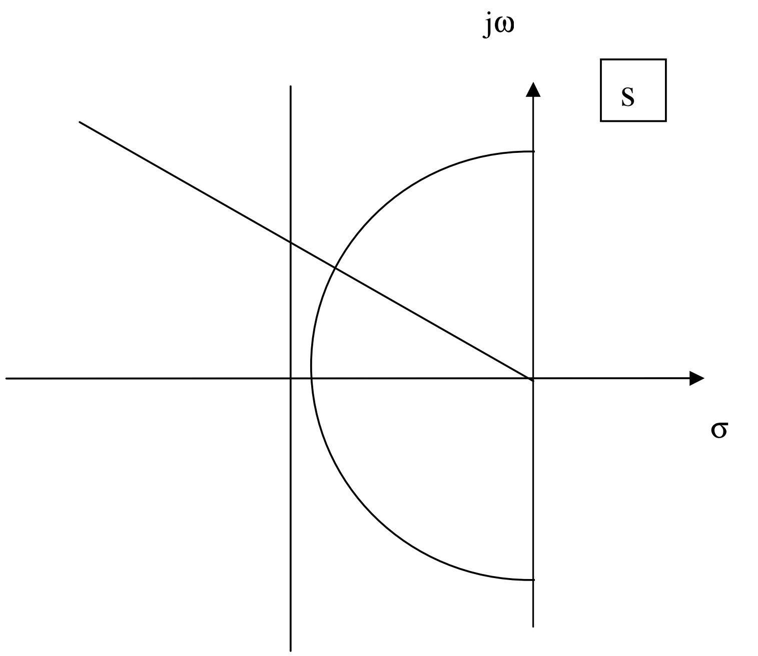
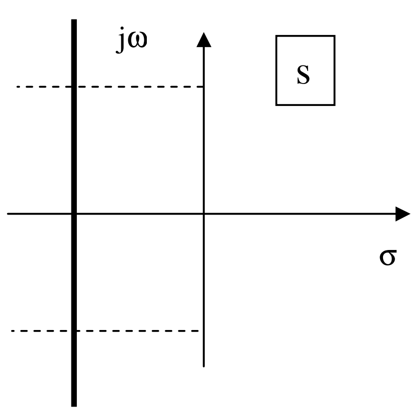
Chapter 7. Pole-Placement Design

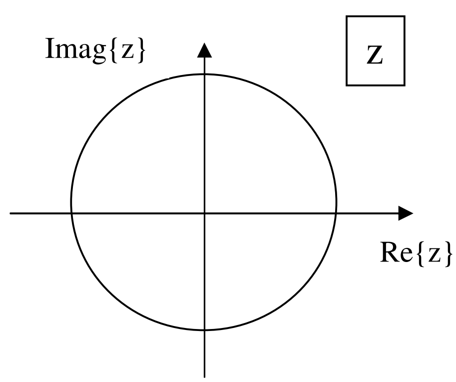
7.1 The Z-Grid Template

The following design loci in the s plane are known:

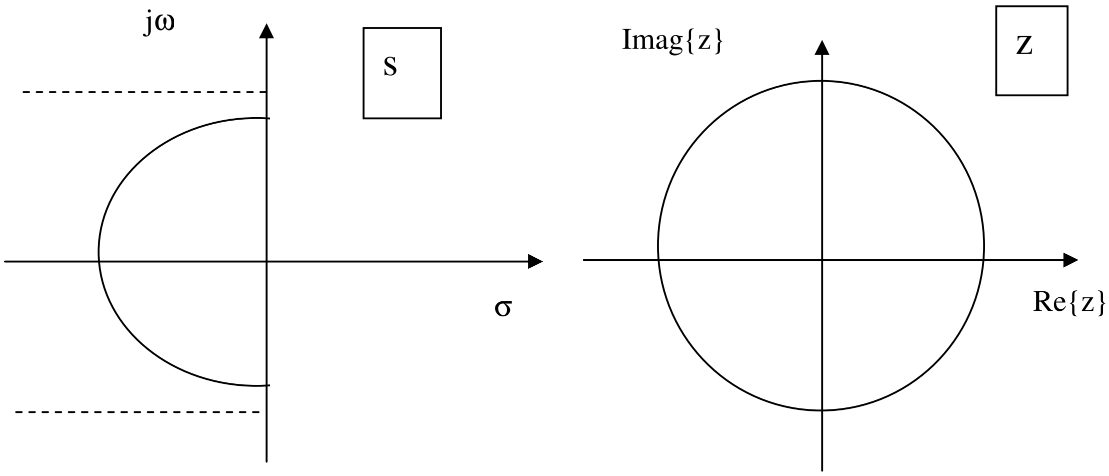


i) Mapping the Settling Time to the Z Plane

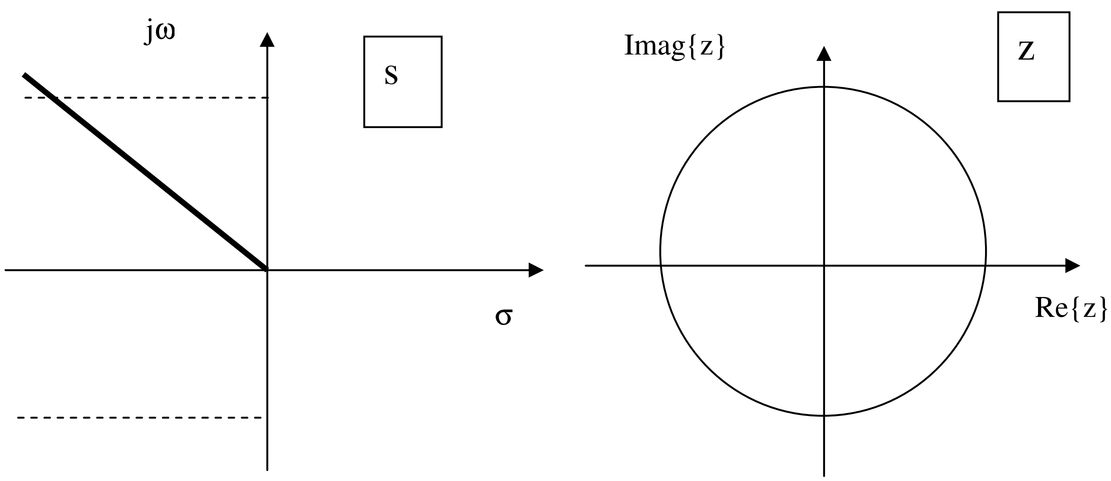




ii) <u>Mapping the Natural frequency loci to the Z Plane</u>



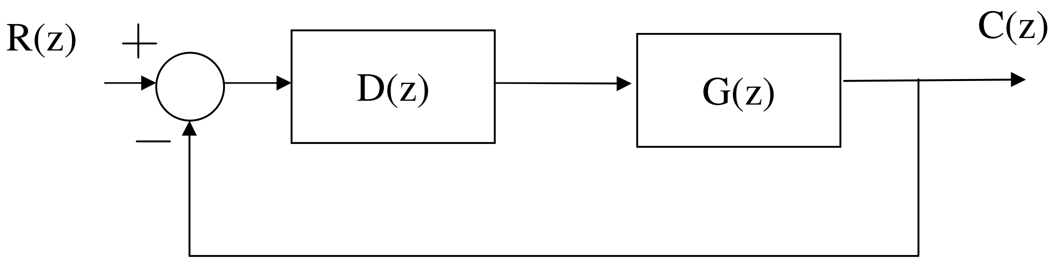
iii) <u>Mapping the Damping Line to the Z Plane</u>



Yields the Z Grid Template:

7.2 Root Locus Design

The closed-loop discrete-time process is:



The characteristic equation is:

Hence the poles of the closed-loop process obey D(z)G(z)=-1

Hence a testpoint z=ζ on the Z plane will be a pole of the closed-loop process if:

Consider now the controller is now factorised:

$$D(z) = KD'(z)$$

Then the poles of the closed-loop process will be a function of the gain controller K. The root locus plot is the locus of the closed-loop poles on the Z plane as K is increased from 0 to∞.

Every point z=ζ on the root locus must obey:

$$\left|KD'(z)G(z)\right|_{z=\varsigma} = 1$$

$$\angle KD'(z)G(z)\Big|_{z=\varsigma} = 180^\circ$$

7.2.1 Rules for Plotting Root Loci

1) There are as many loci as poles.

2) Loci begin on the poles of the OLTF.

3) Loci end on the zeros of the OLTF or at ∞.

4) Plots are symmetrical about the real axis.

5) For large values of z, the loci are asymptotic to straight lines which intersect the real axis at the point, α, where,

$$\alpha = \frac{sum\ of\ \ poles - sum\ of\ \ zeros}{no.\ of\ \ poles - no.\ of\ \ zeros}$$

6) These lines make angles θ with the real axis of:

$$\theta = \frac{(2k+1)\pi}{no.\ of\ \ poles - no.\ of\ \ zeros} \quad , k = 0, 1, 2, ...$$

7) On a given section of the real axis, a locus will exist if the sum of the poles and zeros to the right of the section is an odd number.

8) The angles of departure from complex poles and arrival at complex zeros are found by measuring the angle from the pole (or zero) to all other poles and zeros, and obtaining the residue angle:

angle of departure from pole (or arrival at zero) = residue angle - 180°

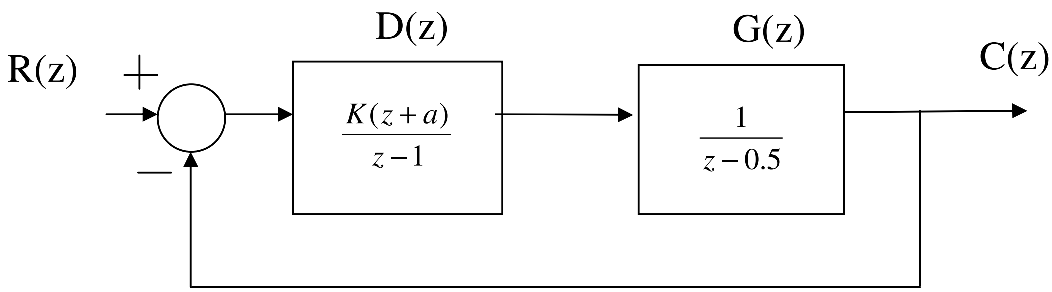
9) The intersection of the locus with the unit circle may be found using Jury's method.

10) If the n OLTF poles are p_1 ,p_2, …..p_n and the m OLTF zeros are z_1, z_2, ..z_m, then the point of departure from the real axis, σ, (known as the breakaway point), must obey:

$$\sum_{i=1}^{n} \frac{1}{\sigma - p_i} = \sum_{j=1}^{m} \frac{1}{\sigma - z_j}$$

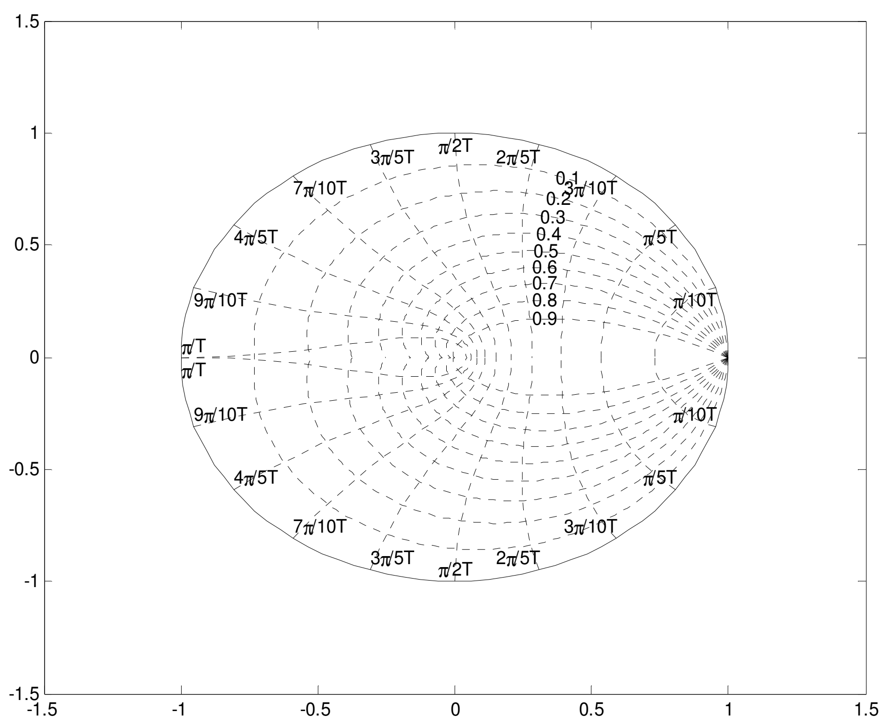
7.2.2 Transient response design via gain adjustment

Consider the example:



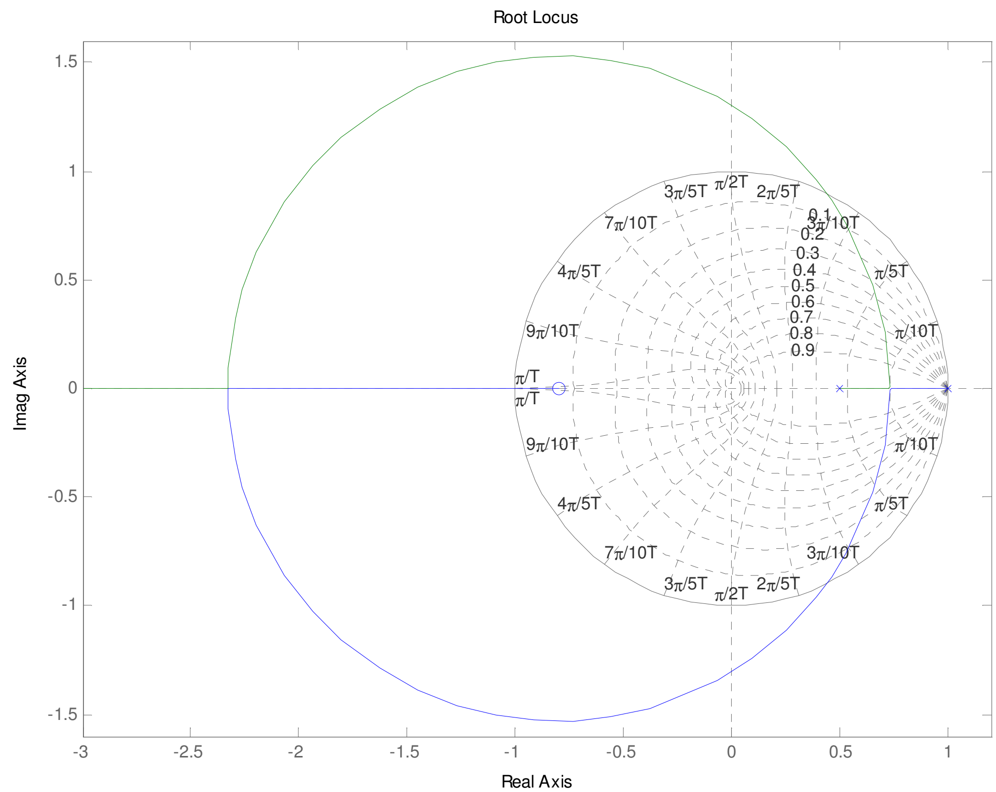
<u>Open-loop Poles:</u>

<u>Open-loop Zeros:</u>



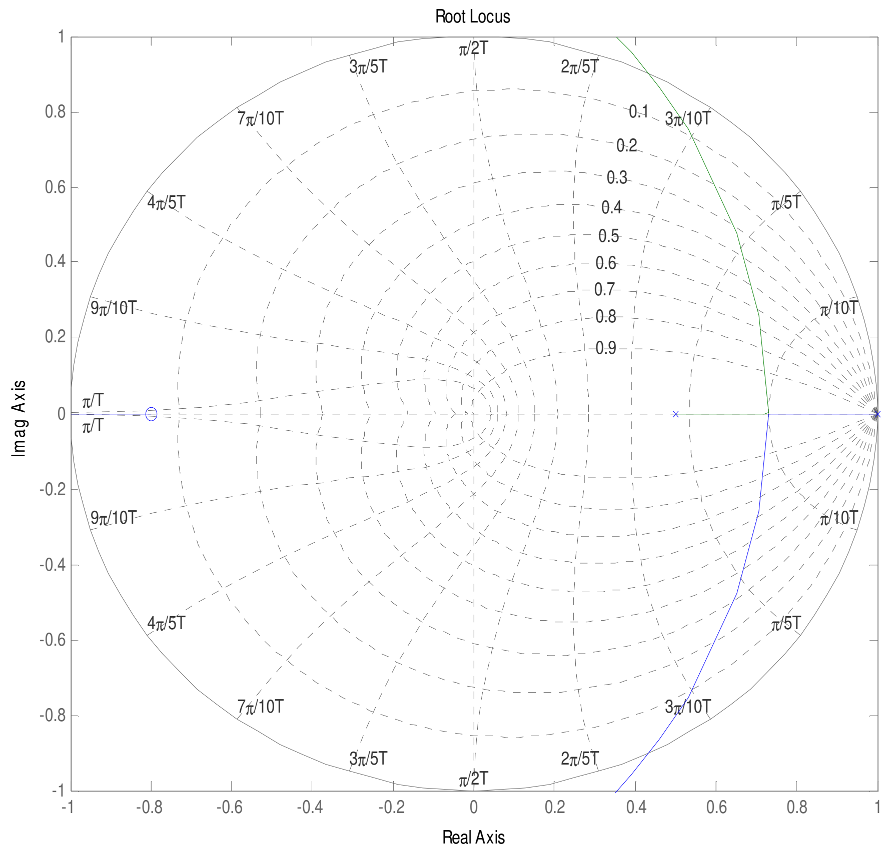
Consider a=0.8 $$D(z)=\frac{K(z+0.8)}{z-1}$$

The root-locus diagram for $G(z)D(z)=\dfrac{K(z+0.8)}{z-1}\dfrac{1}{z-0.5}$ is:



Design K to achieve a closed-loop damping ξ=0.7

Focus in on the unit circle:



Desired poles are:

But we know that:

$$\left|D(z)G(z)\right|_{z=} \qquad =1$$

That is:

$$\left|D(z)G(z)\right|_{z=0.7+j0.2} = 1 =$$

Or using the distances from open-loop poles and zeros:

Tutorial: Simulate the closed-loop process in Simulink and verify that you get the desired peak overshoot for a step setpoint.

What is the value of K for stability?

7.2.3 Designing a Phase-Lead Compensator

Consider the following digital phase lead compensator:

$$D(z) = \frac{K(z-a)}{z-b}$$

Place the zero, z=a, directly under the desired pole locations:-

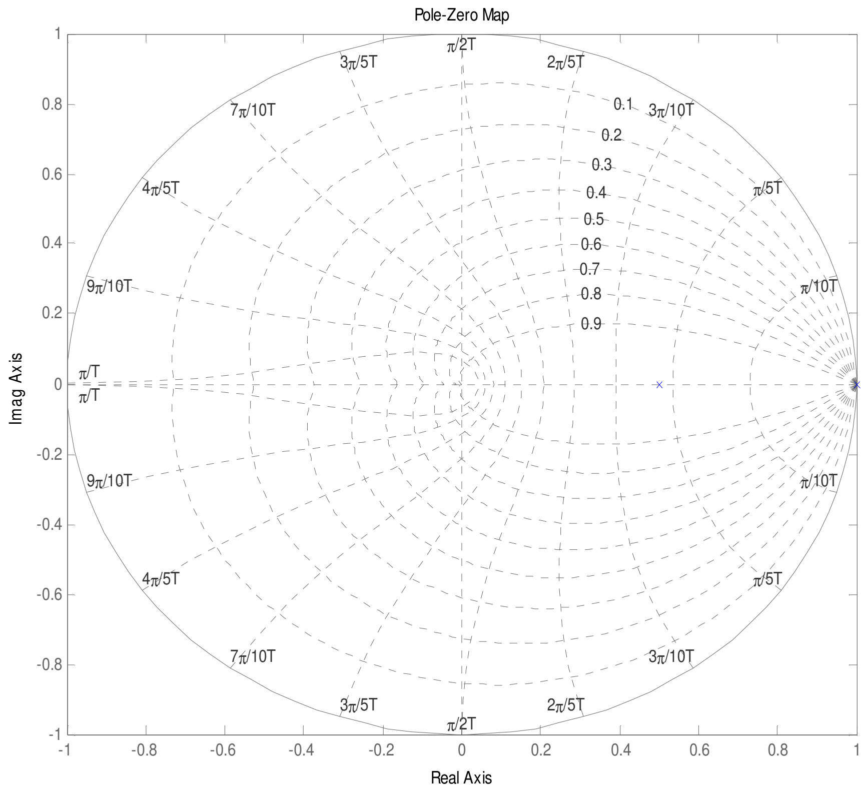
Adjust the pole position b:-

Adjust the gain K:-

EXAMPLE:

$$G(z) = \frac{10}{(z-1)(z-0.5)}$$

Design a phase-lead compensator, with sample time T=0.8s to achieve the following closed-loop specifications:



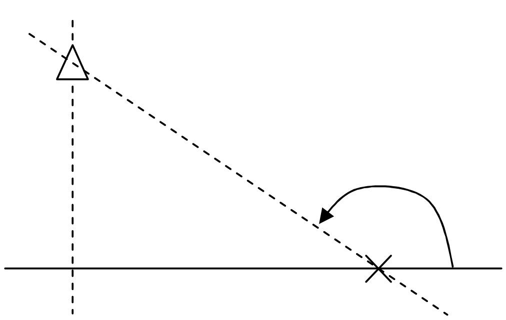
Place the zero of compensator at:

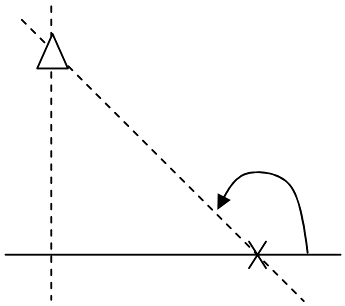
The controller is then

$$D(z) = \frac{K(z-0.4)}{z-b}$$

Place the controller pole so that:

$$ARG(D(z)G(z)\Big|_{z=0.4+j0.35} = 180^\circ$$

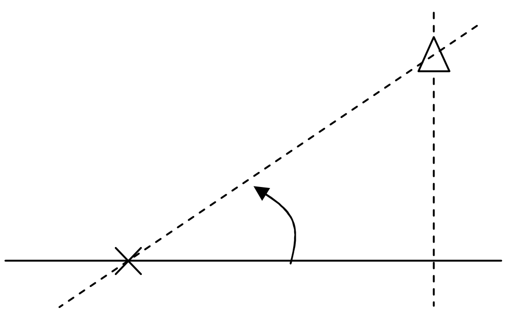




Obviously: $\theta = 90^\circ$

Hence for the root locus to go through the desired point:

$$\phi_1 + \phi_2 + \phi_3 - \theta = 180^\circ$$
$$\phi_3 = 270^\circ - \phi_1 - \phi_2$$



The controller is now:

$$D(z) = \frac{K(z-0.4)}{z+1}$$

Now determine the gain K so that at the desired point:

$$\left| \frac{K(z-0.4)}{z+1} \frac{10}{(z-1)(z-0.5)} \right|_{z=0.4+j0.35} = 1$$

or:

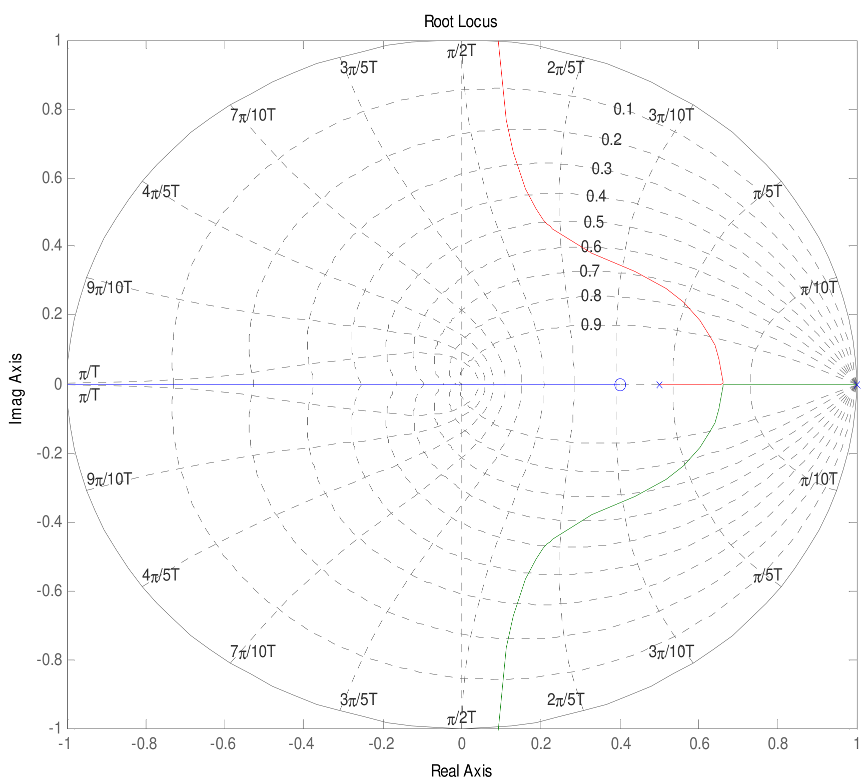
$$\frac{10Kr_1}{R_1R_2R_3} = 1$$

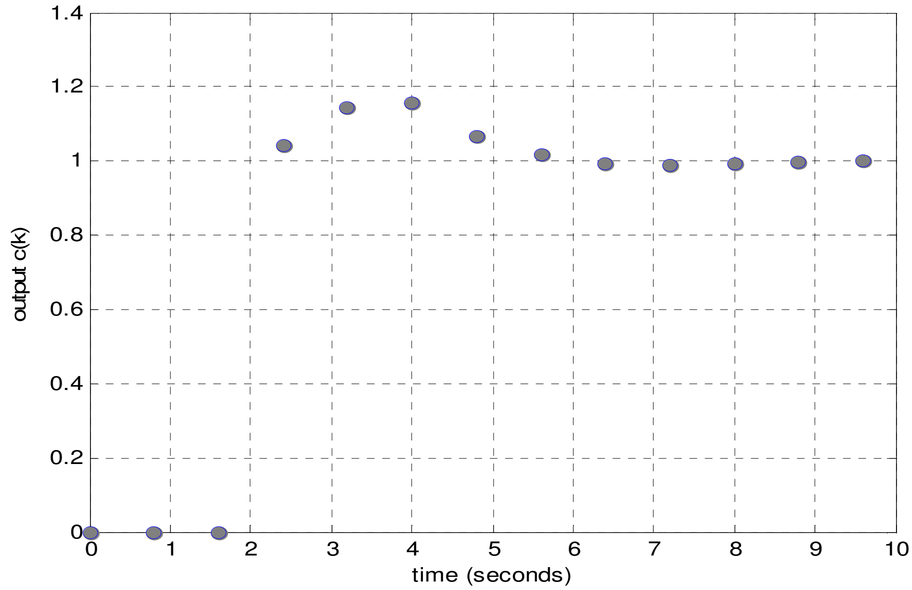
And the controller is then:

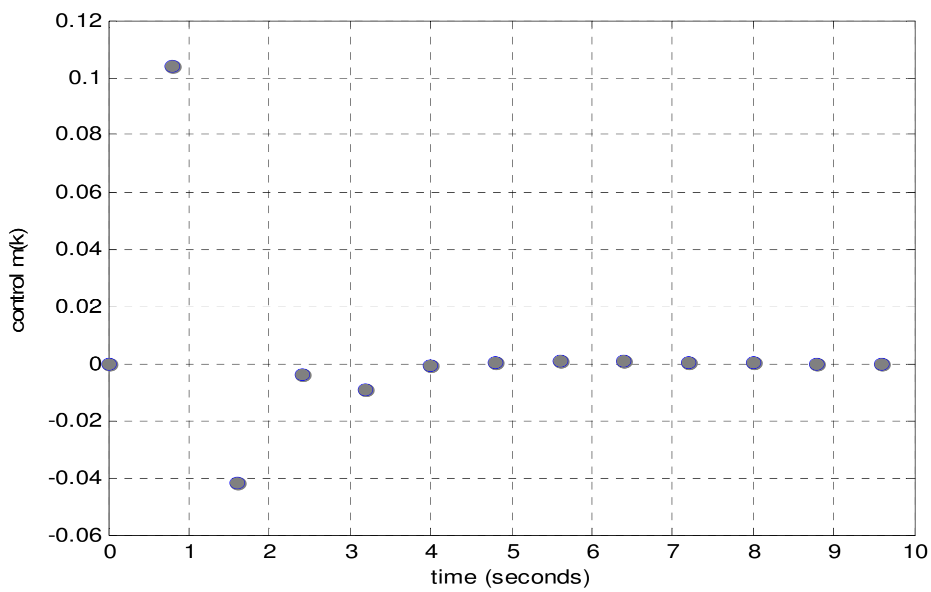
Draw the compensated root locus for D(z)G(z):

$$D(z)G(z) = \frac{0.1(z-0.4)}{z+1}\frac{10}{(z-0.5)(z-1)}$$

The compensated root locus is:







<u>Notes on Matlab:</u>

rlocus:

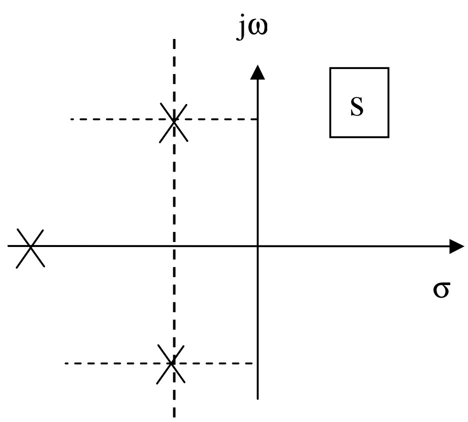
pzmap:

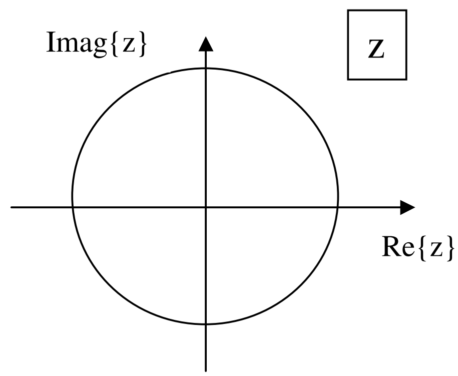
c2d:

d2c:

7.3 Note on dominance

Consider a 3rd order process with poles : s=-c and s=-a±bj





A simple rule of dominance:

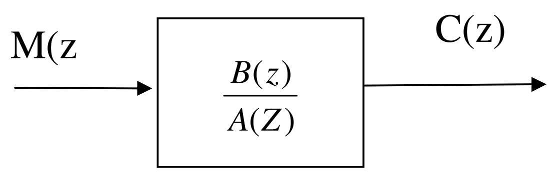
- For the s plane a pole s = –a+bj dominates a pole s= –c+dj if:

- For the z plane a pole $r_1\angle\phi$ dominates a pole $r_2\angle\theta$ if:

7.4 Pole-Placement Design- A polynomial Approach

7.4.1 The QST Control Scheme

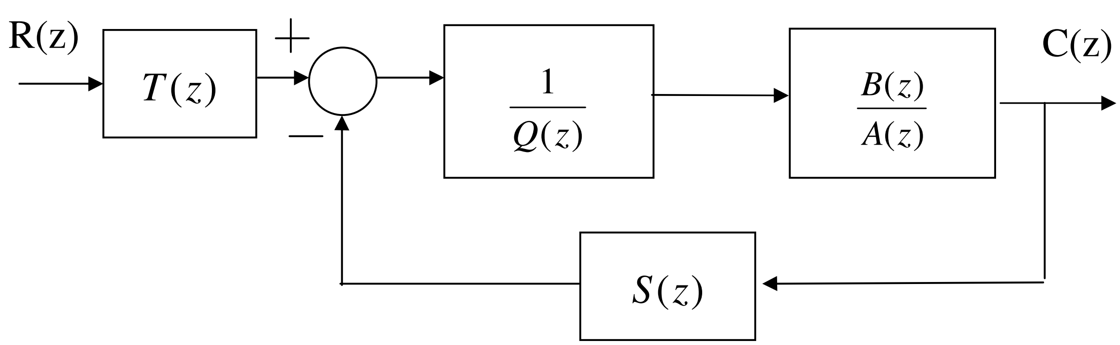
Consider the open-loop process:



Where the process is n^{th} order and:

$$A(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots a_n$$
$$B(z) = b_1 z^{n-1} + b_2 z^{n-2} + \cdots b_m z^{n-m}$$

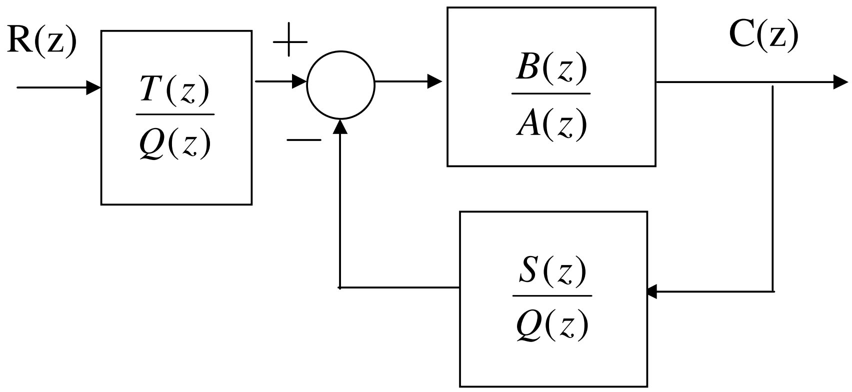
Consider now the closed-loop control scheme:



The control-law is then:

$$M(z) = \frac{1}{Q(z)}\left(T(z)R(z) - S(z)C(z)\right)$$

This of course could be redrawn as:



We now define the following controller polynomials:

$$T(z) = t_0 z^{n_t} + t_1 z^{n_t-1} + t_2 z^{n_t-2} + \cdots + t_{n_t}$$

$$S(z) = s_0 z^{n_s} + s_1 z^{n_s-1} + s_2 z^{n_s-2} + \cdots + s_{n_s}$$

$$Q(z) = z^{n_q} + q_1 z^{n_q-1} + q_2 z^{n_q-2} + \cdots + q_{n_q}$$

For realisability – ie for causal control

$\frac{T(z)}{Q(z)}$ *and* $\frac{S(z)}{Q(z)}$ must both be causal:

The closed-loop transfer function is:

$$\frac{C(z)}{R(z)} = \frac{T(z)}{Q(z)} \frac{\frac{B(z)}{A(z)}}{1 + \frac{B(z)}{A(z)}\frac{S(z)}{Q(z)}} =$$

The characteristic equation for the closed-loop system is:

$$A(z)Q(z) + B(z)S(z) = 0$$

Roots of the characteristic equation give the poles of the closed-loop system.

But how many closed-loop poles?

Remember:

Then: $\deg\left(A(z)Q(z)+B(z)S(z)\right)=$

Hence there are $n+n_q$ poles for the closed-loop system.

7.4.2 The Polynomial Pole-Placement Design Route

The pole-placement design problem is then:

i) Select desired poles:

ii) Specify desired closed-loop characteristic equation:

iii) Design S(z) and Q(z)

iv) Design T(z)

The design equation:

$$A_{cl}(z)=A(z)Q(z)+B(z)S(z)$$

Is an example of a Diophantine Equation

Consider now that we require the closed-loop system to remain as n[th] order dominant.

We could factorise the desired closed-loop characteristic equation as follows:

$$A_{cl}(z) = A_C(z)A_O(z)$$

where:

$A_c(z) =$

$A_o(z) =$

We know from the closed loop transfer function that:

$$C(z) = \frac{B(z)T(z)}{A(z)Q(z) + B(z)S(z)} R(z)$$

when the closed loop poles have been placed:

$$C(z) = \frac{B(z)T(z)}{A_{cl}(z)} R(z) =$$

It is usual to choose T(z) to cancel out the fast poles:

This yields:

$$C(z) = \frac{t_o A_o B(z)}{A_o A_c(z)} R(z) =$$

The gain t_o can now be adjusted to achieve a closed-loop DC gain of unity.

For unity DC gain:

$$\lim_{z\to 1}\frac{t_o B(z)}{A_c(z)} = 1$$

hence:

<u>EXAMPLE:</u>

$$G(z) = \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^2}$$

The Diophantine equation is:

$$A_{cl}(z) = A(z)Q(z) + B(z)S(z)$$

First we will specify a simple zero-order controller:

$$A_{cl}(z) = A(z)Q(z) + B(z)S(z)$$

Which yields:

$$A_{cl}(z) = (z-1)^2\ 1 + (z+1)\ s_0$$

Now try a first order controller:

$$
\begin{aligned}
Q(z) &= z + q_1 \\
S(z) &= \\
T(z) &=
\end{aligned}
$$

The Diophantine equation becomes:

$$A_{cl}(z) = (z^2 - 2z + 1)(z + q_1) + (z + 1)\ (s_0 z + s_1)$$

Now consider the desired closed-loop characteristic equation for a 3rd order process:

$$A_{cl}(z) = z^3 + c_1 z^2 + c_2 z + c_3 = A_o(z) A_c(z)$$

Comparing similar powers of z:

Which could be written in matrix form as:

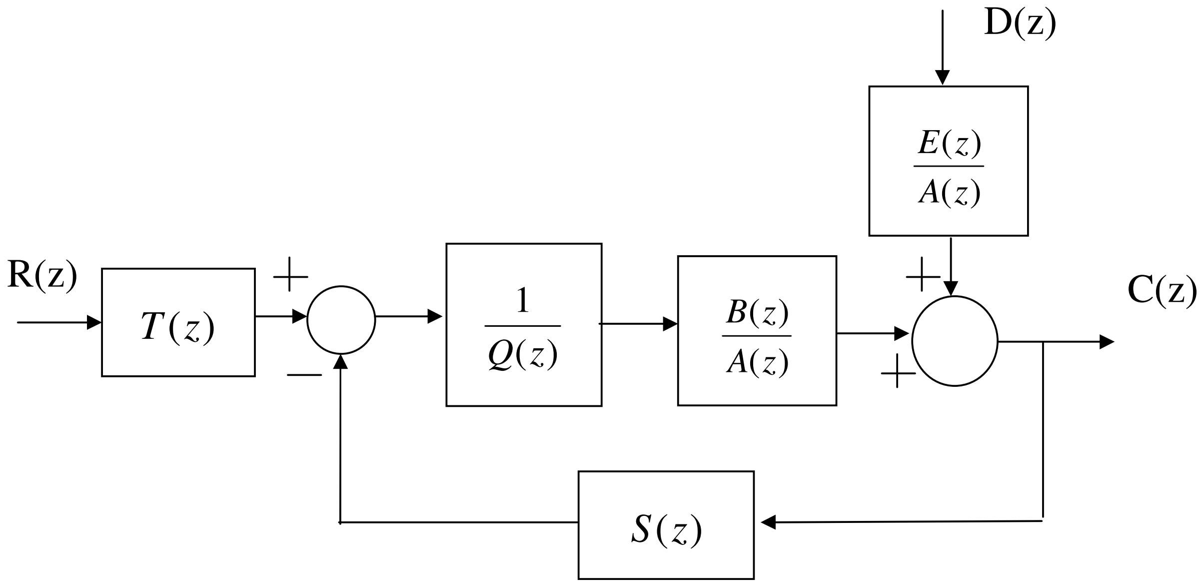
$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} c_1 + 2 \\ c_2 - 1 \\ c_3 \end{bmatrix}$$

Hence the controller parameters are obtained as:

$$\begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0.25 & -0.25 & 0.25 \\ 0.75 & 0.25 & -0.25 \\ -0.25 & 0.25 & 0.75 \end{bmatrix} \begin{bmatrix} c_1 + 2 \\ c_2 - 1 \\ c_3 \end{bmatrix}$$

7.4.3 Steady State Errors

The closed-loop process could be drawn as:



We know that with the choice:

$$T(z) = t_0 A_0(z)$$

Yields a unity DC gain:

But this technique can be sensitive to errors in the B(z) polynomial:

NOTE: Good tracking of the setpoint does not imply good disturbance rejection.

TUTORIAL: Determine the steady-state error for an asymptotically constant disturbance, if the process B/A is "type 0" and if T(z)=(Ac(1)/B(1))A_o (z).

Redo, with B/A as "type 1".

If we need to increase the Type of the process, ie. to introduce integration, we could force a factorisation of Q(z):

$$Q(z) = z^{n_q} + q_1 z^{n_q - 1} + q_2 z^{n_q - 2} + \cdots + q_{n_q}$$

7.4.4 Automated Pole-Placement Design

The Diophantine Equation is:

$$A_{cl}(z) = A(z)Q(z) + B(z)S(z)$$

First assume without loss of generality that:

Hence:

$$\left(z^n + a_1 z^{n-1} + \cdots a_n\right)\left(z^{n-1} + q_1 z^{n-2} + \cdots q_{n-1}\right) \\ + \left(b_1 z^{n-1} + b_2 z^{n-2} + \cdots b_n\right)\left(s_0 z^{n-1} + s_1 z^{n-2} + \cdots s_{n-1}\right)$$

$$=$$

Compare similar powers of z:

$$
\begin{array}{ccc}
z^{2n-1} & : & 1 = \\
z^{2n-2} & : & c_1 = \\
z^{2n-3} & : & c_2 = \\
z^{2n-4} & : & c_3 = \\
\vdots & \vdots & \vdots
\end{array}
$$

$$
\left[\begin{array}{c|c}
\quad & \quad
\end{array}\right]
\left[\begin{array}{c}
q_1 \\ \vdots \\ q_{n-1} \\ \hdashline s_0 \\ \vdots \\ s_{n-1}
\end{array}\right]
=
\left[\begin{array}{c}
\quad
\end{array}\right]
$$

The complete equations are then:

$$
\left[\begin{array}{ccccccc:cccccc}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & b_1 & 0 & 0 & 0 & \cdots & 0 \\
a_1 & 1 & 0 & 0 & 0 & \cdots & 0 & b_2 & b_1 & 0 & 0 & \cdots & 0 \\
a_2 & a_1 & 1 & 0 & 0 & \cdots & 0 & b_3 & b_2 & b_1 & 0 & \cdots & 0 \\
a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 & b_4 & b_3 & b_2 & b_1 & \cdots & 0 \\
a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 & b_5 & b_4 & b_3 & b_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} & a_{n-6} & \cdots & 1 & b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \cdots & 0 \\
a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & a_{n-5} & \cdots & a_1 & b_n & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_1 \\
a_n & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_2 & 0 & b_n & b_{n-1} & b_{n-2} & \cdots & b_2 \\
0 & a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_3 & 0 & 0 & b_n & b_{n-1} & \cdots & b_3 \\
0 & 0 & a_n & a_{n-1} & a_{n-2} & \cdots & a_4 & 0 & 0 & 0 & b_n & \cdots & b_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & a_n & 0 & 0 & 0 & 0 & \cdots & b_n
\end{array}\right]
\left[\begin{array}{c}
q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{n-1} \\ \hdashline s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1}
\end{array}\right]
=
\left[\begin{array}{c}
c_1 - a_1 \\ c_2 - a_2 \\ c_3 - a_3 \\ \vdots \\ c_n - a_n \\ \hdashline c_{n+1} \\ c_{n+2} \\ c_{n+3} \\ \vdots \\ c_{2n-1}
\end{array}\right]
$$

Note the structure of the Sylvester Matrix:

The parameters of the controller polynomials can now be calculated as:

Theory: The Sylvester Matrix is invertible if the polynomials A(z) and B(z) do not have any common factors:

EXAMPLE:

$$G(z) = \frac{z^{-1} + 0.7z^{-2}}{\left(1 - z^{-1}\right)\left(1 - 0.8z^{-1}\right)} =$$

Choose the following polynomials:

$$Q(z) = z + q_1$$
$$S(z) = s_0 z + s_1$$

Third order characteristic equation:

$$A_{cl}(z) = z^3 + c_1 z^2 + c_2 z + c_3$$

The following matrix equation could be written:

$$\left[\begin{array}{c:cc} \quad & \quad & \quad \\ & & \\ & & \end{array}\right]\left[\begin{array}{c} q_1 \\ \hdashline s_0 \\ s_1 \end{array}\right] = \left[\begin{array}{c} \quad \\ \hdashline \quad \end{array}\right]$$

The specification for the closed-loop performance is:

T=0.5seconds ω_n=2rad/s ξ=0.707

Using the template:

Place the fast pole at:

The desired closed loop characteristic equation is:

$$A_{cl}(z) = A_O(z)A_C(z) =$$

Then the controller parameters are given by:

$$\begin{bmatrix} q_1 \\ \hline s_0 \\ s_1 \end{bmatrix} = \left[\begin{array}{c:cc} 1 & 1 & 0 \\ -1.8 & 0.7 & 1 \\ 0.8 & 0 & 0.7 \end{array}\right]^{-1} \begin{bmatrix} 1.07 \\ -0.56 \\ \hline -0.0066 \end{bmatrix} =$$

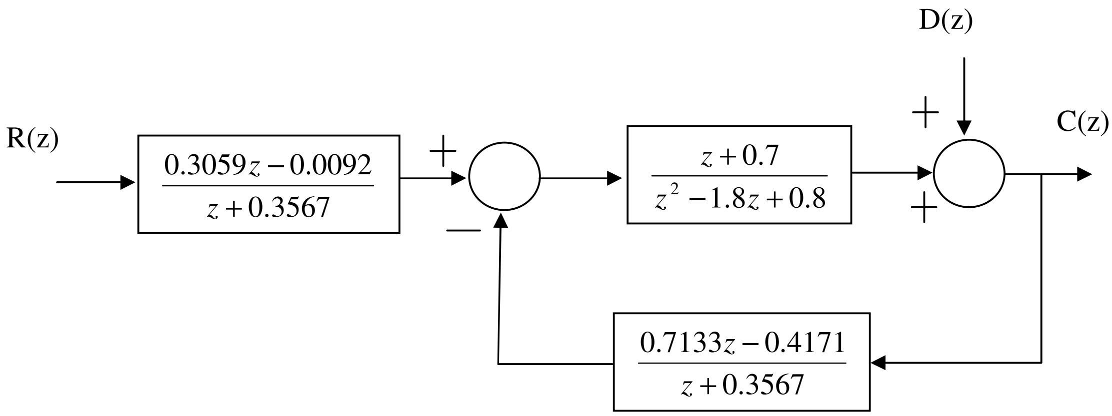
This yields the controller polynomials:

$$Q(z) = z + 0.3567$$
$$S(z) = 0.7133z - 0.4171$$

With the prefilter:

$$T(z) = t_0 A_O = t_0(z - 0.03)$$

where: $$t_0 = \frac{A_C(1)}{B(1)} =$$


D(z)
R(z)
C(z)
+
+
+
−
$\frac{0.3059z - 0.0092}{z + 0.3567}$
$\frac{z + 0.7}{z^2 - 1.8z + 0.8}$
$\frac{0.7133z - 0.4171}{z + 0.3567}$