LECTURE NO. 10

3.3 Trignometric Substitution

Wright State University

$$\int x\sqrt{x^2+1}\,dx$$

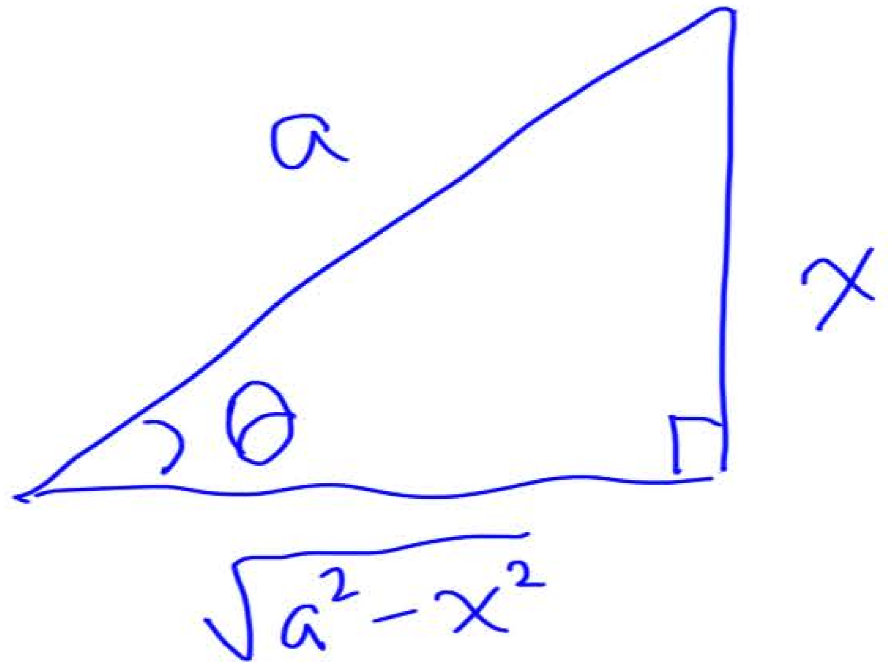
Substitution

$u = x^2+1$

$$\int x\sqrt{x^2+1}\,dx$$

Trig Substitution

$x = \tan\theta$

- In Trig Substitution, we make the original variable x equal some trig function.
- If we see terms like $\sqrt{a^2 - x^2}$, we make $x = a \sin\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2 \cos^2\theta} = a\cos\theta$
- So we can get rid of the square root in the original integral.
- Since $x = a\sin\theta$, $\sin\theta = \frac{x}{a}$. We construct a reference triangle as follows.
- 

$$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$$

$$\tan\theta = \frac{x}{\sqrt{a^2-x^2}}$$

$\int x\sqrt{4-x^2}dx$

we see $\sqrt{4-x^2} = \sqrt{(2)^2 - x^2}$: $x = 2\sin\theta$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

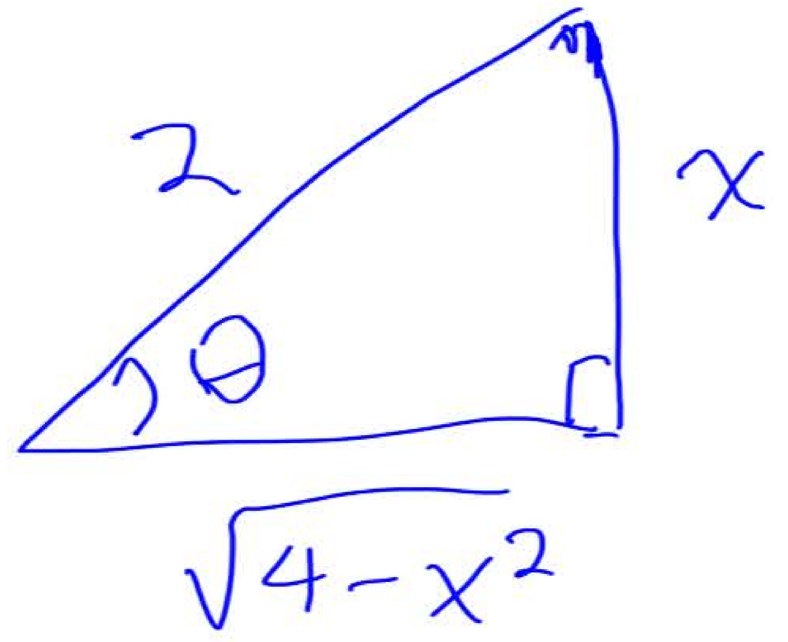
$\frac{dx}{d\theta} = 2\cos\theta$ $dx = 2\cos\theta\, d\theta$

$$\int \underbrace{2\sin\theta}_{x} \underbrace{\sqrt{4-(2\sin\theta)^2}}_{\sqrt{4-x^2}} \underbrace{2\cos\theta\, d\theta}_{dx}$$

$$= \int 2\sin\theta \sqrt{4(1-\sin^2\theta)}\; 2\cos\theta\, d\theta = \int 2\sin\theta \sqrt{4\cos^2\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int 2\sin\theta \cdot 2\cos\theta \cdot 2\cos\theta\, d\theta = \underbrace{8\int \sin\theta \cos^2\theta\, d\theta}_{\text{Trig Integral}}$$

$x = 2\sin\theta \longrightarrow \sin\theta = \frac{x}{2}$



reference triangle

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$

$$8\int \sin\theta \cos^2\theta \, d\theta$$

$u = \cos\theta \qquad \frac{du}{d\theta} = -\sin\theta \qquad d\theta = -\frac{du}{\sin\theta}$

$$8\int \cancel{\sin\theta}\, u^2 \left(-\frac{du}{\cancel{\sin\theta}}\right)$$

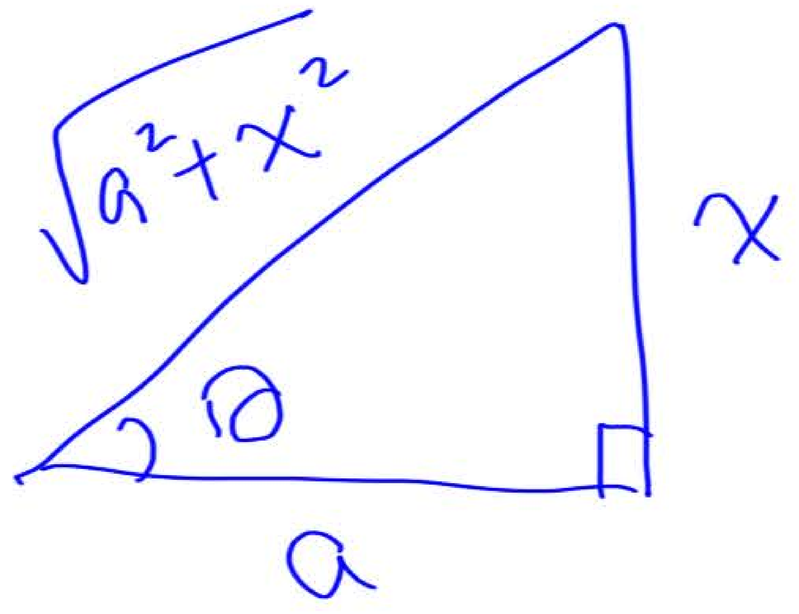
$$-8\int u^2\, du = -8 \cdot \frac{u^3}{3} + C$$

$$= -\frac{8}{3}\cos^3\theta + C = -\frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C$$

$$= -\frac{1}{3}\left(\sqrt{4-x^2}\right)^3 + C$$

FINAL ANSWER

Integral involving $\sqrt{a^2 + x^2}$

- In Trig Substitution, we make the original variable x equal some trig function.
- For $\sqrt{a^2 + x^2}$, we make $x = a\tan\theta$, where $\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
- $\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1 + \tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$
- So we can get rid of the square root in the original integral.
- Since $x = a\tan\theta$, $\tan\theta = \frac{x}{a}$. We construct a reference triangle as follows.
- 

$\sin\theta = \frac{opp}{hyp} = \frac{x}{\sqrt{a^2+x^2}}$

$\cos\theta = \frac{adj}{hyp} = \frac{a}{\sqrt{a^2+x^2}}$

$\csc\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta}$

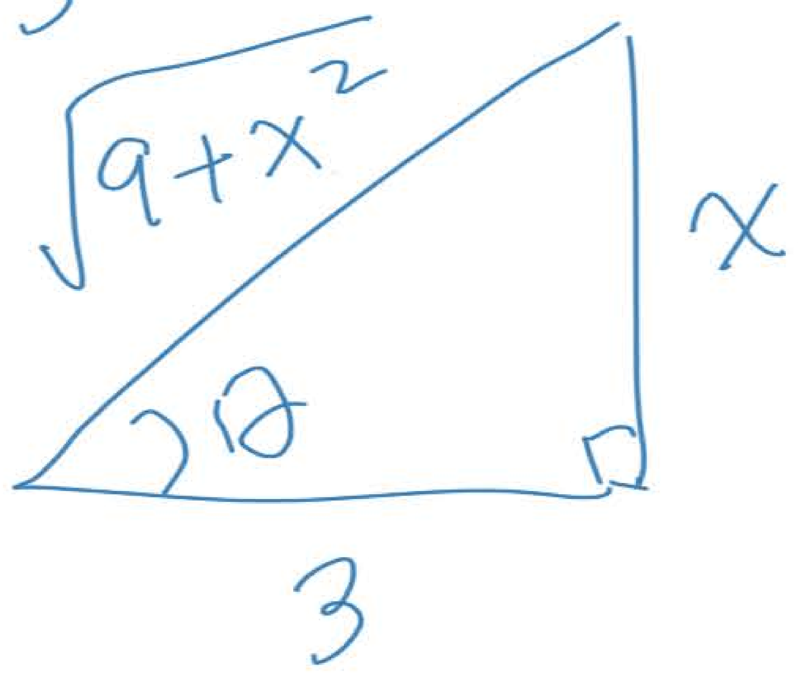
$\int \frac{1}{\sqrt{9+x^2}}dx$

$$\sqrt{9+x^2} = \sqrt{3^2+x^2} \therefore x = 3\tan\theta \qquad \frac{dx}{d\theta} = 3\sec^2\theta \qquad dx = 3\sec^2\theta\, d\theta$$

$$\int \frac{1}{\sqrt{9+9\tan^2\theta}}\, 3\sec^2\theta\, d\theta = \int \frac{1}{\sqrt{9(\underbrace{1+\tan^2\theta}_{\sec^2\theta})}} \cdot 3\sec^2\theta\, d\theta = \int \frac{1}{\sqrt{9\sec^2\theta}} \cdot 3\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\cancel{3}\cancel{\sec\theta}} \cdot \cancel{3}\overset{\sec\theta}{\cancel{\sec^2\theta}}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\tan\theta = \frac{x}{3}$$



$$\sec\theta = \frac{\sqrt{9+x^2}}{3}$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

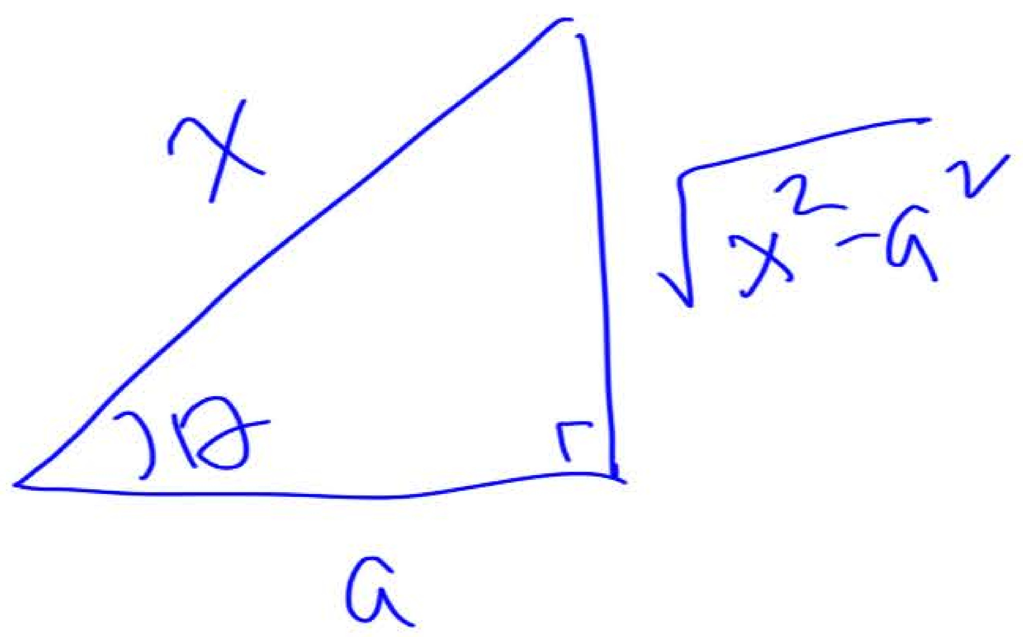
FINAL ANSWER.

Integrals involving $\sqrt{x^2 - a^2}$

- In Trig Substitution, we make the original variable x equal some trig function.
- For $\sqrt{x^2 - a^2}$, we make $x = a\sec\theta$, where $\theta \neq \frac{\pi}{2}$ is between 0 and π.
- $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$
- So we can get rid of the square root in the original integral.
- Since $x = a\sec\theta$, $\sec\theta = \frac{x}{a}$. We construct a reference triangle as follows.

$\sec\theta = \frac{1}{\cos\theta}$

$\cos\theta = \frac{a}{x}$



$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

$\sin\theta = \frac{\sqrt{x^2-a^2}}{x}$

$\int_3^6 \frac{1}{\sqrt{x^2-4}}dx$

$$\int \frac{1}{\sqrt{x^2-4}}dx \qquad \sqrt{x^2-4} = \sqrt{x^2-(2)^2} \qquad x = 2\sec\theta \qquad \frac{dx}{d\theta} = 2\sec\theta\tan\theta \qquad dx = 2\sec\theta\tan\theta\, d\theta$$

$$\int \frac{1}{\sqrt{4\sec^2\theta-4}}\, 2\sec\theta\tan\theta\, d\theta = \int \frac{1}{\sqrt{4(\sec^2\theta-1)}}\, 2\sec\theta\tan\theta\, d\theta = \int \frac{1}{\sqrt{4\tan^2\theta}}\, 2\sec\theta\tan\theta\, d\theta$$

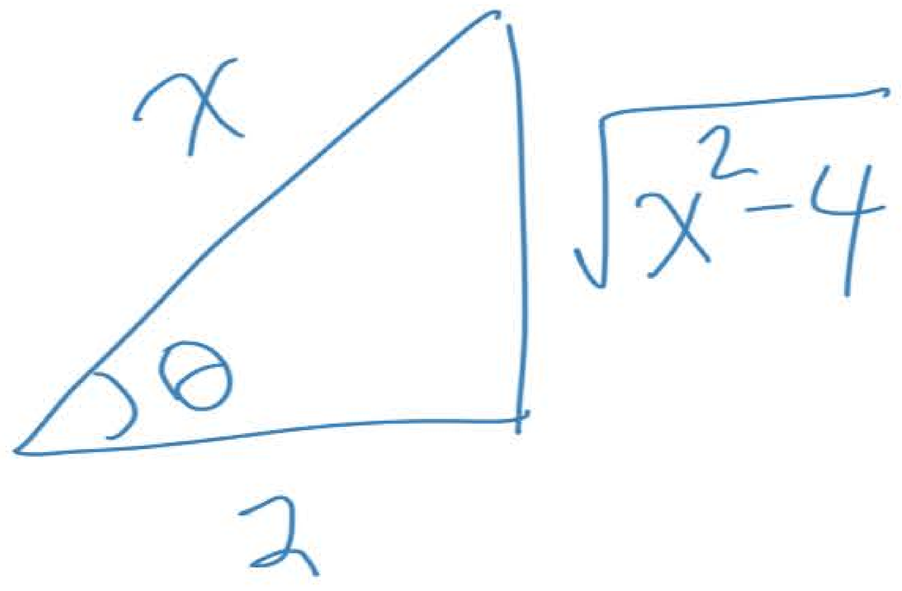
$$= \int \frac{1}{\cancel{2}\cancel{\tan\theta}}\, \cancel{2}\sec\theta\cancel{\tan\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta| + C$$

$$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

$x = 2\sec\theta$

$\sec\theta = \frac{x}{2}$

$\cos\theta = \frac{2}{x}$



$$\boxed{\tan\theta = \frac{\sqrt{x^2-4}}{2}}$$

$$\int_3^6 \frac{1}{\sqrt{x^2-4}}\,dx \xrightarrow[\text{Substitution}]{\text{trig}} \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| \Bigg|_3^6$$

$$\ln\left|3 + \frac{\sqrt{32}}{2}\right| - \ln\left|\frac{3}{2} + \frac{\sqrt{5}}{2}\right|$$

FINAL ANSWER