LECTURE NO. 15

5.2 Infinite Series

Wright State University

Sequence vs Series

- Recall that a sequence is a set of numbers listed in order.
- A sequence $a_n = \frac{1}{n}, n \geq 1$:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

- A series is the sum of all numbers in an infinite sequence.
- An example of series

Summation

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

- Recall that a sequence $\{a_n\}$ is convergent if

$$\lim_{n \to \infty} a_n = \text{a number}$$

- It is much harder to test if a series is convergent.

Partial Sums and Convergence of a Series

- We cannot add infinitely many numbers; we need to use partial sums.
- Given

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

- The partial sum S_n is the sum of the first n terms.
  - $S_1 = a_1$;
  - $S_2 = a_1 + a_2$;
  - $S_3 = a_1 + a_2 + a_3$;
  - $S_4 = a_1 + a_2 + a_3 + a_4$;
  - $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

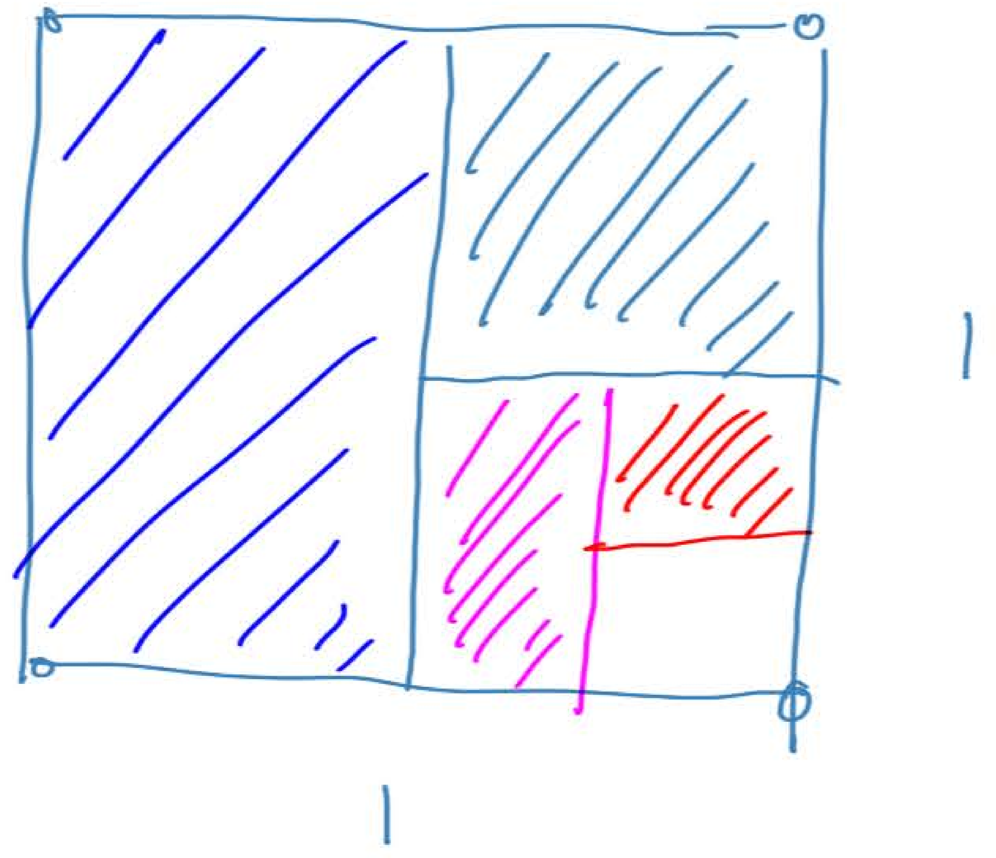
The partial sums S_n form a sequence.

- $$\sum_{n=1}^{\infty} a_n \text{ is convgent if } \lim_{n\to\infty} S_n = \text{a number.}$$

An example on Partial Sum

- Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$



$\to \frac{1}{\infty} = 0$

- Let's compute some partial sums:
  - $S_1 = \frac{1}{2}$
  - $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
  - $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$
  - $S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$
  - The pattern suggests that $S_n = \frac{2^n-1}{2^n}$.

$$\lim_{n\to\infty} \underbrace{\frac{2^n}{2^n}}_{} - \frac{1}{2^n} = \lim_{n\to\infty} 1 - \frac{1}{2^n}$$

$\frac{2^n}{2^n} = 1$

- $$\text{Since } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2^n-1}{2^n} = 1, \text{ the series } \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ is convergent to } 1.$$

More Examples on Partial Sums

- $$\sum_{n=1}^{\infty}(-1)^{n+1} = 1 + (-1) + 1 + (-1) + \cdots$$
  - $S_1 = 1$
  - $S_2 = 1 + (-1) = 0$
  - $S_3 = 1 + (-1) + 1 = 1$
  - $S_4 = 1 + (-1) + 1 + (-1) = 0$
  - The pattern suggests that $S_n = 1$ if n is odd; $S_n = 0$ if n is even.

S_n: 1, 0, 1, 0, 1, 0, 1, 0, - - -

$\lim_{n\to\infty} S_n$ DNE

- Since $\lim_{n\to\infty} S_n$ does not exist, the series $\sum_{n=1}^{\infty}(-1)^{n+1}$ is divergent.

More Examples on Partial Sums

- $$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

  - $S_1 = \frac{1}{2}$
  - $S_2 = \frac{1}{2} + \frac{2}{3} \geq \frac{1}{2} \cdot 2$
  - $S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} \geq \frac{1}{2} \cdot 3$
  - $S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \geq \frac{1}{2} \cdot 4$
  - The pattern suggests that $S_n \geq \frac{1}{2} \cdot n$.

since $\lim_{n \to \infty} \frac{1}{2} n = \infty$,

$\lim_{n \to \infty} S_n \geq \lim_{n \to \infty} \frac{1}{2} n = \infty$

- Since $\lim_{n\to\infty} S_n = \infty$, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent.

- $$\sum_{n=1}^{\infty} ar^{n-1}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

$$S_n = a + ar + ar^2 + ar^3 + \cdots + \underline{ar^{n-1}} \qquad (1)$$

multiply both sides by r in (1)

$$rS_n = \quad ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \qquad (2)$$

(1) − (2) $\quad (1-r)S_n = a - ar^n = \underline{a(1-r^n)}$

Solve for S_n: $S_n = \dfrac{a(1-r^n)}{1-r}$ for $\sum_{n=1}^{\infty} a r^{n-1}$ $(a \neq 0)$

$= a + ar + ar^2 + ar^3 + \cdots$

Case 1: $r = 1$: $\sum_{n=1}^{\infty} a r^n = a + a + a + a + \cdots$ divergent if $a \neq 0$.

Case 2: $r = -1$. $r^n = (-1)^n$ $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{a(1-r^n)}{1-r}$ Does not exist! divergent.

Case 3. $|r| > 1$, (or $r > 1$, or $r < -1$) $\lim_{n\to\infty} S_n = \pm\infty$, divergent!

Case 4: $|r| < 1$, $\lim_{n\to\infty} r^n = 0$ $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{a(1-r^n)}{1-r} = \dfrac{a}{1-r}$. Convergent!

Geometric series $\sum_{n=1}^{\infty} a r^{n-1}$ $\begin{cases} \text{is convergent to } \dfrac{a}{1-r} \text{ if } |r| < 1. \\ \text{divergent otherwise.} \end{cases}$

$\dfrac{\text{first term}}{1 - \text{Common Ratio}}$

Writing Repeated Decimals as Fraction

- $3.5\overline{26} = 3.52626262626\cdots = 3.5 + 0.0\underline{26}\,26\,26\,26 \cdots$

$$= \frac{7}{2} + \left(\underline{0.026} + \underline{0.00026} + 0.0000026 + \cdots\right.$$

Common Ratio $\frac{1}{100}$

Geometric series with $r = \frac{1}{100}$, Convergent to $\frac{\text{first term}}{1 - \text{Common Ratio}}$

$$= \frac{7}{2} + \frac{\frac{26}{1000} \cdot 1000}{\left(1 - \frac{1}{100}\right) 1000} = \frac{7}{2} + \frac{26}{1000 - 10} = \frac{7}{2} + \frac{26}{990}$$

$$2 \overline{)990} \quad 495$$

$$= \frac{3465}{990} + \frac{26}{990} = \frac{3{,}491}{990}$$

FINAL ANSWER!

$$\begin{array}{r} 495 \\ \times\ 7 \\ \hline 3465 \end{array}$$

A Telescope Sum

- $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Use partial fractions, $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right)$$

$$+ \cdots\cdots$$

$$= 1$$

FINAL ANSWER