LECTURE NO. 8

3.1 Integration by Parts

Wright State University

Integration Formulas

$\int \frac{1}{x} dx$

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $\int \frac{1}{x} dx = \ln|x| + C$
- $\int e^x dx = e^x + C$, $\int a^x dx = \frac{a^x}{\ln a} + C$
- $\int \sin x dx = -\cos x + C$, $\int \cos x dx = \sin x + C$
- $\int \sec^2 x dx = \tan x + C$, $\int \csc^2 x dx = -\cot x + C$
- $\int \sec x \tan x dx = \sec x + C$, $\int \csc x \cot x dx = -\csc x + C$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$, $\int \frac{1}{x^2+1} dx = \tan^{-1} x + C$

Two Integration Techniques

- Use these formulas to integrate "term by term".
- The second technique we learned is "Substitution" which comes from Chain Rule for Differentiation.

composite functions

- Today we study Integration by Parts: it comes from Product Rule for Differentiation.

IBP

product of functions

Integration by Parts: the Rule

- Product Rule: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
- Take Integral of Both Sides: $\int [f(x)g(x)]'dx = \int f'(x)g(x) + f(x)g'(x)dx$
- Recall Integral = Antiderivative, so $f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$
- Solve for $\int f(x)g'(x)dx$: $\int \underbrace{f(x)}_{u}\underbrace{g'(x)dx}_{dv} = \underbrace{f(x)}_{u}\underbrace{g(x)}_{v} - \int \underbrace{f'(x)}_{du}\underbrace{g(x)}_{v}dx$
- Let $U = f(x)$ and $V = g(x)$. Then $dU = f'(x)dx$ and $dV = g'(x)dx$
- Integration by Parts Formula: $\int U \cdot dV = U \cdot V - \int V \cdot dU$

IBP

IBP $\int u\, dv = uv - \int v\, du$

$u = x \quad dv = \sin x\, dx$

Take derivative — Take antiderivative

$du = dx \quad v = -\cos x$

$= x(-\cos x) - \int (-\cos x)\, dx$

$= -x\cos x + \int \cos x\, dx$

$= -x\cos x + \sin x + C$

FINAL ANSWER

what if we choose $u = \sin x$, $dv = x\, dx$?

$u = \sin x \quad dv = x\, dx$

$du = \cos x\, dx \quad v = \dfrac{x^2}{2}$

$\int x \sin x\, dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x\, dx$

The new integral we get is more complicated!

$\sin x, \cos x \longrightarrow dv$

$\int x^2 e^{3x} dx$

IBP $\int u\,dv = uv - \int v\,du$

$e^{kx} \xrightarrow{\text{Antiderivative}} \dfrac{e^{kx}}{k} + C$

$u = x^2 \qquad dv = e^{3x}dx$

$du = 2x\,dx \qquad v = \frac{1}{3}e^{3x}$

$$= \frac{1}{3}x^2e^{3x} - \int \frac{1}{3}e^{3x}\cdot 2x\,dx$$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\underbrace{\int x e^{3x}\,dx}_{\text{IBP again!}}$$

$u = x \qquad dv = e^{3x}dx$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}dx\right]$$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{3}\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right] + C$$

$$= \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$$

FINAL ANSWER

OR

$\int x^5 \ln x dx$

$(\ln x)' = \frac{1}{x}$, do we know $\int \ln x dx$? No, we donot!

IBP $\int u dv = uv - \int v du$

$u = \ln x \quad dv = x^5 dx$

$du = \frac{1}{x} dx \quad v = \frac{x^6}{6}$

$= \frac{1}{6} x^6 \ln x - \int \frac{x^6}{6} \cdot \frac{1}{x} dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^5 dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \cdot \frac{x^6}{6} + C$ FINAL ANSWER

IBP $\ln x \longrightarrow u$

you may practice $\int \ln x dx$

using IBP

$u = \ln x \quad dv = dx$

$\cdots \quad v = x$

$\frac{1}{6} x^6 \ln x - \frac{1}{36} \cdot x^6 + C$

IBP $\int u dv = uv - \int v du$

$u = \tan^{-1} x \quad dv = dx$

$du = \frac{1}{x^2+1} dx \quad v = x$

$= x \tan^{-1} x \Big|_0^1 - \int_0^1 x \cdot \frac{1}{x^2+1} dx$

$= (\tan^{-1} 1 - 0) - \int_0^1 \frac{x}{x^2+1} dx$

$= \frac{\pi}{4} - \int_0^1 \frac{x}{x^2+1} dx = \frac{\pi}{4} - \frac{1}{2} \ln 2$

FINAL ANSWER.

$\int_0^1 \frac{x}{x^2+1} dx$ Substitution!

$u = x^2+1 \quad \frac{du}{dx} = 2x \quad dx = \frac{du}{2x}$

$x: 0 \to 1$

$u = x^2+1: 1 \to 2$

$\int_1^2 \frac{\cancel{x}}{u} \frac{du}{2\cancel{x}} = \frac{1}{2} \ln|u| \Big|_1^2$

$= \frac{1}{2} \ln 2 - \frac{1}{2} \ln 1 = \frac{1}{2} \ln 2$

$(\ln 1 = 0)$

Summery on Integrations by Parts

Summary

- The Formula:

$$\int U \cdot dV = U \cdot V - \int V \cdot dU$$

IBP

- Place $\sin x$, $\cos x$, e^x into the dV part when trying IBP.

- Place $\ln x$, $\tan^{-1}(x)$, $\sin^{-1}(x)$ into U part when trying IBP.

IBP $\int u\,dv = u \cdot v - \int v\,du$

$u = e^x$, $dv = \sin(2x)\,dx$

$du = e^x dx$, $v = -\frac{\cos(2x)}{2}$

$$\int e^x \sin(2x)\,dx = -\frac{1}{2} e^x \cos(2x) - \int -\frac{\cos(2x)}{2} \cdot e^x\,dx$$

$$\int e^x \sin(2x)\,dx = -\frac{1}{2} e^x \cos(2x) + \frac{1}{2} \int e^x \cos(2x)\,dx$$

IBP again

$u = e^x$ $\quad dv = \cos(2x)\,dx$

$du = e^x dx$ $\quad v = \frac{\sin(2x)}{2}$

$$\int e^x \sin(2x)\,dx = -\frac{1}{2}e^x\cos(2x) + \frac{1}{2}\left[\frac{1}{2}e^x\sin(2x) - \int \frac{\sin(2x)}{2}e^x\,dx\right]$$

$$\int e^x \sin(2x)\,dx = -\frac{1}{2}e^x\cos(2x) + \frac{1}{4}e^x\sin(2x) - \frac{1}{4}\int e^x\sin(2x)\,dx$$

$$+\frac{1}{4}\int e^x\sin(2x)\,dx \qquad\qquad +\frac{1}{4}\int e^x\sin(2x)\,dx$$

$$\frac{5}{4}\int e^x\sin(2x)\,dx = -\frac{1}{2}e^x\cos(2x) + \frac{1}{4}e^x\sin(2x)$$

$$\int e^x\sin(2x)\,dx = \frac{4}{5}\left(-\frac{1}{2}e^x\cos(2x) + \frac{1}{4}e^x\sin 2x\right) + C$$

FINAL ANSWER

$$\int e^x\sin(2x)\,dx = -\frac{2}{5}e^x\cos(2x) + \frac{1}{5}e^x\sin(2x) + C$$