LECTURE NO. 18

5.5 Alternating Series

Wright State University

- $$\sum_{n=1}^{\infty} a_n$$

- First check if the terms a_n, as a sequence, goes to zero, i.e.,

$$\text{is } \lim_{n\to\infty} a_n = 0?$$

- If no, then the series is divergent by Test for Divergence.
- Otherwise check if the series is a positive series.
- If yes, then try either Comparison Test or Integral Test.
- What if the series is not a positive series?

Alternating Series

- A series is called an alternating series if the terms are alternating between positive and negative values.
- $$\sum_{n=1}^{\infty}(-\frac{1}{2})^n = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n$$
- $$\sum_{n=1}^{\infty}(-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
- An alternating series can be written in one of the following two forms, where $b_n > 0$:

$$\sum_{n=1}^{\infty}(-1)^n \cdot b_n \text{ or } \sum_{n=1}^{\infty}(-1)^{n+1} \cdot b_n$$

Alternating Series Test

- For an alternating series

$$\sum_{n=1}^{\infty}(-1)^n \cdot b_n \text{ or } \sum_{n=1}^{\infty}(-1)^{n+1} \cdot b_n, \text{ if}$$

$= b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$

1) $\lim_{n\to\infty} b_n = 0$ and
2) the sequence $\{b_n\}$ is decreasing,

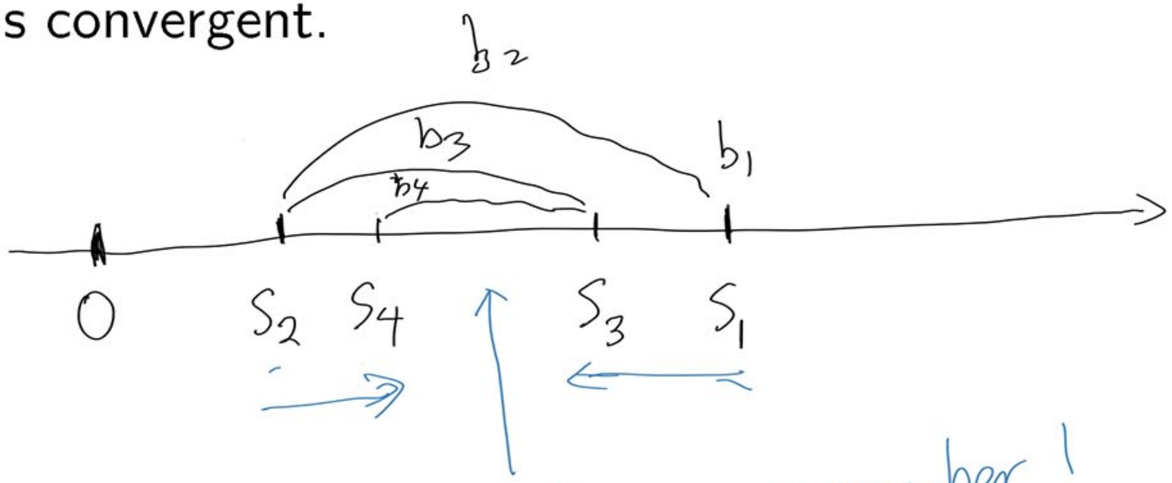
- then the alternating series is convergent.

$S_1 = b_1$

$S_2 = b_1 - b_2$

$S_3 = b_1 - b_2 + b_3$

$S_4 = b_1 - b_2 + b_3 - b_4$



$\lim_{n\to\infty} S_n$ = a number!

An example

$$\sum_{n=1}^{\infty}(-1)^n \cdot \frac{2}{3n+1}$$

$b_n = \frac{2}{3n+1}$

(1) $\lim_{n\to\infty} \frac{2}{3n+1} = 0$ ✓

(2) $\left\{\frac{2}{3n+1}\right\}$ decreasing? Yes ✓

Therefore, the alternating Series $\sum_{n=1}^{\infty}(-1)^n \frac{2}{3n+1}$ is Convergent

by Alternating Series Test!

Remainder of an Alternating Series

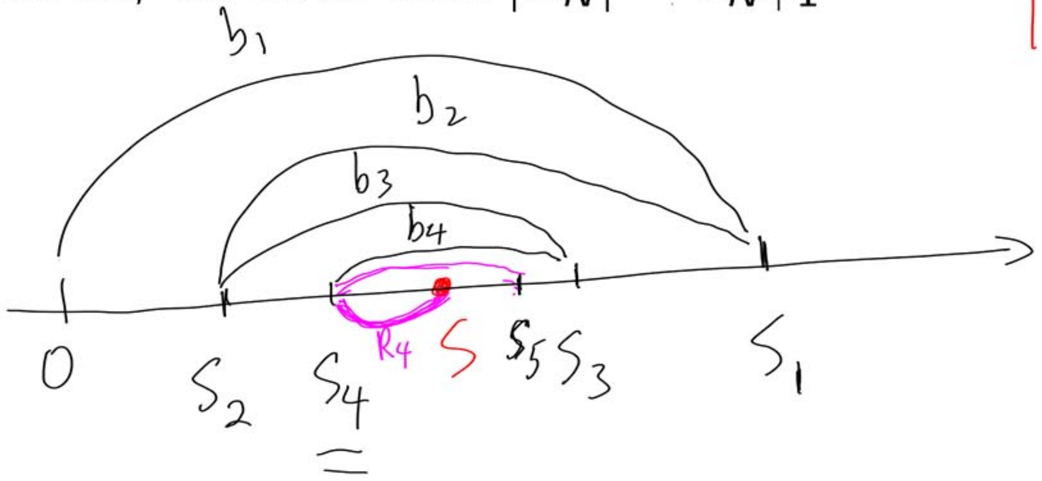
- Suppose that the following alternating series is convergent to S, i.e.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot b_n = S \quad = b_1 - b_2 + b_3 - b_4 + \cdots$$

- Typically we use a partial sum S_N to estimate the value of S.
- The value of $S - S_N$ is called the remainder, denoted by R_N.
- For alternating series, we have that $|R_N| < b_{N+1}$

$|R_{10}| < b_{11}$

$|R_N| < b_{N+1}$



Remainder of an Alternating Series

- Suppose that the following alternating series is convergent to S, i.e.

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- Typically we use a partial sum S_N to estimate the value of S.
- The value of $S - S_N$ is called the remainder, denoted by R_N.
- For alternating series, we have that $|R_N| \leq b_{N+1}$

- If we use S_{10} to estimate the following series, then the error$= |R_{10}| < b_{11} = \frac{1}{11^2} \approx 0.008265$.

Remainder

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cdot \frac{1}{n^2}$$

Absolute Convergence

- Given a series:

$$\sum_{n=1}^{\infty} a_n$$

- It is called absolutely convergent if

$$\text{the series } \sum_{n=1}^{\infty} |a_n| \text{ is convergent.}$$

- Why do we study absolute convergence?
  - After taking absolute value, we get a positive series, so we may apply comparison test or integral test.
  - If the absolute value series is convergent, then the original series must be convergent!
  - A question: Does "Absolute Convergence" imply "convergence" ?
  - Answer: Absolutely!

An example on absolute convergence

$$\sum_{n=1}^{\infty} \frac{\sin n}{n\sqrt{n}}$$

we look at $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin n}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n\sqrt{n}}$ (a positive series)

$\displaystyle\frac{|\sin n|}{n\sqrt{n}} < \frac{1}{n\sqrt{n}}$ Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

(p-series: $p = \frac{3}{2} \Rightarrow$ Convergent!)

① $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n\sqrt{n}}$ is convergent by Simple Comparison Test.

② $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n\sqrt{n}}$ is convergent by absolute convergence.

Three types of series regarding convergence/divergence

- In the following series, the terms donot approach 0, so it is divergent.

$$\sum_{n=1}^{\infty} \frac{n+1}{2n}$$

- The next series is convergent by AST, but it is not absolutely convergent.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} \xrightarrow[\text{value}]{\text{absolute}} \sum_{n=1}^{\infty} \frac{1}{n}$$

p-series $p=1$

divergent.

- The series below is absolutely convergent.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2} \xrightarrow[\text{value}]{\text{absolute}} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

p-series $p=2$

convergent!