EE4002 Control Engineering

B) State-Space Control

Chapter 1. The State Space Modelling Approach

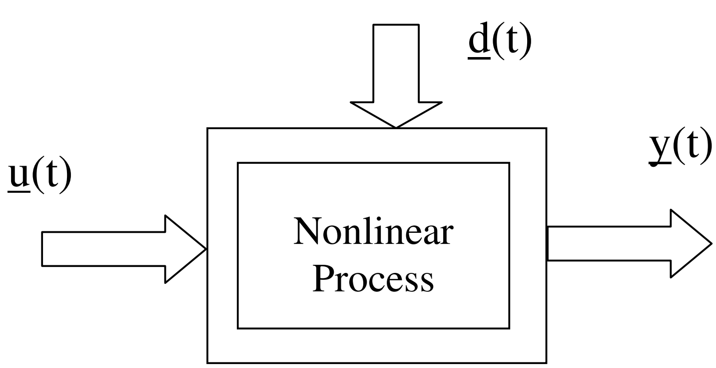
1.1 The State Space Model

Transformation of an N^{th} order multi-input-multi-output system to N first order differential equations:

Benefits

- Robust method of simulating high order differential equations
- Easy analysis of dynamics
- Allows for use of model reduction methods
- Can easily apply advanced control
- Used for estimation

Consider the N^{th} order nonlinear dynamical process:

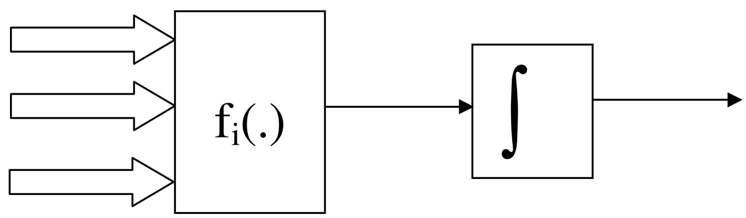


In general this process could be represented by a model consisting of :

It is possible to transform this set of coupled nonlinear differential equations to yield a set of N first order differential equations with states, $\{x_1(t), x_2(t), \ldots x_N(t)\}$.

The differential equation describing the dynamics of the i^{th} state $x_i(t)$ can be written as:

Which could be represented by the following simulation diagram:



The N first order differential equations could be written as:

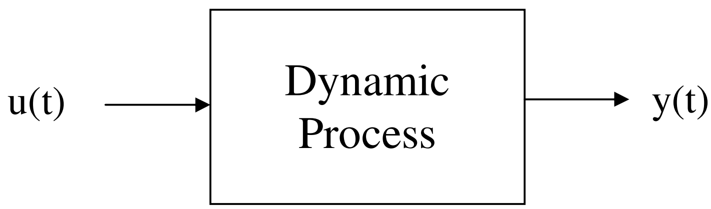
$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \underline{\dot{x}}(t) = \begin{bmatrix} f_1(\underline{x}(t), \underline{u}(t), \underline{d}(t)) \\ f_2(\underline{x}(t), \underline{u}(t), \underline{d}(t)) \\ \vdots \\ f_N(\underline{x}(t), \underline{u}(t), \underline{d}(t)) \end{bmatrix}$$

The output $\underline{y}(t)$ is then generated by:

$$\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_P(t) \end{bmatrix} = \begin{bmatrix} h_1(\underline{x}(t), \underline{u}(t), \underline{d}(t)) \\ h_2(\underline{x}(t), \underline{u}(t), \underline{d}(t)) \\ \vdots \\ h_P(\underline{x}(t), \underline{u}(t), \underline{d}(t)) \end{bmatrix}$$

1.1.1 Some Example State-Space Systems:

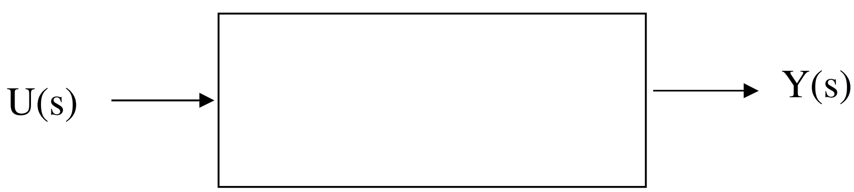
<u>i) Third Order Linear Process</u>



Modelled by the following differential equation:

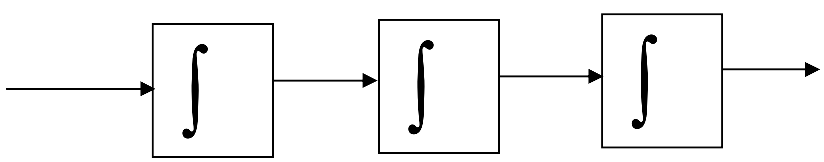
$$\frac{d^3}{dt^3}y(t)+5\frac{d^2}{dt^2}y(t)+3\frac{d}{dt}y(t)+y(t)=u(t)$$

Which could of course be represented by the transfer function model:



Rearrange to yield an expression for the highest derivative:

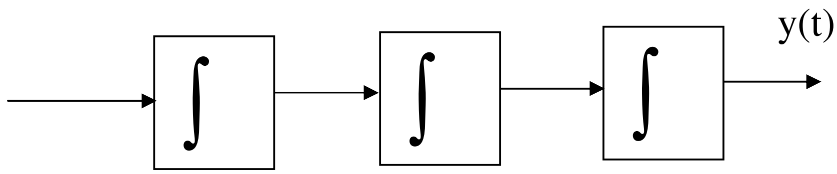
Form lower derivatives then by integration:



But we know that:

$$\dddot{y}(t)=u(t)-5\ddot{y}(t)-3\dot{y}(t)-y(t)$$

This yields the following simulation diagram:



Assign the state variables as the outputs of integrators:

We can now specify the state vector for the process as:

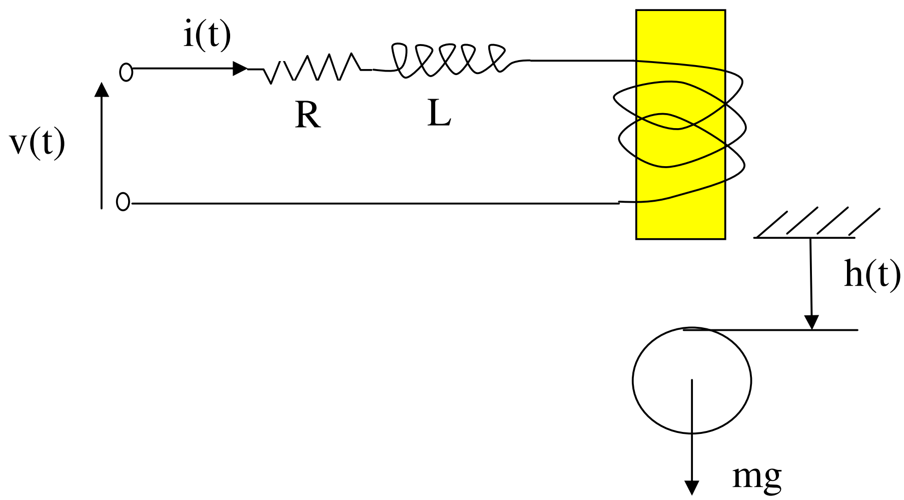
Now this could be written in matrix form as:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \quad \\ \\ \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

ii) Third Order Nonlinear Process

Consider the following magnetic suspension system:



Can be modelled by:

$$m\frac{d^2h(t)}{dt^2} = mg - \frac{Ki^2(t)}{h^2(t)}$$

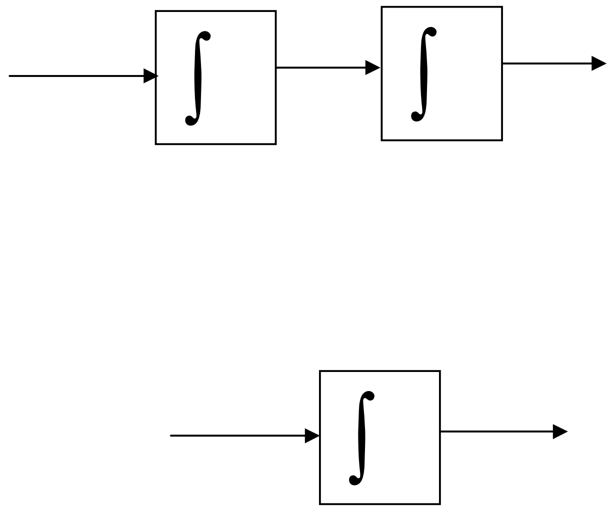
$$L\frac{di(t)}{dt} = v(t) - Ri(t)$$

Rewritten as:

$$\frac{d^2h(t)}{dt^2} = g - \frac{Ki^2(t)}{mh^2(t)}$$

$$\frac{di(t)}{dt} = \frac{1}{L}\left(v(t) - Ri(t)\right)$$

Could be represented by the following simulation diagram:



Note that three integrators are required – system is 3rd order

Now can arbitrarily assign the three states $\{x_1(t), x_2(t), x_3(t)\}$

And define the state vector:

The state-equations are now:

$$\dot{x}_1 = x_2(t)$$

$$\dot{x}_2 = g - \frac{Kx_3^2(t)}{mx_1^2(t)}$$

$$\dot{x}_3 = \frac{1}{L}\left(u(t) - Rx_3(t)\right)$$

with output equation:

$$y(t) = x_1(t)$$

1.2 Derivation of the Linear State Space Model - by Linearisation

Consider first the linearisation of a multivariate function:

$$z = g(\underline{w}) \quad where \quad \underline{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

about the operating point vector:

$$\underline{w}_0 = \begin{bmatrix} w_1' \\ \vdots \\ w_N' \end{bmatrix}$$

Use the first order Taylor's series approximation :

$$z \approx g(\underline{w}_0) + \left.\frac{\partial g}{\partial w_1}\right|_{\underline{w}=\underline{w}_0} (w_1 - w_1') + \left.\frac{\partial g}{\partial w_2}\right|_{\underline{w}=\underline{w}_0} (w_2 - w_2') + \cdots \left.\frac{\partial g}{\delta w_N}\right|_{\underline{w}=\underline{w}_0} (w_N - w_N')$$

which could be written as:

$$(z - z_0) = \begin{bmatrix} \left.\frac{\partial g}{\partial w_1}\right|_{\underline{w}_0} & \left.\frac{\partial g}{\partial w_2}\right|_{\underline{w}_0} & \cdots & \left.\frac{\partial g}{\partial w_N}\right|_{\underline{w}_0} \end{bmatrix} (\underline{w} - \underline{w}_0)$$

Now for simplicity first consider a linearisation of the ith state equation:

$$\frac{dx_i(t)}{dt} = f_i(\underline{x}(t))$$

Could be linearised to yield:

$$(z - z_0) = \left[\left.\frac{\partial f_i}{\partial x_1}\right|_{\underline{x}_0} \quad \left.\frac{\partial f_i}{\partial x_2}\right|_{\underline{x}_0} \quad \cdots \quad \left.\frac{\partial f_i}{\partial x_N}\right|_{\underline{x}_0} \right] (\underline{x} - \underline{x}_0)$$

But:

Hence the linearised equation becomes:

$$\frac{d}{dt}\Delta x_i(t) = \left[\left.\frac{\partial f_i}{\partial x_1}\right|_{\underline{x}_0} \quad \left.\frac{\partial f_i}{\partial x_2}\right|_{\underline{x}_0} \quad \cdots \quad \left.\frac{\partial f_i}{\partial x_N}\right|_{\underline{x}_0} \right] \Delta \underline{x}(t)$$

Now let us expand to include input and disturbance vector:

$$\frac{dx_i(t)}{dt} = f_i\left(\underline{x}(t), \underline{u}(t), \underline{d}(t)\right)$$

Which will have the linearisation about the operating point:

$$\frac{d}{dt}\Delta x_i(t) = \left[\left.\frac{\partial f_i}{\partial x_1}\right|_{op} \quad \left.\frac{\partial f_i}{\partial x_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial f_i}{\partial x_N}\right|_{op} \right] \Delta \underline{x}(t) + \left[\left.\frac{\partial f_i}{\partial u_1}\right|_{op} \quad \left.\frac{\partial f_i}{\partial u_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial f_i}{\partial u_m}\right|_{op} \right] \Delta \underline{u}(t)$$
$$+ \left[\left.\frac{\partial f_i}{\partial d_1}\right|_{op} \quad \left.\frac{\partial f_i}{\partial d_2}\right|_{op} \quad \cdots \quad \left.\frac{\partial f_i}{\partial d_s}\right|_{op} \right] \Delta \underline{d}(t)$$

which could further be written as:

$$\frac{d}{dt}\Delta x_i(t) = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{iN}] \Delta \underline{x}(t) + [b_{i1} \quad b_{i2} \quad \cdots \quad b_{im}] \Delta \underline{u}(t) + [e_{i1} \quad e_{i2} \quad \cdots \quad e_{is}] \Delta \underline{d}(t)$$

This could of course be repeated for all N state equations:

$$\frac{d}{dt}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_i\\ \vdots\\ \Delta x_N\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1N}\\ a_{21} & a_{22} & \cdots & a_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{iN}\\ \vdots & \vdots & \ddots & \vdots\\ a_{N1} & a_{N2} & \cdots & a_{NN}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_i\\ \vdots\\ \Delta x_N\end{bmatrix}+\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1m}\\ b_{21} & b_{22} & \cdots & b_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ b_{i1} & b_{i2} & \cdots & b_{im}\\ \vdots & \vdots & \ddots & \vdots\\ b_{N1} & b_{N2} & \cdots & b_{Nm}\end{bmatrix}\begin{bmatrix}\Delta u_1\\ \Delta u_2\\ \vdots\\ \Delta u_m\end{bmatrix}$$

$$+\begin{bmatrix}e_{11} & e_{12} & \cdots & e_{1s}\\ e_{21} & e_{22} & \cdots & e_{2s}\\ \vdots & \vdots & \ddots & \vdots\\ e_{i1} & e_{i2} & \cdots & e_{is}\\ \vdots & \vdots & \ddots & \vdots\\ e_{N1} & e_{N2} & \cdots & e_{Ns}\end{bmatrix}\begin{bmatrix}\Delta d_1\\ \Delta d_2\\ \vdots\\ \Delta d_s\end{bmatrix}$$

Or in more compact form as:

Now consider the static output equation for the i$^{\text{th}}$ output:

$$y_i(t)=h_i\left(\underline{x}(t),\underline{u}(t),\underline{d}(t)\right)$$

The linearisation about the operating point $\{\underline{x}_0,\underline{u}_0,\underline{d}_0\}$ is:

$$\Delta y_i(t)=\left[\left.\frac{\partial h_i}{\partial x_1}\right|_{op}\quad \left.\frac{\partial h_i}{\partial x_2}\right|_{op}\quad \cdots\quad \left.\frac{\partial h_i}{\partial x_N}\right|_{op}\right]\Delta\underline{x}(t)+\left[\left.\frac{\partial h_i}{\partial u_1}\right|_{op}\quad \left.\frac{\partial h_i}{\partial u_2}\right|_{op}\quad \cdots\quad \left.\frac{\partial h_i}{\partial u_m}\right|_{op}\right]\Delta\underline{u}(t)$$
$$+\left[\left.\frac{\partial h_i}{\partial d_1}\right|_{op}\quad \left.\frac{\partial h_i}{\partial d_2}\right|_{op}\quad \cdots\quad \left.\frac{\partial h_i}{\partial d_s}\right|_{op}\right]\Delta\underline{d}(t)$$

which could be written as:

$$\Delta y_i(t) = \begin{bmatrix} c_{i1} & c_{i2} & \cdots & c_{iN} \end{bmatrix} \Delta \underline{x}(t) + \begin{bmatrix} d_{i1} & d_{i2} & \cdots & d_{im} \end{bmatrix} \Delta \underline{u}(t) + \begin{bmatrix} f_{i1} & f_{i2} & \cdots & f_{is} \end{bmatrix} \Delta \underline{d}(t)$$

This could be repeated for all the P outputs:

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \vdots \\ \Delta y_i \\ \vdots \\ \Delta y_P \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ c_{P1} & c_{P2} & \cdots & c_{PN} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_i \\ \vdots \\ \Delta x_N \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i1} & d_{i2} & \cdots & d_{im} \\ \vdots & \vdots & \ddots & \vdots \\ d_{P1} & d_{P2} & \cdots & d_{Pm} \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \\ \Delta u_m \end{bmatrix}$$

$$+ \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1s} \\ f_{21} & f_{22} & \cdots & f_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i1} & f_{i2} & \cdots & f_{is} \\ \vdots & \vdots & \ddots & \vdots \\ f_{P1} & f_{P2} & \cdots & f_{Ps} \end{bmatrix} \begin{bmatrix} \Delta d_1 \\ \Delta d_2 \\ \vdots \\ \Delta d_s \end{bmatrix}$$

which again could be written in more compact form as:

This yields the following linear state-space model for, a process about a particular operating point:

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t) + E\underline{d}(t)$$
$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) + F\underline{d}(t)$$

1.2.1 Linearisation Examples:

i) The magnetic suspension system

$$m\frac{d^2h(t)}{dt^2} = mg - \frac{Ki^2(t)}{h^2(t)}$$

$$L\frac{di(t)}{dt} = v(t) - Ri(t)$$

Find a linear model about the desired operating airgap h=0.01m

The process parameters are:

L=10mH M=0.05 Kg   g=$10ms^{-2}$ R=1 ohm   K=$2x10^{-4}Nm^2/A^2$

First find the operating point,

From the force equation:

$$m\frac{d^2h(t)}{dt^2} = mg - \frac{Ki^2(t)}{h^2(t)} =$$

From the electrical equation:

$$L\frac{di(t)}{dt} = v(t) - Ri(t) =$$

The operating point vector will now be defined as:

$$\underline{x}_0 = \begin{bmatrix} h_0 \\ \dot{h}_0 \\ i_0 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Now linearise about this operating point vector:

The state equations can be written as:

$$\dot{x}_1 = x_2(t)$$

$$\dot{x}_2 = g - \frac{Kx_3^2(t)}{mx_1^2(t)}$$

$$\dot{x}_3 = \frac{1}{L}\left(u(t) - Rx_3(t)\right)$$

Now define deviations of the states from their operating point values as:

Hence the linearised model is:

$$\frac{d}{dt}\Delta x_1(t) = \left.\frac{\partial f_1}{\partial x_2}\right|_{OP} \Delta x_2(t)$$

$$\frac{d}{dt}\Delta x_2(t) = \left.\frac{\partial f_2}{\partial x_1}\right|_{OP} \Delta x_1(t) + \left.\frac{\partial f_2}{\partial x_3}\right|_{OP} \Delta x_3(t)$$

$$\frac{d}{dt}\Delta x_3(t) = \left.\frac{\partial f_3}{\partial x_3}\right|_{OP} \Delta x_3(t) + \left.\frac{\partial f_2}{\partial u}\right|_{OP} \Delta u(t)$$

But we know that:

$$f_1(x_2) = x_2$$

$$f_2(x_1, x_3) = g - \frac{Kx_3^2}{Mx_1^2}$$

$$f_3(x_3, u) = \frac{u}{L} - \frac{R}{L}x_3$$

This yields the following linearised model about the operating point:

$$\frac{d}{dt}\Delta x_1(t) = \Delta x_2(t)$$

$$\frac{d}{dt}\Delta x_2(t) =$$

$$\frac{d}{dt}\Delta x_3(t) = -100\Delta x_3(t) + 100\Delta u(t)$$

And of course:

$$\Delta y(t) = h(t) - h_0 = \Delta x_1(t)$$

This then could be written in matrix form as:

$$\frac{d}{dt}\begin{bmatrix}\Delta x_1 \\ \Delta x_2 \\ \Delta x_3\end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix}\Delta x_1 \\ \Delta x_2 \\ \Delta x_3\end{bmatrix} + \begin{bmatrix} \quad \\ \\ \end{bmatrix}\Delta u(t)$$

$$\Delta y(t) = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}\begin{bmatrix}\Delta x_1 \\ \Delta x_2 \\ \Delta x_3\end{bmatrix}$$

<u>ii) Permanent magnet DC motor</u>

Could be modelled by the following coupled equations:

$$\frac{di}{dt} = \frac{1}{L}(v(t) - Ri(t) - K_m\omega(t))$$

$$\frac{d\omega}{dt} = \frac{1}{J}(K_M i(t) - B\omega(t) - T_L(t))$$

As you can see these are linear differential equations:

If a tachometer of gain K_T V/rads^{-1} is used to measure the speed, then the output equation could be written as:

Consider now that the motor is driving a nonlinear fan load:

$$T_L(t) = K_F \omega^2(t)$$

The process then would be modelled as:

$$\frac{di}{dt} = \frac{1}{L}(v(t) - Ri(t) - K_m \omega(t))$$

$$\frac{d\omega}{dt} = \frac{1}{J}(K_M i(t) - B\omega(t) - K_F \omega^2(t))$$

Generate a linear state-space model which describes the dynamics of this process close to the operating speed:

First find the operating point:

$$\frac{d\omega}{dt} = \frac{1}{J}(K_M i(t) - B\omega(t) - K_F \omega^2(t)) =$$

Now from the electrical equation:

$$\frac{di}{dt} = \frac{1}{L}(v(t) - Ri(t) - K_m \omega(t)) =$$

Assign the states: $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} i(t) \\ \omega(t) \end{bmatrix}$

and input and output: $u(t) = v(t)$ and $y(t) = v_T(t) = K_T \omega(t)$

hence:

$$\dot{x}_1(t) = \frac{1}{L}(u(t) - Rx_1(t) - K_m x_2(t)) = f_1(x_1, x_2, u)$$

$$\dot{x}_2(t) = \frac{1}{J}(K_M x_1(t) - Bx_2(t) - K_F x_2^2(t)) = f_2(x_1, x_2)$$

$$y(t) = K_T x_2(t) = h(x_2)$$

The linearised model about the operating point is:

$$\frac{d}{dt}\Delta x_1(t) = \left.\frac{\partial f_1}{\partial x_1}\right|_{OP} \Delta x_1(t) + \left.\frac{\partial f_1}{\partial x_2}\right|_{OP} \Delta x_2(t) + \left.\frac{\partial f_1}{\partial u}\right|_{OP} \Delta u(t)$$
$$\frac{d}{dt}\Delta x_2(t) = \left.\frac{\partial f_2}{\partial x_1}\right|_{OP} \Delta x_1(t) + \left.\frac{\partial f_2}{\partial x_2}\right|_{OP} \Delta x_2(t)$$
$$\Delta y(t) = \left.\frac{\partial h}{\partial x_2}\right|_{OP} \Delta x_2(t)$$

Obviously the electrical equation is linear:

$$\dot{x}_1(t) = \frac{1}{L}(u(t) - Rx_1(t) - K_m x_2(t)) = f_1(x_1, x_2, u)$$

Now concentrating on the mechanical equation:

$$\dot{x}_2(t) = \frac{1}{J}(K_M x_1(t) - Bx_2(t) - K_F x_2^2(t)) = f_2(x_1, x_2)$$

And simply for the output equation:

$$y(t) = K_T x_2(t) = h(x_2)$$

This yields the following linear state-space model:

$$\frac{d}{dt}\Delta \underline{x} = \left[\begin{array}{cc} \qquad\qquad & \qquad\qquad \\ & \\ & \end{array} \right] \Delta \underline{x}(t) + \left[\begin{array}{c} \quad \\ \\ \end{array} \right] \Delta u(t)$$

$$\Delta y(t) = \left[\qquad\qquad\qquad \right] \Delta \underline{x}(t)$$