CALCULUS 2

4. Multiple Integrals
(Chapter 15)

Nguyễn Anh Tú
natu@hcmiu.edu.vn

Contents

Multiple Integrals

- Integrals of a function of several variables are called **multiple integrals**.
- We will extend the definition of integrals of function of one variable to functions of two or three variables.
- We first study integrals of functions of two variables, which are also called **double integrals**.

Section 1

Double Integrals

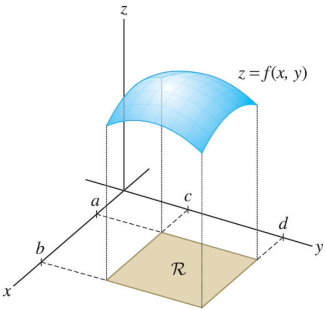
Double Integrals

Suppose $f(x, y) \geq 0$ on the closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 |\ a \leq x \leq b,\ c \leq y \leq d\}.$$

The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f:

$$S = \{(x, y, z) \in \mathbb{R}^3 |\ (x, y) \in R,\ 0 \leq z \leq f(x, y)\}.$$



Question: How to find the volume of S?

Double Integrals

- Divide the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$.
- By drawing lines parallel to the coordinate axes through the endpoints of these subintervals we obtain mn subrectangles

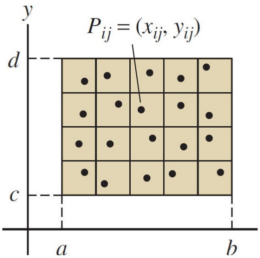
$$\begin{aligned} R_{ij} &= [x_{i-1}, x_i] \times [y_{j-1}, y_j] \\ &= \left\{(x, y) \in \mathbb{R}^2 \middle|\ x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j\right\} \end{aligned}$$

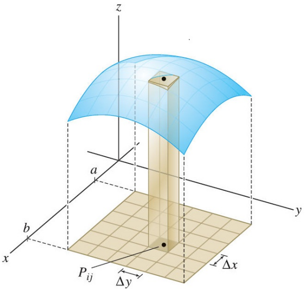
- Each rectangle has area $\Delta A = \Delta x \Delta y$.

Double Integrals

- Choose an arbitrary point $(x_{ij}, y_{ij}) \in R_{ij}$.
- Then we approximate the part of S that lies above each R_{ij} by the rectangular box with base R_{ij} and height $f(x_{ij}, y_{ij})$.
- The volume of each box is

$$f(x_{ij}, y_{ij})\Delta A.$$



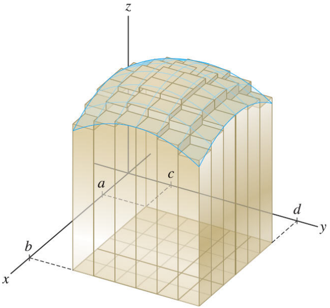


Double Integrals

- Summing up, we get an approximation for the volume of S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta A.$$

- Our intuition tells us that the approximation becomes better as m and n become larger.



Double Integrals

The **double integral** of f over the rectangle R is

$$\iint\limits_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A \tag{1}$$

if this limit exists.

- A function is called **integrable** if the limit in (1) exists.
- Thus, if f is integrable,

$$\boxed{V = \iint\limits_R f(x,y)dA}$$

Recall that

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta x \Delta y = L$$

if for every number $\epsilon > 0$, there is an integer N (depending on ϵ) such that

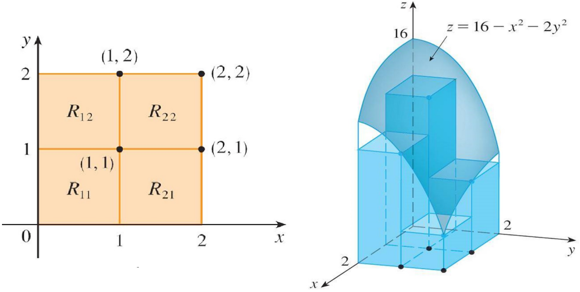
$$\left| L - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta x \Delta y \right| < \epsilon$$

if $m, n \geq N$, regardless of the choice of sample points (x_{ij}, y_{ij}).

Examples

1. If $f(x, y) = c$ is a constant function, find
$$\iint_R f(x, y)dA.$$

2. Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square.



Iterated Integrals

- Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.
- We use the notation
$$\int_c^d f(x, y)dy$$
to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$.
- Now $\int_c^d f(x, y)dy$ is a number that depends on the value of x, so it defines a function of x:
$$A(x) = \int_c^d f(x, y)dy.$$

Iterated Integrals

- If we now integrate the function $A(x)$ with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x,y)dy \right] dx. \qquad (2)$$

The integral on the right side of Equation (2) is called an **iterated integral**.

- Usually the brackets are omitted. Thus

$$\int_a^b \int_c^d f(x,y)dydx \quad \text{or} \quad \int_a^b dx \int_c^d f(x,y)dy.$$

means that we first integrate with respect to y from $y = c$ to $y = d$ and then with respect to x from $x = a$ to $x = b$.

Iterated Integrals

Similarly, the iterated integral

$$\int_c^d dy \int_a^b f(x,y)dx = \int_c^d \int_a^b f(x,y)dxdy$$
$$= \int_c^d \left[\int_a^b f(x,y)dx \right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$.

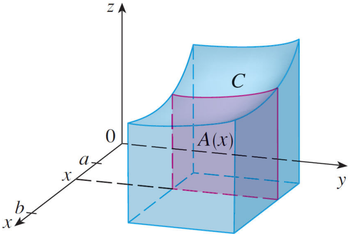
Example: Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 ydydx$

(b) $\int_1^2 \int_0^3 x^2 ydxdy$.

Volume via Iterated Integrals

- Suppose f is continuous and positive on
$$R = [a, b] \times [c, d].$$
- Then the volume of the solid that lies above R and under the surface $z = f(x, y)$ is
$$V = \int_a^b A(x)dx$$
where $A(x)$ is the area of a cross-section of S in the plane through x perpendicular to the x-axis.



Volume via Iterated Integrals

- Recall that

$$A(x) = \int_c^d f(x, y)dy.$$

- Hence

$$V = \int_a^b A(x)dx = \int_a^b \left[\int_c^d f(x, y)dy \right] dx.$$

- And as we seen earlier,

$$V = \iint_R f(x, y)dxdy.$$

- Thus,

$$\iint_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx$$

Fubini's Theorem

If f is continuous on the rectangle $R = [a, b] \times [c, d]$ then

1. $\iint\limits_R f(x, y)dA = \int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$
2. In the special case where $f(x, y) = g(x)h(y)$,

$$\iint\limits_R g(x)h(y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

Example: Evaluate the double integrals

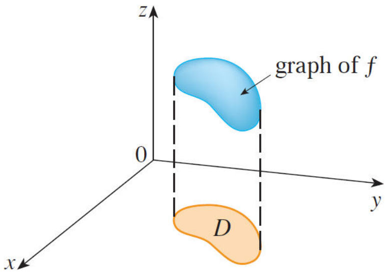
1. $\iint\limits_R (1 - 6x^2y)dA$ where $R = [0, 2] \times [-1, 1]$.
2. $\iint\limits_R e^x \cos y dA$. where $R = [-1, 2] \times [0, \pi/2]$

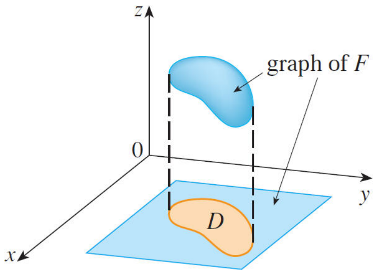
Double Integrals - General Regions

Let $f(x, y)$ be a function defined on a region D. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R.

Then we define a new function F with domain R by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D, \\ 0 & \text{if } (x,y) \in R \text{ but not in } D. \end{cases}$$





Double Integrals - General Regions

Definition

If F is integrable over R, then we define the **double integral of f over D** by

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA$$

where

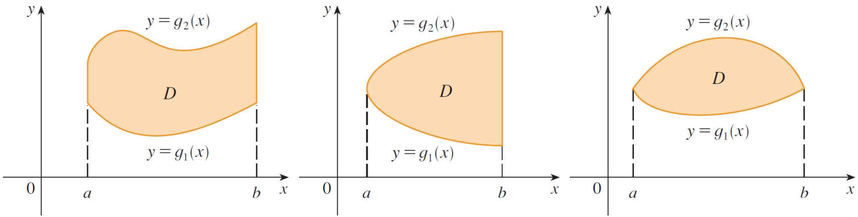
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D, \\ 0 & \text{if } (x,y) \in R \text{ but not in } D. \end{cases}$$

Double Integrals - Type I Regions

A plane region D is said to be of **Type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) |\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

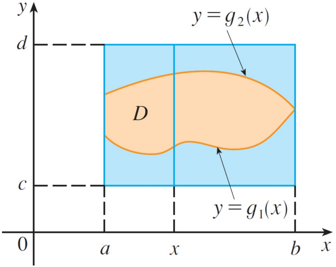
where g_1 and g_2 are continuous on $[a, b]$.



Double Integrals - Type I Regions

- By Fubini's theorem,

$$\iint\limits_R F(x,y)dA = \int_a^b \int_c^d F(x,y)dydx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$



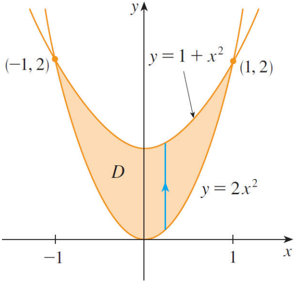
Double Integrals - Type I Regions

Theorem

If f is continuous on a Type I region $D = \{(x, y) | \ a \le x \le b, \ g_1(x) \le y \le g_2(x)\}$ where $g_1(x)$ and $g_2(x)$ are continuous on $[a, b]$, then

$$\iint\limits_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

Example: Evaluate $\iint\limits_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = x^2 + 1$.



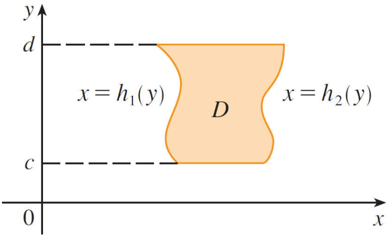
Answer: $\frac{32}{15}$.

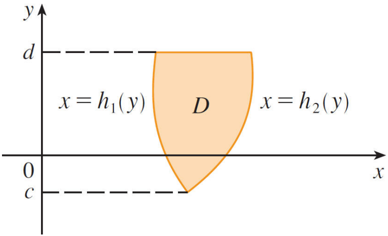
Double Integrals - Type II Regions

A plane regions which can be expressed as

$$D = \{(x, y)|\ c \le y \le d,\ \ h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous, is called a region of **type II**.





Double Integrals - Type II Regions

Theorem

If f is continuous on a region

$$D = \{(x, y)|\ c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

where $h_1(x)$ and $h_2(x)$ are continuous on $[a, b]$, then

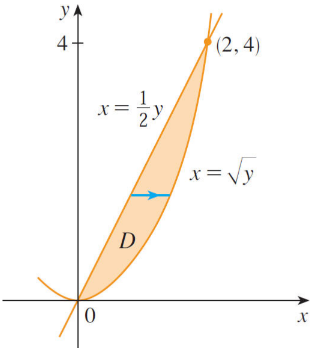
$$\iint\limits_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$$

Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution: D can be written as a type II region:

$$D = \left\{ (x, y) \middle| \; 0 \le y \le 4, \; \frac{y}{2} \le x \le \sqrt{y} \right\}.$$

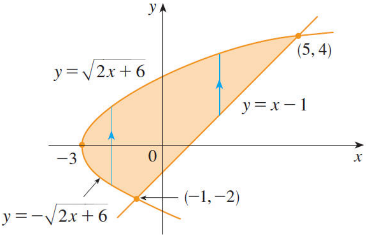


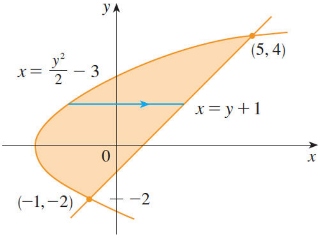
Therefore

$$\begin{aligned} V &= \iint_D (x^2+y^2)\,dA \\ &= \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y^2)\,dx\,dy \\ &= \int_0^4 \left(\frac{x^3}{3}+y^2x\right)\Big|_{x=y/2}^{x=\sqrt{y}} dy \\ &= \int_0^4 \left(\frac{y^{3/2}}{3}+y^{5/2}-\frac{y^3}{24}-\frac{y^3}{2}\right) dy \\ &= \frac{2}{15}y^{5/2}+\frac{2}{7}y^{7/2}-\frac{13}{96}y^4\Big|_0^4 = \frac{216}{35}. \end{aligned}$$

Examples

1. Evaluate $\iint_D xy\,dA$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.
Answer: 36.





2. Evaluate
$$\iint_D e^{y^2}\,dA$$
where D is the region bounded by the lines $y = 2$, $x = 0$, and $y = x/2$. Answer: $e^4 - 1$.

Properties of Double Integrals

Assume that f and g are integrable over D and c is a constant. Then

(a) $\iint\limits_D f(x,y)dA = 0$ if D has zero area.

(b) If $f(x,y) \geq 0$ on D, then $\iint\limits_D f(x,y)dA$ is the volume of the solid lying vertically above D and below the surface $z = f(x,y)$.

(c) $\iint\limits_D 1dA = \mathrm{Area}(D)$.

(d)
$$\iint\limits_D \left[f(x,y) + g(x,y)\right] dA = \iint\limits_D f(x,y)dA + \iint\limits_D g(x,y)dA.$$

(e)
$$\iint\limits_D cf(x,y)dA = c\iint\limits_D f(x,y)dA.$$

Properties of Double Integrals

(f) If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then

$$\iint\limits_D f(x,y)\,dA \geq \iint\limits_D g(x,y)\,dA.$$

(g) If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$, then

$$m\,\mathrm{area}(D) \leq \iint\limits_D f(x,y)\,dA \leq M\,\mathrm{area}(D).$$

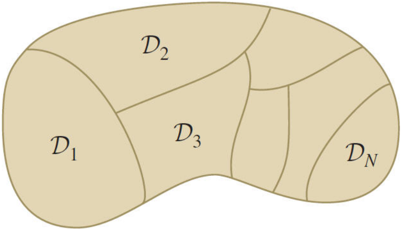
(h)

$$\left|\iint\limits_D f(x,y)\,dA\right| \leq \iint\limits_D |f(x,y)|\,dA.$$

Properties of Double Integrals

(i) If D is the union of domains $D_1, D_2, \ldots, D_N$ that do not overlap except possibly on boundary curves, then

$$\iint\limits_{D} f(x,y)dA = \iint\limits_{D_1} f(x,y)dA + \cdots + \iint\limits_{D_N} f(x,y)dA.$$



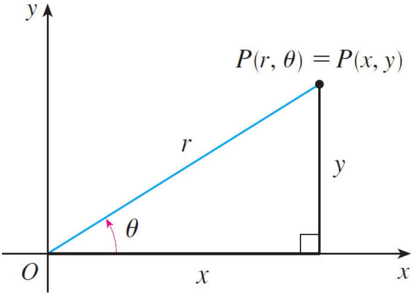
Section 2

Polar Coordinates

Polar Coordinates

Recall that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

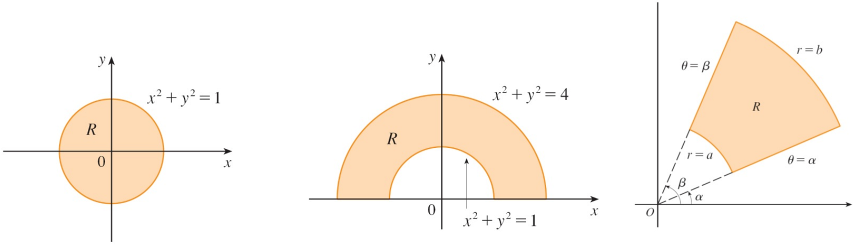
$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$



Polar Rectangles

A polar rectangle is a region of the form

$$R = \{(r, \theta) : a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}.$$

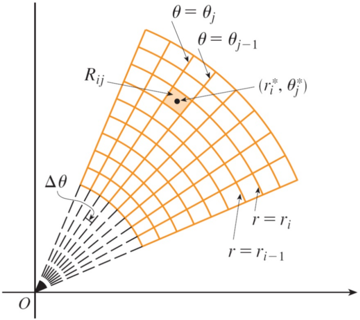


(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$ Polar rectangle

Integrals over Polar Rectangles

In order to compute the double integral $\iint_R f(x,y)dA$ over is a polar rectangle R, we divide

- the interval $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b-a)/m$
- the interval $[\alpha,\beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta-\alpha)/n$.



Integrals over Polar Rectangles

- We obtain mn polar rectangles

$$R_{ij} = \{(r, \theta) : r_{i-1} \leq r \leq r_i,\ \theta_{j-1} \leq \theta \leq \theta_j\}.$$

whose areas are

$$\Delta A_{ij} \approx r_i \Delta r \Delta \theta.$$

- Thus, a double Riemann sum with respect to this polar partition is

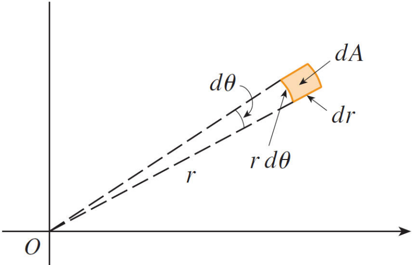
$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta A_{ij} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i \cos \theta_j, r_i \sin \theta_j) r_i \Delta r \Delta \theta.$$

Integrals in Polar Coordinates

If f is continuous on a polar rectangle R given by
$0 \leq a \leq r \leq b,\ \alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

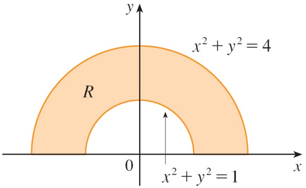
$$\iint\limits_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

Thus, we convert from rectangular to polar coordinates by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $rdrd\theta$.



Example

Compute $\iint\limits_R (3x+4y^2)dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$R = \{(r,\theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Solution:

$$\iint\limits_R (3x+4y^2)dA = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)rdrd\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)rdrd\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\frac{1-\cos(2\theta)}{2})rdrd\theta = \frac{15\pi}{2}.$$

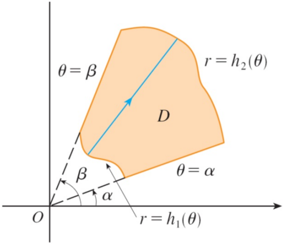
General Polar Domain

If f is continuous on the polar region

$$D = \{(r, \theta) |\ \alpha \leq \theta \leq \beta,\ \ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

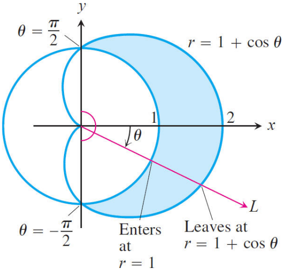
$$\iint\limits_D f(x, y)dA \ = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$$



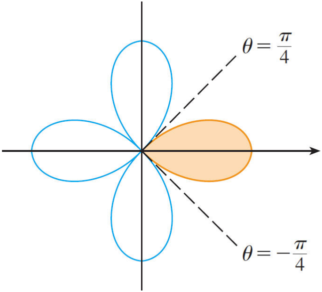
$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

Examples

1. Describe in polar coordinates the region D that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.



2. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = |\cos 2\theta|$.



Solution: A loop is given by the region

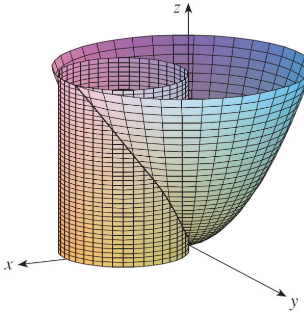
$$D = \left\{(r, \theta) : -\frac{\pi}{4} \le \theta \le \frac{\pi}{4},\ 0 \le r \le \cos 2\theta\right\}.$$

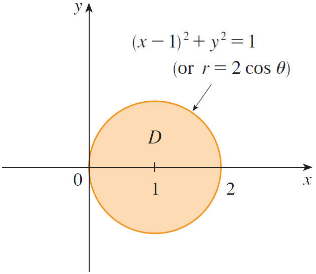
So the area of one loop of the four-leaved rose is

$$\begin{aligned}
A &= \iint\limits_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta \\
&= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(r^2 \Big|_{r=0}^{r=\cos 2\theta} \right) d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta d\theta \\
&= \frac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta) d\theta = \frac{1}{4} \left(\theta + \frac{1}{4} \sin 4\theta \right) \Big|_{-\pi/4}^{\pi/4} \\
&= \frac{\pi}{8} \quad \text{(square units)}.
\end{aligned}$$

Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.





Solution: In polar coordinates, the boundary circle becomes $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \left\{ (r, \theta) \mid -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos\theta \right\}.$$

Therefore,

$$
\begin{aligned}
V &= \iint_D (x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\, r dr d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{1}{4} r^4 \Big|_0^{2\cos\theta} dr d\theta = 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta \\
&= 8\int_0^{\pi/2} \cos^4\theta\, d\theta = 2\int_0^{\pi/2} (1+\cos 2\theta)^2 d\theta \\
&= 2\int_0^{\pi/2} \Big[1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\Big] d\theta \\
&= 2\Big(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\Big)\Big|_0^{\pi/2} = \frac{3\pi}{2}.
\end{aligned}
$$

Application of Double Integrals

- If the **charge density** over a region D is given by $\sigma(x, y)$ then the total charge in D is
$$Q = \iint_D \sigma(x, y) dA.$$
- If a pair of random variables X and Y have **joint density function** $f(x, y)$ then the probability that (X, Y) lies in a region D is
$$P((X, Y) \in D) = \iint_D f(x, y) dA.$$
The expected value of X and Y are
$$\iint_{\mathbb{R}^2} xf(x, y) dA \text{ and } \iint_{\mathbb{R}^2} yf(x, y) dA,$$
respectively.

Application of Double Integrals

- If a lamina occupies a region D has density function (in units of mass per unit area) given by $\rho(x, y)$ then its mass is

$$m = \iint_D \rho(x, y) dA.$$

The moment about the y-axis is

$$M_y = \iint_D x\rho(x, y) dA$$

and about the x-axis is

$$M_x = \iint_D y\rho(x, y) dA.$$

The center of mass is the point

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

Section 3

Triple Integrals

Triple Integrals

- Integral of a function of three variables $f(x, y, z)$ over a rectangular box

$$E = \{(x, y, z)|\ a \leq x \leq b,\ c \leq y \leq d,\ r \leq z \leq s\}$$

is defined in a similar way to the two-dimensional analog.

- The first step is to divide E into sub-boxes by dividing $[a, b]$ into m subintervals, divide $[c, d]$ into n subintervals, $[r, s]$ into p subintervals.
- This results in the division of E into mnp sub-boxes

$$E_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k].$$

Triple Integrals

- Each E_{ijk} has volume $\Delta V = \Delta x \Delta y \Delta z$.
- Choosing arbitrary points $P_{ijk} \in E_{ijk}$, we form **triple Riemann sum**

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f(P_{ijk}) \Delta V,$$

Definition

The triple integral of f over the rectangular box E is

$$\iiint_E f(x, y, z) dV = \lim_{m,n,p \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f(P_{ijk}) \Delta V$$

if this limit exists.

The triple integral always exists if f is continuous.

Fubini's Theorem for Triple Integrals

If f is continuous on the rectangular box

$$E = [a, b] \times [c, d] \times [r, s],$$

then

$$\boxed{\iiint\limits_E f(x, y, z)dV = \int_r^s \int_c^d \int_a^b f(x, y, z)dxdydz}$$

Note There are five other possible orders in which we can integrate, all of which give the same value.

Example: Evaluate the triple integral $\iiint_B xyz^2 dV$, where B is the rectangular box

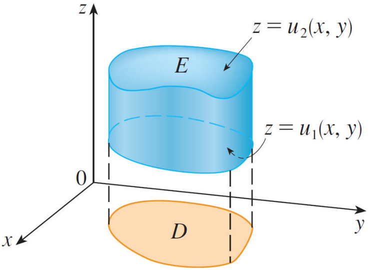
$$B = (x, y, z)|0 \leq x \leq 1, \breve{}1 \leq y \leq 2, 0 \leq z \leq 3.$$

Triple Integrals over Type 1 Domains

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y:

$$E = \{(x, y, z) | \ (x, y) \in D, \ u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the *projection of E onto the xy-plane*.



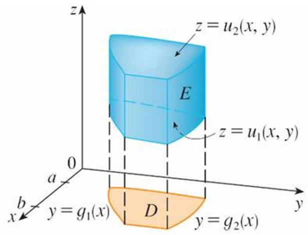
Triple Integrals over Type 1 Domains

- Then

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz \right] dA \qquad (3)$$

- In particular, if $D = \{(x,y) |\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ is a type I region, then

$$E = \{(x,y,z) |\ a \le x \le b,\ g_1(x) \le y \le g_2(x),\\ u_1(x,y) \le z \le u_2(x,y)\}$$



Triple Integrals over Type 1 Domains

- Equation (3) then becomes

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

- Similarly, If D is a type II plane region then

$$E = \big\{(x,y,z) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y),\ u_1(x,y) \le z \le u_2(x,y)\big\}$$

and Equation (3) becomes

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Examples

1. Evaluate
$$I = \iiint_E (x+y+z)dV,$$
where E is bounded by the coordinate planes
$x = 0,\ y = 0,\ z = 0$, and the plane $x + y + z = 1$.

   Answer: $I = \frac{1}{8}$.

2. Find the volume of the tetrahedron T bounded by the planes
$x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

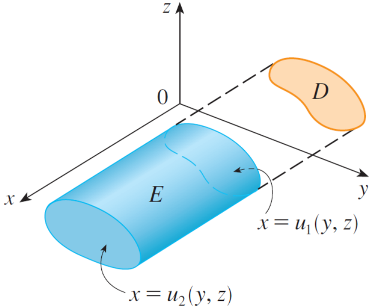
   Answer: $I = \frac{1}{3}$.

Triple Integrals over Type 2 Domains

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) |\ (y, z) \in D,\ u_1(y, z) \leq x \leq u_2(y, z)\}$$

where D is the projection of E onto the yz-plane.



Then

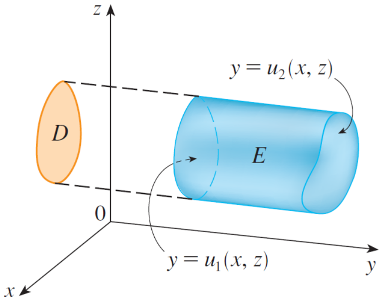
$$\iiint\limits_E f(x, y, z) dV = \iint\limits_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) dx \right) dA$$

Triple Integrals over Type 3 Domains

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) | \ (x, z) \in D, \ u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the yz-plane.

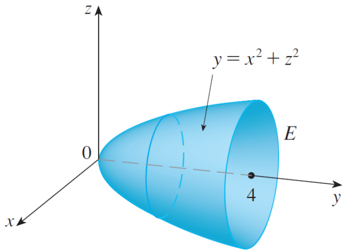


Then

$$\iiint\limits_E f(x, y, z) dV = \iint\limits_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) dy \right) dA$$

Example

Evaluate $\iiint\limits_E \sqrt{x^2+z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.



Solution: The projection D of E onto the xz-plane is the disk $x^2 + z^2 \leq 4$. In polar coordinates,

$$D = \{(r, \theta) |\ 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}.$$

Triple Integrals

Since

$$E = \{(x, y, z)|\ (x, z) \in D,\ x^2 + z^2 \le y \le 4\}.$$

we have

$$\iiint\limits_E \sqrt{x^2+z^2}\,dV = \iint\limits_D \left(\int_{x^2+z^2}^4 \sqrt{x^2+z^2}\,dy\right) dA$$

$$= \iint\limits_D (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA$$

$$= \int_0^{2\pi}\int_0^2 (4 - r^2) r \cdot r\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right) \cdot \left(\int_0^2 (4r^2 - r^4)\,dr\right)$$

$$= 2\pi\left(\frac{4}{3}r^3 - \frac{1}{5}r^5\Big|_0^2\right) = \frac{128}{15}\pi.$$

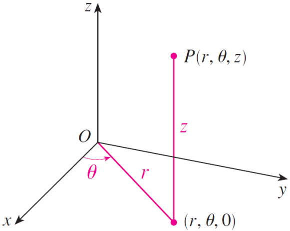
Section 4

Cylindrical Coordinates

Triple Integrals - Cylindrical Coordinates

- In the cylindrical coordinate system, a point $P(x, y, z)$ is represented by the triple (r, θ, z), where r and θ are polar coordinates of the point (x, y)
- To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

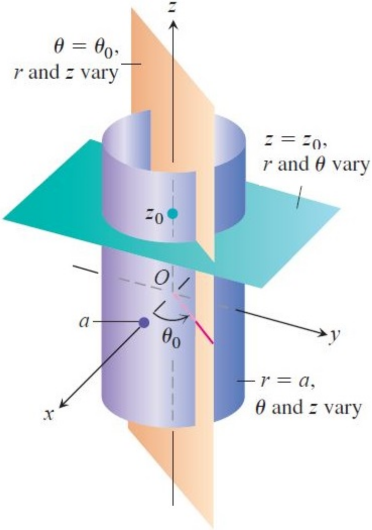


Example: (a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Triple Integrals - Cylindrical Coordinates

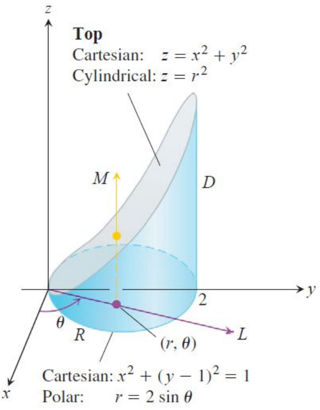
In cylindrical coordinates:

- The equation $r = a$ describes an entire cylinder about the z-axis.
- The z-axis is given by $r = 0$.
- The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.
- The equation $z = z_0$ describes a plane perpendicular to the z-axis.



Example

Find a cylindrical coordinate description for the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.



Integrals in Cylindrical Coordinates

Suppose that E is a type 1 domain

$$E = \left\{(x, y, z) |\ (x, y) \in D,\ u_1(x, y) \leq z \leq u_2(x, y)\right\}$$

where D is given in polar coordinates by

$$D = \left\{(r, \theta) |\ \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\right\}.$$

Then recall that

$$\iiint\limits_E f(x, y, z)dV = \iint_D \int_{u_1(x,y)}^{u_1(x,y)} f(x, y, z)dzdA.$$

Integrals in Cylindrical Coordinates

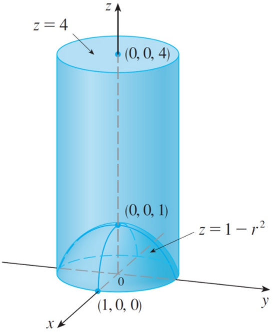
Switching to the polar coordinates, we get

$$\iiint_E f(x, y, z)dV$$
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$$

Examples

1. Evaluate the integral $\iiint\limits_E z dV$, where E is the region inside the sphere $x^2+y^2+z^2=6$ and above the paraboloid $z=x^2+y^2$.
Answer: $\frac{11}{3}\pi$.

2. A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above the paraboloid $z=1-x^2-y^2$. The density is $\rho(x,y,z)=\sqrt{x^2+y^2}$. Find the mass of E.

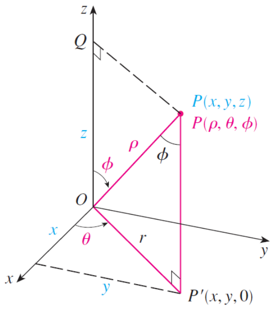


Answer: $\frac{12\pi}{5}$.

Section 5

Spherical Coordinates

Spherical Coordinates



The spherical coordinates (ρ, θ, ϕ) of a point P in space are given by the equations

$$\boxed{x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi}$$

Here $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$, and

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

Integrals in Spherical Coordinates

If E is a spherical wedge given by

$$E = \left\{(\rho, \theta, \phi) |\ a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\right\}$$

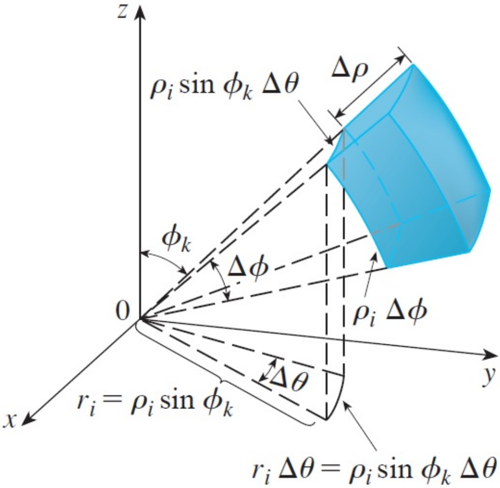
then

$$\boxed{\begin{aligned}&\iiint\limits_E f(x, y, z)dV \\ &= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi\end{aligned}}$$

This is the three dimensional analogue of the formula for integrals in polar coordinates. Note that the part $\rho^2 \sin\phi d\rho d\theta d\phi$ comes from the approximate volume of small spherical wedges (see next slide)

$$\rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi.$$

Integrals in Spherical Coordinates



Integrals in Spherical Coordinates

More generally, if E is defined by

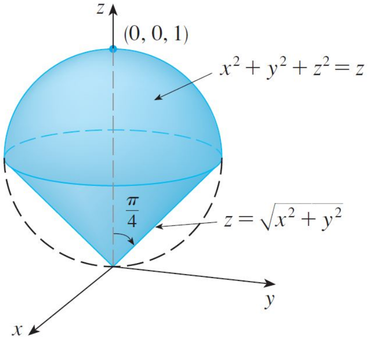
$$\theta_1 \leq \theta \leq \theta_2, \quad \phi_1 \leq \phi \leq \phi_2, \quad \rho_1(\theta, \phi) \leq \rho \leq \rho_2(\theta, \phi)$$

then we have

$$\boxed{\begin{aligned}&\iiint\limits_E f(x, y, z)dV \\ =&\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_2(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi \; d\rho d\theta d\phi\end{aligned}}$$

Example

Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.



Solution: The sphere passes through the origin and has center $(0, 0, \frac{1}{2})$. The description of the solid in spherical coordinates is

$$E = \left\{0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \rho \le \cos\phi\right\}.$$

The volume of E is

$$\begin{aligned} V = \iiint\limits_E 1dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1}{3}\sin\phi\left(\rho^3\Big|_{\rho=0}^{\rho=\cos\phi}\right)d\phi d\theta \\ &= \frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sin\phi\cos^3\phi d\phi d\theta = \cdots = \frac{\pi}{8}. \end{aligned}$$