LECTURE NO. 24

7.2 Calculus of Parametric Curves

Wright State University

Derivative of parametric equations

- If $y = f(x)$, then it is easy to find $\frac{dy}{dx}$, the derivative of y with respect to x.
- Given $x = x(t)$, $y = y(t)$, $a \leq t \leq b$. How do we find $\frac{dy}{dx}$?
- For parametric equations, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}$$

Find $\frac{dy}{dx}$ if $x = 5\cos t,\ y = 5\sin t,\ 0 \le t \le 2\pi$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{(5\sin t)'}{(5\cos t)'} = \frac{5\cos t}{-5\sin t} = -\frac{\cos t}{\sin t} = -\cot(t)$$

Find an equation of the tangent line to the curve $x = 2t + 1$, $y = t^3 - 3t + 5$ at the point $t = 1$.

Equation of tangent < point, slope

point $x = 2 \cdot 1 + 1 = 3$

$y = 1 - 3 + 5 = 3$

$(3, 3)$

$$\text{slope} = \left.\frac{dy}{dx}\right|_{t=1} = \frac{3(1)^2 - 3}{2} = 0$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 3}{2}$$

point-slope equation: $y - 3 = 0(x - 3)$

$y - 3 = 0$

$y = 3$ (a horizontal tangent line)

Arc Length of Parametric Curves

- Recall that, for $y = f(x)$, $a \leq x \leq b$, the arc length is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- For a parametric curve $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, the arc length is given by a similar integral:

$$\text{Arc Length} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Circumference of a circle of radius r.

$x = r\cos t,\ y = r\sin t,\ 0 \le t \le 2\pi.$ $\longrightarrow 2\pi r$

$$\text{Arc Length} = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$\sin^2 t + \cos^2 t = 1$

$$= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt = \int_0^{2\pi} r\sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} r\, dt = rt\Big|_0^{2\pi} = 2\pi r$$

Find the arc length of the curve $x = 3t^2$, $y = 2t^3$, $1 \le t \le 3$.

$$\text{Arc length} = \int_1^3 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \int_1^3 \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_1^3 \sqrt{36t^2 + 36t^4}\, dt$$

$$= \int_1^3 \sqrt{36t^2(1+t^2)}\, dt = \int_1^3 6t\sqrt{1+t^2}\, dt$$

Substitution

$$u = 1+t^2 \qquad \frac{du}{dt} = 2t \qquad dt = \frac{du}{2t} \qquad t: 1 \to 3 \quad u = 1+t^2: 2 \to 10$$

$$\int_2^{10} 6t\sqrt{u}\,\frac{du}{2t} = \int_2^{10} 3u^{\frac{1}{2}}\, du = 3 \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_2^{10}$$

$$= 2(10)^{\frac{3}{2}} - 2(2)^{\frac{3}{2}}$$

Surface Area of Revolution

- Recall that, for $y = f(x)$, $a \leq x \leq b$, the area of the surface generated by revolving around x-axis is

$$\text{Surface Area of Revolution} = \int_a^b 2\pi \underbrace{f(x)}_{y} \sqrt{1 + [f'(x)]^2}\, dx$$

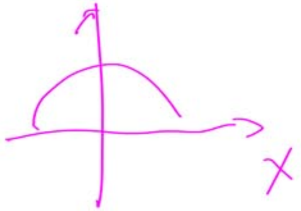
- For a parametric curve $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, the area of the surface generated by revolving around x-axis is

$$\text{Surface Area of Revolution} = \int_a^b 2\pi y(t) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Surface Area of a Sphere of Radius r

$x = r\cos t,\ y = r\sin t,\ 0 \le t \le \pi$

$4\pi r^2$ (Volume $= \frac{4}{3}\pi r^3$)



$$\text{Surface Area} = \int_0^{\pi} 2\pi\, y(t)\sqrt{(x'(t))^2 + [y'(t)]^2}\, dt$$

$$= \int_0^{\pi} 2\pi\, r\sin t\sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt$$

$\sin^2 t + \cos^2 t = 1$

$\cos\pi = -1$

$\cos 0 = 1$

$$= \int_0^{\pi} 2\pi r\sin t \cdot r\, dt$$

$$= \int_0^{\pi} 2\pi r^2 \sin t\, dt = 2\pi r^2 \cdot (-\cos t)\Big|_0^{\pi}$$

$$= 2\pi r^2(-\cos\pi) - 2\pi r^2(-\cos 0)$$

$$= 2\pi r^2 + 2\pi r^2 = 4\pi r^2$$