MTH 2530, MT2 Review, solutions

#1
$$A=\begin{bmatrix}1&4&0&2&-1\\3&12&1&5&5\\2&8&1&3&2\\5&20&2&8&8\end{bmatrix}\begin{matrix}\\-3R_1\\-2R_1\\-5R_1\end{matrix}\rightarrow\begin{bmatrix}1&4&0&2&-1\\0&0&1&-1&8\\0&0&1&-1&4\\0&0&2&-2&13\end{bmatrix}\begin{matrix}\\-R_3\\ \\-2R_3\end{matrix}\rightarrow\begin{bmatrix}1&4&0&2&-1\\0&0&1&-1&4\\0&0&0&0&4\\0&0&0&0&5\end{bmatrix}\begin{matrix}\\ \\ \frac{1}{4}\\ \frac{1}{5}-R_3\end{matrix}$$

$$\rightarrow\begin{array}{c}\begin{bmatrix}1&4&0&2&-1\\0&0&1&-1&4\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\begin{matrix}\leftarrow R_1\\\leftarrow R_2\\\leftarrow R_3\\ \end{matrix}\\ \begin{matrix}\uparrow&\uparrow&\uparrow&\uparrow&\uparrow\\x_1&x_2&x_3&x_4&x_5\end{matrix}\end{array}$$

(a) $B=\{R_1, R_2, R_3\}$; $\dim(\text{Row}A)=3$

(b) $\because$ The Columns #1, 3, 5 have pivots $\therefore$ $\{\vec{a}_1, \vec{a}_3, \vec{a}_5\}$ is linearly independent and it's a basis of ColA; $\dim \text{Col}A=3$.

For $A\vec{x}=\vec{0}$,

(c) let $\vec{X}=(x_1, x_2, x_3, x_4, x_5)$, and let x_2, x_4 be free then: $x_5=0$,

$x_3-x_4+4*0=0 \quad \therefore x_3=x_4$;

$x_1+4x_2+0\cdot x_3+2x_4-1*0 = x_1+4x_2+2x_4=0 \quad \therefore x_1=-4x_2-2x_4$

$$\therefore \vec{X}=\begin{bmatrix}-4x_2-2x_4\\x_2\\x_4\\x_4\\0\end{bmatrix}=x_2\begin{bmatrix}-4\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}-2\\0\\1\\1\\0\end{bmatrix}$$

$\therefore \{(-4,1,0,0,0), (-2,0,1,1,0)\}$ is a basis of Nul A; Dim NulA = 2.

#2 (a)
$$|A-\lambda I|=\begin{vmatrix}2-\lambda&0&1\\0&1-\lambda&0\\1&0&2-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}2-\lambda&1\\1&2-\lambda\end{vmatrix}=(1-\lambda)\left[(\lambda-2)^2-1^2\right]=(1-\lambda)(\lambda-2-1)(\lambda-2+1)$$
$$=-(\lambda-1)^2(\lambda-3)=0 \quad \therefore \lambda_{1,2}=1, \lambda_3=3$$

For $\lambda_{1,2}=1$, Let $\vec{V}=(r,s,t)$, then:

$$\left[A-I\,|\,\vec{0}\right]=\left[\begin{array}{ccc|c}1&0&1&0\\0&0&0&0\\1&0&1&0\end{array}\right]\begin{matrix}\\ \\-R_1\end{matrix}\rightarrow\left[\begin{array}{ccc|c}1&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right] \quad \therefore r+t=0 \therefore r=-t.$$

$$\therefore \vec{V}=\begin{pmatrix}-t\\s\\t\end{pmatrix}=t\begin{pmatrix}-1\\0\\1\end{pmatrix}+s\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \therefore \vec{V}_1=\begin{pmatrix}-1\\0\\1\end{pmatrix}, \vec{V}_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

For $\lambda_3=3$,

$$\left[A-3I\,|\,\vec{0}\right]=\left[\begin{array}{ccc|c}-1&0&1&0\\0&-2&0&0\\1&0&-1&0\end{array}\right]\begin{matrix}\\ /(-2)\\+R_1\end{matrix}\Rightarrow\left[\begin{array}{ccc|c}-1&0&1&0\\0&1&0&0\\0&0&0&0\end{array}\right] \quad \therefore -r+t=0 \therefore r=t, \; s=0$$

$$\therefore \vec{V}=\begin{pmatrix}t\\0\\t\end{pmatrix}=t\begin{pmatrix}1\\0\\1\end{pmatrix} \quad \therefore \vec{V}_3=\begin{pmatrix}1\\0\\1\end{pmatrix} \quad \therefore P=\begin{bmatrix}-1&0&1\\0&1&0\\1&0&1\end{bmatrix}, D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&3\end{bmatrix}$$

(b) $A\vec{u}=\vec{0}$ (c) $E_{\lambda=0}=\text{Nul}A$.

#3. (a). $\vec{u}_1 = \vec{a}_1 = \begin{pmatrix}3\\1\\-1\\3\end{pmatrix}$; $\hat{a}_2 = \left(\frac{\vec{a}_2\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 = \frac{3*(-5)+1-5-21}{9+1+1+9}\vec{u}_1 = \frac{-14-26}{20}\vec{u}_1 = -2\vec{u}_1$

$\therefore \vec{a}_2 - \hat{a}_2 = \begin{pmatrix}-5\\1\\5\\-7\end{pmatrix} + 2\vec{u}_1 = \begin{pmatrix}-5+6\\1+2\\5-2\\-7+6\end{pmatrix} = \begin{pmatrix}1\\3\\3\\-1\end{pmatrix}$ $\therefore \vec{u}_2 = \begin{pmatrix}1\\3\\3\\-1\end{pmatrix}$;

$\hat{a}_3 = \left(\frac{\vec{a}_3\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{a}_3\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2 = \frac{3+1+2+24}{20}\vec{u}_1 + \frac{1+3-6-8}{20}\vec{u}_2 = \frac{3}{2}\vec{u}_1 - \frac{1}{2}\vec{u}_2$

$= \frac{1}{2}*(3\vec{u}_1 - \vec{u}_2) = \frac{1}{2}\begin{pmatrix}9-1\\3-3\\-3-3\\9+1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}8\\0\\-6\\10\end{pmatrix} = \begin{pmatrix}4\\0\\-3\\5\end{pmatrix}$

$\therefore \vec{a}_3 - \hat{a}_3 = \begin{pmatrix}1-4\\1-0\\-2+3\\8-5\end{pmatrix} = \begin{pmatrix}-3\\1\\1\\3\end{pmatrix}$ $\therefore \vec{u}_3 = \begin{pmatrix}-3\\1\\1\\3\end{pmatrix}$

$\therefore \left\{\vec{u}_1 = \begin{pmatrix}3\\1\\-1\\3\end{pmatrix}, \vec{u}_2 = \begin{pmatrix}1\\3\\3\\-1\end{pmatrix}, \vec{u}_3 = \begin{pmatrix}-3\\1\\1\\3\end{pmatrix}\right\}$ is an orthogonal basis for ColA.

(b) $\hat{b} = \left(\frac{\vec{b}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\right)\vec{u}_1 + \left(\frac{\vec{b}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\right)\vec{u}_2 + \left(\frac{\vec{b}\cdot\vec{u}_3}{\vec{u}_3\cdot\vec{u}_3}\right)\vec{u}_3 = \left(\frac{3-1}{20}\right)\vec{u}_1 + \left(\frac{1+3}{20}\right)\vec{u}_2 + \left(\frac{-3+1}{20}\right)\vec{u}_3$

$= \frac{1}{10}\vec{u}_1 + \frac{2}{10}\vec{u}_2 - \frac{1}{10}\vec{u}_3 = \frac{1}{10}*(\vec{u}_1 + 2\vec{u}_2 - \vec{u}_3)$

$= \frac{1}{10}\begin{pmatrix}3+2+3\\1+6-1\\-1+6-1\\3-2-3\end{pmatrix} = \frac{1}{10}\begin{pmatrix}8\\6\\4\\-2\end{pmatrix} = \begin{pmatrix}0.8\\0.6\\0.4\\-0.2\end{pmatrix}$

$\therefore \vec{b} - \hat{b} = \begin{pmatrix}1-0.8\\0-0.6\\1-0.4\\0+0.2\end{pmatrix} = \begin{pmatrix}0.2\\-0.6\\0.6\\0.2\end{pmatrix} = 0.2\begin{pmatrix}1\\-3\\3\\1\end{pmatrix}$ $\therefore \|\vec{b} - \hat{b}\| = 0.2\sqrt{20} = 0.4\sqrt{5}$ is the distance from $\vec{b}$ to ColA.

#4. $A^TA = \begin{bmatrix}1&1&1\\3&-1&1\end{bmatrix}\begin{bmatrix}1&3\\1&-1\\1&1\end{bmatrix} = \begin{bmatrix}3&3\\3&11\end{bmatrix}$; $A^T\vec{b} = \begin{bmatrix}1&1&1\\3&-1&1\end{bmatrix}\begin{bmatrix}5\\1\\0\end{bmatrix} = \begin{bmatrix}6\\14\end{bmatrix}$

$\therefore [A^TA \mid A^T\vec{b}] = \left[\begin{array}{cc|c}3&3&6\\3&11&14\end{array}\right] \begin{array}{l}\frac{1}{3}\\-R_1\end{array} \rightarrow \left[\begin{array}{cc|c}1&1&2\\0&8&8\end{array}\right]$ $\therefore x_2 = 1,\ x_1 + 1 = 2$ $\therefore x_1 = 1$ $\therefore \hat{x} = \begin{pmatrix}1\\1\end{pmatrix}$

$\therefore A\hat{x} = \hat{b} = \begin{bmatrix}1&3\\1&-1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}4\\0\\2\end{bmatrix}$ $\therefore \vec{b} - \hat{b} = \begin{pmatrix}5-4\\1-0\\0-2\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$

$\therefore \|\vec{b} - \hat{b}\| = \sqrt{1+1+4} = \sqrt{6}$ is the least-squares error.