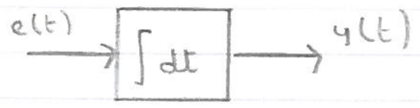
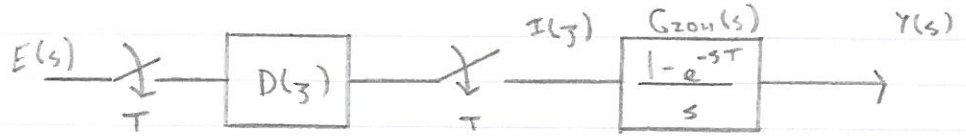
Summer 2006

Q 1.



Digital approximation of analog integrator



$$\int_0^t e(\tau)\,d\tau \approx I(k) = I(k-1) + T/2\,[e(k) + e(k-1)]$$

Taking Z transforms

$$I(z) = z^{-1} I(z) + T/2\,(E(z) + z^{-1} E(z))$$

$$I(z)(1 - z^{-1}) = T/2\, E(z)(1 + z^{-1})$$

$$I(z) = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) E(z)$$

$$\frac{I(z)}{E(z)} = \frac{T}{2}\,\frac{(1+z^{-1})}{(1-z^{-1})} = \frac{T}{2}\left(\frac{z+1}{z-1}\right) = \frac{1}{s}$$

$$s = \frac{2}{T}\,\frac{z-1}{z+1}$$

(b). $m(t) = K_P\left(e(t) + 1/T_I \int_0^t e(\tau)\,d\tau\right)$

$$M(s) = K_P\left(E(s) + \frac{1}{T_I s} E(s)\right)$$

$$M(s) = K_P E(s)\left(1 + \frac{1}{T_I s}\right)$$

$$\Rightarrow \frac{M(s)}{E(s)} = K_P\left[1 + \frac{1}{T_I s}\right]$$

Using Tustin's

$$D(z) = K_P\left[1 + \frac{1}{T_I}\cdot\frac{T}{2}\,\frac{z+1}{z-1}\right]$$

$$= K_P\left[\frac{2T_I(z-1) + T(z+1)}{2T_I(z-1)}\right]$$

$$= K_P\left[\frac{2T_I z - 2T_I + Tz + T}{2T_I(z-1)}\right]$$

$$= \frac{K_P}{2T_I}\left[\frac{T-2T_I+z(T+2T_I)}{z-1}\right]$$

$$= \frac{K_P(T+2T_I)}{2T_I}\left[\frac{\frac{T-2T_I}{T+2T_I}+z}{z-1}\right]$$

$$= K_P\left(1+\frac{T}{2T_I}\right)\left[\frac{z+\frac{\frac{T}{2T_I}-1}{\frac{T}{2T_I}+1}}{z-1}\right]$$

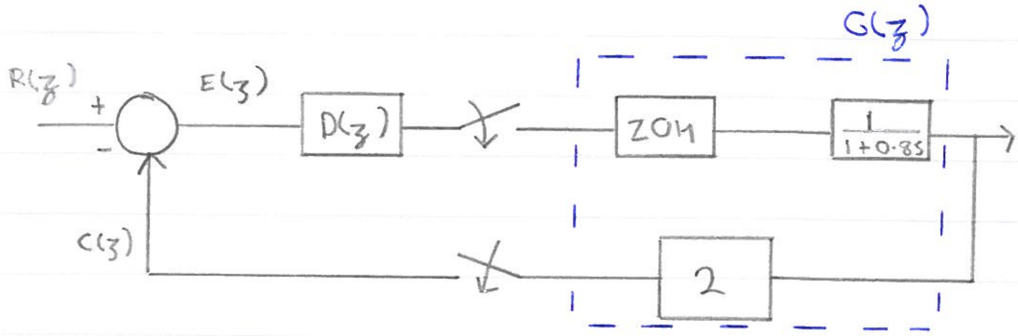
$$= K_P\left(1+\frac{T}{2T_I}\right)\left[\frac{z-\frac{1-\frac{T}{2T_I}}{1+\frac{T}{2T_I}}}{z-1}\right]$$

$$= K_d\left(\frac{z-\gamma}{z-1}\right)$$

(ii) $K_P = 0.81 \quad T_I = 2.5s$

$$K_d = \left(1+\frac{1}{2(2.5)}\right)(0.81) = 0.96$$

$$\gamma = \frac{1-\frac{1}{2(2.5)}}{1+\frac{1}{2(2.5)}} = 0.67$$



$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}\cdot\frac{1}{1+0.8s}\cdot 2\right\}$$

$$= \frac{2}{0.8}\,\mathcal{Z}\left\{\frac{1-e^{-Ts}}{s}\cdot\frac{1}{s+1.25}\right\}$$

$$= 2.5\,(1-z^{-1})\,\mathcal{Z}\left\{\frac{1}{s}\cdot\frac{1}{s+1.25}\right\}$$

From tables

$$\mathcal{Z}\left\{\frac{1}{s(s+a)}\right\} = \frac{1}{a}\;\frac{(1-e^{-aT})\,z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$$

$a = 1.25$
$T = 1$

$$= \frac{1}{2.5}\;\frac{(1-e^{-1.25})\,z^{-1}}{(1-z^{-1})(1-e^{-1.25}z^{-1})}$$

$$\Rightarrow G(z) = \frac{2.5}{1.25}\;\frac{(1-z^{-1})(0.713z^{-1})}{(1-z^{-1})(1-0.287z^{-1})} = \frac{1.427}{z-0.287}$$

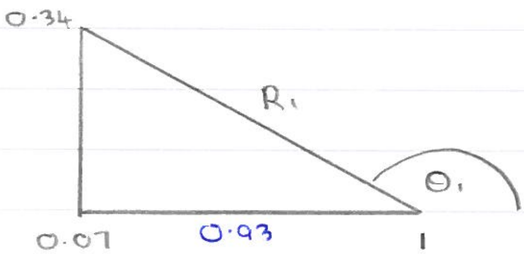
$$D(z) = \frac{0.96(z-0.67)}{z-1}$$

$D(z)$: pole at $+1$ $\qquad$ $G(z)$: pole at 0.287
zero at 0.67

From the root locus plot it can be seen that the locus does not pass through the desired point determined by ω_n and ξ $\Rightarrow$ will not achieve spec

(iii) Desired location
$z = 0.07 + j0.34$ (approx)

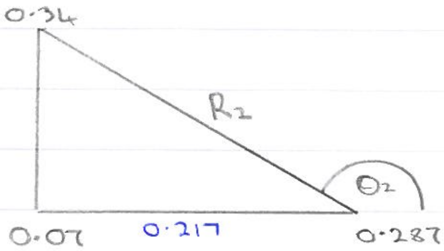
Pole at +1



$$R_1^2 = 0.93^2 + 0.34^2$$
$$\Rightarrow R_1 = 0.403$$

$$\theta_1 = 180 - \tan^{-1}\left(\frac{0.34}{0.93}\right)$$
$$= 159.92^\circ$$

Pole at 0·287



$R_2^2 = 0.217^2 + 0.34^2$

$\Rightarrow R_2 \simeq 0.403$

$\theta_2 = 180 - \tan^{-1}\left(\frac{0.34}{0.217}\right)$

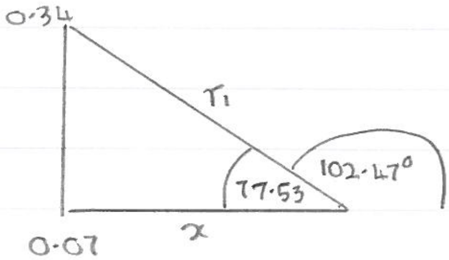
$= 122.547^\circ$

Zero location is unknown

Find angle from $\theta_1 + \theta_2 - \theta_z = 180$

$\Rightarrow \theta_z = \theta_1 + \theta_2 - 180$

$= 159.92^\circ + 122.547 - 180^\circ = 102.47^\circ$



$\tan(77.53) = \frac{0.34}{x}$

$\Rightarrow x = 0.075$

$r_1^2 = 0.34^2 + 0.075^2$

$\Rightarrow r_1 = 0.34$

$|D(z)G(z)|_{z=0.07+j0.34} = 1$

$\cancel{\Rightarrow \left|\frac{0.96(1.427)(0.07+j0.34-0.67)}{(0.07+j0.34-1)(0.07+j0.34-0.287)}\right|}$

$\frac{1.427 K_d r_1}{R_1 R_2} = 1$

$\frac{1.427 K_d (0.34)}{0.403(0.99)} = 1$

$$\Rightarrow K_d = 0.822$$

$$\gamma = 0.145$$

$$\gamma = \frac{1 - \frac{T}{2T_I}}{1 + \frac{T}{2T_I}} = \frac{1 - \frac{1}{2T_I}}{1 + \frac{1}{2T_I}} = \frac{2T_I - 1}{2T_I + 1}$$

$$\Rightarrow 2T_I \gamma + \gamma = 2T_I - 1$$

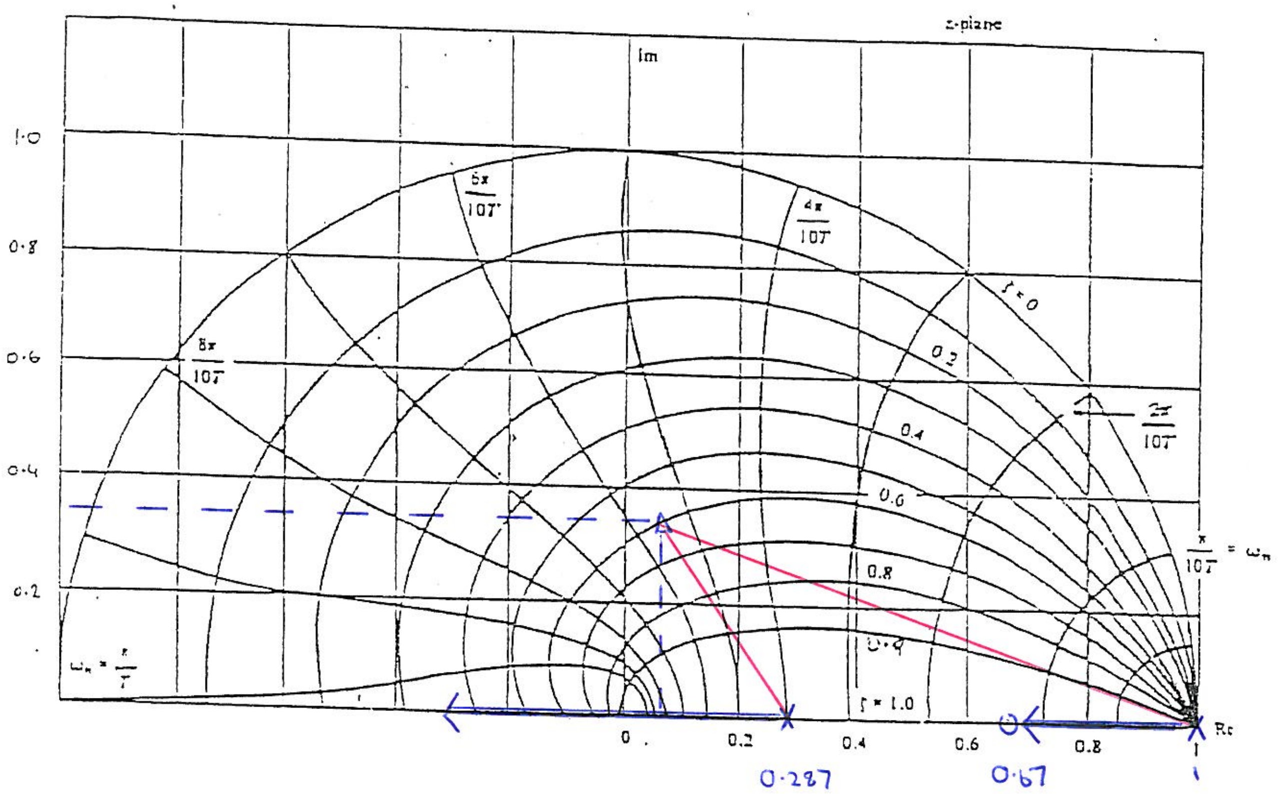
$$2T_I(\gamma - 1) = -(1 + \gamma)$$

$$T_I = \frac{-(1+\gamma)}{2(\gamma - 1)} = \frac{-1.145}{2(0.145-1)} \approx 0.67s$$

$$K_d = 0.822 = \left(1 + \frac{T}{2T_I}\right) K_p$$

$$0.822 = \left(1 + \frac{1}{2(0.67)}\right) K_p$$

$$K_p \approx 0.47$$



Z Plane Design Template

<u>*Please submit with your script*</u>

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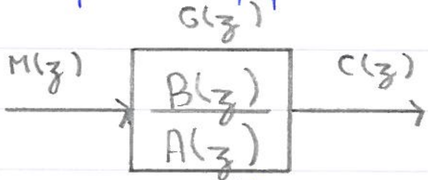
Q3 (a). $G(z) = \dfrac{\gamma z^{-2}}{1 + \alpha z^{-1} + \beta z^{-2}} = \dfrac{\gamma}{z^2 + \alpha z + \beta} = \dfrac{B(z)}{A(z)}$

Consider the open loop process

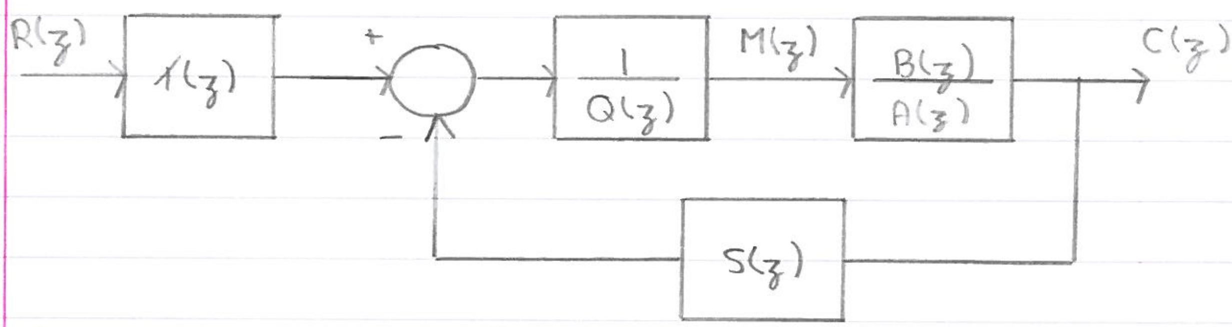


where the process is n^{th} order and:

$A(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n$

$B(z) = b_1 z^{n-1} + b_2 z^{n-2} + \ldots + b_m z^{n-m} \qquad n > m$

Consider the closed-loop control scheme:

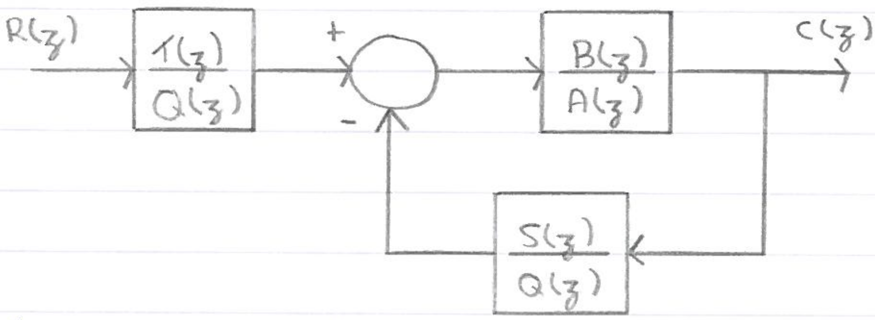


The control-law is then:

$M(z) = \frac{1}{Q(z)}\left(T(z)R(z) - S(z)C(z)\right)$

$= \frac{T(z)}{Q(z)} R(z) - \frac{S(z)}{Q(z)} C(z)$

This could be redrawn as



The closed loop transfer function is given by

$\dfrac{C(z)}{R(z)} = \dfrac{T(z)}{Q(z)} \dfrac{B(z)/A(z)}{1 + \frac{B(z)}{A(z)}\frac{S(z)}{Q(z)}} = \dfrac{B(z)T(z)}{A(z)Q(z) + B(z)S(z)}$

The characteristic equation for the closed-loop system is:

$$A(z)Q(z) + B(z)S(z) = 0$$

The controller polynomials are defined as

$$T(z) = t_0 z^{n_t} + t_1 z^{n_t-1} + t_2 z^{n_t-2} + \dots + t_{n_t}$$

$$S(z) = s_0 z^{n_s} + s_1 z^{n_s-1} + s_2 z^{n_s-2} + \dots + s_{n_s}$$

$$Q(z) = z^{n_q} + q_1 z^{n_q-1} + q_2 z^{n_q-2} + \dots + q_{n_q}$$

For causal control

$T(z)/Q(z)$ and $S(z)/Q(z)$ must both be causal

$\Rightarrow n_q \geq n_t \; ; \; n_q \geq n_s$

The Diophantine Equation is

$$A_{cl}(z) = A(z)Q(z) + B(z)S(z)$$

Assume without loss of generality that:

$\deg(A(z)) = n$; Let $n_q = n_s = n-1$ and $m = n$

$$(z^n + a_1 z^{n-1} + \dots + a_n)(z^{n-1} + q_1 z^{n-2} + \dots + q_{n-1}) + (b_1 z^{n-1} + b_2 z^{n-2} + \dots + b_n)(s_0 z^{n-1} + s_1 z^{n-2} + \dots + s_{n-1})$$
$$= z^{2n-1} + c_1 z^{2n-2} + \dots + c_{2N-1}$$

Comparing similar powers of z

z^{2n-1}: $1 = 1$

z^{2n-2}: $c_1 = a_1 + q_1 + b_1 s_0 \Rightarrow c_1 - a_1 = q_1 + b_1 s_0$

z^{2n-3}: $c_2 = a_2 + q_2 + a_1 q_1 + b_2 s_0 + b_1 s_1$

z^{2n-4}: $c_3 = a_3 + q_3 + a_1 q_2 + a_2 q_1 + b_1 s_2 + b_2 s_1 + b_3 s_0$

$\vdots \qquad \vdots$

$$\left[\begin{array}{c|c} \overbrace{\begin{matrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & 0 \dots & 0 \end{matrix}}^{n-1} & \overbrace{\begin{matrix} b_1 & 0 & \dots & & 0 \\ b_2 & b_1 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & 0 \dots & 0 \end{matrix}}^{n} \end{array}\right] \left[\begin{array}{c} q_1 \\ \vdots \\ q_{n-1} \\ \hline s_0 \\ \vdots \\ s_{n-1} \end{array}\right] \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n-1 \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n \end{matrix} = \left[\begin{array}{c} c_1 - a_1 \\ c_2 - a_2 \\ \vdots \\ c_n - a_n \\ c_{n+1} \\ \vdots \\ c_{2n-1} \end{array}\right]$$

The complete equations are then

$$
\begin{array}{cc}
2n-1 &
\left[\begin{array}{ccccc|cccc}
1 & 0 & 0 & \cdots & 0 & b_1 & 0 & \cdots & 0 \\
a_1 & 1 & 0 & \cdots & 0 & b_2 & b_1 & \cdots & 0 \\
a_2 & a_1 & 1 & \cdots & 0 & b_3 & b_2 & \cdots & 0 \\
a_3 & a_2 & a_1 & \cdots & 0 & b_4 & b_3 & \cdots & 0 \\
a_4 & a_3 & a_2 & \cdots & 0 & b_5 & b_4 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & 0 & \vdots & \vdots & \ddots & 0 \\
a_{n-2} & a_{n-3} & a_{n-4} & \cdots & 1 & b_{n-1} & b_{n-2} & \cdots & 0 \\
a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & b_n & b_{n-1} & \cdots & b_1 \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & 0 & b_n & \cdots & b_2 \\
0 & a_n & a_{n-1} & \cdots & a_3 & 0 & 0 & \cdots & b_3 \\
0 & 0 & 0 & \cdots & a_4 & 0 & 0 & \cdots & b_4 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_n & 0 & 0 & \cdots & b_n
\end{array}\right]
\\
& \underbrace{\qquad\qquad\qquad\qquad}_{n-1} \quad \underbrace{\qquad\qquad\quad}_{n}
\end{array}
\begin{array}{c}
\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{n-1} \\ s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \end{bmatrix} \\
2n-1 \text{ unknowns}
\end{array}
=
\begin{bmatrix} c_1 - a_1 \\ c_2 - a_2 \\ c_3 - a_3 \\ \vdots \\ c_n - a_n \\ c_{n+1} \\ c_{n+2} \\ c_{n+3} \\ \vdots \\ c_{2n-1} \end{bmatrix}
$$

$n = 2$, $a_1 = \alpha$, $b_1 = 0$, $a_2 = \beta$, $b_2 =$

$n_q = n_s = (n-1) = 1$

$Q(z) = z + q_1$

$S(z) = s_0 z + s_1$

$$
\begin{bmatrix} 1 & b_1 & 0 \\ a_1 & b_2 & b_1 \\ a_2 & 0 & b_2 \end{bmatrix}
\begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix}
=
\begin{bmatrix} c_1 - a_1 \\ c_2 - a_2 \\ c_3 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 0 & 0 \\ \alpha & \gamma & 0 \\ \beta & 0 & \gamma \end{bmatrix}
\begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix}
=
\begin{bmatrix} c_1 - \alpha \\ c_2 - \beta \\ c_3 \end{bmatrix}
$$

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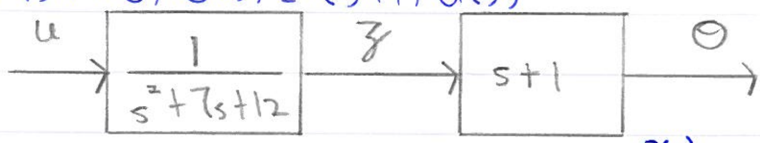
Q 4(a). $\frac{d^2\theta(t)}{dt^2} + 7\frac{d\theta(t)}{dt} + 10\theta(t) = \frac{du(t)}{dt} + u(t)$

$(s^2+7s+10)\,\theta(s) = (s+1)\,U(s)$



$\theta(s)/U(s) = \frac{s+1}{s^2+7s+10}$

$z/u = \frac{1}{s^2+7s+12}$

$\Rightarrow s^2 z + 7sz + 12z = u$

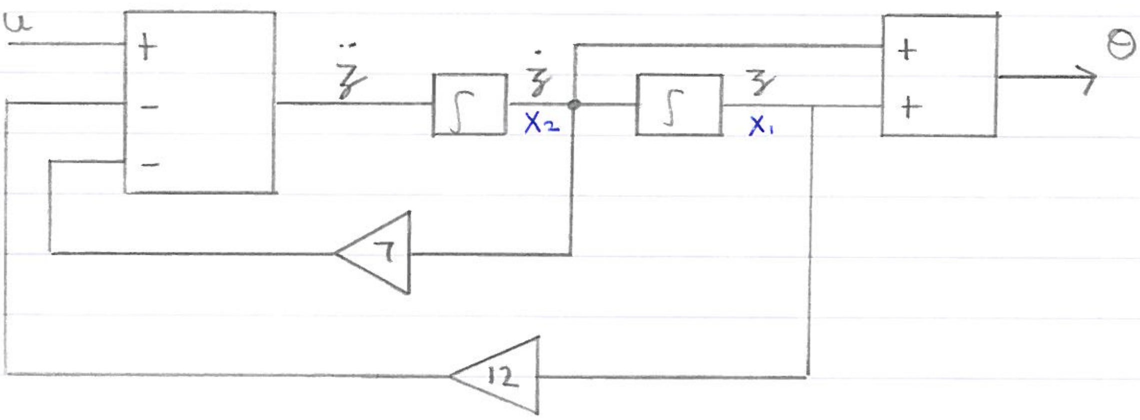
$\Rightarrow \ddot{z} + 7\dot{z} + 12z = u$

$\Rightarrow \ddot{z} = u - 7\dot{z} - 12z$

$\theta/z = s+1$

$\theta = sz + z$

$\Rightarrow \theta = \dot{z} + z$



(ii) $\dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{z} = u - 7\dot{z} - 12z$

$= u - 7x_2 - 12x_1$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(iii) $\theta(0) = 1 \quad \Rightarrow \dot{\theta}(0) = 0$

$\theta(t) = x_1(t) + x_2(t)$

$\theta(0) = x_1(0) + x_2(0)$

$\Rightarrow 1 = x_1(0) + x_2(0)$

$\dot{\theta}(t) = \dot{x}_1(t) + \dot{x}_2(t)$

$\dot{\theta}(0) = \dot{x}_1(0) + \dot{x}_2(0)$

$0 = x_2(0) + u(0) - 7x_2(0) - 12x_1(0)$

$0 = x_2(0) + 0 - 7x_2(0) - 12x_1(0)$

$0 = -6x_2(0) - 12x_1(0)$

$0 = -x_2(0) - 2x_1(0)$

$x_1(0) + x_2(0) = 1$

$-2x_1(0) - x_2(0) = 0$

$-x_1(0) = 1$

$x_1(0) = -1$

$\Rightarrow x_2(0) = 2$

$\underline{x}(t) = \phi(t)\underline{x}(0)$

$\phi(s) = (sI - A)^{-1}$

$$= \left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -12 & -7 \end{pmatrix}\right]^{-1} = \begin{pmatrix} s & -1 \\ 12 & s+7 \end{pmatrix}^{-1}$$

$$= \frac{1}{s(s+7)+12}\begin{pmatrix} s+7 & 1 \\ -12 & s \end{pmatrix}$$

$$= \frac{1}{(s+3)(s+4)}\begin{pmatrix} s+7 & 1 \\ -12 & s \end{pmatrix}$$

$$\phi(t) = L^{-1}\{\phi(s)\} = \begin{pmatrix} L^{-1}\left\{\frac{s+7}{(s+3)(s+4)}\right\} & L^{-1}\left\{\frac{1}{(s+3)(s+4)}\right\} \\ L^{-1}\left\{\frac{-12}{(s+3)(s+4)}\right\} & L^{-1}\left\{\frac{s}{(s+3)(s+4)}\right\} \end{pmatrix}$$

$$\frac{s+7}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$A = \frac{s+7}{s+4}\Big|_{s=-3} = 4 \qquad \Rightarrow \frac{4}{s+3} - \frac{3}{s+4} = 4e^{-3t} - 3e^{-4t}$$
$$B = \frac{s+7}{s+3}\Big|_{s=-4} = -3$$

$$\frac{1}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$A = \frac{1}{s+4}\Big|_{s=-3} = 1 \qquad \Rightarrow \frac{1}{s+3} - \frac{1}{s+4} = e^{-3t} - e^{-4t}$$
$$B = \frac{1}{s+3}\Big|_{s=-4} = -1$$

$$\frac{-12}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$A = \frac{-12}{s+4}\Big|_{s=-3} = -12 \qquad \Rightarrow \frac{12}{s+4} - \frac{12}{s+3} = 12e^{-4t} - 12e^{-3t}$$
$$B = \frac{-12}{s+3}\Big|_{s=-4} = 12$$

$$\frac{s}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$A = \frac{s}{s+4}\Big|_{s=-3} = -3 \qquad \Rightarrow \frac{4}{s+4} - \frac{3}{s+3} = 4e^{-4t} - 3e^{-3t}$$
$$B = \frac{s}{s+3}\Big|_{s=-4} = 4$$

$$\Rightarrow \underline{x}(t) = \begin{pmatrix} 4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t} \\ 12e^{-4t} - 12e^{-3t} & 4e^{-4t} - 3e^{-3t} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{-3t} + e^{-4t} \\ -4e^{-4t} + 6e^{-3t} \end{pmatrix}$$

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Q 4 (b). $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$

$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(i) $\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$

Take Laplace Transforms

$sX(s) - \underline{x}(0) = A\underline{X}(s) + B\underline{U}(s)$

$(sI - A)\underline{X}(s) = \underline{x}(0) + B\underline{U}(s)$

$(sI-A)^{-1}(sI-A)\underline{X}(s) = (sI-A)^{-1}(B\underline{U}(s) + \underline{x}(0))$

$\underline{X}(s) = (sI-A)^{-1}(B\underline{U}(s) + \underline{x}(0))$

$\underline{Y}(s) = C\underline{X}(s) = C(sI-A)^{-1}(\underbrace{B\underline{U}(s)}_{\underline{Y}_{zs}(s)} + \underbrace{\underline{x}(0)}_{\underline{Y}_{zi}(s)})$

$G(s) = Y(s)/U(s) = Y_{zs}(s)/U(s) = C(sI-A)^{-1}B$

$(sI-A)^{-1} = \left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\right]^{-1}$

$= \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1}$

$= \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$

$(sI-A)^{-1}B = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{2}{s+2} \end{bmatrix}$

$C(sI-A)^{-1}B = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{s+1} \\ \frac{2}{s+2} \end{bmatrix} = \frac{1}{s+1} + \frac{4}{s+2} = \frac{5s+6}{(s+1)(s+2)}$

$\Rightarrow G(s) = \frac{5s+6}{(s+1)(s+2)} = \frac{5s+6}{s^2+3s+2}$

(ii) System is controllable if C_x is full rank

$\Rightarrow \det(C_x) \neq 0$

$C_x = [B \mid AB \mid A^2B \mid \cdots \mid A^{N-1}B] \qquad N = 2$

$\Rightarrow C_x = [B \mid AB]$

$AB = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$

$C_x = \begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix}$

$\det(C_x) = 1(-4) + 1(2) = -2 \neq 0$

$\Rightarrow$ system is controllable

(iii) $$G(s) = \frac{5s+6}{s^2+3s+2}$$

Control Canonical Form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y(t) = [6 \;\; 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Controllability Matrix for the CCF

$C_{CCF} = [B_c \mid A_c B_c]$

$$A_c B_c = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\Rightarrow C_{CCF} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$C_{CCF} = T C_x$

$\Rightarrow T = C_{CCF} C_x^{-1}$

$$= \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} -\tfrac{1}{2}\begin{pmatrix} -4 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 2 & -0.5 \\ 1 & -0.5 \end{pmatrix} = \begin{pmatrix} 1 & -0.5 \\ -1 & 1 \end{pmatrix}$$