Martingales

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1 Introductory Example

(a) Assume we are playing a game where we toss a coin and each time we get either 2 if we get head or -1 if we get tails. We can play as many rounds as we want and our amount of money is added up.

- How much money would you be willing to pay to play this game?
- Why?

(b) Assume we look at a stock S that starts off with initial value is $S_0 = 4$. It goes up with probability $1/2$ to $S_1(H) = 8$ and goes down with probability $1/2$ to $S_1(T) = 2$. Assume we need to construct a call option with strike price K and maturity 1 period (so after 1 tick the holder of the call option has the right to buy the stock for K).

- What would be the fair K?
- How much would be a fair price for such a call option?

2 Martingales

What does fair mean in a sense of option pricing? If we want to calculate this for more complex scenarios, we will need to formalise what fair means. This is where martingales comes in. First we define what we understand with collecting information.

Definition 2.1. A sequence $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ of σ-algebras with $\mathcal{F}_n \subset \mathcal{A}$ and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}_0$ is called filtration.

A filtration is just a bunch of σ-algebras that are growing. A σ-algebra $\mathcal{F}_n$) can be seen as the knowledge we know at time n and we learn more with every time step. Note that for a random variable X we write $\sigma(X) := \{X^{-1}(B) : B \in \mathcal{B}_\mathbb{R}\}$ for the σ-algebra that is generated by X. We write $X \in \mathcal{F}$ if and only if $\sigma(X) \subset \mathcal{F}$. We can now define what a martingale is.

Definition 2.2. Let $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be a filtration. A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables is called martingale if for all $n \in \mathbb{N}_0$ it holds that:

(M1) $\mathbb{E}[|X_n|] < \infty$

(M2) $X_n \in \mathcal{F}_n$

(M3) $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$.

The same definition can be done in continuous time.

Definition 2.3. Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be a filtration. A stochastic process X_t is called martingale if for all $t \in [0,\infty)$ it holds that:

(M1) $\mathbb{E}[|X_t|] < \infty$

(M2) $X_t \in \mathcal{F}_t$

(M3) $\mathbb{E}[X_{t+s}|\mathcal{F}_t] = X_t$ for all $s > 0$.

We will use the discrete variant here as it is easier to handle intuitively. A martingale describes what a fair process means in mathematical term. The conditions (M1) and (M2) just mean that the process is not crazy (so we can measure it) and that we observe it. The condition (M3) defines what fair means. It says that knowing the process in the past, moving forward we will expect to have the same capital as we have now. So, we won't win or lose any capital on average. This formalisation seems weird (as one would expect the expectation to be 0 or something for every step) and in the discrete version it is only for the next step, but one can indeed show that

Proposition 2.4. Let $(X_n)_{n\in\mathbb{N}}$ be a martingale with regards to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$. Then for all $n, m \in \mathbb{N}$ it holds that

$$\mathbb{E}[X_{n+m}|\mathcal{F}_n] = X_n,$$

For $n = 0$, we have $\mathbb{E}[X_0] = \mathbb{E}[X_m]$.

That shows that being a martingale means more than our intuitive understanding of being fair. It is hard to find real world examples where the expectations are all the same but it is not a martingale, though. A heuristic way to understand why the definition of the martingale for being fair is more precise is that it takes into the account the whole progression of the process, rather than every individual step.

Now that we understand what being a martingale means it makes a lot of sense to try and formulate stock movements as stochstic processes and construct them so that they are martingales as this makes them fair. Being fair in the context of finance means there is no arbitrage. So, it is very key to construct pricing models in such a way that they are martingales as this insures that we cannot be taken advantage of.

3 Stopping Time

There are a couple of other key concepts what we can do with martingales. An important one is stopping time.

Definition 3.1. Let $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be a filtration. A measurable $\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ is called stopping time (with regards to $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$), if

$$\text{For all } n \in \mathbb{N}_0 : \{\omega \in \Omega : \tau(\omega) = n\} \in \mathcal{F}_n.$$

That looks complicated but the interpretation is actually quite easy: At every step n you know whether you have to stop or not (this is indicated by being in our filtration). This helps to make statements about strategies and certian models as barrier option. We won't go into much details here, but two statements are helpful when working with stopping times.

Theorem 3.2. Let τ be a stopping time and $(X_n)_{n\in\mathbb{N}}$ a martingale. If one of the following statements is true

(i) There exists an $n_0 \in \mathbb{N}$ such that $\tau < n_0$ a.s. (a.s = almost surely; with probability 1)

(ii) $\mathbb{E}[\tau] < \infty$ and there exists an $c > 0$ such that for all $n \in \mathbb{N} : \mathbb{E}[|X_n - X_{n-1}||\mathcal{F}_{n-1}] \leq c$

(iii) $\mathbb{P}(\tau < \infty) = 1$, $\mathbb{E}[|X_\tau|] < \infty$ and $\mathbb{E}[X_n \mathbb{1}_{\tau>n}] \to 0$.

Then it holds that

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$$

This is the optimal sampling theorem. It states that if the stopping time is in some way reached (is finite) the expected returns will still be the same. That means if the game is fair (a martingale) you cannot trick the system. Every strategy (stopping time) you will employ still have the same expected returns.
Another one is the Wald-equality.

Theorem 3.3. Let Y_i be i.i.d. random variables with $\mathbb{E}[|Y_i|] < \infty$ and τ a stopping time with regards to $(\mathcal{F}_n)_{n\in\mathbb{N}_0} = \sigma(Y_1, ..., Y_n)$ and $\mathbb{E}[\tau] < \infty$. Then

$$\mathbb{E}\Big[\sum_{i=1}^{\tau} Y_i\Big] = \mathbb{E}[Y_1]\mathbb{E}[\tau].$$