Control Engineering Summer '08

4 b ii) For the MIMO process: $\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$
$y(t) = C\underline{x}(t)$

The state trajectory is given by:

$$\underline{x}(t) = \phi(t)\underline{x}(0) + \int_0^t \phi(t-\tau) B\underline{u}(\tau)d\tau$$

If initial time is t_0:

$$\underline{x}(t) = \phi(t-t_0)x(t_0) + \int_{t_0}^t \phi(t-\tau) B\underline{u}(\tau)d\tau$$

Over a timestep T, $t_0 = kT$ $t = (k+1)T$

$$x((k+1)T) = \phi((k+1)T - kT)x(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T-\tau)B\underline{u}(\tau)d\tau$$

For ZOH, $u(\tau) = \underline{u}(kT)$ over integral

Substituting $\eta = (k+1)T - \tau$
$d\eta = -d\tau$

$$x((k+1)T) = \phi(T)x(kT) + \int_0^T \phi(\eta)Bd\eta u(k)$$

$\Rightarrow$ Simplify: $(k+1)T \rightarrow (k+1)$, $kT \rightarrow k$

$x(k+1) = A_d\underline{x}(k) + B_d\underline{u}(k)$ $\quad A_d = \phi(T) = e^{AT}$

$y(k) = C\underline{x}k$

$$\phi(T) = I + AT + \frac{AT^2}{2!} + \dots$$
$$\simeq I + AT \quad \text{(as T is small)}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 7t & 1 & 0 \\ 0 & t & 1 \end{bmatrix}$$

$$B_d = \int_0^T \phi(\eta)d\eta = \int_0^T \left(I + \frac{A\eta}{1} + \frac{A^2\eta^2}{2!} + \dots \right)$$
$$= I\eta + A\frac{\eta^2}{2} + A^2\frac{\eta^3}{6} + \dots \Big|_0^T \times B$$
$$\simeq IT \times B$$
$$= \begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} kT \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ The representation is valid.