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EE4002 Control Engineering

B) State-Space Control

Chapter 1. The State Space Modelling Approach

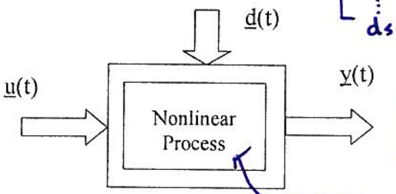
1.1 The State Space Model

Transformation of an N^{th} order multi-input-multi-output system to N first order differential equations: TIME DOMAIN MODEL

Benefits

- Robust method of simulating high order differential equations
- Easy analysis of dynamics
- Allows for use of model reduction methods ↗ LINEAR QUADRATIC REGULATOR
- Can easily apply advanced control → OPTIMAL CONTROL
- Used for estimation → KALMAN FILTERING MULTIVARIATE

Consider the N^{th} order nonlinear dynamical process:



$\begin{bmatrix} d_1 \\ \vdots \\ d_s \end{bmatrix}$ disturbance vector

M inputs $\begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ $\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$ p outputs

In general this process could be represented by a model consisting of : p differential equations (1 each o/p) possibly high order + non linear

THIS MEANS PROCESS IS n^{th} ORDER

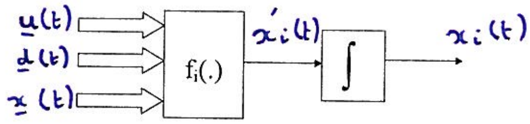
It is possible to transform this set of coupled nonlinear differential equations to yield a set of N first order differential equations with states, $\{x_1(t), x_2(t), \ldots x_N(t)\}$.

STATE VECTOR $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix}$ REPRESENTS THE DYNAMIC STATE OF PROCESS

The differential equation describing the dynamics of the i^{th} state $x_i(t)$ can be written as:

$$\frac{d}{dt} x_i(t) = f_i(\underline{x}(t), \underline{u}(t), \underline{d}(t))$$ general

Which could be represented by the following simulation diagram:



The N first order differential equations could be written as:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \dot{\underline{x}}(t) = \begin{bmatrix} f_1(\underline{x}(t),\underline{u}(t),\underline{d}(t)) \\ f_2(\underline{x}(t),\underline{u}(t),\underline{d}(t)) \\ \vdots \\ f_N(\underline{x}(t),\underline{u}(t),\underline{d}(t)) \end{bmatrix} \Bigg\} \text{ N equations}$$

⇒ This represents the internal dynamics of the process

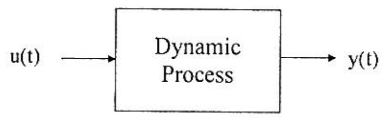
The output $\underline{y}(t)$ is then generated by:

$$\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_P(t) \end{bmatrix} = \begin{bmatrix} h_1(\underline{x}(t),\underline{u}(t),\underline{d}(t)) \\ h_2(\underline{x}(t),\underline{u}(t),\underline{d}(t)) \\ \vdots \\ h_P(\underline{x}(t),\underline{u}(t),\underline{d}(t)) \end{bmatrix} \Bigg\} \text{ p outputs } \therefore \text{ p eqs}$$

THIS IS STATIC

1.1.1 Some Example State-Space Systems:

i) Third Order Linear Process



Modelled by the following differential equation:

$$\frac{d^3}{dt^3}y(t) + 5\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + y(t) = u(t)$$

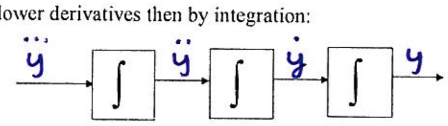
Which could of course be represented by the transfer function model:

$$U(s) \longrightarrow \boxed{\frac{1}{s^3+5s^2+3s+1}} \longrightarrow Y(s)$$

Rearrange to yield an expression for the highest derivative:

$$\dddot{y} = u(t) - y - 3\dot{y} - 5\ddot{y}$$

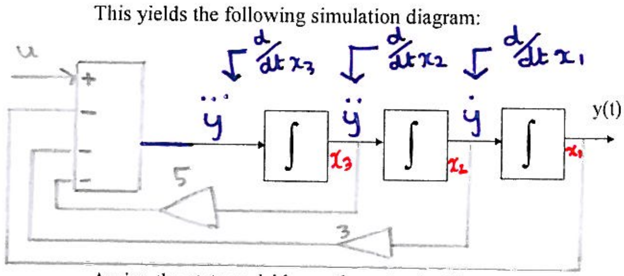
Form lower derivatives then by integration:



But we know that:

$$\dddot{y}(t) = u(t) - 5\ddot{y}(t) - 3\dot{y}(t) - y(t)$$

This yields the following simulation diagram:



Assign the state variables as the outputs of integrators:

$x_1(t), x_2(t), x_3(t)$

We can now specify the state vector for the process as:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Now this could be written in matrix form as:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1\;0\;0]\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

ASIDE

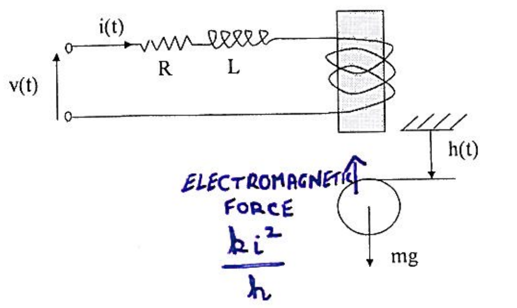
$$\frac{d}{dt}x_1 = \dot{y} = x_2$$

$$\frac{d}{dt}x_2 = \ddot{y} = x_3$$

$$\frac{d}{dt}x_3 = \dddot{y} = u - y - 3\dot{y} - 5\ddot{y} = u - x_1 - 3x_2 - 5x_3$$

ii) Third Order Nonlinear Process

Consider the following magnetic suspension system:



Can be modelled by:

coupled non linear 3rd order model

$$m\frac{d^2h(t)}{dt^2} = mg - \frac{Ki^2(t)}{h(t)}$$ 2nd order (FORCES NEWTON'S LAWS)

$$L\frac{di(t)}{dt} = v(t) - Ri(t)$$ 1st order (electrical)

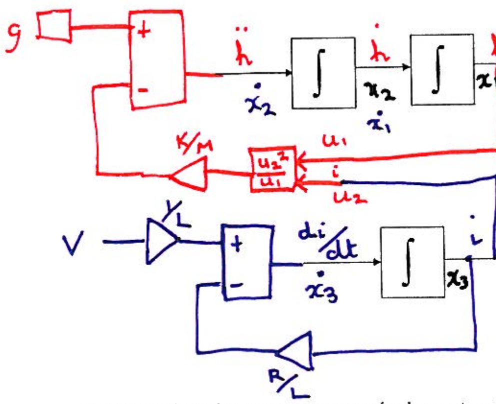
Rewritten as:

$$\frac{d^2h(t)}{dt^2} = g - \frac{Ki^2(t)}{mh(t)}$$

$$\frac{di(t)}{dt} = \frac{1}{L}\left(v(t) - Ri(t)\right)$$

Could be represented by the following simulation diagram:

$$\ddot{h} = g - \frac{Ki^2}{mh}$$



$$\frac{di}{dt} = \frac{1}{L}(v - Ri)$$

Note that three integrators are required – system is 3rd order

Now can arbitrarily assign the three states $\{x_1(t), x_2(t), x_3(t)\}$

And define the state vector:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The state-equations are now:

$$\dot{x}_1 = x_2(t)$$

$$\dot{x}_2 = g - \frac{Kx_3^2(t)}{mx_1(t)}$$ ($x_3^2 \rightarrow i^2$, $x_1 \rightarrow h$)

$$\dot{x}_3 = \frac{1}{L}\left(u(t) - Rx_3(t)\right)$$ ($x_3 \rightarrow i$)

OR

$$\frac{d}{dt}\underline{x} = \underline{f}(\underline{x}, u) = \begin{bmatrix} f_1(\underline{x}, u) \\ f_2(\underline{x}, u) \\ f_3(\underline{x}, u) \end{bmatrix}$$

with output equation:

$$y(t) = x_1(t)$$ $h(t)$

the gap $h(t)$

1.2 Derivation of the Linear State Space Model - by Linearisation

Consider first the linearisation of a multivariate function:

$$z = g(\underline{w}) \quad where \quad \underline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} = g(w_1, w_2, \dots w_N)$$

about the operating point vector:

$$\underline{w}_0 = \begin{bmatrix} w_1' \\ \vdots \\ w_N' \end{bmatrix}$$

Use the first order Taylor's series approximation :

$$z \approx g(\underline{w}_0) + \left.\frac{\partial g}{\partial w_1}\right|_{\underline{w}=\underline{w}_0}(w_1 - w_1') + \left.\frac{\partial g}{\partial w_2}\right|_{\underline{w}=\underline{w}_0}(w_2 - w_2') + \cdots \left.\frac{\partial g}{\partial w_N}\right|_{\underline{w}=\underline{w}_0}(w_N - w_N')$$

CONSTANTS GAINS SENSITIVITIES

which could be written as:

CONSTANTS

$$\underbrace{(z - z_0)}_{\Delta z} = \left[\left.\frac{\partial g}{\partial w_1}\right|_{\underline{w}_0} \quad \left.\frac{\partial g}{\partial w_2}\right|_{\underline{w}_0} \quad \cdots \quad \left.\frac{\partial g}{\partial w_N}\right|_{\underline{w}_0}\right]\underbrace{(\underline{w} - \underline{w}_0)}_{\Delta \underline{w}} \quad \begin{bmatrix} w_1 - w_1' \\ w_2 - w_2' \\ \vdots \\ w_N - w_N' \end{bmatrix}$$

Now for simplicity first consider a linearisation of the i^{th} state equation:

$$z = \frac{dx_i(t)}{dt} = f_i(\underline{x}(t))$$

let $z = \frac{dx_i}{dt}$ $\therefore z_0 = \left.\frac{dx_i}{dt}\right|$ at operating point

Could be linearised to yield: $z = f_i(\underline{x})$

$$\underbrace{(z - z_0)}_{\Delta z} = \underbrace{\left[\left.\frac{\partial f_i}{\partial x_1}\right|_{\underline{x}_0} \quad \left.\frac{\partial f_i}{\partial x_2}\right|_{\underline{x}_0} \quad \cdots \quad \left.\frac{\partial f_i}{\partial x_N}\right|_{\underline{x}_0}\right]}_{\text{CONSTANTS - DEPEND ON } x_0}\underbrace{(\underline{x} - \underline{x}_0)}_{\Delta \underline{x}}$$

But: $z_0 = \left.\frac{dx_i}{dt}\right|_{op}$ we will define op as an equilibrium point $z_0 = 0$

$z = \frac{dx_i}{dt}$

Hence the linearised equation becomes:

$(z - z_0) = \Delta z = \frac{dx_i}{dt}$

$$z - z_0 \quad \frac{d}{dt}\Delta x_i(t) = \left[\left.\frac{\partial f_i}{\partial x_1}\right|_{\underline{x}_0} \quad \left.\frac{\partial f_i}{\partial x_2}\right|_{\underline{x}_0} \quad \cdots \quad \left.\frac{\partial f_i}{\partial x_N}\right|_{\underline{x}_0}\right]\Delta \underline{x}(t)$$

LINEAR EQUATION

SMALL CHANGE IN STATE VECTOR $= \frac{d}{dt}\Delta x_i$

Now let us expand to include input and disturbance vector:

$$\frac{dx_i(t)}{dt} = f_i(\underline{x}(t), \underline{u}(t), \underline{d}(t))$$

Which will have the linearisation about the operating point:

$op = \{\underline{x}_0, \underline{u}_0, \underline{d}_0\}$

$$\frac{d}{dt}\Delta x_i(t) = \left[\left.\frac{\partial f_i}{\partial x_1}\right|_{op} \quad \left.\frac{\partial f_i}{\partial x_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial f_i}{\partial x_N}\right|_{op}\right]\Delta \underline{x}(t) + \left[\left.\frac{\partial f_i}{\partial u_1}\right|_{op} \quad \left.\frac{\partial f_i}{\partial u_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial f_i}{\partial u_m}\right|_{op}\right]\Delta \underline{u}(t)$$
$$+ \left[\left.\frac{\partial f_i}{\partial d_1}\right|_{op} \quad \left.\frac{\partial f_i}{\partial d_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial f_i}{\partial d_s}\right|_{op}\right]\Delta \underline{d}(t)$$

which could further be written as:

$$\frac{d}{dt}\Delta x_i(t) = \left[a_{i1} \quad a_{i2} \quad \cdots \quad a_{iN}\right]\Delta \underline{x}(t) + \left[b_{i1} \quad b_{i2} \quad \cdots \quad b_{im}\right]\Delta \underline{u}(t) + \left[e_{i1} \quad e_{i2} \quad \cdots \quad e_{is}\right]\Delta \underline{d}(t)$$

CONSTANTS CALCULATED AT OPERATING POINT

This could of course be repeated for all N state equations:

$$N\left\{\frac{d}{dt}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_i\\ \vdots\\ \Delta x_N\end{bmatrix}\right. = \overset{N\times N}{\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1N}\\ a_{21} & a_{22} & \cdots & a_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{iN}\\ \vdots & \vdots & \ddots & \vdots\\ a_{N1} & a_{N2} & \cdots & a_{NN}\end{bmatrix}}\overset{\Delta x}{\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_i\\ \vdots\\ \Delta x_N\end{bmatrix}} + \overset{N\times M}{\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1m}\\ b_{21} & b_{22} & \cdots & b_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ b_{i1} & b_{i2} & \cdots & b_{im}\\ \vdots & \vdots & \ddots & \vdots\\ b_{N1} & b_{N2} & \cdots & b_{Nm}\end{bmatrix}}\overset{\Delta u}{\begin{bmatrix}\Delta u_1\\ \Delta u_2\\ \vdots\\ \Delta u_m\end{bmatrix}}$$

$$+\overset{N\times S}{\begin{bmatrix}e_{11} & e_{12} & \cdots & e_{1s}\\ e_{21} & e_{22} & \cdots & e_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ e_{i1} & e_{i2} & \cdots & e_{is}\\ \vdots & \vdots & \ddots & \vdots\\ e_{N1} & e_{N2} & \cdots & e_{Ns}\end{bmatrix}}\overset{\Delta d}{\begin{bmatrix}\Delta d_1\\ \Delta d_2\\ \vdots\\ \Delta d_s\end{bmatrix}}$$

Or in more compact form as:

$$\frac{d}{dt}\Delta \underline{x} = A\Delta\underline{x} + B\Delta\underline{u} + E\Delta\underline{d}$$ INTERNAL DYNAMICS

Now consider the static output equation for the ith output:

$$y_i(t) = h_i\big(\underline{x}(t), \underline{u}(t), \underline{d}(t)\big)$$

The linearisation about the operating point $\{\underline{x}_0, \underline{u}_0, \underline{d}_0\}$ is:

$$\Delta y_i(t) = \left[\left.\frac{\partial h_i}{\partial x_1}\right|_{op} \quad \left.\frac{\partial h_i}{\partial x_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial h_i}{\partial x_N}\right|_{op}\right]\Delta\underline{x}(t) + \left[\left.\frac{\partial h_i}{\partial u_1}\right|_{op} \quad \left.\frac{\partial h_i}{\partial u_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial h_i}{\partial u_m}\right|_{op}\right]\Delta\underline{u}(t)$$
$$+\left[\left.\frac{\partial h_i}{\partial d_1}\right|_{op} \quad \left.\frac{\partial h_i}{\partial d_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial h_i}{\partial d_s}\right|_{op}\right]\Delta\underline{d}(t)$$

which could be written as:

$$\Delta y_i(t) = [\overset{\left.\frac{\partial h_i}{\partial x_i}\right|_{op}}{c_{i1}} \quad c_{i2} \quad \cdots \quad c_{iN}]\Delta\underline{x}(t) + [d_{i1} \quad d_{i2} \quad \cdots \quad d_{im}]\Delta\underline{u}(t) + [f_{i1} \quad \overset{\left.\frac{\partial h_i}{\partial d_2}\right|_{op}}{f_{i2}} \quad \cdots \quad f_{is}]\Delta\underline{d}(t)$$

This could be repeated for all the P outputs:

$$P\left\{\begin{bmatrix}\Delta y_1\\ \Delta y_2\\ \vdots\\ \Delta y_i\\ \vdots\\ \Delta y_P\end{bmatrix}\right. = \overset{P\times N}{\begin{bmatrix}c_{11} & c_{12} & \cdots & c_{1N}\\ c_{21} & c_{22} & \cdots & c_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ c_{i1} & c_{i2} & \cdots & c_{iN}\\ \vdots & \vdots & \ddots & \vdots\\ c_{P1} & c_{P2} & \cdots & c_{PN}\end{bmatrix}}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_i\\ \vdots\\ \Delta x_N\end{bmatrix} + \overset{P\times M}{\begin{bmatrix}d_{11} & d_{12} & \cdots & d_{1m}\\ d_{21} & d_{22} & \cdots & d_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ d_{i1} & d_{i2} & \cdots & d_{im}\\ \vdots & \vdots & \ddots & \vdots\\ d_{P1} & d_{P2} & \cdots & d_{Pm}\end{bmatrix}}\left.\begin{bmatrix}\Delta u_1\\ \Delta u_2\\ \vdots\\ \Delta u_m\end{bmatrix}\right\}M$$

$$+\overset{P\times s}{\begin{bmatrix}f_{11} & f_{12} & \cdots & f_{1s}\\ f_{21} & f_{22} & \cdots & f_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ f_{i1} & f_{i2} & \cdots & f_{is}\\ \vdots & \vdots & \ddots & \vdots\\ f_{P1} & f_{P2} & \cdots & f_{Ps}\end{bmatrix}}\left.\begin{bmatrix}\Delta d_1\\ \Delta d_2\\ \vdots\\ \Delta d_s\end{bmatrix}\right\}S$$

which again could be written in more compact form as:

$$\Delta\underline{y} = C\Delta\underline{x} + D\Delta\underline{u} + F\Delta\underline{d}$$

This yields the following linear state-space model for, a process about a particular operating point:

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t) + E\underline{d}(t)$$ → DESCRIBES INTERNAL DYNAMICS

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) + F\underline{d}(t)$$ → MAPS STATES TO OUTPUTS

drop the Δ notation

1.2.1 Linearisation Examples:

i) The magnetic suspension system

$$m\frac{d^2h(t)}{dt^2} = mg - \frac{Ki^2(t)}{h(t)}$$

$$L\frac{di(t)}{dt} = v(t) - Ri(t)$$

Find a linear model about the desired operating airgap h=0.01m

The process parameters are:

L=10mH M=0.05 Kg g=10ms^{-2} R=1 ohm K=0.01Nm/A^2

First find the operating point, EQUILIBRIUM POINT

From the force equation:

$$\frac{\overset{0.01}{K}i_o^2}{\underset{0.01}{h_o}} = \overset{0.05}{m}g - 10$$

$$m\cancel{\frac{d^2h(t)}{dt^2}}^{\,0} = mg - \frac{Ki^2(t)}{h(t)} =$$

$i_o = 0.707$ A

From the electrical equation:

$$L\cancel{\frac{di(t)}{dt}}^{\,0} = v(t) - \overset{1\Omega}{R}\underset{0.707A}{i(t)} = 0.707\ V = V_o$$

The operating point vector will now be defined as:

$$\underline{x}_0 = \begin{bmatrix}h_0\\ \dot{h}_0\\ i_0\end{bmatrix} = \begin{bmatrix}0.01\\ 0\\ .707\end{bmatrix}$$ N.B AND $U_o = V_o = 0.707$

OPERATING POINT STATE VECTOR

Now linearise about this operating point vector:

The state equations can be written as:

$$\dot{x}_1 = x_2(t) = f_1(x_2)$$

$$\dot{x}_2 = g - \frac{Kx_3^2(t)}{mx_1(t)} = f_2(x_1, x_3)$$

$$\dot{x}_3 = \frac{1}{L}\big(u(t) - Rx_3(t)\big) = f_3(x_3, u)$$

Now define deviations of the states from their operating point values as: $\Delta\underline{x} = \begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{bmatrix} = \underline{x} - \underline{x}_o$ $\quad \Delta u = u - u_o$

Hence the linearised model is:

$$\frac{d}{dt}\Delta x_1(t) = \left.\frac{\partial f_1}{\partial x_2}\right|_{OP}\Delta x_2(t)$$

$$\frac{d}{dt}\Delta x_2(t) = \left.\frac{\partial f_2}{\partial x_1}\right|_{OP}\Delta x_1(t) + \left.\frac{\partial f_2}{\partial x_3}\right|_{OP}\Delta x_3(t)$$

$$\frac{d}{dt}\Delta x_3(t) = \left.\frac{\partial f_3}{\partial x_3}\right|_{OP}\Delta x_3(t) + \left.\frac{\partial f_3}{\partial u}\right|_{OP}\Delta u(t)$$

But we know that: OP ⇒ $x_1 = 0.01$ $\quad x_2 = 0$ $\quad x_3 = 0.707$ $\quad u = 0.707$

$$f_1(x_2) = x_2 \qquad \frac{\partial f_1}{\partial x_2} = 1 \text{ everywhere}$$

$$f_2(x_1, x_3) = g - \frac{Kx_3^2}{Mx_1} \qquad \frac{\partial f_2}{\partial x_1} = \frac{Kx_3^2}{Mx_1^2} \Rightarrow \left.\frac{\partial f_2}{\partial x_1}\right|_{op} = \frac{K(0.707)^2}{M(0.01)^2} = 999.69$$

$$f_3(x_3, u) = \frac{u}{L} - \frac{R}{L}x_3 \qquad \frac{\partial f_2}{\partial x_3} = \frac{-2Kx_3}{Mx_1} \Rightarrow \left.\frac{\partial f_2}{\partial x_3}\right|_{op} = -28.3$$

$$\frac{\partial f_3}{\partial u} = \frac{1}{L} = 100 \text{ everywhere!} \qquad \frac{\partial f_3}{\partial x_3} = \frac{-R}{L} = -100 \text{ everywhere}$$

This yields the following linearised model about the operating point:

$$\frac{d}{dt}\Delta x_1(t) = 1\Delta x_2(t)$$
$$\frac{d}{dt}\Delta x_2(t) = 999.69\Delta x_1(t) - 28.28\Delta x_3(t)$$
$$\frac{d}{dt}\Delta x_3(t) = -100\Delta x_3(t) + 100\Delta u(t)$$

And of course:

$$\Delta y(t) = h(t) - h_0 = \Delta x_1(t)$$

This then could be written in matrix form as:

$$\frac{d}{dt}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 999.69 & 0 & -28.28\\ 0 & 0 & -100\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 100\end{bmatrix}\Delta u(t)$$

$$\Delta y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\end{bmatrix}$$

ii) Permanent magnet DC motor

Could be modelled by the following coupled equations:

$$\frac{di}{dt} = \frac{1}{L}(v(t) - Ri(t) - K_m\omega(t))$$
$$\frac{d\omega}{dt} = \frac{1}{J}(K_M i(t) - B\omega(t) - T_L(t))$$

2 states $\to i, \omega$
1 input $\to v$
1 dist $\to T_L$

As you can see these are linear differential equations:

$$\frac{d}{dt}\begin{bmatrix} i\\ \omega\end{bmatrix} = \underbrace{\begin{bmatrix}-\frac{R}{L} & -\frac{K_m}{L}\\ \frac{K_m}{J} & -\frac{B}{J}\end{bmatrix}}_{2\times 2}\begin{bmatrix} i\\ \omega\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}}_{2\times 1} V + \underbrace{\begin{bmatrix}0\\ -\frac{1}{J}\end{bmatrix}}_{2\times 1} T_L$$

$$\dot{\underline{x}} = A\underline{x} + Bu + Ed$$

If a tachometer of gain K_T V/rads^{-1} is used to measure the speed, then the output equation could be written as:

$$V_T(t) = \underbrace{\begin{bmatrix}0 & K_T\end{bmatrix}}_{C}\begin{bmatrix} i\\ \omega\end{bmatrix}$$

Consider now that the motor is driving a nonlinear fan load:

$$T_L(t) = K_F\omega^2(t)$$

The process then would be modelled as:

$$\frac{di}{dt} = \frac{1}{L}(v(t) - Ri(t) - K_m\omega(t))$$

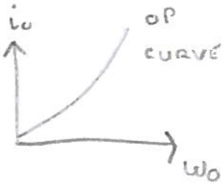
THIS IS NOW NON LINEAR $\to$
$$\frac{d\omega}{dt} = \frac{1}{J}(K_M i(t) - B\omega(t) - \underbrace{K_F\omega^2(t)}_{T_L})$$
NO LONGER AN INDEPENDENT INPUT

Generate a linear state-space model which describes the dynamics of this process close to the operating speed:

First find the operating point: i_o, ω_o, V_o

EQUILIB
$$\cancel{\frac{d\omega}{dt}} = \frac{1}{J}(K_M i(t) - B\omega(t) - K_F\omega^2(t)) = 0$$

$$K_m i_o = B\omega_o + K_f\omega_o^2$$
$$i_o = \frac{B\omega_o + K_f\omega_o^2}{K_m}$$



Now from the electrical equation:

$$\cancel{\frac{di}{dt}} = \frac{1}{L}(v(t) - Ri(t) - K_m\omega(t)) \Rightarrow V_o - Ri_o - K_m\omega_o = 0$$

$$V_o = \left(K_m + \frac{RB}{K_m}\right)\omega_o + \frac{RK_f}{K_m}\omega_o^2$$

Assign the states: $\underline{x}(t) = \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}i(t)\\ \omega(t)\end{bmatrix}$

and input and output: $\underbrace{\boxed{u(t) = v(t)}}_{\text{INPUT}}$ and $\underbrace{\boxed{y(t) = v_T(t) = K_T\omega(t)}}_{\text{OUTPUT}}$

hence:

LINEAR $\quad \dot{x}_1(t) = \frac{1}{L}(u(t) - Rx_1(t) - K_m x_2(t)) = f_1(x_1, x_2, u)$

NON LINEAR $\quad \dot{x}_2(t) = \frac{1}{J}(K_M x_1(t) - Bx_2(t) - K_F x_2^2(t)) = f_2(x_1, x_2)$

LINEAR $\quad y(t) = K_T x_2(t) = h(x_2)$

The linearised model about the operating point is:

$$\frac{d}{dt}\Delta x_1(t) = \left.\frac{\partial f_1}{\partial x_1}\right|_{OP}\Delta x_1(t) + \left.\frac{\partial f_1}{\partial x_2}\right|_{OP}\Delta x_2(t) + \left.\frac{\partial f_1}{\partial u}\right|_{OP}\Delta u(t)$$
$$\frac{d}{dt}\Delta x_2(t) = \left.\frac{\partial f_2}{\partial x_1}\right|_{OP}\Delta x_1(t) + \left.\frac{\partial f_2}{\partial x_2}\right|_{OP}\Delta x_2(t)$$
$$\Delta y(t) = \left.\frac{\partial h}{\partial x_2}\right|_{OP}\Delta x_2(t)$$

Obviously the electrical equation is linear:

$$\dot{x}_1(t) = \frac{1}{L}(u(t) - Rx_1(t) - K_m x_2(t)) = f_1(x_1, x_2, u)$$

$\frac{\partial f_1}{\partial x_1} = \frac{-R}{L}$ $\quad \frac{\partial f_1}{\partial x_2} = -\frac{K_m}{L}$ $\quad \frac{\partial f_1}{\partial u} = \frac{1}{L}$ everywhere!

Now concentrating on the mechanical equation:

$$\dot{x}_2(t) = \frac{1}{J}(K_M x_1(t) - Bx_2(t) - K_F x_2^2(t)) = f_2(x_1, x_2)$$

$$\frac{\partial f_2}{\partial x_1} = \frac{K_m}{J}$$

$$\frac{\partial f_2}{\partial x_2} = \frac{-B}{J} - \frac{2K_f x_2}{J} \Rightarrow \left.\frac{\partial f_2}{\partial x_2}\right|_{OP} = \frac{-B}{J} - \frac{2K_f\omega_o}{J}$$

And simply for the output equation:

$$y(t) = K_T x_2(t) = h(x_2) \qquad \frac{\partial h}{\partial x_2} = K_T$$

This yields the following linear state-space model:

$$\Delta\underline{x} = \begin{bmatrix} i(t) - i_o\\ \omega(t) - \omega_o\end{bmatrix}$$

$$\frac{d}{dt}\Delta\underline{x} = \begin{bmatrix}-\frac{R}{L} & -\frac{K_m}{L}\\ \frac{K_m}{J} & -\frac{B}{J} - \frac{2K_f\omega_o}{J}\end{bmatrix}\Delta\underline{x}(t) + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}\Delta u(t)$$

$$\Delta y(t) = \begin{bmatrix}0 & K_T\end{bmatrix}\Delta\underline{x}(t)$$

$$\Delta y = V_T(t) - K_T\omega_o$$ OPERATING POINT TACHO O/P

$$\Delta u = v(t) - V_o$$