EE4011: RFIC Design

Noise Circles
Designing with Gain/Noise Tradeoff

Two-port Noise Formulas

The noise factor of a transistor amplifier can be expressed as

$$F = F_{\min} + \frac{R_N}{G_S}\left|Y_S - Y_{opt}\right|^2$$

The quantities are as follows:

$Y_S = G_S + jB_S$	the source admittance as seen by the transistor input
$Y_{opt} = G_{opt} + jB_{opt}$	the source admittance that gives the minimum noise factor F_{min}
F_{min}	the minimum noise factor, obtained when $Y_S = Y_{opt}$
R_N	the equivalent noise resistance of the transistor
G_S	the real part of the source admittance

This states that the noise factor of a transistor depends on the source admittance as well as on four noise parameters of the device, F_{min}, R_N, G_{opt} and B_{opt}. These parameters vary with frequency and bias point so they are usually tabulated for a small number of bias points and frequencies on the data sheets of discrete devices.

This formula was presented in “Representation of Noise in Linear Twoports” by H.A. Haus et al., Proceedings of the Institute of Radio Engineers (IRE), Vol. 48, Issue 1, 1960, pp 69-74 and has been widely used ever since.

Putting Reflection Coefficients into the Formula

We want to represent noise on a Smith chart which is a plot of reflection coefficients so the formula has to be rearranged to use reflection coefficients:

$$\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0} \Rightarrow Z_S = Z_0 \frac{1+\Gamma_S}{1-\Gamma_S} \qquad Y_S = \frac{1}{Z_S} = \frac{1}{Z_0}\frac{1-\Gamma_S}{1+\Gamma_S} \qquad Y_{opt} = \frac{1}{Z_0}\frac{1-\Gamma_{opt}}{1+\Gamma_{opt}}$$

Now, Γ_{opt} is the source reflection coefficient which gives rise to the minimum noise factor F_{min}.

$$\left|Y_S - Y_{opt}\right|^2 = \left|\frac{1}{Z_0}\frac{1-\Gamma_S}{1+\Gamma_S} - \frac{1}{Z_0}\frac{1-\Gamma_{opt}}{1+\Gamma_{opt}}\right|^2 = \frac{1}{Z_0^2}\left|\frac{(1-\Gamma_S)(1+\Gamma_{opt}) - (1-\Gamma_{opt})(1+\Gamma_S)}{(1+\Gamma_S)(1+\Gamma_{opt})}\right|^2$$

$$= \frac{1}{Z_0^2}\left|\frac{2(\Gamma_{opt} - \Gamma_S)}{(1+\Gamma_S)(1+\Gamma_{opt})}\right|^2 = \frac{4}{Z_0^2}\frac{\left|\Gamma_S - \Gamma_{opt}\right|^2}{\left|1+\Gamma_S\right|^2\left|1+\Gamma_{opt}\right|^2}$$

Writing G_S in terms of the source reflection coefficient:

$$G_S = \mathrm{Re}\{Y_S\} = \frac{Y_S + Y_S^*}{2} = \frac{1}{2Z_0}\left(\frac{1-\Gamma_S}{1+\Gamma_S} + \frac{1-\Gamma_S^*}{1+\Gamma_S^*}\right)$$

$$= \frac{1}{2Z_0}\frac{(1-\Gamma_S)(1+\Gamma_S^*) + (1-\Gamma_S^*)(1+\Gamma_S)}{(1+\Gamma_S)(1+\Gamma_S^*)} = \frac{1}{Z_0}\frac{1-\left|\Gamma_S\right|^2}{\left|1+\Gamma_S\right|^2}$$

Noise Figure Parameter N_i

Putting the new expressions for G_S and $|Y_S - Y_{opt}|^2$ into the formula for F_{min} gives:

$$F = F_{\min} + \frac{4R_N}{Z_0}\frac{\left|\Gamma_S - \Gamma_{opt}\right|^2}{\left(1-\left|\Gamma_S\right|^2\right)\left|1+\Gamma_{opt}\right|^2} \Rightarrow \frac{\left|\Gamma_S - \Gamma_{opt}\right|^2}{\left(1-\left|\Gamma_S\right|^2\right)} = \frac{F - F_{\min}}{4R_N / Z_0}\left|1+\Gamma_{opt}\right|^2$$

For a given bias point and frequency, F_{min}, Γ_{opt} and R_N are constant. If we consider the noise factor F to be constant as well, all the RHS of the expression above can be used to define another constant, N_i, the noise factor parameter i.e.

$$N_i = \frac{F - F_{\min}}{4R_N / Z_0}\left|1+\Gamma_{opt}\right|^2 \Rightarrow \frac{\left|\Gamma_S - \Gamma_{opt}\right|^2}{\left(1-\left|\Gamma_S\right|^2\right)} = N_i \Rightarrow \left|\Gamma_S - \Gamma_{opt}\right|^2 = N_i\left(1-\left|\Gamma_S\right|^2\right)$$

$$\Rightarrow \left(\Gamma_S - \Gamma_{opt}\right)\left(\Gamma_S^{\;*} - \Gamma_{opt}^{\;*}\right) = N_i\left(1-\left|\Gamma_S\right|^2\right) \Rightarrow \Gamma_S\Gamma_S - \Gamma_{opt}\Gamma_S^{\;*} - \Gamma_{opt}^{\;*}\Gamma_S + \Gamma_{opt}\Gamma_{opt}^{\;*} = N_i - N_i\left|\Gamma_S\right|^2$$

$$\Rightarrow \left|\Gamma_S\right|^2\left(N_i + 1\right) - \Gamma_{opt}\Gamma_S^{\;*} - \Gamma_{opt}^{\;*}\Gamma_S = N_i - \left|\Gamma_{opt}\right|^2$$

$$\Rightarrow \left|\Gamma_S\right|^2 - \frac{\Gamma_{opt}^{\;*}}{N_i + 1}\Gamma_S - \frac{\Gamma_{opt}}{N_i + 1}\Gamma_S^{\;*} = \frac{N_i - \left|\Gamma_{opt}\right|^2}{N_i + 1}$$

Noise Circles

$$\left|\Gamma_S\right|^2 - \frac{\Gamma_{opt}^{\;*}}{N_i+1}\Gamma_S - \frac{\Gamma_{opt}}{N_i+1}\Gamma_S^{\;*} = \frac{N_i - \left|\Gamma_{opt}\right|^2}{N_i+1}$$

The formula looks like $\left|\Gamma\right|^2 - C^*\Gamma - C\Gamma^* + \left|C\right|^2 = R^2$ except $|C|^2$ has to be added to each side:

$$\left|\Gamma_S\right|^2 - \frac{\Gamma_{opt}^{\;*}}{N_i+1}\Gamma_S - \frac{\Gamma_{opt}}{N_i+1}\Gamma_S^{\;*} + \frac{\left|\Gamma_{opt}\right|^2}{\left(N_i+1\right)^2} = \frac{N_i - \left|\Gamma_{opt}\right|^2}{N_i+1} + \frac{\left|\Gamma_{opt}\right|^2}{\left(N_i+1\right)^2}$$

Tidying up the R.H.S. gives

$$\left|\Gamma_S\right|^2 - \frac{\Gamma_{opt}^{\;*}}{N_i+1}\Gamma_S - \frac{\Gamma_{opt}}{N_i+1}\Gamma_S^{\;*} + \frac{\left|\Gamma_{opt}\right|^2}{\left(N_i+1\right)^2} = \frac{N_i\left(N_i+1-\left|\Gamma_{opt}\right|^2\right)}{\left(N_i+1\right)^2}$$

This defines a circle on the reflection coefficient plane (the Smith chart) with centre and radius given by:

$$C_{Fi} = \frac{\Gamma_{opt}}{N_i+1} \qquad R_{Fi} = \frac{\sqrt{N_i\left(N_i+1-\left|\Gamma_{opt}\right|^2\right)}}{\left(N_i+1\right)}$$

Thus points of constant noise factor (or noise figure) form circles on the Smith chart and these circles can be used to see how the noise figure degrades as the source reflection coefficient moves away from Γ_{opt}.

Plotting the Noise Circles

Circles can be plotted by selecting a range of noise factors, F_i, and calculating N_i for each value and then calculating C_{Fi} and R_{Fi}.

$$N_i = \frac{F_i - F_{\min}}{4R_N / Z_0} \left|1 + \Gamma_{opt}\right|^2 \qquad C_{Fi} = \frac{\Gamma_{opt}}{N_i + 1} \qquad R_{Fi} = \frac{\sqrt{N_i\left(N_i + 1 - \left|\Gamma_{opt}\right|^2\right)}}{\left(N_i + 1\right)}$$

In the formulas the noise factor F is a ratio so if it is specified in dB (i.e. a noise figure) it has to be converted to a ratio first i.e.

$$F_{dB} = 10 * \log_{10}(F_{ratio}) \Rightarrow F_{ratio} = 10^{\frac{F_{dB}}{10}}$$

When $F_i = F_{\min}$, $N_i = 0$, $C_{Fi} = \Gamma_{opt}$ and $R_{Fi} = 0$ i.e. the minimum noise occurs at only one point, $\Gamma_S = \Gamma_{opt}$.

Example

A transistor has the following noise parameters measured using a system with a characteristic impedance of 50Ω:

$$F_{\min} = 2\ dB$$

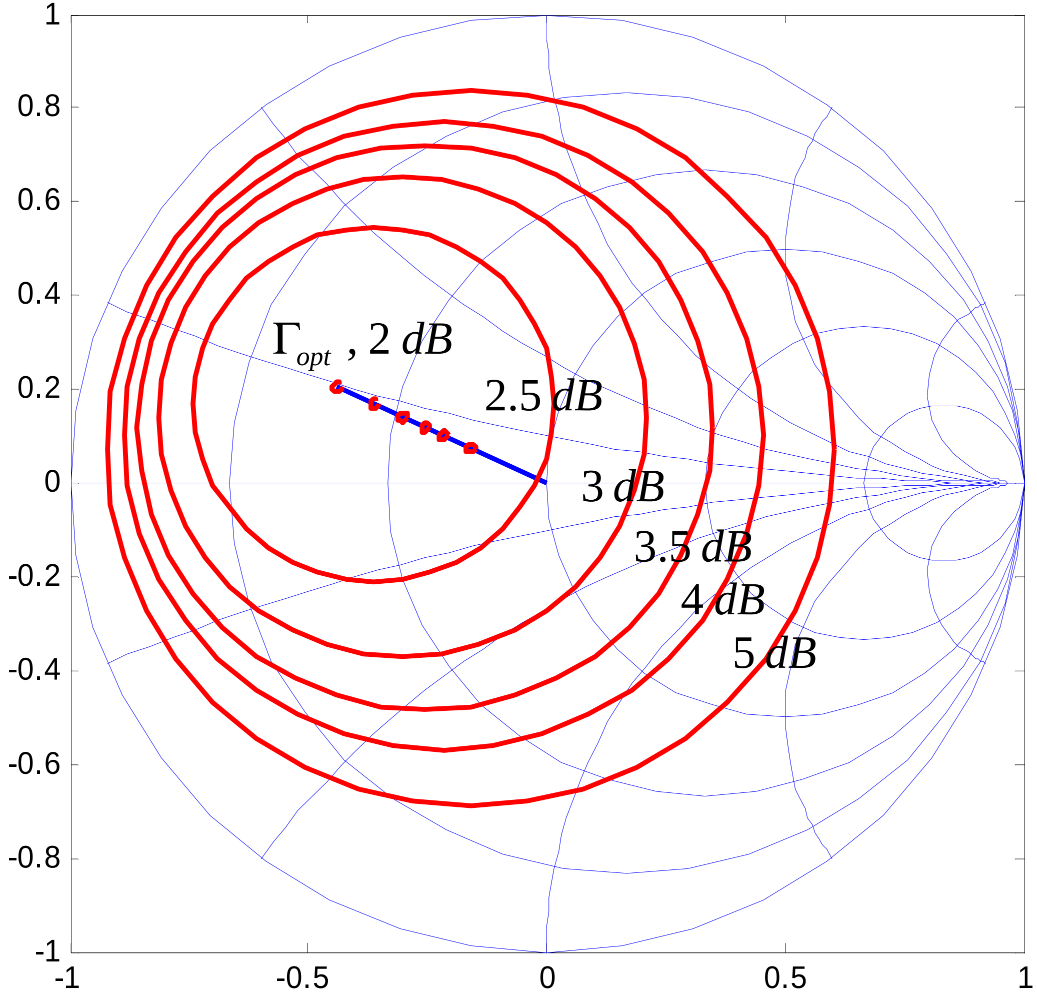
$$\Gamma_{opt} = 0.485\angle 155°$$

$$R_N = 4\ \Omega$$

Plot the constant noise circles for noise figures of 2.5dB, 3.0dB, 3.5dB, 4.0dB and 5.0dB.

Example Noise Circles

dB F_i	ratio F_i	N_i	C_{Fi}	R_{Fi}
2.5	1.78	0.22	$0.40\angle 155°$	0.38
3	2.00	0.46	$0.33\angle 155°$	0.51
3.5	2.24	0.73	$0.28\angle 155°$	0.60
4	2.51	1.03	$0.24\angle 155°$	0.67
5	3.16	1.76	$0.18\angle 155°$	0.76



Note: as the radius of the noise circle is increasing the noise is also increasing.

Trade-off of Gain vs. Noise

A transistor has the following parameters:

$$s_{11} = 0.60\angle -60° \quad s_{11}^{*} = 0.60\angle 60°$$

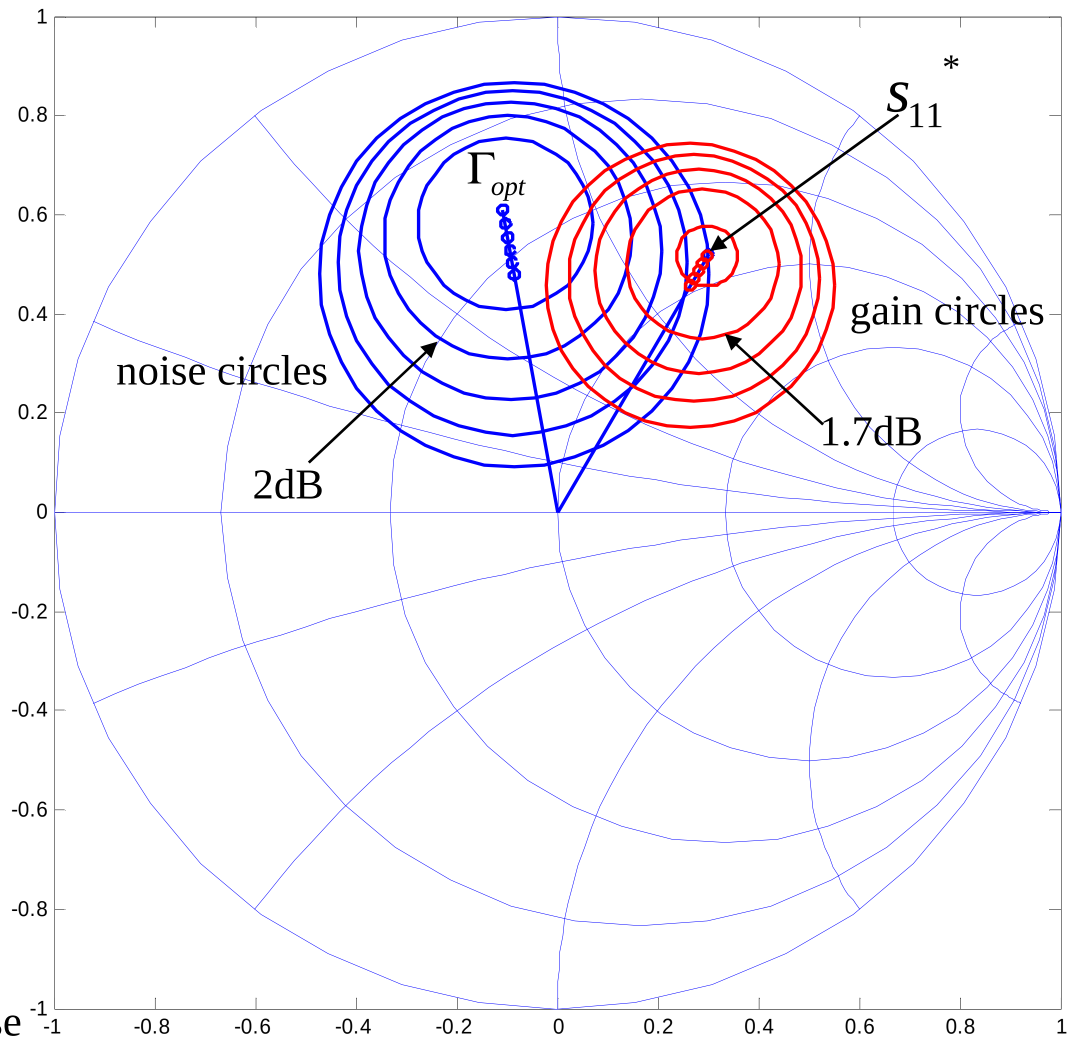
$$F_{\min} = 1.6\ dB$$

$$\Gamma_{opt} = 0.62\angle 100°$$

$$R_N = 20\ \Omega$$

$$(Z_0 = 50\Omega)$$

To obtain maximum gain from the source term, Γ_S should be set to s_{11}^*. But to obtain minimum noise, Γ_S should be set to Γ_{opt}. In this case maximum gain and minimum noise cannot be achieved simultaneously and the gain and noise circles show the tradeoffs that exist.



In the plot the innermost noise circle is 1.8dB and the noise increases in 0.2dB steps. The innermost gain circle is 1.9dB and the gain decreases in 0.2dB steps. As an example of the tradeoffs, if the noise figure cannot exceed 2dB then the maximum gain that can be achieved is 1.7dB.

Designing for High Gain/Low-Noise - 1

The examples so far have involved designing for maximum gain or targeting a specific gain, in the absence of noise considerations. In many applications the amplifier noise figure is also specified as a design goal so this has to be taken into account – we may have to design for maximum gain or design for a specific gain and also meet a minimum noise criterion.

A BJT has the following s-parameters and noise parameters at 1GHz (in a 50Ω system). Draw sample gain and noise circles for the input matching network to illustrate the trade-offs between gain and noise.

$$s_{11} = 0.707\angle -155^\circ \quad s_{12} = 0 \quad s_{21} = 5.00\angle 180^\circ \quad s_{22} = 0.51\angle -20^\circ$$

$$F_{\min} = 3\,dB \quad \Gamma_{opt} = 0.45\angle 180^\circ \quad R_N = 4\,\Omega$$

Stability check:

$$\Delta = s_{11}s_{22} - s_{12}s_{21} = 0.36\angle -175^\circ \quad K = \frac{1 - |s_{11}|^2 - |s_{22}|^2 + |\Delta|^2}{2|s_{12}s_{21}|} = \infty$$

$$K > 1 \quad and \quad |\Delta| < 1$$

The device is unconditionally stable so the input and output matching networks can be designed without reference to stability.

Designing for High Gain/Low-Noise - 2

$$G_{S,\max} = \frac{1}{1-|s_{11}|^2} = \frac{1}{1-|0.707|^2} = 2 = 3dB$$

$$G_0 = |5|^2 = 25 = 14dB$$

$$G_{L,\max} \frac{1}{1-|s_{22}|^2} = \frac{1}{1-|0.51|^2} = 1.35 = 1.3dB$$

The maximum unilateral transducer gain is

$$G_{TU,\max,dB} = G_{S,\max,dB} + G_{0,dB} + G_{L,\max,dB} = 3dB + 14dB + 1.3dB = 18.3dB$$

Because s_{12}=0, there is no error involved in the unilateral approximation for gain i.e. M=0.

The maximum source gain is 3dB so it is interesting to plot source gain circles in steps of 0.1dB from 2dB to 2.9dB
The minimum noise figure is 3dB so we will draw noise circles from 3.1dB to 4dB.

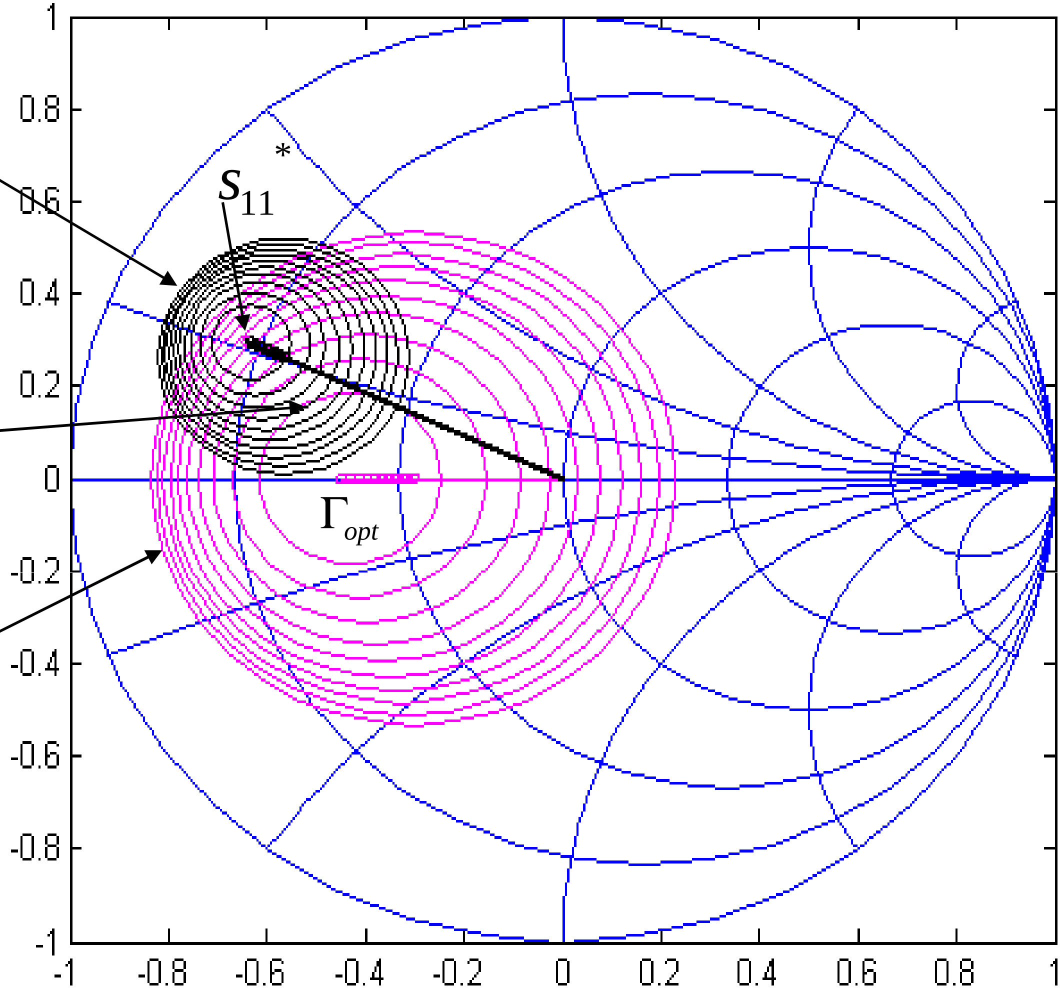
What is the maximum source gain that can be achieved if the noise figure cannot exceed 3.1dB?

Designing for High Gain/Low-Noise - 3

Gain circles from 2.0dB (outer circle) to 2.9dB (inner circle) in 0.1dB steps

The 2.6dB gain circle overlaps slightly with the 3.1dB noise circle so a gain of 2.6dB is the best that can be achieved if the noise figure cannot exceed 3.1dB

Noise circles from 3.1dB (inner circle) to 4dB (outer circle) in 0.1dB steps.



Noise does not place any restrictions on the load matching network so this can still be designed for maximum gain if desired. The maximum gain that can be achieved that still meets the noise requirement is then:

$$G_{TU} = 2.6dB + 14dB + 1.3dB = 17.9dB$$

Designing for High Gain/Low-Noise - 4

The source gain circle for $G_{S,dB} = 2.6$

$$G_{S,dB} = 10\log_{10}(G_S) \Rightarrow G_S = 10^{\frac{G_{S,dB}}{10}} = 10^{\frac{2.6}{10}} = 1.820 \qquad g_s = \frac{G_S}{G_{S,\max}} = \frac{1.820}{2} = 0.910$$

$$|C_S| = \frac{g_s |s_{11}|}{1-|s_{11}|^2(1-g_s)} = \frac{0.910 \times |0.707|}{1-|0.707|^2(1-0.910)} = 0.67$$

$$R_S = \frac{\sqrt{1-g_s}\left(1-|s_{11}|^2\right)}{1-|s_{11}|^2(1-g_s)} = \frac{\sqrt{1-0.910}\left(1-|0.707|^2\right)}{1-|0.707|^2(1-0.910)} = 0.16$$

The centre of the 2.6dB source gain circle is a distance 0.67 along the line joining the origin and the point s_{11}^* and its radius is 0.16

The noise circle for F_i = 3.1dB

$$F_{dB} = 10 * \log_{10}(F_{ratio}) \Rightarrow F_{ratio} = 10^{\frac{F_{dB}}{10}} \qquad F_{\min,ratio} = 10^{\frac{3}{10}} = 1.9953 \qquad F_{i,ratio} = 10^{\frac{3.1}{10}} = 2.0417$$

$$N_i = \frac{F_{i,ratio} - F_{\min,ratio}}{4R_N / Z_0} \left|1+\Gamma_{opt}\right|^2 = \frac{2.0417 - 1.9953}{4 \times 4/50} |1-0.45|^2 = 0.0439$$

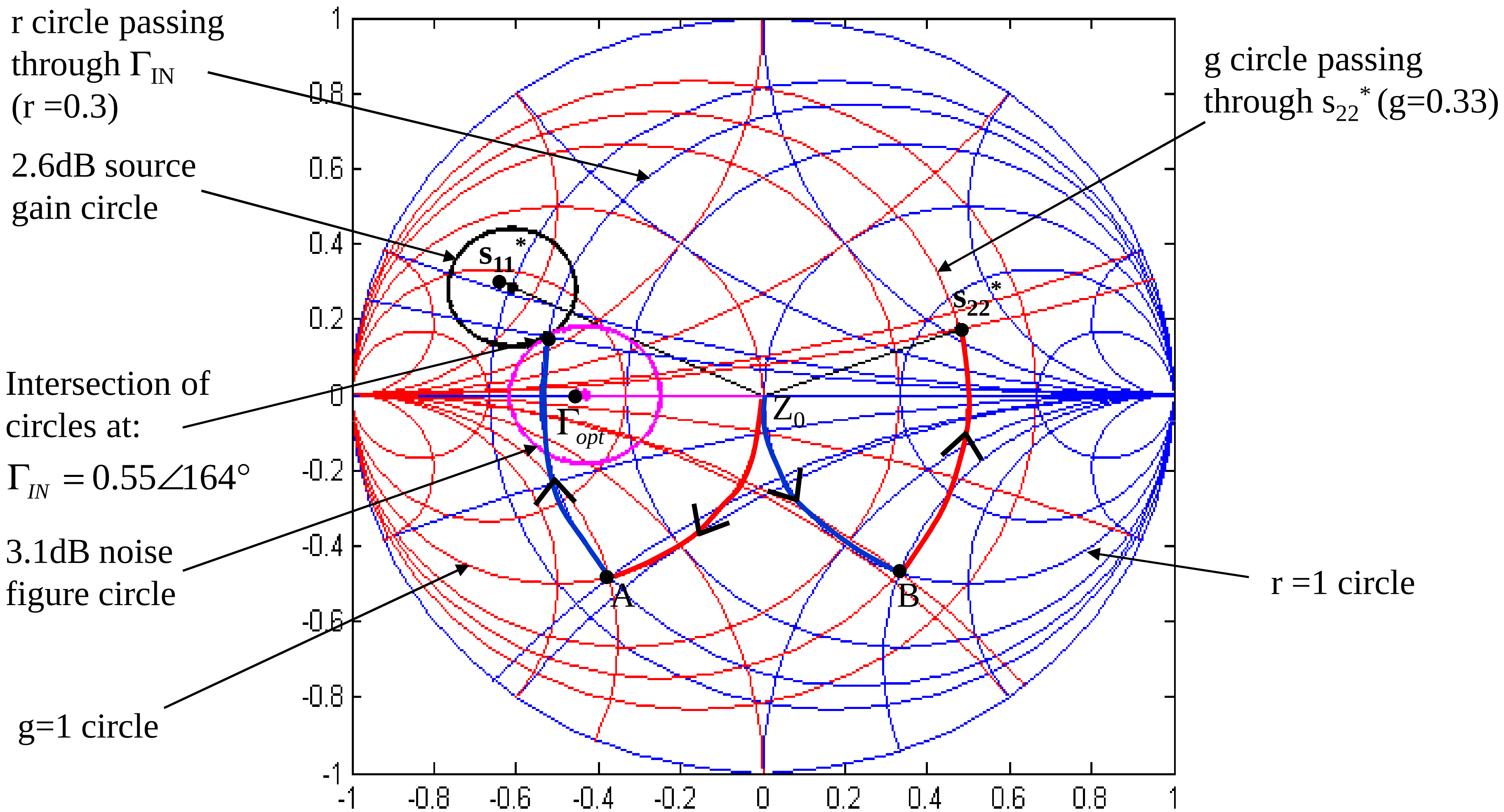
$$\Gamma_{opt} = 0.45\angle 180° = -0.45$$

$$C_{Fi} = \frac{\Gamma_{opt}}{N_i + 1} = \frac{-0.45}{0.0439+1} = -0.43 = 0.43\angle 180°$$

$$R_{Fi} = \frac{\sqrt{N_i\left(N_i + 1 - |\Gamma_{opt}|^2\right)}}{(N_i+1)} = \frac{\sqrt{0.0439\left(0.0439+1-|0.45|^2\right)}}{(0.0439+1)} = 0.18$$

The centre of the 3.1dB noise circle is at Γ = -0.43 and it has a radius of 0.18.

Designing for High Gain/Low-Noise - 5



The goal of the input matching network is to shift $\Gamma=0$ (Z_0) to the point $\Gamma=\Gamma_{IN}$

The goal of the output matching network is to shift $\Gamma=0$ (Z_0) to the point s_{22}^* as before

Designing for High Gain/Low-Noise - 6

Input Matching Element Values

Moving from Z_0 (Γ=0) to point A: Clockwise on conductance circle – shunt capacitor

susceptance at Z_0: b = 0
susceptance at A: b = 1.54

$$C = \frac{|\Delta b|}{2\pi f Z_0} = \frac{|1.54|}{2\pi \times 1 \times 10^9 \times 50} = 4.91 pF$$

Moving from A to Γ_{IN}: Clockwise on resistance circle – series inductor

reactance at A: x = -0.46
reactance at Γ_{IN}: x = 0.13

$$L = \frac{Z_0 |\Delta x|}{2\pi f} = \frac{50 \times |0.59|}{2\pi \times 1 \times 10^9} = 4.69 nH$$

Output Matching Element Values

Moving from Z_0 (Γ=0) to point B: Anti-clockwise on resistance circle – series capacitor

reactance at Z_0: x = 0
reactance at B: x = -1.41

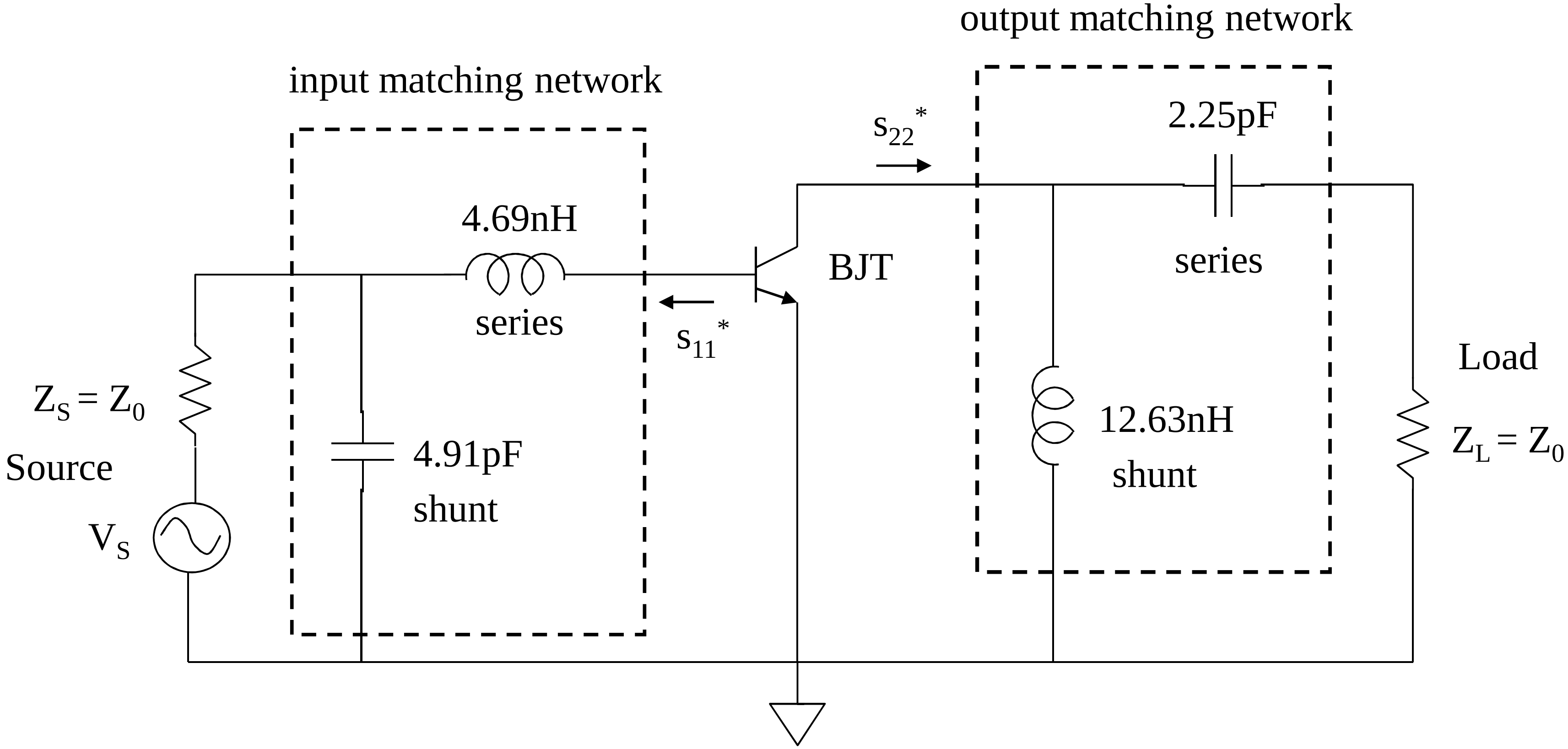
$$C = \frac{1}{2\pi f |\Delta x| Z_0} = \frac{1}{2\pi \times 1 \times 10^9 \times |-1.41| \times 50} = 2.25 pF$$

Moving from B to s_{22}^*: Anti-clockwise on conductance circle – shunt inductor

susceptance at B: b = 0.47
susceptance at s_{22}^*: b = -0.16

$$L = \frac{Z_0}{2\pi f |\Delta b|} = \frac{50}{2\pi \times 1 \times 10^9 \times |-0.63|} = 12.63 nH$$

Designing for High Gain/Low-Noise - 7



Note: This example concerned designing for the highest gain possible but without exceeding a specified noise figure. In some cases it might be necessary to design for a specific gain also without exceeding a given noise figure. In that case, it may be necessary to use the load gain circles to find an appropriate load matching network to give a particular load gain as opposed to just matching the load for maximum gain as was done here.

An example to try out

A low-noise GaAs FET has the following s-parameters and noise parameters measured at V_{DS}=3V and I_{DS}=10mA at 16GHz with a 50Ω system:

$$s_{11} = 0.57\angle -111^\circ \quad s_{12} = 0.01\angle -31^\circ \quad s_{21} = 2.09\angle -42^\circ \quad s_{22} = 0.47\angle -69^\circ$$

$$F_{\min} = 2.5\,dB \quad \Gamma_{opt} = 0.48\angle 175^\circ \quad R_N = 4\,\Omega$$

Design input and output matching networks to give a power gain of 9dB and a noise figure of 4.5dB, assuming 50Ω source and load impedances.

Note : The Smith Chart designs shown in the lectures have used Smith Charts drawn by a Matlab program for clarity – please try all these designs on "real" Smith chart paper for yourself and repeat all the calculations on your own calculators to ensure that you follow each step and that you can read the correct values of reactance and susceptance from the Smith Chart and convert these to inductance or capacitance values at a given frequency.