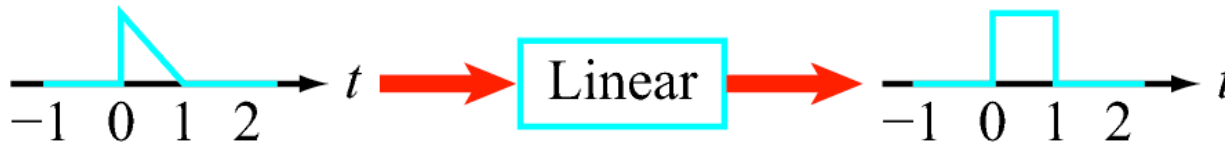
# Scaling Property



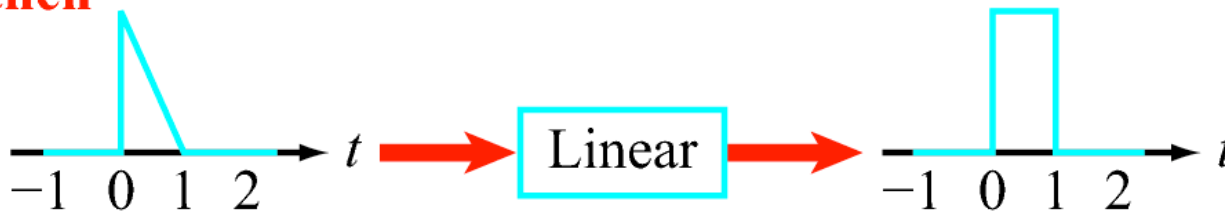
then the system is *scalable* (has the scaling property) if



**If**



**then**



# Scaling Property (cont.)

□ **Example 1:** System described by Diff. Eq.

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 3y = 4 \frac{dx}{dt} + 5x. \quad (2.1)$$

Upon replacing  $x(t)$  with  $c x(t)$  and  $y(t)$  with  $c y(t)$  in all terms, we end up with

$$\frac{d^2}{dt^2} (cy) + 2 \frac{d}{dt} (cy) + 3(cy) = 4 \frac{d}{dt} (cx) + 5(cx).$$

Since  $c$  is constant, we can rewrite the expression as

$$c \left[ \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 3y \right] = c \left[ 4 \frac{dx}{dt} + 5x \right], \quad (2.2)$$

which is identical to the original equation, but multiplied by the constant  $c$ . Hence, since the response to  $c x(t)$  is  $c y(t)$ , the system is scalable and has the scaling property.

# Scaling Property (cont.)

□ **Example 2:** System described by Diff. Eq.

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 3y = 4 \frac{dx}{dt} + 5x + 6$$

Because of the constant term on the far right-hand side of the equation, this system is **NOT scalable**, and therefore **NOT linear**.

# Additivity Property

If the system responses to  $N$  inputs  $x_1(t), x_2(t), \dots, x_N(t)$  are respectively  $y_1(t), y_2(t), \dots, y_N(t)$ , then the system is *additive* if

$$\sum_{i=1}^N x_i(t) \quad \longrightarrow \quad \boxed{\text{System}} \quad \longrightarrow \quad \sum_{i=1}^N y_i(t). \quad (2.4)$$

That is, *the response to the sum is the sum of the responses*.

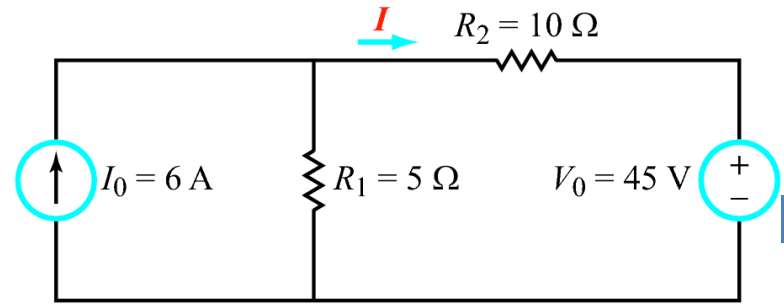
► The combination of scalability and additivity is also known as the *superposition principle*. ◀

# Superposition Example

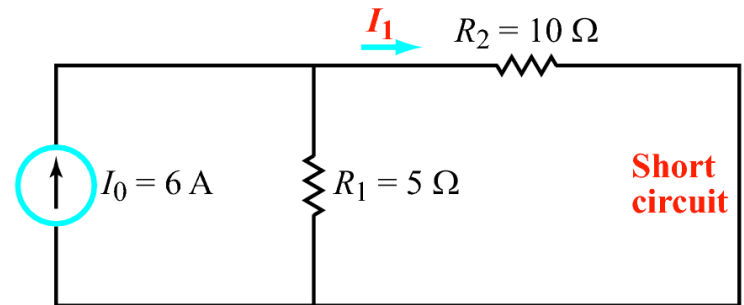
$$I_1 = \frac{I_0 R_1}{R_1 + R_2} = \frac{6 \times 5}{5 + 10} = 2 \text{ A.}$$

$$I_2 = \frac{-V_0}{R_1 + R_2} = \frac{-45}{5 + 10} = -3 \text{ A.}$$

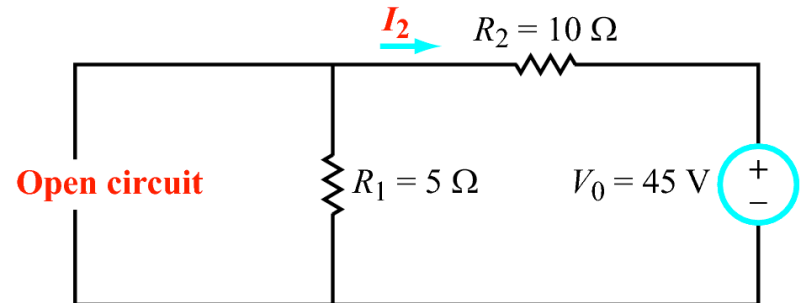
$$I = I_1 + I_2 = 2 - 3 = -1 \text{ A.}$$



(a) **Original circuit**



(b) **Source  $I_0$  alone generates  $I_1$**

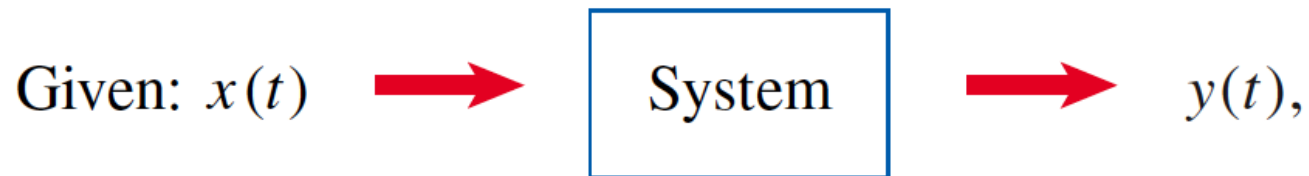


(c) **Source  $V_0$  alone generates  $I_2$**

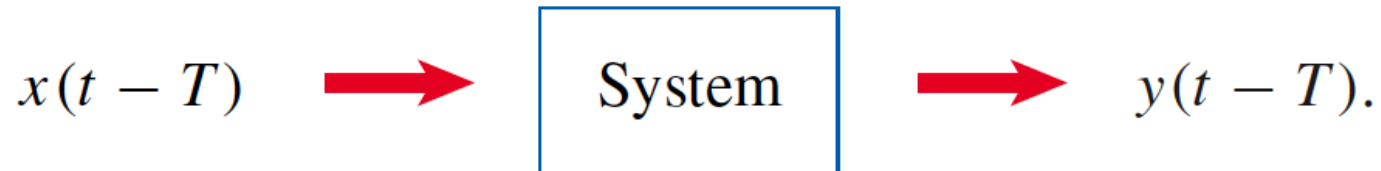


# Time-Invariant Systems

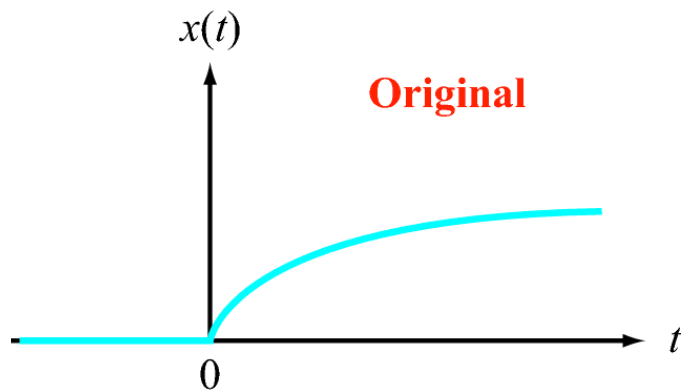
► A system is *time-invariant* if delaying the input signal  $x(t)$  by any constant  $T$  generates the same output  $y(t)$ , but delayed by exactly  $T$ . ◀



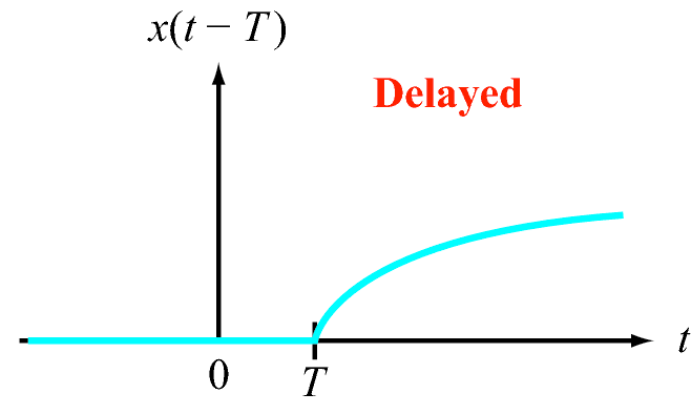
then the system is *time-invariant* if



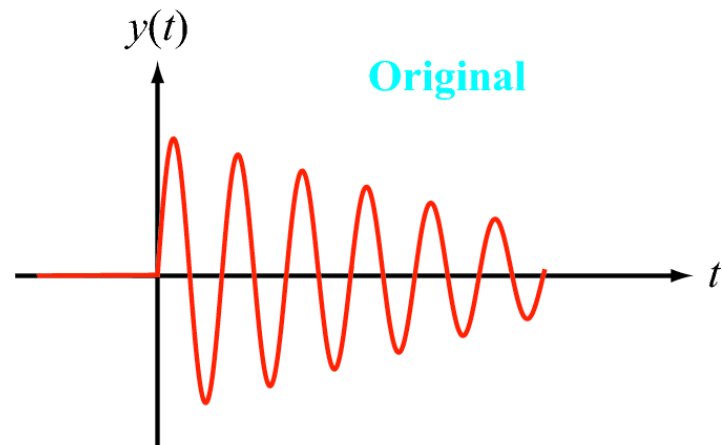
# Examples of Time-Invariant Systems



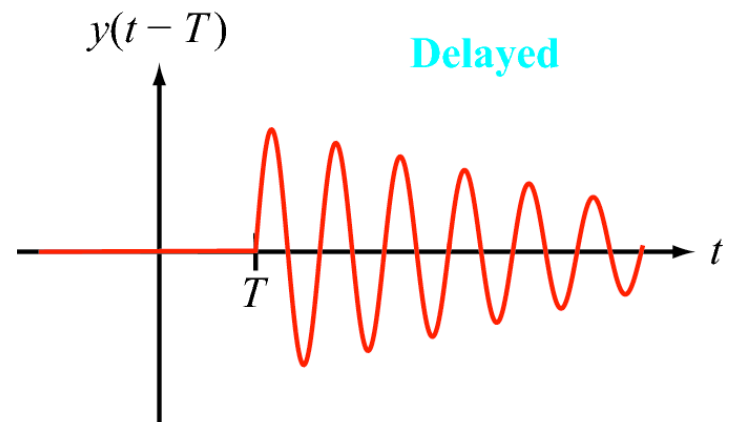
(a)



(c)



(b)



(d)

# More Examples

## □ Time-Invariant

(a)  $y_1(t) = 3 \frac{d^2 x}{dt^2},$

(b)  $y_2(t) = \sin[x(t)].$

(c)  $y_3(t) = \frac{x(t+2)}{x(t-1)}$

**Note 1:** Systems b and c are time-invariant, but not linear.

## Not Time-Invariant

(d)  $y_4(t) = t x(t),$

(e)  $y_5(t) = x(t^2),$

(f)  $y_6(t) = x(-t).$

**Note 2:** Systems d to f are linear, but not time-invariant.

# Impulse & Step Responses

The *impulse response*  $h(t)$  of a system is (logically enough) the response of the system to an impulse  $\delta(t)$ . Similarly, the *step response*  $y_{\text{step}}(t)$  is the response of the system to a unit step  $u(t)$ . In symbolic form:



and



# Static and Dynamic Systems

## □ Static System ( Memoryless)

► A system for which the output  $y(t)$  at time  $t$  depends only on the input  $x(t)$  at time  $t$  is called a *static* or *memoryless* system. ◀

An example of such a system is

$$y(t) = \frac{\sin[x(t)]}{x^2(t) + 1}.$$

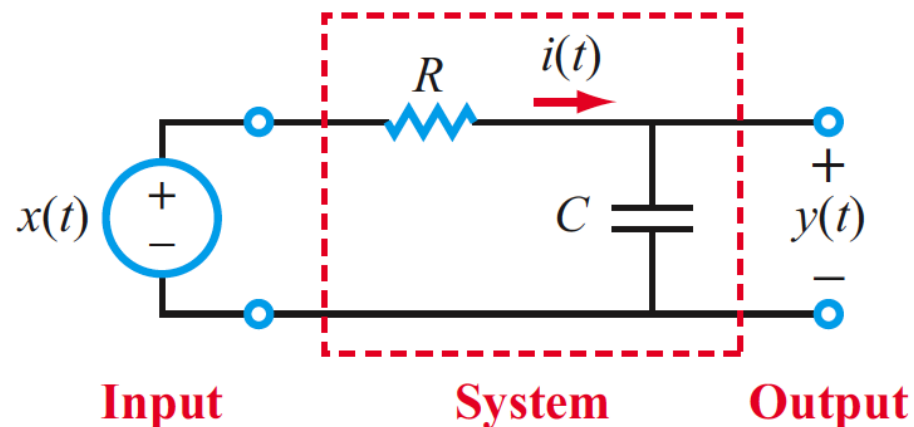
## □ Dynamic System

Output  $y(t)$  depends on past (or future) as well as present values of input  $x(t)$ .

# Computing $h(t)$ and $y_{\text{step}}(t)$ of RC Circuit

$$R i(t) + y(t) = x(t)$$

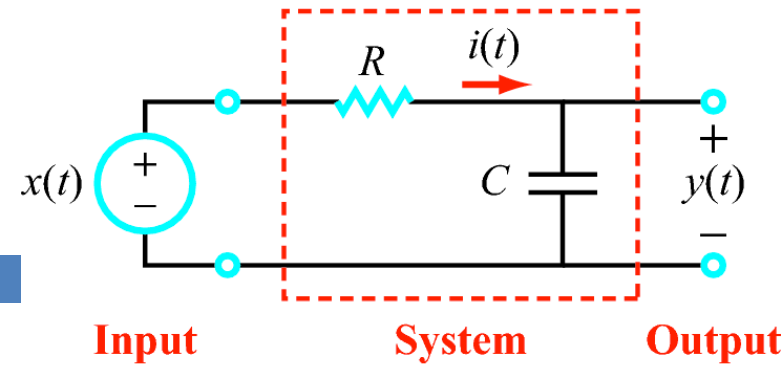
$$i(t) = C \frac{dy}{dt}$$



(a) RC circuit

$$\frac{dy}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

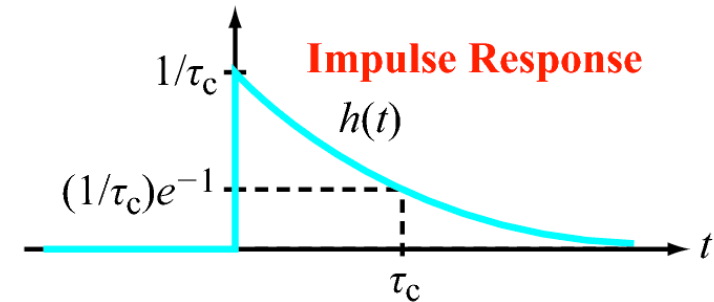
# Impulse & Step Responses



(a) RC circuit

$$h(t) = \frac{1}{\tau_c} e^{-t/\tau_c} u(t).$$

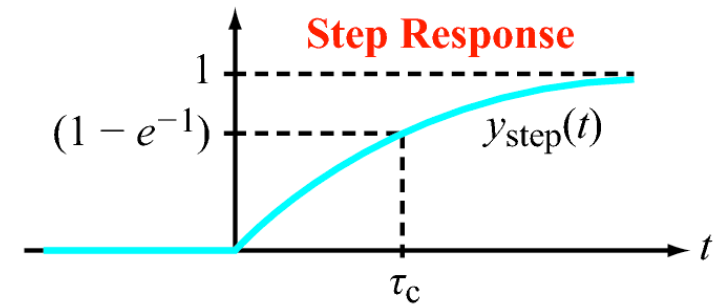
(impulse response of the RC circuit)



(b)  $h(t)$

$$y_{\text{step}}(t) = [1 - e^{-t/\tau_c}] u(t).$$

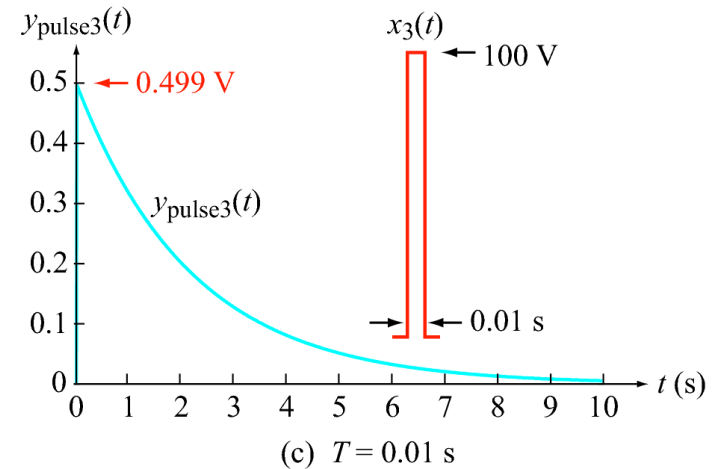
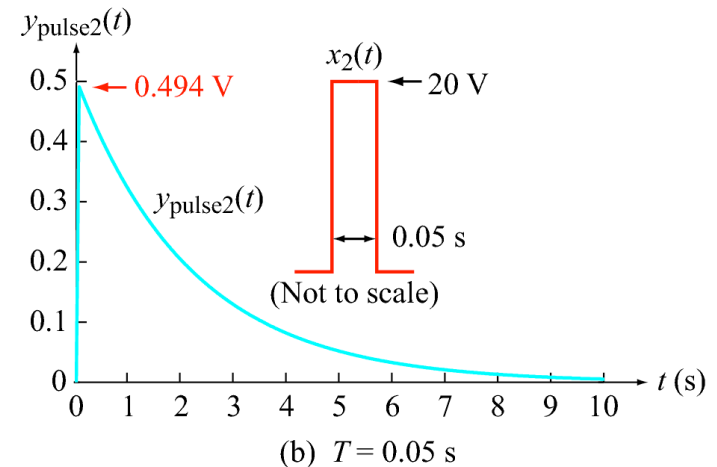
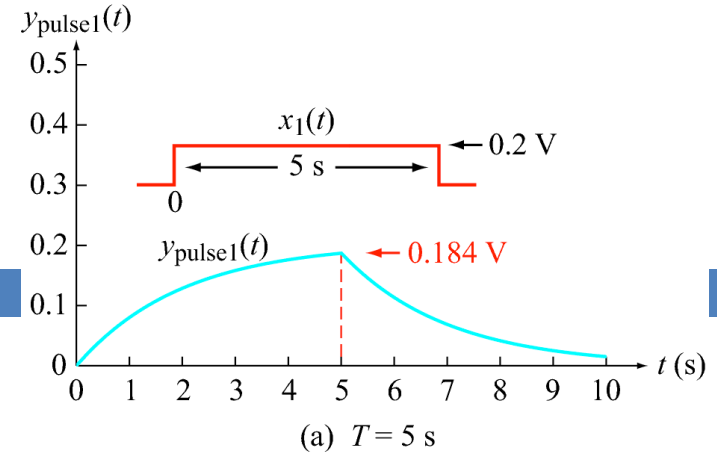
(step response of the RC circuit)



(c)  $y_{\text{step}}(t)$

# Simulating Impulse Response for Circuit with $RC=2s$

For a perfect impulse at the input,  $h(t)=0.5$  at  $t=0$ .





# Impulse Response From Step Response

**Step 1:** Physically *measure* the step response  $y_{\text{step}}(t)$ .

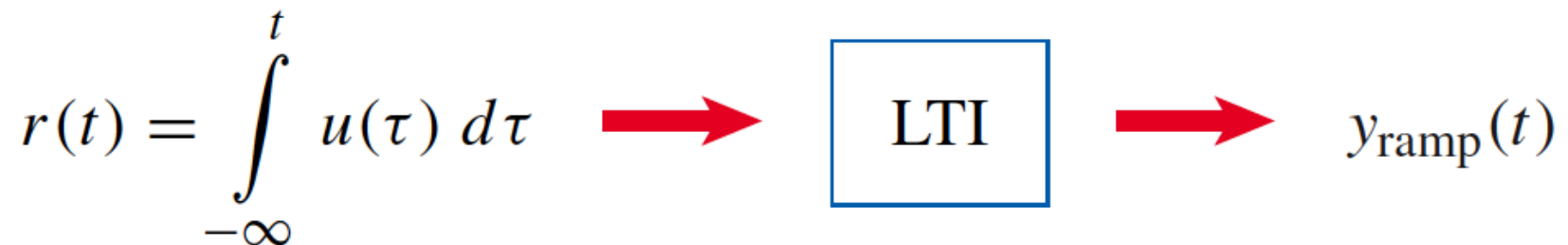
**Step 2:** Differentiate it to obtain

$$h(t) = \frac{dy_{\text{step}}}{dt} .$$



# Ramp Response From Step Response

$$r(t) = \int_{-\infty}^t u(\tau) d\tau$$



# RC Circuit Example

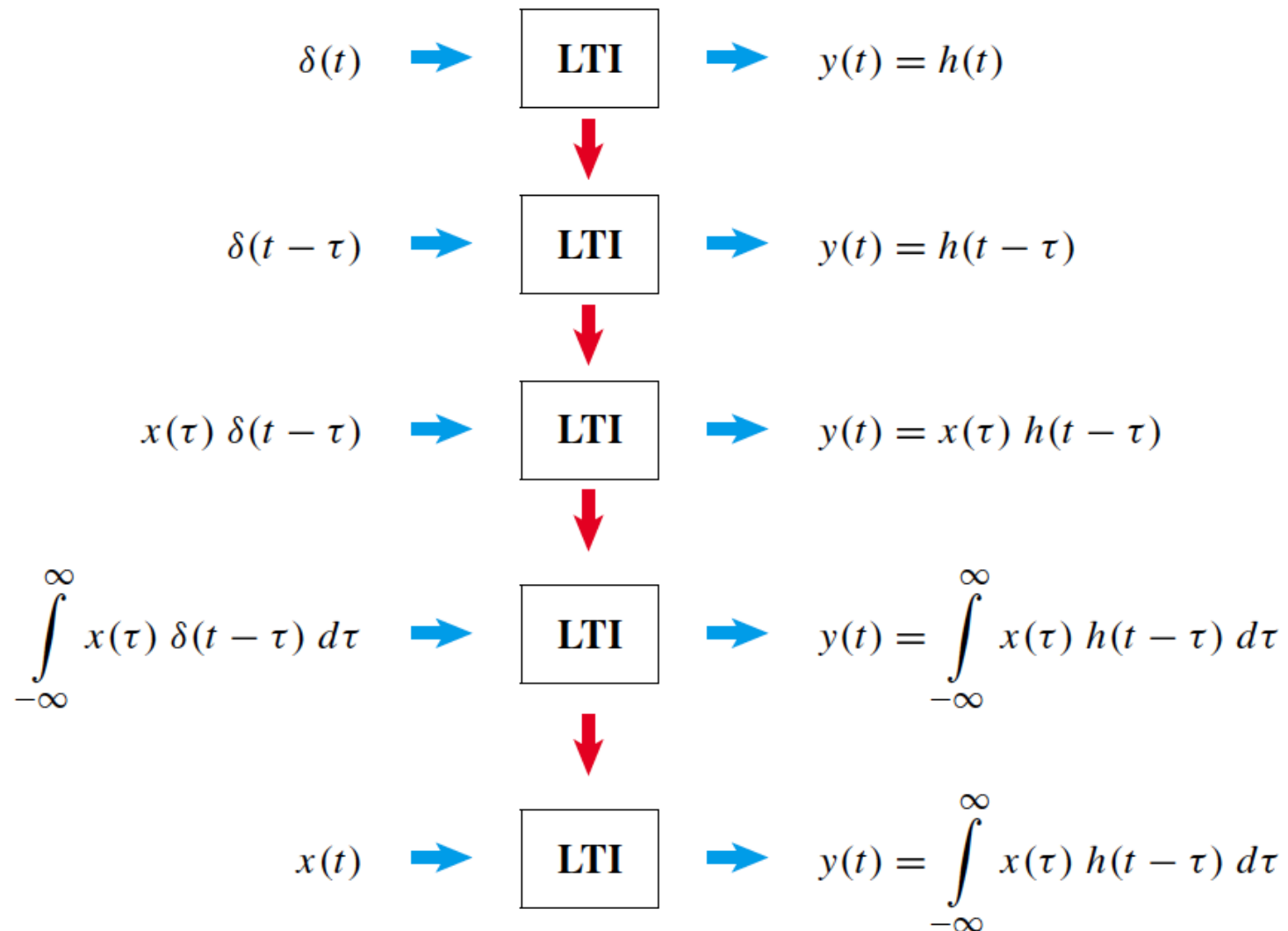
$$y_{\text{step}}(t) = [1 - e^{-t/\tau_c}] u(t).$$

(step response of the RC circuit)

$$\begin{aligned} y_{\text{ramp}}(t) &= \int_{-\infty}^t y_{\text{step}}(\tau) d\tau \\ &= \int_{-\infty}^t (1 - e^{-\tau/\tau_c}) u(\tau) d\tau \\ &= \int_0^t (1 - e^{-\tau/\tau_c}) d\tau \\ &= [t - \tau_c(1 - e^{-t/\tau_c})] u(t). \end{aligned}$$

# Convolution

## LTI System with Zero Initial Conditions



# Use of Convolution

► The response  $y(t)$  of an LTI system with impulse response  $h(t)$  to *any* input  $x(t)$  can be computed *explicitly* using the *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = h(t) * x(t).$$

**(convolution integral)** (2.30)

All initial conditions must be zero. ◀

- Convolution is **Commutative**:

$$x(t) * h(t) = h(t) * x(t)$$

- **For Causal Signals and Systems**

$$\begin{aligned} y(t) &= u(t) \int_0^t x(\tau) h(t - \tau) d\tau \\ &= u(t) \int_0^t x(t - \tau) h(\tau) d\tau. \end{aligned}$$

**(causal signals and systems)**

# Response of Circuit with $RC=1$ s to Triangular Pulse

## Solution:

The input signal, measured in volts, is given by

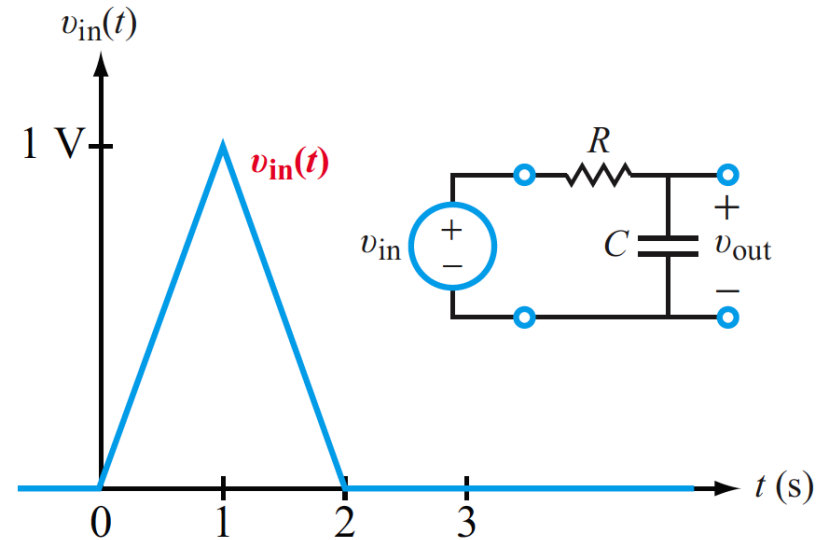
$$v_{\text{in}}(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ t & \text{for } 0 \leq t \leq 1 \text{ s}, \\ 2 - t & \text{for } 1 \leq t \leq 2 \text{ s}, \\ 0 & \text{for } t \geq 2 \text{ s}, \end{cases}$$

and according to Eq. (2.17), the impulse response for  $\tau_c = 1$  is

$$\begin{aligned} h(t) &= \frac{1}{\tau_c} e^{-t/\tau_c} u(t) \\ &= e^{-t} u(t). \end{aligned}$$

with

$$h(t - \tau) = e^{-(t-\tau)} u(t - \tau) = \begin{cases} 0 & \text{for } t < \tau, \\ e^{-(t-\tau)} & \text{for } t > \tau. \end{cases}$$



(a) Triangular pulse

$$\begin{aligned} v_{\text{out}}(t) &= v_{\text{in}}(t) * h(t) \\ &= \int_0^t v_{\text{in}}(\tau) h(t - \tau) d\tau \end{aligned}$$

(1)  $t < 0$ :

The lowest value that the integration variable  $\tau$  can assume is zero. Therefore, when  $t < 0$ ,  $t < \tau$  and  $h(t - \tau) = 0$ . Consequently,

$$v_{\text{out}}(t) = 0 \quad \text{for } t < 0.$$

(2)  $0 \leq t \leq 1 \text{ s}$ :

$$h(t - \tau) = e^{-(t-\tau)}, \quad v_{\text{in}}(\tau) = \tau,$$

and

$$\begin{aligned} v_{\text{out}}(t) &= \int_0^t \tau e^{-(t-\tau)} d\tau \\ &= e^{-t} + t - 1, \quad \text{for } 0 \leq t \leq 1 \text{ s.} \end{aligned}$$

(3)  $1 \text{ s} \leq t \leq 2 \text{ s}$ :

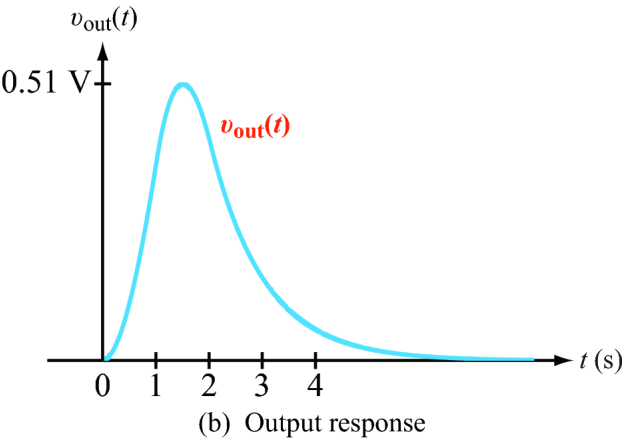
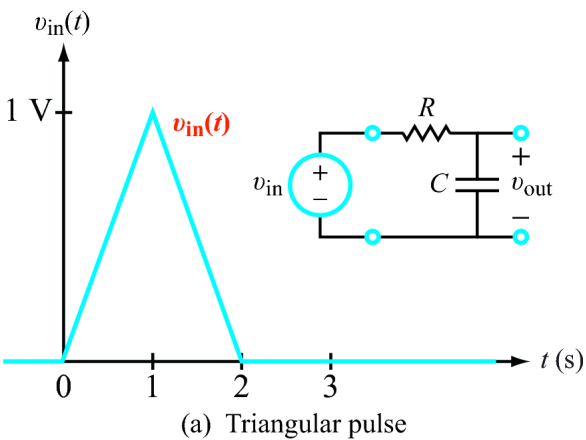
$$v_{\text{in}}(\tau) = \begin{cases} \tau & \text{for } 0 \leq \tau \leq 1 \text{ s,} \\ 2 - \tau & \text{for } 1 \text{ s} \leq \tau \leq 2 \text{ s,} \end{cases}$$

and

$$\begin{aligned} v_{\text{out}}(t) &= \int_0^1 \tau e^{-(t-\tau)} d\tau + \int_1^t (2 - \tau) e^{-(t-\tau)} d\tau \\ &= (1 - 2e)e^{-t} - t + 3, \quad \text{for } 1 \text{ s} \leq t \leq 2 \text{ s.} \end{aligned}$$

(4)  $t \geq 2 \text{ s}$ :

$$\begin{aligned} v_{\text{out}}(t) &= \int_0^1 \tau e^{-(t-\tau)} d\tau + \int_1^2 (2 - \tau) e^{-(t-\tau)} d\tau \\ &= (1 - 2e + e^2)e^{-t} \quad \text{for } t \geq 2 \text{ s.} \end{aligned}$$





# Useful Convolution Formula

## Convolution Integral

For functions  $x(t)$  and  $h(t)$  given by

$$x(t) = f_1(t) u(t - T_1) \quad (2.51a)$$

and

$$h(t) = f_2(t) u(t - T_2), \quad (2.51b)$$

where  $f_1(t)$  and  $f_2(t)$  are any constants or time-dependent signals and  $T_1$  and  $T_2$  are any non-negative numbers, their convolution is

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= u(t) \int_0^t x(t - \tau) h(\tau) d\tau \\ &= \int_0^t f_1(t - \tau) f_2(\tau) u(t - T_1 - \tau) u(\tau - T_2) d\tau \\ &= \left[ \int_{T_2}^{t-T_1} f_1(t - \tau) f_2(\tau) d\tau \right] u(t - T_1 - T_2). \quad (2.52) \end{aligned}$$

# RC Circuit Response to Rectangular Pulse

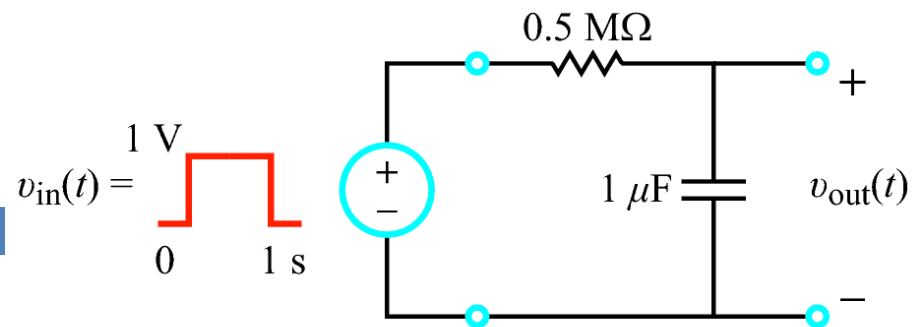
## Solution:

The time constant of the RC circuit is  $\tau_c = RC = (0.5 \times 10^6) \times 10^{-6} = 0.5$  s. In view of Eq. (2.17), the impulse response of the circuit is

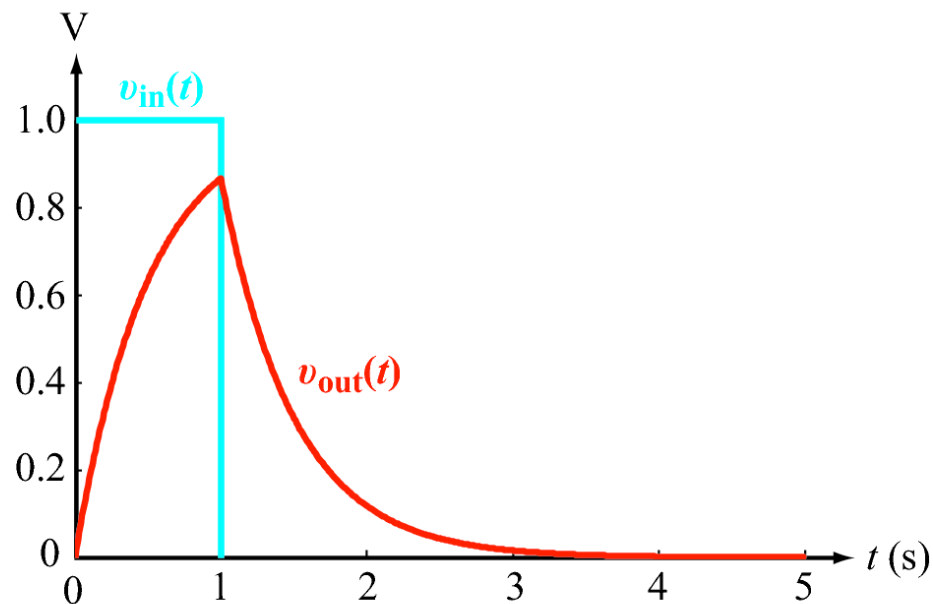
$$h(t) = \frac{1}{\tau_c} e^{-t/\tau_c} u(t) = 2e^{-2t} u(t). \quad (2.53)$$

The input voltage is

$$v_{\text{in}}(t) = [u(t) - u(t - 1)] \text{ V}. \quad (2.54)$$



(a) RC lowpass filter



(b) Output response

$$v_{\text{out}}(t) = v_{\text{in}}(t) * h(t)$$

$$= u(t) \int_0^t v_{\text{in}}(\tau) h(t - \tau) d\tau$$

$$= u(t) \int_0^t [u(\tau) - u(\tau - 1)]$$

$$\times 2e^{-2(t-\tau)} u(t - \tau) d\tau$$

$$= u(t) \int_0^t 2e^{-2(t-\tau)} u(\tau) u(t - \tau) d\tau$$

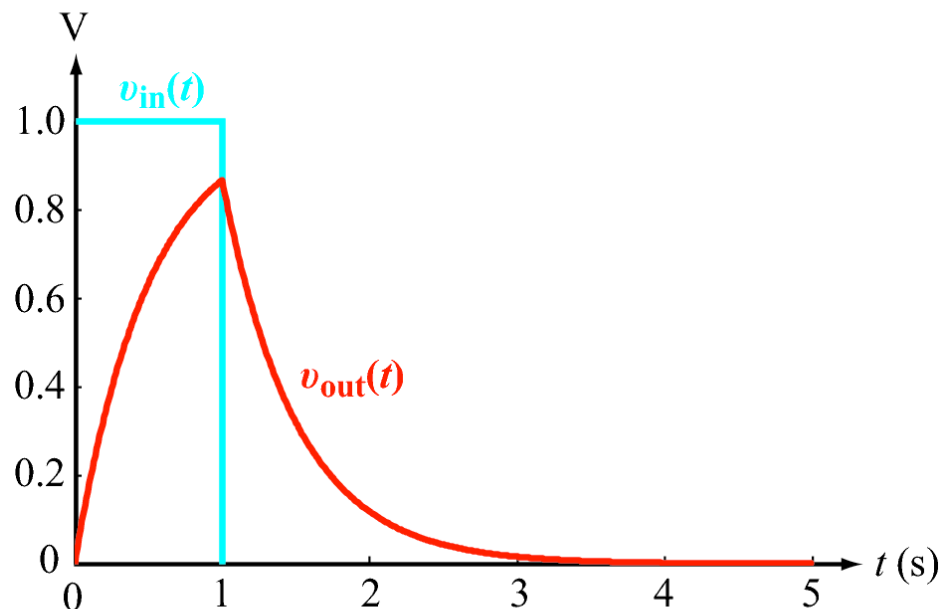
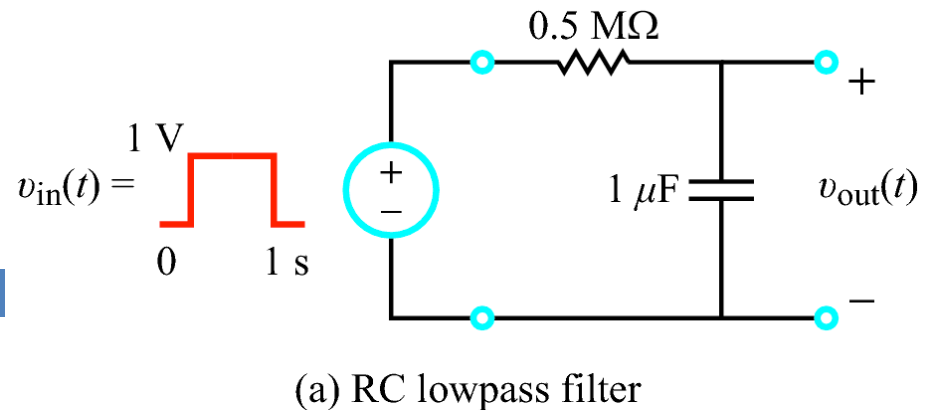
$$- u(t) \int_0^t 2e^{-2(t-\tau)} u(\tau - 1) u(t - \tau) d\tau.$$

$$= \left[ \int_0^t 2e^{-2(t-\tau)} d\tau \right] u(t)$$

$$- \left[ \int_1^t 2e^{-2(t-\tau)} d\tau \right] u(t - 1)$$

$$= \frac{2}{2} e^{-2(t-\tau)} \Big|_0^t u(t) - \frac{2}{2} e^{-2(t-\tau)} \Big|_1^t u(t - 1)$$

$$= [1 - e^{-2t}] u(t) - [1 - e^{-2(t-1)}] u(t - 1) \text{ V,}$$



# Graphical Convolution Technique

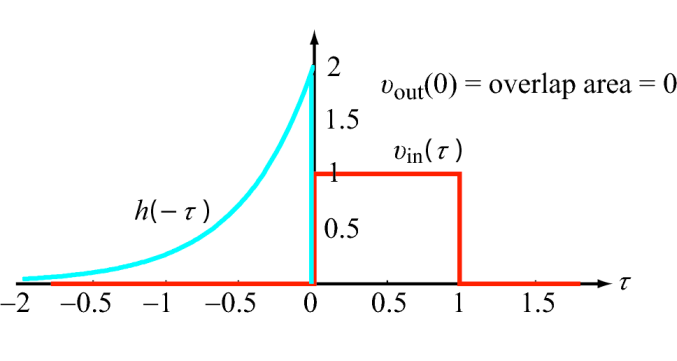
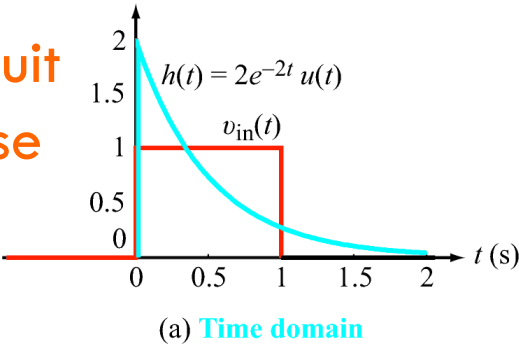
**Step 1:** On the  $\tau$ -axis, display  $x(\tau)$  and  $h(-\tau)$  with the latter being a folded image of  $h(\tau)$  about the vertical axis.

**Step 2:** Shift  $h(-\tau)$  to the right by a small increment  $t$  to obtain  $h(t - \tau) = h(-(\tau - t))$ .

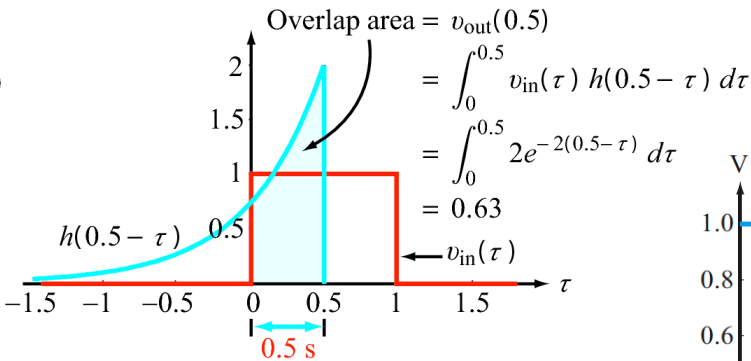
**Step 3:** Determine the product of  $x(\tau)$  and  $h(t - \tau)$  and integrate it over the  $\tau$ -domain from  $\tau = 0$  to  $\tau = t$  to get  $y(t)$ . The integration is equal to the area overlapped by the two functions.

**Step 4:** Repeat steps 2 and 3 for each of many successive values of  $t$  to generate the complete response  $y(t)$ .

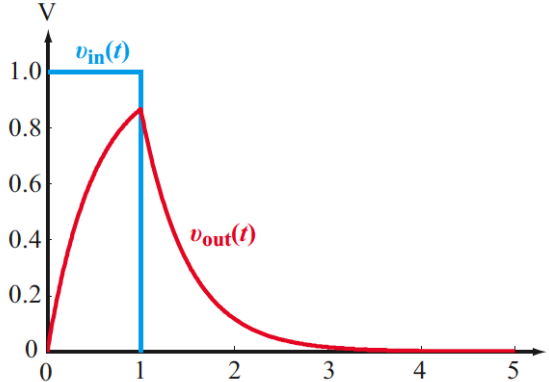
# Example: RC Circuit Excited by a Pulse



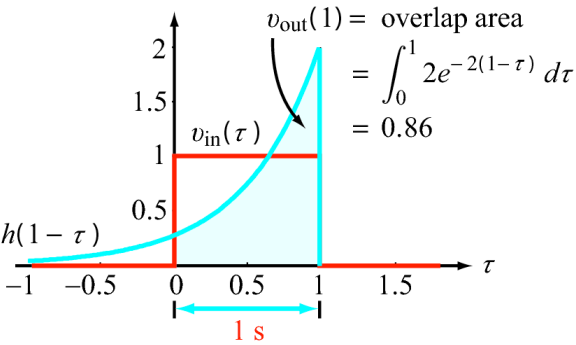
(b) Convolution @ t = 0



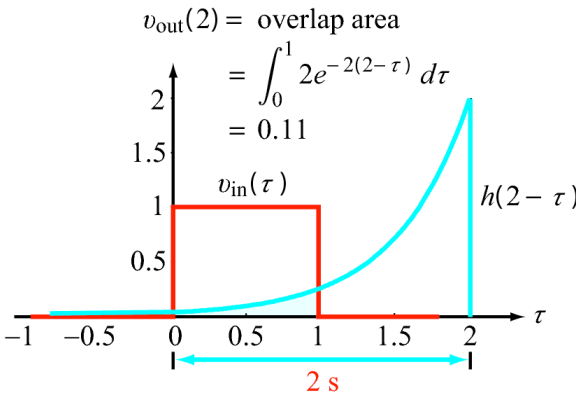
(c) Convolution @ t = 0.5 s



(b) Output response

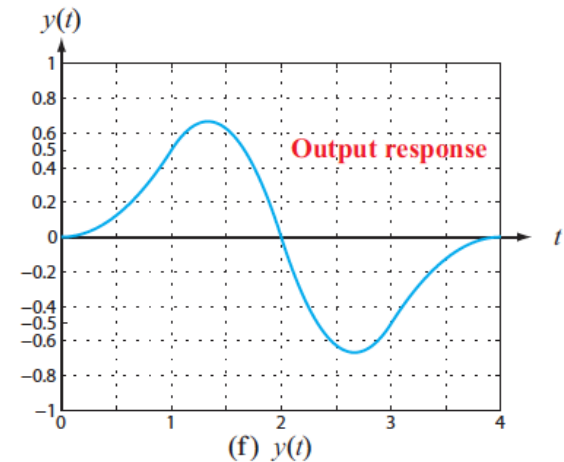
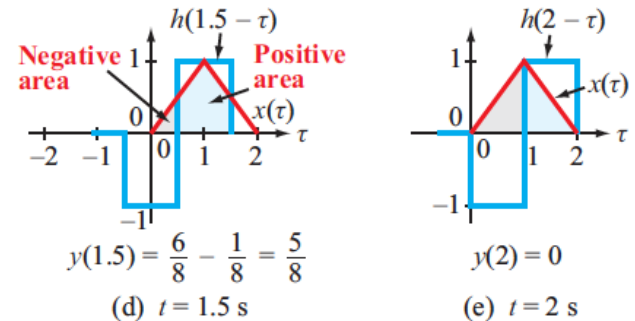
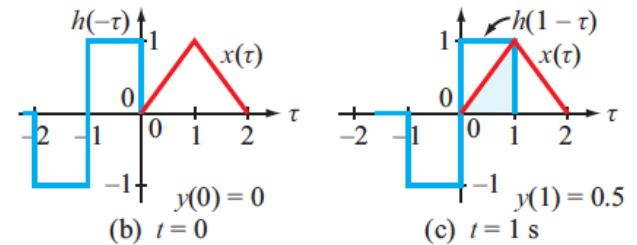
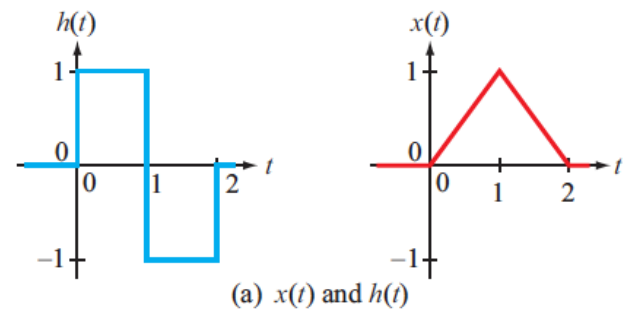


(d) Convolution @ t = 1 s

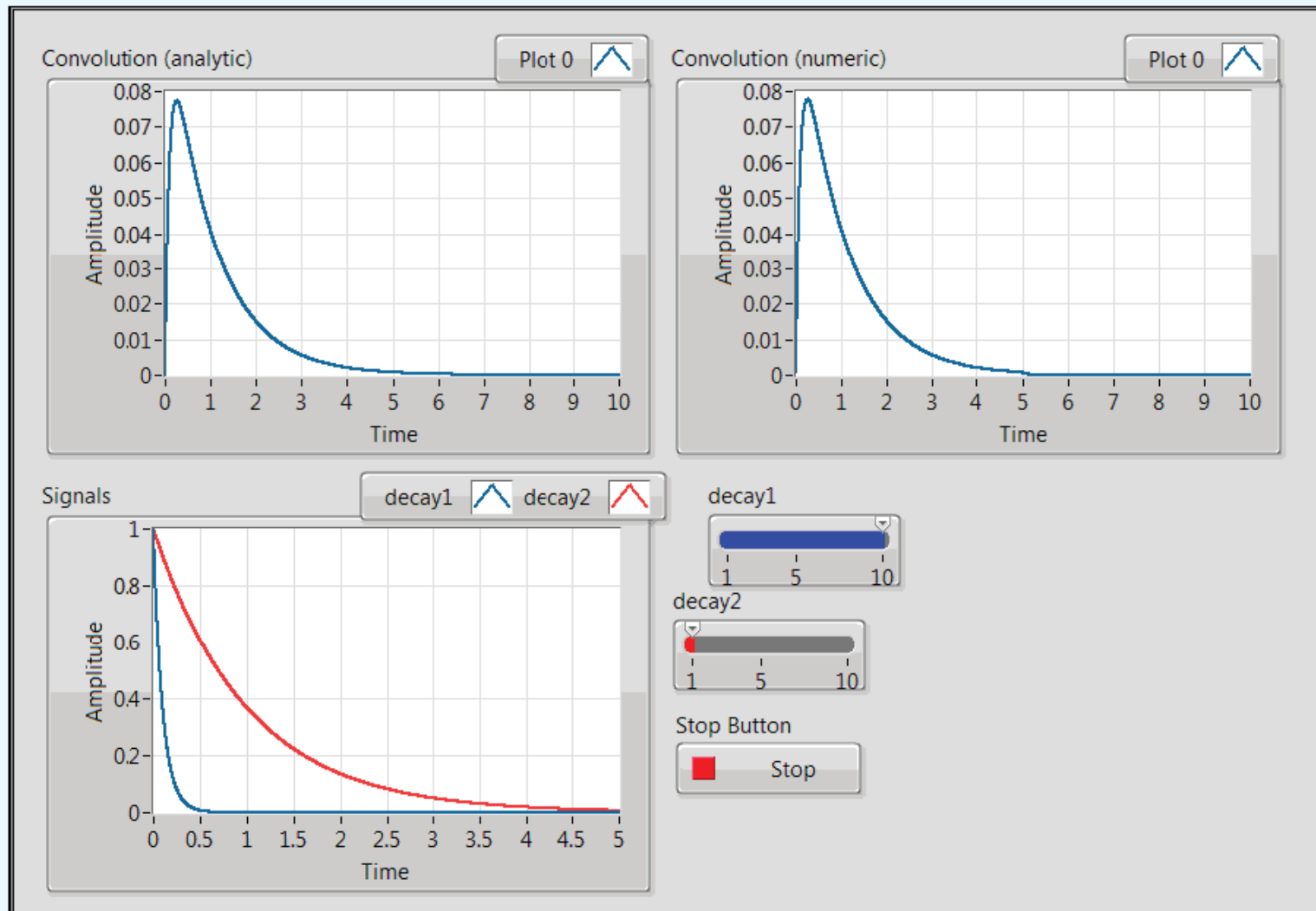


(e) Convolution @ t = 2 s

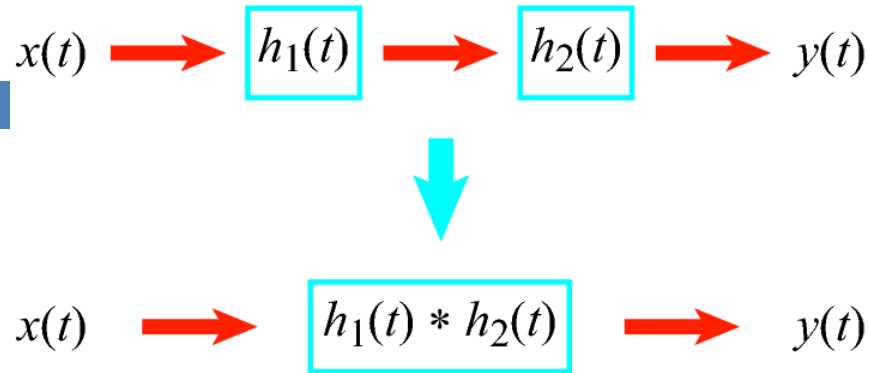
□ Convolution of  
1 cycle of  
square wave  
with Triangle



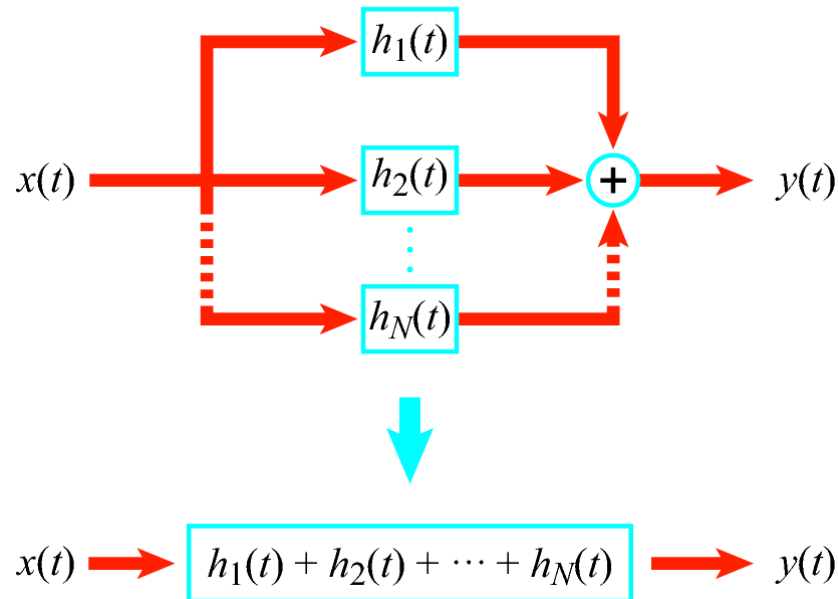
**Module 2.1 Convolution of Exponential Functions** This module computes the convolution of  $e^{-at} u(t)$  and  $e^{-bt} u(t)$ . The values of exponents  $a$  and  $b$  are selectable.



## Systems Connected In-Series



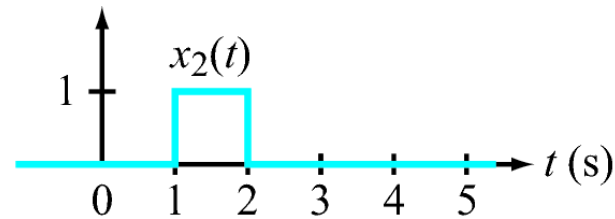
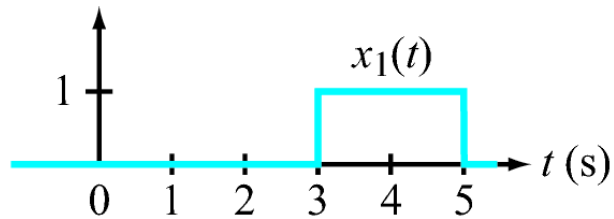
## Systems Connected In-Parallel



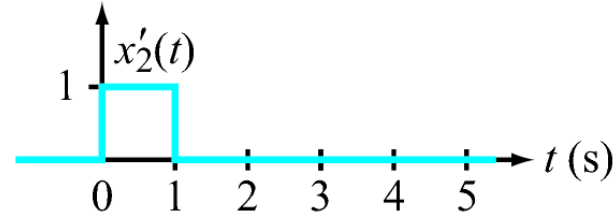
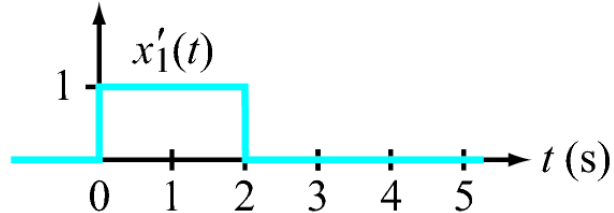


$$h(t - T_1) * x(t - T_2) = y(t - T_1 - T_2)$$

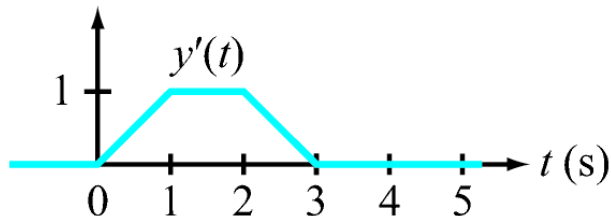
**(time-shift property)**



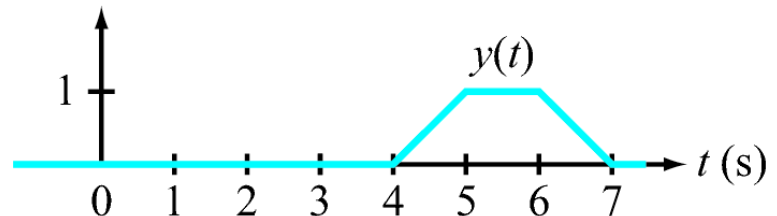
(a) **Original pulses**



(b) **Pulses shifted to start at  $t = 0$**



(c) **Convolution of  $x'_1(t)$  with  $x'_2(t)$**



(d)  **$y'(t)$  delayed by 4 s gives  $y(t)$**

**Table 2-1:** Convolution properties.

<b>Convolution Integral</b>	$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau$
• <b>Causal Systems:</b>	Replace lower limit with 0
Property	Description
<b>1. Commutative</b>	$x(t) * h(t) = h(t) * x(t)$
<b>2. Associative</b>	$[g(t) * h(t)] * x(t) = g(t) * [h(t) * x(t)]$
<b>3. Distributive</b>	$x(t) * [h_1(t) + \dots + h_N(t)] = x(t) * h_1(t) + \dots + x(t) * h_N(t)$
<b>4. Causal * Causal = Causal</b>	$y(t) = u(t) \int_0^t h(\tau) x(t - \tau) d\tau$
<b>5. Time-shift</b>	$h(t - T_1) * x(t - T_2) = y(t - T_1 - T_2)$
<b>6. Convolution with Impulse</b>	$x(t) * \delta(t - T) = x(t - T)$
<b>7. Width</b>	Width of $y(t)$ = width of $x(t)$ + width of $h(t)$
<b>8. Area</b>	Area of $y(t)$ = area of $x(t)$ $\times$ area of $h(t)$
<b>9. Convolution with <math>u(t)</math></b>	$y(t) = x(t) * u(t) = \int_{-\infty}^t x(\tau) d\tau$ (Ideal integrator)
<b>10a. Differentiation</b>	$\left(\frac{d^m x}{dt^m}\right) * \left(\frac{d^n h}{dt^n}\right) = \frac{d^{m+n} y}{dt^{m+n}}$
<b>10b. Integration</b>	$\int_{-\infty}^t y(\tau) d\tau = x(t) * \left[\int_{-\infty}^t h(\tau) d\tau\right] = \left[\int_{-\infty}^t x(\tau) d\tau\right] * h(t)$

## 2-6.1 Causality

We define a *causal system* as a system for which the present value of the output  $y(t)$  can only depend on present and past values of the input  $\{x(\tau), \tau \leq t\}$ . For a noncausal system, the present output could depend on future inputs. Noncausal systems are also called *anticipatory* systems, since they anticipate the future.

A physical system must be causal, because a noncausal system must have the ability to see into the future! For example, the noncausal system  $y(t) = x(t + 2)$  must know the input two seconds into the future to deliver its output at the present time. This is clearly impossible in the real world.

► An LTI system is causal *if and only if* its impulse response is a causal function:  $h(t) = 0$  for  $t < 0$ . ◀

# Stability

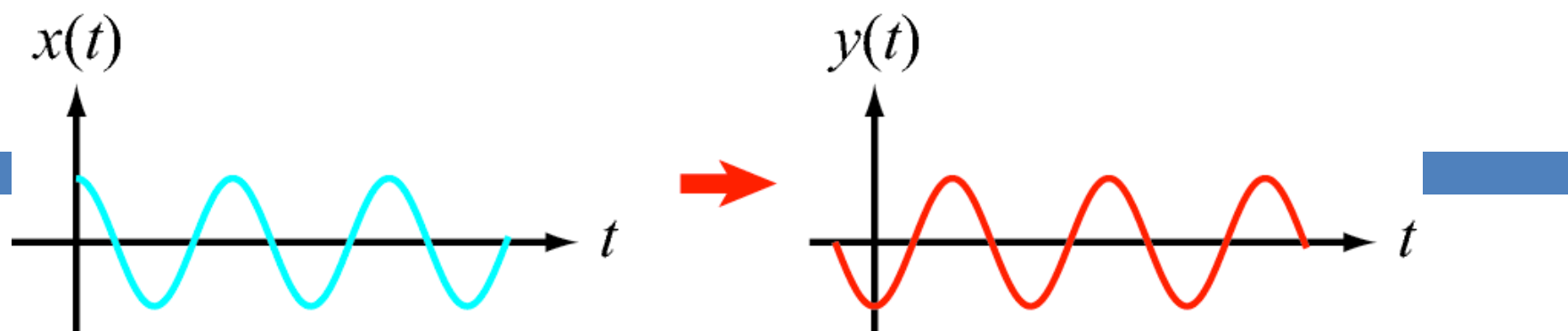
A signal  $x(t)$  is *bounded* if there is a constant  $C$  so that  $|x(t)| \leq C$  for all  $t$ . Examples of bounded signals include:

- $\cos(3t)$ ,  $7e^{-2t} u(t)$ , and  $e^{2t} u(1 - t)$ .

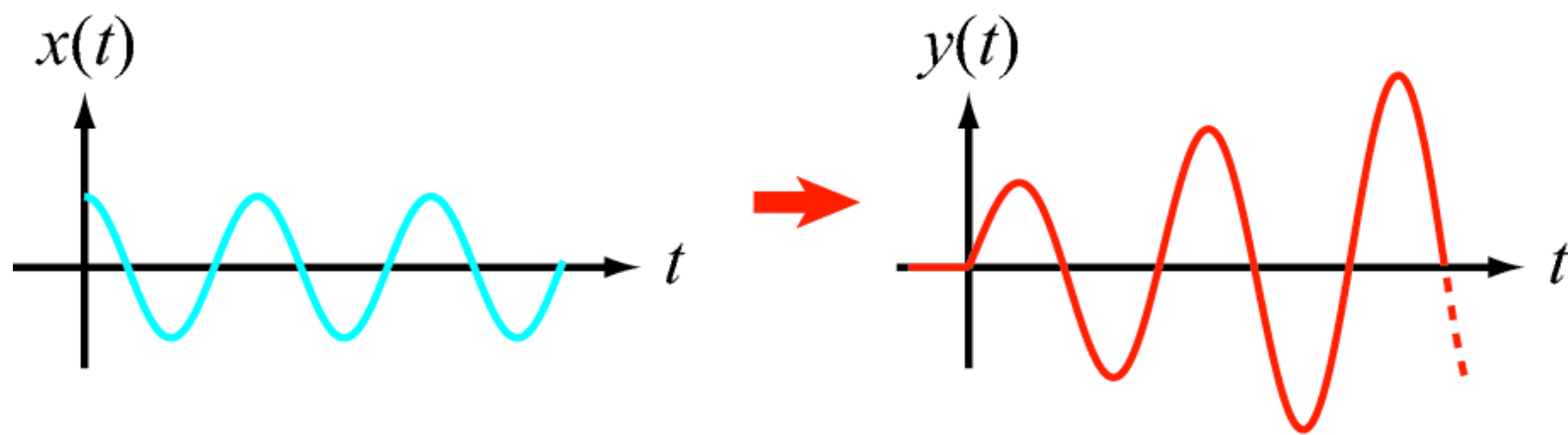
Examples of unbounded signals include:

- $t^2$ ,  $e^{2t} u(t)$ ,  $e^{-t}$ , and  $1/t$ .

A system is *BIBO (bounded input/bounded output) stable* every bounded input  $x(t)$  results in a bounded output  $y(t)$ ,



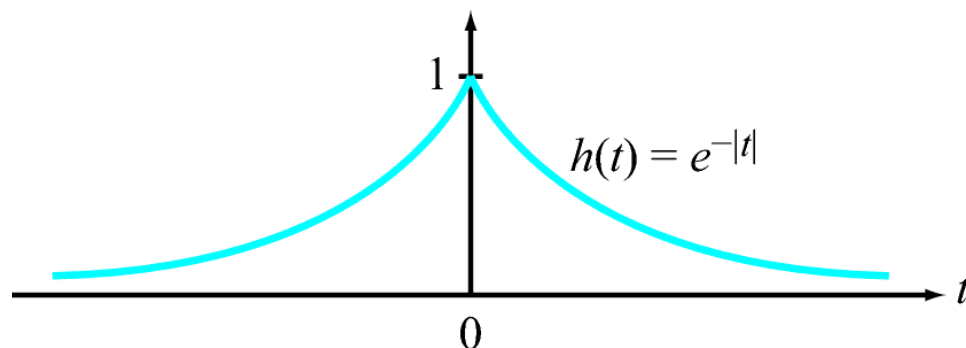
(a) BIBO-stable system



(b) Non-BIBO-stable system

► An LTI system is BIBO stable *if and only if* its impulse response  $h(t)$  is *absolutely integrable* (i.e., if  $\int_{-\infty}^{\infty} |h(t)| dt$  is finite). ◀

Example



$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} e^{-|t|} dt = 2 \int_0^{\infty} e^{-t} dt = 2.$$

Hence, the system is BIBO stable.

# 2-6.4 BIBO Stability of System with Decaying Exponentials

Consider a causal system with an impulse response

$$h(t) = Ce^{\gamma t} u(t), \tag{2.93}$$

where  $C$  is a finite constant and  $\gamma$  is, in general, a finite complex coefficient given by

$$\gamma = \alpha + j\beta, \quad \alpha = \Re[\gamma], \quad \text{and} \quad \beta = \Im[\gamma]. \tag{2.94}$$

Such a system is BIBO stable if and only if  $\alpha < 0$  (i.e.,  $h(t)$  is a one-sided exponential with an exponential coefficient whose real part is negative). To verify the validity of this statement, we test to see if  $h(t)$  is absolutely integrable. Since  $|e^{j\beta t}| = 1$  and  $e^{\alpha t} > 0$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \int_0^{\infty} |Ce^{\alpha t} e^{j\beta t}| dt \\ &= |C| \int_0^{\infty} e^{\alpha t} dt. \end{aligned}$$

(a)  $\alpha < 0$

If  $\alpha < 0$ , we can rewrite it as  $\alpha = -|\alpha|$  in the exponential, which leads to

$$\int_{-\infty}^{\infty} |h(t)| dt = |C| \int_0^{\infty} e^{-|\alpha|t} dt = \frac{|C|}{|\alpha|} < \infty. \tag{2.96}$$

Hence,  $h(t)$  is absolutely integrable and the system is BIBO stable.

(b)  $\alpha \geq 0$

If  $\alpha \geq 0$ , Eq. (2.95) becomes

$$\int_{-\infty}^{\infty} |h(t)| dt = |C| \int_0^{\infty} e^{\alpha t} dt \rightarrow \infty,$$

thereby proving that the system is not BIBO stable when  $\alpha \geq 0$ .

► By extension, for any positive integer  $N$ , an impulse response composed of a linear combination of  $N$  exponential signals

$$h(t) = \sum_{i=1}^N C_i e^{\gamma_i t} u(t) \quad (2.97)$$

*is absolutely integrable, and its LTI system is BIBO stable, if and only if all of the exponential coefficients  $\gamma_i$  have negative real parts.* This is a fundamental attribute of LTI system theory. ◀



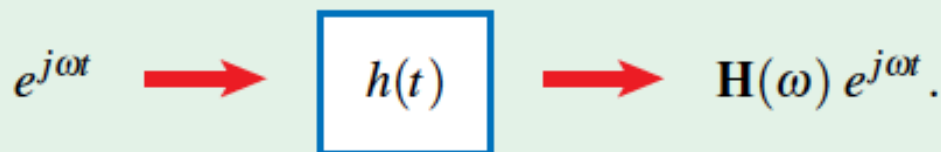
## 2-7.1 LTI System Response to Complex Exponential Signals

Let us examine the response of an LTI system to a complex exponential  $x(t) = Ae^{j\omega t}$ . Since the system is LTI, we may (without loss of generality) set  $A = 1$ . Hence,

$$\begin{aligned} y(t) &= h(t) * x(t) = h(t) * e^{j\omega t} = \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)} d\tau \\ &= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \\ &= \mathbf{H}(\omega) e^{j\omega t}, \end{aligned} \quad (2.100)$$

where the *frequency response function*  $\mathbf{H}(\omega)$  is defined as

$$\mathbf{H}(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau. \quad (2.101)$$



# LTI Response to Sinusoid

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{H}(\omega) e^{j\omega t}.$$

$$e^{-j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow \mathbf{H}(-\omega) e^{-j\omega t}.$$

$$x(t) = \cos \omega t = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]. \quad (2.115)$$

Using the additivity property of LTI systems, we can determine the system's response to  $\cos \omega t$  by adding the responses expressed by Eqs. (2.102) and (2.105):

$$\begin{array}{ccccc} \frac{1}{2} e^{j\omega t} & \longrightarrow & \boxed{h(t)} & \longrightarrow & \frac{1}{2} \mathbf{H}(\omega) e^{j\omega t} \\ + & & & & + \\ \frac{1}{2} e^{-j\omega t} & \longrightarrow & \boxed{h(t)} & \longrightarrow & \frac{1}{2} \mathbf{H}(-\omega) e^{-j\omega t}. \end{array} \quad (2.116)$$

Since  $h(t)$  is real, Eq. (2.107) stipulates that

$$\mathbf{H}(-\omega) = \mathbf{H}^*(\omega).$$

If we define

$$\mathbf{H}(\omega) = |\mathbf{H}(\omega)|e^{j\theta},$$

$$\mathbf{H}(-\omega) = |\mathbf{H}(\omega)|e^{-j\theta},$$

where  $\theta = \angle \mathbf{H}(\omega)$ , the sum of the two outputs in becomes

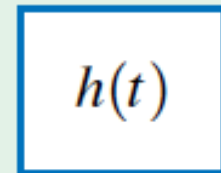
$$\begin{aligned} y(t) &= \frac{1}{2} [\mathbf{H}(\omega) e^{j\omega t} + \mathbf{H}(-\omega) e^{-j\omega t}] \\ &= \frac{1}{2} |\mathbf{H}(\omega)| [e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}] \\ &= |\mathbf{H}(\omega)| \cos(\omega t + \theta). \end{aligned}$$

In summary,

$$\begin{aligned} y(t) &= |\mathbf{H}(\omega)| \cos(\omega t + \theta) \\ &\quad (\text{for } x(t) = \cos \omega t). \end{aligned}$$

In symbolic form:

$$A \cos(\omega t + \phi)$$



$$A|\mathbf{H}(\omega)| \cos(\omega t + \theta + \phi).$$

**Example 2-13: Response of an LCCDE to a Sinusoidal Input**

$$5 \cos(4t + 30^\circ) \rightarrow \boxed{\frac{dy}{dt} + 3y = 3\frac{dx}{dt} + 5x} \rightarrow ?$$

**Solution:** Inserting  $x(t) = e^{j\omega t}$  and  $y(t) = \mathbf{H}(\omega) e^{j\omega t}$  in the specified LCCDE gives

$$j\omega \mathbf{H}(\omega) e^{j\omega t} + 3\mathbf{H}(\omega) e^{j\omega t} = 3j\omega e^{j\omega t} + 5e^{j\omega t}.$$

Solving for  $\mathbf{H}(\omega)$  gives

$$\mathbf{H}(\omega) = \frac{5 + j3\omega}{3 + j\omega}.$$

The angular frequency  $\omega$  of the input signal is 4 rad/s. Hence,

$$\mathbf{H}(4) = \frac{5 + j12}{3 + j4} = \frac{13e^{j67.38^\circ}}{5e^{j53.13^\circ}} = \frac{13}{5} e^{j14.25^\circ}.$$

Application of Eq. (2.120) with  $A = 5$  and  $\phi = 30^\circ$  gives

$$y(t) = 5 \times \frac{13}{5} \cos(4t + 14.25^\circ + 30^\circ) = 13 \cos(4t + 44.25^\circ).$$

## 2-8 Impulse Response of Second-Order LCCDEs

Many physical systems are described by second-order LCCDEs of the form

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y(t) = b_1 \frac{dx}{dt} + b_2 x(t), \quad (2.121)$$

where  $a_1$ ,  $a_2$ ,  $b_1$ , and  $b_2$  are constant coefficients.

### 2-8.1 LCCDE with No Input Derivatives

For simplicity, we start by considering a version of Eq. (2.121) without the  $dx/dt$  term, and then we use the result to treat the more general case in the next subsection.

For  $b_1 = 0$  and  $b_2 = 1$ , Eq. (2.121) becomes

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y(t) = x(t). \quad (2.122)$$

## 2-8.1 LCCDE with No Input Derivatives

For simplicity, we start by considering a version of Eq. (2.121) without the  $dx/dt$  term, and then we use the result to treat the more general case in the next subsection.

For  $b_1 = 0$  and  $b_2 = 1$ , Eq. (2.121) becomes

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y(t) = x(t). \tag{2.122}$$

### Step 1: Roots of Characteristic Equation

Assuming  $y(t)$  has a general solution of the form  $y(t) = Ae^{st}$ , substitution in the homogeneous form of Eq. (2.122)—i.e., with  $x(t) = 0$ —leads to the *characteristic equation*:

$$s^2 + a_1s + a_2 = 0. \tag{2.123}$$

If  $p_1$  and  $p_2$  are the roots of Eq. (2.123), then

$$s^2 + a_1s + a_2 = (s - p_1)(s - p_2), \tag{2.124}$$

which leads to

$$p_1 + p_2 = -a_1, \quad p_1 p_2 = a_2,$$

and

$$p_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2},$$
$$p_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}.$$

Roots  $p_1$  and  $p_2$  are

- (a) real if  $a_1^2 > 4a_2$ ,
- (b) complex conjugates if  $a_1^2 < 4a_2$ , or
- (c) identical if  $a_1^2 = 4a_2$ .

## Step 2: Two Coupled First-Order LCCDEs

The original differential equation given by Eq. (2.122) now can be rewritten as

$$\frac{d^2 y}{dt^2} - (p_1 + p_2) \frac{dy}{dt} + (p_1 p_2) y(t) = x(t), \quad (2.127a)$$

which can in turn be cast in the form

$$\left[ \frac{d}{dt} - p_1 \right] \left[ \frac{d}{dt} - p_2 \right] y(t) = x(t). \quad (2.127b)$$

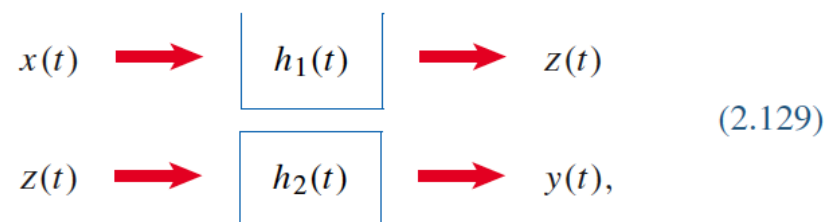
Furthermore, we can split the *second-order differential equation* into *two coupled first-order equations* by introducing an intermediate variable  $z(t)$ :

$$\frac{dz}{dt} - p_1 z(t) = x(t) \quad (2.128a)$$

and

$$\frac{dy}{dt} - p_2 y(t) = z(t). \quad (2.128b)$$

These coupled first-order LCCDEs represent a *series (or cascade) connection* of LTI systems, each described by a first-order LCCDE. In symbolic form, we have



where  $h_1(t)$  and  $h_2(t)$  are the impulse responses corresponding to Eqs. (2.128a and b), respectively.

### Step 3: Impulse Response of Cascaded LTI Systems

By comparison with Eq. (2.10) and its corresponding impulse response, Eq. (2.17), we conclude that

$$h_1(t) = e^{p_1 t} u(t) \quad (2.130a)$$

and

$$h_2(t) = e^{p_2 t} u(t). \quad (2.130b)$$

Using convolution property #2 in Table 2-1, the impulse response of the series connection of two LTI systems is the convolution of their impulse responses. Utilizing property #3 in Table 2-2, the combined impulse response becomes

$$\begin{aligned} h_c(t) &= h_1(t) * h_2(t) \\ &= e^{p_1 t} u(t) * e^{p_2 t} u(t) \\ &= \left[ \frac{1}{p_1 - p_2} \right] [e^{p_1 t} - e^{p_2 t}] u(t). \end{aligned} \quad (2.131)$$



# Summary for LCCDE with No Input Derivative

□ Diff. Eq.:

$$\frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y(t) = x(t).$$

□ Impulse Response:

$$\begin{aligned} h_c(t) &= h_1(t) * h_2(t) \\ &= e^{p_1 t} u(t) * e^{p_2 t} u(t) \\ &= \left[ \frac{1}{p_1 - p_2} \right] [e^{p_1 t} - e^{p_2 t}] u(t). \end{aligned}$$

□ Roots:

$$p_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2},$$

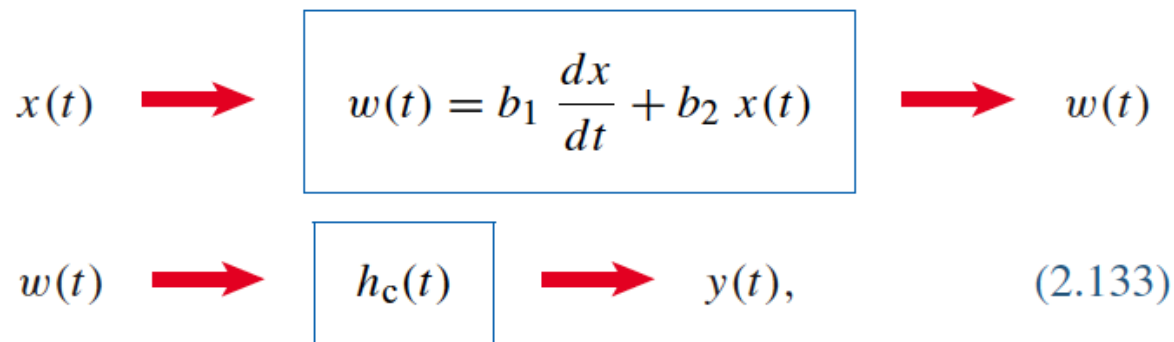
$$p_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2}.$$

## 2-8.2 LCCDE with Input Derivative

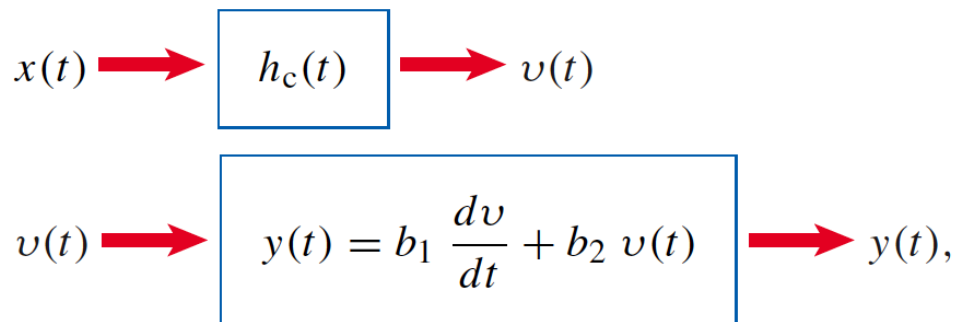
We now consider the more general case of a second-order LCCDE that contains a first-order derivative on the input side of the equation

$$\frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y(t) = b_1 \frac{dx}{dt} + b_2 x(t). \quad (2.132)$$

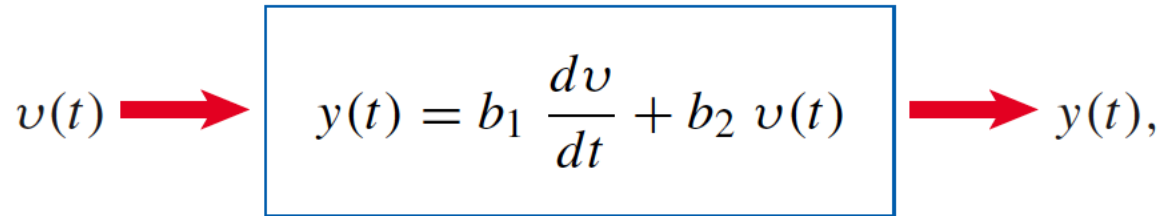
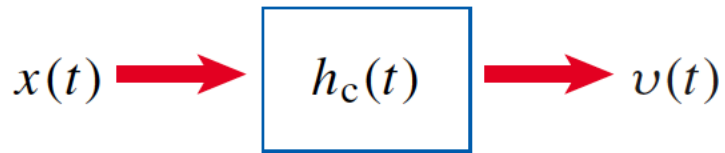
By defining the right-hand side of Eq. (2.132) as an intermediate variable  $w(t)$ , the system can be represented as



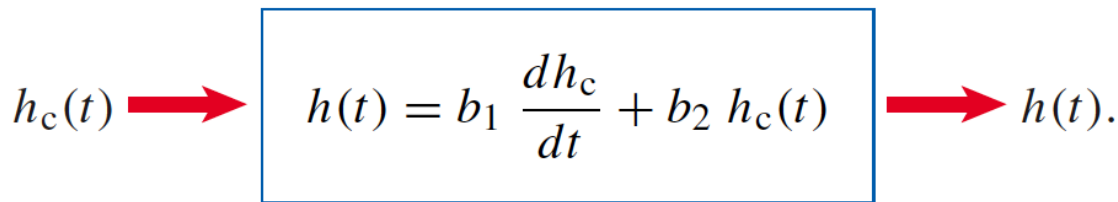
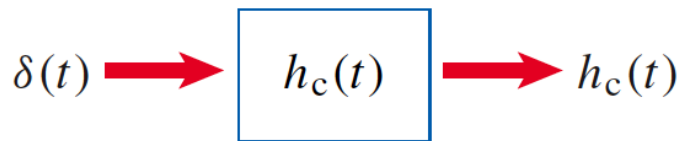
Convolution is commutative, so we can reverse the order of the two systems



Where  $v(t)$  is a new intermediate variable



By definition, when  $x(t) = \delta(t)$ , the output  $y(t)$  becomes the impulse response  $h(t)$  of the overall system. That is, if we set  $x(t) = \delta(t)$ , which results in  $v(t) = h_c(t)$  and  $y(t) = h(t)$ , the system becomes



where  $h_c(t)$  is the impulse response of the LCCDE with no input derivative

$$h_c(t) = \left[ \frac{1}{p_1 - p_2} \right] [e^{p_1 t} - e^{p_2 t}] u(t)$$

# Solution for Impulse Response of LCCDE With Input Derivative

Finally, the impulse response  $h(t)$  of the overall system is

$$\begin{aligned} h(t) &= b_1 \frac{dh_c}{dt} + b_2 h_c(t) \\ &= \left[ b_1 \frac{d}{dt} + b_2 \right] \left[ \frac{1}{p_1 - p_2} \right] [e^{p_1 t} - e^{p_2 t}] u(t) \\ &= \frac{b_1 p_1 + b_2}{p_1 - p_2} e^{p_1 t} u(t) - \frac{b_1 p_2 + b_2}{p_1 - p_2} e^{p_2 t} u(t). \quad (2.136) \end{aligned}$$

Having established in the form of Eq. (2.136) an explicit expression for the impulse response of the general LCCDE given by Eq. (2.132), we can now determine the response  $y(t)$  to any causal input excitation  $x(t)$  by evaluating

$$y(t) = u(t) \int_0^t h(\tau) x(t - \tau) d\tau. \quad (2.137)$$

# Parameters of Second-Order LCCDE

$$\begin{aligned} h(t) &= b_1 \frac{dh_c}{dt} + b_2 h_c(t) \\ &= \left[ b_1 \frac{d}{dt} + b_2 \right] \left[ \frac{1}{p_1 - p_2} \right] [e^{p_1 t} - e^{p_2 t}] u(t) \\ &= \frac{b_1 p_1 + b_2}{p_1 - p_2} e^{p_1 t} u(t) - \frac{b_1 p_2 + b_2}{p_1 - p_2} e^{p_2 t} u(t) \end{aligned}$$

Recall from an earlier section the conclusion:

► By extension, for any positive integer  $N$ , an impulse response composed of a linear combination of  $N$  exponential signals

$$h(t) = \sum_{i=1}^N C_i e^{\gamma_i t} u(t) \quad (2.97)$$

*is absolutely integrable, and its LTI system is BIBO stable, if and only if all of the exponential coefficients  $\gamma_i$  have negative real parts.* This is a fundamental attribute of LTI system theory. ◀

Hence,  $p_1$  and  $p_2$  should have negative real parts in order for the system to be BIBO stable

# Parameters of Second-Order LCCDE

$$p_1 = -\frac{a_1}{2} + \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2} , \quad (2.138a)$$

$$p_2 = -\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - a_2} . \quad (2.138b)$$

- (a) If both  $p_1$  and  $p_2$  are real, distinct, and negative, Eq. (2.138) leads to the conclusion that  $a_1^2 > 4a_2$ ,  $a_1 > 0$ , and  $a_2 > 0$ .
- (b) If  $p_1$  and  $p_2$  are complex conjugates with negative real parts, it follows that  $a_1^2 < 4a_2$ ,  $a_1 > 0$ , and  $a_2 > 0$ .
- (c) If  $p_1$  and  $p_2$  are real, equal, and negative, then  $a_1^2 = 4a_2$ ,  $a_1 > 0$ , and  $a_2 > 0$ .

► The LTI system described by the LCCDE Eq. (2.132) is BIBO stable if and only if  $a_1 > 0$  and  $a_2 > 0$ . ◀

We now introduce three new non-negative, physically meaningful parameters:

$$\alpha = \frac{a_1}{2} = \textit{attenuation coefficient} \quad (\text{Np/s}), \quad (2.139a)$$

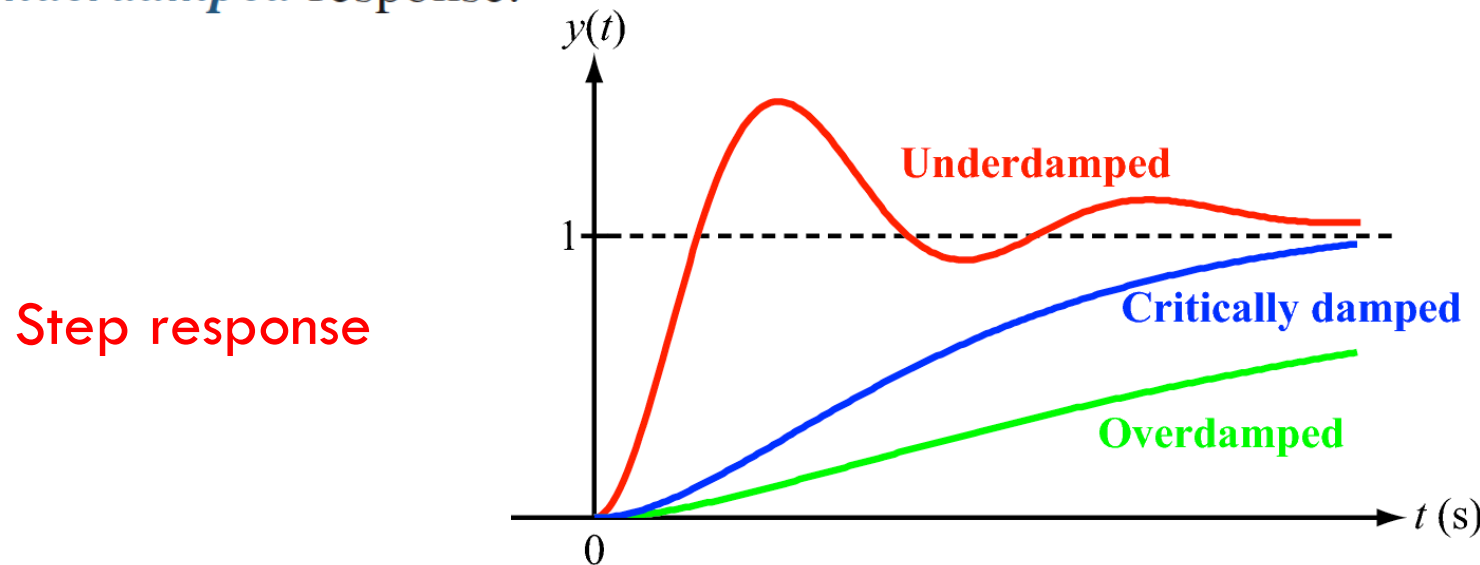
$$\omega_0 = \sqrt{a_2} = \textit{undamped natural frequency} \quad (\text{rad/s}), \quad (2.139b)$$

$$\xi = \frac{\alpha}{\omega_0} = \frac{a_1}{2\sqrt{a_2}} = \textit{damping coefficient} \quad (\text{unitless}).$$

(a)  $\xi > 1 \rightarrow \textit{overdamped}$  response, (2.139c)

(b)  $\xi = 1 \rightarrow \textit{critically damped}$  response,

(c)  $\xi < 1 \rightarrow \textit{underdamped}$  response.



**Table 2-3:** Impulse and step responses of second-order LCCDE.

**LCCDE**  $\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2y = b_1 \frac{dx}{dt} + b_2x$

$$\alpha = \frac{a_1}{2}, \quad \omega_0 = \sqrt{a_2}, \quad \xi = \frac{\alpha}{\omega_0}, \quad p_1 = \omega_0[-\xi + \sqrt{\xi^2 - 1}], \quad p_2 = \omega_0[-\xi - \sqrt{\xi^2 - 1}]$$

**Overdamped Case**  $\xi > 1$

$$h(t) = A_1 e^{p_1 t} u(t) + A_2 e^{p_2 t} u(t)$$

$$y_{\text{step}}(t) = \left[ \frac{A_1}{p_1} (e^{p_1 t} - 1) + \frac{A_2}{p_2} (e^{p_2 t} - 1) \right] u(t)$$

$$A_1 = \frac{b_1 p_1 + b_2}{p_1 - p_2}, \quad A_2 = \frac{-(b_1 p_2 + b_2)}{p_1 - p_2}$$

**Underdamped Case**  $\xi < 1$

$$h(t) = [B_1 \cos \omega_d t + B_2 \sin \omega_d t] e^{-\alpha t} u(t)$$

$$y_{\text{step}}(t) = \frac{1}{\alpha^2 + \omega_d^2} \{ [-(B_1 \alpha + B_2 \omega_d) \cos \omega_d t + (B_1 \omega_d + B_2 \alpha) \sin \omega_d t] e^{-\alpha t} + (B_1 \alpha + B_2 \omega_d) \} u(t)$$

$$B_1 = b_1, \quad B_2 = \frac{b_2 - b_1 \alpha}{\omega_d}, \quad \omega_d = \omega_0 \sqrt{1 - \xi^2}$$

**Critically Damped Case**  $\xi = 1$

$$h(t) = (C_1 + C_2 t) e^{-\alpha t} u(t)$$

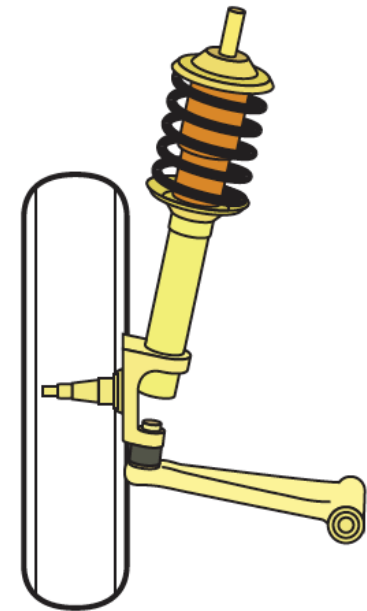
$$y_{\text{step}}(t) = \left[ \left( \frac{C_1}{\alpha} + \frac{C_2}{\alpha^2} \right) (1 - e^{-\alpha t}) - \frac{C_2}{\alpha} t e^{-\alpha t} \right] u(t)$$

$$C_1 = b_1, \quad C_2 = b_2 - \alpha b_1$$

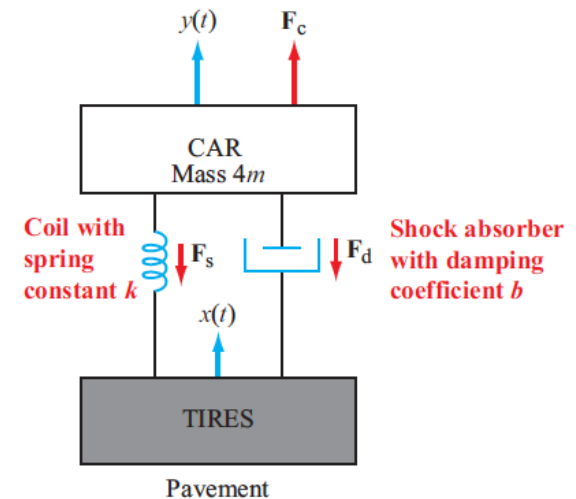


# Car Suspension System

- $x(t)$  = input = vertical displacement of the pavement, defined relative to a reference ground level.
- $y(t)$  = output = vertical displacement of the car chassis from its equilibrium position.
- $m$  = *one-fourth* of the car's mass, because the car has four wheels.
- $k$  = *spring constant* or *stiffness* of the coil.
- $b$  = *damping coefficient* of the shock absorber.



(a) Damping system



(b) Model

Figure 2-26: Car suspension system model.

# Force Equation

$$\mathbf{F}_s = -k(y - x)$$

$$\mathbf{F}_d = -b \frac{d}{dt} (y - x). \quad (2.156)$$

Using Newton's law,  $\mathbf{F}_c = ma = m(d^2y/dt^2)$ , the force equation is

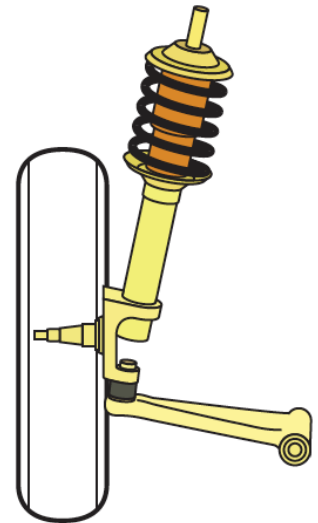
$$\mathbf{F}_c = \mathbf{F}_s + \mathbf{F}_d \quad (2.157)$$

or

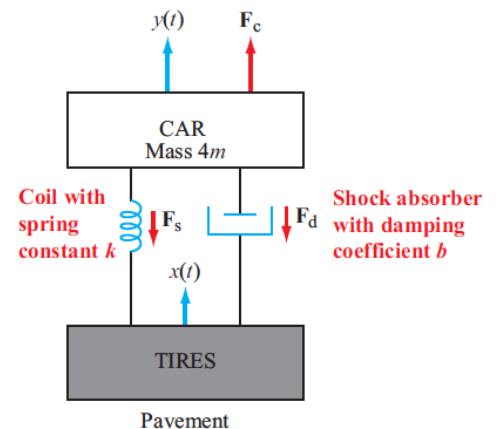
$$m \frac{d^2y}{dt^2} = -k(y - x) - b \frac{d}{dt} (y - x),$$

which can be recast as

$$\frac{d^2y}{dt^2} + \frac{b}{m} \frac{dy}{dt} + \frac{k}{m} y = \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x.$$



(a) Damping system



(b) Model

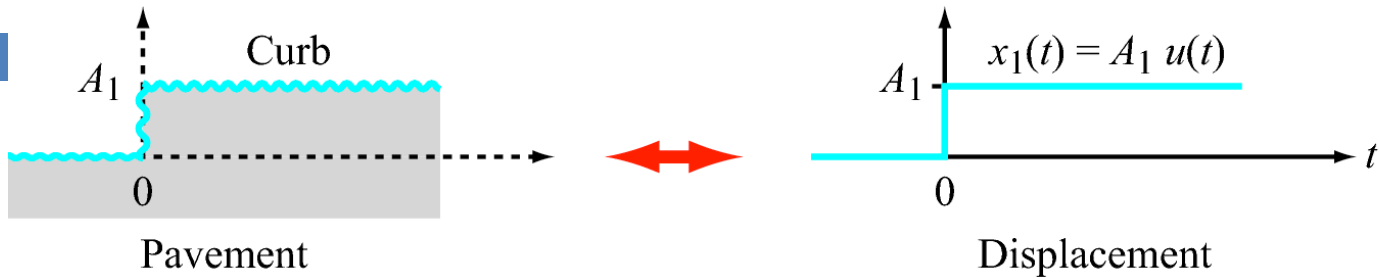
Figure 2-26: Car suspension system model.

# Typical Car Parameters

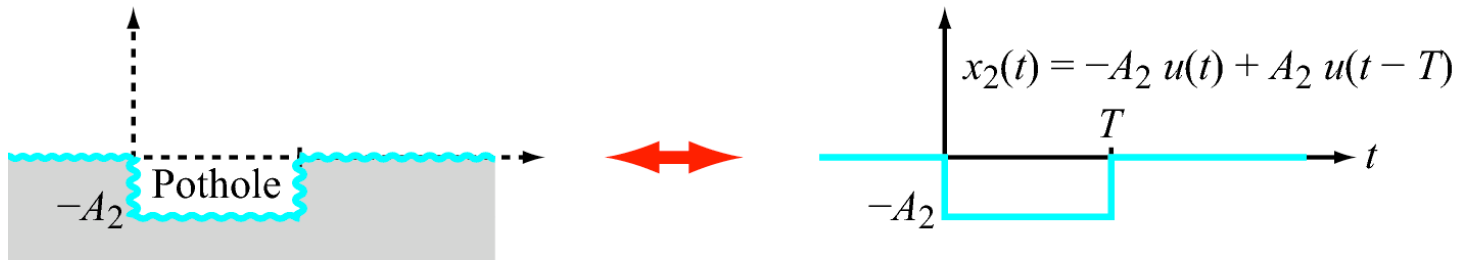
Typical values for a small automobile are:

- $m = 250$  kg for a car with a total mass of one metric ton (1000 kg); each wheel supports one-fourth of the car's mass.
- $k = 10^5$  N/m; it takes a force of 1000 N to compress the spring by 1 cm.
- $b = 10^4$  N·s/m; a vertical motion of 1 m/s incurs a resisting force of  $10^4$  N.

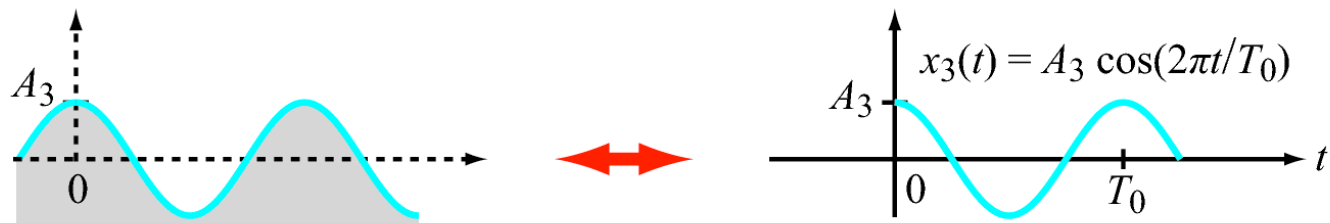
# Pavement Models



(a) Curb

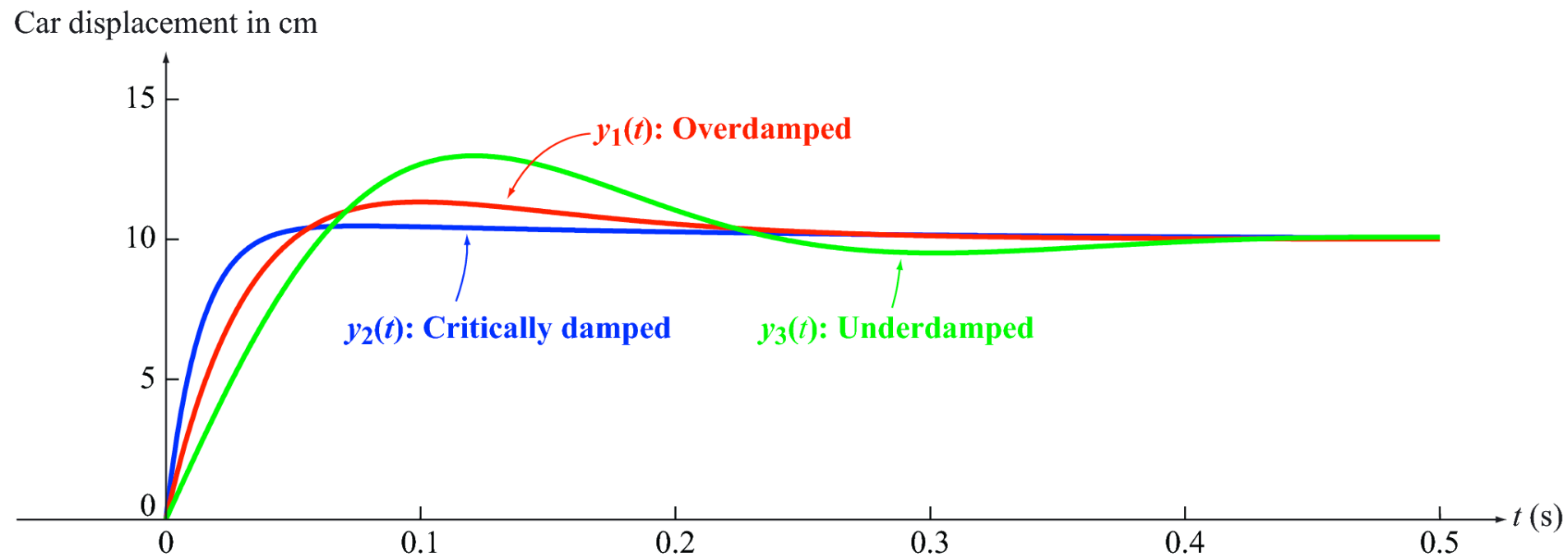


(b) Pothole

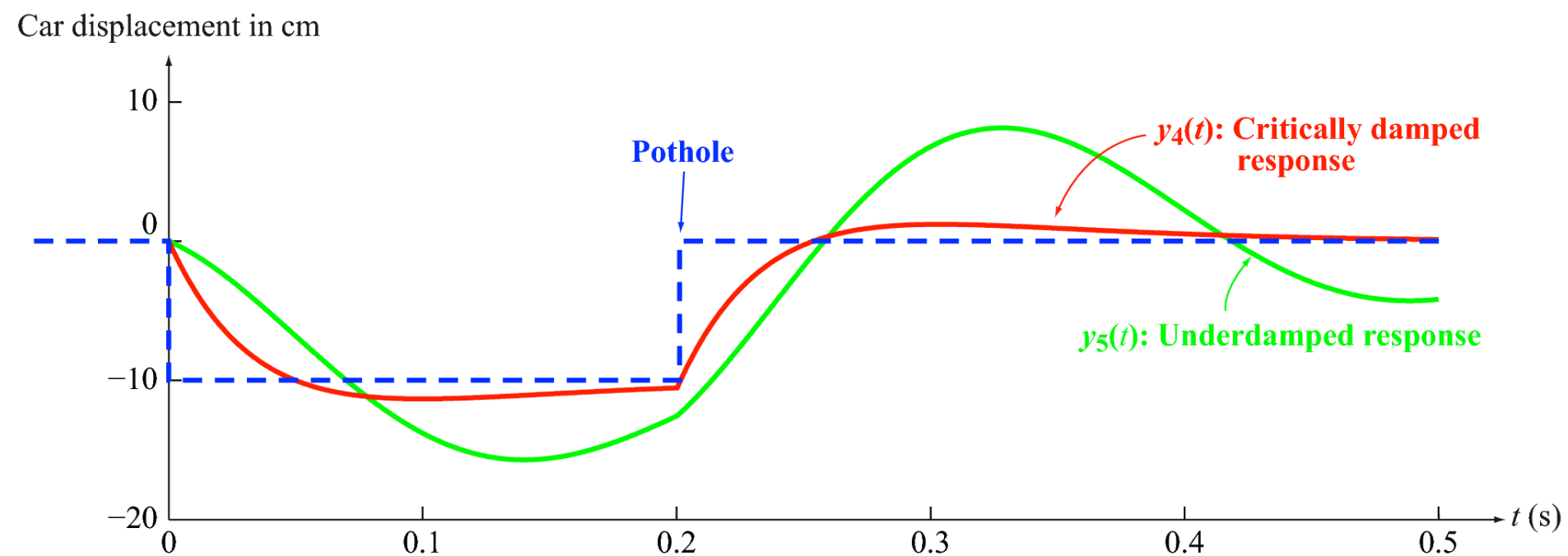


(c) Wavy pavement

# Car's Response to Driving over 10-cm Curb



# Car's Response to Driving over 10-cm Deep Pothole



**Module 2.2 Automobile Suspension Response** Select curb, pothole, or wavy pavement. Then, select the pavement characteristics, the automobile's mass, and its suspension's spring constant and damping coefficient.

