Section 2/3

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Additive White Gaussian Noise Channel.

In a continuous (analogue) channel an information source produces a continuous signal $x(t)$. The set of possible signals is considered as an "ensemble of waveforms" generated by some ergodic random proccess.

ergodic - all of the process' ~~(relevant)~~ relevant statistics can be obtained by observing a single sample of the process.

It is further assumed that $x(t)$ has a finite bandwidth and is therefore, completely characterised by its periodic sample values (i.e. Shannon's sampling theorem). Thus, at any sampling instant, the collection of possible sample values constitutes a continuous random variable x described by its probability density function $f_x(x)$.

Differential Entropy

The average amount of information per sample value of $x(t)$ is measured by:-

$$H(x) = -\int_{-\infty}^{\infty} f_x(x) \log_2 [f_x(x)]\, dx$$

b/sample.

This is known as the "differential" entropy of x.
The average mutual info in a continuous channel is defined by (in analogy with the discrete case):-

$$I(x;y) = H(x) - H(x|y)$$
$$= H(y) - H(y|x)$$

where

$$H(y) = -\int_{-\infty}^{\infty} f_y(y) \log_2 [f_y(y)]\, dy$$

and

$$H(x|y) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y) \log_2 [f_{x|y}(x|y)]\, dx\, dy$$

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$$H(y/x) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y) \log_2 [f_y(y/x)]\, dx\, dy$$

Additive white Gaussian Noise Channel.

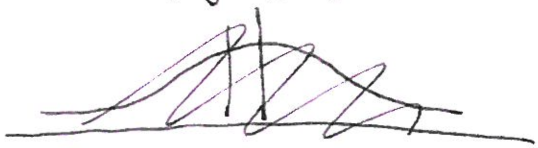
In the additive white Gaussian noise (AWGN) channel, the channel output y is given by:-

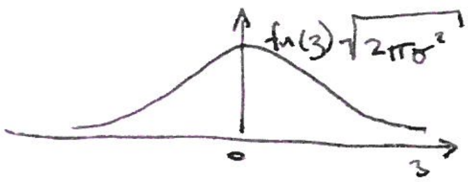
$$y = x + n$$

x: channel input

n: additive band limited white Gaussian noise with zero mean and variance σ^2..

n has the pdf: $f_n(z) = e^{-z^2/2\sigma^2}$





It can be shown that (shannon) the capacity C_s of an AWGN channel is given by:-

$$C_s = \max_{\{f_x(x)\}} [I(x;y)] = \frac{1}{2}\log_2\left(1+\frac{S}{N}\right) \text{ b/sample.}$$

where S/N is the signal power to noise power ratio at the channel output

If the channel bandwidth B Hz is fixed, then the output $y(t)$ is also a band-limited signal completely characterised by its periodic sample values taken at the Nyquist rate of 2B samples/s. Then, the capacity C (b/s) of the AWGN channel is:-

$$C = 2BC_s = B\log_2\left[1 + \frac{S}{N}\right] \text{ b/s}$$

Shannon Hartley Law for an average power limited input S

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Given the probabilities of correct and incorrect transmission, the capacity of the channel can be calculated, i.e.

$$C_s = \underbrace{\max}_{P(x_i)}\left[I(X;Y)\right]$$

Hence, we now must address the calculation of the individual probabilities themselves.

Statistical Decision Theory

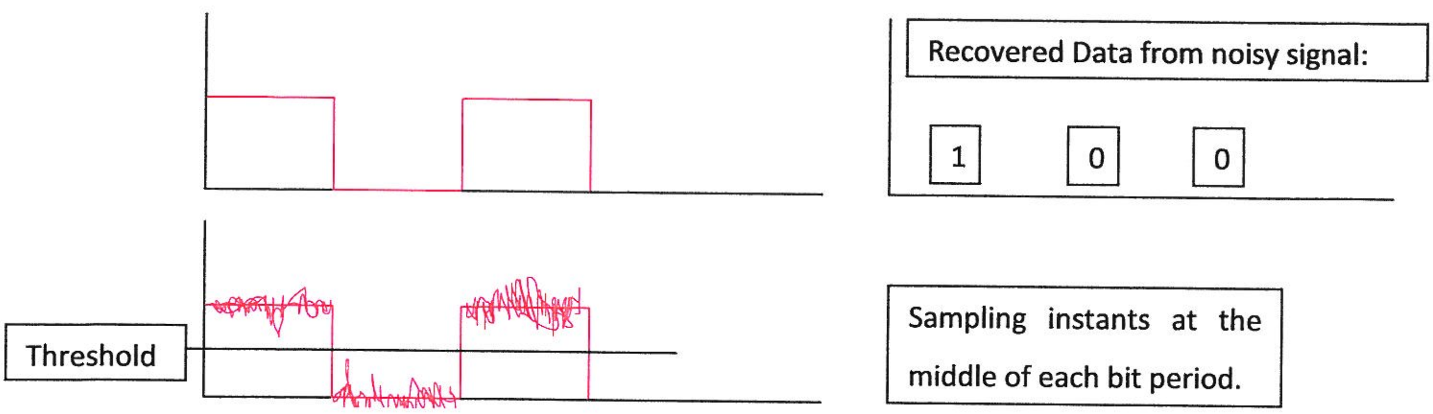
Initially, we will deal with a particularly simple case, to establish ideas and techniques. Subsequently, we will address the more general question of optimised receiver design (minimising the probability of error P_e).

In the first case, the receiver will take a single sample during each bit signal interval and then decides that zero was transmitted or one was transmitted. Clearly, two error types can occur, namely: decide 1 when 0 was transmitted and decide 0 when 1 was transmitted. To minimise the total probability of error, the minimisation must be based on both error types.

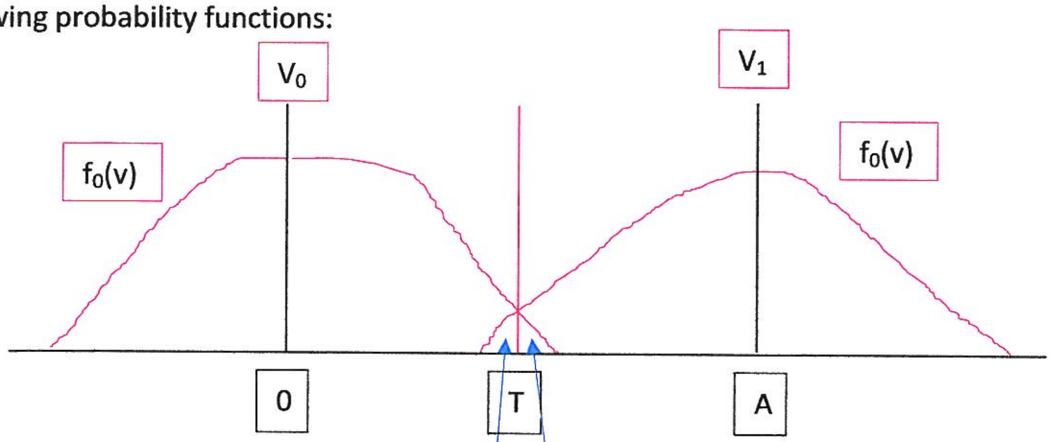
The decision rule (implemented in the receiver) is base on partitioning off values of the received voltage v into two regions V_0 and V_1. The boundary between the regions (i.e. the threshold) is chosen to minimise the total error probability. To illustrate this, we assume P_0 denotes the probability of transmitting 0 and P_1 the probability of transmitting 1. Clearly, we know that probabilities add to 1:

$$P_0 + P_1 = 1$$

The probability that the voltage sample V will fall into sample V_0 when logic 1 was sent is:



We assume our noise is additive white Gaussian noise (AWGN) and so we can draw the following probability functions:



Threshold (equiprobable case)

<u>End Aside</u>

$$\int_{V_0} f_1(v)\,dv$$

Where $f_1(v)$ is the *pdf* of the received voltage when a 1 is sent. The probability that v will fall into the region V_1 when a 0 is sent is:

$$\int_{V_1} f_0(v)\,dv$$

Where $f_0(v)$ is the *pdf* of the received voltage when 0 is sent. The overall probability of error is then:

$$P_e = P_1\int_{V_0} f_1(v)\,dv + P_0\int_{V_1} f_0(v)\,dv$$

Probability of sending 1

Probability of getting it wrong.

Probability of sending 0

Since the union of the regions V_0 and V_1 covers all possible values of V (i.e. $-\infty \leq v \leq \infty$), hence

$$\int_{V_0 \cup V_1} f_1(v)\,dv = 1 = \int_{V_0} f_1(v)\,dv + \int_{V_1} f_1(v)\,dv$$

$$\Rightarrow P_e = P_1 + \int_{V_1} \left(P_0 f_0(v) - P_1 f_1(v)\right) dv$$

We wish to make the integral as negative as possible to reduce the probability of error.

$$\Rightarrow P_1 f_1(v) > P_0 f_0(v)$$

The required decision rule is:

$$\frac{f_1(v)}{f_0(v)} > \frac{P_0}{P_1}$$

$$\lambda = \frac{f_1(v)}{f_0(v)}, \quad \text{known as the "likelihood ratio".}$$

In our case:

$$f_0(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v^2/2\sigma^2}$$

$$f_1(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(v-A)^2/2\sigma^2}$$

since 0 volts represents 0 and A volts represents 1.

The region V_1 is defined by all values of v for which $\lambda > P_o/P_1$.

$$\Rightarrow \frac{e^{-(v-A)^2/2\sigma^2}}{e^{-v^2/2\sigma^2}} > \frac{P_0}{P_1}$$

Taking logarithms we have:

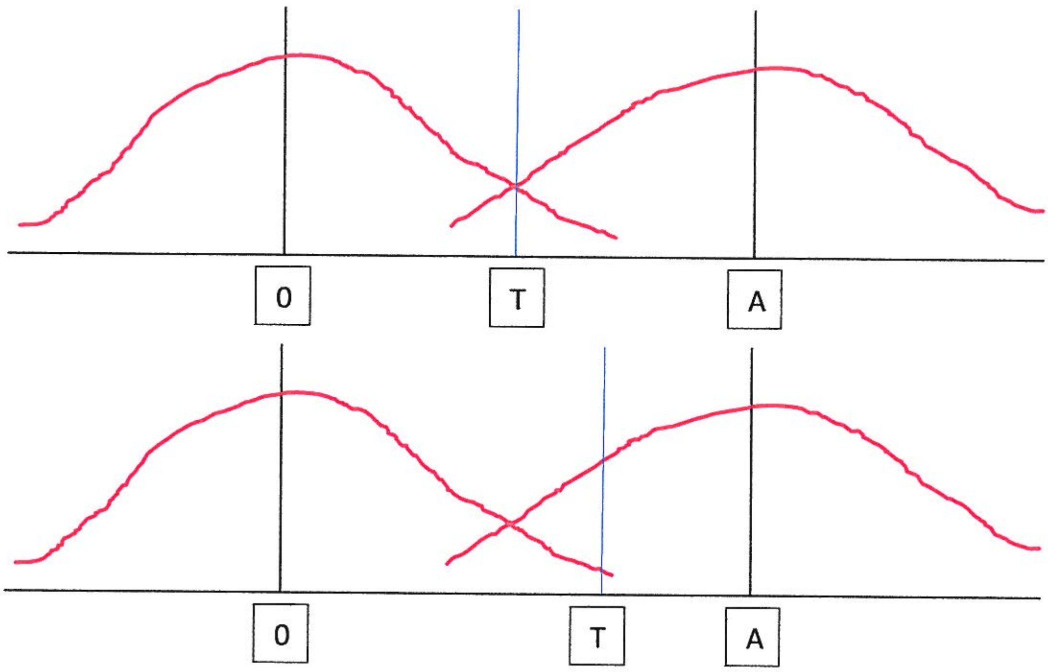
$$v^2 - (v-A)^2 > 2\sigma^2 \ln\left(\frac{P_0}{P_1}\right)$$

$$\Rightarrow V > \frac{A}{2} + \frac{\sigma^2}{A} \ln\left(\frac{P_0}{P_1}\right)$$

Hence the boundary between the regions V_0 and V_1 is:

$$T = \frac{A}{2} + \frac{\sigma^2}{A} \ln\left(\frac{P_0}{P_1}\right)$$

If $P_0 = P_1$ then the threshold $T = {}^{A}/_{2}$ (i.e. midway between 0v and Av) this is shown below:



"Bias in favour of the more popular symbol" in case 2.

Once the decision threshold has been established, the total probability of error can be calculated. Here, we assume $P_0=P_1$. If binary 0 is sent, then the probability that it will be received as a 1 is given by:

$$P_{e0} = \int_{A/2}^{\infty} \frac{e^{-v^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dv$$

i.e. probability that the noise exceeds ${}^{A}/_{2}$. If a 1 is sent, the probability it will be received as a 0 is:-

$$P_{e1} = \int_{-\infty}^{A/2} \frac{e^{-(v-A)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dv$$

$$= \int_{-\infty}^{-A/2} \frac{e^{-v^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dv$$

Given the symmetry of the Gaussian *pdf* we have $P_{e0}=P_{e1}$. Hence the total probability of error, P_e is:

$$\begin{aligned}P_e &= P_0 P_{e0} + P_1 P_{e1}\\ &= P_{e1}(P_0 + P_1), \text{ since } P_{e0} = P_{e1}\\ \Rightarrow P_e &= P_{e1}\\ &= \int_{-\infty}^{-A/2} \frac{e^{-v^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dv\end{aligned}$$

To manipulate this into a standard form we can write:

$$P_e = \underbrace{\int_{-\infty}^{0} \frac{e^{-v^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dv}_{\frac{1}{2}} - \int_{-A/2}^{0} \frac{e^{-v^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dv$$

$$\Rightarrow P_e = \tfrac{1}{2} - \int_{0}^{A/2} \frac{e^{-v^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dv$$

If we substitute

$y = \dfrac{v}{\sqrt{2\sigma^2}}$, this becomes

$$P_e = \tfrac{1}{2} - \tfrac{1}{\sqrt{\pi}}\int_{0}^{\frac{A}{2\sigma\sqrt{2}}} e^{-y^2}\,dy$$

Now, the error function $\operatorname{erf}(x)$ is defined by:

$$erf(x) = \tfrac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-y^2}\,dy$$

$$\Rightarrow P_e = \tfrac{1}{2}\left[1 - erf\left(\frac{A}{2\sigma\sqrt{2}}\right)\right]$$

P_e depends solely on the ratio of peak pulse voltage A to the *rms* noise voltage σ. From the graph, when $A/_\sigma$ = 17.4dB (i.e. voltage ratio17.4:1), $P_e = 10^{-4}$, i.e. one bit in 10^4 will be in error (on average). If $A/_\sigma$ is increased to 21dB, P_e drops to $P_e = 10^{-8}$. Hence, a very large decrease in error probability occurs for a change of only 3.6dB in $A/_\sigma$. This decrease is much smaller below 14dB (for $A/_\sigma$). Hence a "thresholding" effect occurs for values of $A/_\sigma$ around 18dB. Usually, $P_e = 10^{-5}$ is a common target.