LECTURE NO. 16

5.3 The Divergence and Integral Tests

Wright State University

Test for Divergence

- Given
$$\sum_{n=0}^{\infty} a_n$$
- The Partial Sum $S_n = a_1 + a_2 + a_3 + \cdots + a_n$.
- Suppose the series is convergent. Then
$$\lim_{n\to\infty} S_n = S$$
.
- Consider $S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$. Not hard to see that
$$\lim_{n\to\infty} S_{n-1} = S$$
.
- Note that $a_n = S_n - S_{n-1}$. So
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = 0$$

If $\sum_{n=0}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Contrapositive

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ is divergent!

Test for Divergence

Two Examples

- $$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0$$

$\frac{1}{n} \to 0$

By Test for Divergence, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent!

- $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$\lim_{n\to\infty} \frac{1}{n} = 0$: Test for Divergence does not tell us anything!

The Harmonic Series

- $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 = \underline{1 + \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}} > \frac{3}{2} + \frac{1}{2}$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{3}{2} + \frac{1}{2} \cdot 2$

$S_{16} > \frac{3}{2} + \frac{1}{2} \cdot 3$

$S_{32} > \frac{3}{2} + \frac{1}{2} \cdot 4$

$\lim_{n \to \infty} S_n = \infty$

divergent !

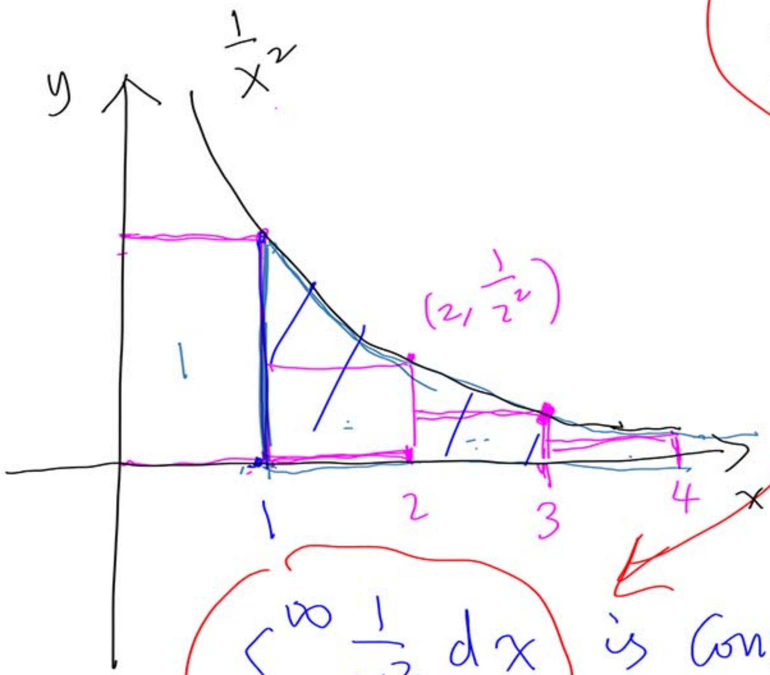
Integral Test

- $$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$\frac{1}{x^2}$

add up to a finite number!

The series is convergent!



$\int_1^{\infty} \frac{1}{x^2}\, dx$ is convergent.

Use Integral Test on the Harmonic Series

- $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$\frac{1}{x}$

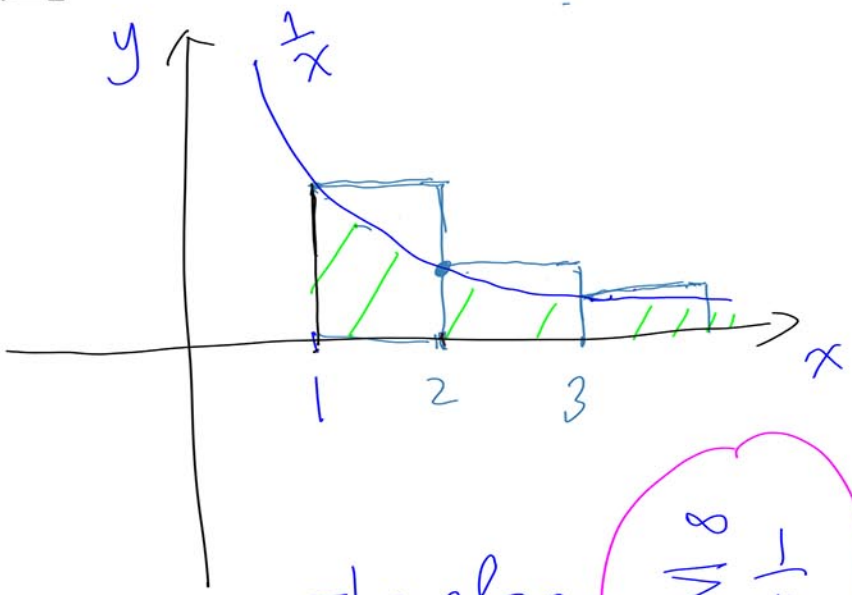
Integral Test

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$

① we need $a_n > 0$,
i.e, a positive series

② $\{a_n\}$ decreasing

$\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ have the same Convergent / divergent property.



$\int_1^{\infty} \frac{1}{x}\,dx$

$\ln|x| \Big|_1^{\infty} = \infty$

divergent

Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Another Example on Integral Test

- $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} = 1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \cdots$$

Integral Test: $\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx = \int_1^\infty \frac{1}{\sqrt{u}} \frac{du}{2}$

$u = 2x-1 \quad \frac{du}{dx} = 2 \quad dx = \frac{du}{2}$

$x: 1 \to \infty$

$u = 2x-1 : 1 \to \infty$

$= \frac{1}{2} \int_1^\infty u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\, 2 \cdot u^{\frac{1}{2}} \Big|_1^\infty$

$= u^{\frac{1}{2}} \Big|_1^\infty = \infty$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$ is divergent.

p-series

- For $p > 0$, the p-series is

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

- Examples of p-series:
  - $$\sum_{n=1}^{\infty} \frac{1}{n} \qquad (p=1)$$
  - $$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad (p=2)$$
  - $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad (p=\frac{1}{2})$$

When is a p-series convegent?

- By Integral Test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent if and only if the p-integral } \int_1^{\infty} \frac{1}{x^p} dx \text{ is convergent.}$$

.

- Recall that
  - $\int_1^{\infty} \frac{1}{x^p} dx$ is convergent if $p > 1$;
  - $\int_1^{\infty} \frac{1}{x^p} dx$ is divergent if $p \leq 1$.

- Therefore, the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent} & \text{if } p > 1 \\ \text{divergent} & \text{if } p \leq 1 \end{cases}$$

Summary

- Test for Divergence applies to any series.
- $$\sum_{n=1}^{\infty} a_n \text{ is divergent if } \lim_{n\to\infty} a_n \neq 0$$
  .
- In the case that $\lim_{n\to\infty} a_n = 0$, Test for Divergence is inconclusive.
- Integral Test only applies to positive series.
- Say $a_n = f(n)$, then
  $$\sum_{n=1}^{\infty} a_n \text{ is convergent if and only if } \int_1^{\infty} f(x)dx \text{ is convergent.}$$

They share the convergent/divergent properties!