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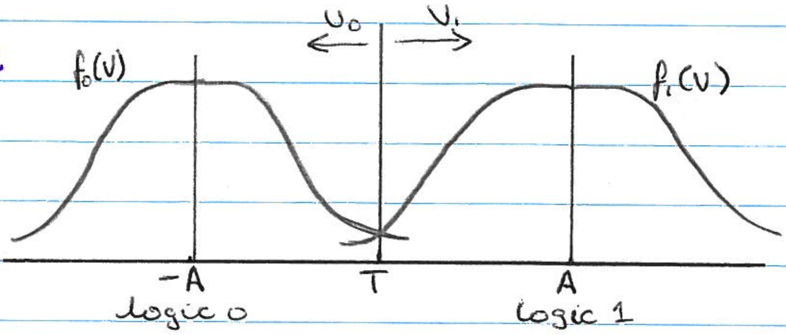
5. a let $A = A\sin\omega t$
where
$t = \frac{T}{4} + \Delta t$
$= \frac{T+2\Delta t}{4}$



Probability of sending a 1 & receiving a zero:

$$P_1 \int_{v_0} f_1(v)\,dv$$

Probability of sending a 0 & receiving a 1:

$$P_0 \int_{v_1} f_0(v)\,dv$$

Overall:

$$P_e = P_1 \int_{v_0} f_1(v)\,dv + P_0 \int_{v_1} f_0(v)\,dv$$
$$= P_0 \int_{v_1} f_0(v)\,dv + P_1\left(1 - \int_{v_1} f_1(v)\,dv\right)$$
$$= P_1 + \int_{v_1} \left(P_0 f_0(v)\,dv - P_1 f_1(v)\,dv\right)$$

$$f_0(V) = \frac{e^{\frac{-(V+A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \qquad f_1(V) = \frac{e^{\frac{-(V-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

To minimise P_e, integral should be as '-'ive as possible.

$$\Rightarrow \frac{f_1}{f_0} > \frac{P_0}{P_1} \qquad \Rightarrow \frac{e^{\frac{-(V-A)^2}{2\sigma^2}}}{e^{-\frac{(V+A)^2}{2\sigma^2}}} = \frac{P_0}{P_1}$$

$$-(V-A)^2 + (V+A)^2 = 2\sigma^2 \ln\left(\frac{P_0}{P_1}\right)$$
$$2Av \cancel{-A^2} + 2AV \cancel{+A^2}$$
$$4AV = 2\sigma^2 \ln\left(\frac{P_0}{P_1}\right) \Rightarrow V = \frac{\sigma^2}{2A} \ln\left(\frac{P_0}{P_1}\right)$$

Subbing in for A:

$$V = \frac{\sigma^2}{2A\sin\left(\omega \frac{T+2\Delta t}{4}\right)} \ln\left(\frac{P_0}{P_1}\right)$$

V is the threshold voltage.

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5 b) We know from a) that:

$$P_e = P_1 + \int_{v_1} (P_0 f_0(v) - P_1 f_1(v))\,dv$$
$$= (1-P_0) + \int_{v_1} (P_0 f_0(v) - [1-P_0] f_1(v))\,dv$$

Using $A = A\sin(\omega t)$ $\qquad T = D$

$$t = \frac{T}{2} + \Delta t = \frac{T + 2\Delta t}{4}$$

$$\int f_0(v) = \int_T^{\infty} \frac{e^{\frac{-(v+A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv \qquad \int f_1(v) = \int_T^{\infty} \frac{e^{\frac{-(v-A)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv$$

$$= \frac{1}{2} - \int_0^{T+A} \frac{e^{\frac{-v^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv \qquad = \frac{1}{2} + \int_0^{A-T} \frac{e^{\frac{-v^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv$$

Consider: $\displaystyle\int_0^{\alpha} \frac{e^{\frac{-v^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$ $\quad$ Let $u = \frac{v}{\sqrt{2\sigma^2}}$, $dv = \sqrt{2\sigma^2}\,du$

$$\Rightarrow \int_0^{\frac{\alpha}{\sqrt{2\sigma^2}}} \frac{e^{-u}}{\sqrt{\pi}}\,du = \frac{1}{\sqrt{\pi}} \int_0^{\frac{\alpha}{\sqrt{2\sigma^2}}} e^{u}\,du = \frac{1}{2}\operatorname{erf}\left[\frac{\alpha}{\sqrt{2\sigma^2}}\right]$$

$$\therefore P_e = (1-P_0) + P_0\left[\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left[\frac{T+A}{\sqrt{2\sigma^2}}\right]\right] - P_1\left[\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left[\frac{A-T}{\sqrt{2\sigma^2}}\right]\right]$$

$$= 1 - \cancel{P_0} + \cancel{\frac{P_0}{2}} - \frac{P_0}{2}\operatorname{erf}\left[\frac{T+A}{\sqrt{2\sigma^2}}\right] - \frac{1}{2} + \cancel{\frac{P_0}{2}} - \frac{(1-P_0)}{2}\operatorname{erf}\left[\frac{A-T}{\sqrt{2\sigma^2}}\right]$$

$$\boxed{P_e = \frac{1}{2}\left[1 - \left(P_0 \operatorname{erf}\left[\frac{D + A\sin(\omega\frac{T+2\Delta t}{4})}{\sqrt{2\sigma^2}}\right] + (1-P_0)\operatorname{erf}\left[\frac{A\sin(\omega\frac{T+2\Delta t}{4}) - D}{\sqrt{2\sigma^2}}\right]\right)\right]}$$

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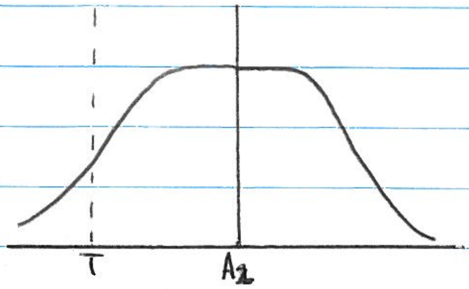
5. $P_e = P_1 \int_{V_0} f_1(v)\,dv + P_0 \int_{V_1} f_0(v)\,dv$

$= P_1 + \int_{V_0} \left(P_0 f_0(v) - P_1 f_1(v)\right) dv$

$P_e = (1-P_0) + \int_{V_0} \left(P_0 f_0(v) - (1-P_0) f_1(v)\right) dv$

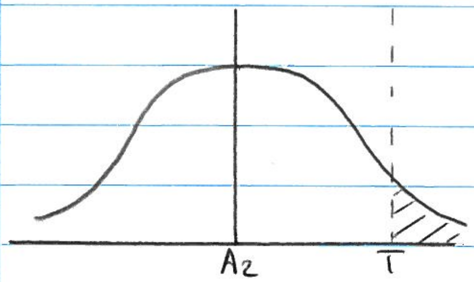
Consider $\int_T^{\infty} f_1(v)\,dv = \int_T^{\infty} \frac{e^{\frac{-(v-A_2)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv$

$= \frac{1}{2} + \int_0^{A_1-T} \frac{e^{\frac{-v^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv$



Consider $\int_T^{\infty} f_0(v)\,dv = \int_T^{\infty} \frac{e^{\frac{-(v-A_2)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv$

$= \frac{1}{2} - \int_0^{T-A_2} \frac{e^{\frac{-v^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv$



Consider $\int_0^{\alpha} \frac{e^{\frac{-v^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv$ Let $u = \frac{v}{\sqrt{2\sigma^2}} \Rightarrow dv = \sqrt{2\sigma^2}\,du$

$\Rightarrow \int_0^{\alpha} \frac{e^{\frac{-v^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\,dv = \frac{1}{\sqrt{\pi}} \int_0^{\frac{\alpha}{\sqrt{2\sigma^2}}} e^{-u^2}\,du = \frac{1}{2}\,\mathrm{erf}\left[\frac{\alpha}{\sqrt{2\sigma^2}}\right]$

$P_e = 1 - \cancel{P_0} + P_0\left[\frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left[\frac{T-A_2}{\sqrt{2\sigma^2}}\right]\right] - (1-P_0)\left[\cancel{\frac{1}{2}} + \frac{1}{2}\,\mathrm{erf}\left[\frac{A_1-T}{\sqrt{2\sigma^2}}\right]\right]$

$P_e = \frac{1}{2}\left(1 - \left(P_0\,\mathrm{erf}\left[\frac{T-A_2}{\sqrt{2\sigma^2}}\right] + (1-P_0)\,\mathrm{erf}\left[\frac{A_1-T}{\sqrt{2\sigma^2}}\right]\right)\right)$

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5 b) i) $P_e = \frac{1}{2}\left[1-\left(P_0 \operatorname{erf}\left[\frac{T-A_2}{\sqrt{2\sigma^2}}\right] + (1-P_0)\operatorname{erf}\left[\frac{A_1-T}{\sqrt{2\sigma^2}}\right]\right)\right]$

$= \frac{1}{2}\left[1-\left(P_0[0.9998] + (1-P_0)[0.9998]\right]\right.$

$= \frac{1}{2}\left[1-(0.9998)\right] = 1\times10^{-4}$, P_0 cancels

ii) $T = \frac{A_1+A_2}{2} + \frac{\sigma^2}{A_1-A_2}\ln\left[\frac{P_0}{1-P_0}\right]$

a) $T = \frac{\sigma^2}{5}\ln\left(\frac{0.65}{0.35}\right)$

$= 0.0557V$

$P_e = \frac{1}{2}\left[1-\left(P_0\operatorname{erf}[2.694] + (1-P_0)\operatorname{erf}[2.577]\right)\right]$

$= \frac{1}{2}\left[1-\left(0.65\times0.999861 + 0.35\times0.999732\right)\right]$

$= 9.2\times10^{-5}$

b) $T = \frac{\sigma^2}{5}\ln(3)$

$= 0.0988V$

$P_e = \frac{1}{2}\left[1-\left(0.75\times\operatorname{erf}[2.739] + 0.25\times\operatorname{erf}[2.531]\right)\right]$

$= \frac{1}{2}\left[1-\left(0.75\times0.999893 + 0.25\times[0.999656]\right)\right]$

$= 8.31\times10^{-5}$

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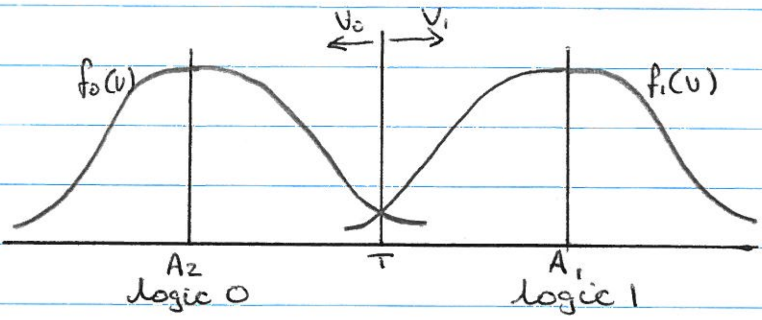
5a)



Probability that a zero is sent & a 1 is received:

$$P_0 \int_{v_1} f_0(v)\,dv$$

Probability that a 1 is sent & a zero is received:

$$P_1 \int_{v_0} f_1(v)\,dv$$

$$\text{Overall } P_e = P_0 \int_{v_1} f_0(v)\,dv + P_1 \int_{v_0} f_1(v)\,dv$$

$$= P_0 \int_{v_1} f_0(v)\,dv + P_1 \left(1 - \int_{v_1} f_1(v)\,dv\right)$$

$$= P_1 + \int_{v_1} \left(P_0 f_0(v) - P_1 f_1(v)\right) dv$$

To minimise P_e, we make the integral as -ive as possible:

$$\frac{f_1(v)}{f_0(v)} > \frac{P_0}{P_1}$$

where

$$f_0(v) = \frac{e^{\frac{-(v-A_2)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \qquad f_1(v) = \frac{e^{\frac{-(v-A_1)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

$$\frac{f_1(v)}{f_2(v)} = \frac{e^{\frac{-(v-A_1)^2}{2\sigma^2}}}{e^{\frac{-(v-A_2)^2}{2\sigma^2}}} = \frac{P_0}{P_1} \Rightarrow \frac{-(v-A_1)^2}{2\sigma^2} + \frac{(v-A_2)^2}{2\sigma^2} = \ln\left[\frac{P_0}{P_1}\right]$$

$$2A_1 v - A_1^2 - 2A_2 v + A_2^2 = 2\sigma^2 \ln\left[\frac{P_0}{P_1}\right]$$

$$2v(A_1 - A_2) - (A_1^2 - A_2^2) =$$

$$(A_1 - A_2)\,2v = (A_1^2 - A_2^2) + 2\sigma^2 \ln\left[\frac{P_0}{P_1}\right] \Rightarrow 2v =$$

$$2v = (A_1 + A_2) + \frac{2\sigma^2}{A_1 - A_2} \ln\left[\frac{P_0}{P_1}\right] \Rightarrow v = \frac{(A_1 + A_2)}{2} + \frac{\sigma^2}{(A_1 - A_2)} \ln\left[\frac{P_0}{P_1}\right]$$