LECTURE NO. 9

3.2 Trigonometric Integrals

Wright State University

Substitutions

$u = \cos x \quad \frac{du}{dx} = -\sin x \quad dx = -\frac{du}{\sin x}$

$\int \sin^{\cancel{3}\,2} x \; u^2 \left(-\frac{du}{\cancel{\sin x}}\right)$

$-\int \sin^2 x \; u^2 \, du$

Recall that $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x = 1 - u^2$

$-\int (1-u^2) u^2 \, du$

$-\int u^2 - u^4 \, du$

$-\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$

$-\left(\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right) + C$. FINAL ANSWER!

why $u = \sin x$ does not work?

$u = \sin x \quad \frac{du}{dx} = \cos x \quad dx = \frac{du}{\cos x}$

$\int u^3 \cos^2 x \frac{du}{\cos x} \qquad \cos^2 x = 1 - \sin^2 x = 1 - u^2$

$\int u^3 \cos x \, du \qquad \cos x = \pm\sqrt{1-u^2}$

must change into u-terms.

Too Complicated, avoid!

$\int \sin^5 x \cos^3 x dx$

Substitution $u = \sin x$ $\quad \frac{du}{dx} = \cos x \quad dx = \frac{du}{\cos x}$

$$\int u^5 \cos^3 x \frac{du}{\cos x} = \int u^5 \cos^2 x \, du$$

$\cos^2 x = 1 - \sin^2 x = 1 - u^2$

$$= \int u^5 (1 - u^2) \, du$$

$$= \int u^5 - u^7 \, du = \frac{u^6}{6} - \frac{u^8}{8} + C$$

$$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$$

FINAL ANSWER.

If both sinx and cosx are raised to an even power, we will use double-angle identity

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\frac{1-\cos(4x)}{2}$$

double-angle

$$\int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} dx = \frac{1}{4}\int 1-\cos^2(2x)\, dx = \frac{1}{4}\int \sin^2(2x)\, dx$$

$$= \frac{1}{4}\int \frac{1-\cos(4x)}{2} dx = \frac{1}{8}\int 1-\cos(4x)\, dx = \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C$$

FINAL ANSWER.

Summary on $\int \sin^m x \cos^n x dx$

- If $\sin x$ is raised to an odd power, then $u = \cos x$ would work.
- If $\cos x$ is raised to an odd power, then $u = \sin x$ would work.
- If both $\sin x$ and $\cos x$ are raised to an even power, then use the double-angle identity:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Integrals involving $\tan x$ and $\sec x$

- $$\int \sec^2 x dx = \tan x + C$$

- $$\int \sec x \tan x dx = \sec x + C$$

- $$\int \tan x dx = \ln|\sec x| + C$$

  $\int \frac{\sin x}{\cos x} dx \quad u = \cos x$

- $$\int \sec x dx = \ln|\sec x + \tan x| + C$$

  $\rightarrow$ will be given if needed.

$\int \tan^3 x \sec^4 x dx$

Substitution $u = \tan x$ $\quad \frac{du}{dx} = \sec^2 x \quad dx = \frac{du}{\sec^2 x}$

$$\int u^3 \sec^4 x \frac{du}{\sec^2 x}$$

$$= \int u^3 \sec^2 x \, du$$

Recall: $\sec^2 x = \tan^2 x + 1$

$\sec^2 x = u^2 + 1$

$$= \int u^3 (u^2+1) \, du$$

$$= \int u^5 + u^3 \, du$$

$$= \frac{u^6}{6} + \frac{u^4}{4} + C = \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C$$

FINAL ANSWER.

practice:

Try to use $u = \sec x$!

$(\sec x)' = \sec x \tan x$

$\int \tan^3 x \sec^5 x dx$

$u = \sec x \qquad \frac{du}{dx} = \sec x \tan x \qquad dx = \frac{du}{\sec x \tan x}$

$$\int \tan^{\cancel{3}\,2} x \; u^5 \frac{du}{\underbrace{\sec x}_{u} \cancel{\tan x}} = \int \tan^2 x \; u^4 \, du$$

$$\begin{aligned} \tan^2 x &= \sec^2 x - 1 \\ \tan^2 x &= u^2 - 1 \end{aligned}$$

$$= \int (u^2 - 1)\, u^4 \, du$$

$$= \int u^6 - u^4 \, du = \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \boxed{\frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C}$$

FINAL ANSWER

$$\int \tan^4 x\, dx = \int \tan^2 x \underbrace{\tan^2 x}_{\sec^2 x - 1} dx = \int \tan^2 x (\sec^2 x - 1)\, dx$$

$\Leftarrow$ separate a $\tan^2 x$

$$= \int \tan^2 x \sec^2 x - \tan^2 x\, dx$$

$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$u = \tan x$

$$\int \tan^2 x\, dx$$

$$= \int \sec^2 x - 1\, dx$$

$$= \tan x - x + C$$

$$\frac{1}{3}\tan^3 x - (\tan x - x) + C$$

$\nwarrow$ FINAL ANSWER

$\int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x \, dx}_{dv}$ IBP $\int u\,dv = uv - \int v\,du$ $\int \sec x\,dx = \ln|\sec x + \tan x|$

$u = \sec x \quad dv = \sec^2 x\,dx$

$du = \sec x \tan x\,dx \quad v = \tan x$

$\sec^2 x - 1$

$= \sec x \tan x - \int \tan x \cdot \sec x \tan x\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$

$\int \sec^3 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx = \sec x \cdot \tan x + \ln|\sec x + \tan x|$

$\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$ FINAL ANSWER!

Summary on $\int \tan^m x \sec^n x dx$

- If $\sec x$ is raised to an even power, then $u = \tan x$ would work.

tanx is

- If both $\tan x$ and $\sec x$ are raised to an odd power, then $u = \sec x$ would work.
- For $\int \tan^m x dx$, replace a $tan^2 x$ by $\sec^2 x - 1$.
- For $\int \sec^n x dx$, use Integration by Parts.

Separate a $\sec^2 x$ out, then use IBP!

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