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Q1 (c). Assume $h_5 = h_6$ and so model has settled

$$Y(z) = h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_4 z^{-4} + h_5 z^{-5} + h_6 z^{-6}$$

$$G(z) = g_1 z^{-1} + g_2 z^{-2} + g_3 z^{-3} + g_4 z^{-4} + g_5 z^{-5} + g_6 z^{-6}$$

$$U(z) = \frac{1}{1-z^{-1}}$$

$$Y(z) = U(z)G(z)$$

$$\Rightarrow G(z) = \frac{Y(z)}{U(z)}$$

$$= (1-z^{-1})(h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_4 z^{-4} + h_5 z^{-5} + h_6 z^{-6})$$

$$= h_1 z^{-1} + (h_2 - h_1)z^{-2} + (h_3 - h_2)z^{-3} + (h_4 - h_3)z^{-4} + (h_5 - h_4)z^{-5} + (h_6 - h_5)z^{-6}$$

$$\Rightarrow Y(z) = (h_1 z^{-1} + (h_2-h_1)z^{-2} + (h_3-h_2)z^{-3} + (h_4-h_3)z^{-4} + (h_5-h_4)z^{-5} + (h_6-h_5)z^{-6})U(z)$$

$$\Rightarrow y(k) = h_1 u(k-1) + (h_2-h_1)u(k-2) + (h_3-h_2)u(k-3) + (h_4-h_3)u(k-4) + (h_5-h_4)u(k-5) + (h_6-h_5)u(k-6)$$

$$y(k+1) = h_1 u(k)$$

$$y(k+2) = h_1 u(k+1) + (h_2-h_1)u(k)$$

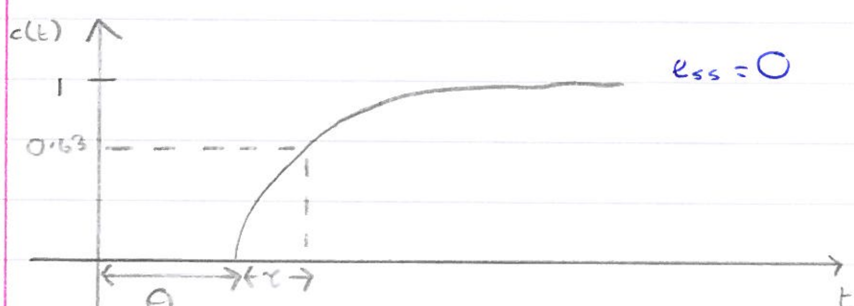
$$y(k+3) = h_1 u(k+2) + (h_2-h_1)u(k+1) + (h_3-h_2)u(k)$$

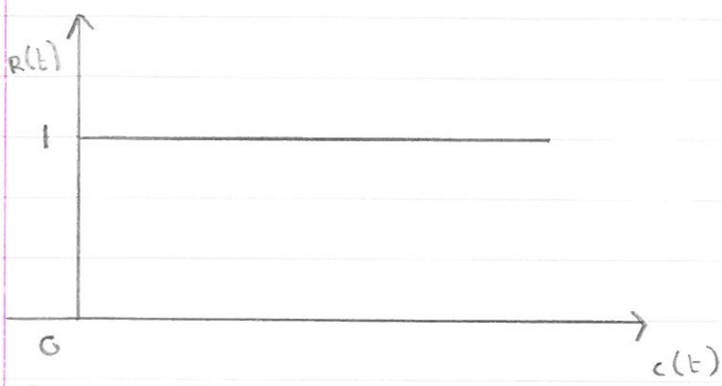
$$y(k+4) = h_1 u(k+3) + (h_2-h_1)u(k+2) + (h_3-h_2)u(k+1) + (h_4-h_3)u(k)$$

$$\begin{bmatrix} y(k+1) \\ y(k+2) \\ y(k+3) \\ y(k+4) \end{bmatrix} = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ h_2-h_1 & h_1 & 0 & 0 \\ h_3-h_2 & h_2-h_1 & h_1 & 0 \\ h_4-h_3 & h_3-h_2 & h_2-h_1 & h_1 \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \\ u(k+3) \end{bmatrix} + \underline{y}_f(k)$$

$$\underline{y} = \underline{h}\,\underline{u} + \underline{y}_f = \underline{R} \Rightarrow \underline{h}\,\underline{u} = \underline{R} - \underline{y}_f \Rightarrow \underline{u} = \underline{h}^{-1}(\underline{R} - \underline{y}_f)$$

Q2 (a). Specify a desired step-response for the continuous time signal $c(t)$





First order response - time constant λ
time delay Θ

$$c(t) = \begin{cases} 0 & \text{for } t < \Theta \\ 1 - e^{-\frac{t-\Theta}{\lambda}} & \text{for } t \geq \Theta \end{cases}$$

$$C(s) = L\{c(t)\} = \frac{e^{-\Theta s}}{s(1+\lambda s)}$$

Choose N as $\Theta \approx NT$, $\Theta > NT$

$$C(s) = \frac{e^{-NTs}}{s(1+\lambda s)}$$

From Z transform tables

$$C(z) = \frac{(1-e^{-T/\lambda})z^{-(N+1)}}{(1-z^{-1})(1-e^{-T/\lambda}z^{-1})}$$

$$R(z) = \frac{1}{1-z^{-1}}$$

Using the control design equation

$$D(z) = \frac{1}{G(z)} \frac{\frac{C(z)}{R(z)}}{1-C(z)/R(z)}$$

$$= \frac{1}{G(z)} \frac{\frac{(1-e^{-T/\lambda})z^{-(N+1)}}{1-e^{-T/\lambda}z^{-1}}}{1-\frac{(1-e^{-T/\lambda})z^{-(N+1)}}{1-e^{-T/\lambda}z^{-1}}}$$

$$D(z) = \frac{1}{G(z)} \frac{(1-e^{-T/2}) z^{-(N+1)}}{1 - e^{-T/2} z^{-1} - (1-e^{-T/2}) z^{-(N+1)}}$$

$$= \frac{1}{G(z)} \frac{(1-\alpha) z^{-(N+1)}}{1 - \alpha z^{-1} - (1-\alpha) z^{-(N+1)}}$$

This could be factorised as $D(z) = \frac{1}{G(z)} \frac{(1-\alpha) z^{-(N+1)}}{(1-z^{-1}) Q(z)}$

↑ INTEGRAL ACTION

- Possible oscillation in control signal $c(t)$
- Any pole of $D(z)$ close to the unit circle and close to $z=-1$ $\Rightarrow$ undamped oscillation with a frequency close to $f_s/2$ $\Rightarrow$ hidden

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Q3(a)

$$\frac{Y(z)}{U(z)} = G(z) = \frac{z^{-d}(b_1 z^{-1} + b_2 z^{-2} + \ldots + b_m z^{-m})}{1 - a_1 z^{-1} - a_2 z^{-2} - \ldots - a_n z^{-n}}$$

$$\hat{y}(k+1) = \hat{a}_1 y(k) + \hat{a}_2 y(k-1) + \ldots + \hat{a}_n y(k-n+1) + \hat{b}_1 u(k-d) + \hat{b}_2 u(k-d-1) + \ldots + \hat{b}_m u(k-d-m+1)$$

The first valid test equation is then:

$$\hat{y}(m+d) = \hat{a}_1 y(m+d-1) + \hat{a}_2 y(m+d-2) + \ldots + \hat{a}_n y(m+d-n) + \hat{b}_1 u(m-1) + \hat{b}_2 u(m-2) + \ldots + \hat{b}_m u(0)$$

We can repeat this to generate output estimates over the valid data set

$$
N \quad
\begin{bmatrix} \hat{y}(m+d) \\ \hat{y}(m+d+1) \\ \hat{y}(m+d+2) \\ \vdots \\ \hat{y}(N_1-1) \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cc} \qquad\qquad n \qquad\qquad & \qquad\qquad M \qquad\qquad \end{array} \\
\left[\begin{array}{ccc|ccc}
y(m+d-1) & \cdots & y(m+d-n) & u(m-1) & \cdots & u(0) \\
y(m+d) & \cdots & y(m+d-n+1) & u(m) & \cdots & u(1) \\
y(m+d+1) & \cdots & y(m+d-n+2) & u(m+1) & \cdots & u(2) \\
\vdots & & \vdots & \vdots & & \\
y(N_1-2) & \cdots & y(N_1-n-1) & u(N_1-d-2) & \cdots & u(N_1-d-m-1)
\end{array}\right]
\end{array}
\left[\begin{array}{c} \hat{a}_1 \\ \hat{a}_2 \\ \vdots \\ \hat{a}_n \\ \hline \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_m \end{array}\right]
\begin{array}{c} n \\ \\ \\ M \end{array}
$$

This could be rewritten as:

$$\underline{\hat{y}}(k) = \phi(k)\,\underline{\hat{\Theta}}(k)$$

Define the Least-Squares cost function

$$J = \sum_{i=m+d}^{N_1-1} e^2(i) \qquad \text{where} \quad e(i) = y(i) - \hat{y}(i)$$

$$\underline{E} = \begin{bmatrix} e(m+d) \\ e(m+d+1) \\ \vdots \\ e(N_1-1) \end{bmatrix} = \begin{bmatrix} y(m+d) \\ y(m+d+1) \\ \vdots \\ y(N_1-1) \end{bmatrix} - \begin{bmatrix} \hat{y}(m+d) \\ \hat{y}(m+d+1) \\ \vdots \\ \hat{y}(N_1-1) \end{bmatrix}$$

$$\underline{E} = \underline{y}(k) - \hat{\underline{y}}(k)$$

$$J = \sum_{i=m+d}^{N_1-1} e^2(i) = \underline{E}^T\underline{E} = (\underline{y}(k) - \hat{\underline{y}}(k))^T(\underline{y}(k) - \hat{\underline{y}}(k))$$

$$\Rightarrow J = (\underline{y}(k) - \phi(k)\hat{\underline{\Theta}}(k))^T(\underline{y}(k) - \phi(k)\hat{\underline{\Theta}}(k)) \qquad (A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

$$J = (\underline{y}(k)^T - \hat{\underline{\Theta}}(k)^T\phi(k)^T)(\underline{y}(k) - \phi(k)\hat{\underline{\Theta}}(k))$$

$$= \underline{y}(k)^T\underline{y}(k) \underbrace{- \underline{y}(k)^T\phi(k)\hat{\underline{\Theta}}(k) - \hat{\underline{\Theta}}(k)^T\phi(k)^T\underline{y}(k)}_{\text{SAME}} + \hat{\underline{\Theta}}(k)^T\phi(k)^T\phi(k)\hat{\underline{\Theta}}(k)$$

$$\Rightarrow J = \hat{\underline{\Theta}}(k)^T\underbrace{\phi(k)^T\phi(k)}_{M}\hat{\underline{\Theta}}(k) - \underbrace{2\underline{y}(k)^T\phi(k)}_{G}\hat{\underline{\Theta}}(k) + \underbrace{\underline{y}(k)^T\underline{y}(k)}_{J_0}$$

$$J = \hat{\underline{\Theta}}(k)^T M \hat{\underline{\Theta}}(k) - G\hat{\underline{\Theta}}(k) + J_0$$

This will be minimised when

$$\hat{\underline{\Theta}}(k)^T = \tfrac{1}{2} G M^{-1} = \tfrac{1}{2}(2\underline{y}(k)^T\phi(k))(\phi(k)^T\phi(k))^{-1}$$

$$\hat{\underline{\Theta}}(k)^T = \underline{y}(k)\phi(k)[\phi(k)^T\phi(k)]^{-1}$$

$$\Rightarrow \hat{\underline{\Theta}}(k) = (\phi(k)^T\phi(k))^{-1}\phi(k)^T\underline{y}(k)$$

(b). $$\frac{V_T(s)}{V(s)} = \frac{Ke^{-sT_d}}{1+s\tau}$$

$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-sT}}{s} \cdot \frac{Ke^{-sT_d}}{1+s\tau}\right\}$$

$$= (1-z^{-1})z^{-1}\frac{K}{\tau}\mathcal{Z}\left\{\frac{1}{s(s+1/\tau)}\right\}$$

$$= \cancel{(1-z^{-1})}z^{-1}\frac{K}{\cancel{\tau}}\frac{1}{1/\tau}\frac{(1-e^{-T/\tau})z^{-1}}{\cancel{(1-z^{-1})}(1-e^{-T/\tau}z^{-1})}$$

$T = 1$

$$\Rightarrow G(z) = \frac{K(1-e^{-1/\tau})z^{-2}}{1-e^{-1/\tau}z^{-1}} = \frac{b z^{-2}}{1-az^{-1}}$$

$$\Rightarrow (1-az^{-1})y(z) = b z^{-2} u(z)$$

$$y(k) = ay(k-1) + bu(k-1)$$

$$\Rightarrow \hat{V}_T(2) = aV_T(1) + bu(0)$$

$$\hat{V}_T(3) = aV_T(2) + bu(1)$$

$$\vdots$$

$$\hat{V}_T(10) = aV_T(9) + bu(8)$$

$$\hat{V}_T(k) = \phi(k)\hat{\underline{\Theta}}(k)$$

$$\phi(k) = \begin{bmatrix} 0 & -1 \\ -3.9 & 1 \\ 1.5 & 3 \\ 12.7 & -1 \\ 3.8 & -2 \\ -5.6 & 0 \\ -3.4 & 1 \\ 1.9 & 1 \\ 5.1 & -1 \end{bmatrix}$$

$$\hat{\underline{\Theta}}(k)_{LS} = (\phi(k)^T\phi(k))^{-1}\phi(k)^T \underline{V}_T(k)$$

$$\phi(k)^T\phi(k) = \begin{bmatrix} 265.73 & -26.3 \\ -26.3 & 19 \end{bmatrix}$$

$$(\phi(k)^T\phi(k))^{-1} = \frac{1}{265.73(19)+26.3(-26.3)}\begin{bmatrix} 19 & 26.3 \\ 26.3 & 265.73 \end{bmatrix}$$

$$= \frac{1}{4357.18}\begin{bmatrix} 19 & 26.3 \\ 26.3 & 265.73 \end{bmatrix}$$

$\phi(k)^T v_T(k)$

$$\begin{bmatrix} 0 & -3.9 & 1.5 & 12.7 & 3.8 & -5.6 & -3.4 & 1.9 & 5.1 \\ -1 & 1 & 3 & -1 & -2 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -3.9 \\ 1.5 \\ 12.7 \\ 3.8 \\ -5.6 \\ -3.4 \\ 1.9 \\ 5.1 \\ -0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 57.86 \\ 58.8 \end{bmatrix}$$

$$\underline{\hat{\Theta}}(k)_{LS} = \frac{1}{4357.18} \begin{bmatrix} 19 & 26.3 \\ 26.3 & 265.73 \end{bmatrix} \begin{bmatrix} 57.86 \\ 58.8 \end{bmatrix} = \begin{bmatrix} 0.61 \\ 3.94 \end{bmatrix}$$

$a = 0.61$ $\qquad$ $b = 3.94$

$e^{-1/\tau} = 0.61$ $\qquad$ $K(1 - e^{-1/\tau}) = 3.94$

$\Rightarrow \tau = 2.02$ $\qquad$ $\Rightarrow K = 10.1$

$T_{s2\%} = 10s$

$4/\xi\omega_n = 10$

$\Rightarrow \omega_n = 0.667$

$\omega_n = x\pi/10T$

$0.667 = x\pi/10(1)$

$x = 2.1$

From Z plane design template

$z = 0.58 \pm 0.32j = 0.66 \angle \pm 28.99°$

Fast pole at 0.125 $(0.66)^5$

$$TF = \frac{b z^{-2}}{1 - a z^{-1}} = \frac{b}{z^2 - a z} = \frac{B(z)}{A(z)}$$

$n = 2 \Rightarrow n_q = n_s = n - 1 = 1$

$Q(z) = z + q_1$

$S(z) = s_0 z + s_1$

$A_{cl}(z) = A(z)Q(z) + B(z)S(z)$

$= (z^2 - az)(z + q_1) + b(s_0 z + s_1)$

$= z^3 + q_1 z^2 - a z^2 - a q_1 z + b s_0 z + b s_1$

$= z^3 + (q_1 - a) z^2 + (b s_0 - a q_1) z + b s_1$

$A_{cl}(z) = z^3 + c_1 z^2 + c_2 z + c_3$

$= (z - 0.58 + j0.32)(z - 0.58 - j0.32)(z - 0.125)$

$= ((z - 0.58)^2 - (j0.32)^2)(z - 0.125)$

$A_c(z)$

$= (z^2 - 1.16z + 0.3364 + 0.1024)(z - 0.125)$ $A_0(z)$

$= (z^2 - 1.16z + 0.4388)(z - 0.125)$

$= z^3 - 0.125 z^2 - 1.16 z^2 + 0.145 z + 0.4388 z - 0.05485$

$= z^3 - 1.285 z^2 + 0.5838 z - 0.05485$

$c_1 = -1.285$

$c_2 = 0.5838$

$c_3 = -0.05485$

$q_1 - a = c_1 \Rightarrow q_1 = c_1 + a$

$b s_0 - a q_1 = c_2$

$b s_1 = c_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ -a & b & 0 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} c_1 + a \\ c_2 \\ c_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.61 & 3.94 & 0 \\ 0 & 0 & 3.94 \end{bmatrix} \begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} -0.675 \\ 0.585 \\ -0.055 \end{bmatrix}$$

$$\begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.61 & 3.94 & 0 \\ 0 & 0 & 3.94 \end{bmatrix}^{-1} \begin{bmatrix} -0.675 \\ 0.585 \\ -0.055 \end{bmatrix}$$

$$M^{-1} = \frac{1}{\det(M)} (\operatorname{adj}(M))$$

$$\det(M) = 1(3.94^2) - 0 + 0 = 15.524$$

$$M^T = \begin{bmatrix} 1 & -0.61 & 0 \\ 0 & 3.94 & 0 \\ 0 & 0 & 3.94 \end{bmatrix}$$

$$\operatorname{adj}(M) = \begin{bmatrix} \begin{vmatrix} 3.94 & 0 \\ 0 & 3.94 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & 3.94 \end{vmatrix} & \begin{vmatrix} 0 & 3.94 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} -0.61 & 0 \\ 0 & 3.94 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 3.94 \end{vmatrix} & \begin{vmatrix} 1 & -0.61 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -0.61 & 0 \\ 3.94 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -0.61 \\ 0 & 3.94 \end{vmatrix} \end{bmatrix}$$

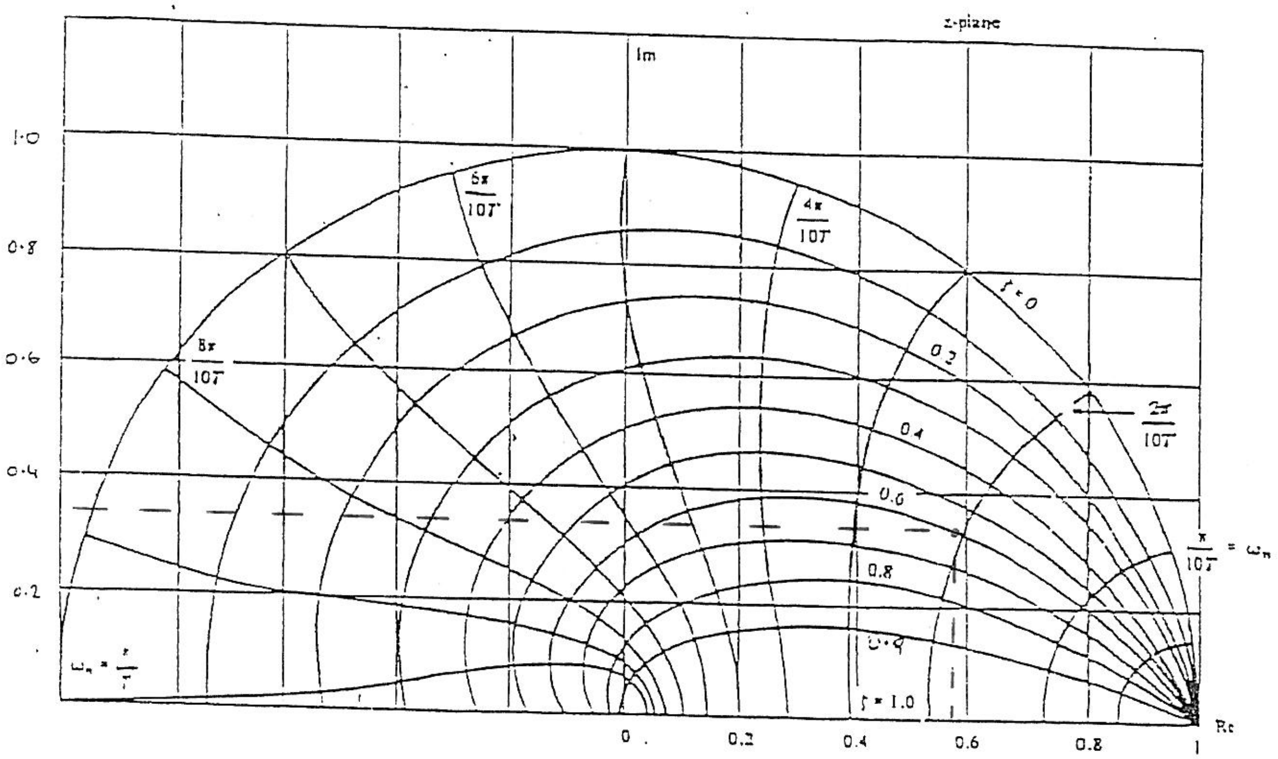
$$\operatorname{adj}(M) = \begin{bmatrix} 15.524 & 0 & 0 \\ 2.403 & 3.94 & 0 \\ 0 & 0 & 4.55 \end{bmatrix}$$

$$\Rightarrow M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0.155 & 0.254 & 0 \\ 0 & 0 & 0.293 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.155 & 0.254 & 0 \\ 0 & 0 & 0.293 \end{bmatrix} \begin{bmatrix} -0.675 \\ 0.585 \\ -0.055 \end{bmatrix} = \begin{bmatrix} -0.675 \\ 0.045 \\ -0.014 \end{bmatrix}$$

$$T(z) = t_0 A_0 = t_0(z - 0.125)$$

$$t_0 = \frac{A_c(1)}{B(1)} = \frac{1 - 1.16 + 0.4388}{3.94} = 0.07$$



Z Plane Design Template

<u>Please submit with your script</u>

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Q5(a) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{A}{\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{B}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} u(t)$

$y(t) = \underset{C}{\begin{bmatrix} 3 & 3 \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(i) $G(s) = C(sI-A)^{-1}B + \cancel{D}^{0}$

$(sI-A) = \begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix}$

$(sI-A)^{-1} = \frac{1}{(s+3)(s+1)}\begin{bmatrix} s+1 & 0 \\ 0 & s+3 \end{bmatrix}$

$C(sI-A)^{-1} = \frac{1}{(s+3)(s+1)}\begin{bmatrix} 3 & 3 \end{bmatrix}\begin{bmatrix} s+1 & 0 \\ 0 & s+3 \end{bmatrix}$

$= \frac{1}{(s+3)(s+1)}\begin{bmatrix} 3s+3 & 3s+9 \end{bmatrix}$

$C(sI-A)^{-1}B = \frac{3}{(s+3)(s+1)}\begin{bmatrix} s+1 & s+3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{3(s+1+s+3)}{(s+3)(s+1)} = \frac{3(2s+4)}{(s+3)(s+1)} = \frac{6s+12}{s^2+4s+3}$

(ii) $C_x = [B \,\vdots\, AB]$

$= \left[\begin{matrix} 1 \\ 1 \end{matrix} \,\vdots\, \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$

$= \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}$

Controllable if $\det C_x \neq 0$

$\det C_x = 1(-1) - (-3)(1) = 2 \neq 0$

$\Rightarrow$ controllable

$O_x = \begin{bmatrix} C \\ \hdashline CA \end{bmatrix}$ $\qquad CA = \begin{bmatrix} 3 & 3 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -9 & -3 \end{bmatrix}$

$$O_x = \begin{bmatrix} 3 & 3 \\ -9 & -3 \end{bmatrix}$$

$\det(O_x) = 3(-3) - 3(-9) = -36 \neq 0$

$\Rightarrow$ observable

(iii) $\dfrac{Y(s)}{U(s)} = G(s) = \dfrac{6s+12}{s^2+4s+3}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 12 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$C_z = [B \,\vdots\, AB] = \left[\begin{matrix} 0 \\ 1 \end{matrix} \;\vdots\; \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}$$

$C_z = T C_x$

$C_z C_x^{-1} = T C_x C_x^{-1}$

$T = C_z C_x^{-1}$

$$C_x^{-1} = \frac{1}{-1+3}\begin{bmatrix} -1 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} -0.5 & 1.5 \\ -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} -0.5 & 1.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 0.5 \\ 1.5 & -0.5 \end{bmatrix}$$

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Q6 (a). $\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t) + E\underline{d}(t)$

$\underline{y}(t) = C\underline{x}(t)$

The full-state estimator (Luenberger Observer) is:

$\frac{d}{dt}\hat{\underline{x}} = A\hat{\underline{x}} + B\underline{u} + G(\underline{y}(t) - \hat{\underline{y}}(t))$

$\hat{\underline{y}}(t) = C\hat{\underline{x}}(t)$

$\Rightarrow \frac{d}{dt}\hat{\underline{x}} = A\hat{\underline{x}} + B\underline{u} + GC(\underline{x}(t) - \hat{\underline{x}}(t))$

Define the state-estimation error vector

$\underline{e}(t) = \underline{x}(t) - \hat{\underline{x}}(t)$

$\frac{d}{dt}\underline{e}(t) = \frac{d}{dt}\underline{x}(t) - \frac{d}{dt}\hat{\underline{x}}(t)$

$\Rightarrow \dot{\underline{e}}(t) = A\underline{x}(t) + \cancel{B\underline{u}(t)} + E\underline{d}(t) - A\hat{\underline{x}} - \cancel{B\underline{u}} - GC(\underline{x}(t) - \hat{\underline{x}}(t))$

$= A(\underline{x}(t) - \hat{\underline{x}}(t)) - GC(\underline{x}(t) - \hat{\underline{x}}(t)) + E\underline{d}(t)$

$= (A - GC)(\underline{x}(t) - \hat{\underline{x}}(t)) + E\underline{d}(t)$

$= (A - GC)\underline{e}(t) + E\underline{d}(t)$

As $t \to \infty$

$\dot{\underline{e}}(t) \to 0$

$e \to e_{ss}$

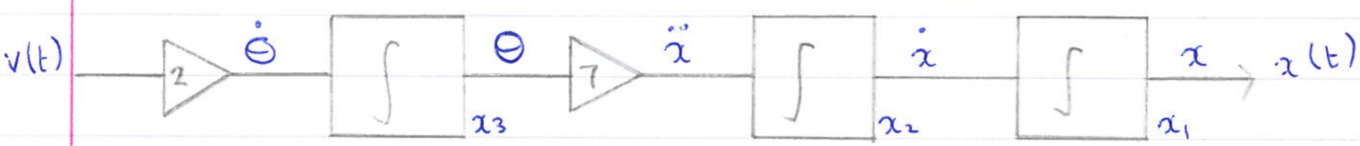
$\Rightarrow 0 = (A - GC)e_{ss} + Ed_\infty$

$-(A - GC)e_{ss} = Ed_\infty$

$e_{ss} = -(A - GC)^{-1}Ed_\infty$

(b) $\frac{d\theta}{dt} = K_v(t) = 2v(t)$

$\frac{d^2x}{dt^2} = 7\theta$



$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}_{B} v(t)$$

$$C_{des}(s) = \det(sI - A + BK)$$

$$\det\left[\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} (k_1 \; k_2 \; k_3)\right]$$

$$\det\left[\begin{pmatrix} s & -1 & 0 \\ 0 & s & -7 \\ 0 & 0 & s \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2k_1 & 2k_2 & 2k_3 \end{pmatrix}\right]$$

$$\det\begin{pmatrix} s & -1 & 0 \\ 0 & s & -7 \\ +2k_1 & +2k_2 & s+2k_3 \end{pmatrix} = s\begin{vmatrix} s & -7 \\ 2k_2 & s+2k_3 \end{vmatrix} + 1\begin{vmatrix} 0 & -7 \\ 2k_1 & s+2k_3 \end{vmatrix} + 0$$

$$= s[s(s+2k_3)+7(2k_2)] + 7(2k_1)$$

$$= s(s^2 + 2k_3 s + 14k_2) + 14k_1$$

$$= s^3 + 2k_3 s^2 + 14k_2 s + 14k_1$$

$$C_{des}(s) = (s^2 + 2\xi\omega_n s + \omega_n^2)(s + p_1)$$

$$= (s^2 + 2(0.7)s + 1)(s + p_1)$$

$$= (s^2 + 1.4s + 1)(s + p_1) \qquad p_1 \text{ - fast pole}$$

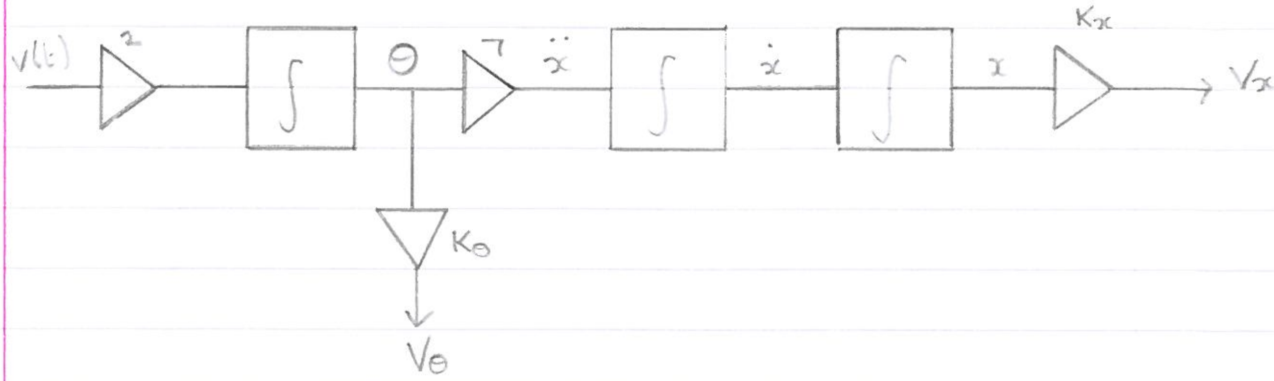
$$= (s^2 + 1.4s + 1)(s + 4)$$

$$\Rightarrow C_{des}(s) = s^3 + 4s^2 + 1.4s^2 + 5.6s + s + 4$$

$$= s^3 + 5.4s^2 + 6.6s + 4$$

$2k_3 = 5.4$	$14k_2 = 6.6$	$14k_1 = 4$
$k_3 = 2.7$	$k_2 = 0.47$	$k_1 = 0.29$

(ii)



$$\frac{d^2x}{dt^2} = 7\theta = 7\,\frac{V_\theta}{K_\theta} = \left(\frac{7}{K_\theta}\right)V_\theta$$

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 7/K_\theta \end{bmatrix} V_\theta$$

$$V_x = \begin{bmatrix} K_x & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

The full state estimator is

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix} = (A - GC)\begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix} + BV_\theta + GV_x$$

$$G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$A - GC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}\begin{bmatrix} K_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -K_x g_1 & 1 \\ -K_x g_2 & 0 \end{bmatrix} = \begin{bmatrix} -5g_1 & 1 \\ -5g_2 & 0 \end{bmatrix} = F$$

Poles of estimator are given by:

$$\det(sI - F) = 0$$

$$sI - F = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -5g_1 & 1 \\ -5g_2 & 0 \end{pmatrix} = \begin{pmatrix} s + 5g_1 & -1 \\ 5g_2 & s \end{pmatrix}$$

$$\det(sI - F) = (s + 5g_1)s + 5g_2$$
$$= s^2 + 5g_1 s + 5g_2$$

Choose fast estimator poles

$\Rightarrow s = -5$ twice

$C_{des}(s) = (s+5)^2 = s^2 + 10s + 25$

$5g_1 = 10 \qquad 5g_2 = 25$

$g_1 = 2 \qquad g_2 = 5$

$$\Rightarrow F = \begin{bmatrix} -10 & 1 \\ -25 & 0 \end{bmatrix}$$

So observer is

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -25 & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix} + \begin{bmatrix} 0 \\ 3.5 \end{bmatrix} V_\theta + \begin{bmatrix} 2 \\ 5 \end{bmatrix} V_x$$