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Chapter 2. State-Space Theory

2.1 Relationship between the State-Space and Transfer Function Representations

Consider the linear N[th] order process, with M inputs and P outputs, (for simplicity lets first assume no disturbances):

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t) \quad (1)$$
$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \quad (2)$$

Now apply the Laplace transform: to (1)

INITIAL CONDITION IN TIME DOMAIN

First some revision: $L\{\frac{d}{dt} x_i(t)\} = s X_i(s) - X_i(0)$

$$L\{\underline{\dot{x}}(t)\} = \begin{bmatrix} L\{\dot{x}_1(t)\} \\ L\{\dot{x}_2(t)\} \\ \vdots \\ L\{\dot{x}_N(t)\} \end{bmatrix} = \begin{bmatrix} sX_1(s) \\ sX_2(s) \\ \vdots \\ sX_N(s) \end{bmatrix} - \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_N(0) \end{bmatrix}$$

$L\{\underline{\dot{x}}\} = s\underline{X}(s) - \underline{x}(0)$ $\quad \underline{X}(s) = L\{\underline{x}(t)\}$

Hence the Laplace transform of equation (1) yields:

$s\underline{X}(s) - \underline{x}(0) = A\underline{X}(s) + B\underline{U}(s)$ (3) $\quad \underline{U}(s) = L\{\underline{u}(t)\}$

Similarly the Laplace transform of equation (2) yields:

$$\underline{Y}(s) = C\underline{X}(s) + D\underline{U}(s)$$

Rearranging equation (3):

$$s\underline{X}(s) - A\underline{X}(s) = B\underline{U}(s) + \underline{x}(0)$$

NXN identity

$(s\underset{}{I} - A)\underline{X}(s) = B\underline{U}(s) + \underline{x}(0)$

Can now solve for the state by premulyiplying both sides of equation (4) by $(sI-A)^{-1}$:

$$(sI - A)^{-1}(sI - A)\underline{X}(s) = (sI - A)^{-1}(B\underline{U}(s) + \underline{x}(0))$$

$\underline{X}(s) = (sI - A)^{-1}[B\underline{U}(s) + \underline{x}(0)]$

Of course we know that: $\underline{Y}(s) = C\underline{X}(s) + D\underline{U}(s)$

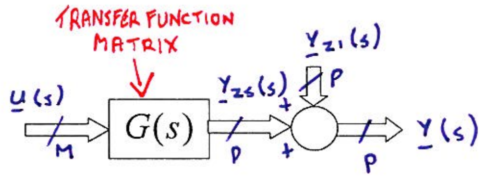
Then we get an expression for the output in terms of the input and the initial state vector:

$$\underline{Y}(s) = C\underbrace{\boxed{(sI - A)^{-1}(B\underline{U}(s) + \underline{x}(0))}}_{}^{\underline{X}(s)} + D\underline{U}(s)$$

which could be written as:

$$\underline{Y}(s) = \underbrace{(C(sI - A)^{-1}B + D)\underline{U}(s)}_{\underline{Y}_{zs}(s)} + \underbrace{C(sI - A)^{-1}\underline{x}(0)}_{\underline{Y}_{zi}(s)}$$

This could be represented by the following block diagram:



$\underline{Y}_{zi}(s) \Rightarrow$ zero input response (free)

$\underline{Y}_{zs}(s) \Rightarrow$ zero state response (forced)

Hence the MIMO transfer function of the process is:

$$G(s) = C(sI - A)^{-1}B + D$$

defined as $\underline{Y} = G\underline{u}$ not $G = Y/u$

Now if it is assumed that: $\underline{x}(0) = 0$

$\underline{Y}(s) = \underline{Y}_{zs}(s) = G(s)\underline{U}(s)$

Or:

$$P\left\{\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_P(s) \end{bmatrix} = \overbrace{\begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1M}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2M}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{P1}(s) & G_{P2}(s) & \cdots & G_{PM}(s) \end{bmatrix}}^{M} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_M(s) \end{bmatrix}\right\}M$$

PXM MATRIX OF TRANSFER FUNCTIONS

Consider now an expression for the i[th] output:

$$Y_i(s) = G_{i1}(s)U_1(s) + G_{i2}(s)U_2(s) + \cdots G_{iM}(s)U_M(s)$$

PRINCIPLE OF SUPERPOSITION

Note: the efficiency of the state-space model:

The state-space model of an N[th] order process with P outputs and M inputs requires: N states

A transfer function matrix model of the same process requires PxM separate transfer functions: each possibly of order N

at worst NXPXM integrators

\# STATE SPACE IS THE MINIMAL REPRESENTATION

2.1.1 Determining the Poles of a State-Space Model

First some matrix revision: $R^{-1}_{(N\times N)} = \dfrac{adj(R)}{\det(R)}$ ← (NXN)

Hence we can write:

$$(sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)}$$

NXN MATRIX

SCALAR POLYNOMIAL IN S

and the transfer function matrix is:

$$G(s) = C(sI - A)^{-1}B + D = \frac{C\,adj(sI - A)\,B}{\det(sI - A)} + D$$
$$= \frac{C\,adj(sI - A)\,B + \det(sI - A)D}{\det(sI - A)}$$

MATRIX PXM $F(s)$ OF POLYNOMIALS

POLYNOMIAL $E(s)$

Hence G(s) could be written as:

$F_{ij}(s)$ - scalar polynomial in s

$$G(s) = \overbrace{\begin{bmatrix} \frac{F_{11}(s)}{E(s)} & \frac{F_{12}(s)}{E(s)} & \cdots & \frac{F_{1M}(s)}{E(s)} \\ \frac{F_{21}(s)}{E(s)} & \frac{F_{22}(s)}{E(s)} & \cdots & \frac{F_{2M}(s)}{E(s)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F_{P1}(s)}{E(s)} & \frac{F_{P2}(s)}{E(s)} & \cdots & \frac{F_{PM}(s)}{E(s)} \end{bmatrix}}^{M} \Big\} P$$

all TF elements have a common denominator

The characteristic equation of each transfer function of G(s) is the same $E(s) = 0$

Hence the N poles of the N[th] order process are given by the N roots of the characteristic polynomial:

$$\det(sI - A) = 0$$

Tutorial: Determine the transfer function and the poles of :

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&-1&-1\\0&0&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\1\\1\end{bmatrix}u(t)$$ 1 input

$$\underline{y}(t)=\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

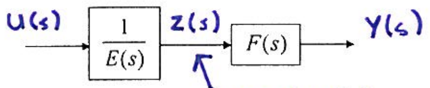
3 states ∴ 3 poles

2.2 How to Obtain the State-Space Representation from the Transfer Function

Consider for simplicity the SISO Nth order process:

$$\frac{Y(s)}{U(s)}=G(s)=\frac{f_r s^r+f_{r-1}s^{r-1}+\cdots+f_0}{s^N+e_{N-1}s^{N-1}+\cdots+e_0}=\frac{F(s)}{E(s)}$$

SINCE G(s) IS LINEAR



INTRODUCE A DUMMY VARIABLE

Therefore we can write:

$$\frac{Z(s)}{U(s)}=\frac{1}{E(s)}=\frac{1}{s^N+e_{N-1}s^{N-1}+\cdots+e_0}$$

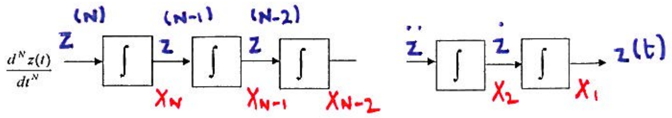
Which could be written as:

$$s^N Z(s)=U(s)-e_{N-1}s^{N-1}Z(s)-\cdots-e_1 sZ(s)-e_0 Z(s)$$

Taking the inverse Laplace transform yields the Nth order differential equation:

$$\frac{d^N}{dt^N}z(t)=u(t)-e_{N-1}\frac{d^{N-1}}{dt^{N-1}}z(t)\ldots-e_0 z(t)$$

Transform this into N first order equations:



Now assign the state vector:

$$\underline{x}(t)=\begin{bmatrix}x_1(t)\\x_2(t)\\\vdots\\x_N(t)\end{bmatrix}=\begin{bmatrix}z(t)\\\dot{z}(t)\\\vdots\\z^{(N-1)}(t)\end{bmatrix}$$ OUTPUTS OF N INTEGRATORS

Then:

$$\frac{d}{dt}\underline{x}(t)=\frac{d}{dt}\begin{bmatrix}z(t)\\\dot{z}(t)\\\vdots\\z^{(N-1)}(t)\end{bmatrix}=\begin{bmatrix}\dot{z}(t)\\\ddot{z}(t)\\\vdots\\z^{N}(t)\end{bmatrix}=\begin{bmatrix}x_2\\x_3\\\vdots\\x_N\\\hline z^{(N)}\end{bmatrix}$$

But we know that:

$$\frac{d^N z(t)}{dt^N}=U(s)-e_{N-1}\frac{d^{N-1}z(t)}{dt^{N-1}}-\cdots-e_1\frac{dz(t)}{dt}-e_0 z(t)$$

$$=u(t)-e_{N-1}X_N-\ldots-e_1X_2-e_0X_1$$

Could be written in matrix form as:

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\\\vdots\\x_N(t)\end{bmatrix}=\begin{bmatrix}0&1&0&0&\cdots&0\\0&0&1&0&\cdots&0\\0&0&0&1&\cdots&0\\\vdots&&&&&\\0&0&0&0&\cdots&1\\\hline -e_0&-e_1&-e_2&\cdots&&-e_{N-1}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\\vdots\\x_N(t)\end{bmatrix}+\begin{bmatrix}0\\0\\0\\\vdots\\0\\\hline 1\end{bmatrix}u(t)$$

The output equation was given by:

$$Y(s)=\boxed{F(s)Z(s)}=(f_r s^r+f_{r-1}s^{r-1}+\cdots+f_0)Z(s)$$

Taking inverse Laplace transforms yields:

$$y(t)=f_r\frac{d^r}{dt^r}z(t)+f_{r-1}\frac{d^{r-1}}{dt^{r-1}}z(t)+\ldots+f_0 z(t)$$

But: $z(t)=x_1$ $\quad z^{(1)}=x_2$ $\quad z^{(2)}=x_3$ $\quad z^{(r-1)}=x_r$ $\quad z^{(r)}=x_{r+1}$

Hence the output equation is now:

$$y(t)=f_0x_1+f_1x_2+\ldots+f_rx_{r+1}$$

$$y(t)=[f_0\ f_1\ \ldots\ f_r \mid 0\ 0\ \ldots\ 0]\begin{bmatrix}x_1(t)\\x_2(t)\\\vdots\\x_{r+1}(t)\\\hline x_{r+2}(t)\\\vdots\\x_N(t)\end{bmatrix}$$

2.2.1 The Control Canonical Form

Hence the SISO Nth order process:

N.B. NORMALISED

$$\frac{Y(s)}{U(s)}=G(s)=\frac{f_r s^r+f_{r-1}s^{r-1}+\cdots+f_0}{s^N+e_{N-1}s^{N-1}+\cdots+e_0}=$$

can be represented by the SISO state-space equations:

$$\dot{\underline{x}}(t)=A\underline{x}(t)+Bu(t)$$
$$y(t)=C\underline{x}(t)$$

$$\underline{e}=\begin{bmatrix}e_0\\\vdots\\e_{N-1}\end{bmatrix}$$

where:

$$A=\begin{bmatrix}0&1&0&0&\cdots&0\\0&0&1&0&\cdots&0\\0&0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\cdots&1\\-e_0&-e_1&-e_2&-e_3&\cdots&-e_{N-1}\end{bmatrix}=\begin{bmatrix}\underline{0}_{N-1}&I_{N-1}\\\hline -\underline{e}^T\end{bmatrix}$$

NULL VECTOR

all coeffs of denom of G(s) are in here

$$B=\begin{bmatrix}0\\0\\0\\\vdots\\0\\1\end{bmatrix}=\begin{bmatrix}\underline{0}_{N-1}\\\hline 1\end{bmatrix}$$

$$C=[\underbrace{f_0\ f_1\ \cdots\ f_r}_{r+1}\mid 0\ \cdots\ 0]=[\underline{f}^T \mid \underline{0}^T_{N-r-1}]$$

$$\underline{f}=\begin{bmatrix}f_0\\\vdots\\f_r\end{bmatrix}$$

*This is called the CONTROL CANONICAL FORM

Example: $G(s) = \dfrac{s+2}{s(s+1)(s+3)} = \dfrac{\overset{b_1}{1}s + \overset{b_0}{2}}{s^3 + \underset{\uparrow a_2}{4}s^2 + \underset{\uparrow a_1}{3}s + \underset{\uparrow a_0}{0}}$

Modelled as the following control canonical form:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \underline{x}(t)$$

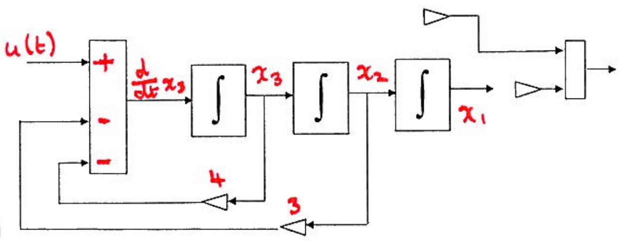
3rd order
$N = 3$

where: $\underline{x}(t) = [x_1(t) \quad x_2(t) \quad x_3(t)]^T$

$\dfrac{dx_1}{dt} = x_2 \quad \dfrac{dx_2}{dt} = x_3 \quad \dfrac{dx_3}{dt} = \dot{x}_3 = 0x_1 - 3x_2 - 4x_3 + u$

$y = 2x_1 + x_2$

Could be represented as the following simulation diagram:



BUT: Remember the tutorial question of section 2.1.1:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \underline{x}(t)$$

$G(s) = \dfrac{s+2}{s(s+1)(s+3)}$

INCONSISTENT??

TWO STATE SPACE MODELS HAVE THE SAME INPUT/OUTPUT BEHAVIOUR. BUT THERE IS ACTUALLY AN INFINITE NUMBER OF STATE SPACE

First some revision: $(QR)^{-1} = R^{-1}Q^{-1}$

The transfer function : $G_2(s) = CT^{-1}(sI - TAT^{-1})^{-1}TB$

$$G_2(s) = C(sIT - TAT^{-1}T)^{-1}TB = C(sIT - TA)^{-1}TB$$

which then could be written as:

$$G_2(s) = C(T^{-1}sIT - T^{-1}TA)^{-1}B = C(T^{-1}sIT - A)^{-1}B$$
$$= C(sT^{-1}IT - A)^{-1}B$$
$$= C(sI - A)^{-1}B$$

Tutorial:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x}(t)$$

i) Determine the transfer function: G(s)=Y(s)/U(s)
ii) Transform the state equations using :

$$T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

iii) Determine the transfer function of the transformed process
iv) Represent each system as a simulation diagram

2.3 State Transformation Theory

Consider the linear MIMO process: N^{th} order

$$\left.\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) \end{aligned}\right\} \text{ORIGINAL REPRESENTATION}$$

Consider an arbitrary transformation of the state vector $\underline{x}(t)$ to a new state vector $\underline{z}(t)$:

$\underline{z}(t) = T\underline{x}(t)$

T is square invertible (non-singular)

$\Rightarrow \underline{x}(t) = T^{-1}\underline{z}(t)$

Rewrite the state equations: $\dot{\underline{x}} = T^{-1}\dot{\underline{z}}$

ORIGINAL

$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$

$T^{-1}\dot{\underline{z}} = AT^{-1}\underline{z} + B\underline{u}(t)$

$\underline{y}(t) = C\underline{x}(t)$

$\underline{y}(t) = CT^{-1}\underline{z}$

The transformed state-equations are then:

$$\dot{\underline{z}}(t) = TAT^{-1}\underline{z}(t) + TB\underline{u}(t)$$
$$\underline{y}(t) = CT^{-1}\underline{z}(t)$$

$\dot{\underline{z}} = A_2\underline{z} + B_2\underline{u}$

$\underline{y} = C_2\underline{z}$

$A_2 = TAT^{-1}$

$B_2 = TB$

$C_2 = CT^{-1}$

Just to prove that this is the same system, only represented differently internally, consider the transfer functions:

Orig.: $G(s) = C(sI - A)^{-1}B$ Trans.: $G_2(s) = C_2(sI - A_2)^{-1}B_2$

$G_2(s) = CT^{-1}(sI - TAT^{-1})^{-1}TB$

2.4 Solution of the State-Space Equations

Consider the state-equation:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$

Taking Laplace transforms yields:

$$\underline{X}(s) = (sI - A)^{-1}(B\underline{U}(s) + \underline{x}(0))$$

$\underline{X}(s) = (sI - A)^{-1}B\underline{u} + (sI - A)^{-1}\underline{x}(0)$

Now solve for the state trajectory using the inverse Laplace transform: $\underline{x}(t) = L^{-1}\{\underline{X}(s)\}$

Revision: $L^{-1}\{W(s)V(s)\} = w(t) \otimes v(t)$

$= \int_0^t w(t-\tau)v(\tau)\,d\tau$

Now if we define : $\Phi(s) = (sI - A)^{-1}$

SQUARE MATRIX $N \times N$ "TRANSITION MATRIX"

Then: $\Phi(t) = L^{-1}\{\Phi(s)\}$
$= L^{-1}\{(sI - A)^{-1}\}$

We can now write:

$$\underline{X}(s) = \Phi(s)\underline{x}(0) + \underbrace{\Phi(s)}_{W(s)}\underbrace{B\underline{U}(s)}_{V(s)}$$

Taking inverse Laplace transforms yields: $\underline{x}(t) = \Phi(t)\underline{x}(0) + L^{-1}\{W(s)V(s)\}$

Which yields the solution of the state-trajetory in the time-domain as: $\underline{x}(t) = \Phi(t)\underline{x}(0) + w(t) \otimes v(t)$

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \int_0^t \underbrace{\Phi(t-\tau)}_{w(t-\tau)}\underbrace{B\underline{u}(\tau)}_{v(\tau)}\,d\tau$$

Now if we assume that D=0 then: $\underline{y}(t) = C\underline{x}(t)$

$$\underline{y}(t) = \underbrace{C\Phi(t)\underline{x}(0)}_{\text{ZERO INPUT}} + \underbrace{\int_0^t C\Phi(t-\tau)Bu(\tau)\,d\tau}_{\text{ZERO STATE RESPONSE}}$$

But since: $\Phi(s) = (sI - A)^{-1}$

Then: $C\Phi(s)B = C(sI-A)^{-1}B = G(s)$ ← TRANSFER FUNCTION MATRIX

$= G(t) = L^{-1}\{C\Phi(s)B\} = C\Phi(t)B$ IMPULSE RESPONSE MATRIX

The output equation could be rewritten then as:

$$\underline{y}(t) = C\Phi(t)\underline{x}(0) + \int_0^t G(t-\tau)\underline{u}(\tau)d\tau$$

$C\Phi(t-\tau)B$

If we just concentrate on the zero-state response:

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_i(t) \\ \vdots \\ y_P(t) \end{bmatrix} = \int_0^t \begin{bmatrix} g_{11}(t-\tau) & g_{12}(t-\tau) & g_{13}(t-\tau) & \cdots & g_{1M}(t-\tau) \\ \vdots & & & & \vdots \\ g_{i1}(t-\tau) & g_{i2}(t-\tau) & g_{i3}(t-\tau) & \cdots & g_{iM}(t-\tau) \\ \vdots & & & & \vdots \\ g_{P1}(t-\tau) & g_{P2}(t-\tau) & g_{P3}(t-\tau) & \cdots & g_{PM}(t-\tau) \end{bmatrix} \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ \vdots \\ u_M(\tau) \end{bmatrix} d\tau$$

An expression for the i^{th} output $y_i(t)$ is:

$$y_i(t) = \int_0^t g_{i1}(t-\tau)u_1(\tau)d\tau + \int_0^t g_{i2}(t-\tau)u_2(\tau)d\tau + \cdots + \int_0^t g_{iM}(t-\tau)u_M(\tau)d\tau$$

PRINCIPLE OF SUPERPOSITION

2.5 How to Calculate the Transition Matrix $\Phi(t)$

2.5.1 The Laplace-Transform Method

$$\Phi(t) = L^{-1}\{(sI-A)^{-1}\}$$

Example: $\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$

$sI - A$

$$\Phi(s) = (sI-A)^{-1} = \left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\right]^{-1}$$

$$= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \Rightarrow \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\Phi(t) = L^{-1}\{\Phi(s)\} = \begin{pmatrix} L^{-1}\left\{\frac{s+3}{(s+2)(s+1)}\right\} & L^{-1}\left\{\frac{1}{(s+2)(s+1)}\right\} \\ L^{-1}\left\{\frac{-2}{(s+2)(s+1)}\right\} & L^{-1}\left\{\frac{s}{(s+2)(s+1)}\right\} \end{pmatrix}$$

Take each term in turn and use partial fractions:

Eg. $L^{-1}\left\{\frac{s+3}{(s+2)(s+1)}\right\} = L^{-1}\left\{\frac{2}{s+1} + \frac{-1}{s+2}\right\}$

Repeating for the other three terms yields:

$$\Phi(t) = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

2.5.2 Matrix Exponential Method

Consider first the zero-input response: $u(t) = 0 \ \forall t$

SOLUTION $\boxed{\underline{x}(t) = \Phi(t)\underline{x}(0)}$

which is of course the solution to:

$\dot{\underline{x}} = A\underline{x}$

Revision: Consider first the scalar differential equation:

$$\dot{x}(t) = ax(t)$$

If the initial condition is x(0), then this is solved to yield:

$x(t) = e^{at}x(0)$

Now considering: $\dot{\underline{x}}(t) = A\underline{x}(t)$

Propose the solution: $\underline{x}(t) = e^{At}\underline{x}(0)$

Proof: We know the solution is $\underline{x}(t) = \Phi(t)\underline{x}(0)$

Hence we can write:

Now look at $\dot{\underline{x}} = A\underline{x}$

$$\dot{\underline{x}}(t) = \frac{d}{dt}(\Phi(t)\underline{x}(0)) = \frac{d\Phi}{dt}\underline{x}(0) \qquad \dot{\underline{x}} = A\underline{x} = A\Phi(t)\underline{x}(0)$$

$$\ddot{\underline{x}}(t) = \frac{d^2}{dt^2}(\Phi(t)\underline{x}(0)) = \frac{d^2\Phi}{dt^2}\underline{x}(0) \qquad \ddot{\underline{x}} = A\dot{\underline{x}} = AA\Phi(t)\underline{x}(0)$$

$$\dddot{\underline{x}}(t) = \frac{d^3}{dt^3}(\Phi(t)\underline{x}(0)) = \frac{d^3\Phi}{dt^3}\underline{x}(0) \qquad \dddot{\underline{x}} = A\ddot{\underline{x}} = AAA\Phi(t)\underline{x}(0)$$

$$d\Phi/dt = A\Phi(t) \qquad d^2\Phi/dt^2 = A^2\Phi(t) \qquad d^3\Phi/dt^3 = A^3\Phi(t)$$

$$d^i\Phi/dt^i = A^i\Phi(t)$$

In fact we could write:

$$\frac{d^i}{dt^i}\Phi(t) = A^i\Phi(t)$$

This will be true if:

$$\Phi(t) = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

We will now define the matrix exponential function as:

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

Example: $\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$

Approximate $\Phi(t)$ to four terms using the matrix exponential method:

$$e^{At} \approx \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}_{A}\frac{t}{1!} + \underbrace{\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}}_{A^2}\frac{t^2}{2!} + \underbrace{\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}}_{A^3}\frac{t^3}{3!}$$

$$\approx \begin{pmatrix} 1 - \frac{t^2}{2} + \frac{2t^3}{6} & t - \frac{2t^2}{2} + \frac{3t^3}{6} \\ -t + \frac{2t^2}{2} - \frac{3t^3}{6} & 1 - 2t + \frac{3t^2}{2} - \frac{4t^3}{6} \end{pmatrix}$$

THIS IS AN APPROXIMATION - IT IS ONLY USEFUL IF t IS SMALL

2.5.3 Using Cayley Hamilton Theory

The Cayley Hamilton theory states that any NxN matrix A obeys:

$$A = \beta_{i0}I + \beta_{i1}A + \beta_{i2}A^2 + \cdots \beta_{i\,N-1}A^{N-1}$$

ASIDE A MATRIX OBEYS ITS OWN CHARACTERISTIC EQUATION

$\det(\lambda I - A) = 0$

poly: $\lambda^N + a_{N-1}\lambda^{N-1} + \ldots + a_0 = 0$

$A^N + a_{N-1}A^{N-1} + \ldots + a_0 I = 0$

Example: $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ $N=2$

Then: $\overset{N+1}{A^3} = \beta_{10}I + \beta_{11}A$

$$A^3 = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} \beta_{10} & 0 \\ 0 & \beta_{10} \end{bmatrix} + \begin{bmatrix} 0 & \beta_{11} \\ -\beta_{11} & -2\beta_{11} \end{bmatrix}$$

WE KNOW THIS

$\beta_{10} = 2$ $\beta_{11} = 3$

Now consider the matrix exponential function:

$$\Phi(t) = e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots$$ ∞ SERIES

By Cayley Hamilton theory:

$i=0$ $A^N = \beta_{00}I + \beta_{01}A + \beta_{02}A^2 + \cdots + \beta_{0(N-1)}A^{N-1}$

$i=1$ $A^{N+1} = \beta_{10}I + \beta_{11}A + \beta_{12}A^2 + \cdots + \beta_{1(N-1)}A^{N-1}$

$i=2$ $A^{N+2} = \beta_{20}I + \beta_{21}A + \beta_{22}A^2 + \cdots + \beta_{2(N-1)}A^{N-1}$

$\vdots$ $\vdots$

$i=\infty$ $A^{N+\infty} = \beta_{\infty 0}I + \beta_{\infty 1}A + \beta_{\infty 2}A^2 + \cdots + \beta_{\infty(N-1)}A^{N-1}$

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots \frac{A^{N-1}t^{N-1}}{(N-1)!}$$
$$+ \underbrace{(\beta_{00}I + \beta_{01}A + \beta_{02}A^2 + \cdots + \beta_{0(N-1)}A^{N-1})}_{A^N}\frac{t^N}{N!}$$
$$+ \underbrace{(\beta_{10}I + \beta_{11}A + \beta_{12}A^2 + \cdots + \beta_{1(N-1)}A^{N-1})}_{A^{N+1}}\frac{t^{N+1}}{(N+1)!}$$
$$+ \underbrace{(\beta_{20}I + \beta_{21}A + \beta_{22}A^2 + \cdots + \beta_{2(N-1)}A^{N-1})}_{A^{N+2}}\frac{t^{N+2}}{(N+2)!}$$
$$\vdots$$
$$+ \underbrace{(\beta_{\infty 0}I + \beta_{\infty 1}A + \beta_{\infty 2}A^2 + \cdots + \beta_{\infty(N-1)}A^{N-1})}_{A^{N+\infty}}\frac{t^{N+\infty}}{(N+\infty)!}$$

Bringing all similar terms in A together:

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2 + \cdots \alpha_{N-1}(t)A^{N-1}$$

N.B. THIS IS FINITE

$\alpha_0(t) = I + \beta_{00}\frac{t^N}{N!} + \beta_{10}\frac{t^{N+1}}{(N+1)!} + \beta_{20}\frac{t^{N+2}}{(N+2)!} \ldots$ SCALAR ∞ SERIES

$\alpha_1(t) = \frac{t}{1!} + \frac{\beta_{01}t^N}{N!} + \frac{\beta_{11}t^{N+1}}{(N+1)!} + \ldots$

How to find the scalar functions of time:

From a singular variable decomposition of A, then every eigenvalue λ_i of A must also satisfy:

REPLACE A WITH λ_i

$$e^{\lambda_i t} = \alpha_0(t) + \alpha_1(t)\lambda_i + \alpha_2(t)\lambda_i^2 + \cdots \alpha_{N-1}(t)\lambda_i^{N-1}$$

Revision: The eigenvalues of A are the roots of:

$$\det(\lambda I - A) = 0$$

eigenvalues $\Rightarrow \lambda_1 \ldots \lambda_i \ldots \lambda_N$

If we have N distinct eigenvalues of A: $\lambda_1, \lambda_2, \ldots \lambda_N$

$$\left.\begin{aligned} e^{\lambda_1 t} &= \alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots \alpha_{N-1}(t)\lambda_1^{N-1} \\ &\vdots \\ e^{\lambda_N t} &= \alpha_0(t) + \alpha_1(t)\lambda_N + \alpha_2(t)\lambda_N^2 + \cdots \alpha_{N-1}(t)\lambda_N^{N-1} \end{aligned}\right\} \text{N simultaneous equations}$$

SOLVE THESE TO YIELD N FUNCTIONS $\alpha_0(t) \ldots \alpha_{N-1}(t)$

Example:

$$\dot{x}(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x(t)$$

The eigenvalues of A are roots of: 2 eigenvalues λ_1, λ_2

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = 0$$ $\lambda_1 = 3$ $\lambda_2 = -1$

$(\lambda-1)^2 - 4 = 0$ $\lambda^2 - 2\lambda - 3 = 0$ $(\lambda-3)(\lambda+1) = 0$

Since N=2 then we can write: $e^{At} = \alpha_0(t)I + \alpha_1(t)A$ NEED TO SOLVE FOR THESE

This equation must be obeyed by the eigenvalues of A:

MUST BE OBEYED BY λ_1, λ_2

$$e^{3t} = \alpha_0(t) + \alpha_1(t)(3)$$
$$e^{-t} = \alpha_0(t) + \alpha_1(t)(-1)$$

$(e^{3t} - e^{-t}) = 3\alpha_1(t) + \alpha_1(t) = 4\alpha_1(t) \Rightarrow \alpha_1(t) = \frac{e^{3t} - e^{-t}}{4}$

$\alpha_0(t) = \frac{e^{3t} + 3e^{-t}}{4}$

Now: $$e^{At} = \underbrace{\frac{1}{4}(e^{3t} + 3e^{-t})}_{\alpha_0(t)}I + \underbrace{\frac{1}{4}(e^{3t} - e^{-t})}_{\alpha_1(t)}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \frac{e^{3t} + e^{-t}}{2} & \frac{e^{3t} - e^{-t}}{2} \\ \frac{e^{3t} - e^{-t}}{2} & \frac{e^{3t} + e^{-t}}{2} \end{pmatrix}$$

2.6 The Discrete State Space Equations

Consider the MIMO process:

CONTINUOUS TIME
$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$
$$\underline{y}(t) = C\underline{x}(t)$$

The state trajectory is given by:

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \int_0^t \Phi(t-\tau)B\underline{u}(\tau)d\tau$$

Where the initial state at time 0 is $\underline{x}(0)$

Now consider that the initial time is t_0, with initial state: $\underline{x}(t_0)$

Then the trajectory could be rewritten as:

$$\underline{x}(t) = \Phi(t-t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t-\tau)B\underline{u}(\tau)d\tau$$

Consider what happens to the state vector over a time step T: $t_0 = kT$ $t = (k+1)T$

$$\underline{x}(\overset{t}{(k+1)T}) = \Phi(\overset{t-t_0}{(k+1)T - kT})\underline{x}(\overset{t_0}{kT}) + \int_{\underset{t_0}{kT}}^{\overset{t}{(k+1)T}} \Phi(\overset{t-\tau}{(k+1)T - \tau})B\underline{u}(\tau)d\tau$$

CONSTANT

$$\underline{x}((k+1)T) = \Phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \Phi((k+1)T - \tau)B\underline{u}(\tau)d\tau$$

If we assume that a zero-order hold (ZOH) is utilised:

$\Rightarrow \underline{u}(t) = \underline{u}(kT)$ for $kT \le t < (k+1)T$

Then:

THIS IS CONSTANT OVER INTEGRAL

$$\underline{x}((k+1)T) = \Phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \Phi((k+1)T-\tau)Bd\tau\,\underline{u}(kT)$$

Now make the following substitution: $\eta = (k+1)T - \tau$

$d\eta = -d\tau$

$$\underline{x}((k+1)T) = \Phi(T)\underline{x}(kT) - \int_{T}^{0} \Phi(\eta)Bd\eta\,\underline{u}(kT)$$

$\tau = kT$ $\quad \eta = (k+1)T - \tau$

$\eta = T$

Remembering that:

$$\int_0^a f(x)\,dx = -\int_a^0 f(x)\,dx$$

And simplifying the notation:

$(k+1)T \rightarrow (k+1)$

$kT \rightarrow k$

$$\underline{x}(k+1) = \Phi(T)\underline{x}(k) + \int_0^T \Phi(\eta)Bd\eta\,\underline{u}(k)$$

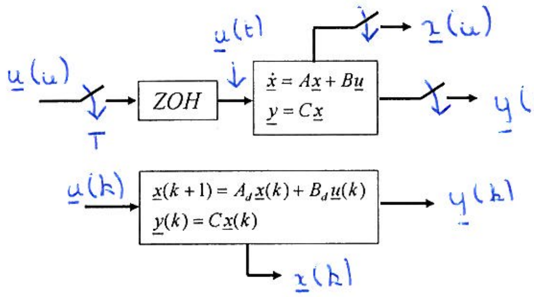
This yields the discrete-time state-space equations:

$$\underline{x}(k+1) = A_d\underline{x}(k) + B_d\underline{u}(k)$$
$$\underline{y}(k) = C\underline{x}(k)$$

$A_d = \Phi(T) \Rightarrow e^{AT}$

$B_d = \int_0^T \Phi(\eta)B\,d\eta$

where:



Tutorial:

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u(t)$$
$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

i) Determine the discrete-time state-space representation, assuming a ZOH, and sample time T=0.2 seconds.
ii) If $u(t) = 1 \quad \forall t > 0^+, \quad x_1(0) = x_2(0) = 1$, determine and sketch the responses for the states x_1, x_2, and the output y(t).
iii) Use Simulink to compare the responses of both the continuous and discrete process models.

2.6.1 Discrete Transfer Function Matrix

Consider the MIMO discrete time system: $Z\{\underline{x}(k+1)\} = z\underline{X}(z)$

$$\underline{x}(k+1) = A_d\underline{x}(k) + B_d\,\underline{u}(k)$$
$$\underline{y}(k) = C\underline{x}(k)$$

Taking Z transforms:

$$z\underline{X}(z) = A_d\underline{X}(z) + B_d\underline{U}(z)$$
$$\underline{Y}(z) = C\underline{X}(z)$$

Hence we can write:

$$(zI - A_d)\underline{X}(z) = B_d\underline{U}(z)$$
$$\therefore \underline{X}(z) = (zI - A_d)^{-1}B_d\underline{U}(z)$$

Which yields the following:

$$\underline{Y}(z) = C\underbrace{(zI - A_d)^{-1}B_d\underline{U}(Z)}_{\underline{X}(z)}$$

TRANSFER FUNCTION MATRIX IS : $C(zI - A_d)^{-1}B_d$

The poles of the discrete time process on the Z plane are given by the roots of the characteristic equation:

$$\det(zI - A_d) = 0$$

2.6.2 Realisation of Digital Filters

note

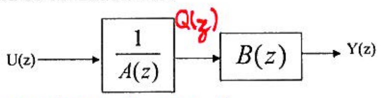
pure time delay is easily modelled in discrete domain

Consider the discrete-time transfer function for a SISO process:

INCLUDING DELAY

$$\frac{Y(z)}{U(z)} = G(z) = \frac{z^{-d}\left(b_1z^{-1} + b_2z^{-2} + \cdots + b_mz^{-m}\right)}{1 - a_1z^{-1} - a_2z^{-2} - \cdots - a_nz^{-n}} = \frac{B(z)}{A(z)}$$

This could be constructed as:



Hence: $A(z)Q(z) = U(z)$

$$\overbrace{\left(1 - a_1z^{-1} - a_2z^{-2} - \cdots - a_nz^{-n}\right)}^{A(z)}Q(z) = U(z)$$

$$Q(z) = U(z) + a_1z^{-1}Q(z) + a_2z^{-2}Q(z) + \ldots + a_nz^{-n}Q(z)$$

Taking inverse Z transforms yields:

$$q(k) = a_1q(k-1) + a_2q(k-2) + \cdots + a_nq(k-n) + u(k)$$

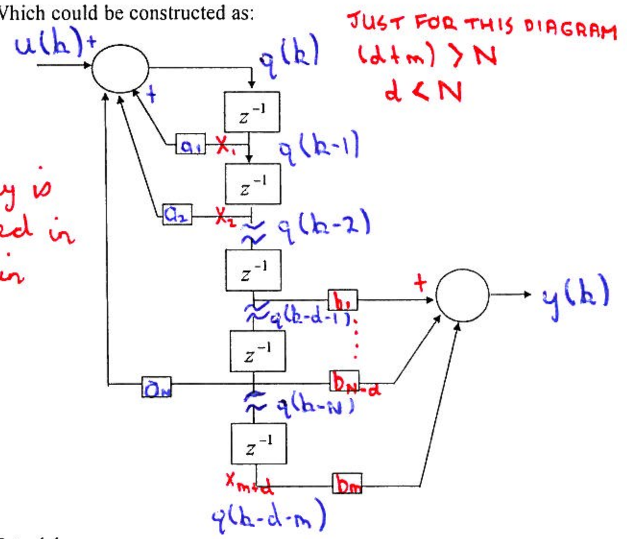
For the output equation : $B(z)Q(z) = Y(z)$

$$z^{-d}\overbrace{\left(b_1z^{-1} + b_2z^{-2} + \cdots + b_mz^{-m}\right)}^{B(z)}Q(z) = Y(z)$$

$$y(k) = b_1q(k-d-1) + b_2q(k-d-2) + \ldots + b_mq(k-d-m)$$

Which could be constructed as:

JUST FOR THIS DIAGRAM $(d+m) > N$, $d < N$



Tutorial:

Use this diagram to develop the following control-canonical state-space representation from the transfer function given above:

$$\begin{bmatrix} x_1(k+1)\\ x_2(k+1)\\ \vdots \\ x_n(k+1) \\ \vdots \\ x_{m+d}(k+1)\end{bmatrix} = \left[\begin{array}{cccc|ccc} a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ \hline & & \underbrace{I_{m+d-1}}_{\text{IDENTITY}} & & & & 0_{m+d-1}\end{array}\right]\begin{bmatrix} x_1(k)\\ x_2(k)\\ \vdots \\ x_n(k) \\ \vdots \\ x_{m+d}(k)\end{bmatrix} + \begin{bmatrix}1 \\ \hline 0_{m+d-1}\end{bmatrix}u(k)$$

$$y(k) = \left[\underline{0}_d^T \,\vdots\, b_1 \quad b_2 \quad \cdots \quad b_m\right]\underline{x}(k)$$