The state trajectory is given by:

$$\underline{x}(t) = \phi(t)\underline{x}(0) + \int_0^t \phi(t-\tau)B\underline{u}(\tau)d\tau$$

where the initial state at time 0 is $\underline{x}(0)$

Now consider that the initial time is t_0, with initial state $\underline{x}(t_0)$

Then the trajectory could be rewritten as:

$$\underline{x}(t) = \phi(t-t_0)\underline{x}(t_0) + \int_{t_0}^t \phi(t-\tau)B\underline{u}(\tau)d\tau$$

Consider what happens to the state vector over a time step T:

$t_0 = kT$

$t = (k+1)T$

$$\underline{x}((k+1)T) = \phi((k+1)T - kT)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T-\tau)B\underline{u}(\tau)d\tau$$

$$\underline{x}((k+1)T) = \phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T-\tau)B\underline{u}(\tau)d\tau$$

If we assume that a zero-order hold is utilised

$\Rightarrow \underline{u}(t) = \underline{u}(kT)$ for $kT \leq t < (k+1)T$

$$\underline{x}((k+1)T) = \phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T-\tau)B\,d\tau\,\underline{u}(kT)$$

Now make the following substitution

$\eta = (k+1)T - \tau$

$d\eta = -d\tau$

$$\underline{x}((k+1)T) = \phi(t)\underline{x}(kT) - \int_T^0 \phi(\eta)B\,d\eta\,\underline{u}(kT)$$

$(k+1)T \rightarrow (k+1)$

$kT \rightarrow k$

$$\underline{x}(k+1) = \phi(T)\underline{x}(k) + \int_0^T \phi(\eta)B\,d\eta\,\underline{u}(k)$$

This yields discrete time state space equations

$$\underline{x}(k+1) = A_d\,\underline{x}(k) + B_d\,\underline{u}(k)$$

$$\underline{y}(k) = C\,\underline{x}(k)$$

$$A_d = \phi(T) = e^{AT} = I + AT + \frac{A^2T^2}{2!} + \dots$$

T is small

$$\Rightarrow A_d = I + AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 7T & 0 & 0 \\ 0 & T & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 7T & 1 & 0 \\ 0 & T & 1 \end{bmatrix}$$

$$B_d = \int_0^T \phi(\eta) B \, d\eta$$

$$= \int_0^T \left(I + A\eta + \frac{A^2\eta^2}{2} + \dots\right) B \, d\eta$$

$$= \left(I\eta + A\frac{\eta^2}{2} + A^2\frac{\eta^3}{6} + \dots\right)\Big|_0^T B$$

$$= \left(IT + A\frac{T^2}{2} + \dots\right) B$$

$$= ITB \quad (T \text{ is small})$$

$$= \begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} KT \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \theta(k+1) \\ \dot{x}(k+1) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 7T & 1 & 0 \\ 0 & T & 1 \end{bmatrix} \begin{bmatrix} \theta(k+1) \\ \dot{x}(k+1) \\ x(k+1) \end{bmatrix} + \begin{bmatrix} KT \\ 0 \\ 0 \end{bmatrix} v(k)$$

$$\begin{bmatrix} V_\theta(k) \\ V_2(k) \end{bmatrix} = \begin{bmatrix} K_\theta & 0 & 0 \\ 0 & 0 & K_x \end{bmatrix} \begin{bmatrix} \theta(k) \\ \dot{x}(k) \\ x(k) \end{bmatrix}$$

$$\det(zI - A_d) = 0$$

$$zI - A_d = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 7T & 1 & 0 \\ 0 & T & 1 \end{bmatrix} = \begin{bmatrix} z-1 & 0 & 0 \\ -7T & z-1 & 0 \\ 0 & -T & z-1 \end{bmatrix}$$

$$\det(zI - A_d) = (z-1)\begin{vmatrix} z-1 & 0 \\ -T & z-1 \end{vmatrix}$$
$$= (z-1)(z-1)(z-1)$$

$\det(zI - A_d) = 0$

$(z-1)^3 = 0$

$\Rightarrow$ 3 equal poles at $z = 1$

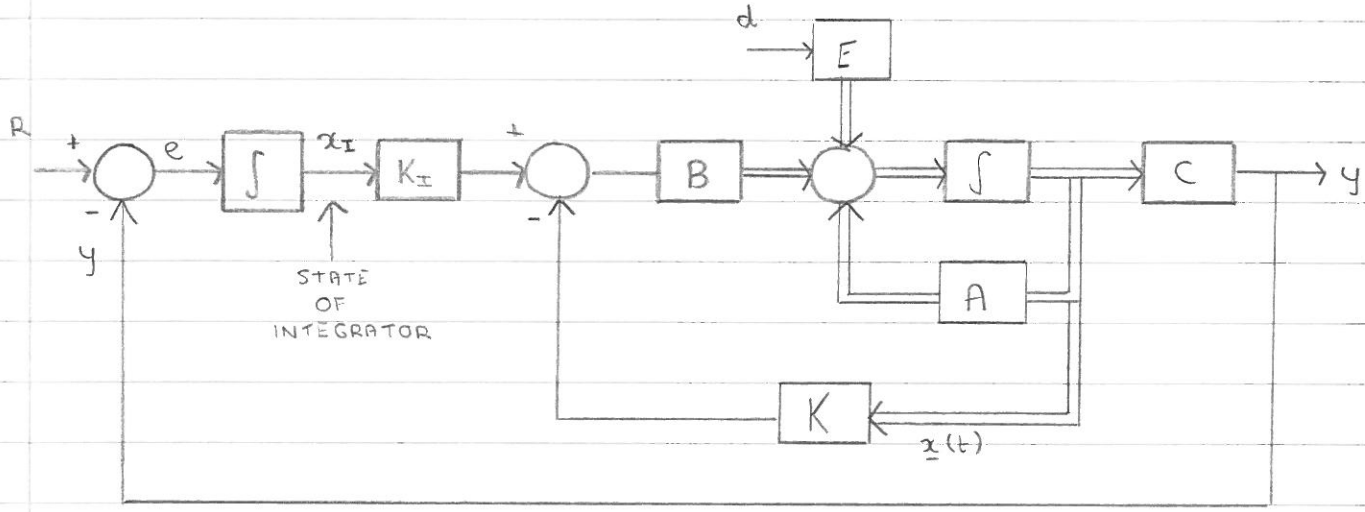
Q5(a). $\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$

$y(t) = C\underline{x}(t)$

The state control law with integral action is

$$u(t) = -K\underline{x}(t) + K_I\int_0^t e(\tau)\,d\tau$$

Where $e(t) = R(t) - y(t)$



Introduce another state

$$x_I(t) = \int_0^t e(\tau)\,d\tau$$

$u(t) = -K\underline{x}(t) + K_I x_I$

This yields closed loop state equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B(-K\underline{x}(t) + K_I x_I(t))$$
$$= (A - BK)\underline{x}(t) + BK_I x_I(t)$$

$$x_I(t) = \int_0^t e(\tau)\,d\tau$$

$$\Rightarrow \dot{x}_I(t) = e(t) = R(t) - y(t) = R(t) - C\underline{x}(t)$$

Assign a new state vector $\underline{z}(t) = \begin{bmatrix} \underline{x}(t) \\ x_I(t) \end{bmatrix} \begin{matrix} \updownarrow N \\ \updownarrow 1 \end{matrix}$

Closed loop equations can be written as

$$\frac{d}{dt}\begin{bmatrix} \underline{x}(t) \\ x_I(t) \end{bmatrix} = \left[\begin{array}{c|c} A-BK & BK_I \\ -C & 0 \end{array}\right]\begin{bmatrix} \underline{x}(t) \\ x_I(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} R(t)$$

$$\dot{\underline{z}} = A_2 \underline{z} + B_2 R(t)$$

Poles of closed loop system are given by roots of:

$$\det(sI - A_2) = 0$$

$$sI - A_2 = \begin{bmatrix} sI_N & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} A-BK & BK_I \\ -C & 0 \end{bmatrix}$$

$$\Rightarrow \det\begin{bmatrix} sI_N - A + BK & -BK_I \\ C & s \end{bmatrix} = 0$$

(b). $m \, {}^{d^2h}/_{dt^2} = mg - {}^{Kv^2(t)}/_{h(t)}$

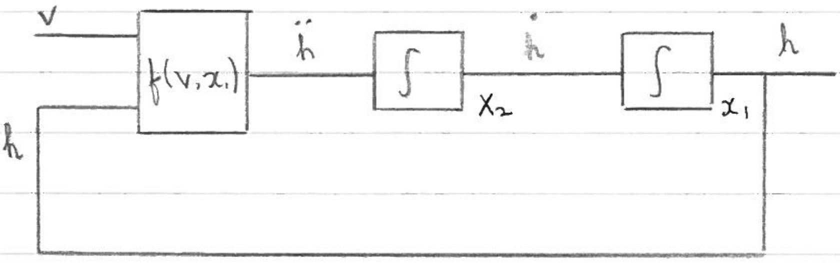
$m = 0.02\,kg \quad K = 2\times10^{-5}\,NmV^{-2} \quad g = 10\,ms^{-2}$

Find the operating point

$$0 = mg - K\,{}^{v^2(t)}/_{h(t)}$$

$$0 = 0.02(10) - 2\times10^{-5}\,{}^{V_0^2}/_{0.015}$$

$$V_0 = 12.25\,V$$



2 states

$$\dot{x}_2 = f(v, x_1)$$

$$\Delta\dot{x}_2 = {}^{\partial f}/_{\partial x_1}\big|_{op}\,\Delta x_1 + {}^{\partial f}/_{\partial v}\big|_{op}\,\Delta v$$

$$\frac{d^2h}{dt^2} = g - \frac{Kv^2(t)}{h(t)}$$

$$\alpha = \frac{\partial f}{\partial x_1} = \frac{Kv^2}{mh^2} = \frac{2\times10^{-5}(150)}{0.02(0.015)^2} = 667$$
$$\beta = \frac{\partial f}{\partial v} = \frac{-2Kv}{mh} = \frac{-4\times10^{-5}(12.25)}{0.02(0.015)} = -1.63$$

$$\Delta\dot{x}_2 = \alpha\Delta x_1 + \beta\Delta v$$
$$= 667\Delta x_1 - 1.63\Delta v$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 667 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.63 \end{bmatrix} v$$

Slope of graph $= \dfrac{5.5 - 3}{0.015 - 0.02} = -500$

$$y = -500h$$
$$\Rightarrow y = \begin{bmatrix} -500 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Use the control law $v(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + K_I\int_0^t (R_{x_1}(\tau) - y(\tau))\,d\tau$

$$\dot{\underline{x}}_I(t) = R_{x_1}(t) - C\underline{x}(t)$$
$$= R_{x_1}(t) - \begin{bmatrix} -500 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det\begin{bmatrix} sI_N - A + BK & -BK_I \\ C & s \end{bmatrix} = 0$$

$$\det\left[\begin{array}{c|c} \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 667 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1.63 \end{pmatrix}\begin{pmatrix} k_1 & k_2 \end{pmatrix} & -\begin{pmatrix} 0 \\ -1.63K_I \end{pmatrix} \\ -500 \quad 0 & s \end{array}\right] = 0$$

$$\det\left[\begin{array}{cc|c} \begin{pmatrix} s & -1 \\ -667 & s \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1.63k_1 & -1.63k_2 \end{pmatrix} & & \begin{matrix} 0 \\ 1.63K_I \end{matrix} \\ -500 \quad 0 & & s \end{array}\right] = 0$$

$$\det\begin{bmatrix} s & -1 & 0 \\ -667-1.63k_1 & s-1.63k_2 & 1.63K_I \\ -500 & 0 & s \end{bmatrix} = 0$$

$$s\begin{vmatrix} s-1.63k_2 & 1.63K_I \\ 0 & s \end{vmatrix} + \begin{vmatrix} -667-1.63k_1 & 1.63K_I \\ -500 & s \end{vmatrix} = 0$$

$$s\left[s(s-1.63k_2)\right] + s(-667-1.63k_1) + 1.63K_I(500) = 0$$
$$s^3 - 1.63k_2 s^2 + (-667-1.63k_1)s + 815K_I = 0$$

Want second order dominant response with
$\xi = 0.707 \quad \omega_N = 200$ rad/s

$$s^2 + 2\xi\omega_N + \omega_N^2$$
$$s^2 + 2(0.707)(200) + 200^2$$
$$s^2 + 282.8s + 40000$$

Place controller pole further out left
$$C_{des}(s) = (s+400)(s^2+282.8s+40000)$$
$$= s^3 + 682.8s^2 + 153120s + 16000000$$

$$815K_I = 16000000 \Rightarrow K_I = 19632$$
$$-667-1.63k_1 = 153120$$
$$k_1 = -94348$$
$$-1.63k_2 = 682.8 \Rightarrow k_2 = 419$$

$$\Rightarrow v(t) = 94348x_1 + 419x_2 + 19632\int_0^t (R_{x_1}(\tau) - \omega(\tau))\,d\tau$$

30/4/09

Summer 2008

Q6 (a). $\underline{u}(t) = -K\hat{\underline{x}}(t)$ $\qquad \frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t)$

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) - BK\hat{\underline{x}}(t)$$

State estimation error

$$\underline{e}(t) = \underline{x}(t) - \hat{\underline{x}}(t)$$

$$\Rightarrow \hat{\underline{x}}(t) = \underline{x}(t) - \underline{e}(t)$$

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) - BK(\underline{x}(t) - \underline{e}(t))$$
$$= (A - BK)\underline{x}(t) + BK\underline{e}(t)$$

$$\frac{d}{dt}\underline{e}(t) = \frac{d}{dt}\underline{x}(t) - \frac{d}{dt}\hat{\underline{x}}(t)$$

Luenberger Observer

$$\frac{d}{dt}\hat{\underline{x}} = A\hat{\underline{x}} + B\underline{u} + G(\underline{y}(t) - \hat{\underline{y}}(t))$$
$$\hat{\underline{y}}(t) = C\hat{\underline{x}}(t)$$

$$\Rightarrow \frac{d}{dt}\underline{e}(t) = A\underline{x}(t) + B\underline{u}(t) - A\hat{\underline{x}} - B\underline{u} - G(\underline{y}(t) - \hat{\underline{y}}(t))$$

$$= A(\underline{x}(t) - \hat{\underline{x}}(t)) - G(\underline{y}(t) - \hat{\underline{y}}(t))$$
$$= A\underline{e}(t) - GC(\underline{x}(t) - \hat{\underline{x}}(t))$$
$$= (A - GC)\underline{e}(t)$$

$$\frac{d}{dt}\underline{x}(t) = (A - BK)\underline{x}(t) + BK\underline{e}(t)$$
$$\frac{d}{dt}\underline{e}(t) = (A - GC)\underline{e}(t)$$

$$\frac{d}{dt}\begin{bmatrix}\underline{x}(t) \\ \underline{e}(t)\end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - GC \end{bmatrix}\begin{bmatrix}\underline{x}(t) \\ \underline{e}(t)\end{bmatrix}$$ 2N STATES

(ii) Poles given by:

$$\det\left(\underset{\substack{\uparrow \\ 2N\times 2N \\ \text{IDENTITY}}}{sI} - \begin{pmatrix} A - BK & BK \\ 0 & A - GC \end{pmatrix}\right) = 0$$

$$\det\left[\begin{pmatrix} sI_N & 0 \\ 0 & sI_N \end{pmatrix} - \begin{pmatrix} A - BK & BK \\ 0 & A - GC \end{pmatrix}\right] = 0$$

$$\det\begin{pmatrix} sI_N - A + BK & -BK \\ 0 & sI_N - A + GC \end{pmatrix} = 0$$

$$\underset{A_c(s)}{|sI_N - A + BK|}\;\underset{A_o(s)}{|sI_N - A + GC|} = 0$$

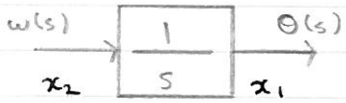
The estimator doesn't affect the position of the regulator poles
=> design K for regulator to place N closed loop lopes assuming the states are available
=> design G for estimator to provide these states with desired error dynamics

(b) $\frac{\omega(s)}{u(s)} = \frac{2K_A}{1+0.3s}$

$\omega(s) + 0.3 s\,\omega(s) = 2K_A U(s)$

$\omega(t) + 0.3 \frac{d\omega}{dt} = 0.8(2)\,u(t)$

$\frac{d\omega}{dt} = \frac{16}{3}u(t) - \frac{10}{3}\omega(t)$



$\frac{dx_1}{dt} = x_2$

$\frac{dx_2}{dt} = \frac{16}{3}u(t) - \frac{10}{3}\omega(t)$

$= \frac{16}{3}K_1 R_\theta - \frac{80}{3}K_1 x_1 - \frac{16}{3}K_2 x_2 - \frac{10}{3}x_2$

$= \frac{16}{3}K_1 R_\theta - \frac{80}{3}K_1 x_1 - \frac{1}{3}(16K_2 + 10)x_2$

Aside

$u(t) = K_1(R_\theta(t) - K_\theta x_1) - K_2 K_\omega x_2$

$= K_1 R_\theta - K_1 K_\theta x_1 - K_2 K_\omega x_2$

$= K_1 R_\theta - 5K_1 x_1 - K_2 x_2$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-80K_1}{3} & -\frac{1}{3}(16K_2+10) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{16}{3}K_1 \end{bmatrix} R_\theta$$

$$\underline{y} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\det(sI - A) = 0$

$$\det\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \frac{-80K_1}{3} & -\frac{1}{3}(16K_2+10) \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} s & -1 \\ \frac{80}{3}K_1 & s + \frac{1}{3}(16K_2+10) \end{pmatrix} = 0$$

$s(s + \frac{1}{3}(16K_2+10)) + \frac{80}{3}K_1 = 0$

$s^2 + \frac{1}{3}(16K_2+10)s + \frac{80}{3}K_1$

$C_{des}(s) = (s+10)^2$

$= s^2 + 20s + 100$

$\frac{1}{3}(16K_2+10) = 20$ $\qquad \frac{80}{3}K_1 = 100$

$\Rightarrow K_2 = 3.125$ $\qquad K_1 = 3.75$

(b). Full order Luenberger observer

$\frac{d}{dt}\hat{\underline{x}} = (A - GC)\hat{\underline{x}} + B\underline{u} + G\underline{y}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{B}{\begin{bmatrix} 0 \\ 20 \end{bmatrix}} R_\theta$$

$$\underline{y} = \overset{C}{\begin{bmatrix} 1 & 0 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A - GC = \begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -100 & -20 \end{bmatrix} - \begin{bmatrix} g_1 & 0 \\ g_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -g_1 & 1 \\ -100-g_2 & -20 \end{bmatrix} = F$$

Poles of estimator are given by:

$\det(sI - F) = 0$

$$sI - F = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -g_1 & 1 \\ -100-g_2 & -20 \end{pmatrix} = \begin{pmatrix} s+g_1 & -1 \\ 100+g_2 & s+20 \end{pmatrix}$$

$$\det(sI - F) = (s+g_1)(s+20) + (100+g_2)$$
$$= s^2 + 20s + g_1 s + 20g_1 + 100 + g_2$$
$$= s^2 + (g_1+20)s + (20g_1 + g_2 + 100)$$

Estimator error dynamics much faster than dominant state dynamics

$\Rightarrow s = 5(-10) = -50$ twice

$$C_{des}(s) = (s+50)^2$$
$$= s^2 + 100s + 2500$$

$g_1 + 20 = 100$

$g_1 = 80$

$20g_1 + g_2 + 100 = 2500$

$20(80) + g_2 + 100 = 2500$

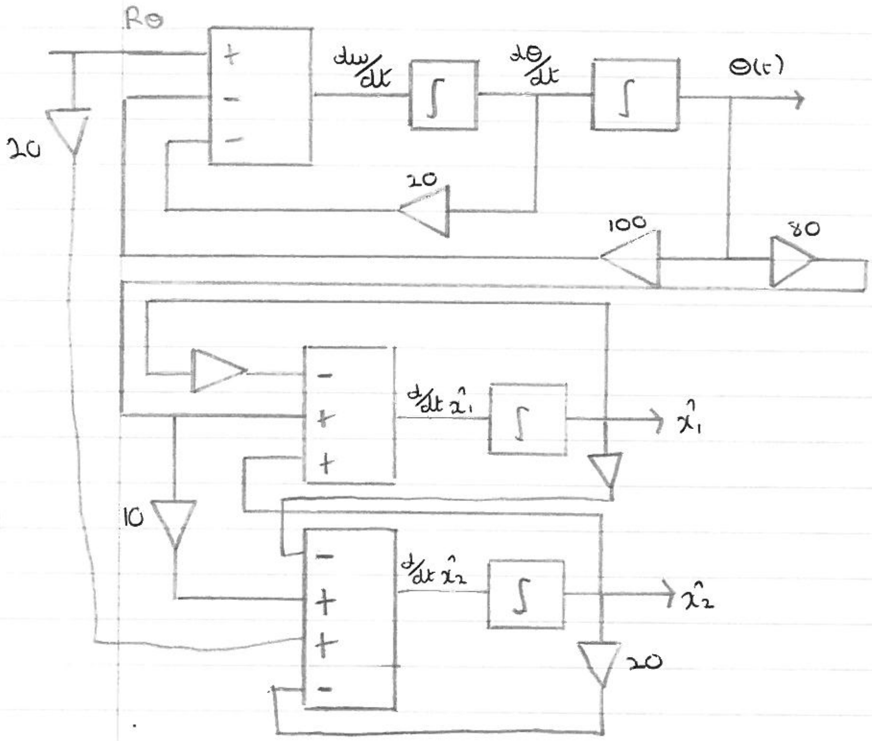
$g_2 = 800$

$$\frac{d}{dt}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -80 & 1 \\ -900 & -20 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 20 \end{bmatrix} R_\theta + \begin{bmatrix} 80 \\ 800 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -80 & 1 \\ -900 & -20 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 20 \end{bmatrix} R_\theta + \begin{bmatrix} 80 & 0 \\ 800 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\frac{d}{dt} x_1 = x_2 \Rightarrow \frac{d\theta}{dt} = \omega$

$\frac{d}{dt} x_2 = -100x_1 - 20x_2 + 20R_\theta \Rightarrow \frac{d\omega}{dt} = -100\theta(t) - 20\omega(t) + 20R_\theta$



$\frac{d}{dt}\hat{x}_1 = -80\hat{x}_1 + \hat{x}_2 + 80x_1$
$= -80\hat{x}_1 + \hat{x}_2 + 80\theta(t)$

$\frac{d}{dt}\hat{x}_2 = -900\hat{x}_1 - 20\hat{x}_2 + 20R_\theta + 800\theta(t)$