Midterm Exam1, MTH 2530, October 13, 2022

Show all of your work for full credit.

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1. (a) (5 points) Write down the corresponding augmented matrix of the given system:

$$\begin{cases} x_1 - 2x_2 = 1 \\ x_1 - 2x_3 = 3 \end{cases} : x\begin{pmatrix}1\\1\end{pmatrix} + x_2\begin{pmatrix}-2\\0\end{pmatrix} + x_3\begin{pmatrix}0\\-2\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 3 \end{array}\right]$$

(b) (15 points) Find the solutions in the vector form **x**. Use back-substitution to select the free variables. Provide a geometric description of the solution set.

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 3 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 2 & -2 & 2 \end{array}\right] \xrightarrow{R_2/2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

free $x_3$ $\quad 1(x_2) - 1(x_3) = 1 \qquad 1(x_1) - 2(1+x_3) = 1$

$x_2 = x_3 + 1 \qquad x_1 - 2 - 2x_3 = 1$

$x_1 = 2x_3 + 3$

$$\vec{x} = \begin{pmatrix} 2x_3 + 3 \\ x_3 + 1 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix}2\\1\\1\end{pmatrix} + \begin{pmatrix}3\\1\\0\end{pmatrix}$$

line in $\mathbb{R}^3$ that passes through $(3,1,0)$ in the direction of $(2,1,1)$

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$.

(a) (15 points) Determine if the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Justify your answer.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

not linearly independent,

(b) (5 points) Is $\mathbf{v}_3$ in the Span$\{\mathbf{v}_1, \mathbf{v}_2\}$? Justify your answer.

Yes, $v_3$ is linearly dependent on $v_1, v_2$

3. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, and let $T : R^3 \to R^3$ be a linear transformation that maps $\mathbf{x}$ into $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$.

(a) (5 points) Find a matrix $A$ such that $T(\mathbf{x})$ is $A\mathbf{x}$ for each $\mathbf{x}$.

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 3 & -5 & -5 \end{bmatrix}$$

$\cancel{0}$ (b) (5 points) What is the range of $T$?

Span of $v_1, v_2$

(c) (10 points) Find a vector $\mathbf{x} \in R^3$ such that $T(\mathbf{x}) = \begin{bmatrix} 1 \\ 2 \\ 11 \end{bmatrix}$. Use back-substitution to select the free variables.

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 3 & -5 & -5 & 11 \end{array}\right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 4 & -8 & 8 \end{array}\right] \downarrow R_3 - 4R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free $x_3$ $\quad x_2 - 2(x_3) = 2$

$x_2 = 2x_3 + 2$

$x_1 - 3(2x_3 + 2) + x_3 = 1$

$x_1 - 6x_3 - 6 + x_3 = 1$

$x_1 = 5x_3 + 7$

$$\vec{x} = \begin{pmatrix} 5x_3 + 7 \\ 2x_3 + 2 \\ x_3 \end{pmatrix}$$

for $x_3 = 0$ $\quad \vec{x} = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix}$

4. (a) (5 points) Find the matrix $A$ such that $A\mathbf{x} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (10 points) Define a linear transformation $T$ on $R^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the image of $\mathbf{x} = (1, 2, 3)$. Find the range of $T$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2+0 \\ 0+2+3 \\ 0+0+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$$

range = Span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(c) (10 points) Is $T$ invertible? Justify your answer.

Yes, it has 3 pivot points as a 3x3 matrix

5. (a) (10 points) Calculate the determinant of

$$A = \begin{bmatrix} 2 & -2 & 0 & 2 \\ 5 & -1 & 0 & -6 \\ 2 & 0 & 1 & 7 \\ -6 & 3 & 0 & 0 \end{bmatrix} \xrightarrow{R_2+3R_1} \begin{vmatrix} 2 & -2 & 0 & 2 \\ 11 & -7 & 0 & 0 \\ 2 & 0 & 1 & 7 \\ -6 & 3 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -2 & 0 & 2 \\ 11 & -7 & 0 & 0 \\ 2 & 0 & 1 & 7 \\ -6 & 3 & 0 & 0 \end{vmatrix} = (-1)^{3+3}\, 1 \begin{vmatrix} 2 & -2 & 2 \\ 11 & -7 & 0 \\ -6 & 3 & 0 \end{vmatrix} = (-1)^{1+3}\, 2 \begin{vmatrix} 11 & -7 \\ -6 & 3 \end{vmatrix} = 2(11\cdot 3 - -6\cdot -7)$$

$$= 2(33-42)$$

$$= 2(-9) = -18$$

(b) (5 points) Is $A$ invertible? What is the determinant of $A^{-1}$ if $A$ is invertible?

Yes, $\frac{1}{-18}$