Ordinary Differential Equations (ODEs)

May 20, 2022

ODEs is a huge topic. Luckily, we don't need to know a lot. We will not care about existence or uniqueness of solutions but rather focus on a couple very important types of ODEs that have easy analytical solutions. We will cover:

- ODEs with seperated variables
- Linear ODEs
- Homogeneous ODEs
- Linear ODEs higher order with constant coefficients

1 ODEs with seperated variables

2 Linear ODEs

3 Homogeneous ODEs

4 Higher order linear ODEs with constant coefficients

Linear ODEs of higher order can be a pain. A linear ODE of order $n \in \mathbb{N}$ looks like that

$$g(x) = \sum_{i=0}^{n} y^{(i)} f_i(x), \tag{1}$$

where y is the function of interest and $f_i(x), g(x)$ are arbitrary functions. They are solved by reverting them to a system of equations of order 1 that is then solved by similar tricks as before and some linear algebra. We will not focus on that here. Instead if the coefficients $f_i(x) = a_i \in \mathbb{R}$ are constants, there is a simple solution for (1). We will look at two cases:

- $g(x) = 0$ (homogeneous)
- $g(x) \neq 0$ (non-homogeneous)

4.1 Homogeneous Case

In case $g(x) = 0$, we can make the Ansatz $y(x) = e^{\lambda x}$. If we plug that in into (1) and divide by $e^{\lambda x}$, we get

$$0 = \sum_{i=0}^{n} c_i \lambda^i, \tag{2}$$

since $\frac{\partial}{\partial x} e^{\lambda x} = \lambda e^{\lambda x}$. It can be shown that one can formulate a solution for y with the roots $\lambda_1, ..., \lambda_n$ (From linear algebra we know that such a polynomial function will have n roots).
A solution can then be obtained in the following way. We define building blocks $b_j(x)$ of the solution by

$$b_j(x) = e^{x\lambda_j}$$

if λ_j is a simple root of (2) and

$$b_j(x) = e^{x\lambda_j}, b_{j+1}(x) = xe^{x\lambda_j}, ..., b_{j+m-1}(x) = x^{m-1}e^{x\lambda_j}$$

if λ_j is a multiple root of order m of (2). A solution of (1) is then formed by multiplying these building blocks by any constant C_j and adding them up

$$y(x) = \sum_{i=1}^{n} C_i b_i(x),$$

where $b_i(x)$ are the building blocks and C_i are constants. Note that I was a bit sloppy with the notation here. If λ_j is a multiple root of order k then we have k building blocks for it. We just multiply all building blocks by a constant and add them up.

That is nice! The only tricky thing is to find the roots of (2). This is a thing we need to practice here.

Trick!

If $n = 2$ this is an easy task. If $n > 2$ the trick is to guess the root and use polynomial division. To guess the root look at all real divisors of c_0 as all roots must be factors of it.

Examples

1) Simple roots with $n = 3$
Consider

$$y''' - 2y'' + 2y' - y = 0.$$

This translates to

$$\lambda^3 - 2\lambda^2 + 2\lambda - 1 = 0. \tag{3}$$

Here $c_0 = 1$. So, we just have to check whether 1 or -1 is a root. Indeed, $\lambda_1 = 1$ works (verify by plugging in). That means that $(\lambda - 1)$ must be a factor of $(\lambda^3 - 2\lambda^2 + 2\lambda - 1)$. Doing the polynomial division we get

$$(\lambda^3 - 2\lambda^2 + 2\lambda - 1) : (\lambda - 1) = \lambda^2 - \lambda + 1.$$

Solving the quadratic equation

$$\lambda^2 - \lambda + 1 = 0,$$

we get $\lambda_2 = \frac{1+i\sqrt{3}}{2}$ and $\lambda_3 = \frac{1-i\sqrt{3}}{2}$. A solution is then

$$y(x) = Ae^x + Be^{\frac{1+i\sqrt{3}}{2}x} + Ce^{\frac{1-i\sqrt{3}}{2}x},$$

where $A, B, C \in \mathbb{R}$ are arbitrary constants.

2) Simple roots with $n = 4$
Consider

$$y'''' + y' = 0.$$

This translates to

$$\lambda^4 + \lambda = 0. \tag{4}$$

By factoring out λ we get

$$\lambda^3 + 1 = 0, \tag{5}$$

so $\lambda_1 = 0$. Again we have $c_0 = 1$. This time $\lambda_2 = -1$. This gives again

$$(\lambda^3 + 1) : (\lambda + 1) = \lambda^2 - \lambda + 1.$$

So, $\lambda_3 = \frac{1+i\sqrt{3}}{2}$ and $\lambda_4 = \frac{1-i\sqrt{3}}{2}$.A solution is then

$$y(x) = A + Be^{-x} + Ce^{\frac{1+i\sqrt{3}}{2}x} + De^{\frac{1-i\sqrt{3}}{2}x},$$

where $A, B, C, D \in \mathbb{R}$ are arbitrary constants.

3) Multiple roots with $n = 2$
Consider

$$y'' - 4y' + 4y = 0.$$

This translates to

$$\lambda^2 - 4\lambda + 4 = 0. \tag{6}$$

We have $c_0 = 4$, so $2, -2, 1$ or -1 are possible. It helps to quickly consider these to see that 2 works. Otherwise just solve with the formula for the roots of quadratic functions.In any case, we get that $\lambda_1 = \lambda_2 = 2$ is a multiple root of order 2. A solution is then

$$y(x) = Ae^{2x} + Bxe^{2x},$$

where $A, B \in \mathbb{R}$ are arbitrary constants.

Note! Most of the time one wants real value solutions instead of complex ones as in examples 1) and 2). There is a simple trick using Euler's Formula to transform this solutions. Real value building blocks that arise from roots of the form $a + bi$ and $a - bi$ can be obtained by writing

$$e^{ax} sin(bx), e^{ax} cos(bx)$$

instead of $e^{(a+bi)x}, e^{(a-bi)x}$. The proof for that has to do with basis transformation. The building blocks b_j are building a basis of the vector space of all solutions for y. We won't go into the theory here. Just know that above transformation is allowed.
The real valued solutions for examples 1) and 2) are

$$y(x) = Ae^x + Be^{\frac{1}{2}x} sin\Big(\frac{\sqrt{3}}{2}\Big) + Ce^{\frac{1}{2}x} cos\Big(\frac{\sqrt{3}}{2}\Big)$$

and

$$y(x) = A + Be^{-x} + Ce^{\frac{1}{2}x} sin\Big(\frac{\sqrt{3}}{2}\Big) + De^{\frac{1}{2}x} cos\Big(\frac{\sqrt{3}}{2}\Big),$$

respectively.

4.2 Non-Homogeneous Case