EE4011: RFIC Design

Analysis of Amplifier Gain

Typical Steps in a Single-Transistor RF Amplifier Design

1. Select transistor based on s-parameters, noise figure, power level, process technology, etc
2. Check the stability – stability factor K, input/output stability circles
3. Check gain – gain circles
4. Check noise – noise circles
5. Design input and output matching networks (and DC biasing)
6. Iterate if necessary

There are many formulas used to describe gain in an RF/microwave amplifier – we'll derive a few of them in this section. Many design constraints can be represented by circles on the Smith Chart – we'll derive the formulas for some of these as well. This section on Amplifier Design is based on "traditional" microwave textbooks such as:

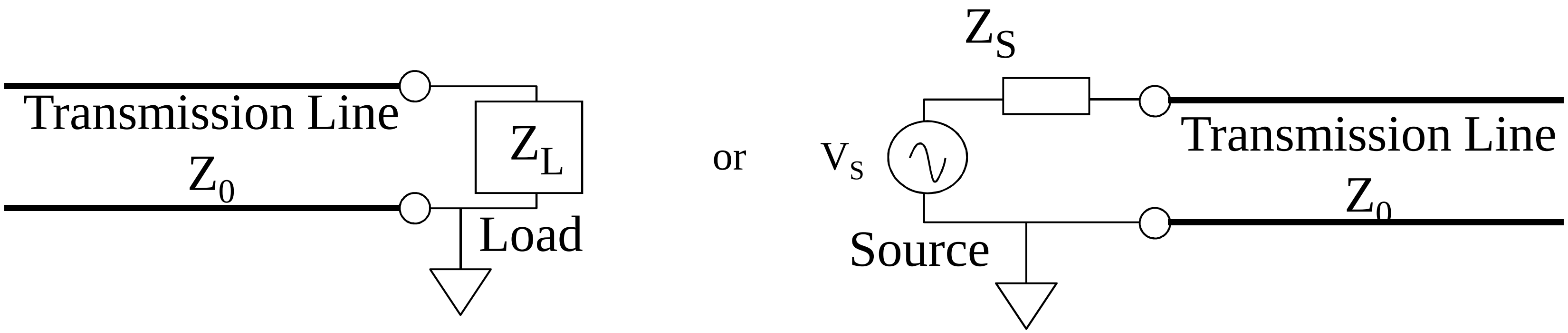
"High-frequency Circuit Design and Measurements", Peter C.L. Yip, Chapman and Hall, 1990

"RF Circuit Design", Chris Bowick, Newnes, 1st Ed. 2002, 2nd Ed. 2007

"Microwave Circuit Analysis and Amplifier Design", Samuel. Y. Liao, Prentice-Hall, 1987

"Microwave Engineering", David. M. Pozar, Wiley, 2nd Ed., 1998

Recall the reflection coefficient



Where a source (or load) with an impedance Z_S (Z_L) is connected to a transmission line with a characteristic impedance Z_0 the source (or load) reflection coefficient is defined as follows:

$$\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0} \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

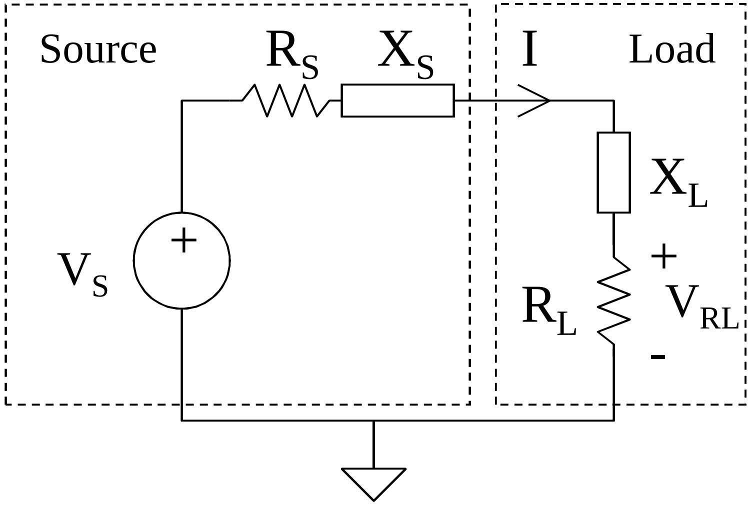
Usually when a source or load is connected to a transmission line, it is desirable to have no reflection i.e. to set the "effective" source or load impedances to be equal to the characteristic impedance of the transmission line.

Although, the reflection coefficient was defined using a lumped element connected to a transmission line, the concept of reflection coefficient is useful even if all lumped components are being used, especially when "matching" circuits to each other to ensure maximum power transfer.

Matching for Maximum Power Transfer

Here a source with both resistive and reactive elements is driving a load with resistive and reactive elements. The load power is the power dissipated in the load resistance – how is this maximised?

$$I = \frac{V_S}{(R_S + R_L) + j(X_S + X_L)}$$



$$V_{RL} = IR_L = \frac{V_S R_L}{(R_S + R_L) + j(X_S + X_L)}$$

$$P_L = \frac{|V_{RL}|^2}{R_L} = \frac{V_S^2 R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}$$

$$\frac{dP_L}{dX_L} = -\frac{V_S^2 R_L}{\left[(R_S + R_L)^2 + (X_S + X_L)^2\right]^2} 2(X_S + X_L)$$

$$\frac{dP_L}{dX_L} = 0 \Rightarrow X_L = -X_S \quad \text{for maximum } P_L$$

Note: for these power formulas V_S and V_{RL} are r.m.s. quantities

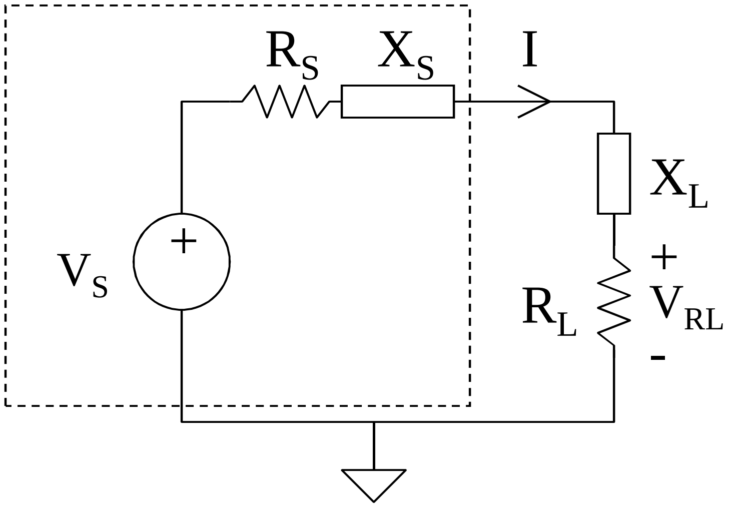
$$\text{when} \quad X_L = -X_S: \quad P_L = \frac{V_S^2 R_L}{(R_S + R_L)^2} \quad \text{and} \quad \frac{dP_L}{dR_L} = 0 \Rightarrow R_L = R_S \Rightarrow P_L = \frac{V_S^2}{4R_S}$$

for maximum P_L

Conjugate Matching

Power is dissipated in the resistive part of the load. To maximize this the load reactance should be the negative of the source reactance and the load resistance should be equal to the source resistance i.e. the load impedance should be equal to the complex conjugate of the source impedance.

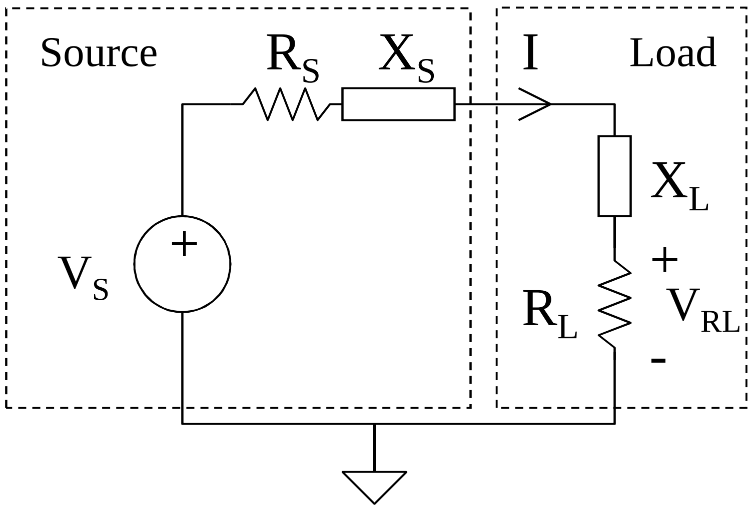
$$R_L = R_S \quad X_L = -X_S \quad Z_S = R_S + jX_S \quad Z_L = R_L + jX_L = R_S - jX_S = Z_S^*$$



This scenario is called conjugate matching. It is possible to achieve conjugate matching by making the load inductive if the source is capacitive or vice versa because capacitive and inductive reactances have opposite signs:

$$Z_L = j\varpi L \quad Z_C = \frac{1}{j\varpi C} = -\frac{j}{\varpi C}$$

Real and Imaginary Parts of Reflection Coefficients



Although there isn't a transmission line in this circuit, the reflection coefficients of both the source and the load can still be defined w.r.t. a characteristic impedance Z_0 as before:

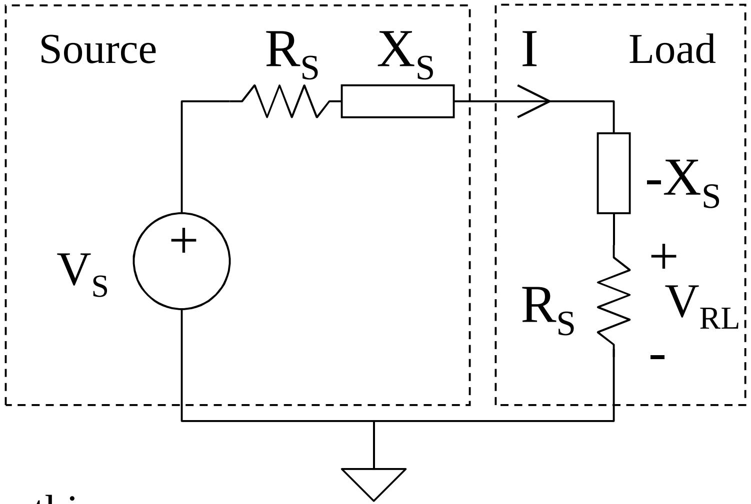
$$\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0} \qquad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Assuming Z_0 is real the reflection coefficients are:

$$\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0} = \frac{(R_S - Z_0) + jX_S}{(R_S + Z_0) + jX_S} = \frac{\left(R_S{}^2 - Z_0{}^2 + X_S{}^2\right)}{(R_S + Z_0)^2 + X_S{}^2} + j\frac{2X_S Z_0}{(R_S + Z_0)^2 + X_S{}^2}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L} = \frac{\left(R_L{}^2 - Z_0{}^2 + X_L{}^2\right)}{(R_L + Z_0)^2 + X_L{}^2} + j\frac{2X_L Z_0}{(R_L + Z_0)^2 + X_L{}^2}$$

Conjugate Matching in Terms of Reflection Coefficients



For maximum power transfer to the load, the load should be conjugately matched i.e.

$$Z_L = Z_S^* = R_S - jX_S$$

In this case:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(R_S - Z_0) - jX_S}{(R_S + Z_0) - jX_S} = \frac{\left(R_L^{\,2} - Z_0^{\,2} + X_S^{\,2}\right)}{(R_L + Z_0)^2 + X_S^{\,2}} - j\frac{2X_S Z_0}{(R_L + Z_0)^2 + X_S^{\,2}} = \Gamma_S^*$$

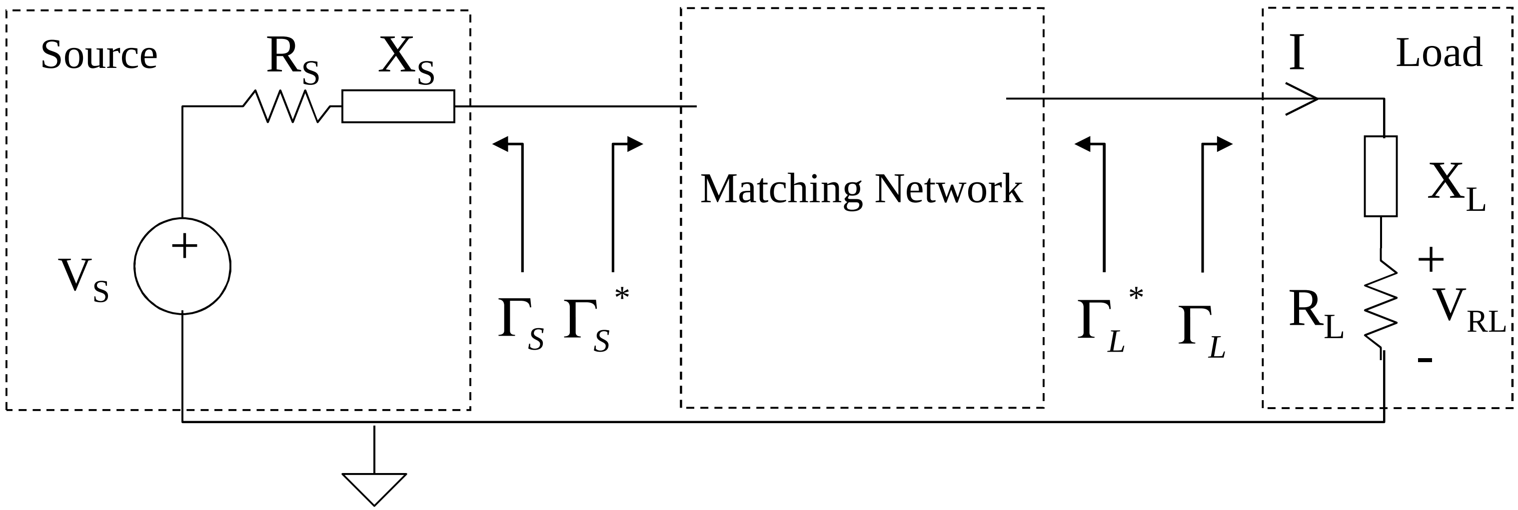
i.e. if matching for maximum power transfer, the load impedance should be the complex conjugate of the source impedance and this also implies that the load reflection coefficient will be the complex conjugate of the source reflection coefficient (assuming real Z_0)

$$Z_L = Z_S^* \Rightarrow \Gamma_L = \Gamma_S^*$$

Because the Smith Chart is "all about" reflection coefficients, we can use it to design matching networks even if we're dealing with lumped elements and not transmission lines.

Matching Networks

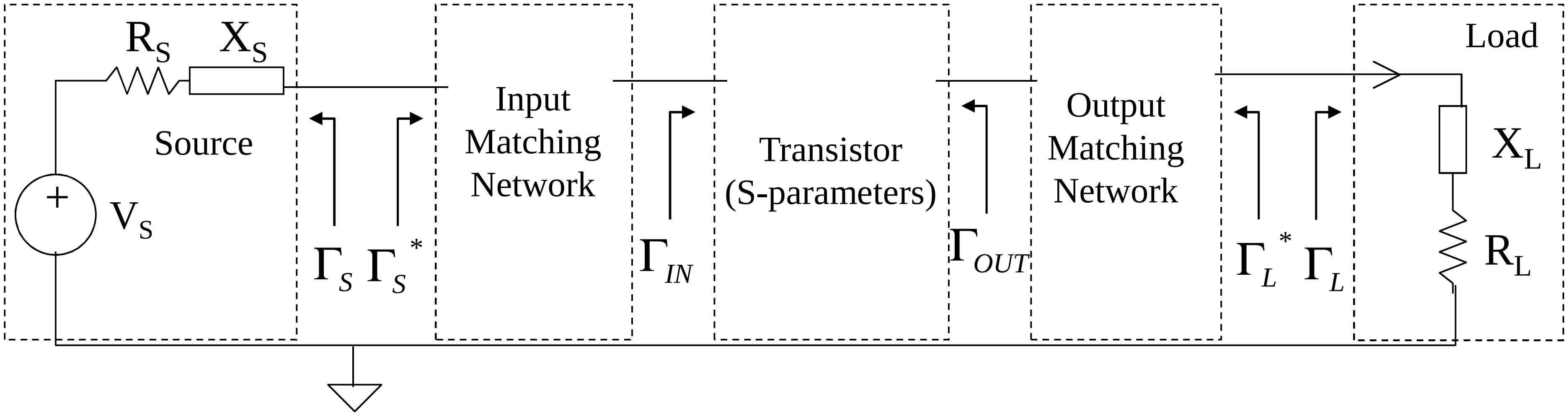
In practical situations, the source and load impedances are often fixed and cannot be changed. In this case a conjugate match is achieved by putting a matching network between the source and the load so that the effective impedance (or reflection coefficient) as “seen” by the source or load is the desired conjugate value:



The reflection coefficient looking into the source is Γ_S. For conjugate matching, the reflection coefficient looking into the matching network from the source side must be Γ_S^*.
The reflection coefficient looking into the load is Γ_L. For conjugate matching, the reflection coefficient looking into the matching network from the load must be Γ_L^*.
Looking from the source, the matching network “transforms” the load reflection coefficient to be equal to Γ_S^* or looking from the load, the matching network “transforms” the source reflection coefficient to be equal to Γ_L^*.

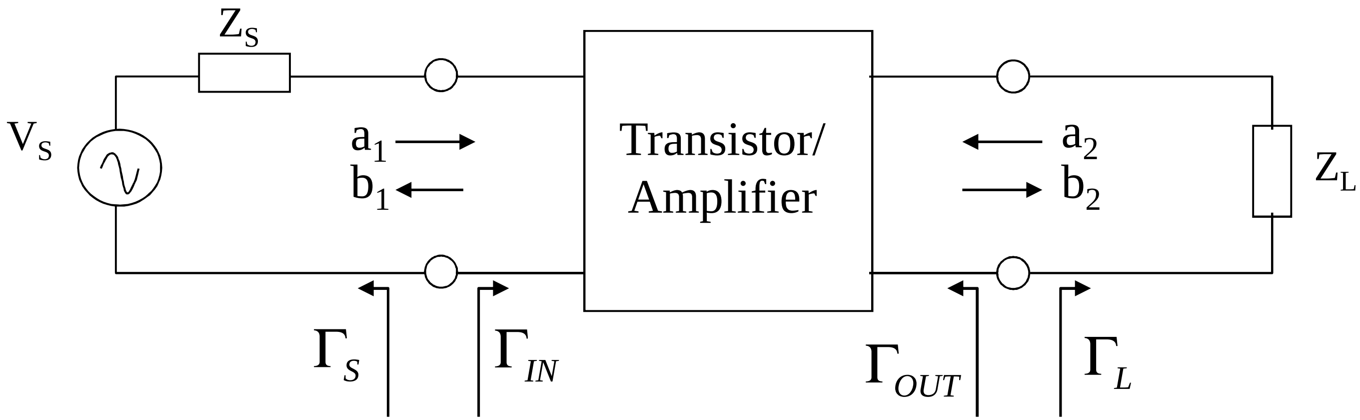
Matching a Transistor Amplifier

A transistor used as an RF amplifier will have a certain input impedance (or input reflection coefficient) and a certain output impedance (or output reflection coefficient) at the frequency of operation. Therefore, an input matching network will be needed to match the input of the transistor to the source and an output matching network will be needed to match the output of the transistor to the load.

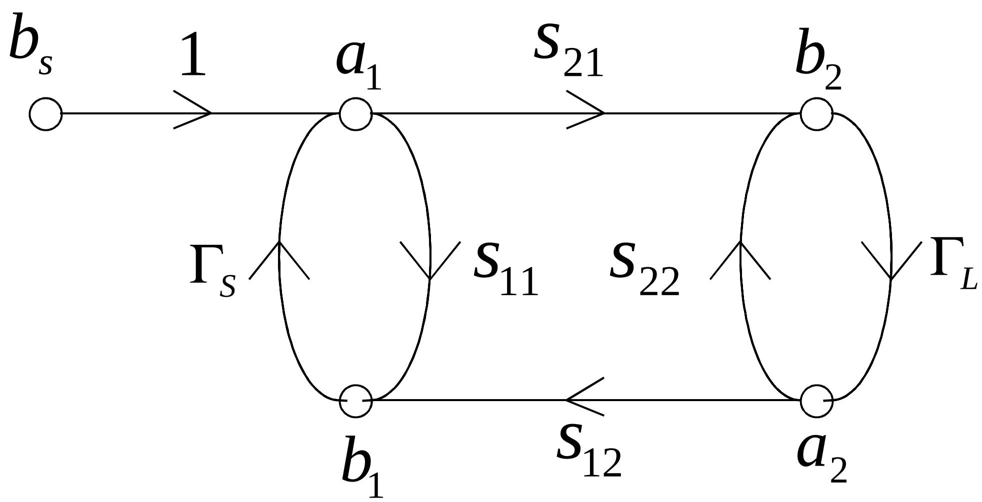


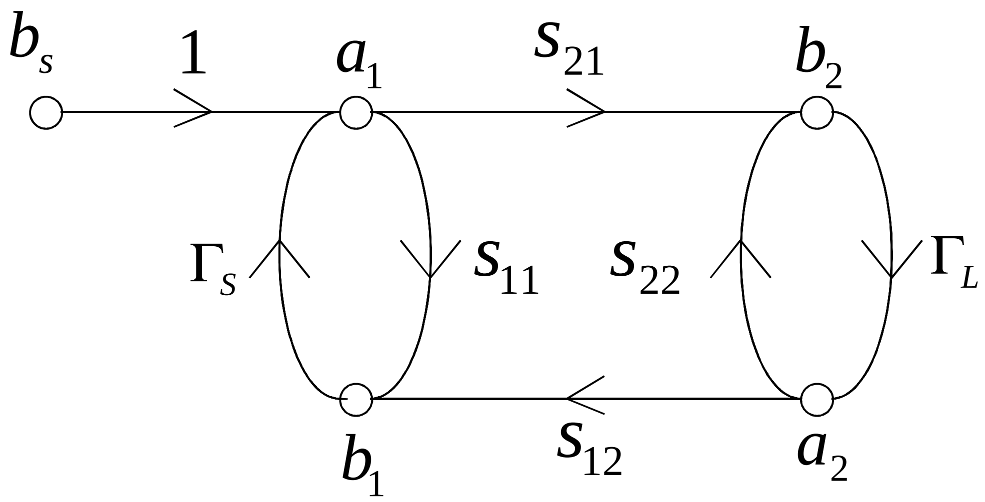
The matching networks should not dissipate any power and therefore are designed with reactive components only i.e. inductors and capacitors.

Formulas for Gain



A signal-flow graph of the circuit shows the relationships between the incident and reflected voltage components including the net flow of power into the system from the source. The nodes on the signal-flow graph represent the variables and the lines represent the relationships between the variables.





Working back from the load, the incident and reflected voltages can be expressed as functions of the wave travelling from the source b_s:

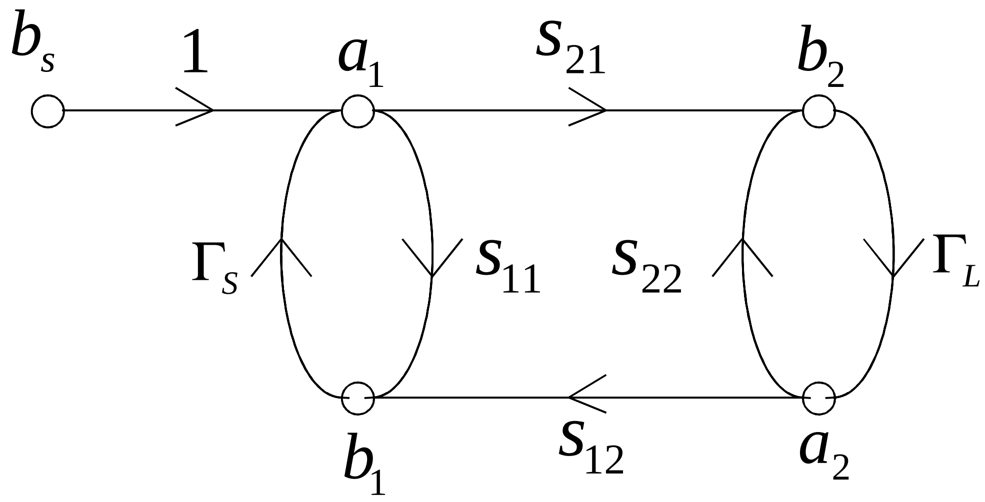
$$a_2 = \Gamma_L b_2$$

$$b_2 = s_{21}a_1 + s_{22}a_2 = s_{21}a_1 + s_{22}\Gamma_L b_2 \Rightarrow b_2 = \frac{s_{21}a_1}{1 - s_{22}\Gamma_L} \Rightarrow a_2 = \frac{s_{21}\Gamma_L a_1}{1 - s_{22}\Gamma_L}$$

$$b_1 = s_{11}a_1 + s_{12}a_2 = s_{11}a_1 + \frac{s_{12}s_{21}\Gamma_L a_1}{1 - s_{22}\Gamma_L} = \frac{(s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L)a_1}{1 - s_{22}\Gamma_L}$$

$$a_1 = b_s + \Gamma_S b_1 = b_s + \frac{(s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L)\Gamma_S a_1}{1 - s_{22}\Gamma_L}$$

$$\Rightarrow a_1 = \frac{(1 - s_{22}\Gamma_L)b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L}$$



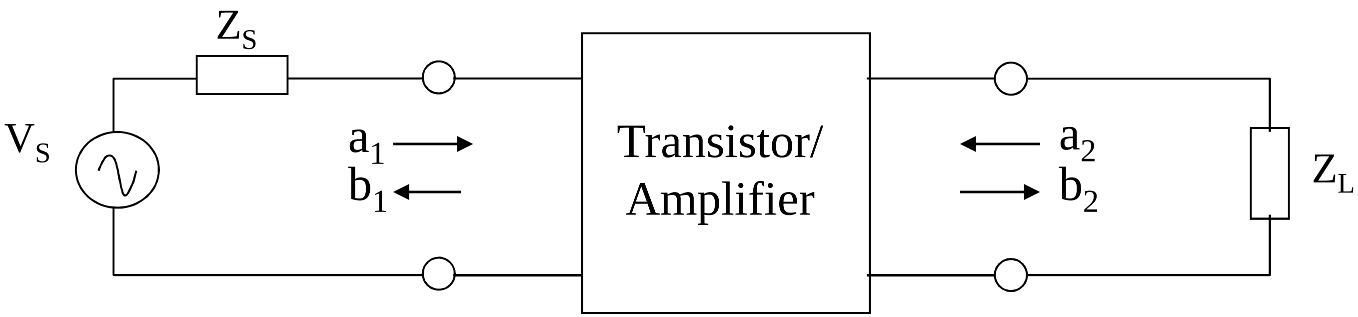
$$a_1 = \frac{(1 - s_{22}\Gamma_L)b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L}$$

$$b_1 = \frac{(s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L)a_1}{1 - s_{22}\Gamma_L} = \frac{(s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L)b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L}$$

$$a_2 = \frac{s_{21}\Gamma_L a_1}{1 - s_{22}\Gamma_L} = \frac{s_{21}\Gamma_L b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L}$$

$$b_2 = \frac{s_{21}a_1}{1 - s_{22}\Gamma_L} = \frac{s_{21}b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L}$$

Operating Power Gain (G_P)



The operating power gain (or also just called the power gain) is the ratio of the power delivered to the load to the power delivered by the source to the network:

$$G_P = \frac{P_{OUT}}{P_{IN}} = \frac{\frac{1}{2}|b_2|^2 - \frac{1}{2}|a_2|^2}{\frac{1}{2}|a_1|^2 - \frac{1}{2}|b_1|^2} = \frac{|b_2|^2 - |a_2|^2}{|a_1|^2 - |b_1|^2}$$

$$= \frac{|s_{21}|^2 - |s_{21}\Gamma_L|^2}{|1 - s_{22}\Gamma_L|^2 - |s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L|^2}$$

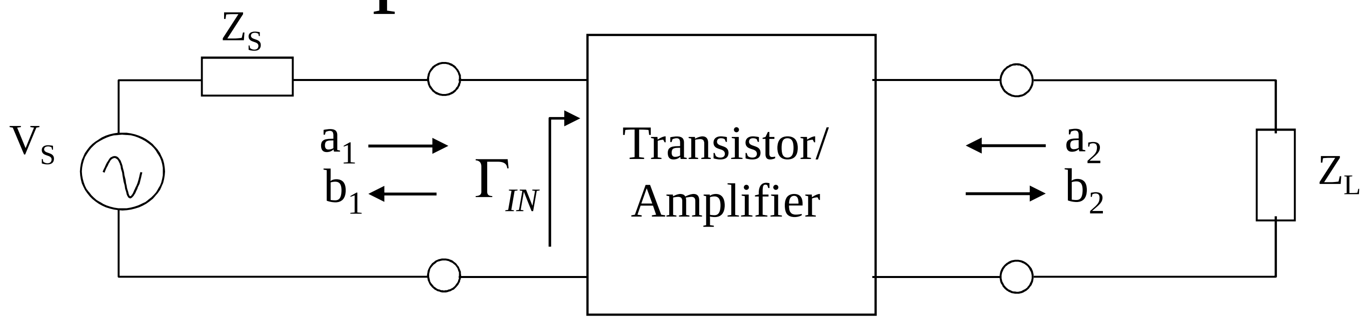
$$= \frac{|s_{21}|^2\left(1 - |\Gamma_L|^2\right)}{|1 - s_{22}\Gamma_L|^2 - |s_{11}(1 - s_{22}\Gamma_L) + s_{12}s_{21}\Gamma_L|^2}$$

$$= \frac{|s_{21}|^2\left(1 - |\Gamma_L|^2\right)}{|1 - s_{22}\Gamma_L|^2 - |s_{11} - \Delta\Gamma_L|^2} \quad where \quad \Delta = s_{11}s_{22} - s_{12}s_{21}$$

Note the formula for G_P does not depend on the source impedance (Z_S, Γ_S) but does depend on the load impedance (Z_L, Γ_L)

Note Δ is a complex number

The input reflection coefficient



The reflection coefficient at the input of the network can be found from

$$\Gamma_{IN} = \frac{b_1}{a_1} = \frac{s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L}{1 - s_{22}\Gamma_L} = \frac{s_{11}(1 - s_{22}\Gamma_L) + s_{12}s_{21}\Gamma_L}{1 - s_{22}\Gamma_L}$$

$$= s_{11} + \frac{s_{12}s_{21}\Gamma_L}{1 - s_{22}\Gamma_L}$$

So, unfortunately the reflection coefficient (and the impedance) seen looking into the input of a two-port network such as a transistor amplifier depends on the load reflection coefficient (load impedance). This complicates the design of the matching networks. A unilateral two-port is a two-port for which s_{12}=0. In this simple case:

$$\Gamma_{IN} = s_{11} \qquad \text{for a unilateral two-port}$$

In a unilateral transistor, the input reflection coefficient does not depend on the load and only depends on the transistor parameter s_{11}. In this case the design of the matching networks in easier.

Putting Γ_{IN} into the formula for G_P

The input reflection coefficient can be used as a convenient "shorthand" for a bunch of terms:

$$\Gamma_{IN} = s_{11} + \frac{s_{12}s_{21}\Gamma_L}{1-s_{22}\Gamma_L} = \frac{s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L}{1-s_{22}\Gamma_L} \Rightarrow s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L = \Gamma_{IN}\left(1-s_{22}\Gamma_L\right)$$

This can be used to tidy up the formula for G_P:

$$G_P = \frac{|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{|1-s_{22}\Gamma_L|^2 - |s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L|^2}$$

$$= \frac{|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{|1-s_{22}\Gamma_L|^2 - |(1-s_{22}\Gamma_L)\Gamma_{IN}|^2} = \frac{|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{|1-s_{22}\Gamma_L|^2\left(1-|\Gamma_{IN}|^2\right)}$$

$$\frac{1}{1-|\Gamma_{IN}|^2}|s_{21}|^2\frac{1-|\Gamma_L|^2}{|1-s_{22}\Gamma_L|^2}$$

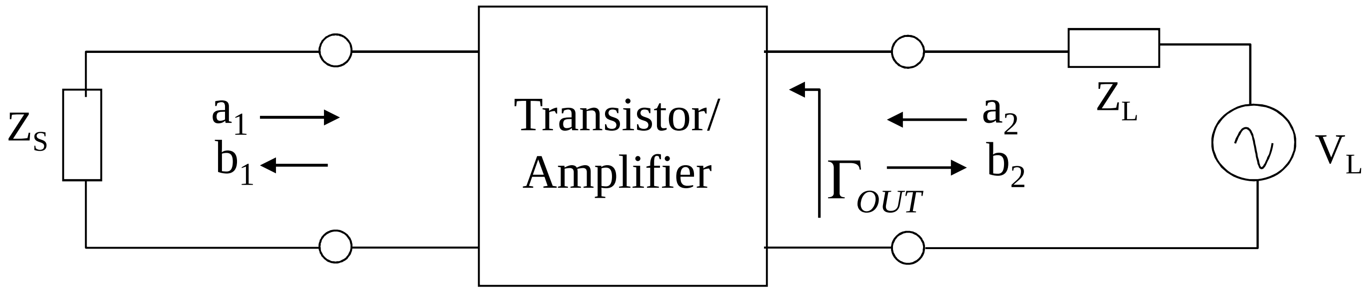
This shows that the power gain of the circuit depends on three elements:

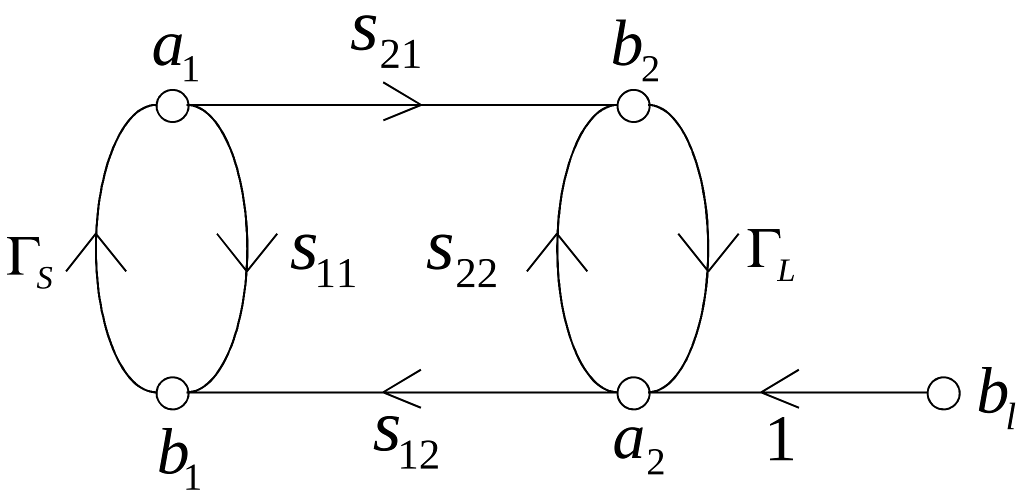
1. The gain factor of the transistor, s_{21}
2. The input reflection coefficient (which depends on s_{11} as well as s_{12} and Γ_L)
3. The load side of the network i.e. s_{22} and the load reflection coefficient Γ_L

To get maximum gain, it is necessary to maximise all 3 terms simultaneously.

The output reflection coefficient

A formula for the output reflection coefficient is obtained by considering the load as the signal source and the source as the load, with the same transistor s-parameters:





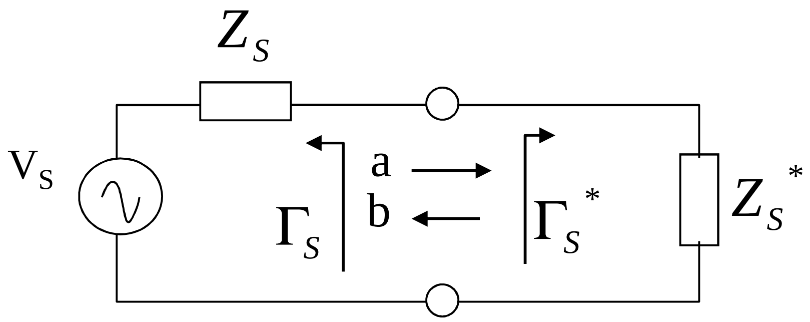
Working through the signal flow diagram gives:

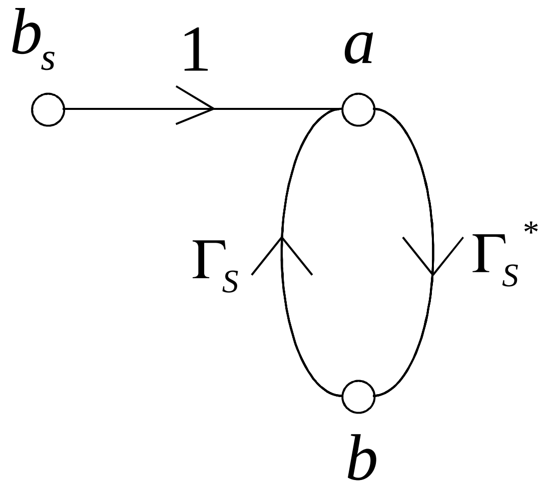
$$\Gamma_{OUT} = \frac{b_2}{a_2} = s_{22} + \frac{s_{12}s_{21}\Gamma_S}{1 - s_{11}\Gamma_S}$$

This time the reflection coefficient looking from the load into the output of the two-port depends on the source reflection coefficient unless the two-port is unilateral. For a unilateral two-port the output reflection coefficient is just s_{22}.

Power Available from the Source, P_{AVS}

The maximum power that can be delivered by a source into a load is achieved when the load impedance is the complex conjugate of the source impedance. This power is called the power available from the source, P_{AVS}. Looking at a source and load in this situation:





$$b = \Gamma_S^* a$$

$$a = b_s + \Gamma_S b = b_s + \Gamma_S \Gamma_S^* a = b_s + |\Gamma_S|^2 a$$

$$\Rightarrow a = \frac{b_s}{1 - |\Gamma_S|^2} \quad b = \frac{\Gamma_S^* b_s}{1 - |\Gamma_S|^2}$$

$$P_{AVS} = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 = \frac{1}{2}\frac{|b_s|^2}{\left(1 - |\Gamma_S|^2\right)^2} - \frac{1}{2}\frac{|\Gamma_S^*|^2 |b_s|^2}{\left(1 - |\Gamma_S|^2\right)^2} = \frac{1}{2}\frac{|b_s|^2}{1 - |\Gamma_S|^2}$$

Transducer Power Gain, G_T

The transducer power gain is the ratio of the power delivered to the load to the power available from the source i.e.

$$G_T = \frac{P_{OUT}}{P_{AVS}} = \frac{\frac{1}{2}|b_2|^2 - \frac{1}{2}|a_2|^2}{\frac{1}{2}|a|^2 - \frac{1}{2}|b|^2} = \frac{|b_2|^2 - |a_2|^2}{|a|^2 - |b|^2}$$

just derived:

$$a = \frac{b_s}{1 - |\Gamma_S|^2} \qquad b = \frac{\Gamma_S^* b_s}{1 - |\Gamma_S|^2}$$

Remember previous formulas for a_2 and b_2:

$$a_2 = \frac{s_{21}\Gamma_L b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L} = \frac{s_{21}\Gamma_L b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L}$$

$$b_2 = \frac{s_{21}b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L} = \frac{s_{21}b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L}$$

$$G_T = \frac{\dfrac{|s_{21}b_s|^2}{|1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L|^2} - \dfrac{|s_{21}\Gamma_L b_s|^2}{|1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L|^2}}{\dfrac{|b_s|^2}{\left(1 - |\Gamma_S|^2\right)^2} - \dfrac{|\Gamma_S^* b_s|^2}{\left(1 - |\Gamma_S|^2\right)^2}}$$

Some gymnastics:

$$G_T = \frac{\dfrac{|s_{21}b_s|^2}{|1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L|^2} - \dfrac{|s_{21}\Gamma_L b_s|^2}{|1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L|^2}}{\dfrac{|b_s|^2}{\left(1 - |\Gamma_S|^2\right)^2} - \dfrac{|\Gamma_S^{*} b_s|^2}{\left(1 - |\Gamma_S|^2\right)^2}}$$

$$= \frac{\dfrac{|s_{21}|^2|b_s|^2}{|1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L|^2} - \dfrac{|s_{21}|^2|\Gamma_L|^2|b_s|^2}{|1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L|^2}}{\dfrac{|b_s|^2}{\left(1 - |\Gamma_S|^2\right)^2} - \dfrac{|\Gamma_S|^2|b_s|^2}{\left(1 - |\Gamma_S|^2\right)^2}}$$

$$= \frac{|s_{21}|^2\left(1 - |\Gamma_L|^2\right)\left|1 - |\Gamma_S|^2\right|^2}{|1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L|^2\left(1 - |\Gamma_S|^2\right)} = \frac{\left(1 - |\Gamma_S|^2\right)|s_{21}|^2\left(1 - |\Gamma_L|^2\right)}{|1 - s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L|^2}$$

Note: $|\Gamma_S^{*}| = |\Gamma_S|$

Rearranging the formula for G_T

$$G_T = \frac{\left(1-|\Gamma_S|^2\right)|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{\left|1-s_{11}\Gamma_S - s_{22}\Gamma_L + \Delta\Gamma_S\Gamma_L\right|^2} = \frac{\left(1-|\Gamma_S|^2\right)|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{\left|1-s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L\right|^2}$$

Looking at the denominator of the second expression:

$$1-s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L = 1-\Gamma_S\left(s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L\right) - s_{22}\Gamma_L$$

Recall the “shortcut” with Γ_{IN}:

$$s_{11} - s_{11}s_{22}\Gamma_L + s_{12}s_{21}\Gamma_L = \Gamma_{IN}\left(1-s_{22}\Gamma_L\right)$$

Putting this into the denominator of the second term gives:

$$1-\Gamma_S\Gamma_{IN}\left(1-s_{22}\Gamma_L\right) - s_{22}\Gamma_L = 1-s_{22}\Gamma_L - \Gamma_S\Gamma_{IN}\left(1-s_{22}\Gamma_L\right) = \left(1-s_{22}\Gamma_L\right)\left(1-\Gamma_S\Gamma_{IN}\right)$$

Then the expression for G_T becomes

$$G_T = \frac{\left(1-|\Gamma_S|^2\right)|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{\left|1-s_{22}\Gamma_L\right|^2\left|1-\Gamma_S\Gamma_{IN}\right|^2} = \frac{1-|\Gamma_S|^2}{\left|1-\Gamma_S\Gamma_{IN}\right|^2}|s_{21}|^2\frac{1-|\Gamma_L|^2}{\left|1-s_{22}\Gamma_L\right|^2}$$

Using Γ_{OUT} in the Formula for G_T

$$G_T = \frac{\left(1-|\Gamma_S|^2\right)|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{\left|1-s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L\right|^2}$$

Looking at the denominator:

$$1-s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L = 1-\Gamma_L\left(s_{22} - s_{11}s_{22}\Gamma_S + s_{12}s_{21}\Gamma_S\right) - s_{11}\Gamma_S$$

The formula for Γ_{OUT} can be used to simplify the last expression:

$$\Gamma_{OUT} = s_{22} + \frac{s_{12}s_{21}\Gamma_S}{1-s_{11}\Gamma_S} = \frac{s_{22} - s_{22}s_{11}\Gamma_S + s_{12}s_{21}\Gamma_S}{1-s_{11}\Gamma_S} \Rightarrow s_{22} - s_{11}s_{22}\Gamma_S + s_{12}s_{21}\Gamma_S = \Gamma_{OUT}\left(1-s_{11}\Gamma_S\right)$$

The denominator then becomes:

$$1-\Gamma_L\Gamma_{OUT}\left(1-s_{11}\Gamma_S\right) - s_{11}\Gamma_S = 1-s_{11}\Gamma_S - \Gamma_L\Gamma_{OUT}\left(1-s_{11}\Gamma_S\right) = \left(1-s_{11}\Gamma_S\right)\left(1-\Gamma_L\Gamma_{OUT}\right)$$

Giving an expression for G_T:

$$G_T = \frac{\left(1-|\Gamma_S|^2\right)|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{|1-s_{11}\Gamma_S|^2|1-\Gamma_L\Gamma_{OUT}|^2} = \frac{1-|\Gamma_S|^2}{|1-s_{11}\Gamma_S|^2}|s_{21}|^2\frac{1-|\Gamma_L|^2}{|1-\Gamma_L\Gamma_{OUT}|^2}$$

Transducer Power Gain, G_T: One gain, 3 common ways of writing the formula

$$G_T = \frac{\left(1-\left|\Gamma_S\right|^2\right)\left|s_{21}\right|^2\left(1-\left|\Gamma_L\right|^2\right)}{\left|1-s_{11}\Gamma_S-s_{22}\Gamma_L+\Delta\Gamma_S\Gamma_L\right|^2}$$

The formula with the source and load reflection coefficients and all the transistor s-parameters

$$G_T = \frac{1-\left|\Gamma_S\right|^2}{\left|1-\Gamma_S\Gamma_{IN}\right|^2}\left|s_{21}\right|^2\frac{1-\left|\Gamma_L\right|^2}{\left|1-s_{22}\Gamma_L\right|^2}$$

This formula emphasises the importance of the input reflection coefficient of the two-port

$$G_T = \frac{1-\left|\Gamma_S\right|^2}{\left|1-s_{11}\Gamma_S\right|^2}\left|s_{21}\right|^2\frac{1-\left|\Gamma_L\right|^2}{\left|1-\Gamma_L\Gamma_{OUT}\right|^2}$$

This formula emphasises the importance of the output reflection coefficient of the two-port

$$\Delta = s_{11}s_{22}-s_{12}s_{21} \qquad \Gamma_{IN} = s_{11}+\frac{s_{12}s_{21}\Gamma_L}{1-s_{22}\Gamma_L} \qquad \Gamma_{OUT} = s_{22}+\frac{s_{12}s_{21}\Gamma_S}{1-s_{11}\Gamma_S}$$

Unilateral Transducer Power Gain, G_{TU}

If the two-port is unilateral i.e. s_{12}=0 then the input reflection coefficient is equal to s_{11} and the output reflection coefficient is equal to s_{22} i.e. the load has no influence on the input reflection coefficient and the source has no influence on the output reflection coefficient.

$$\Gamma_{IN} = s_{11} + \frac{s_{12}s_{21}\Gamma_L}{1-s_{22}\Gamma_L} = s_{11} \quad if \quad s_{12} = 0 \qquad \Gamma_{OUT} = s_{22} + \frac{s_{12}s_{21}\Gamma_S}{1-s_{11}\Gamma_S} = s_{22} \quad if \quad s_{12} = 0$$

The transducer power gain in this case is called the unilateral transducer power gain:

$$G_{TU} = \frac{1-\left|\Gamma_S\right|^2}{\left|1-\Gamma_S\Gamma_{IN}\right|^2}\left|s_{21}\right|^2 \frac{1-\left|\Gamma_L\right|^2}{\left|1-s_{22}\Gamma_L\right|^2} = \frac{1-\left|\Gamma_S\right|^2}{\left|1-s_{11}\Gamma_S\right|^2}\left|s_{21}\right|^2 \frac{1-\left|\Gamma_L\right|^2}{\left|1-s_{22}\Gamma_L\right|^2}$$

i.e.

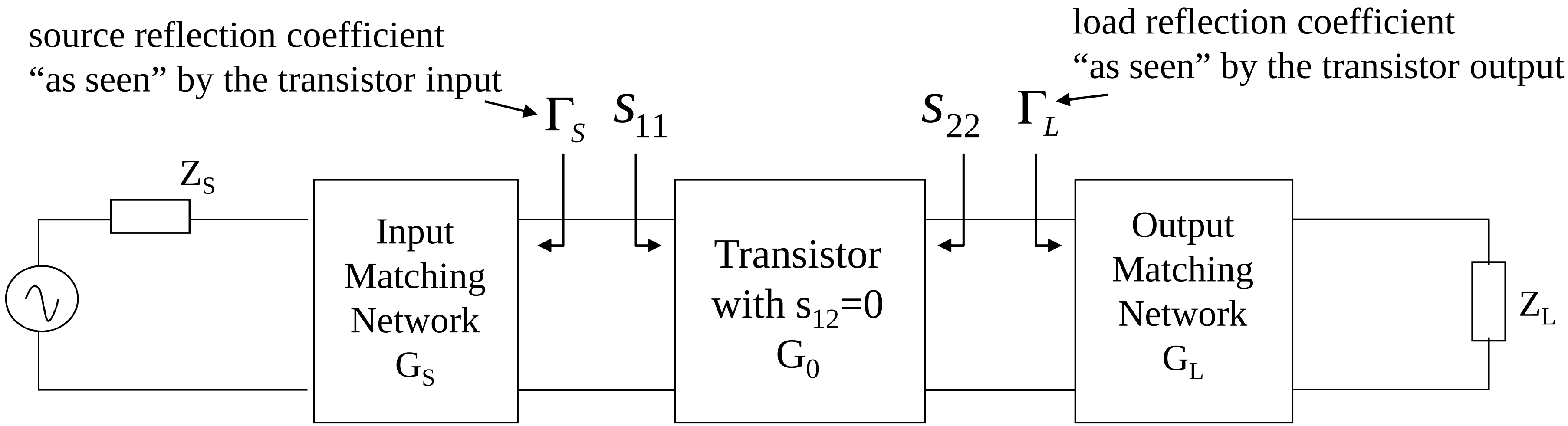
$$G_{TU} = G_S G_O G_L \quad G_S = \frac{1-\left|\Gamma_S\right|^2}{\left|1-s_{11}\Gamma_S\right|^2} \quad G_O = \left|s_{21}\right|^2 \quad G_L = \frac{1-\left|\Gamma_L\right|^2}{\left|1-s_{22}\Gamma_L\right|^2}$$

or

$$\left(G_{TU}\right)_{dB} = \left(G_S\right)_{dB} + \left(G_O\right)_{dB} + \left(G_L\right)_{dB}$$

G_{TU} is determined by 3 quantities: G_S which depends on the input side only, G_0 which depends on the transistor gain factor s_{21}, and G_L which depends on the output side only.

Maximum Unilateral Transducer Power Gain, $G_{TU,max}$



Maximum unilateral transducer gain can be achieved by making a conjugate match between the source and the transistor input and between the load and the transistor output i.e.

$$\Gamma_S = s_{11}^* \quad \Gamma_L = s_{22}^*$$

$$\Rightarrow G_{TU} = G_{TU,\max} = \frac{1-|\Gamma_S|^2}{|1-s_{11}\Gamma_S|^2}|s_{21}|^2\frac{1-|\Gamma_L|^2}{|1-s_{22}\Gamma_L|^2} = \frac{1-|s_{11}^*|^2}{|1-s_{11}s_{11}^*|^2}|s_{21}|^2\frac{1-|s_{22}^*|^2}{|1-s_{22}s_{22}^*|^2}$$

$$G_{TU,\max} = \frac{1-|s_{11}|^2}{\left|1-|s_{11}|^2\right|^2}|s_{21}|^2\frac{1-|s_{22}|^2}{\left|1-|s_{22}|^2\right|^2} = \frac{1}{1-|s_{11}|^2}|s_{21}|^2\frac{1}{1-|s_{22}|^2}$$

Note:
$$s_{11}s_{11}^* = |s_{11}|^2$$
$$s_{22}s_{22}^* = |s_{22}|^2$$

Unilateral Figure of Merit - 1

If the transistor is unilateral the transducer gain is G_{TU} as seen before:

$$G_{TU} = \frac{1-|\Gamma_S|^2}{|1-s_{11}\Gamma_S|^2}|s_{21}|^2\frac{1-|\Gamma_L|^2}{|1-s_{22}\Gamma_L|^2}$$

However, if the transistor isn't unilateral the transducer gain is the more general formula seen before:

$$G_T = \frac{\left(1-|\Gamma_S|^2\right)|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{|1-s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L|^2} = \frac{\left(1-|\Gamma_S|^2\right)|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{|(1-s_{11}\Gamma_S)(1-s_{22}\Gamma_L)-s_{12}s_{21}\Gamma_S\Gamma_L|^2}$$

The difference between the unilateral and non-unilateral gains can be expressed as a ratio which gives the fractional error in predicting gain when a unilateral approximation is used for a non-unilateral device:

$$\frac{G_T}{G_{TU}} = \frac{|1-s_{11}\Gamma_S|^2|1-s_{22}\Gamma_L|^2}{|(1-s_{11}\Gamma_S)(1-s_{22}\Gamma_L)-s_{12}s_{21}\Gamma_S\Gamma_L|^2} = \frac{1}{|1-X|^2} \qquad X = \frac{s_{12}s_{21}\Gamma_S\Gamma_L}{(1-s_{11}\Gamma_S)(1-s_{22}\Gamma_L)}$$

X is a complex number and the fractional gain error will thus be bounded by the following inequality:

$$\frac{1}{(1+|X|)^2} < \frac{G_T}{G_{TU}} < \frac{1}{(1-|X|)^2}$$

Unilateral Figure of Merit - 2

If the amplifier is designed assuming it is unilateral then to give maximum gain the source and load reflection coefficients will be set to:

$$\Gamma_S = s_{11}^* \qquad \Gamma_L = s_{22}^*$$

In this case

$$X = \frac{s_{12}s_{21}\Gamma_S\Gamma_L}{(1 - s_{11}\Gamma_S)(1 - s_{22}\Gamma_L)} = \frac{s_{12}s_{21}s_{11}^*s_{22}^*}{(1 - s_{11}s_{11}^*)(1 - s_{22}s_{22}^*)} = \frac{s_{12}s_{21}s_{11}^*s_{22}^*}{\left(1 - |s_{11}|^2\right)\left(1 - |s_{22}|^2\right)}$$

The **unilateral figure of merit** is defined as the magnitude of X in this case i.e.

$$M = |X| = \left| \frac{s_{12}s_{21}s_{11}^*s_{22}^*}{\left(1 - |s_{11}|^2\right)\left(1 - |s_{22}|^2\right)} \right| = \frac{|s_{11}||s_{12}||s_{21}||s_{22}|}{\left(1 - |s_{11}|^2\right)\left(1 - |s_{22}|^2\right)}$$

$$s_{11}s_{11}^* = |s_{11}|^2$$

$$s_{22}s_{22}^* = |s_{22}|^2$$

$$|s_{11}^*| = |s_{11}|$$

$$|s_{22}^*| = |s_{22}|$$

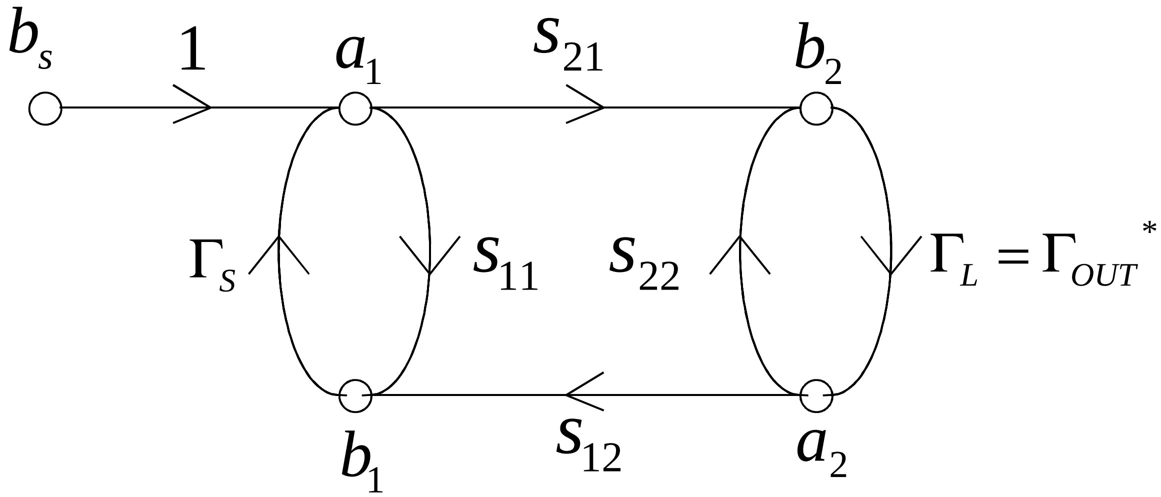
and the error is bounded by

$$\frac{1}{(1+M)^2} < \frac{G_T'}{G_{TU,\max}} < \frac{1}{(1-M)^2}$$

where G_T' is G_T calculated assuming $\Gamma_S = s_{11}^*$ and $\Gamma_L = s_{22}^*$

If these error bounds are within a few tenths of a dB then the unilateral approximation is considered acceptable

Power Available from the Network, P_{AVN}



The output power available from the network is the maximum power which can be delivered by the network into the load i.e. the output power when the load is conjugately matched to the output, $\Gamma_L=\Gamma_{OUT}^*$

From the initial analysis we had:

$$a_2 = \frac{s_{21}\Gamma_L b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L} \quad b_2 = \frac{s_{21} b_s}{1 - s_{11}\Gamma_S - s_{22}\Gamma_L + s_{11}s_{22}\Gamma_S\Gamma_L - s_{12}s_{21}\Gamma_S\Gamma_L}$$

The denominator terms were previously simplified by using Γ_{OUT} i.e.

$$a_2 = \frac{s_{21}\Gamma_L b_s}{(1 - s_{11}\Gamma_S)(1 - \Gamma_L\Gamma_{OUT})} \quad b_2 = \frac{s_{21} b_s}{(1 - s_{11}\Gamma_S)(1 - \Gamma_L\Gamma_{OUT})}$$

Available Power Gain, G_A (1)

$$a_2 = \frac{s_{21}\Gamma_L b_s}{(1 - s_{11}\Gamma_S)(1 - \Gamma_L \Gamma_{OUT})} \quad b_2 = \frac{s_{21} b_s}{(1 - s_{11}\Gamma_S)(1 - \Gamma_L \Gamma_{OUT})}$$

Now if we set $\Gamma_L = \Gamma_{OUT}{}^*$ we get:

$$a_2' = \frac{s_{21}\Gamma_L b_s}{(1 - s_{11}\Gamma_S)(1 - \Gamma_L \Gamma_{OUT})} = \frac{s_{21}\Gamma_{OUT} b_s}{(1 - s_{11}\Gamma_S)\left(1 - \Gamma_{OUT}{}^* \Gamma_{OUT}\right)} = \frac{s_{21}\Gamma_{OUT}{}^* b_s}{(1 - s_{11}\Gamma_S)\left(1 - |\Gamma_{OUT}|^2\right)}$$

$$b_2' = \frac{s_{21} b_s}{(1 - s_{11}\Gamma_S)(1 - \Gamma_L \Gamma_{OUT})} = \frac{s_{21} b_s}{(1 - s_{11}\Gamma_S)\left(1 - \Gamma_{OUT}{}^* \Gamma_{OUT}\right)} = \frac{s_{21} b_s}{(1 - s_{11}\Gamma_S)\left(1 - |\Gamma_{OUT}|^2\right)}$$

The available output power from the network is then $P_{AVN} = \frac{1}{2}|b_2'|^2 - \frac{1}{2}|a_2'|^2$

The **available power gain**, G_A, is defined as ratio of the power available from the network to the power available from the source i.e.

$$G_A = \frac{P_{AVN}}{P_{AVS}} = \frac{\frac{1}{2}|b_2'|^2 - \frac{1}{2}|a_2'|^2}{\frac{1}{2}|a|^2 - \frac{1}{2}|b|^2} = \frac{|b_2'|^2 - |a_2'|^2}{|a|^2 - |b|^2}$$

Where a and b are the incident and reflected voltages determined earlier to represent the power available from the source.

Available Power Gain, G_A (2)

$$a_2' = \frac{s_{21}\Gamma_{OUT}^{*} b_s}{(1 - s_{11}\Gamma_S)\left(1 - |\Gamma_{OUT}|^2\right)} \qquad b_2' = \frac{s_{21} b_s}{(1 - s_{11}\Gamma_S)\left(1 - |\Gamma_{OUT}|^2\right)}$$

The formulas derived earlier for a and b were: $a = \dfrac{b_s}{1 - |\Gamma_S|^2} \qquad b = \dfrac{\Gamma_S^{*} b_s}{1 - |\Gamma_S|^2}$

$$G_A = \frac{|b_2'|^2 - |a_2'|^2}{|a|^2 - |b|^2} = \frac{\left|\dfrac{s_{21} b_s}{(1 - s_{11}\Gamma_S)\left(1 - |\Gamma_{OUT}|^2\right)}\right|^2 - \left|\dfrac{s_{21}\Gamma_{OUT}^{*} b_s}{(1 - s_{11}\Gamma_S)\left(1 - |\Gamma_{OUT}|^2\right)}\right|^2}{\left|\dfrac{b_s}{1 - |\Gamma_S|^2}\right|^2 - \left|\dfrac{\Gamma_S^{*} b_s}{1 - |\Gamma_S|^2}\right|^2}$$

$$= \frac{|s_{21}|^2\left(1 - |\Gamma_{OUT}|^2\right)\left(1 - |\Gamma_S|^2\right)^2}{|1 - s_{11}\Gamma_S|^2\left(1 - |\Gamma_{OUT}|^2\right)^2\left(1 - |\Gamma_S|^2\right)}$$

$$= \frac{1 - |\Gamma_S|^2}{|1 - s_{11}\Gamma_S|^2}|s_{21}|^2 \frac{1}{1 - |\Gamma_{OUT}|^2}$$

The available power gain, G_A, depends on the source impedance (Z_S,Γ_S) but not on the load impedance (Z_L,Γ_L).

The main gain formulas again

Operating Power Gain: $$G_P = \frac{|s_{21}|^2\left(1-|\Gamma_L|^2\right)}{|1-s_{22}\Gamma_L|^2 - |s_{11}-\Delta\Gamma_L|^2} \quad where \quad \Delta = s_{11}s_{22} - s_{12}s_{21}$$

Transducer Power Gain: $$G_T = \frac{1-|\Gamma_S|^2}{|1-\Gamma_S\Gamma_{IN}|^2}|s_{21}|^2\frac{1-|\Gamma_L|^2}{|1-s_{22}\Gamma_L|^2} \quad where \quad \Gamma_{IN} = s_{11} + \frac{s_{12}s_{21}\Gamma_L}{1-s_{22}\Gamma_L}$$

Unilateral Transducer Power Gain: $$G_{TU} = \frac{1-|\Gamma_S|^2}{|1-s_{11}\Gamma_S|^2}|s_{21}|^2\frac{1-|\Gamma_L|^2}{|1-s_{22}\Gamma_L|^2}$$

Maximum Unilateral Transducer Power Gain: $$G_{TU,\max} = \frac{1}{1-|s_{11}|^2}|s_{21}|^2\frac{1}{1-|s_{22}|^2}$$

Unilateral Figure of Merit $$M = \frac{|s_{11}||s_{12}||s_{21}||s_{22}|}{\left(1-|s_{11}|^2\right)\left(1-|s_{22}|^2\right)} \qquad \frac{1}{(1+M)^2} < \frac{G_T{}'}{G_{TU,\max}} < \frac{1}{(1-M)^2}$$

Available Power Gain: $$G_A = \frac{1-|\Gamma_S|^2}{|1-s_{11}\Gamma_S|^2}|s_{21}|^2\frac{1}{1-|\Gamma_{OUT}|^2} \quad where \quad \Gamma_{OUT} = s_{22} + \frac{s_{12}s_{21}\Gamma_S}{1-s_{11}\Gamma_S}$$

Sample Numbers to Check Calculations

Using the formulas on the previous page with the following values:

$$s_{11} = 0.52\angle -145^\circ \quad s_{12} = 0.03\angle 20^\circ \quad s_{21} = 2.56\angle 170^\circ \quad s_{22} = 0.48\angle -20^\circ$$
$$\Gamma_S = 0.43\angle 70^\circ \quad \Gamma_L = 0.45\angle 80^\circ$$

All the gains are power ratios so they are converted to dB using 10*log10(ratio)

$$G_P = 8.83 = 9.46\,dB$$
$$G_T = 5.67 = 7.53\,dB$$
$$G_{TU} = 5.49 = 7.40\,dB$$
$$G_T' = 12.51 = 10.97\,dB$$
$$G_{TU,\max} = 11.67 = 10.67\,dB$$
$$M = 0.0341$$
$$0.9351 < \frac{G_T'}{G_{TU,\max}} < 1.0719 \Rightarrow -0.29\,dB < \left(G_T'\right)_{dB} - \left(G_{TU,\max}\right)_{dB} < 0.30\,dB$$
$$G_A = 7.42 = 8.71\,dB$$