22/4/09 Summer 2008

Q1 (a) The process is stable as the impulse response settles to zero.

$g(0) = 0$ $\qquad$ $g(7) = -0.15$

$g(1) = 1.1$ $\qquad$ $g(8) = 0.02$

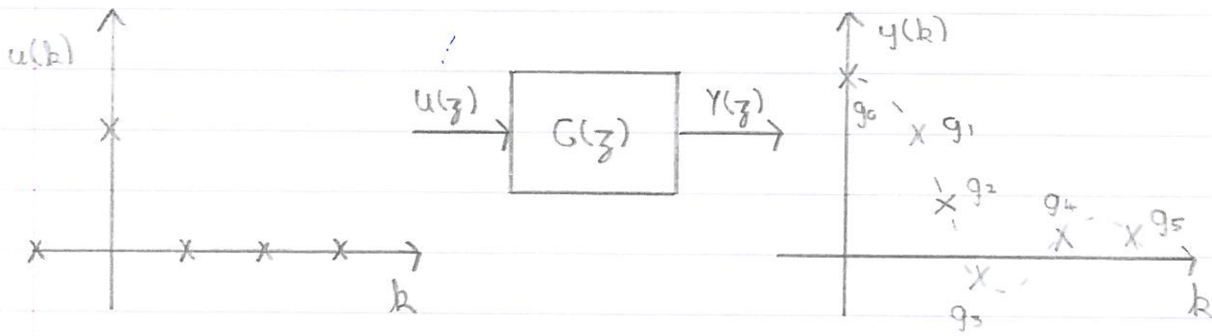
$g(2) = 0.8$ $\qquad$ $g(9) = 0.01$

$g(3) = -0.1$ $\qquad$ $g(10) = 0$

$g(4) = -0.4$ $\qquad$ $g(11) = 0$

$g(5) = 0.15$

$g(6) = 0.1$



$$Y(z) = G(z).1$$
$$= G(z)$$
$$= \mathcal{Z}\{g(k)\}$$

$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k}$$

$$= g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots$$

$$Y(z) = G(z)U(z) = (g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots)U(z)$$
$$\Rightarrow y(k) = g_0 u(k) + g_1 u(k-1) + g_2 u(k-2) + \ldots$$

Consider the unit step response of a discrete system

$$Y(z) = G(z)U(z) = (g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots)\frac{1}{1-z^{-1}}$$
$$= h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots$$

$$\Rightarrow (1-z^{-1})(h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots + h_{N-1} z^{-N+1} + h_N z^{-N} + h_n z^{-N-1}) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots$$
$$\Rightarrow h_0 + (h_1 - h_0) z^{-1} + (h_2 - h_1) z^{-2} + \ldots = g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots$$

$\Rightarrow h_0 = g_0 \qquad h_i = g_i + h_{i-1}$

$h_0 = g_0 = 0$

$h_1 = g_1 + h_0 = 1.1$

$h_2 = g_2 + h_1 = 1.9$

$h_3 = g_3 + h_2 = 1.8$

$h_4 = g_4 + h_3 = 1.4$

$h_5 = g_5 + h_4 = 1.55$

$h_6 = g_6 + h_5 = 1.65$

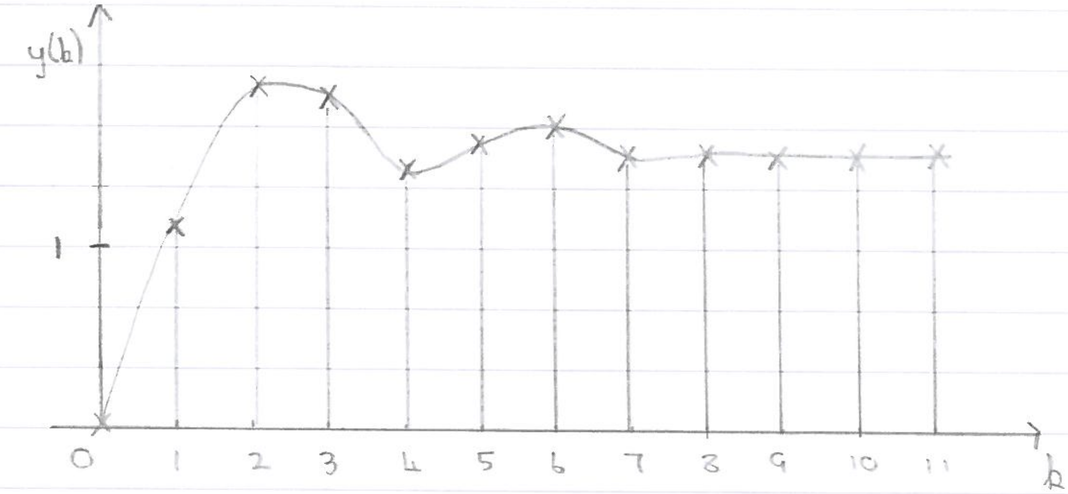
$h_7 = g_7 + h_6 = 1.5$

$h_8 = g_8 + h_7 = 1.52$

$h_9 = g_9 + h_8 = 1.53$

$h_{10} = g_{10} + h_9 = 1.53$

$h_{11} = g_{11} + h_{10} = 1.53$



Stability

$G(z) = 1.1z^{-1} + 0.8z^{-2} - 0.1z^{-3} + \ldots$

$$
\begin{array}{r|l}
 & 0.91z - 0.662 + 0.564z^{-1} \\
\hline
1.1z^{-1} + 0.8z^{-2} - 0.1z^{-3} & 1 \\
 & \underline{1 + 0.728z^{-1} - 0.091z^{-2}} \\
 & -0.728z^{-1} + 0.091z^{-2} \\
 & \underline{-0.728z^{-1} - 0.5296z^{-2} + 0.0662} \\
 & 0.621z^{-2} - 0.0662 \\
 & \vdots
\end{array}
$$

$$G(z) = \frac{1}{0.91z - 0.662 + 0.564z^{-1}} = \frac{1.1z}{z^2 - 0.73z + 0.62}$$

Stable if poles are inside the unit circle

$$z = \frac{0.73 \pm \sqrt{0.73^2 - 4(0.62)}}{2}$$

$z = 0.365 \pm j\,0.698$

$|z| = 0.78 < 1$

$\Rightarrow$ process is stable

30/4/09

Summer 2008

Q 1 (b). $s = \frac{2}{T}\frac{z-1}{z+1}$

$$m(t) = K_P\left(e(t) + \frac{1}{T_I}\int_0^t e(\tau)\,d\tau + T_D \frac{de(t)}{dt}\right)$$

$$M(s) = K_P\left(E(s) + \frac{E(s)}{T_I s} + T_D s E(s)\right)$$

$$\frac{M(s)}{E(s)} = K_P\left(1 + \frac{1}{T_I s} + T_D s\right)$$

Using Tustin's

$$D(z) = K_P\left[1 + \frac{1}{T_I}\frac{T}{2}\frac{z+1}{z-1} + T_D\frac{2}{T}\frac{z-1}{z+1}\right]$$

$$= K_P\left[\frac{2TT_I(z-1)(z+1) + T(z+1)T(z+1) + 2T_D(z-1)2T_I(z-1)}{2TT_I(z-1)(z+1)}\right]$$

$$= K_P\left[\frac{2TT_I(z^2-1) + T^2(z^2+2z+1) + 4T_DT_I(z^2-2z+1)}{2TT_I(z^2-1)}\right]$$

$$= K_P\left[\frac{(2TT_I + T^2 + 4T_DT_I)z^2 + (2T^2 - 8T_DT_I)z + (T^2 - 2TT_I + 4T_DT_I)}{2TT_I(z^2-1)}\right]$$

$$2TT_I(z^2-1)M(z) = K_P\left[(2TT_I+T^2+4T_DT_I)z^2 + (2T^2-8T_DT_I)z + (T^2-2TT_I+4T_DT_I)\right]E(z)$$

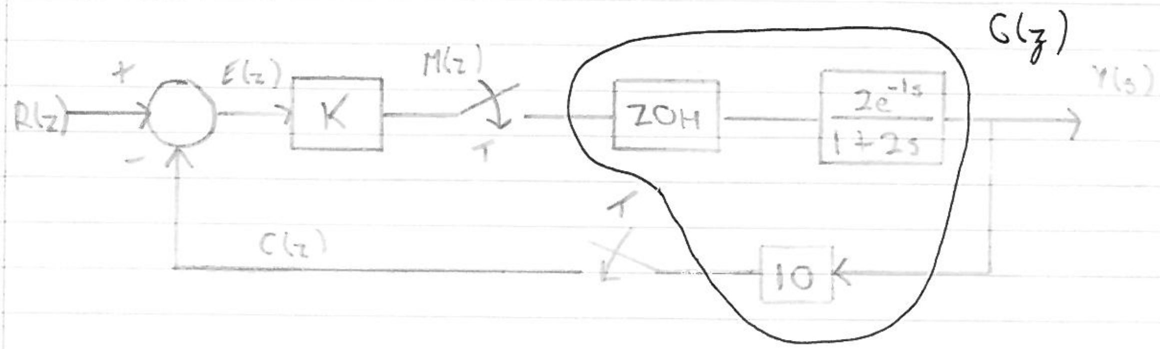
$$(1-z^{-2})M(z) = \left[\underbrace{\frac{K_P(2TT_I+T^2+4T_DT_I)}{2TT_I}}_{\alpha} + \underbrace{\frac{K_P(2T^2-8T_DT_I)}{2TT_I}}_{\beta}z^{-1} + \underbrace{\frac{K_P(T^2-2TT_I+4T_DT_I)}{2TT_I}}_{\gamma}z^{-2}\right]E(z)$$

$$m(k) = m(k-2) + \alpha e(k) + \beta e(k-1) + \gamma e(k-2)$$

- PID controller has a pole at -1 in the z plane
- For any positive gain value the pole will move to the left outside of the unit circle

=> unstable.

(c).



$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-sT}}{s}\cdot\frac{2e^{-s}}{1+2s}\cdot 10\right\}$$

$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-s}}{s} \cdot \frac{2e^{-s}}{1+2s} \cdot 10\right\}$$

$$= 10(1-z^{-1})z^{-1}\,\mathcal{Z}\left\{\frac{1}{s(s+0.5)}\right\}$$

$$= 10(1-z^{-1})z^{-1}\,\frac{1}{0.5}\,\frac{(1-e^{-0.5})z^{-1}}{\cancel{(1-z^{-1})}(1-e^{-0.5}z^{-1})}$$

$$= \frac{7.869\,z^{-2}}{1-0.61\,z^{-1}}$$

$$= \frac{7.869}{z^2-0.61z}$$

$$z^2 - 0.61z = 0$$

$$z(z-0.61) = 0$$

$$z = 0 \quad \text{and} \quad z = 0.61$$

$$\sum_{i=1}^{n}\frac{1}{\sigma - p_i} = \sum_{j=1}^{m}\frac{1}{\sigma - z_i}$$

$$\frac{1}{\sigma} + \frac{1}{\sigma - 0.61} = 0$$

$$\sigma - 61 + \sigma = 0$$

$$2\sigma = 0.61$$

$$\sigma = 0.305$$

As $K\uparrow$ $\xi\downarrow$ $\omega_n\uparrow$

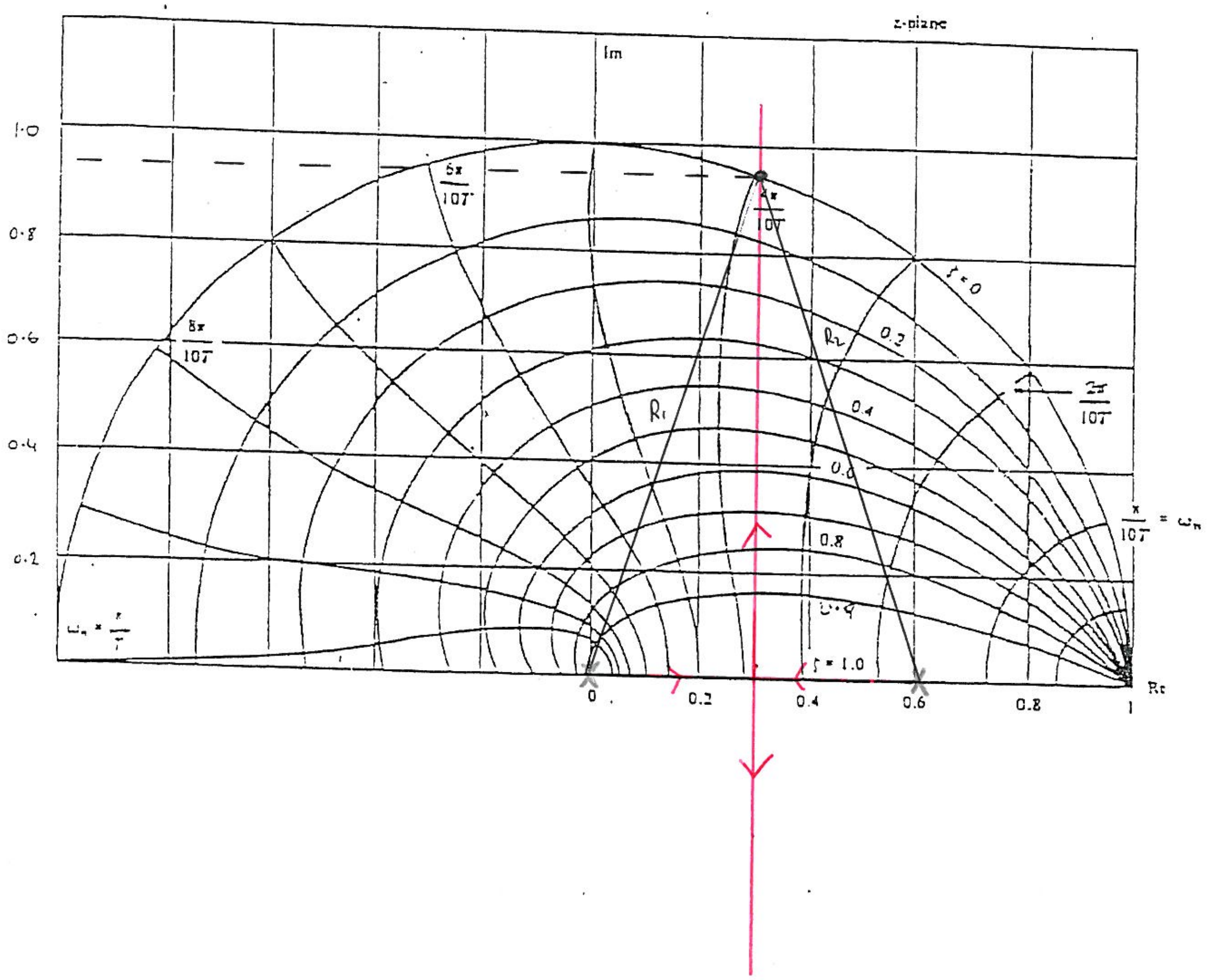
Marginal stability when cut the unit circle

$$|D(z)G(z)|_{z=0.305\pm0.93j} = 1$$

$$\Rightarrow \left.\frac{7.869K}{z^2-0.61z}\right|_{z=0.305+0.93j} = 1$$

$$\frac{7.869K}{|0.305+0.93j||-0.305+0.93j|} = 1$$

$$\Rightarrow K = 0.12$$



Z Plane Design Template

__Please submit with your script__

Alternatively

$R_1^2 = 0.305^2 + 0.93^2$

$\Rightarrow R_1 = 0.98$

Similarly $R_2 = 0.98$

$$\frac{7.869K}{0.98 \times 0.98} = 1$$

$K = 0.12$

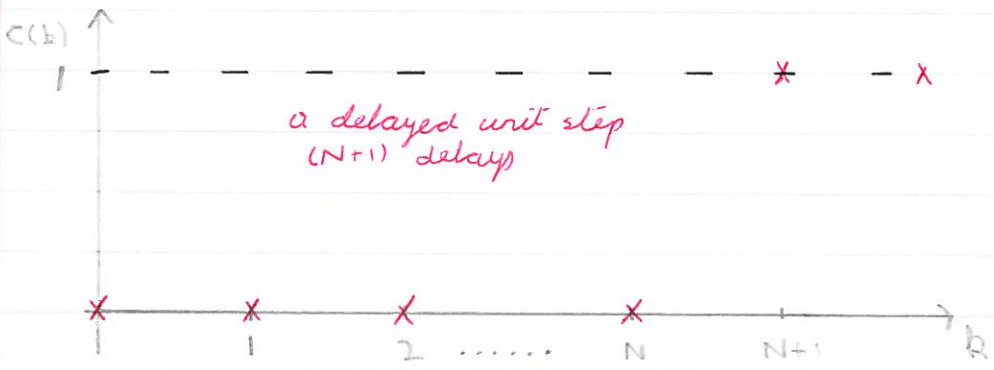
$\Rightarrow$ range of K for stability $0 < K < 0.12$

Q2(a). Specification: settling time is finite, rise time is minimum, zero steady state error } for a step change in set point

Consider the process $G(s)H(s) = {}^{B(s)}/_{A(s)}\, e^{-(NT+\theta)s}$

$$G_z = Z_m\left\{ \frac{1-e^{-sT}}{s} e^{-NTs} \frac{B(s)}{A(s)} \right\}_{m=1-\theta/T}$$

If the process is now under deadbeat control, then the response to a unit step in the setpoint is:



For the unit step setpoint:

$$R(k) = \begin{cases} 0 & \text{for } k < 0 \\ 1 & \text{for } k \geq 0 \end{cases}$$

$C(z) = z^{-(N+1)} R(z)$

$C(z)/R(z) = z^{-(N+1)}$

The controller $D(z)$ which will achieve this deadbeat response is:

$$D(z) = \frac{1}{G(z)} \frac{z^{-(N+1)}}{1 - z^{-(N+1)}}$$

$$= \frac{1}{G(z)} \frac{1}{z^{N+1} - 1}$$

(ii) $D(z)$

24/4/09

Summer 2008

Q 3 (a). $$\frac{Y(z)}{U(z)} = G(z) = \frac{z^{-d}(b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m})}{1 - a_1 z^{-1} - a_2 z^{-2} - \dots - a_n z^{-n}}$$

$$\hat{y}(k+1) = \hat{a}_1 y(k) + \hat{a}_2 y(k-1) + \dots + \hat{a}_n y(k-n+1) + \hat{b}_1 u(k-d) + \hat{b}_2 u(k-d-1) + \dots + \hat{b}_m u(k-d-m+1)$$

The first valid "test" equation is then:

$$\hat{y}(m+d) = \hat{a}_1 y(m+d-1) + \hat{a}_2 y(m+d-2) + \dots \hat{a}_n y(m+d-n) + \hat{b}_1 u(m-1) + \hat{b}_2 u(m-2) + \dots \hat{b}_m u(0)$$

This can be repeated to get output estimates over the valid dataset

$$\underbrace{\begin{bmatrix} \hat{y}(m+d) \\ \hat{y}(m+d+1) \\ \hat{y}(m+d+2) \\ \vdots \\ \hat{y}(N_1-2) \\ \hat{y}(N_1-1) \end{bmatrix}}_{\hat{\underline{Y}}(k)} = \underbrace{\left[\begin{array}{ccc|ccc} y(m+d-1) & \cdots & y(m+d-n) & u(m-1) & \cdots & u(0) \\ y(m+d) & \cdots & y(m+d-n+1) & u(m) & \cdots & u(1) \\ y(m+d+1) & \cdots & y(m+d-n+2) & u(m+1) & \cdots & u(2) \\ \vdots & & & \vdots & & \vdots \\ y(N_1-3) & \cdots & y(N_1-n-2) & u(N_1-d-3) & \cdots & u(N_1-d-m-2) \\ y(N_1-2) & \cdots & y(N_1-n-1) & u(N_1-d-2) & \cdots & u(N_1-d-m-1) \end{array}\right]}_{\phi(k)} \underbrace{\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_n \\ \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_m \end{bmatrix}}_{\hat{\underline{\Theta}}(k)} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} M \end{matrix}$$

$$\hat{\underline{Y}}(k) = \phi(k)\hat{\underline{\Theta}}(k)$$

Define Least-Squares cost function

$$J = \sum_{i=m+d}^{N_1-1} e^2(i) \quad \text{where } e(i) = y(i) - \hat{y}(i)$$

$$\underline{E} = \begin{bmatrix} e(m+d) \\ e(m+d+1) \\ \vdots \\ e(N_1-1) \end{bmatrix} = \begin{bmatrix} y(m+d) \\ y(m+d+1) \\ \vdots \\ y(N_1-1) \end{bmatrix} - \begin{bmatrix} \hat{y}(m+d) \\ \hat{y}(m+d+1) \\ \vdots \\ \hat{y}(N_1-1) \end{bmatrix}$$

$$\underline{E} = \underline{Y}(k) - \hat{\underline{Y}}(k)$$

$$J = \underline{E}^T \underline{E}$$

$$J = (\underline{Y}(k) - \underline{\hat{Y}}(k))^T(\underline{Y}(k) - \underline{\hat{Y}}(k))$$

$$\Rightarrow J = (\underline{Y}(k) - \phi(k)\underline{\hat{\Theta}}(k))^T(\underline{Y}(k) - \phi(k)\underline{\hat{\Theta}}(k))$$

$$(A+B)^T = A^T + B^T$$
$$(AB)^T = B^T A^T$$

$$\Rightarrow [\underline{Y}(k) - \phi(k)\underline{\hat{\Theta}}(k)]^T = \underline{Y}(k)^T - (\phi(k)\underline{\hat{\Theta}}(k))^T = \underline{Y}(k)^T - \underline{\hat{\Theta}}(k)^T\phi(k)^T$$

$$J = [\underline{Y}(k)^T - \underline{\hat{\Theta}}(k)^T\phi(k)^T][\underline{Y}(k) - \phi(k)\underline{\hat{\Theta}}(k)]$$

$$= \underline{Y}(k)^T\underline{Y}(k) - \underline{Y}(k)^T\phi(k)\underline{\hat{\Theta}}(k) - \underline{\hat{\Theta}}(k)^T\phi(k)^T\underline{Y}(k) + \underline{\hat{\Theta}}(k)^T\phi(k)^T\phi(k)\underline{\hat{\Theta}}(k)$$

$$= \underline{\hat{\Theta}}(k)^T \overset{M}{\phi(k)^T\phi(k)}\,\underline{\hat{\Theta}}(k) - \overset{G}{2\underline{Y}(k)^T\phi(k)}\,\underline{\hat{\Theta}}(k) + \overset{J_0}{\underline{Y}(k)^T\underline{Y}(k)}$$

This will be minimised when:

$$\underline{\hat{\Theta}}_{LS}^T = \tfrac{1}{2}GM^{-1} = \tfrac{1}{2}(2\underline{Y}(k)^T\phi(k))(\phi(k)^T\phi(k))^{-1}$$

$$\underline{\hat{\Theta}}_{LS}^T = \underline{Y}(k)^T\phi(k)(\phi(k)^T\phi(k))^{-1}$$

$$\Rightarrow \underline{\hat{\Theta}}(k) = \underline{(\phi(k)^T\phi(k))^{-1}\phi(k)^T\underline{Y}(k)}$$

$$\underline{Y}(k+1) = \begin{bmatrix} y(m+d) \\ y(m+d+1) \\ \vdots \\ y(k) \\ \hline y(k+1) \end{bmatrix} = \begin{bmatrix} \underline{Y}(k) \\ \hline y(k+1) \end{bmatrix}$$

$$\underline{\hat{Y}}(k+1) = \begin{bmatrix} \hat{y}(m+d) \\ \hat{y}(m+d+1) \\ \vdots \\ \hat{y}(k) \\ \hline \hat{y}(k+1) \end{bmatrix} = \begin{bmatrix} \underline{\hat{Y}}(k) \\ \hline \hat{y}(k+1) \end{bmatrix}$$

where

$$\hat{y}(k+1) = \hat{a}_1 y(k) + \dots \hat{a}_n y(k-n+1) + \hat{b}_1 u(k-d) + \dots \hat{b}_m u(k-d-m+1)$$

$$\hat{y}(k+1) = \underline{\Psi}^T(k+1)\,\underline{\hat{\Theta}}(k) \qquad \underline{\Psi}(k+1) = \begin{bmatrix} y(k) \\ \vdots \\ y(k-n+1) \\ \hline u(k-d) \\ \vdots \\ u(k-d-m+1) \end{bmatrix} \qquad \underline{\hat{\Theta}}(k) = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \\ \hline \hat{b}_1 \\ \vdots \\ \hat{b}_m \end{bmatrix}$$

The estimator equation can then be written as

$$\begin{bmatrix} \hat{y}(m+d) \\ \hat{y}(m+d+1) \\ \vdots \\ \hat{y}(k) \\ \hat{y}(k+1) \end{bmatrix} = \begin{bmatrix} y(m+d-1) & \cdots & y(m+d-n) & u(m-1) & \cdots & u(0) \\ y(m+d) & \cdots & y(m+d-n+1) & u(m) & \cdots & u(1) \\ \vdots & & & \vdots & & \\ y(k-1) & \ddots & y(k-n) & u(k-d-1) & \cdots & u(k-d-m) \\ y(k) & \cdots & y(k-n+1) & u(k-d) & \cdots & u(k-d-m+1) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_n \\ \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_m \end{bmatrix}$$

$$\underline{\hat{Y}}(k+1) = \begin{bmatrix} \hat{Y}(k) \\ \hline \hat{y}(k+1) \end{bmatrix} = \begin{bmatrix} \phi(k) \\ \hline \underline{\Psi}^T(k+1) \end{bmatrix} \underline{\hat{\Theta}}(k)$$

Consider

$$\phi(k+1)^T \phi(k+1) = \begin{bmatrix} \phi(k) \\ \underline{\Psi}^T(k+1) \end{bmatrix}^T \begin{bmatrix} \phi(k) \\ \underline{\Psi}^T(k+1) \end{bmatrix}$$

$$= \left[\phi(k)^T \,\vdots\, \underline{\Psi}(k+1) \right] \begin{bmatrix} \phi(k) \\ \hline \underline{\Psi}^T(k+1) \end{bmatrix}$$

$$= \phi(k)^T \phi(k) + \underline{\Psi}(k+1)\,\underline{\Psi}(k+1)^T$$

Define:

$P(k) = (\phi(k)^T \phi(k))^{-1}$

$P(k+1) = (\phi(k+1)^T \phi(k+1))^{-1}$

$$\Rightarrow P(k+1) = \underline{\underline{(P(k)^{-1} + \underline{\Psi}(k+1)\underline{\Psi}(k+1)^T)^{-1}}}$$

$(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$

$A = P(k)^{-1}$

$B = \underline{\Psi}(k+1)$

$C = 1$

$D = \underline{\Psi}(k+1)^T$

$$\Rightarrow P(k+1) = P(k) - P(k)\underline{\Psi}(k+1)[1+\underline{\Psi}(k+1)^T P(k)\underline{\Psi}(k+1)]^{-1}\underline{\Psi}(k+1)^T P(k)$$

But $[1+\underline{\Psi}(k+1)^T P(k)\underline{\Psi}(k+1)]^{-1} = \dfrac{1}{1+\underline{\Psi}(k+1)^T P(k)\underline{\Psi}(k+1)}$

$$P(k+1) = P(k) - \frac{P(k)\underline{\Psi}(k+1)\underline{\Psi}(k+1)^T P(k)}{1+\underline{\Psi}(k+1)^T P(k)\underline{\Psi}(k+1)}$$

$$\begin{aligned}\hat{\underline{\Theta}}(k+1) &= (\phi(k+1)^T\phi(k+1))^{-1}\phi(k+1)^T\underline{Y}(k+1)\\ &= P(k+1)\phi(k+1)^T\underline{Y}(k+1)\end{aligned}$$

$$\begin{aligned}\phi(k+1)^T\underline{Y}(k+1) &= \begin{bmatrix}\phi(k)\\ \underline{\Psi}^T(k+1)\end{bmatrix}^T\begin{bmatrix}\underline{Y}(k)\\ y(k+1)\end{bmatrix}\\ &= \begin{bmatrix}\phi(k)^T & \underline{\Psi}(k+1)\end{bmatrix}\begin{bmatrix}\underline{Y}(k)\\ y(k+1)\end{bmatrix}\\ &= \phi(k)^T\underline{Y}(k) + \underline{\Psi}(k+1)y(k+1)\end{aligned}$$

$$\hat{\underline{\Theta}}(k+1) = \left[P(k) - \frac{P(k)\underline{\Psi}(k+1)\underline{\Psi}(k+1)^T P(k)}{1+\underline{\Psi}(k+1)^T P(k)\underline{\Psi}(k+1)}\right]\left[\phi(k)^T\underline{Y}(k) + \underline{\Psi}(k+1)y(k+1)\right]$$

4/5/09

Summer 2008

Q4 (a) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{A}{\begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{B}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} u(t)$

$y(t) = \overset{C}{[2 \;\; 3]}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$G(s) = C(sI-A)^{-1}B$

$(sI-A) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} s+2 & 0 \\ 0 & s+4 \end{pmatrix}$

$(sI-A)^{-1} = \frac{1}{(s+2)(s+4)}\begin{pmatrix} s+4 & 0 \\ 0 & s+2 \end{pmatrix}$

$C(sI-A)^{-1}B = \frac{1}{(s+2)(s+4)}(2 \;\; 3)\begin{pmatrix} s+4 & 0 \\ 0 & s+2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \frac{1}{(s+2)(s+4)}(2 \;\; 3)\begin{pmatrix} s+4 \\ s+2 \end{pmatrix}$

$= \frac{2(s+4)+3(s+2)}{(s+2)(s+4)}$

$\Rightarrow G(s) = \frac{5s+14}{s^2+6s+8}$

$C_x = [B \,\vdots\, AB]$
$= \begin{bmatrix} 1 & -2 \\ 1 & -4 \end{bmatrix}$

$AB = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

$\det(C_x) = -4+2 = -2$
$\det(C_x) \neq 0$
$\Rightarrow$ controllable

$O_x = \begin{bmatrix} C \\ \hdashline CA \end{bmatrix}$

$CA = (2 \;\; 3)\begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$

$= (-4 \;\; -12)$

$O_x = \begin{bmatrix} 2 & 3 \\ -4 & -12 \end{bmatrix}$

$\det(O_x) = -24+12 = -12$
$\det(O_x) \neq 0$
$\Rightarrow$ observable

$G(s) = \dfrac{5s+14}{s^2+6s+8}$

CCF

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u(t)$$

$$y(t) = \overset{C}{[14 \;\; 5]}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$AB = \begin{pmatrix} 0 & 1 \\ -8 & -6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

$C_z = [B \,\vdots\, AB] = \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}$

$C_z = T C_x$
$T = C_z C_x^{-1}$

$$C_x^{-1} = \frac{1}{-4+2}\begin{bmatrix} -4 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0.5 & -0.5 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & -1 \\ -0.5 & 2 \end{bmatrix}$$

24/4/09 Summer 2008

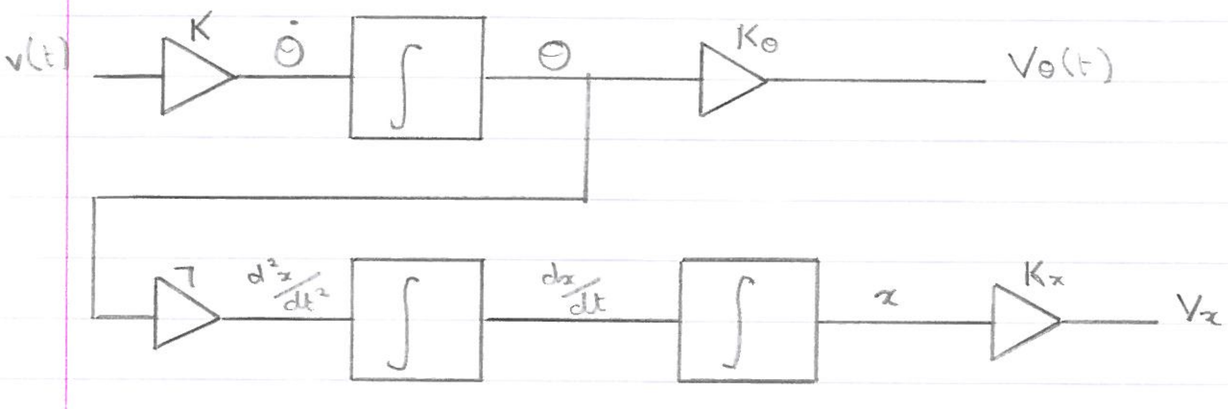
Q4(b). $\frac{d\theta(t)}{dt} = K\,v(t)$

$\frac{d^2x}{dt^2} = 7\,\theta(t)$

$\frac{dx}{dt} = \frac{dx}{dt}$

$V_\theta(t) = K_\theta\,\theta(t)$

$V_x(t) = K_x\,x(t)$



$G(s) = \frac{7KK_x}{s^3}$

Zero input response

$\Rightarrow \underline{x}(t) = \phi(t)\,\underline{x}(0)$

$\phi(s) = (sI - A)^{-1}$

$$= \begin{bmatrix} s & 0 & 0 \\ -7 & s & 0 \\ 0 & -1 & s \end{bmatrix}^{-1}$$

$$\left[\begin{array}{ccc|ccc} s & 0 & 0 & 1 & 0 & 0 \\ -7 & s & 0 & 0 & 1 & 0 \\ 0 & -1 & s & 0 & 0 & 1 \end{array}\right]$$

$\downarrow \times \frac{1}{s}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{s} & 0 & 0 \\ -\frac{7}{s} & 1 & 0 & 0 & \frac{1}{s} & 0 \\ 0 & -\frac{1}{s} & 1 & 0 & 0 & \frac{1}{s} \end{array}\right]$$

$\downarrow R_2 + \frac{7}{s}R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{s} & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{s^2} & \frac{1}{s} & 0 \\ 0 & -\frac{1}{s} & 1 & 0 & 0 & \frac{1}{s} \end{array}\right]$$

$$\downarrow R_3 + \tfrac{1}{s} R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Bigg| \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ \frac{7}{s^2} & \frac{1}{s} & 0 \\ \frac{7}{s^3} & \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \phi(s) = (sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ \frac{7}{s^2} & \frac{1}{s} & 0 \\ \frac{7}{s^3} & \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}$$

$$\Rightarrow \phi(t) = \begin{bmatrix} 1 & 0 & 0 \\ 7t & 1 & 0 \\ 3.5t^2 & t & 1 \end{bmatrix}$$

$$\underline{y}(t) = C\underline{x}(t)$$
$$= C\phi(t)\underline{x}(0)$$

$$\begin{bmatrix} V_\theta(t) \\ V_x(t) \end{bmatrix} = \begin{bmatrix} K_\theta & 0 & 0 \\ 0 & 0 & K_x \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 7t & 1 & 0 \\ 3.5t^2 & t & 1 \end{bmatrix} \begin{bmatrix} \theta_0 \\ 0 \\ x_0 \end{bmatrix}$$

$$\begin{bmatrix} V_\theta(t) \\ V_x(t) \end{bmatrix} = \begin{bmatrix} K_\theta & 0 & 0 \\ 0 & 0 & K_x \end{bmatrix} \begin{bmatrix} \theta_0 \\ 7t\theta_0 \\ 3.5t^2\theta_0 + x_0 \end{bmatrix}$$

$$\Rightarrow V_x(t) = K_x(3.5\theta_0 t^2 + x_0)$$

(ii) Consider the MIMO process:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$
$$\underline{y}(t) = C\underline{x}(t)$$