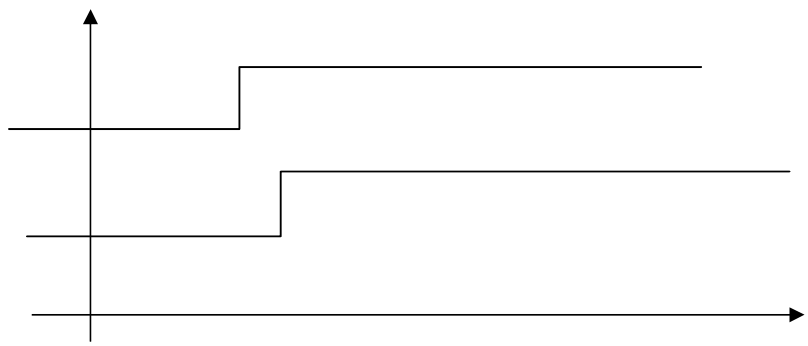
Chapter 4. Design of State-Space Servo-Controllers

4.1 Introducing the Reference Signal

It is desired that the process output vector should follow a specified vector of setpoints:



Consider the SISO process:

$$\dot{\underline{x}} = A\underline{x} + Bu$$
$$y = C\underline{x}$$

For a steady state desired output y_{ss} we have:

Propose the control-law:

$$u(t) = u_{ss} - K\left(\underline{x}(t) - \underline{x}_{ss}\right)$$

In the steady state:

$$\underline{O} = A\underline{x}_{ss} + Bu_{ss}$$
$$y_{ss} = C\underline{x}_{ss}$$

Let us propose the simple relationships:

$$u_{ss} = N_u r_{ss} \qquad\qquad \underline{x}_{ss} = N_x r_{ss}$$

This implies:

$$\underline{O} = AN_x r_{ss} + BN_u r_{ss}$$
$$y_{ss} = CN_x r_{ss}$$

Of course if we want the steady state output to be $r_{ss:}$

Then the gains N_x and N_u can be determined as:

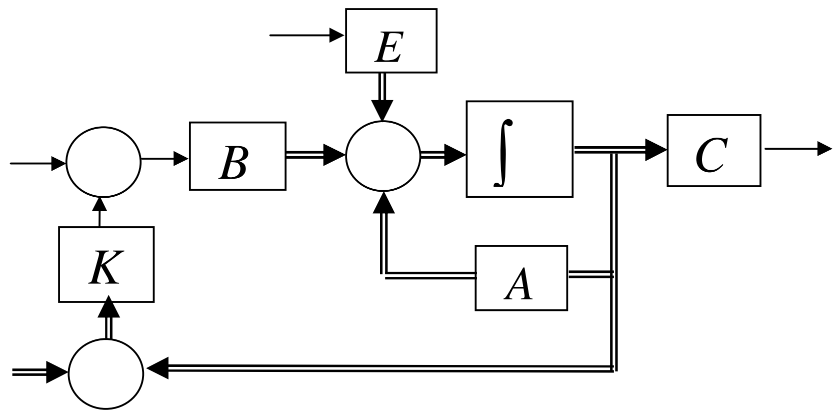
$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

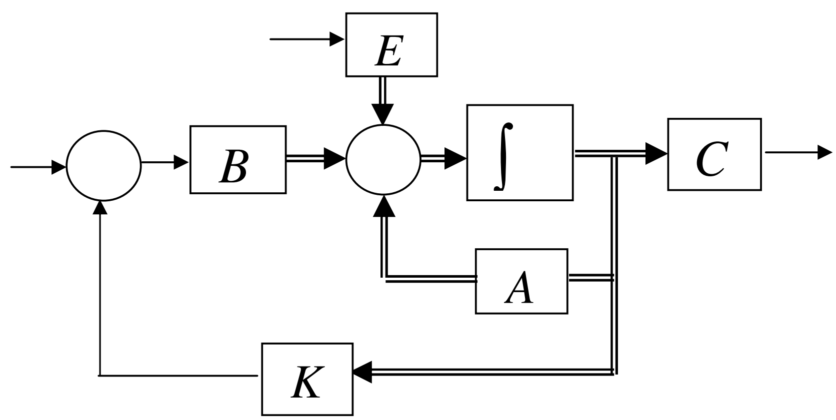
The control-law is then:

$$u(t) = N_u r(t) - K\left(x(t) - N_x r(t)\right)$$

Which could of course be rewritten as:

This yields the following control structures:





Tutorial: Introduce a setpoint signal to the motor speed controller developed earlier – test in Simulink.

Some of the problems of this technique include:

- Does not increase the system type:

- Gains designed to reduce e_{ss} for setpoint changes:

4.2 State-Space Control with Integral Action

Consider again the SISO process:

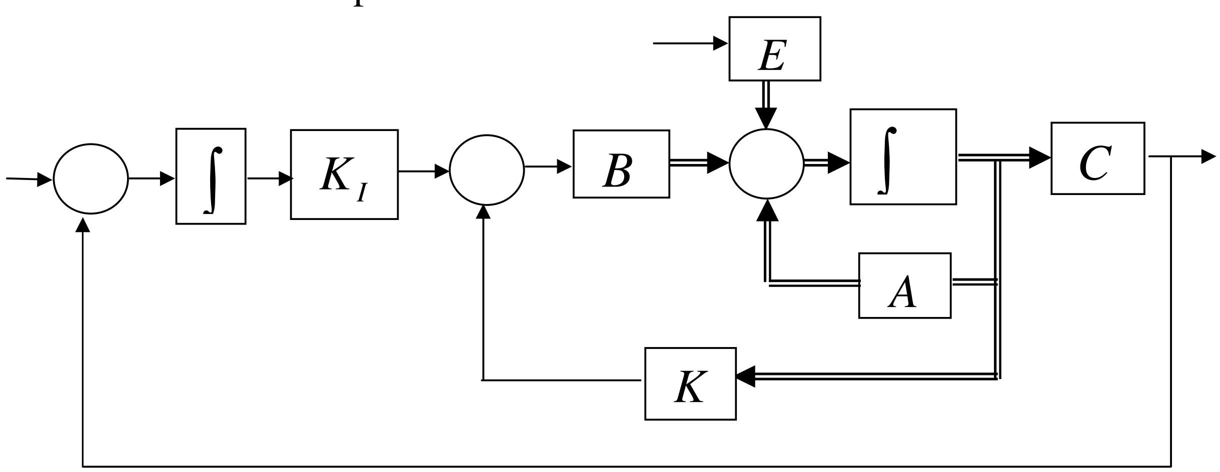
$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$y(t) = C\underline{x}(t)$$

The state control-law with integral action is:

$$u(t) = -K\underline{x}(t) + K_I \int_0^t e(\tau)d\tau$$

where:

This could be represented as:



Introduce another state:

$$x_I(t) = \int_0^t e(\tau)d\tau$$

The control-law become:

$$u(t) = -K\underline{x}(t) + K_I x_I$$

This yields the closed-loop state equation:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\left(-K\,\underline{x}(t) + K_I x_I(t)\right)$$

But we know:

$$x_I(t) = \int_0^t e(\tau)d\tau$$

Hence the complete closed loop system can be represented by the coupled equations:

$$\dot{\underline{x}}(t) = \left(A - BK\right)\underline{x}(t) + BK_I x_I(t)$$
$$\dot{x}_I(t) = -C\,\underline{x}(t) + r(t)$$

Assign a new state vector:

$$\underline{z}(t) = \left[\begin{array}{c} \underline{x}(t) \\ \hdashline x_I(t) \end{array}\right]$$

The closed-loop equations can be written more compactly as:

$$\frac{d}{dt}\left[\begin{array}{c} \\ \hdashline \\ \end{array}\right] = \left[\begin{array}{c:c} & \\ \hdashline & \\ \end{array}\right]\left[\begin{array}{c} \\ \hdashline \\ \end{array}\right] + \left[\begin{array}{c} \\ \hdashline \\ \end{array}\right] r(t)$$

The poles of the closed-loop system are given by the roots of:

$$\det\left(sI - A_2\right) = 0$$

Determine the gains to place the N+1 closed-loop poles to obtain the following characteristic equation:

$$C_{des}(s) = s^{N+1} + C_N s^N + \cdots C_1 s + C_0$$

<u>Proof of Integral Action</u>:

Consider an asymptotically constant setpoint signal:

Since the closed-loop system is stable – the states must converge to steady-state values:

$$\underline{0} = (A - BK)\underline{x}_{ss} + BK_I x_{I\,ss}$$
$$0 = -C\underline{x}_{ss} + r_{ss}$$

hence:

EXAMPLE: The DC Motor

$$\frac{d}{dt}\begin{bmatrix}\omega(t)\\ i(t)\end{bmatrix} = \begin{bmatrix}0 & 50\\ -200 & -200\end{bmatrix}\begin{bmatrix}\omega(t)\\ i(t)\end{bmatrix} + \begin{bmatrix}0\\ 200\end{bmatrix}v(t) + \begin{bmatrix}-50\\ 0\end{bmatrix}T_L(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\omega(t)\\ i(t)\end{bmatrix}$$

Use the control-law

$$v(t) = -\begin{bmatrix}k_1 & k_2\end{bmatrix}\begin{bmatrix}\omega(t)\\ i(t)\end{bmatrix} + K_I\int_0^t \left(r_\omega(\tau) - y(\tau)\right)d\tau$$

Then :

Hence the closed loop state-equation is:

$$\frac{d}{dt}\left[\begin{array}{c}\omega(t)\\ i(t)\\ \hline x_I(t)\end{array}\right]=\left[\begin{array}{cc|c}0 & 50 & 0\\ -200-200k_1 & -200-200k_2 & 200K_I\\ \hline -1 & 0 & 0\end{array}\right]\left[\begin{array}{c}\omega(t)\\ i(t)\\ \hline x_I(t)\end{array}\right]+\left[\begin{array}{c}0\\ 0\\ \hline 1\end{array}\right]r(t)+\left[\begin{array}{c}-50\\ 0\\ \hline 0\end{array}\right]T_L(t)$$

The poles of the closed-loop system are given by roots of:

$$\det\left(\left[\begin{array}{cc|c}s & 0 & 0\\ 0 & s & 0\\ \hline 0 & 0 & s\end{array}\right]-\left[\begin{array}{cc|c}0 & 50 & 0\\ -200-200k_1 & -200-200k_2 & 200K_I\\ \hline -1 & 0 & 0\end{array}\right]\right)=0$$

which yields:

$$\det\left(\left[\begin{array}{cc|c}s & -50 & 0\\ 200+200k_1 & s+200+200k_2 & -200K_I\\ \hline 1 & 0 & s\end{array}\right]\right)=0$$

the closed-loop characteristic equation is then:

Now we still want a second order dominant response with :

This can be achieved by placing the controller pole further out left:

The desired closed-loop characteristic equation is:

$$C_{des}(s) = (s+400)(s^2 + 282.8s + 40000)$$
$$= s^3 + 682.8s^2 + 153120s + 16000000$$

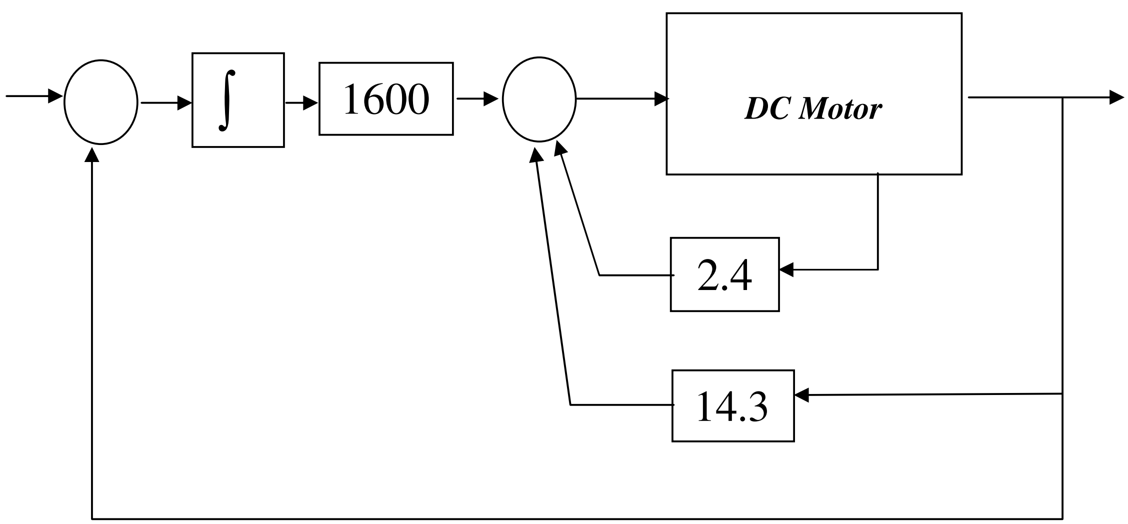
Compare with the closed-loop characteristic equation:

$$10000K_I = 16000000$$
$$10000(1+k_1) = 153120$$
$$200(1+k_2) = 682.8$$

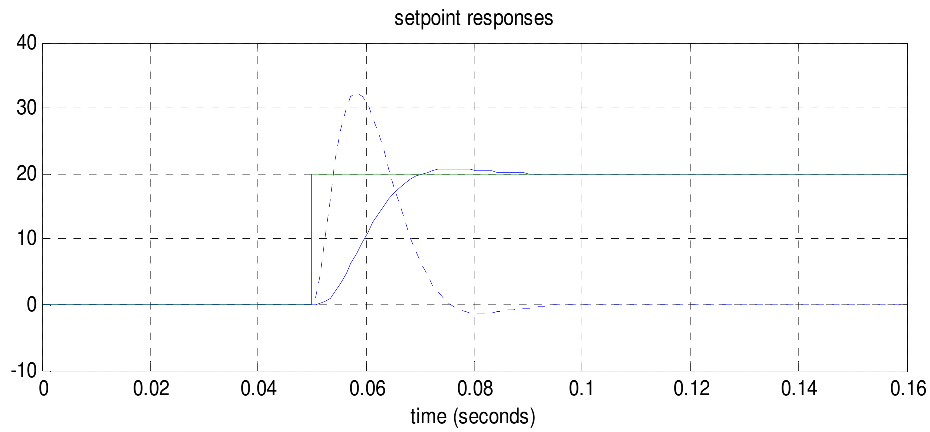
This yields the controller:

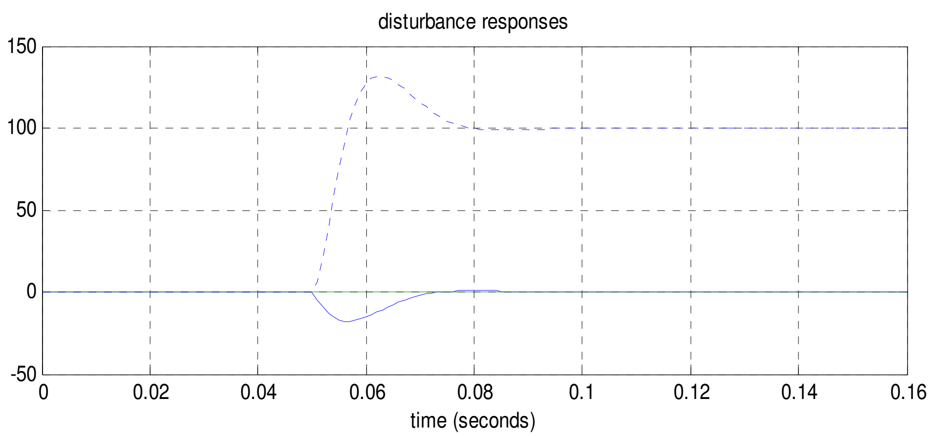
$$v(t) = -14.3\omega(t) - 2.4i(t) + 1600\int_0^t (r_\omega(\tau) - \omega(\tau))d\tau$$

which could be built as follows:



The closed-loop responses are:





4.2.1 Use of Ackermans Method to Design Controllers with Integral Action

Define the open-loop equations as:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$\dot{x}_I(t) = r(t) - C\underline{x}(t)$$

Or in more compact form as:

$$\frac{d}{dt}\left[\begin{array}{c} \\ \hdashline \\ \end{array}\right] = \left[\begin{array}{c:c} \quad & \quad \\ \hdashline \quad & \quad \\ \end{array}\right]\left[\begin{array}{c} \\ \hdashline \\ \end{array}\right] + \left[\begin{array}{c} \\ \hdashline \\ \end{array}\right]u(t) + \left[\begin{array}{c} \\ \hdashline \\ \end{array}\right]r(t)$$

Which is under the control:

$$u(t) = -K\underline{x}(t) + K_I x_I$$

or:

$$u(t) = -\left[K \;\vdots\; -K_I\right]\underline{z}(t)$$

We can then use Ackermann's formula:

$$\left[K \;\vdots\; -K_I\right] = \begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix} C_z^{-1} C_{des}(A_2)$$

where:

$$C_{des}(s) = s^{N+1} + C_N s^N + \cdots C_1 s + C_0 = 0$$

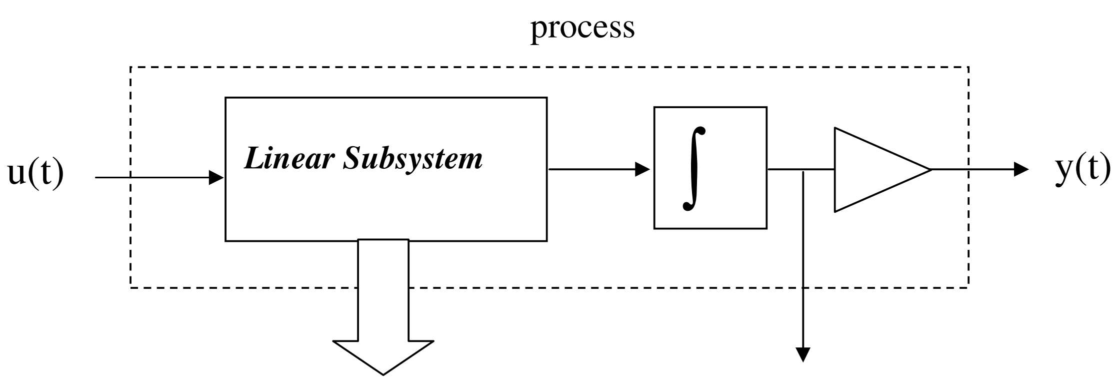
and: $$C_z = \left[B_2 \;\vdots\; A_2 B_2 \;\vdots\; {A_2}^2 B_2 \;\vdots\; \cdots \;\vdots\; {A_2}^N B_2\right]$$

4.2.2 Use of Inherent Integral Action

Consider the SISO process model:

$$\dot{\underline{x}} = A\underline{x} + Bu$$
$$y = C\underline{x}$$

of the special form:



The state of the N[th] order process is then: $\underline{x}(t) = \begin{bmatrix} x_1 \\ \hline \underline{x}_2 \end{bmatrix}$

The output is: $y(t) = \left[c_1 \;\vdots\; 0 \quad 0 \quad \cdots \quad 0\right]\underline{x}(t)$

We want the output y(t) to track the setpoint r(t). Consider that r(t) is asymptotically constant:

Consider the standard tracking controller structure:

$$u(t) = N_u r(t) - K\left(\underline{x}(t) - N_x r(t)\right)$$

with the choice:

The control-law becomes:

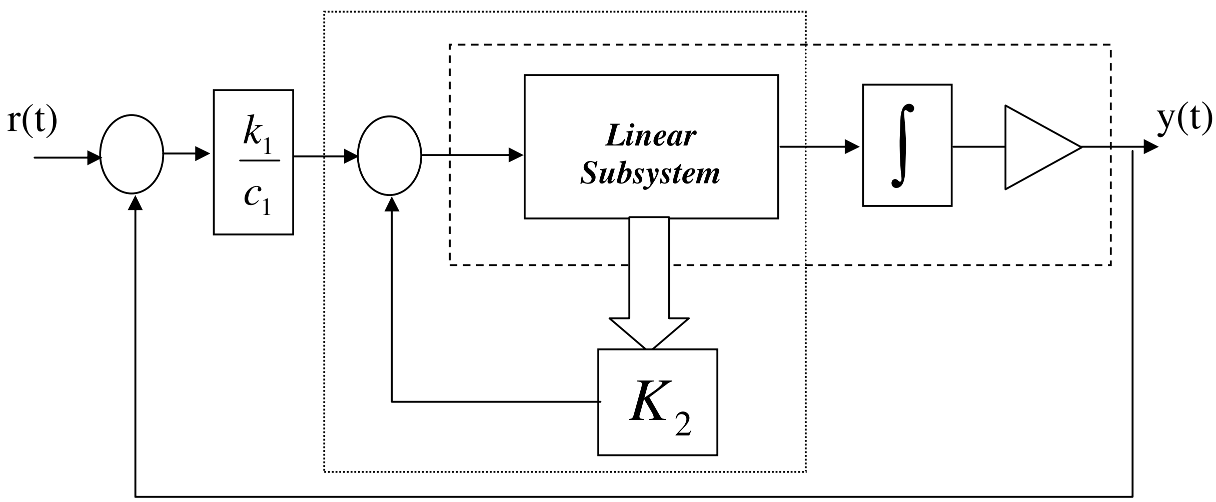
$$u(t) = -K\left(\underline{x}(t) - \begin{bmatrix} 1/c_1 \\ \hdashline 0 \\ \vdots \\ 0 \end{bmatrix} r(t)\right)$$

Or:

Which could be written as:

$$u(t) = -K\begin{bmatrix} \dfrac{y(t)-r(t)}{c_1} \\ \hdashline x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = -\left[\begin{array}{c:c} \dfrac{k_1}{c_1} & K_2 \end{array}\right]\begin{bmatrix} y(t)-r(t) \\ \hdashline \underline{x}_2(t) \end{bmatrix}$$

This could be represented by:



Proof of Integral Action: