**OLLSCOIL NA hÉIREANN, CORCAIGH**
THE NATIONAL UNIVERSITY OF IRELAND, CORK

COLÁISTE NA hOLLSCOILE, CORCAIGH
UNIVERSITY COLLEGE, CORK

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**SUMMER EXAMINATIONS, 2009**

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**B.E. DEGREE (ELECTRICAL)**

CONTROL ENGINEERING
EE4002

Professor C. Delabie
Professor P. Murphy
Dr. G. Lightbody

Time allowed: *3 hours*

Answer *four* questions
All questions carry equal marks

The use of departmental approved non-programmable calculators is permitted

**1.**

(a) Consider the following closed-loop system, where the sample time is T,

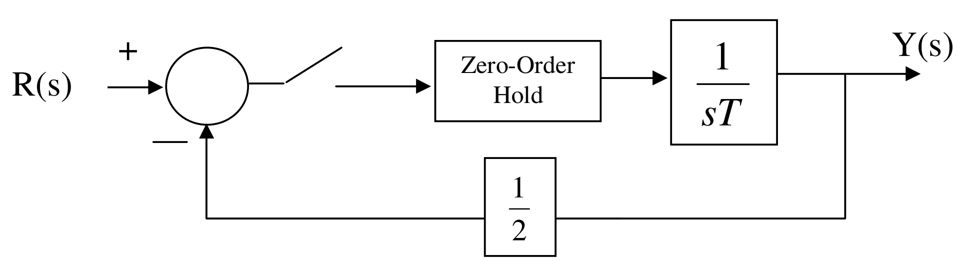


Fig. 1.1: Discrete-time control system

Sketch the response for the continuous signal *y(t)* for a unit step in the setpoint *r(t).*

[5 marks]

(b) Derive Tustins's transformation.

A certain continuous controller C(s) has been designed,

$$C(s) = \frac{M(s)}{E(s)} = \frac{1}{s+a}$$

Use Tustin's transformation to develop a difference equation representation of this controller for implementation on a digital computer with sample-time *T*.

By comparison with the matched-pole-zero method, derive the following first-order Padé approximation,

$$e^{-aT} \approx \frac{1-\dfrac{Ta}{2}}{1+\dfrac{Ta}{2}}$$

[8 Marks]

(c) A certain SISO discrete-time process has input *u(k)* and output *y(k)*. The response of this system to a unit step input is given in Fig. 1.2.

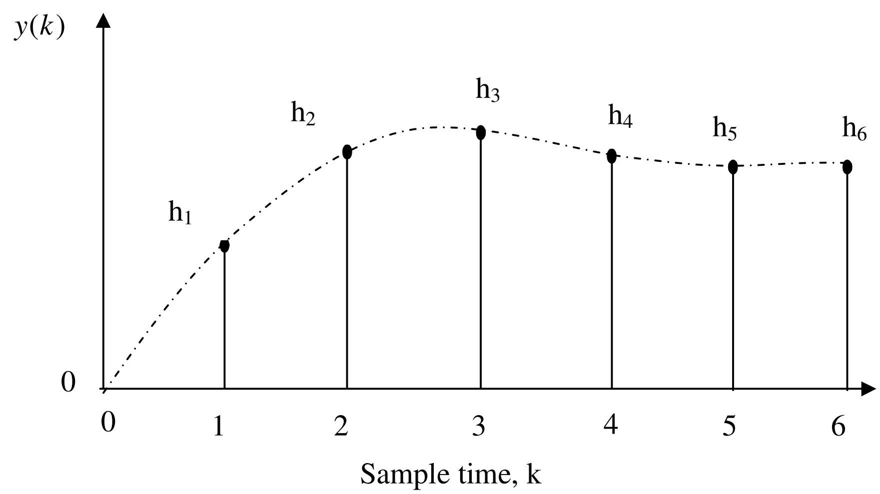


Fig. 1.2: Discrete-time unit step response

Determine an expression for the discrete time transfer function Y(z)/U(z).

The discrete-time step response is commonly used in predictive control to provide a predictive model of the process. If we are currently at the $k^{th}$ sampling instant, then show from Fig. 1.2, that the output prediction over the next four steps into the future can be written as,

$$\begin{bmatrix} y(k+1) \\ y(k+2) \\ y(k+3) \\ y(k+4) \end{bmatrix} = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ h_2 - h_1 & h_1 & 0 & 0 \\ h_3 - h_2 & h_2 - h_1 & h_1 & 0 \\ h_4 - h_3 & h_3 - h_2 & h_2 - h_1 & h_1 \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \\ u(k+3) \end{bmatrix} + \underline{y}_f(k).$$

Where the vector $\underline{y}_f(k)$ contains the free response, caused by control actions in the past (before the current k$^{th}$ sampling instant).

The desired setpoint over the next four samples is,

$$\underline{r}(k) = \begin{bmatrix} r(k+1) & r(k+2) & r(k+3) & r(k+4) \end{bmatrix}^T.$$

Determine (without actually solving) a mathematical expression for the vector of controls $\underline{u}(k) = \begin{bmatrix} u(k) & u(k+1) & u(k+2) & u(k+3) \end{bmatrix}^T$ that will drive the process output exactly to the setpoint over the next four samples.

[12 Marks]

**2.**

(a) A certain digital controller has been designed as:

$$D(z) = \frac{K(z-\alpha)}{z^2(z-\beta)(z-\gamma)}$$

Show how this controller could be realised using four delay blocks.

[5 Marks]

(b) Consider the following closed-loop discrete-time process,

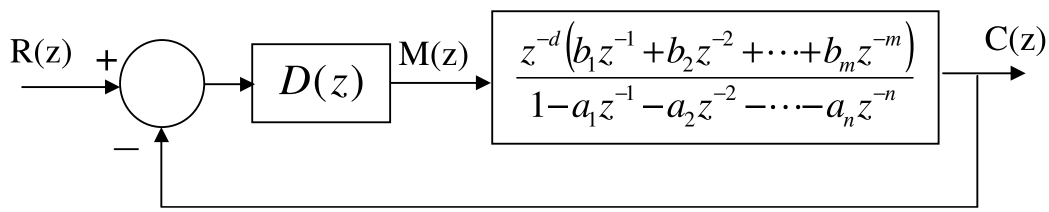


Fig. 2.1: Closed-loop Discrete Time Process

Show that the following Kalman controller could be designed for this process:

$$D(z) = \frac{1 - \sum_{i=1}^{n} a_i z^{-i}}{\sum_{j=1}^{m} b_j \left(1 - z^{-d-j}\right)}$$

Sketch the closed-loop response for both the controller output sequence *m(k)* and the process output *c(k)*, for a unit-step in the setpoint signal *r(k)*.

What are the key benefits and potential drawbacks of this controller design method?

[12 Marks]

(c) A closed-loop position control scheme for a single-link robotic manipulator is shown below. The controller gain K is designed in the continuous domain to achieve some desired closed-loop performance.

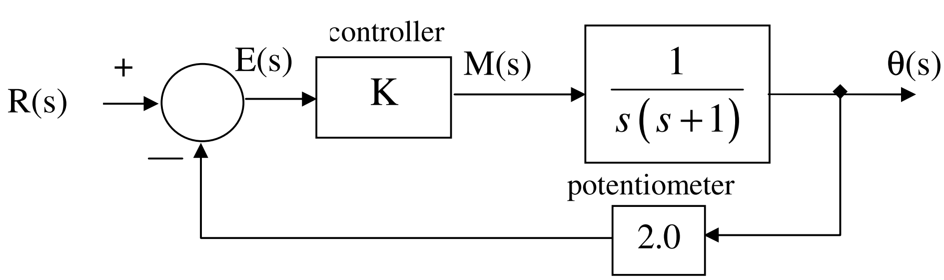


Fig. 2.2: Closed-loop motor speed control

By use of root-locus plots, show how the closed-loop dynamic performance for the digital implementation of this controller may be very different from that designed for in the continuous domain. The sample time is $T$=0.1 seconds.

[8 marks]

**3.**

In order to emphasise more recent information, "forgetting" can be incorporated within the least squares algorithm. A common choice for the least squares cost function over N valid test points is then:

$$J\left(\underline{\hat{\theta}}(k)\right) = \sum_{i=0}^{N-1} \lambda^i e(k-i)^2$$

Where, the forgetting factor $\lambda \le 1$, and e(k) is the prediction error.

(i) Derive in full, the following least-squares algorithm with forgetting, for the identification of the parameters $\underline{\hat{\theta}}(k)$, of a discrete-time transfer function. Here $\Phi(k)$ is a matrix of input and output data, and the vector $\underline{y}(k)$ contains N valid samples of the process output, up to the current k$^{th}$ sample, *y(k)*.

$$\underline{\hat{\theta}}(k) = \left(\Phi(k)^T \Lambda_N \Phi(k)\right)^{-1} \Phi(k)^T \Lambda_N \underline{Y}(k)$$

Where the weighting matrix for N valid points is the diagonal matrix, defined as:

$$\Lambda_N = \begin{bmatrix} \lambda^{N-1} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & & & 0 \\ 0 & & \lambda^2 & & \vdots \\ \vdots & & & \lambda & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

[13 Marks]

(ii) If a square matrix *P(k)* is now defined as $P(k) = \left(\Phi(k)^T \Lambda_N \Phi(k)\right)^{-1}$, derive the following update equation for $P^{-1}(k+1)$ from process data up to the (k+1)$^{th}$ sample, where the vector $\underline{\psi}(k+1)$ contains process input and output data sampled up to the (k+1)$^{th}$ sample,

$$P^{-1}(k+1) = \lambda P^{-1}(k) + \underline{\psi}(k+1)\underline{\psi}^T(k+1)$$

use Householders Matrix Inversion Lemma,

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1} DA^{-1},$$

to derive the following update equation:

$$P(k+1) = \frac{1}{\lambda}\left[ P(k) - \frac{P(k)\underline{\psi}(k+1)\underline{\psi}^T(k+1)P(k)}{\lambda + \underline{\psi}^T(k+1)P(k)\underline{\psi}(k+1)} \right].$$

[12 Marks]

**4.**

(a) A certain second-order SISO process can be modelled as:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the transfer function of this process, $G(s)=Y(s)/U(s)$.

Is this system representation controllable and observable?

[7 Marks]

(b) Consider the following state-space equation,

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t).$$

Develop fully the following solution for the state trajectory $\underline{x}(t)$,

$$\underline{x}(t) = e^{At}\left(\underline{x}(0) + \int_0^t e^{-A\tau}B\underline{u}(\tau)d\tau\right),$$

where $\underline{x}(0)$ is the initial state, and $e^{At}$ is the matrix exponential.

If the sample time T is small, and a Zero-order-Hold is assumed on the input, derive the following discrete-time approximation of this process,

$$\frac{\Delta\underline{x}(k+1)}{T} = A\underline{x}(k) + B\underline{u}(k)$$

where,

$$\Delta\underline{x}(k+1) = \underline{x}(k+1) - \underline{x}(k)$$

[9 Marks]

(c) A classical control scheme for a general DC motor based positioning system is shown in Fig. 4.1.

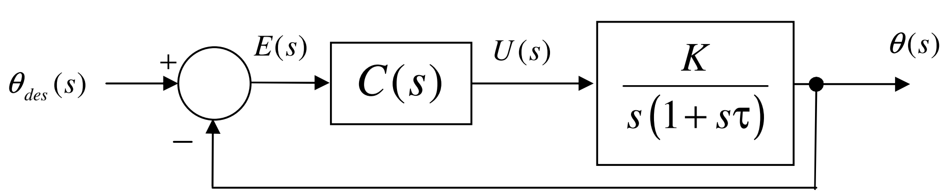


Fig. 4.1: Closed-loop position control system

Here the following PD controller C(s) is used;

$$C(s) = K_C\left(s + z\right).$$

The setpoint is usually constant. Use the state-space design method to provide the following design equations for the PD controller:

$$z = \frac{\alpha^2\tau}{2\alpha\tau - 1}$$
$$K_C = \frac{2\alpha\tau - 1}{K}$$

Here the desired closed loop poles are both placed at $s = -\alpha$, where, $\frac{1}{2\tau} \le \alpha \le \frac{1}{\tau}$

[9 Marks]

**5.**

(a) Consider the following N[th] order open-loop process with a single input *u(t),* a single output *y(t)* and a single unmeasured disturbance *d(t)*,

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + Bu(t) + Ed(t)$$
$$y(t) = C\underline{x}(t).$$

If there is no measurement of the disturbance, but it is known that, $\lim_{t\to\infty} d(t) = d_\infty$, show that the steady state estimation error vector, for a Luenberger observer is:

$$\underline{e}_{ss} = \lim_{t\to\infty}\left(\underline{x}(t) - \hat{\underline{x}}(t)\right) = -\left(A - GC\right)^{-1} Ed_\infty,$$

where G is the Luenberger observer gain matrix.

[6 marks]

(b) Consider the following ball-on-beam apparatus consisting of a rigid beam, free to rotate in one plane about its central pivot. A servo-motor is used to rotate the beam. There are two parallel guide rails, on which a steel ball sits.

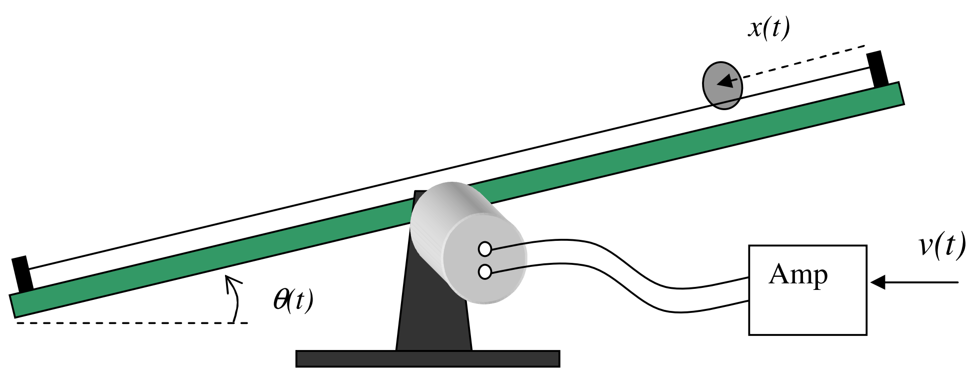


Fig.5.1: Ball-on-Beam Apparatus

Two sensors are available. The first is a simple rotary potentiometer that is used to provide a measure of the beam angle $\theta(t)$. The second sensor provides a measurement of the ball position *x(t),* using the wire guide rails as a linear potentiometer.

The servo-motor dynamics are so fast that the rotation of the beam can be described by the following first-order differential equation:

$$\frac{d\theta(t)}{dt} = Kv(t).$$

The gains of the linear and rotary potentiometers are $K_x$ and $K_\theta$. respectively

If the moment of inertia, about the axis of rotation, of the ball of mass $m$ and radius $r$, is $J={}^2/_5\, mr^2$, basic rotational mechanics yields the following expression for the linear acceleration:

$$\frac{d^2x}{dt^2} = 7\theta(t).$$

(i) Assume first that all the states of this third order model are available and that the gain $K$=5Vrad$^{-1}$s. Design a state-space position controller, that will meet the following specifications.

- Zero steady-state error for a constant desired ball position
- Closed-loop poles are selected to ensure second-order dominance.
- In response to a step change in the desired ball position, the peak overshoot in ball position is specified to be 15%, and the settling time is specified as $Ts_{2\%} = 2$ seconds.

[10 marks]

(ii) If we note that there is a decoupling of the beam dynamics from the ball dynamics, it is possible to build a simplified second-order observer to estimate the ball velocity from just the potentiometer output voltages $v_x(t)$ and $v_\theta(t)$.

The potentiometer gains are $K_x$ =2V/m and $K_\theta$= 1V/radian.

Design a second-order Luenberger Observer to provide an estimate of the ball velocity for use in the controller designed in part ii) above.

[9 marks]

**6.**

(a) A certain process can be modelled by the transfer function:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{K\left(1+s\tau_1\right)}{\left(1+s\tau_2\right)\left(1+s\tau_3\right)}$$

Develop fully a simulation diagram for the Observer Canonical representation of this process.

[5 Marks]

(b) Consider the following N$^{th}$ order open-loop process, with single input *u(t)*, single output *y(t)* , and state-vector $\underline{x}(t)$,

$$\frac{d}{dt}\underline{x}(t)=A\,\underline{x}(t)+B\,u(t)$$
$$y(t)=C\,\underline{x}(t)$$

The state vector is not measured directly, but is estimated as $\hat{\underline{x}}(t)$ using a full-state Luenberger observer with estimator gain matrix *G*.

The following control-law is utilised, where *r(t)* is the setpoint signal.

$$u(t)=-K\,\hat{\underline{x}}(t)+Nr(t)$$

(i) Develop fully the following representation of the closed loop system,

$$\frac{d}{dt}\begin{bmatrix}\underline{x}(t)\\ \hline \underline{e}(t)\end{bmatrix}=\left[\begin{array}{c|c}A-BK & BK\\ \hline 0 & A-GC\end{array}\right]\begin{bmatrix}\underline{x}(t)\\ \hline \underline{e}(t)\end{bmatrix}+\begin{bmatrix}BN\\ \hline \underline{0}\end{bmatrix}r(t)$$

where the estimation error $\underline{e}(t)$ is defined as, $\underline{e}(t)=\underline{x}(t)-\hat{\underline{x}}(t)$

Use this representation to explain the "Separation Principle", and how this principle is applied in state-space control design.

[10 Marks]

(ii) Show that the closed-loop system could be represented by the following classical T,Q,S realisation.

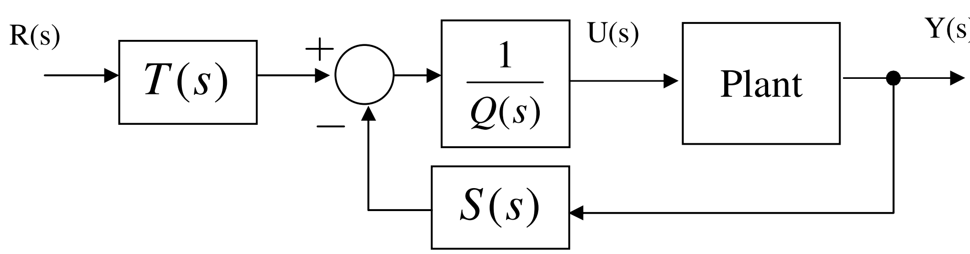


Fig. 6.1: Classical T,Q,S representation of closed-loop system

Where $Q(s)$, $T(s)$ and $S(s)$ are polynomials in $s$.

If $Q(s)$ is,

$$Q(s) = \det\left(sI - A + GC + BK\right)$$

Determine expressions for the feedback polynomial $S(s)$ and the pre-filter polynomial $T(s)$.

[10 Marks]