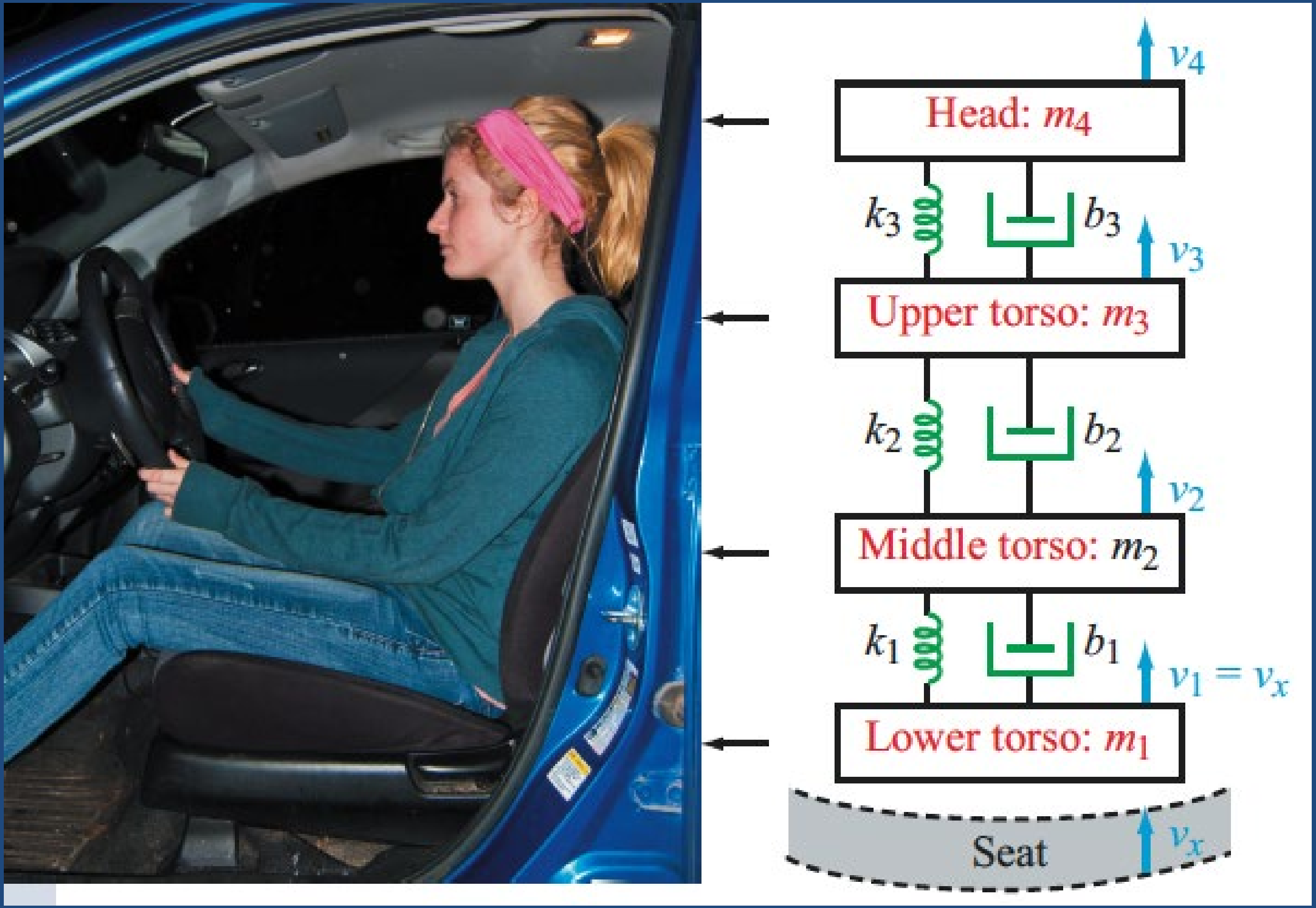

4.APPLICATIONS OF THE LAPLACE TRANSFORM

4 Applications of the Laplace Transform

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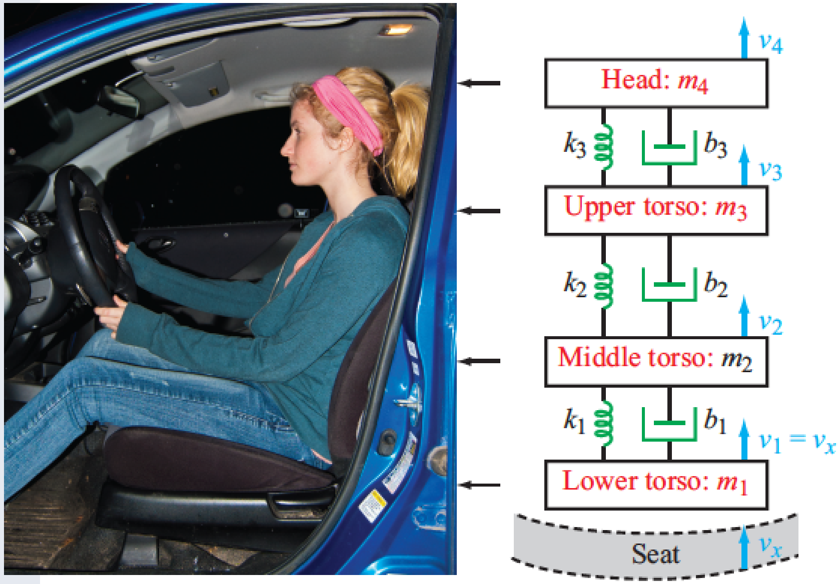
Objectives

Learn to:

- Use s-domain circuit element models to analyze electric circuits.
- Use electromechanical analogues to simulate and analyze mechanical systems.
- Use op-amp circuits to implement systems.
- Develop system realizations that conform to specified transfer functions.
- Employ feedback control techniques to improve system performance and stability



The Laplace-transform tools learned in the previous chapter are now applied to model and solve a wide variety of ***mechanical and thermal systems***, including how to compute the movement of a passenger's head as the car moves over curbs and other types of pavements, and how to design ***feedback loops*** to control ***motors*** and heating systems.

s-Domain Circuit Element Models

▶ The s-domain transformation of circuit elements incorporates initial conditions associated with any energy storage that may have existed in capacitors and inductors at $t = 0^-$. ◀

$$v = Ri \quad \longleftrightarrow \quad \mathbf{V} = R\mathbf{I}.$$

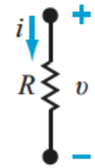
$$v = L\frac{di}{dt} \quad \longleftrightarrow \quad \mathbf{V} = \mathbf{s}L\mathbf{I} - L\,i(0^-).$$

$$i = C\frac{dv}{dt} \quad \longleftrightarrow \quad \mathbf{I} = \mathbf{s}C\mathbf{V} - C\,v(0^-),$$

Table 4-1: Circuit models for R, L, and C in the s-domain.

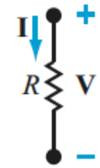
<table>
<tr><th>Time-Domain</th><th colspan="3">s-Domain</th></tr>
<tr><td>

Resistor



$\upsilon = Ri$

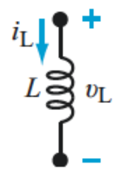
</td><td colspan="3">



$\mathbf{V} = R\mathbf{I}$

</td></tr>
<tr><td>

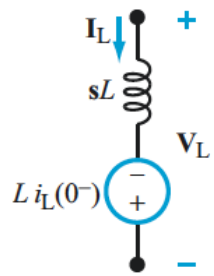
Inductor



$\upsilon_L = L\,\dfrac{di_L}{dt}$

$i_L = \dfrac{1}{L}\displaystyle\int_{0^-}^{t} \upsilon_L\,dt' + i_L(0^-)$

</td><td>

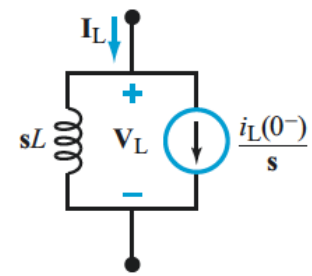


$\mathbf{V}_L = \mathbf{s}L\mathbf{I}_L - L\,i_L(0^-)$

</td><td>

OR

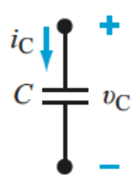
</td><td>



$\mathbf{I}_L = \dfrac{\mathbf{V}_L}{\mathbf{s}L} + \dfrac{i_L(0^-)}{\mathbf{s}}$

</td></tr>
<tr><td>

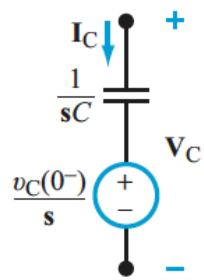
Capacitor



$i_C = C\,\dfrac{d\upsilon_C}{dt}$

$\upsilon_C = \dfrac{1}{C}\displaystyle\int_{0^-}^{t} i_C\,dt' + \upsilon_C(0^-)$

</td><td>

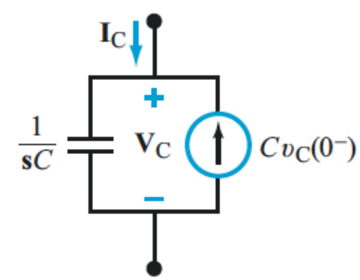


$\mathbf{V}_C = \dfrac{\mathbf{I}_C}{\mathbf{s}C} + \dfrac{\upsilon_C(0^-)}{\mathbf{s}}$

</td><td>

OR

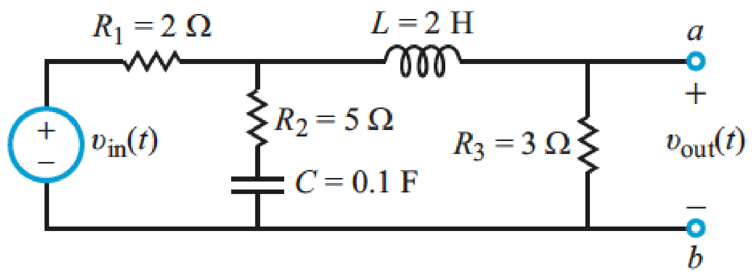
</td><td>



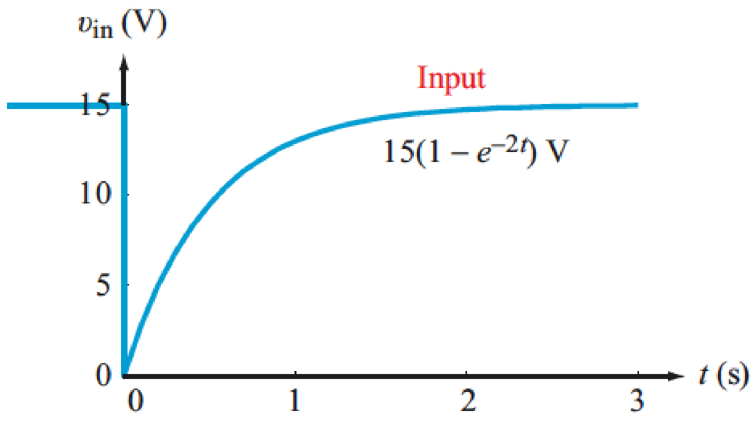
$\mathbf{I}_C = \mathbf{s}C\mathbf{V}_C - C\,\upsilon_C(0^-)$

</td></tr>
</table>

Example 4-1: Interrupted Voltage Source

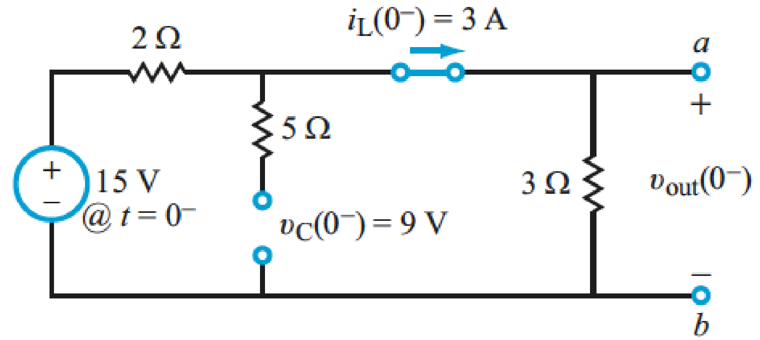


(a) Time domain



(b) Waveform of $v_{in}(t)$

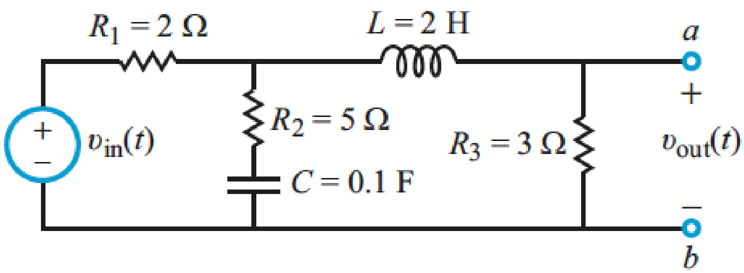
Initial Conditions:



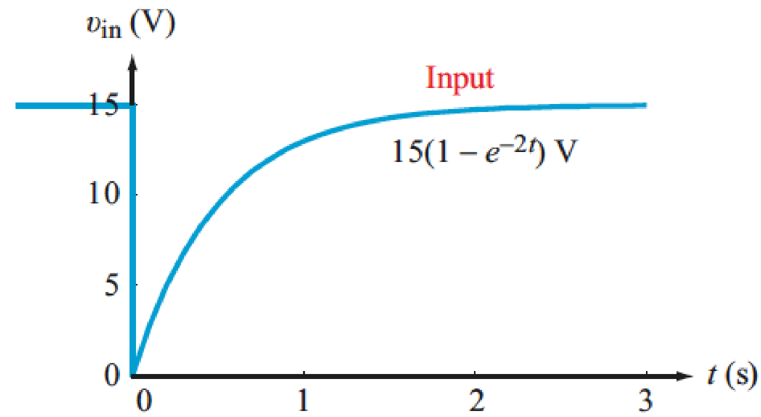
(d) At $t = 0^-$

$$v_C(0^-) = 9\text{ V},\quad i_L(0^-) = 3\text{ A},\ \text{and}\ v_{out}(0^-) = 9\text{ V}$$

Example 4-1: Interrupted Voltage Source



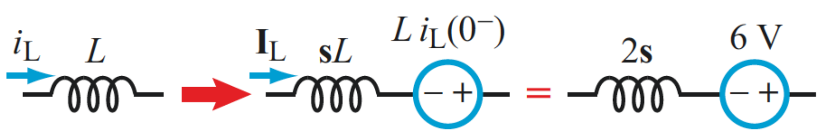
(a) Time domain



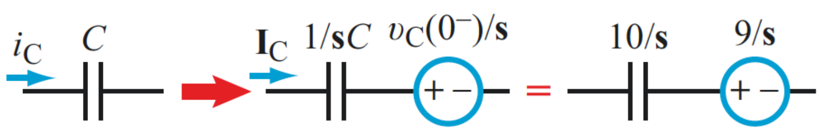
(b) Waveform of $v_{in}(t)$

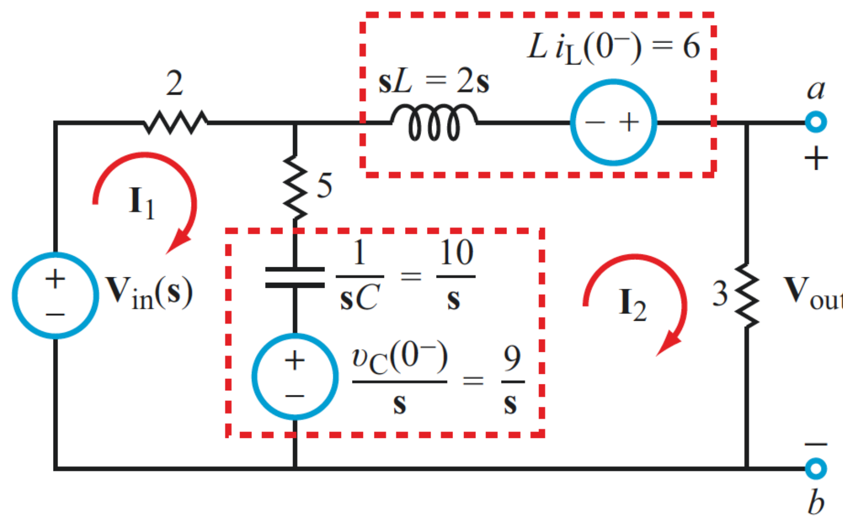
$$\mathbf{V}_{in}(\mathbf{s}) = \frac{15}{\mathbf{s}} - \frac{15}{\mathbf{s}+2}$$

Time Domain **s-Domain**



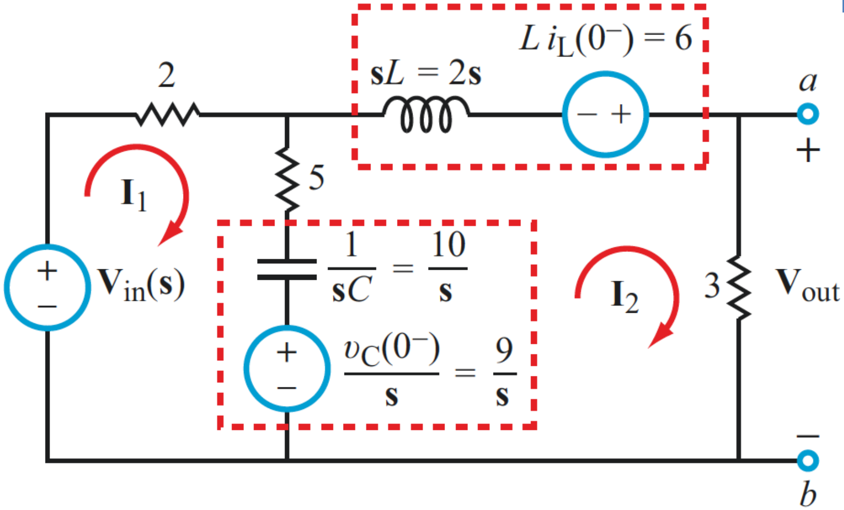
and





(e) s-domain

Example 4-1: Interrupted Voltage Source



(e) s-domain

By inspection, the mesh-current equations for loops 1 and 2 are given by

$$\left(2+5+\frac{10}{\mathbf{s}}\right)\mathbf{I}_1-\left(5+\frac{10}{\mathbf{s}}\right)\mathbf{I}_2=\mathbf{V}_{\text{in}}-\frac{9}{\mathbf{s}} \tag{4.13}$$

and

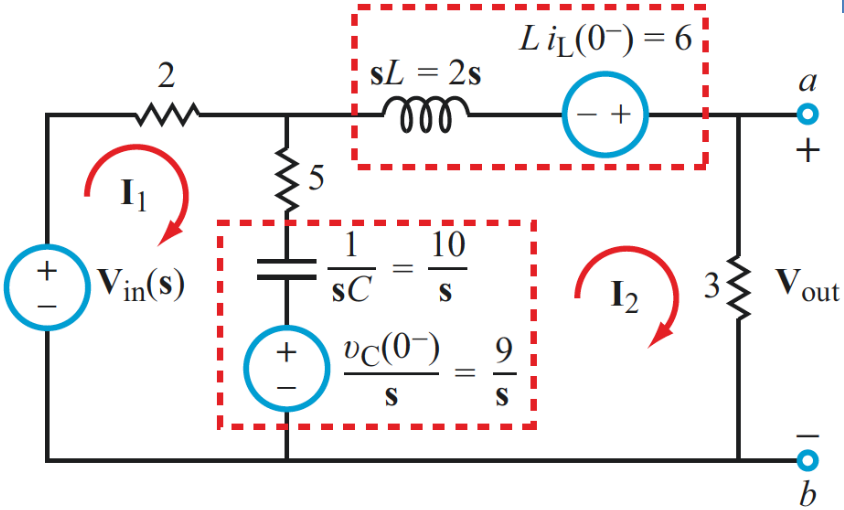
$$-\left(5+\frac{10}{\mathbf{s}}\right)\mathbf{I}_1+\left(3+5+2\mathbf{s}+\frac{10}{\mathbf{s}}\right)\mathbf{I}_2=\frac{9}{\mathbf{s}}+6. \tag{4.14}$$

$$\mathbf{V}_{\text{in}}(\mathbf{s})=\frac{15}{\mathbf{s}}-\frac{15}{\mathbf{s}+2}$$

Simultaneous solution leads to:

$$\mathbf{I}_2=\frac{42\mathbf{s}^3+162\mathbf{s}^2+306\mathbf{s}+300}{\mathbf{s}(\mathbf{s}+2)(14\mathbf{s}^2+51\mathbf{s}+50)}$$
$$=\frac{42\mathbf{s}^3+162\mathbf{s}^2+306\mathbf{s}+300}{14\mathbf{s}(\mathbf{s}+2)(\mathbf{s}^2+51\mathbf{s}/14+50/14)}.$$

Example 4-1: Interrupted Voltage Source



(e) s-domain

$$I_2 = \frac{42s^3 + 162s^2 + 306s + 300}{s(s+2)(14s^2 + 51s + 50)}$$

$$= \frac{42s^3 + 162s^2 + 306s + 300}{14s(s+2)(s^2 + 51s/14 + 50/14)}.$$

$$= \frac{42s^3 + 162s^2 + 306s + 300}{14s(s+2)(s+1.82+j0.5)(s+1.82-j0.5)}$$

Partial fraction expansion:

$$I_2 = \frac{3}{s} + \frac{5.32e^{-j90^\circ}}{s+1.82+j0.5} + \frac{5.32e^{j90^\circ}}{s+1.82-j0.5}$$

Laplace Transform pairs:

$$\frac{3}{s} \leftrightarrow 3\,u(t),$$

and from property #3 of Table 3-3, we have

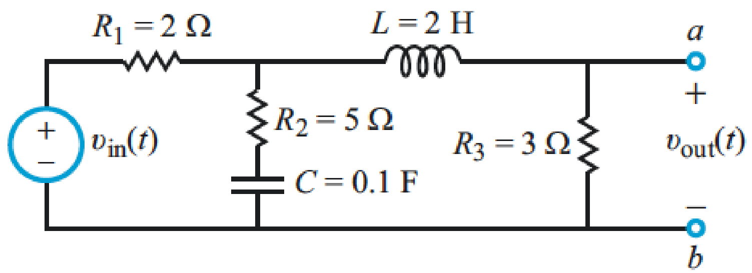
$$\frac{Ae^{j\theta}}{s+a+jb} + \frac{Ae^{-j\theta}}{s+a-jb} \leftrightarrow 2Ae^{-at}\cos(bt-\theta)\,u(t).$$

Time-domain current:

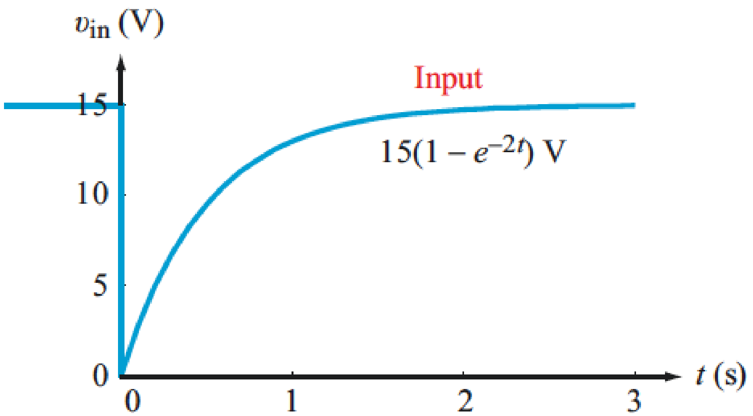
$$i_2(t) = [3 + 10.64e^{-1.82t}\cos(0.5t + 90^\circ)]\,u(t)$$

$$= [3 - 10.64e^{-1.82t}\sin 0.5t]\,u(t)\text{ A},$$

Example 4-1: Interrupted Voltage Source

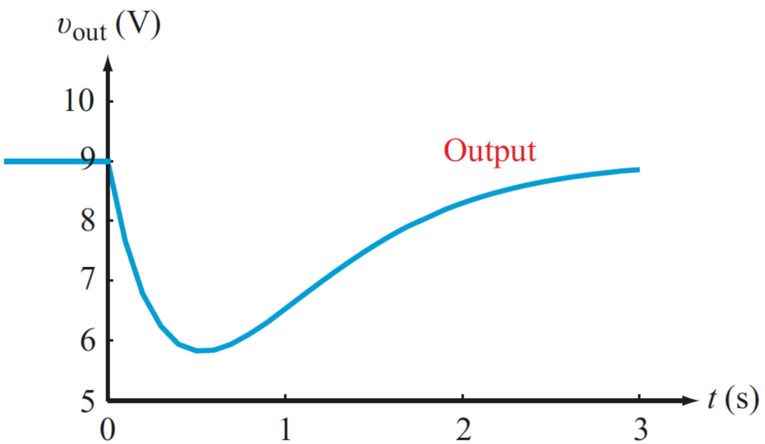


(a) Time domain



(b) Waveform of $v_{in}(t)$

$$v_{out}(t) = 3i_2(t)$$
$$= [9 - 31.92e^{-1.82t} \sin 0.5t]\, u(t) \text{ V}.$$



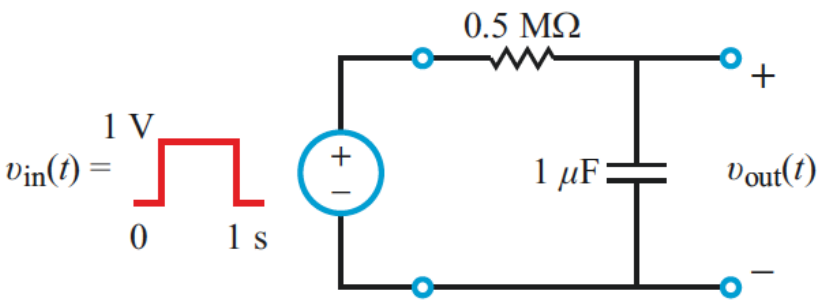
(c) Waveform of $v_{out}(t)$

Example 4-4: Lowpass Filter Response to a Rectangular Pulse

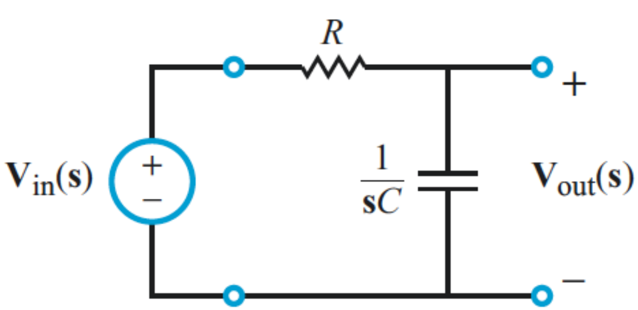
Solution:

With $R = 0.5$ MΩ and $C = 1\ \mu$F, the product is $RC = 0.5$ s. Voltage division in the **s**-domain (Fig. 4-4(b)) leads to

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{V}_{\text{out}}(\mathbf{s})}{\mathbf{V}_{\text{in}}(\mathbf{s})} = \frac{1/\mathbf{s}C}{R + 1/\mathbf{s}C} = \frac{1/RC}{\mathbf{s} + 1/RC} = \frac{2}{\mathbf{s}+2}. \tag{4.43}$$



(a) RC lowpass filter



(b) **s**-domain

The rectangular pulse is given by

$$v_{\text{in}}(t) = [u(t) - u(t-1)]\ \text{V},$$

and with the help of Table 3-2, its **s**-domain counterpart

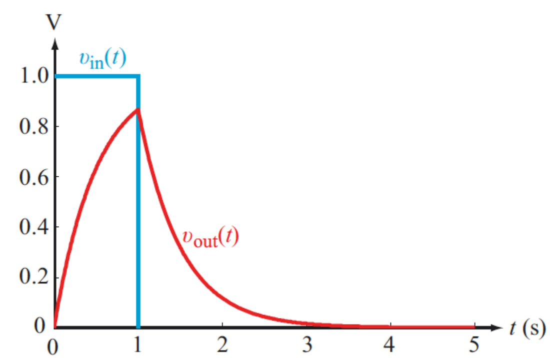
$$\mathbf{V}_{\text{in}}(\mathbf{s}) = \left[\frac{1}{\mathbf{s}} - \frac{1}{\mathbf{s}}\, e^{-\mathbf{s}}\right]\ \text{V}.$$

Hence,

$$\begin{aligned}\mathbf{V}_{\text{out}}(\mathbf{s}) &= \mathbf{H}(\mathbf{s})\ \mathbf{V}_{\text{in}}(\mathbf{s}) \\ &= 2(1 - e^{-\mathbf{s}})\left[\frac{1}{\mathbf{s}(\mathbf{s}+2)}\right].\end{aligned}$$

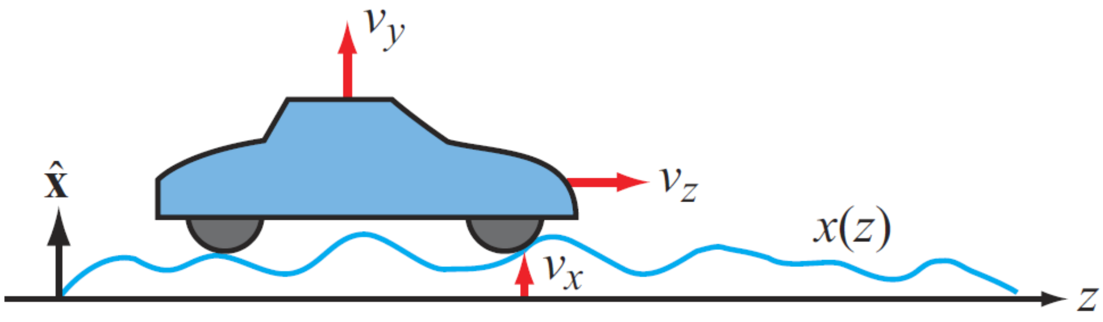
$$\mathbf{V}_{\text{out}}(\mathbf{s}) = \frac{1}{\mathbf{s}} - \frac{1}{\mathbf{s}+2} - \frac{1}{\mathbf{s}}\, e^{-\mathbf{s}} + \frac{1}{\mathbf{s}+2}\, e^{-\mathbf{s}}.$$

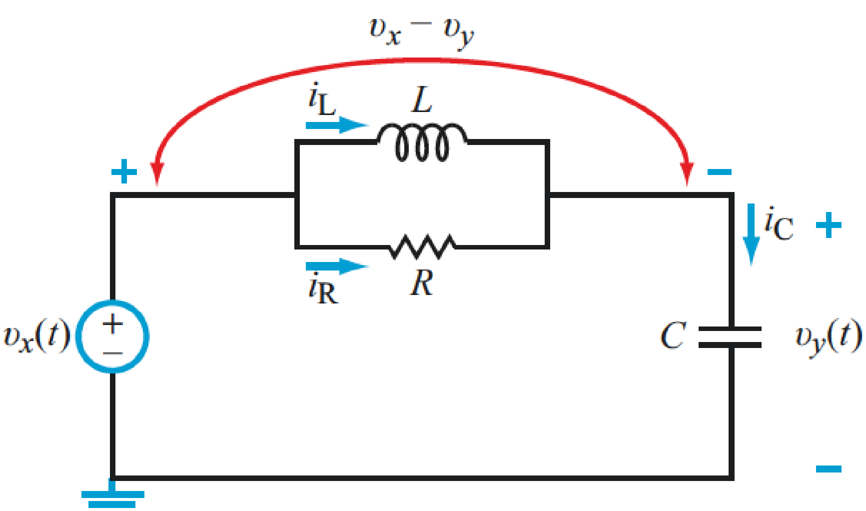
$$v_{\text{out}}(t) = \Big[[1 - e^{-2t}]\, u(t) - [1 - e^{-2(t-1)}]\, u(t-1)\Big]\ \text{V}.$$



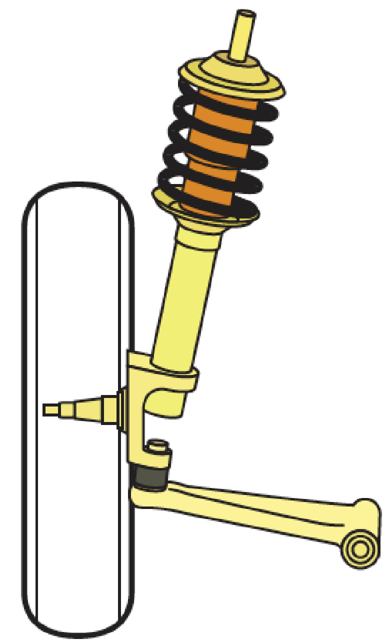
(c) Output response

Electromechanical Analogs

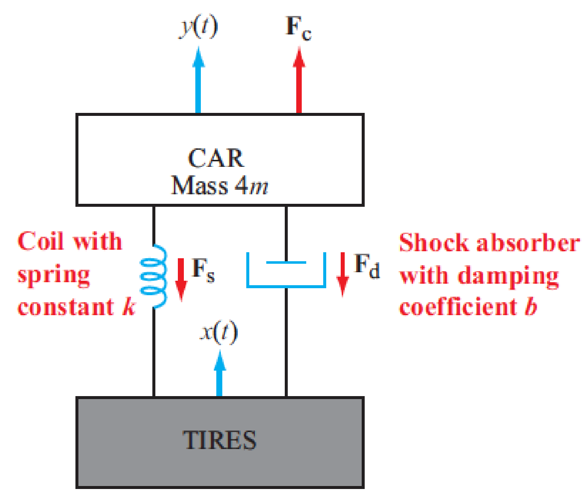




(b) Electrical system



(a) Damping system



(b) Model

Figure 2-26: Car suspension system model.

Table 4-2: Mechanical-electrical analogue.

Mechanical		Electrical
• Force **F** **F** is positive when pointing upwards	⟷	Current i i is positive when entering positive voltage terminal of device
• Vertical velocity v v is positive when car or tire is moving upwards	⟷	Voltage υ υ's positive terminal is where i enters device
• Mass m (1/4 of car's mass) $F_c = m \dfrac{dv_y}{dt}$	⟷	Capacitance C $i_C = C \dfrac{d\upsilon_y}{dt}$
• Spring constant k $F_s = k \int_0^t (v_x - v_y)\, d\tau$	⟷	$1/L$: Inverse of inductance $i_L = \dfrac{1}{L} \int_0^t (\upsilon_x - \upsilon_y)\, d\tau$
• Damping coefficient b $F_d = b(v_x - v_y)$	⟷	$1/R$: Inverse of resistance (conductance) $i_R = \dfrac{1}{R} (\upsilon_x - \upsilon_y)$
• $F_c = F_s + F_d$	⟷	$i_C = i_L + i_R$

SMD-RLC Analysis Procedure

Step 1: Replace each mass with a capacitor with one terminal connected to a node and the other to ground.

Step 2: Replace each spring with an inductor with $L = 1/k$, where k is the spring's stiffness coefficient.

- If the spring connects two masses, its equivalent inductor connects to their equivalent capacitors at their non-ground terminals.
- If the spring connects a mass to a stationary surface, its equivalent inductor should be connected between the capacitor's non-ground terminal and ground.
- If one end of the spring connects to a moving surface, the corresponding terminal of its equivalent inductor should be connected to a voltage source.

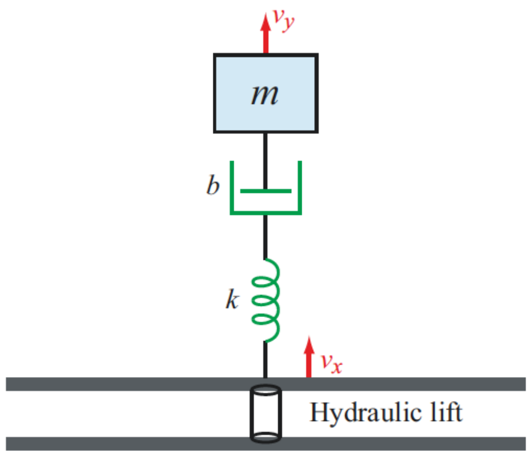
Step 3: Replace each damper with a resistor with $R = 1/b$. Connection rules are the same as for springs.

Step 4: Analyze the RLC circuit using the **s**-domain technique described in Section 4-2.

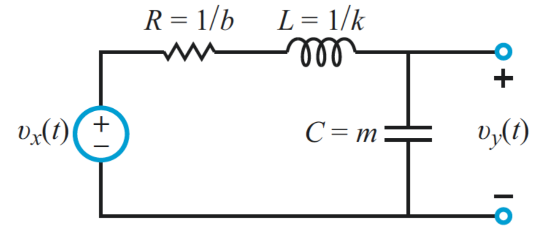
The solution of the RLC circuit provides expressions for the voltages across capacitors, corresponding to the velocities of their counterpart masses in the mechanical system. Displacement of a mass or its acceleration can be obtained by integrating or differentiating its velocity $v(t)$, respectively.

Hydraulic Lift Example

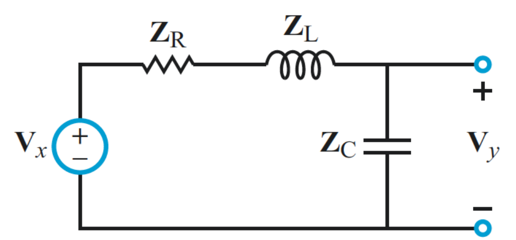
. The lift was used to raise the platform by 4 m at a constant speed of 0.5 m/s. Determine the corresponding vertical speed and displacement of the mass m, given that $m = 150$ kg, $k = 1200$ N/m, and $b = 200$ N·s/m.



(a) Series system



(b) Equivalent circuit



$\mathbf{Z}_R = 1/b, \;\; \mathbf{Z}_L = \mathbf{s}/k, \;\; \mathbf{Z}_C = 1/\mathbf{s}m$

(c) **s**-domain circuit

0.5 m/s over a distance of 4 m, which corresponds to a travel time of $4/0.5 = 8$ s, $v_x(t)$ is a rectangle waveform given by

$$v_x(t) = 0.5[u(t) - u(t-8)] \text{ m/s.} \tag{4.60}$$

Using entries #2 and #2a in Table 3-2, the Laplace transform of $v_x(t)$ is

$$\mathbf{V}_x = \frac{0.5}{\mathbf{s}} - \frac{0.5}{\mathbf{s}}\, e^{-8\mathbf{s}}. \tag{4.61}$$

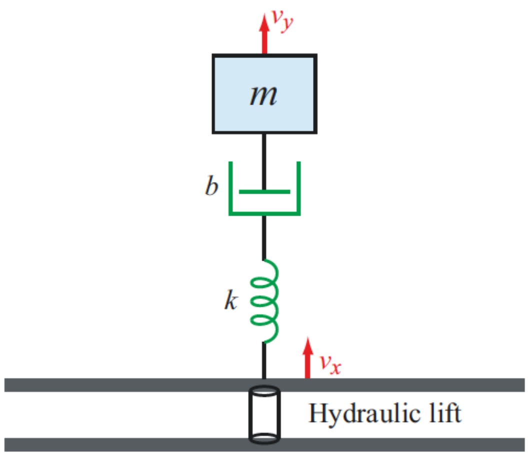
$$\mathbf{V}_y = \frac{\mathbf{V}_x \mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}} = \frac{1/\mathbf{s}m}{\frac{1}{b} + \frac{\mathbf{s}}{k} + \frac{1}{\mathbf{s}m}}\, \mathbf{V}_x$$

$$= \frac{k/m}{\mathbf{s}^2 + \frac{k}{b}\,\mathbf{s} + \frac{k}{m}}\, \mathbf{V}_x$$

$$= \frac{8}{\mathbf{s}^2 + 6\mathbf{s} + 8}\, \mathbf{V}_x.$$

Hydraulic Lift Example

The lift was used to raise the platform by 4 m at a constant speed of 0.5 m/s. Determine the corresponding vertical speed and displacement of the mass m, given that $m = 150$ kg, $k = 1200$ N/m, and $b = 200$ N·s/m.



(a) Series system

$$\mathbf{V}_y = \frac{4}{\mathbf{s}(\mathbf{s}^2 + 6\mathbf{s} + 8)} - \frac{4e^{-8\mathbf{s}}}{\mathbf{s}(\mathbf{s}^2 + 6\mathbf{s} + 8)}$$

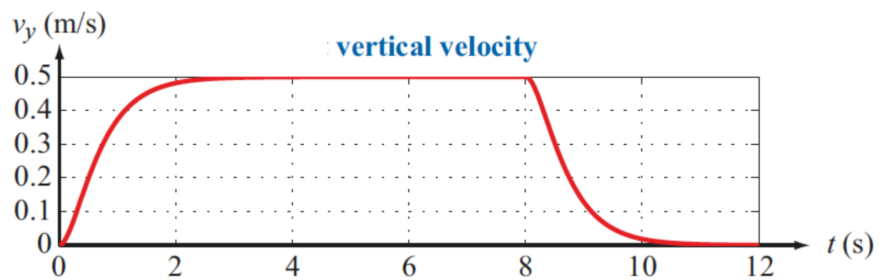
$$= \frac{4}{\mathbf{s}(\mathbf{s}+2)(\mathbf{s}+4)} - \frac{4e^{-8\mathbf{s}}}{\mathbf{s}(\mathbf{s}+2)(\mathbf{s}+4)}$$

$$v_{y_1}(t) = [0.5 - e^{-2t} + 0.5e^{-4t}]\, u(t)$$

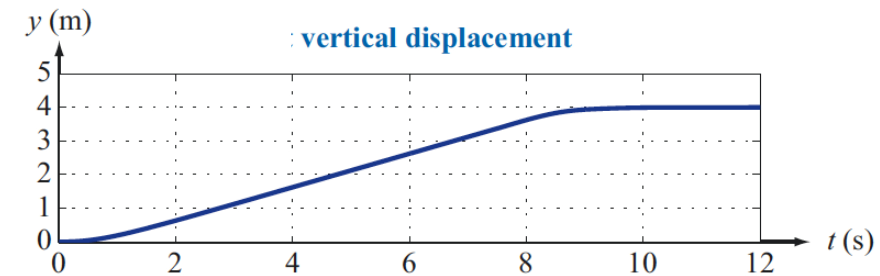
$$v_{y_2}(t) = -[0.5 - e^{-2(t-8)} + 0.5e^{-4(t-8)}]\, u(t-8),$$

$$v_y(t) = v_{y_1}(t) + v_{y_2}(t).$$

$$y(t) = \int_0^t v_y(t)\, d\tau.$$

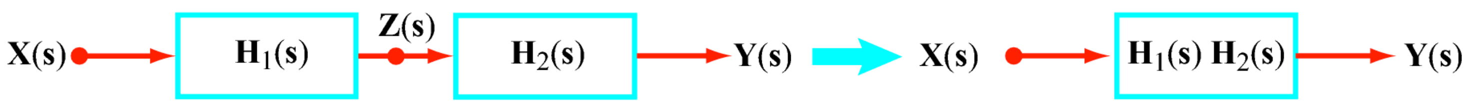


(d) Vertical velocity $v_y(t)$

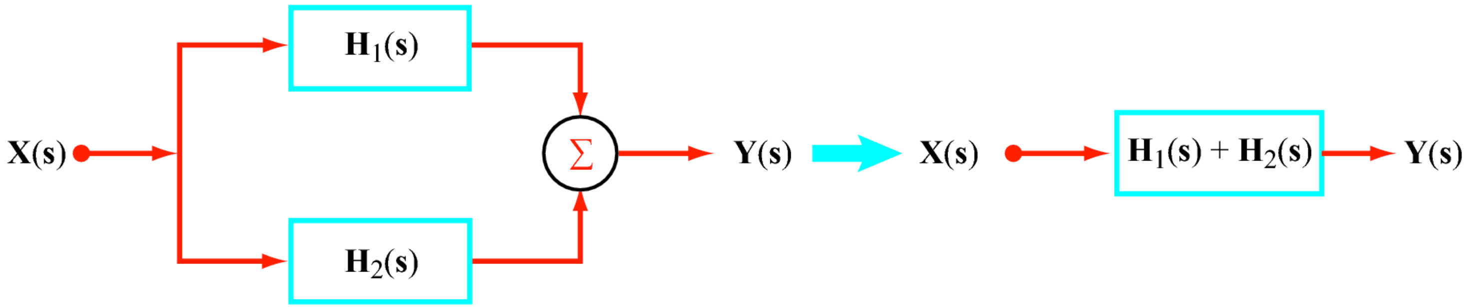


(e) Vertical displacement $y(t)$

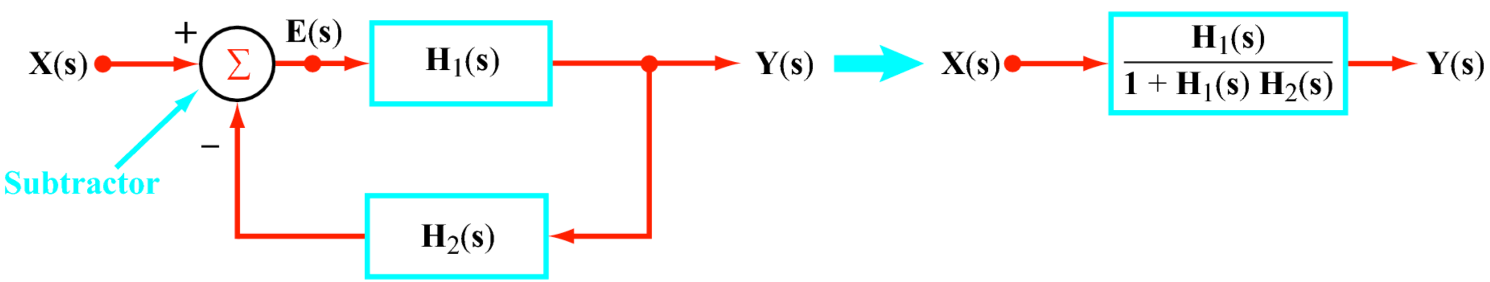
System Configurations



(a) Two cascaded systems



(b) Two parallel systems



(c) Negative feedback system

System Synthesis– Direct Forms I & II Topologies

Given:

$$\mathbf{H}(\mathbf{s}) = \frac{b_0\mathbf{s}^3 + b_1\mathbf{s}^2 + b_2\mathbf{s}}{\mathbf{s}^3 + a_1\mathbf{s}^2 + a_2\mathbf{s} + a_3}$$

procedure entails rewriting the expression in terms of inverse powers of $\mathbf{s}$:

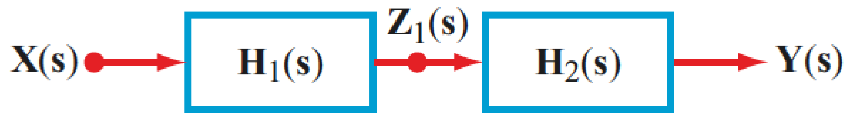
$$\begin{aligned}\mathbf{H}(\mathbf{s}) &= \frac{b_0\mathbf{s}^3 + b_1\mathbf{s}^2 + b_2\mathbf{s}}{\mathbf{s}^3 + a_1\mathbf{s}^2 + a_2\mathbf{s} + a_3} \cdot \frac{1/\mathbf{s}^3}{1/\mathbf{s}^3} \\ &= \left(b_0 + \frac{b_1}{\mathbf{s}} + \frac{b_2}{\mathbf{s}^2}\right)\left(\frac{1}{1 + \frac{a_1}{\mathbf{s}} + \frac{a_2}{\mathbf{s}^2} + \frac{a_3}{\mathbf{s}^3}}\right) \\ &= \mathbf{H}_1(\mathbf{s})\ \mathbf{H}_2(\mathbf{s}),\end{aligned} \tag{4.89}$$

with

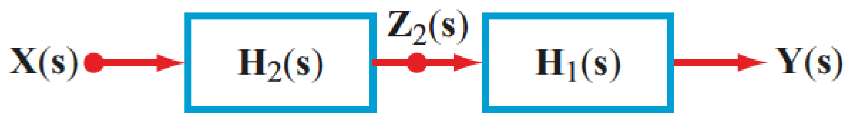
$$\mathbf{H}_1(\mathbf{s}) = b_0 + \frac{b_1}{\mathbf{s}} + \frac{b_2}{\mathbf{s}^2} \tag{4.90a}$$

and

$$\mathbf{H}_2(\mathbf{s}) = \left(1 + \frac{a_1}{\mathbf{s}} + \frac{a_2}{\mathbf{s}^2} + \frac{a_3}{\mathbf{s}^3}\right)^{-1}. \tag{4.90b}$$

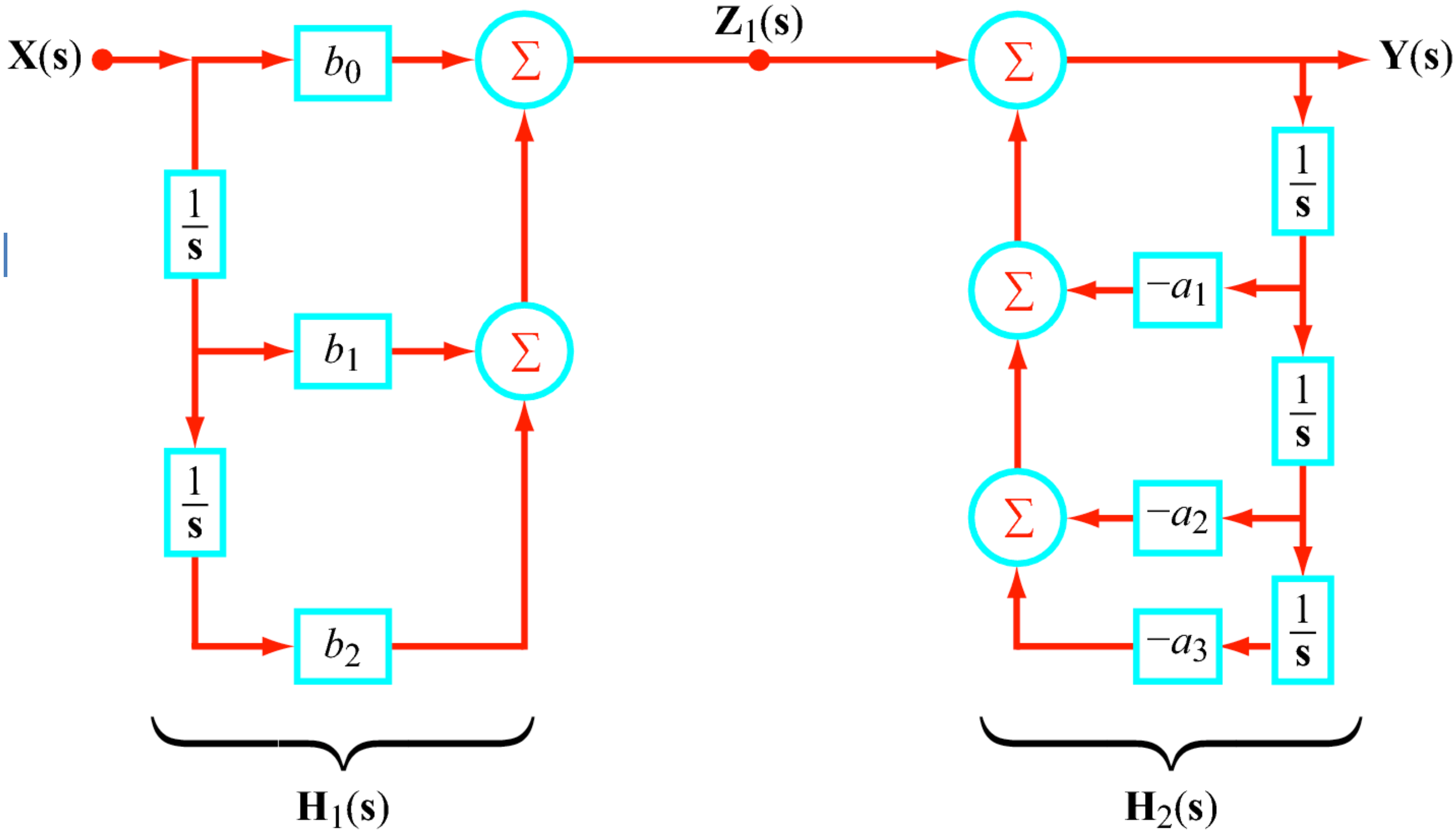


(a) **DFI** realization topology



(b) **DFII** realization topology

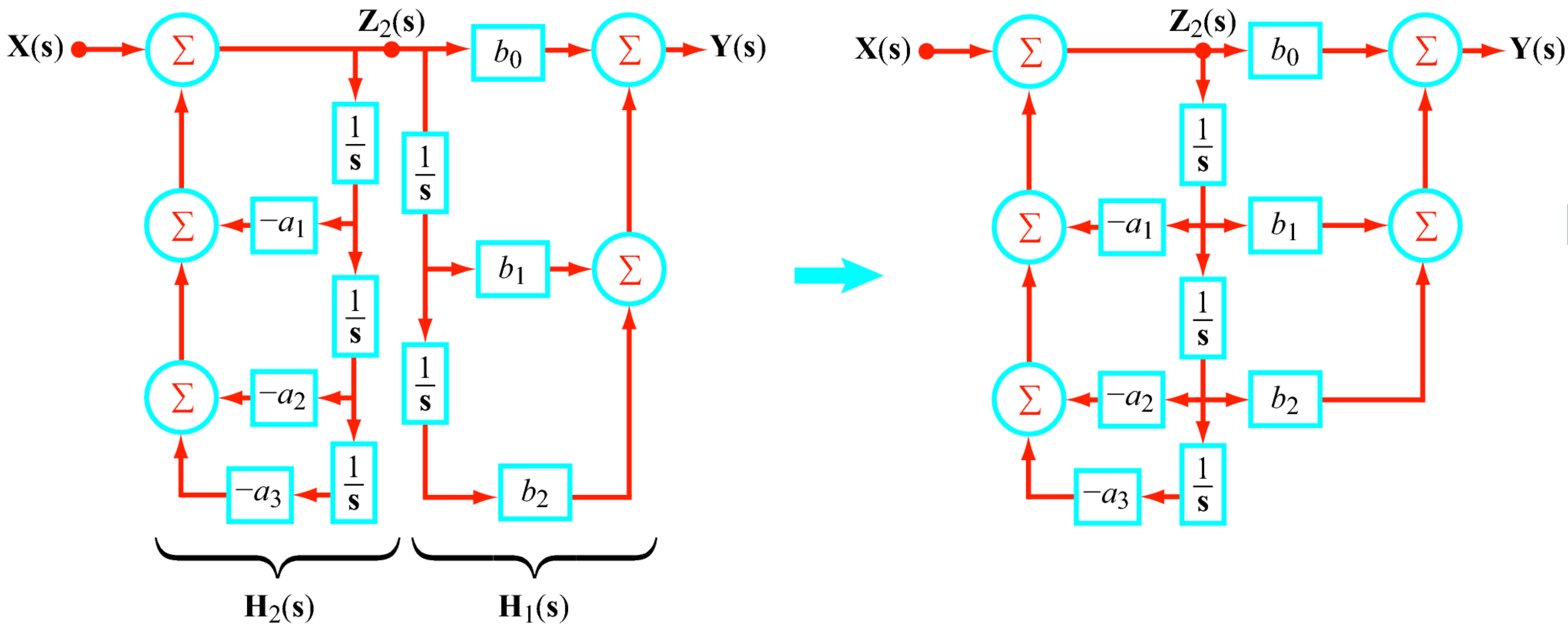
Figure 4-19: In the DFI process, $\mathbf{H}_1(\mathbf{s})$ is realized ahead of $\mathbf{H}_2(\mathbf{s})$, whereas the reverse is the case for the DFII process.



$$H_1(s) = b_0 + \frac{b_1}{s} + \frac{b_2}{s^2}$$

Direct Form I Topology

$$H_2(s) = \left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}\right)^{-1}$$



(a) **Separate integrators** (b) **Common integrators**

$$\mathbf{H}_1(\mathrm{s}) = b_0 + \frac{b_1}{\mathrm{s}} + \frac{b_2}{\mathrm{s}^2}$$

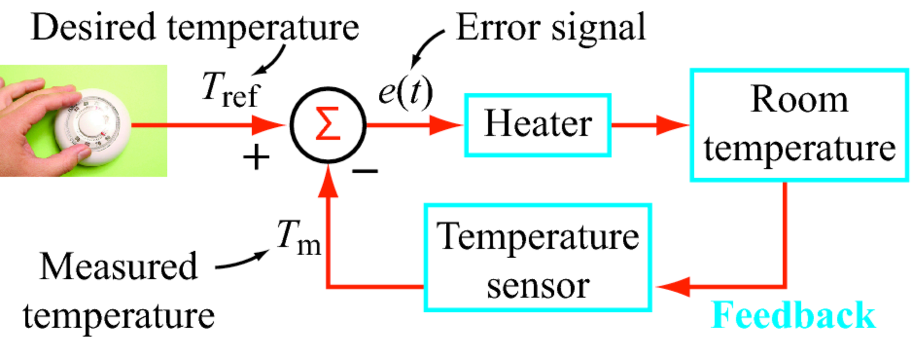
DFII topology requires fewer operations to implement than DFI

Direct Form II Topology

$$\mathbf{H}_2(\mathrm{s}) = \left(1 + \frac{a_1}{\mathrm{s}} + \frac{a_2}{\mathrm{s}^2} + \frac{a_3}{\mathrm{s}^3}\right)^{-1}$$

Control Theory

Temperature Control Example



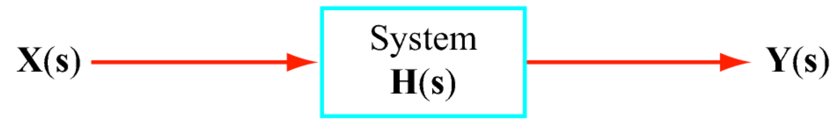
$$\mathbf{Y}(s) = \mathbf{H}(s)\ \mathbf{E}(s)$$
$$= \mathbf{H}(s)[\mathbf{X}(s) - \mathbf{G}(s)\ \mathbf{Y}(s)],$$

which leads to

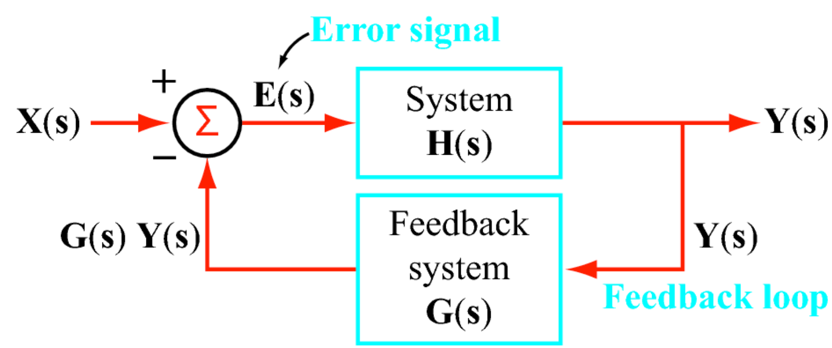
$$\mathbf{Y}(s) = \frac{\mathbf{H}(s)\ \mathbf{X}(s)}{1 + \mathbf{G}(s)\ \mathbf{H}(s)}. \tag{4.100}$$

The output-to-input ratio of the closed-loop system is called the ***closed-loop transfer function*** $\mathbf{Q}(s)$. From Eq. (4.100), we obtain

$$\mathbf{Q}(s) = \frac{\mathbf{Y}(s)}{\mathbf{X}(s)} = \frac{\mathbf{H}(s)}{1 + \mathbf{G}(s)\ \mathbf{H}(s)}. \tag{4.101}$$

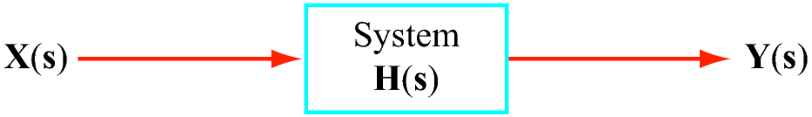


(a) **Open-loop system**

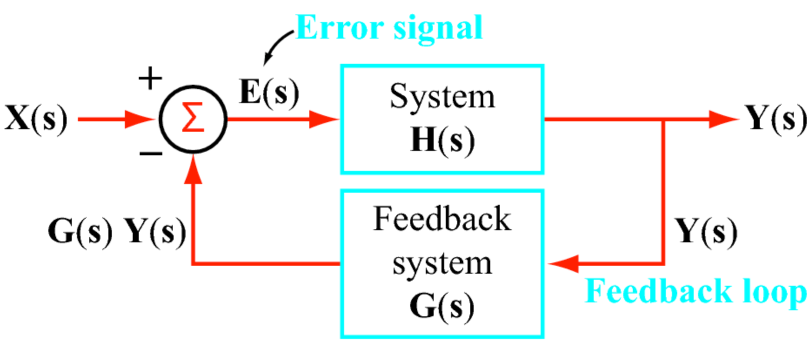


(b) **Closed-loop system**:

System Stabilization



(a) Open-loop system



(b) Closed-loop system:

Consider the first-order system:

$$\mathbf{H}(\mathbf{s}) = \frac{A}{\mathbf{s} - p_1} \qquad \text{with } p_1 > 0. \tag{4.106}$$

The open-loop transfer function $\mathbf{H}(\mathbf{s})$ has a single pole at $\mathbf{s} = p_1$, and since $p_1 > 0$, the pole resides in the RHP

Hence, the system is unstable.

Let's introduce Proportional Feedback with:

$$\mathbf{G}(\mathbf{s}) = K$$

Hence:

$$\mathbf{Q}(\mathbf{s}) = \frac{\mathbf{H}(\mathbf{s})}{1 + \mathbf{G}(\mathbf{s})\,\mathbf{H}(\mathbf{s})} = \frac{A/(\mathbf{s} - p_1)}{1 + \dfrac{KA}{\mathbf{s} - p_1}} = \frac{A}{\mathbf{s} - p_1 + KA}$$

The pole of Q(s) is at $p_1 - KA$. *By choosing* $KA > p_1$, the pole location moves to the OLHP, thereby converting the unstable open-loop system into a stable closed-loop system.