4/2/09 Brian Fitzgibbon

Chapter 3. Introduction to State-Space Control

3.1 Continuous Time Regulator Design

What is a regulator? A controller used to improve the dynamics about a desired operating point

Consider for simplicity the SISO process:

N^{th} order

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t) + Ed(t)$$
$$y(t) = C\underline{x}(t)$$

NB disturbance

The open-loop dynamic behaviour of the plant to changes in the disturbance d(t) is given by the transfer function:

$$Y(s)/D(s) = C(sI-A)^{-1}E$$

The poles of this transfer function dictate the dynamics of the open-loop process to changes in d(t):

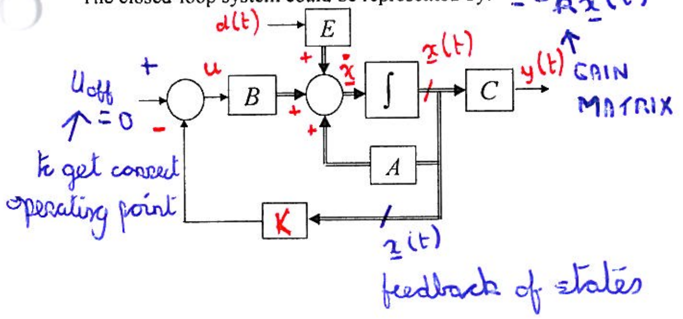
N poles are the roots of $\det(sI-A)=0$

Assume a regulator control law:

$$u(t) = -[k_1 \quad k_2 \quad \cdots \quad k_n]\underline{x}(t)$$
$$= -k_1x_1(t) - k_2x_2(t) \ldots - k_nx_n(t)$$

The closed-loop system could be represented by: $= -\underline{k}\underline{x}(t)$

GAIN MATRIX



The closed loop state equation is then: $u = -\underline{k}\underline{x}$

$$\dot{\underline{x}}(t) = A\underline{x}(t) - BK\underline{x}(t) + Ed(t)$$
$$\dot{\underline{x}}(t) = (A-BK)\underline{x}(t) + Ed(t)$$
$$y(t) = C\underline{x}(t)$$

$\dot{\underline{x}} = A_{cl}\underline{x} + Ed$
$y = C\underline{x}$

Which yields the following closed loop transfer function:

$$G_D^{cl}(s) = \frac{Y(s)}{D(s)} = C(sI - A_{cl})^{-1}E =$$

The poles of the closed-loop system are given then by the roots of the closed-loop characteristic equation:

$$\det(sI - A + BK) = 0$$

N.B. closed loop process has N poles

For a specified closed-loop performance we will specify the closed-loop poles to be placed at:

$s = P_1 \ldots P_n$

to achieve some desired performance

ξ, PO%, $T_{s2\%}$, etc

This yields the desired characteristic equation:

$$C_{des}(s) = (s-p_1)(s-p_2)\cdots(s-p_N) = 0$$

Hence we choose the gain matrix K so that:

$$\det(sI - A + BK) = C_{des}(s)$$

↑ CHOOSE K

Example:

A DC motor is modelled by the following equations:

$$\frac{d\omega}{dt} = \frac{1}{J}(K_m i(t) - B\omega(t) - T_L(t))$$
$$\frac{di}{dt} = \frac{1}{L}(v(t) - K_m\omega(t) - Ri(t))$$

Where: B=0, J=0.02Kgm^2, K_m=1NmA^{-1}, R=1Ω, L=5mH

The open-loop state-space model is then:

$$\frac{d}{dt}\begin{bmatrix}\omega\\ i\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 50\\ -200 & -200\end{bmatrix}}_{A}\begin{bmatrix}\omega\\ i\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 200\end{bmatrix}}_{B}V + \underbrace{\begin{bmatrix}-50\\ 0\end{bmatrix}}_{E}T_L$$

Tutorial:

Show for the open-loop system:

$$\frac{\Omega(s)}{T_L(s)} = \frac{-(5s+1000)}{(s+100)^2}$$

The open-loop poles are obviously s=-100 twice

Suggest the following regulator:

$$u(t) = -[k_1 \quad k_2]\underline{x}(t)$$

$\underline{x} = \begin{bmatrix}\omega\\ i\end{bmatrix}$

$\Rightarrow v(t) = -k_1\omega(t) - k_2 i(t)$

Then the closed loop poles are given by the roots of:

$$\det(sI - A + BK) = 0$$

$\begin{bmatrix}0 & 0\\ 200K_1 & 200K_2\end{bmatrix}$

$$\det\left[\underbrace{\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix}}_{sI} - \underbrace{\begin{pmatrix}0 & 50\\ -200 & -200\end{pmatrix}}_{A} + \underbrace{\begin{pmatrix}0\\ 200\end{pmatrix}}_{B}\underbrace{(k_1 \quad k_2)}_{K}\right] = 0$$

$$\det\begin{bmatrix}s & -50\\ 200+200k_1 & s+200+200k_2\end{bmatrix} = 0$$

Which yields the closed-loop characteristic equation:

$$s^2 + 200(1+K_2)s + 10000(1+K_1)$$

Now we must specify the desired characteristic equation $C_{des}(s)$:

Assume the following 2nd order structure:

$$C_{des}(s) = s^2 + 2\xi\omega_n s + \omega_n^2$$

For this example we will choose: $\xi = 0.707$ $\omega_N = 200$ Rad/s

$$C_{des}(s) = s^2 + 282.8s + 40000$$

To achieved the desired pole locations:

$200(1+K_2) = 282.8 \Rightarrow K_2 = 0.41$

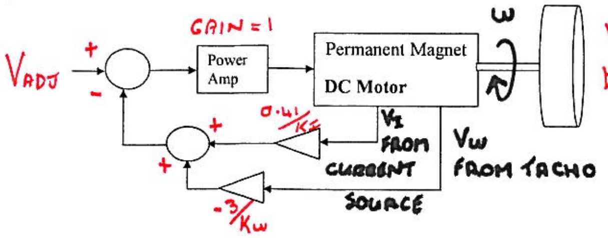
$10000(1+K_1) = 40000 \Rightarrow K_1 = 3$

This yields the following regulator:

$$u(t) = -[3 \quad 0.41]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$
$$v(t) = -3\omega(t) - 0.41\,i(t)$$

OR IN REAL LIFE $v(t) = -\frac{3}{K_\omega}V_\omega - \frac{0.41}{K_I}V_I$

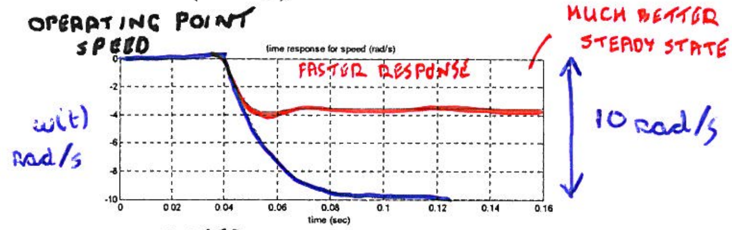
Could be built as follows:

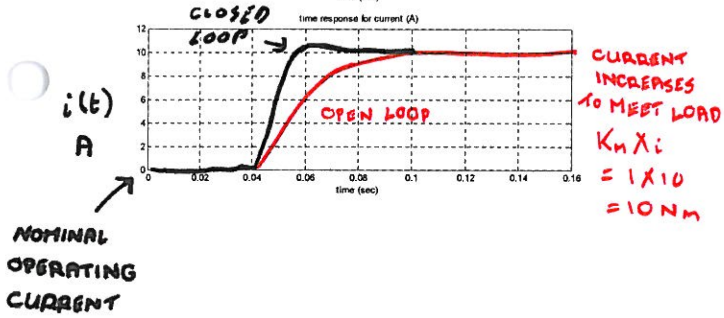


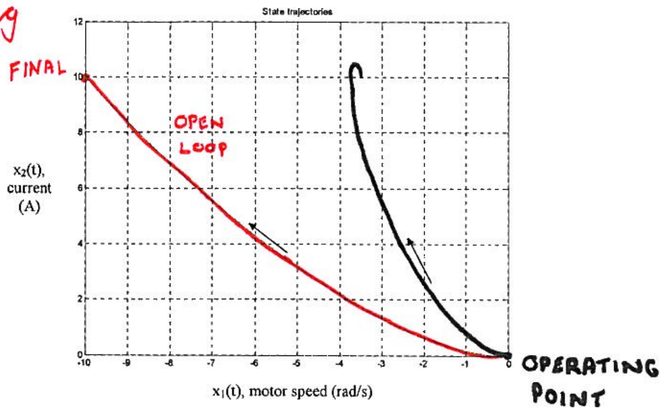
V_{ADJ} is used to set the operating speed

Tutorial: Use the state-space technique to design the following PD speed controller, to achieve the performance highlighted above:

$$m(t) = K\left(e(t) + T_d \frac{de}{dt}\right) \quad \text{where} \quad e(t) = r(t) - \omega(t)$$







TRAJECTORY OF THE STATE VECTOR

3.2 Regulator Design for High Order Processes

The state-space pole-placement design method proposed above is difficult to solve for high order processes:

However consider the N[th] order SISO process in control canonical form:

$$A = \left[\begin{array}{c|ccccc} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \hline -e_0 & -e_1 & -e_2 & -e_3 & \cdots & -e_{N-1} \end{array}\right] \qquad B = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \end{array}\right]$$

$$C = \left[\begin{array}{cccc|ccc} f_0 & f_1 & \cdots & f_r & 0 & \cdots & 0 \end{array}\right]$$

If the following regulator is used:

$$u(t) = -K\underline{x}(t) = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}\underline{x}(t)$$

Then the closed-loop state equation becomes:

$$\dot{\underline{x}} = (A - BK)\underline{x}$$

Lets look at the matrix product BK:

null matrix (N-1) ROWS N cols

$$BK = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 & \cdots & k_N \end{bmatrix} = \left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \\ \hline k_1 & k_2 & \cdots & k_N \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline K \end{array}\right]$$

Hence we can write:

N-1 ROWS

$$(A - BK) = \left[\begin{array}{c|c} O_{N-1} & I_{N-1} \\ \hline \multicolumn{2}{c}{-\underline{e}^T} \end{array}\right] - \left[\begin{array}{c} O_{N-1} \\ \hline K \end{array}\right] = \left[\begin{array}{c|c} O_{N-1} & I_{N-1} \\ \hline \multicolumn{2}{c}{-\underline{e}^T - K} \end{array}\right]$$

Hence the characteristic equation of the closed-loop system is:

$$-\underline{e}^T - K = \begin{bmatrix} -e_0 - K_1 & -e_1 - K_2 & \cdots & -e_{N-1} - K_N \end{bmatrix}$$

$$s^N + (e_{N-1} + K_N)s^{N-1} + \ldots + (e_0 + K_1) = 0$$

Now if the desired characteristic equation is:

N th order

$$C_{des}(s) = s^N + C_{N-1}s^{N-1} + \cdots C_1 s + C_0 = 0$$

It is then easy to choose the gains $k_1, \ldots k_N$, to obtain the desired characteristic equation:

$$e_{N-1} + k_N = C_{N-1} \Rightarrow K_N = C_{N-1} - e_{N-1}$$

$$e_{N-2} + k_{N-1} = C_{N-2} \Rightarrow K_{N-1} = C_{N-2} - e_{N-2}$$

$$\vdots$$

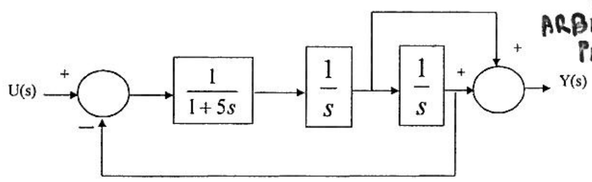
$$e_0 + k_1 = C_0 \Rightarrow K_1 = C_0 - e_0$$

NOTE: EASY DESIGN !!!

IT IS OBVIOUS THAT IF THE SYSTEM WAS IN CONTROL CANONICAL FORM YOU CAN DESIGN A REGULATOR TO PLACE THE CLOSED LOOP POLES ANYWHERE

ARBITRARY PERFORMANCE

Tutorial: Design a regulator for the following system to place the three closed-loop poles at s=-10.



Even if the process is not even in control canonical form, it would at first glance seem trivial to design a regulator for even a high order process.

Consider the SISO process:

ORIGINAL REPRESENTATION
$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$y(t) = C\underline{x}(t)$$

Could easily transform this to a control canonical format:

FIRST GET TF DIRECTLY

$$G(s) = C(sI - A)^{-1}B \rightarrow \begin{matrix} \dot{\underline{x}}_2(t) = A_2\underline{x}_2(t) + B_2u(t) \\ y(t) = C_2\underline{x}_2(t) \end{matrix}$$

Then design controller for control canonical format:

BUT! Control will be $u(t) = -K\underline{x}_2(t)$
Based on new vector $\underline{x}_2$
$\underline{x}_2$ may have no physical meaning

Tutorial: A certain chemical reactor can be represented by the following state-space equations:

$$\frac{d}{dt}\begin{bmatrix} C(t) \\ T(t) \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 2 & -10 \end{bmatrix}\begin{bmatrix} C(t) \\ T(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}q(t)$$
$$C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} C(t) \\ T(t) \end{bmatrix}$$

a) Represent the process in control canonical format

b) Design a state-space regulator to place the closed-loop poles at s=-10 twice using, i) A control canonical form of the model, ii) using the original state-space model.

c) Comment on the practicality and the realisation of each of the controllers.

3.2.1 Design of High Order Regulators Using Transformation Theory

It may be possible to transform the original state-space equations using the transformation, $\underline{z} = T\underline{x}$

ORIGINAL PROCESS REPRESENTATIONS
$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$y(t) = C\underline{x}(t)$$

Into control canonical form:

$$\dot{\underline{z}}(t) = A_C\underline{z}(t) + B_Cu(t) \qquad \dot{\underline{z}}(t) = TAT^{-1}\underline{z}(t) + TBu(t)$$
$$y(t) = C_C\underline{z}(t) \qquad y(t) = CT^{-1}\underline{z}(t)$$

Where T is chosen so that : Transform in CCF

That is:

$$TAT^{-1} = A_C = \left[\begin{array}{c|c} \underline{O}_{N-1} & I_{N-1} \\ \hline \multicolumn{2}{c}{-\underline{e}^T} \end{array}\right]$$

we get these from $G(s) = C(sI-A)^{-1}B$

$$TB = B_C = \left[\begin{array}{c} \underline{O}_{N-1} \\ \hline 1 \end{array}\right]$$

Now design the regulator: BASED ON THE TRANSFORM STATES

$u(t) = -K_C\underline{z}(t)$

If the desired characteristic equation is:

$$C_{des}(s) = s^N + C_{N-1}s^{N-1} + \cdots C_1s + C_0 = 0$$

Then the gain matrix is:

$$K_C = [C_0 - e_0 \quad C_1 - e_1 \quad \ldots \quad C_{N-1} - e_{N-1}]$$

Now we know that: $\underline{z}(t) = T\underline{x}(t)$

Hence the control law for the original system is:

$$u = -K_C\underline{z}(t) = -K_CT\underline{x}(t) = -K\underline{x}(t)$$

where $K = K_CT$

Tutorial:

$$\frac{d}{dt}\underline{x}(t) = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t)$$

Determine T to obtain the control canonical form.

- shows how difficult this could be

3.3 Controllability

There are two common definitions of controllability of the linear MIMO process:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t):$$

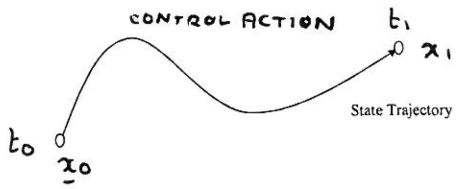
1) A Frequency Domain Definition

This process is controllable using the regulator $\underline{u}(t) = -K\underline{x}(t)$ if the gain matrix K can be selected to place the closed-loop poles anywhere on the s plane.

Important Note: We know that a system which is in CCF, you can place poles anywhere => CCF model is controllable => if a system can be transformed into CCF then it is also controllable

2) A Time Domain Definition

Consider the possible trajectory through the state-space:



The system is controllable, if for any $\underline{x}_0$ and $\underline{x}_1$, there exists a piecewise continuous control signal $\underline{u}(t)$, that will operate between times t_0 and t_1 to drive the state from any $\underline{x}_0$ at time t_0 to state $\underline{x}_1$ at time t_1

3.3.1 Derivation of the Controllability Matrix

We will derive this test for controllability from the time domain definition and that we know the solution to the state trajectory at time t is given by:

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \int_0^t \Phi(t-\tau)B\underline{u}(\tau)d\tau$$

$$\underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau$$

Without loss of generality, we can express the time domain definition as: The system model is controllable if given any state $\underline{x}(0)$ at the time 0, there exists a piecewise continuous control that will drive the state $\underline{x}(t)$ to the origin at some finite time t_1

Hence using the state-trajectory equation we can write: $\boxed{t = t_1}$

$$\underline{0} = e^{At_1}\underline{x}(0) + \int_0^{t_1} e^{A(t_1-\tau)}B\underline{u}(\tau)d\tau$$

$$\Rightarrow e^{At_1}\underline{x}(0) = -\int_0^{t_1} e^{At_1}e^{-A\tau}B\underline{u}(\tau)d\tau$$

Is there a solution for the control $\underline{u}(t)$ over time 0 to t_1 which will ensure that:

$$e^{At_1}\underline{x}(0) = -e^{At_1}\int_0^{t_1} e^{-A\tau}B\underline{u}(\tau)d\tau$$

$$\underline{x}(0) = -\int_0^{t_1} e^{-A\tau}B\underline{u}(\tau)d\tau$$

Expand the matrix exponential:

$$e^{-A\tau} = I - A\tau + \frac{A^2\tau^2}{2!} - \frac{A^3\tau^3}{3!}\cdots$$

Then:

$$\int_0^{t_1} e^{-A\tau}B\underline{u}(\tau)d\tau = \int_0^{t_1}\overbrace{\left(I - A\tau + \frac{A^2\tau^2}{2!}\cdots\right)}^{e^{-A\tau}}B\underline{u}(\tau)d\tau$$

$$= B\int_0^{t_1}\underline{u}(\tau)d\tau + AB\int_0^{t_1}-\tau\underline{u}(\tau)d\tau + A^2B\int_0^{t_1}\frac{\tau^2}{2!}\underline{u}(\tau)d\tau + A^3B\int_0^{t_1}-\frac{\tau^3}{3!}\underline{u}(\tau)d\tau\cdots$$

Hence we could write in matrix form:

$$\underline{x}(0) = -\int_0^{t_1} e^{-A\tau}B\underline{u}(\tau)d\tau = -[B\ AB\ A^2B \cdots]\begin{bmatrix}\int_0^{t_1}\underline{u}(\tau)d\tau \\ -\int_0^{t_1}\tau\underline{u}(\tau)d\tau \\ \int_0^{t_1}\frac{\tau^2}{2!}\underline{u}(\tau)d\tau \\ -\int_0^{t_1}\frac{\tau^3}{3!}\underline{u}(\tau)d\tau \\ \vdots\end{bmatrix}$$

N STATES

Define the controllability matrix as:

$$C_x = [B \mid AB \mid A^2B \mid A^3B \mid \cdots \mid A^{N-1}B]$$

BY CAYLEY HAMILTON THERE IS NO NEED TO HOLD ANY MORE COLS.

Now since there are N elements in the initial state: $\underline{x}(0)$:

We need N linearly independent equations $\underline{x}(0) = -C_xQ$

If C_x was square then of course we could solve for Q as follows:

$$Q = -C_x^{-1}\underline{x}(0)$$

which is easily solvable if $\det C_x \neq 0$

N^{th} order process

C_x is $N \times N$

i.e. C_x is full rank N

If however C_x is non square: multiinput process M inputs

$\underline{x}(0) = -C_xQ$ → we have N equations $N \times M$ unknowns in Q

Which is solvable if: Rank $C_x = N$

Since C_x is not square the test for rank is: count the number of non zero eigenvalues of $C_x^TC_x$

For controllability we need N non zero eigenvalues

Hence a linear MIMO process is controllable if and only if:

$$\text{Rank}[B \mid AB \mid A^2B \mid \dots A^{N-1}B] = N \leftarrow \text{ORDER OF PROCESS}$$

Example 1: $\dot{x}(t) = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t)$ ← SINGLE INPUT

$C_x = [B \mid AB]$ $N = 2$

$AB = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix} \Rightarrow C_x = \begin{bmatrix}0 & 1\\ 1 & -3\end{bmatrix}$

$\det C_x = -1 \neq 0 \Rightarrow$ Rank $C_x = 2 \Rightarrow$ controllable

Example 2: $\dot{x}(t) = \begin{bmatrix}5 & 0 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}x(t) + \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}u(t)$

2 INPUTS

$$C_x = [B \mid AB \mid A^2B \mid \cdots] = \begin{bmatrix}1\,0 & 5\,0 & \\ 0\,0 & 1\,0 & A^2B\\ 0\,1 & 0\,1 & \end{bmatrix} \qquad AB = \begin{bmatrix}5 & 0 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$$

$$\underset{A}{\begin{bmatrix}5 & 0 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}}\underset{AB}{\begin{bmatrix}5 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}} = \begin{bmatrix}25 & 0\\ 7 & 0\\ 0 & 1\end{bmatrix}$$

$$C_x = \begin{bmatrix}1 & 0 & 5 & 0 & 25 & 0\\ 0 & 0 & 1 & 0 & 7 & 0\\ 0 & 1 & 0 & 1 & 0 & 1\end{bmatrix}$$

$\text{eig}(C_x^TC_x) = \{0\ 0\ 0\ 1.9\ 5.8\ 6.12\}$

3 non zero eigenvalues. system order is 3 ⇒ controllable

3.3.2 How Controllability is related to the State-Space Model and has nothing to do with the transfer function

Consider the following transfer function, where the zero z is unknown:

$$G(s) = \frac{s-z}{(s+3)(s+4)}$$

This system has the following control-canonical representation;

$$\dot{\underline{x}}_c = \begin{bmatrix}0 & 1\\ -12 & -7\end{bmatrix}\underline{x}_c + \begin{bmatrix}0\\1\end{bmatrix}u$$

$$\begin{array}{cc} B & AB \\ \downarrow & \downarrow \end{array}$$

With the controllability matrix: $C_c = \begin{bmatrix} 0 & 1 \\ 1 & -7 \end{bmatrix}$ $\det C_c = -1 \neq 0$

$\Rightarrow$ Rank $C_c = 2$

Which is controllable Rank $C_c = N$

Now consider the "Observer Canonical Form" of the same system:

$$\underline{\dot{x}}_O = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \underline{x}_O + \begin{bmatrix} 1 \\ -z \end{bmatrix} u$$

The controllability matrix for this realisation is:

$$C_O = \begin{array}{c} \begin{array}{cc} B & AB \end{array} \\ \begin{bmatrix} 1 & -7-z \\ -z & -12 \end{bmatrix} \end{array}$$

So the observer canonical form is controllable if: $\det C_o \neq 0$

$\det C_o = -12 + z(-7-z) = -12 - 7z \quad z^2$

you will lose controllability when

$z^2 + 7z + 12 = 0$

$z \rightarrow -3$

$z \rightarrow -4$

3.3.3 Controllability and the State Transformation

Consider the N^{th} order M input linear process:

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t);$$

The controllability matrix is:

$$C_x = \left[B \mid AB \mid A^2B \mid \cdots \mid A^{N-1}B\right]$$

Now consider the transformation: $\underline{z} = T\underline{x}$

This yields the transformed state-space equations:

$$\underline{\dot{z}} = \underbrace{TAT^{-1}}_{A_2}\underline{z} + \underbrace{TB}_{B_2}\underline{u}$$

The controllability matrix of the transformed system is:

$$C_z = \left[B_2 \mid A_2B_2 \mid A_2^2B_2 \mid \cdots \mid A_2^{N-1}B_2\right]$$

But:

$$B_2 = TB$$
$$A_2B_2 = TAT^{-1}TB = TAB$$
$$A_2^2B_2 = TAT^{-1}TAT^{-1}TB = TA^2B$$
$$A_2^3B_2 = TAT^{-1}TAT^{-1}TAT^{-1}TB = TA^3B$$

Hence: $C_z = \left[TB \mid TAB \mid TA^2B \mid \cdots \mid TA^{N-1}B\right] = TC_x$

<u>Note</u>: Since T is non-singular, (Full Rank), then the transformation $\underline{z}=T\underline{x}$ does not contribute to or take away from a process models controllability.

if C_x is full rank $\rightarrow$ so is C_z

if C_x is rank deficient $\rightarrow$ so is C_z

<u>There is another way to look at it</u>: Consider that we wish to transform our system using $\underline{z}=T\underline{x}$ into control canonical form:

We know the CCF is controllable

$\Rightarrow C_z$ is full rank

But T cannot make an uncontrollable model controllable

$\therefore$ Transformation to CCF can only happen if the original model was controllable

3.4 Design of high Order Regulators Using the Controllability Matrix

Consider the design of a state space regulator for the N^{th} Order SISO process:

$$\underline{\dot{x}} = A\underline{x} + Bu$$
$$y = C\underline{x}$$

First form the controllability matrix based on state vector $\underline{x}$

$$C_x = \left[B \mid AB \mid A^2B \mid \cdots \mid A^{N-1}B\right]$$

Next determine the open-loop transfer function:

$$G(s) = C(sI - A)^{-1}B$$

Use G(s) to directly write down the control-canonical state-space format:

DIRECTLY $\underline{\dot{z}} = A_C\underline{z} + B_Cu$

BY INSPECTION $y = C_C\underline{z}$

Determine the controllability matrix for the CCF:

$$C_z = \left[B_C \mid A_CB_C \mid A_C^2B_C \mid \cdots \mid A_C^{N-1}B_C\right]$$

Design the regulator for the control canonical form: $u(t) = -K_z\underline{z}(t)$

Determine the transformation T:

$C_z = TC_x$

$C_zC_x^{-1} = TC_xC_x^{-1}$

$T = C_zC_x^{-1}$

Finally determine the controller gain matrix K:

$K = K_zT = K_zC_zC_x^{-1}$

to yield controller $u = -K\underline{x}(t)$

3.4.1 Ackermann's Gain Formula

Can only be used for single-input systems:

$$\underline{\dot{x}} = A\underline{x} + Bu$$

Assume the control-law:

$$u = -K\underline{x}(t)$$

Form the desired characteristic equation:

$$C_{des}(s) = s^N + C_{N-1}s^{N-1} + \cdots C_1s + C_0 = 0$$

Form the controllability matrix

$$C_x = \left[B \mid AB \mid A^2B \mid \cdots \mid A^{N-1}B\right]$$

CHECK THIS FIRST FOR FULL RANK N

Ackermann's gain formula is:

$$K = \underbrace{[0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1]}_{N} C_x^{-1}C_{des}(A)$$

where:

$C_{des}(A) = A^N + C_N + A^{N-1} + \ldots + CA + C_oI$

<u>Tutorial:</u>

$$\underline{\dot{x}} = \begin{bmatrix} -14 & 10 & -22 \\ 13 & 10 & 23 \\ 1 & 0 & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}u$$
$$y = [1 \;\; 1 \;\; 1]\underline{x}$$

a) Determine the transformation z=Tx which will convert this system into CCF

Design a control-law to place the poles at s=-3, -3±j

b) Repeat the design using Ackermann's formula.

3.5 Regulator Design for Multi-Input Systems

* can't use Ackerman's Equation

Consider the multi-input system: $\underline{\dot{x}} = A\underline{x} + B\underline{u}$ ← M INPUTS

↑ N STATES

With the regulator: $\underline{u} = -K\underline{x}$

The control gain matrix is:

$$K = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{M1} & k_{M2} & \cdots & k_{MN} \end{bmatrix}$$

M X N gains to "tune"

With the design equation:

$$\det(sI - A + BK) = s^N + C_{N-1}s^{N-1} + \cdots C_1 s + C_0$$

only N equations but M X N gains

∴ Solution for k is not unique

This can be dealt with in the following ways:

1) Fix some of the gains in K to predefined values:-

e.g. to zero to leave N tunable gains

2) Instead of pole-placement use the flexibility of having MxN gains to assign the complete eigenstructure of the process.

3) Use an optimisation approach – e.g. LQR

Design K to minimise

J =

$$\frac{K(s+z_1)}{(s+P_1)(s+P_2)}$$

$$Ae^{-P_1 t} + Be^{-P_2 t}$$