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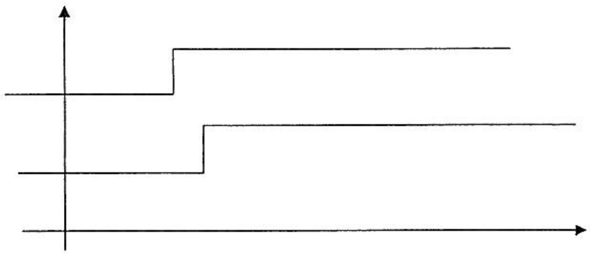
Chapter 4. Design of State-Space Servo-Controllers

4.1 Introducing the Reference Signal

It is desired that the process output vector should follow a specified vector of setpoints:



Consider the SISO process:

$$\underline{\dot{x}} = A\underline{x} + Bu$$
$$y = C\underline{x}$$

For a steady state desired output y_{ss} we have:

Propose the control-law:

$$u(t) = u_{ss} - K(\underline{x}(t) - \underline{x}_{ss})$$

In the steady state:

$$\underline{O} = A\underline{x}_{ss} + Bu_{ss}$$
$$y_{ss} = C\underline{x}_{ss}$$

Let us propose the simple relationships:

$$u_{ss} = N_u r_{ss} \qquad \underline{x}_{ss} = N_x r_{ss}$$

This implies:

$$\underline{O} = AN_x r_{ss} + BN_u r_{ss}$$
$$y_{ss} = CN_x r_{ss}$$

Of course if we want the steady state output to be r_{ss}:

Then the gains N_x and N_u can be determined as:

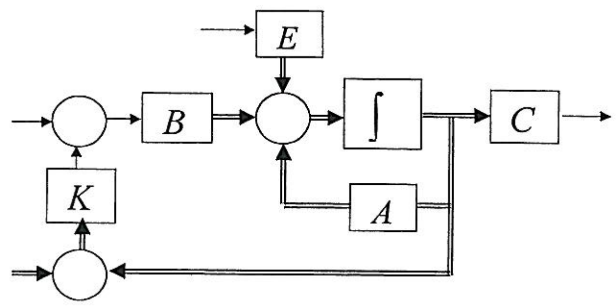
$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

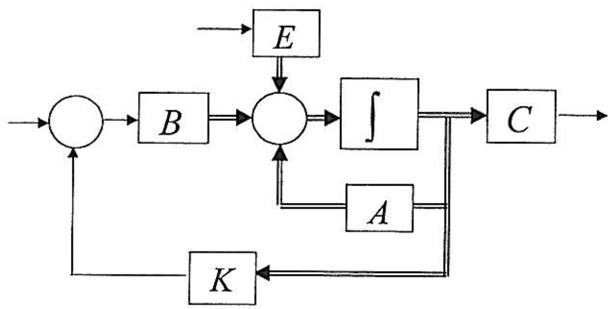
The control-law is then:

$$u(t) = N_u r(t) - K(x(t) - N_x r(t))$$

Which could of course be rewritten as:

This yields the following control structures:





Tutorial: Introduce a setpoint signal to the motor speed controller developed earlier – test in Simulink.

Some of the problems of this technique include:

- Does not increase the system type:

- Gains designed to reduce e_{ss} for setpoint changes:

4.2 State-Space Control with Integral Action

Consider again the SISO process:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$y(t) = C\underline{x}(t)$$

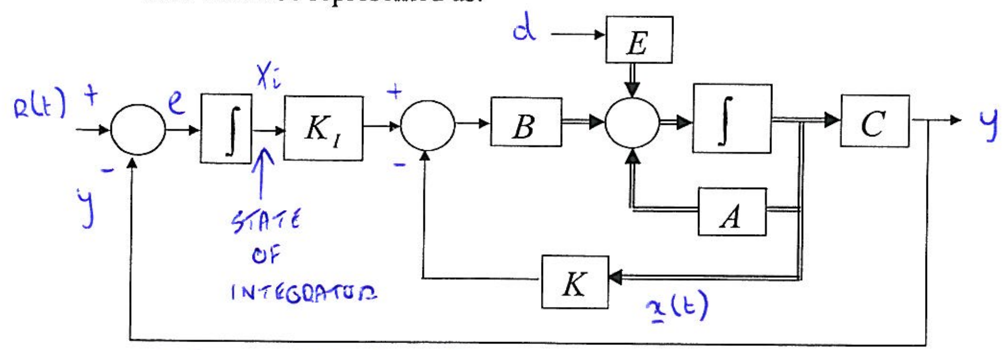
The state control-law with integral action is:

$$u(t) = -K\underline{x}(t) + K_I \int_0^t e(\tau)d\tau$$

where: $e(t) = r(t) - y(t)$

we want $y(t)$ to track setpoint $r(t)$

This could be represented as:



we have introduced an outer feedback loop

Introduce another state:

$$x_I(t) = \int_0^t e(\tau)d\tau$$ OUTPUT OF CONTROLLER INTEGRATOR

The control-law become:

$$u(t) = -K\underline{x}(t) + K_I x_I$$

This yields the closed-loop state equation:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underbrace{\left(-K\underline{x}(t) + K_I x_I(t)\right)}_{u(t)}$$

But we know:

$$x_I(t) = \int_0^t e(\tau)d\tau$$

$\therefore \dot{x}_I(t) = e(t) = r(t) - y(t) = r(t) - C\underline{x}(t)$

Hence the complete closed loop system can be represented by the coupled equations:

N+1 states

$$\begin{cases} \dot{\underline{x}}(t) = (A - BK)\underline{x}(t) + BK_I x_I(t) & \leftarrow \text{PROCESS} \\ \dot{x}_I(t) = -C\underline{x}(t) + r(t) & \leftarrow \text{CONTROLLER} \end{cases}$$

Assign a new state vector:

$$\underline{z}(t) = \begin{bmatrix} \underline{x}(t) \\ x_I(t) \end{bmatrix} \begin{matrix} \updownarrow N \\ \updownarrow 1 \end{matrix} \in \mathbb{R}^{N+1}$$

The closed-loop equations can be written more compactly as:

CLOSED LOOP

$$\frac{d}{dt}\begin{bmatrix} \underline{X}(t) \\ X_I \end{bmatrix} = \begin{bmatrix} A-BK & BK_I \\ -C & 0 \end{bmatrix}\begin{bmatrix} \underline{X}(t) \\ X_I \end{bmatrix} + \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix} r(t)$$

$$\dot{\underline{z}} = A_2\underline{z} + B_2 r(t)$$

The poles of the closed-loop system are given by the roots of:

$$\det(sI - A_2) = 0$$

N.B: There are (N+1) closed loop poles

Determine the gains to place the N+1 closed-loop poles to obtain the following characteristic equation:

N.B.

$$C_{des}(s) = s^{N+1} + C_N s^N + \cdots C_1 s + C_0$$

Proof of Integral Action: Assume closed-loop stability

Consider an asymptotically constant setpoint signal:

"step like" $\lim_{t\to\infty} r(t) = R_{ss}$

Since the closed-loop system is stable – the states must converge to steady-state values: reach equilibrium

$\lim_{t\to\infty} \dot{\underline{x}} = 0 \rightarrow \underline{0} = (A - BK)\underline{x}_{ss} + BK_I x_{I\,ss}$

$\lim_{t\to\infty} \dot{X}_I = 0 \rightarrow 0 = -C\underline{x}_{ss} + r_{ss}$

$x(t) \rightarrow x_{ss}$

$r(t) \rightarrow R_{ss}$

$y(t) \rightarrow y_{ss}$

$y_{ss} = Cx_{ss}$

hence: $0 = -C\underline{x}_{ss} + R_{ss}$

$0 = -y_{ss} + R_{ss}$

$y_{ss} = R_{ss} \quad e_{ss} = 0$

EXAMPLE: The DC Motor

$$\frac{d}{dt}\begin{bmatrix} \omega(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} 0 & 50 \\ -200 & -200 \end{bmatrix}\begin{bmatrix} \omega(t) \\ i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 200 \end{bmatrix} v(t) + \begin{bmatrix} -50 \\ 0 \end{bmatrix} T_L(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \omega(t) \\ i(t) \end{bmatrix}$$

SPEED CONTROL $y(t) = \omega(t)$

Use the control-law

$$v(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} \omega(t) \\ i(t) \end{bmatrix} + K_I \int_0^t \underbrace{(r_\omega(\tau) - y(\tau))}_{\text{SPEED ERROR}} d\tau$$

Then :

$$\dot{x}_I = R_\omega(t) - \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \omega \\ i \end{bmatrix}$$

Hence the closed loop state-equation is:

A-BK   BK_I

$$\frac{d}{dt}\begin{bmatrix}\omega(t)\\ i(t)\\ \hdashline x_I(t)\end{bmatrix}=\left[\begin{array}{cc:c}0 & 50 & 0\\ -200-200k_1 & -200-200k_2 & 200K_I\\ \hdashline -1 & 0 & 0\end{array}\right]\begin{bmatrix}\omega(t)\\ i(t)\\ \hdashline x_I(t)\end{bmatrix}+\begin{bmatrix}0\\0\\ \hdashline 1\end{bmatrix}r(t)+\begin{bmatrix}-50\\0\\ \hdashline 0\end{bmatrix}T_L(t)$$

-C

The poles of the closed-loop system are given by roots of:

3rd order

sI - A₂

$$\det\left(\left[\begin{array}{cc:c}s & 0 & 0\\ 0 & s & 0\\ \hdashline 0 & 0 & s\end{array}\right]-\left[\begin{array}{cc:c}0 & 50 & 0\\ -200-200k_1 & -200-200k_2 & 200K_I\\ \hdashline -1 & 0 & 0\end{array}\right]\right)=0$$

which yields:

$$\det\left(\left[\begin{array}{cc:c}s & -50 & 0\\ 200+200k_1 & s+200+200k_2 & -200K_I\\ \hdashline 1 & 0 & s\end{array}\right]\right)=0$$

+ - +

the closed-loop characteristic equation is then:

$$1\begin{vmatrix}-50 & 0\\ s+200+200k_2 & -200k_I\end{vmatrix}+s\begin{vmatrix}s & -50\\ 200+200k_I & s+200+200k_2\end{vmatrix}=0$$

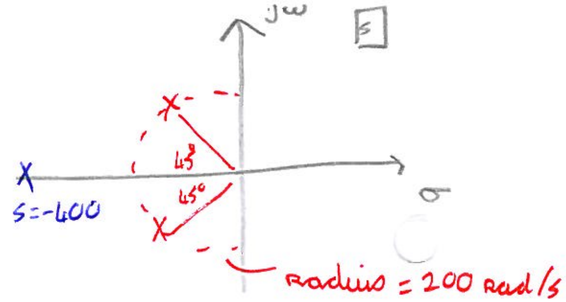
$$s^3+200(1+k_2)s^2+10{,}000(1+k_I)s+10000k_I=0$$

Now we still want a second order dominant response with :

$\xi = 0.707 \qquad \omega_N = 200 \text{ rad/s}$

This can be achieved by placing the controller pole further out left:

CONTROLLER POLE IS FURTHER OUT LEFT



The desired closed-loop characteristic equation is:

FAST   SLOW $s^2+2\xi\omega_N s+\omega_N^2$

$$C_{des}(s)=(s+400)(s^2+282.8s+40000)$$
$$=s^3+682.8s^2+153120s+16000000$$

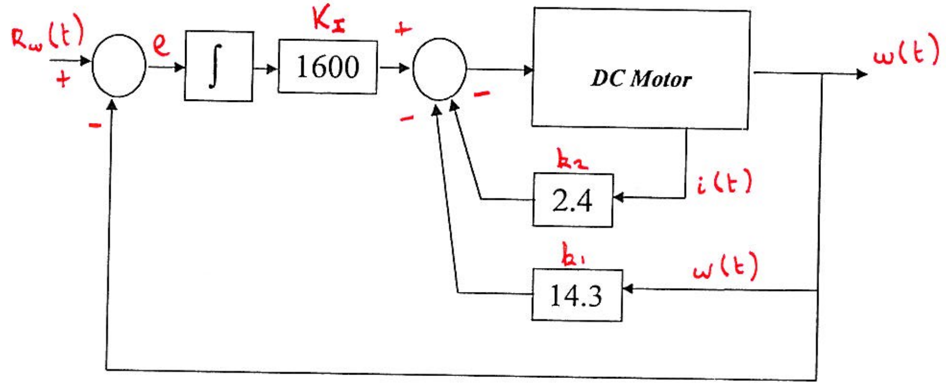
Compare with the closed-loop characteristic equation:

$10000K_I = 16000000$ ⇒ $K_I = 1600$
$10000(1+k_1)=153120$ ⇒ $k_1 = 14.3$
$200(1+k_2)=682.8$ ⇒ $k_2 = 2.4$

This yields the controller:

$$v(t)=-14.3\omega(t)-2.4i(t)+1600\int_0^t\left(r_\omega(\tau)-\omega(\tau)\right)d\tau$$

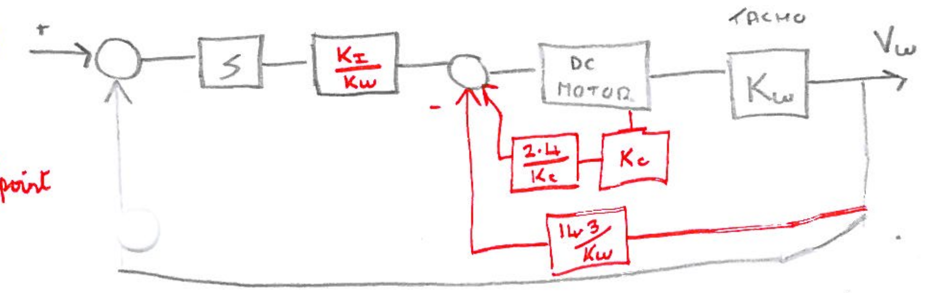
which could be built as follows:



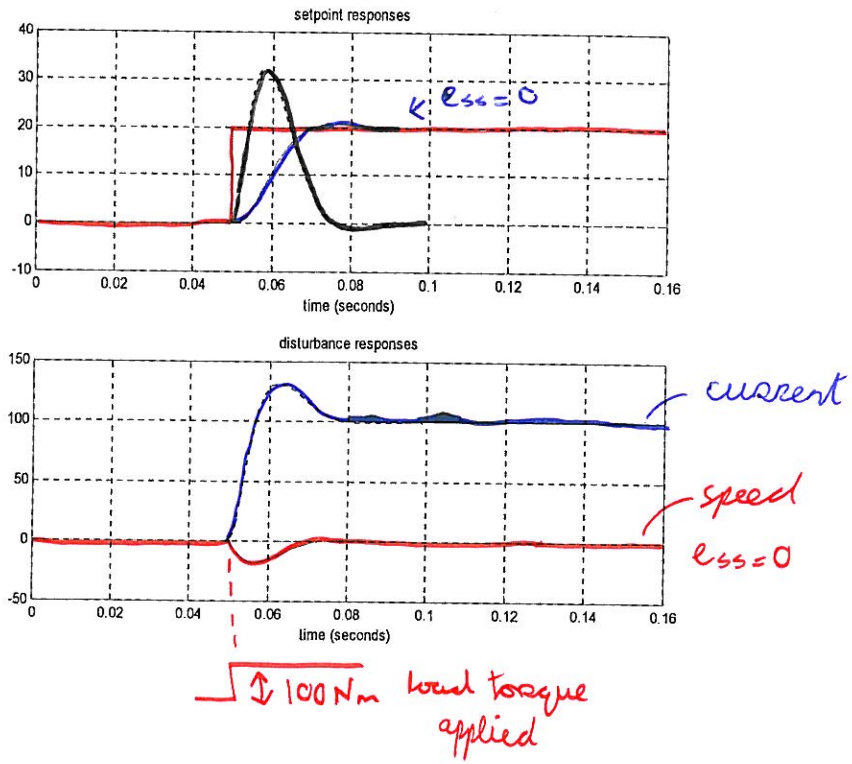
In practice

Rω(t) in volts

Kω X speed setpoint



The closed-loop responses are:



4.2.1 Use of Ackermans Method to Design Controllers with Integral Action

Define the open-loop equations as:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$\dot{x}_I(t) = r(t) - C\underline{x}(t)$$

Or in more compact form as:

OPEN LOOP EQUATIONS

$$\frac{d}{dt}\begin{bmatrix} \underline{x} \\ \hline x_I \end{bmatrix} = \begin{bmatrix} \overset{\longleftrightarrow N}{A_{N\updownarrow}} & \underline{0} \\ \hline -C & 0 \end{bmatrix}\begin{bmatrix} \underline{x} \\ \hline x_I \end{bmatrix} + \begin{bmatrix} B \\ \hline 0 \end{bmatrix} u(t) + \begin{bmatrix} \underline{0} \\ \hline 1 \end{bmatrix} r(t)$$

$$\dot{\underline{z}} = A_2\underline{z} + B_2u + G_R$$

Which is under the control:

$$u(t) = -K\underline{x}(t) + K_I x_I \qquad u = -k_2 \underline{z}(t)$$

or:

$$u(t) = -\left[K \mid -K_I\right]\underline{z}(t)$$

We can then use Ackermann's formula:

$$k_2 = \left[K \mid -K_I\right] = \underbrace{\left[0 \quad 0 \quad \cdots \quad 0 \quad 1\right]}_{N+1} C_z^{-1} C_{des}(A_2)$$

where:

$$C_{des}(s) = s^{N+1} + C_N s^N + \cdots C_1 s + C_0 = 0 \qquad A_2 = \begin{bmatrix} A & \underline{0} \\ -C & 0 \end{bmatrix}$$

and:

$$C_z = \left[B_2 \mid A_2B_2 \mid A_2^{\,2}B_2 \mid \cdots \mid A_2^{\,N}B_2\right]$$

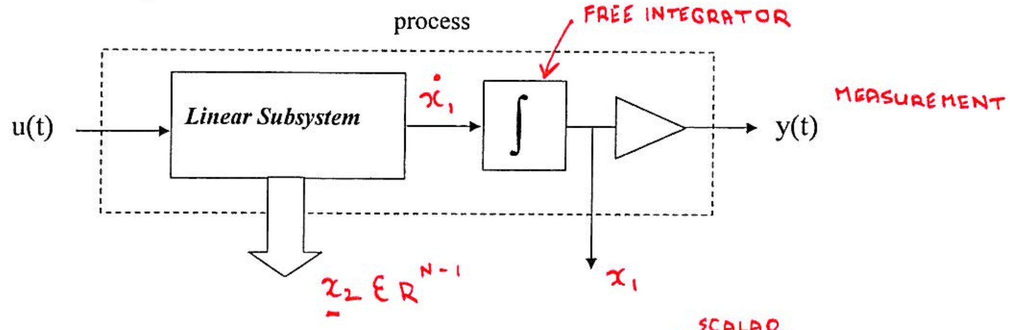
$$\begin{bmatrix} B \\ \hline 0 \end{bmatrix} \qquad \begin{bmatrix} A & \underline{0} \\ -C & 0 \end{bmatrix}$$

4.2.2 Use of Inherent Integral Action

Consider the SISO process model:

$$\dot{\underline{x}} = A\underline{x} + Bu$$
$$y = C\underline{x}$$

of the special form:



The state of the N^th order process is then: $\underline{x}(t) = \begin{bmatrix} x_1 \\ \underline{x}_2 \end{bmatrix}$ SCALAR N N-1

The output is: $y(t) = [c_1 \mid 0 \quad 0 \quad \cdots \quad 0]\underline{x}(t)$

We want the output y(t) to track the setpoint r(t). Consider that r(t) is asymptotically constant:

$\lim_{t\to\infty} R(t) = R_{ss}$ $\lim_{t\to\infty} y(t) \to R_{ss}$

Consider the standard tracking controller structure:

$$u(t) = \underbrace{N_u r(t)}_{u_{ss}} - K(\underline{x}(t) - \underbrace{\underline{N}_x r(t)}_{x_{ss}})$$

with the choice:

$$N_u = 0 \qquad N_x = \begin{bmatrix} 1/c_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The control-law becomes:

$$u(t) = -K\left(\underline{x}(t) - \underbrace{\begin{bmatrix} 1/c_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{x_{ss}} r(t)\right)$$

units of $x_1(t)$ volts

$$R_x = \frac{R(t)}{c_1}$$

Or:

$$u(t) = -K\begin{bmatrix} x_1 - R_x \\ \underline{x}_2 \\ \vdots \\ x_N \end{bmatrix}$$

Which could be written as:

$x_1 = \frac{y(t)}{c_1}$

$$u(t) = -K\begin{bmatrix} \frac{y(t)-r(t)}{c_1} \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = -\left[\frac{k_1}{c_1} \mid K_2\right]\begin{bmatrix} y(t)-r(t) \\ \underline{x}_2(t) \end{bmatrix}$$

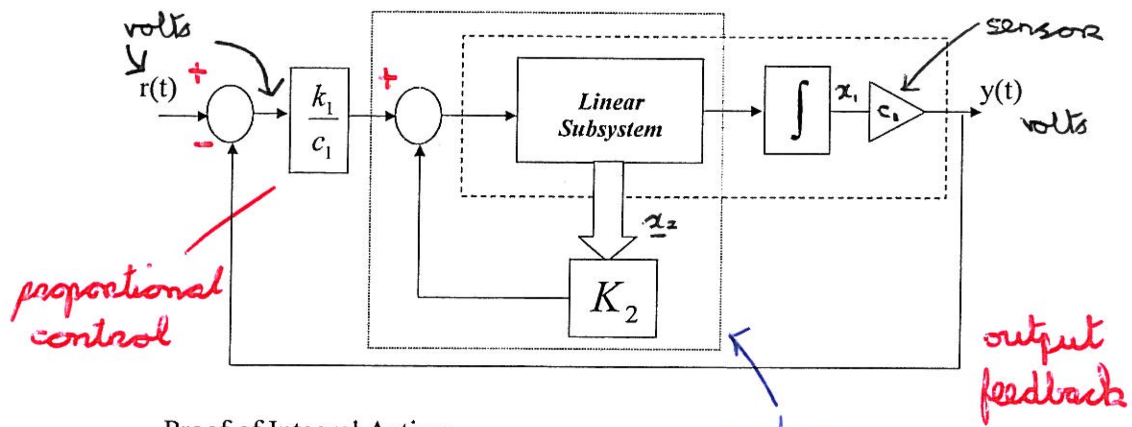
New K matrix

$k = [k_1 \mid K_2]$

in terms of output rather than x_1

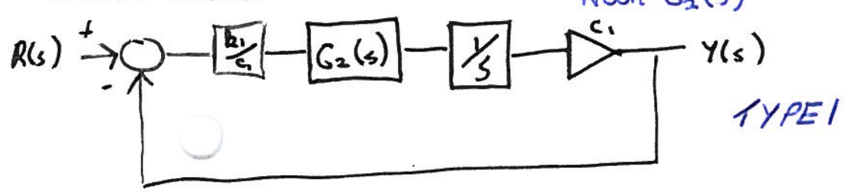
$$= \frac{k_1}{c_1}(R-y) - K_2\underline{x}_2$$

This could be represented by:



Proof of Integral Action:

Process is stable



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