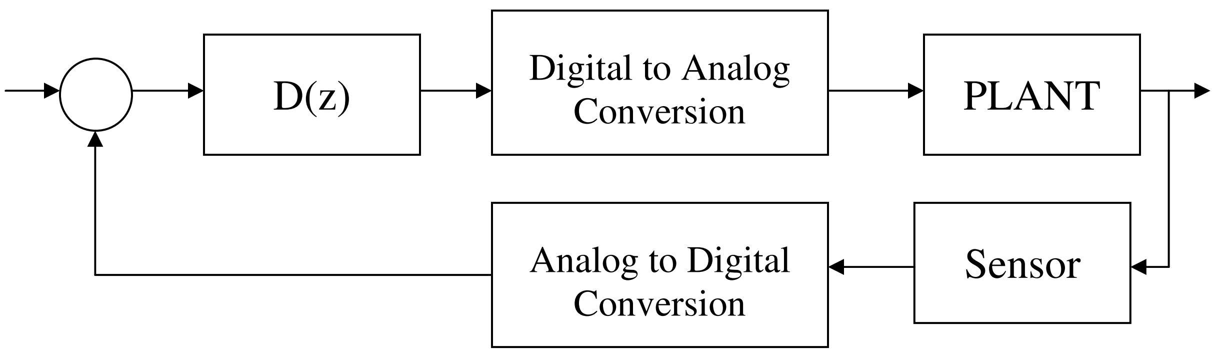
EE4002 Control Engineering

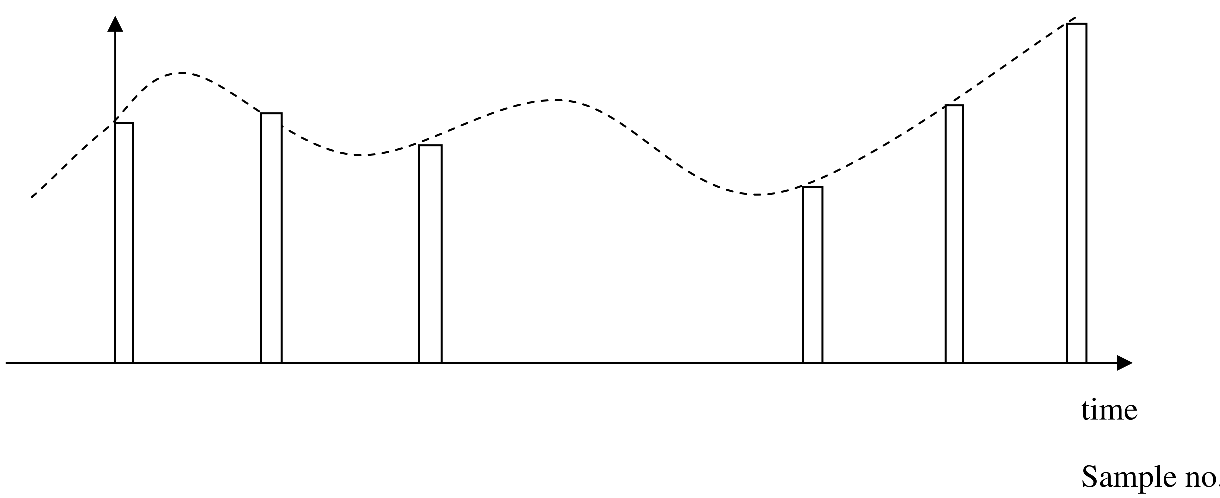
A) Digital Control Systems

Chapter 1. The Basics

Implementation of the control algorithm on a digital computer:



The signal c(t) is sampled, with sampling period T:



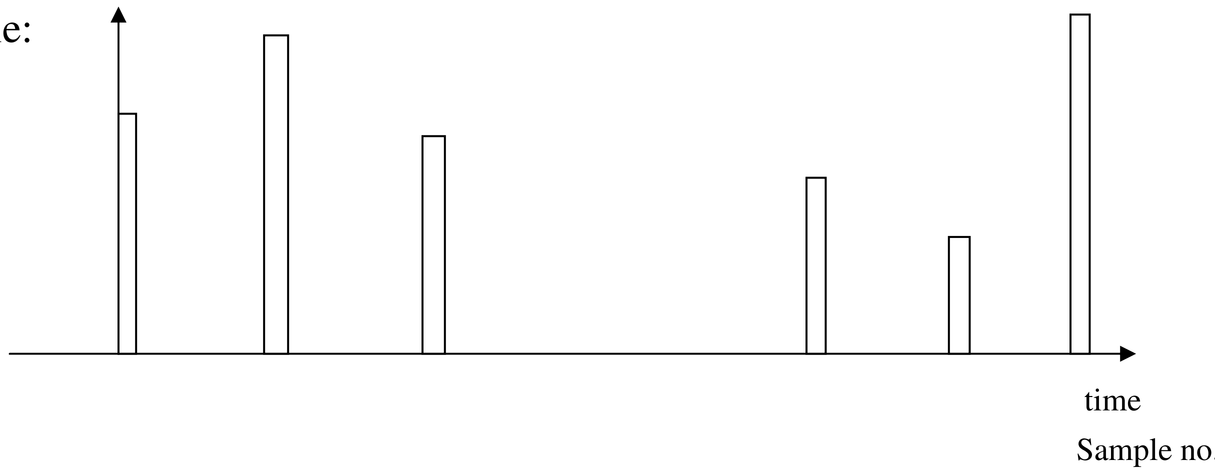
The Setpoint is read every T:

$$e(kT) = r(kT) - c(kT)$$

Control algorithm processes the error e(kT) to generate m(kT)
Digital to Analog convertor (DAC) converts this binary word representation of m(kT) to an analog voltage.

It is usual to hold the DAC output voltage constant between samples:

ie:



A typical digital proportional control algorithm could be:

```
While True Do
    Increment k
    Sample c(t)
    Read setpoint r(kT)
    Generate error, e(kT)=r(kT)-c(kT)
    Calculate control, m(kT)=Ke(kT)
    Convert to analog+hold
    Wait until period T elapses
End
```

1.1 Basic Approximation of Analog Controllers on a Digital Computer

Design the controller C(s) in the s plane – assuming a continuous system

Consider the continuous time PID controller:

$$m(t) = K\left(e(t) + \frac{1}{T_I}\int_0^t e(\tau)d\tau + T_d \frac{de}{dt} \right)$$

- Digital P Control

- Digital PI Control

$$m(t) = K\left(e(t) + \frac{1}{T_I}\int_0^t e(\tau)d\tau \right)$$

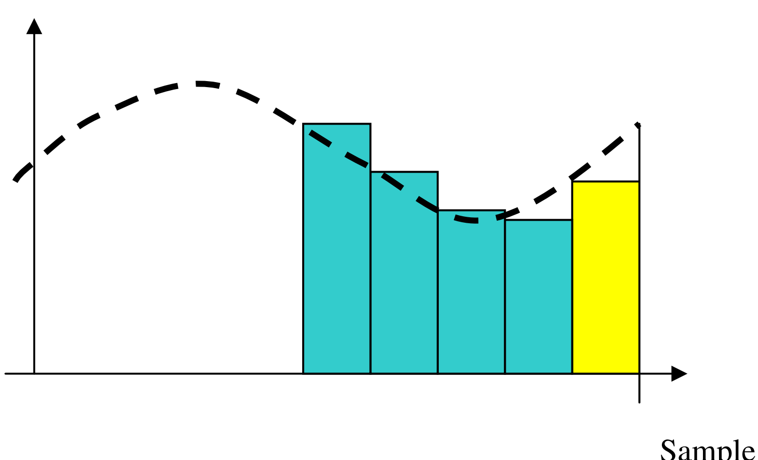
If we approximate the integral at the k^th sample instant as:

$$\int_0^{kT} e(\tau)d\tau \approx I(kT)$$

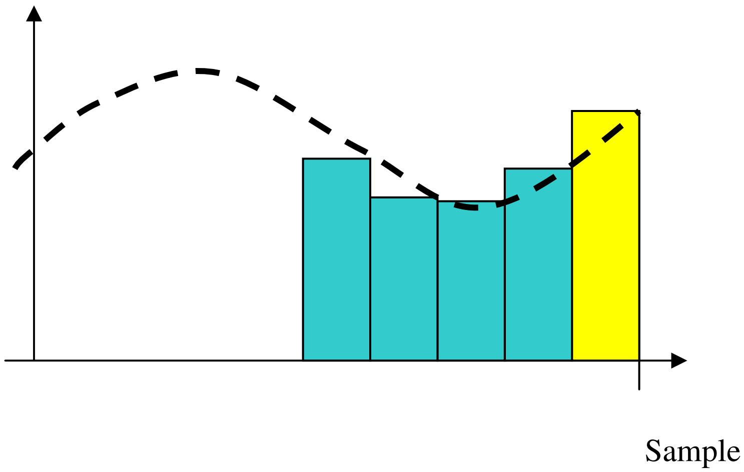
then the algorithm becomes:

There are many ways to get I(kT):

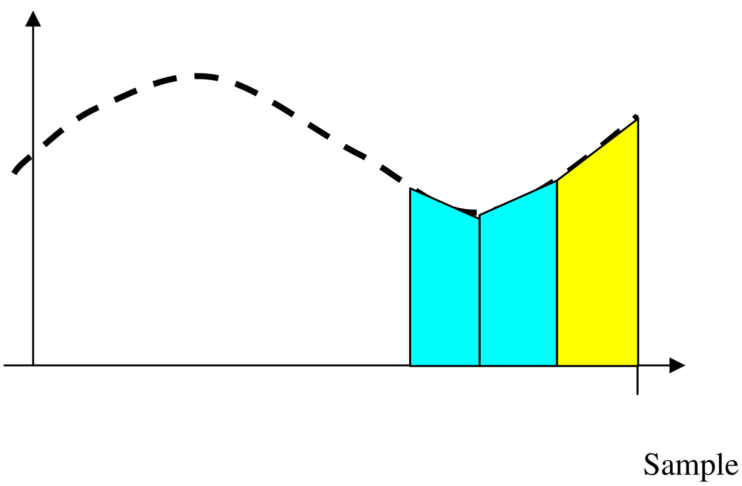
<u>i) Euler's Method</u>



ii) Backward Difference Method

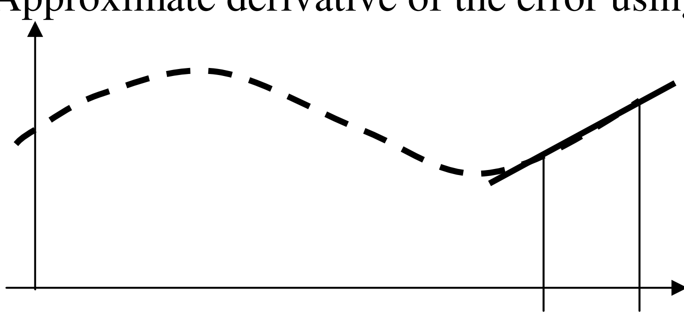


iii) Trapezium Method



- Digital PID Control

Approximate derivative of the error using finite difference:

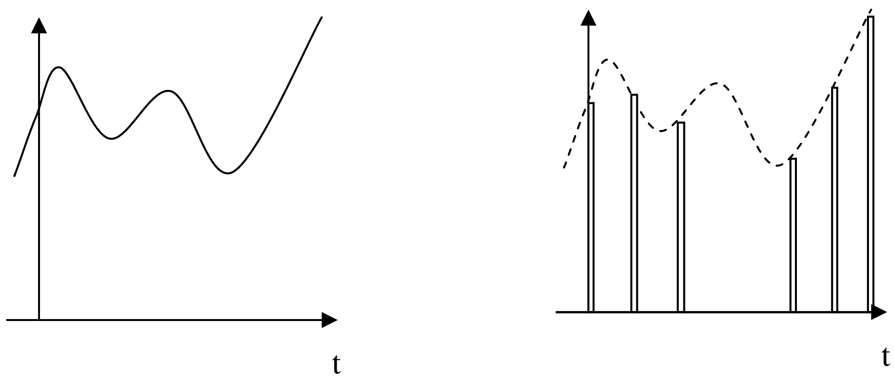


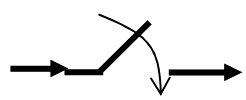
Consider the PID algorithm approximated using the forward difference algorithm:

$$
\begin{aligned}
e(kT) &= r(kT) - c(kT) \\
I(kT) &= I((k-1)T) + Te(kT) \\
D(kT) &= \frac{e(kT) - e((k-1)T)}{T} \\
m(kT) &= K\left(e(kT) + \frac{1}{T_I} I(kT) + T_d D(kT) \right)
\end{aligned}
$$

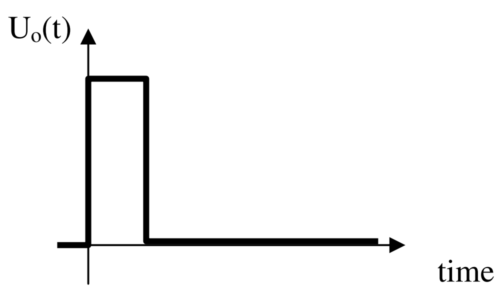
Chapter 2. The Z Transform

Consider a signal e(t) that has been sampled with period T:





First we will define the unit pulse $U_o(t)$ as:



Consider now the product of this unit pulse signal with the continuous signal e(t):

The sampled (or pulsed) signal $e^*(t)$ could then be written as:

$$e^*(t) = e(0)\cdot U_o(t) + e(T)\cdot U_o(t-T) + e(2T)\cdot U_o(t-2T) + \cdots$$

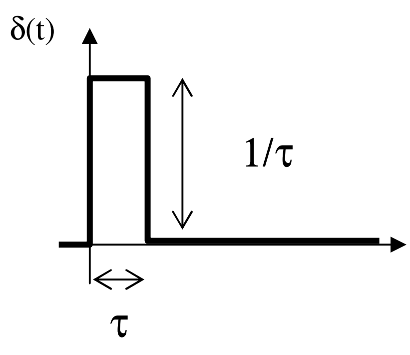
Which could be written as:

$$e^*(t) = \sum_{k=0}^{\infty} e(kT)\cdot U_o(t-kT)$$

Consider now the Laplace transform of the pulsed signal $e^*(t)$:

$$\mathcal{L}\{e^*(t)\} = \mathcal{L}\left\{\sum_{k=0}^{\infty} e(kT)\cdot U_o(t-kT)\right\}$$

But we know the Laplace transform of the unit impulse:
Revision:



Hence we can get the Laplace transform for $U_o(t)$ as:

Now by use of the ***real shift theorem***:

$$\mathcal{L}\{U_o(t-kT)\} = \tau e^{-skT}$$

Hence:

$$\mathcal{L}\{e^*(t)\} = \sum_{k=0}^{\infty} e(kT)\cdot\mathcal{L}\{U_o(t-kT)\} =$$

Now we will define the Z transform as follows:

With the inverse transform:

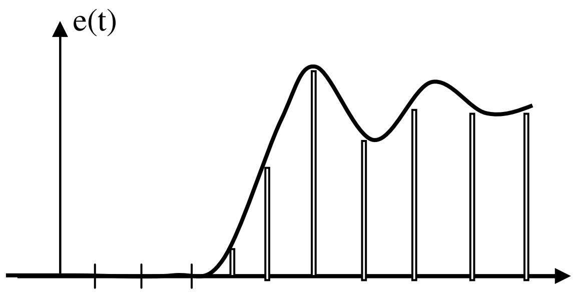
Then we can write:

The Z transform of a sampled sequence e(kT) is then defined as:

$$Z\{e(kt)\} = \sum_{k=0}^{\infty} e(kT)z^{-k}$$

<u>Examples:</u>

1)



$$E(z) = Z\{e(kt)\} = \sum_{k=0}^{\infty} e(kT)z^{-k}$$

2) The unit discrete pulse U_o (kT):

$$U_o(kT) = \begin{cases} 1 & for \quad k = 0 \\ 0 & for \quad k \neq 0 \end{cases}$$

$$U_o(z) = \mathrm{Z}\{U_o(kT)\} = \sum_{k=0}^{\infty} U_o(kT) z^{-k} =$$

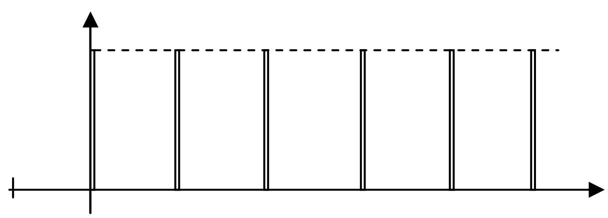
3) The time shifted unit discrete pulse:

$$U_o((k-m)T) = \begin{cases} 1 & for \quad k = m \\ 0 & for \quad k \neq m \end{cases}$$

$$\mathrm{Z}\{U_o((k-m)T)\} = \sum_{k=0}^{\infty} U_o((k-m)T) z^{-k} =$$

4) The unit step signal u(kT)

$$u(kT) = \begin{cases} 1 & for \quad k \geq 0 \\ 0 & for \quad k < 0 \end{cases}$$



$$U(z) = \mathrm{Z}\{u(kT)\} = \sum_{k=0}^{\infty} u(kT) z^{-k} =$$

5) (Tutorial) Unit ramp signal

Show that the Z transform of the following ramp signal,

$$r(t) = \begin{cases} \alpha t & for \quad t \geq 0 \\ 0 & for \quad t < 0 \end{cases}$$

sampled with sampling time T, is:

$$R(z) = \mathrm{Z}\{r(kT)\} = \frac{\alpha T z}{(z-1)^2}$$

6) (Tutorial) Show that the Z transform of the exponential signal,

$$f(t) = \begin{cases} Ke^{-at} & for \quad t \geq 0 \\ 0 & for \quad t < 0 \end{cases}$$

sampled with sampling time T, is:

$$F(z) = \mathrm{Z}\{f(kT)\} = \frac{Kz}{z - e^{-aT}}$$

Chapter 3. Spectrum of Sampled Signals

The sampled signal,

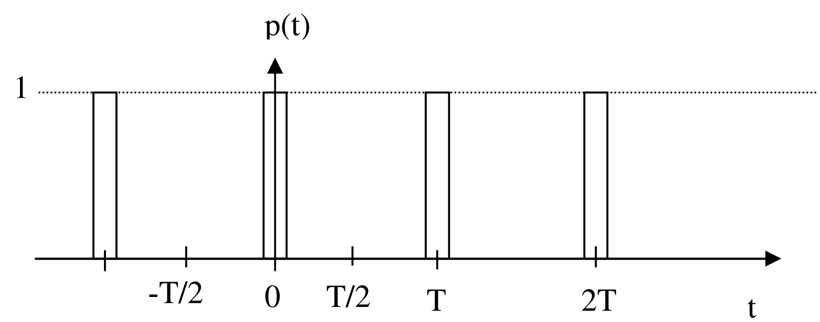
$$e^{*}(t) = \sum_{k=0}^{\infty} e(kT) \cdot U_o(t - kT)$$

Could be rewritten as:

$$e^{*}(t) = e(t) \sum_{k=0}^{\infty} U_o(t - kT) =$$

Where:

$$p(t) = \sum_{k=0}^{\infty} U_o(t - kT)$$



This periodic waveform could be represented by the Fourier Series expansion:

$$p(t) = \sum_{n=0}^{\infty} C_n e^{jn\omega_s t}$$

where the nth Fourier Coefficient is:

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-jn\omega_s t}$$

Using the identities:

$$\left.\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \end{aligned}\right\}$$

Then:

Note that:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} =$$

Hence:

$$\lim_{\tau \to 0} C_n =$$

and we can write an expression for p(t) as:

$$p(t) = \frac{\tau}{T}\sum_{n=0}^{\infty} e^{jn\omega_s t}$$

and the sampled (pulsed) signal is then:

$$e^*(t) = e(t)\cdot p(t) = \frac{\tau}{T}e(t)\sum_{n=0}^{\infty} e^{jn\omega_s t}$$

Now if we define:

$$E^*(j\omega) = F\left\{e^*(t)\right\} = \frac{\tau}{T}F\left\{\sum_{n=0}^{\infty} e^{jn\omega_s t}\cdot e(t)\right\}$$

and:

Now making use of the Complex Shift theorem:

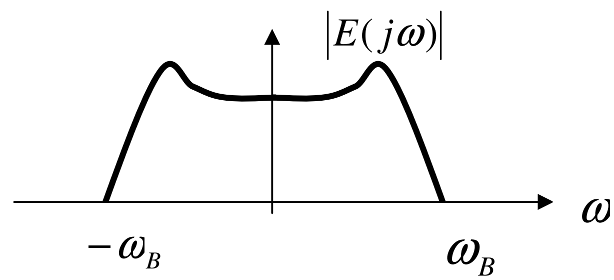
$$F\left\{g(t)e^{jat}\right\} = G\big(j(\omega - a)\big)$$

Then we can write for the sampled signal:

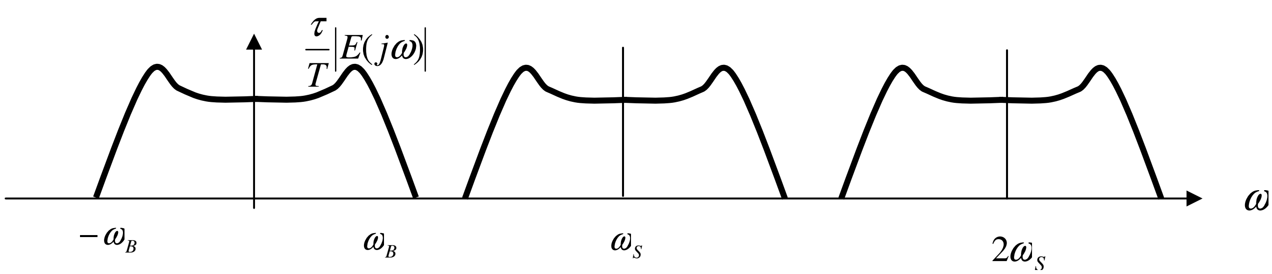
$$E^*(j\omega) = \frac{\tau}{T}\sum_{n=0}^{\infty} E\big(j(\omega - n\omega_s)\big)$$

Now if the spectrum for the signal e(t) looks like:

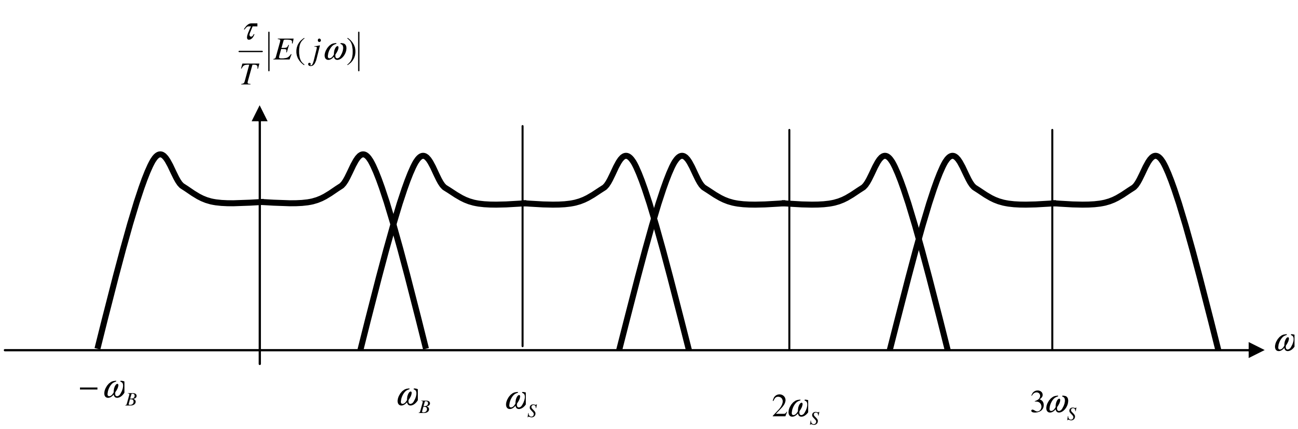
Consider that the spectrum of e(t) is:



Then the spectrum of the sampled signal e*(t) is:

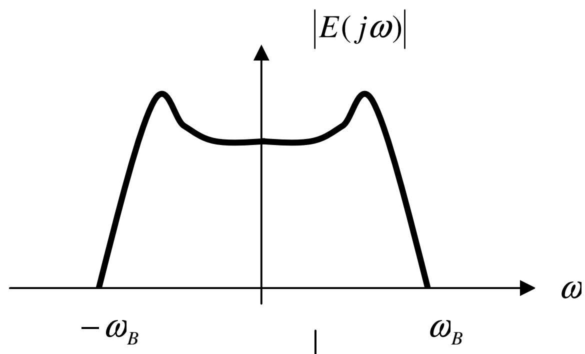


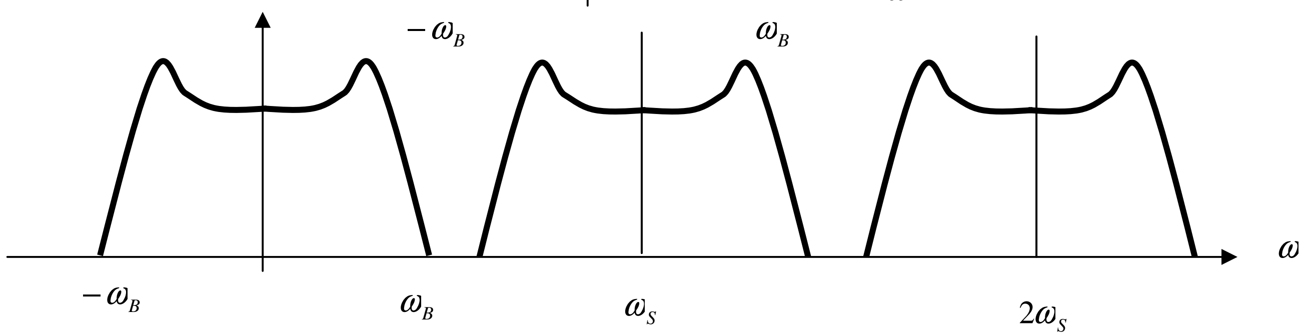
3.1 Shannon's Sampling Theorem



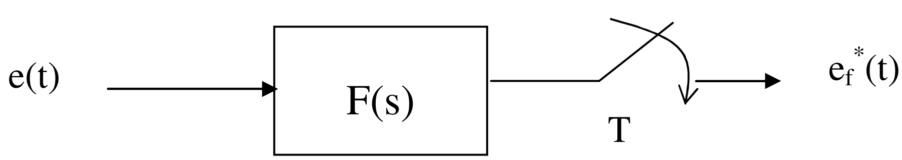
Shannon's Sampling Theorem: For a continuous time signal e(t) with $|E(j\omega)| = 0$ for $|\omega| \ge \omega_B$, then the sampling frequency should be chosen as $\omega_S \ge 2\omega_B$ to ensure that aliasing does not occur.

In Practice, there is not a finite spectrum to e(t) due to noise:

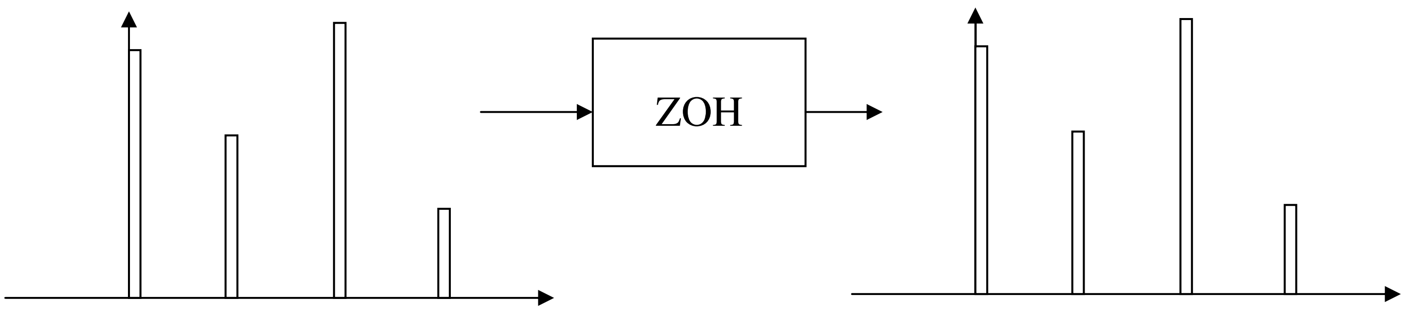




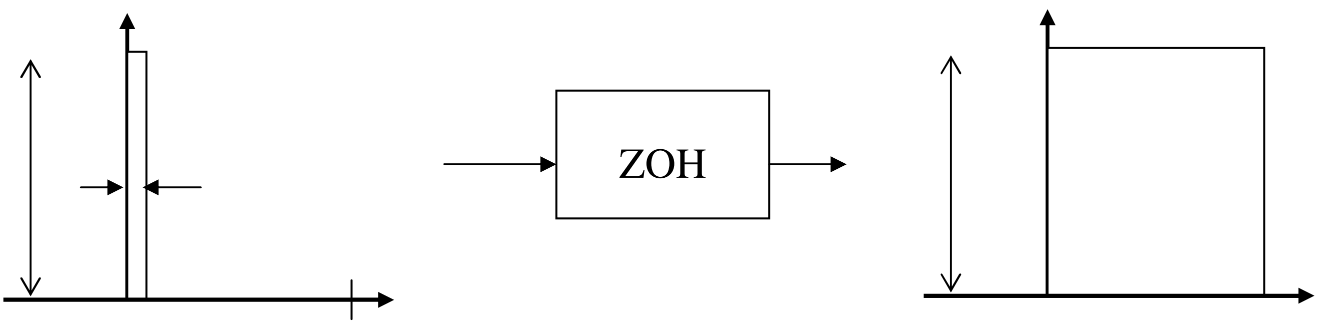
Essential to prefilter the signal e(t), before sampling to avoid large aliasing errors:

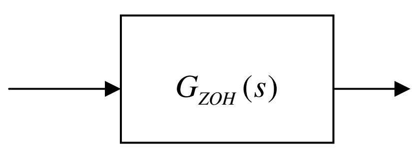


3.2 Transfer Function of a Zero-Order Hold: $G_{ZOH}(s)$



Consider the impulse response of the ZOH





The transfer function $G_{ZOH}(s)$ is obtained by the Laplace transform of the impulse response:

$$G_{ZOH}(s) = L\{g_{ZOH}(t)\} = \int_0^{\infty} g_{ZOH}(t) e^{-st} dt$$

Hence the following block diagram can be drawn for the ZOH:

$$M^*(s) \longrightarrow \boxed{\frac{1-e^{-sT}}{s\tau}} \longrightarrow U(s)$$

The frequency response is determined as G(jω):

$$G_{ZOH}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega\tau}$$

Using the identity,

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Then:

$$G_{ZOH}(j\omega) = \frac{1-\cos\omega T + j\sin\omega T}{j\omega\tau} = \left|G_{ZOH}(j\omega)\right| \angle \arg\left(G_{ZOH}(j\omega)\right)$$

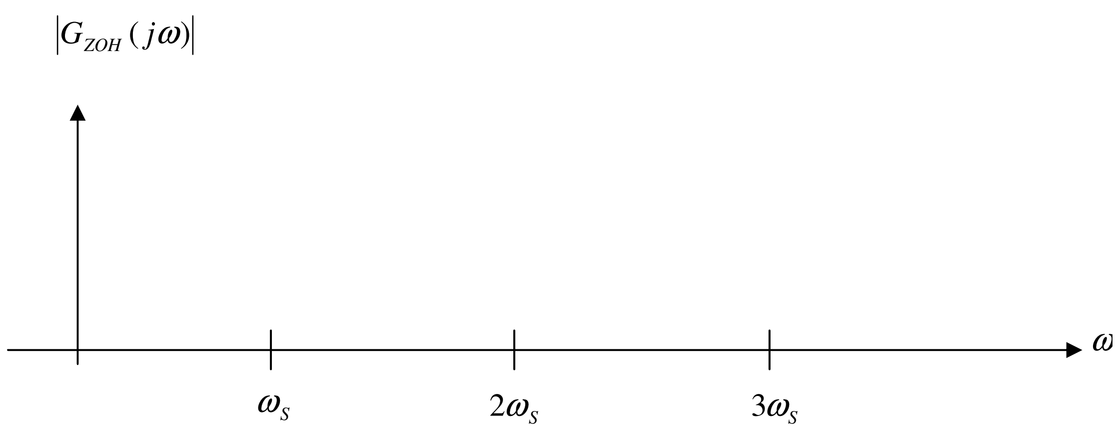
Using the identity:

$$1-\cos\theta = 2\sin^2\left(\theta/2\right)$$

then:

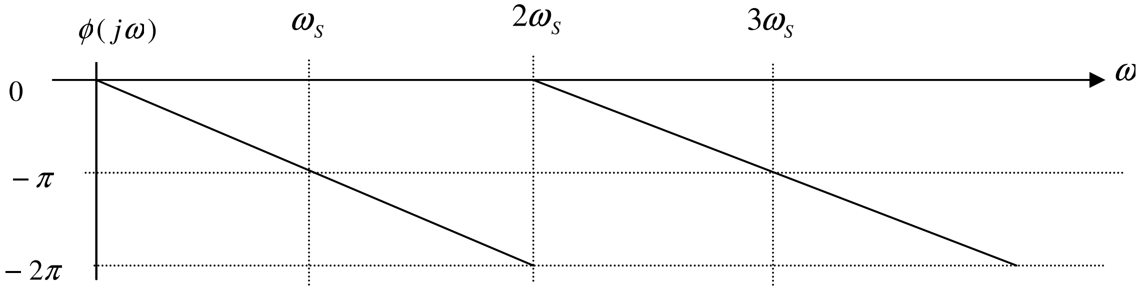
$$\left|G_{ZOH}(j\omega)\right| = \frac{\sqrt{2(1-\cos\omega T)}}{\omega\tau} =$$

which has the gain frequency response plot:

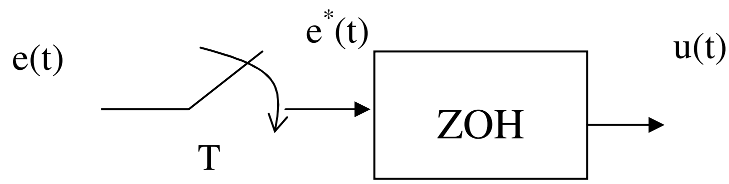


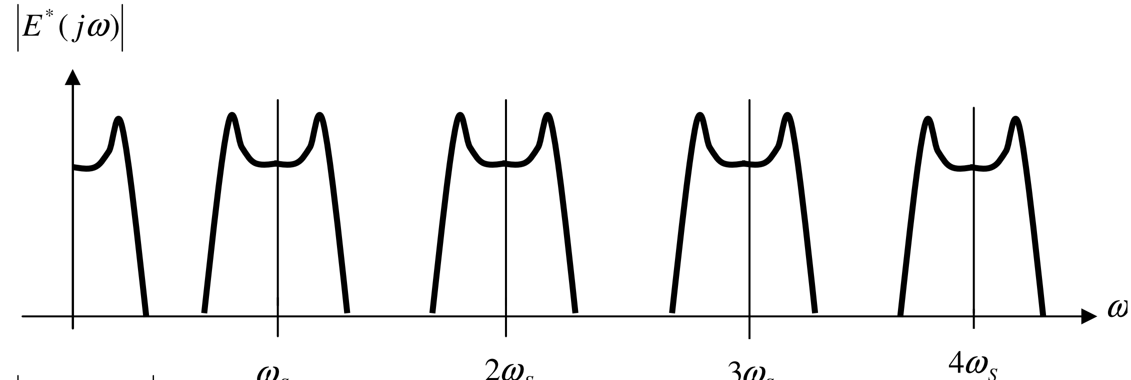
Tutorial: Show that the phase is given by:

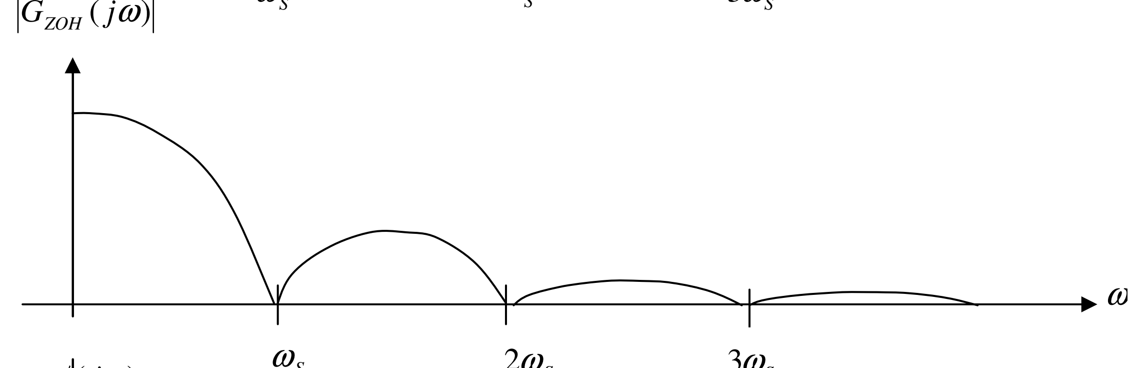
$$\phi(j\omega) = \angle G_{ZOH}(j\omega) = -\frac{\omega}{\omega_S}\pi \quad radians$$

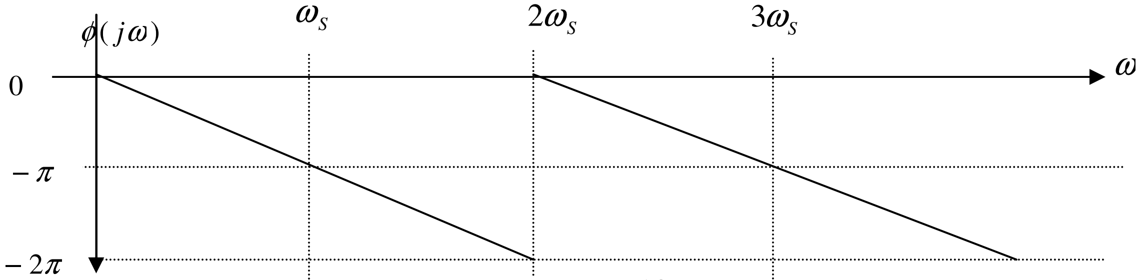


3.3 The Effect of Sampling+Hold on the Spectrum

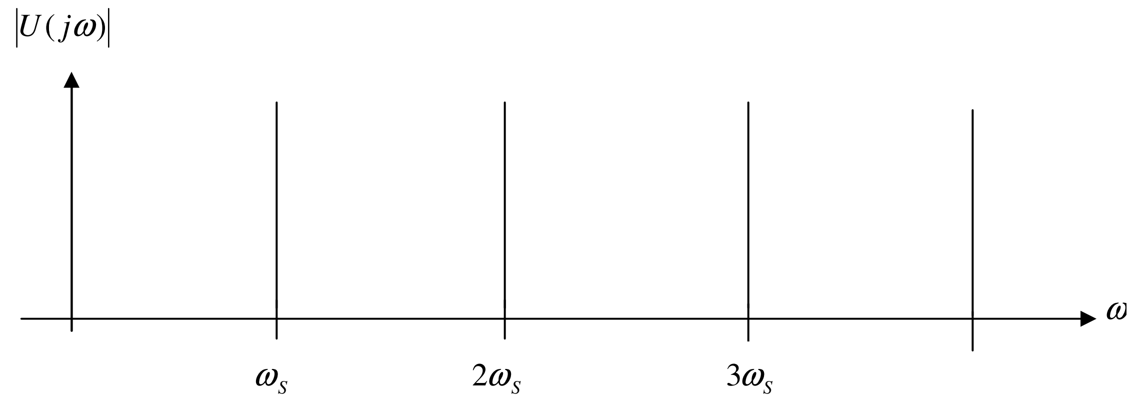




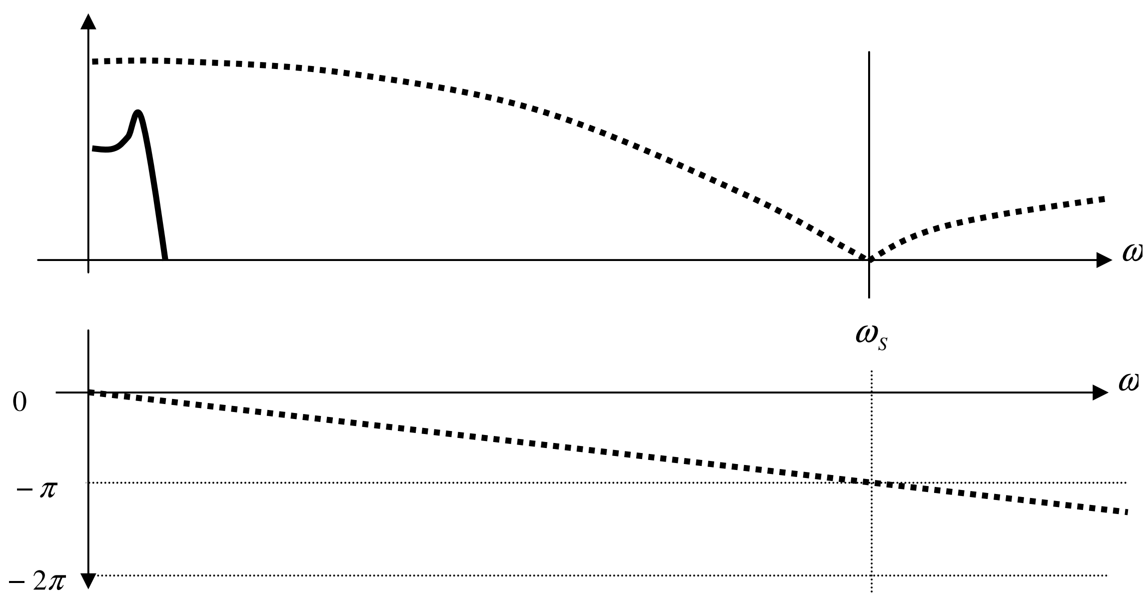




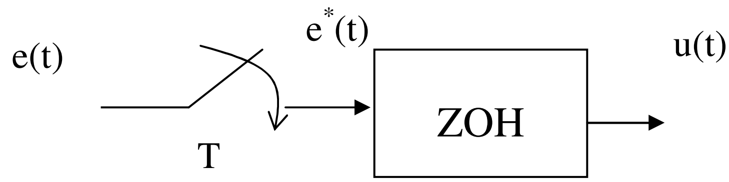
This yields the spectrum for the control input as:



3.4 The Benefits of Oversampling



Now lets get rid of the pulse-width τ:



The Laplace transform of the sampled signal $e^{*}(t)$ is:

$$E^{*}(s) = \boldsymbol{L}\left\{e^{*}(t)\right\} = \tau \sum_{k=0}^{\infty} e(kT)e^{-skT}$$

And the transfer function of the zero order hold is:

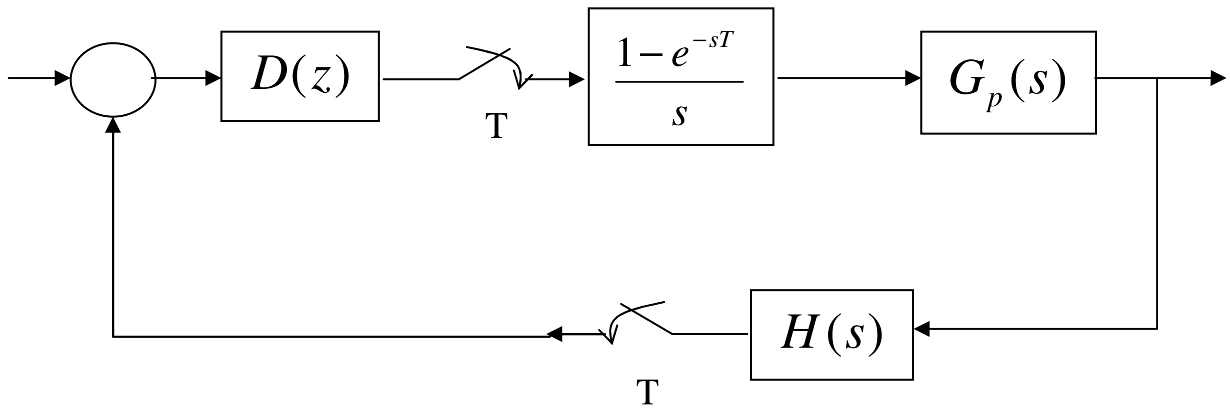
$$G_{ZOH}(s) = \frac{1-e^{-sT}}{s\tau}$$

We will now redefine the following as:

$$\boldsymbol{L}\left\{e^{*}(t)\right\} = \sum_{k=0}^{\infty} e(kT)e^{-skT} \quad =$$

$$G_{ZOH}(s) = \frac{1-e^{-sT}}{s}$$

This will allow the following block diagram to be drawn for a process under digital control:



4.1 The Discrete Time Transfer Function

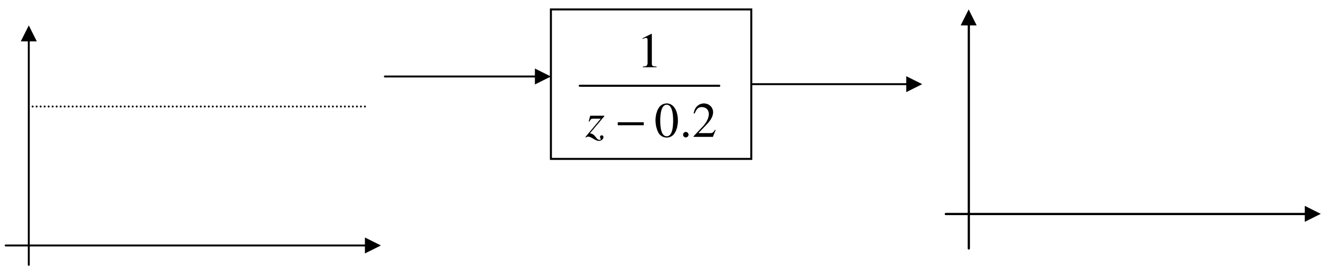
Where;

$$\frac{Y(z)}{U(z)} = G(z)$$

In general for an nth order discrete system we can write:

$$G(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}}$$

Consider the following discrete time system excited by a unit step sequence:



The response can be solved using a number of methods – here we will look at two:

<u>i) Partial Fractions + Tables</u>

The input is a unit step:

$$U(z) = \frac{1}{1-z^{-1}} =$$

The output is given in the Z domain as:

$$Y(z) = G(z)U(z) =$$

Now consider Y(z)/z:

$$Y(z)/z = \frac{1}{(z-1)(z-0.2)} =$$

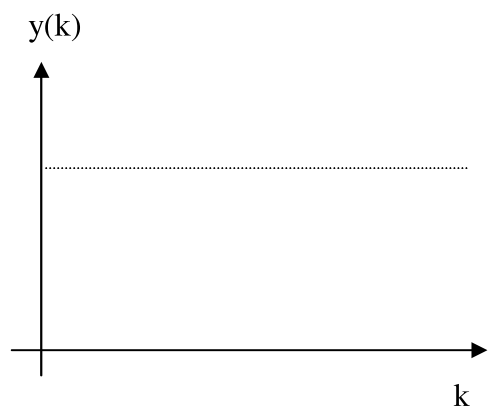
By partial fractions, we can write:

$$Y(z)/z = \frac{\quad}{(z-1)} + \frac{\quad}{(z-0.2)}$$

Or

$$Y(z) = \frac{\quad}{(z-1)} + \frac{\quad}{(z-0.2)}$$

Using the Z transform tables:



ii) Solving Difference Equations

$$Y(z) = G(z)U(z) = \frac{1}{z-0.2}\ U(z)$$

which could be rewritten as:

$$(z-0.2)Y(z)=U(z)$$

Remembering the definition of the z operator:

$$\mathbf{Z}\{f(k-1)\}=z^{-1}F(z)$$

Taking inverse Z transforms yields:

If the initial conditions are defined as:

Then this difference equation can be solved over the time interval of interest:

Sample k	0	1	2	3	4	5
U(k)						
Y(k)						

Generating a difference equation from the general transfer function:

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}} U(z)$$

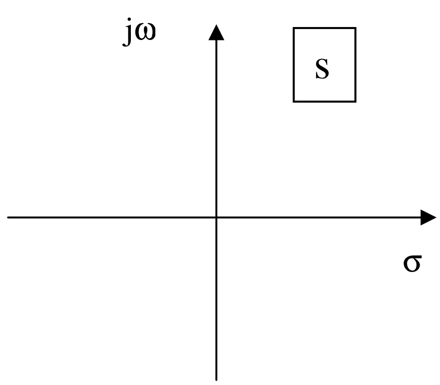
$$(1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}) U(z)$$

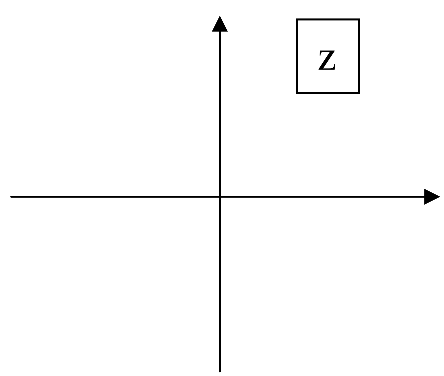
Taking inverse Z transforms, yields:

4.2 Stability of Discrete Transfer Functions

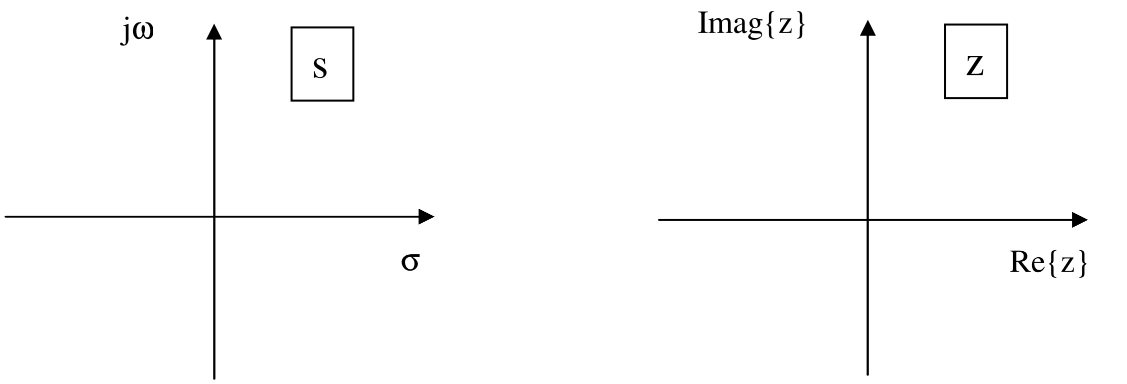
Consider the mapping from the s to the z planes:

$$z = e^{sT}$$

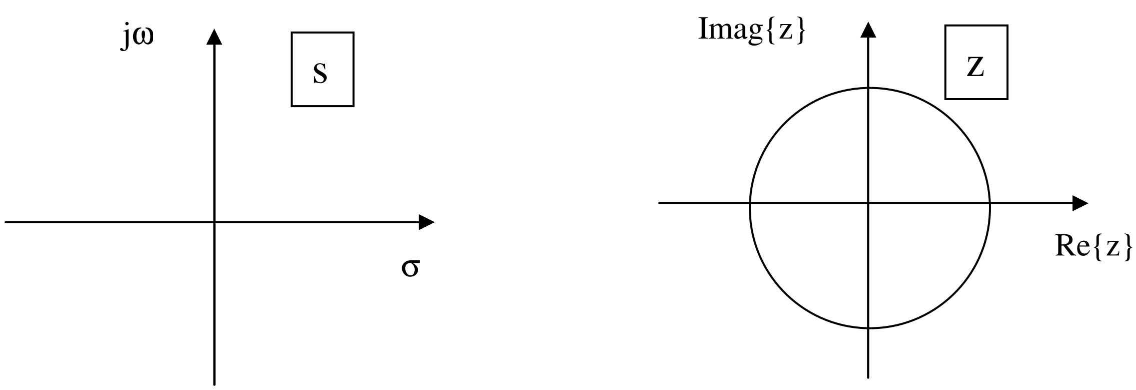




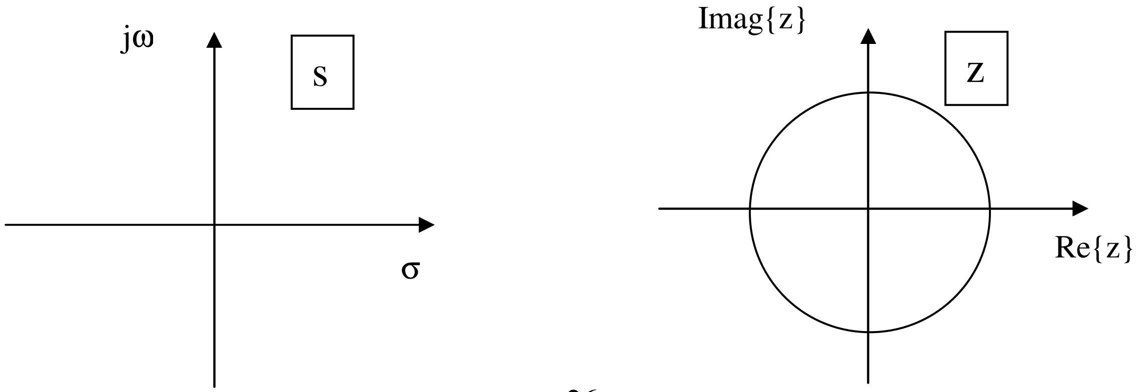
Consider now the mapping of the imaginary axis s=jω from the s plane to the z plane:



Consider now all points on the left hand side of the s plane:



And all points to the right hand side are mapped as follows:



This leads to the following definition for the stability of a discrete time transfer function G(z):

$$G(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}} = \frac{K(z - z_1)(z - z_2)\cdots(z - z_q)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

- There are n poles:
- There are q zeros:

For high order characteristic polynomials we could use:

4.3 Steady-State Performance

$$Y(z) = G(z)U(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}} U(z)$$

Use the final value theorem:

$$\underset{k \to \infty}{Lim}\, y(k) = \underset{z \to 1}{Lim}(z - 1)Y(z)$$

Consider that u(k) is the step input:

Then:

$$y_{final} = \lim_{z \to 1} (z-1) \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}} \frac{Az}{z-1}$$

The steady state (DC) gain of the transfer function is:

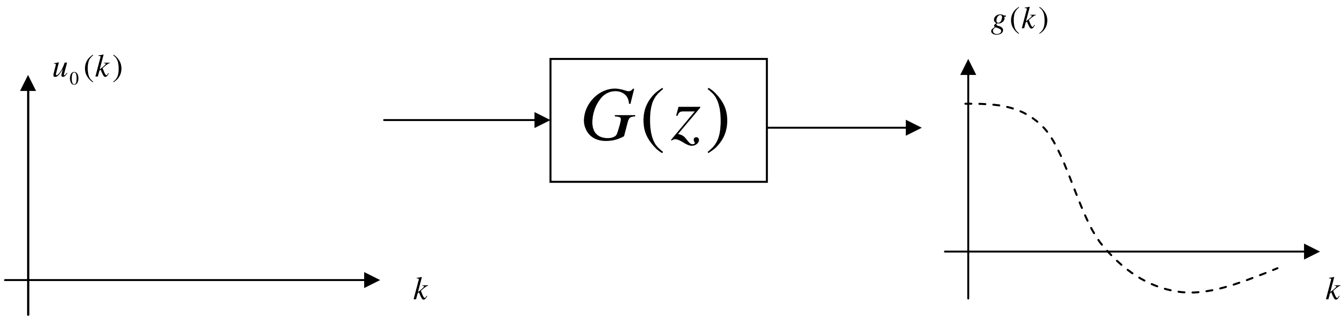
$$\frac{y_{final}}{A} =$$

4.4 The Transfer Function and Discrete Convolution

$$Y(z) = G(z)U(z)$$

$$y(k) = \mathbf{Z}^{-1}\{G(z)U(z)\} =$$

Consider the following impulse response:



Hence:

$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k}$$

Then:

$$Y(z) = G(z)U(z) = \left(g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots\right)U(z)$$

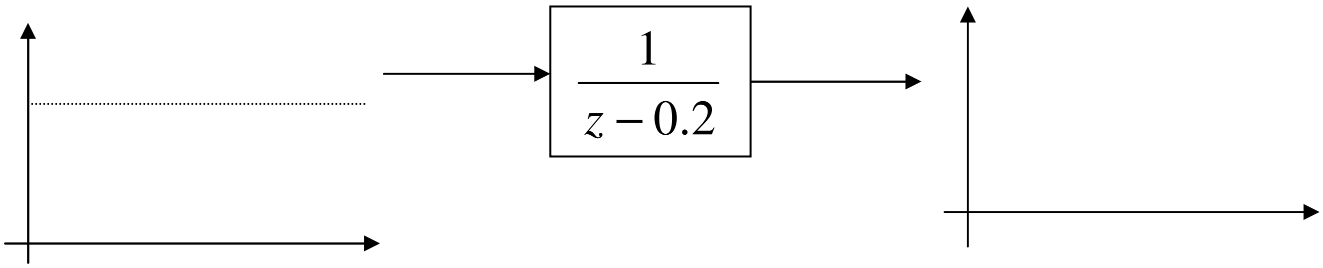
Taking inverse Z transforms yields:

This leads to the discrete-time convolution model:

$$y(k) = \sum_{i=0}^{\infty} g_i u(k-i)$$

Example:

Consider the following discrete time system excited by a unit step sequence:



First determine the impulse response g(k):

$$z-0.2\overline{)\quad 1\qquad\qquad}$$

Hence we get G(z) as an infinite power series

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + g_3 z^{-3} + \cdots g_i z^{-i} + \cdots$$

Consider the unit step input:

Then since:

$$y(k) = \sum_{i=0}^{\infty} g_i u(k-i)$$

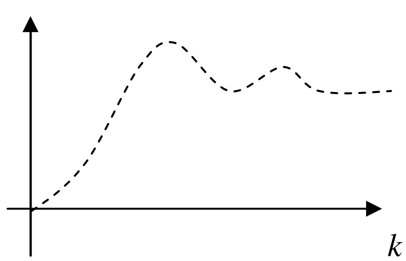
$$y(0) = g_0 u(0) =$$

$$y(1) = g_0 u(1) + g_1 u(0) =$$

$$y(2) = g_0 u(2) + g_1 u(1) + g_2 u(0) =$$

<u>Relationship between impulse and step response models</u>

Consider the unit step response of a discrete system:



$$Y(z) = G(z)U(z) = \left(g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots\right)\frac{1}{1-z^{-1}}$$

hence we could write:

$$(1-z^{-1})\left(h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots h_{N-1} z^{-N+1} + h_N z^{-N} + h_N z^{-N-1} + h_N z^{-N-2} \cdots\right)$$
$$= g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots$$