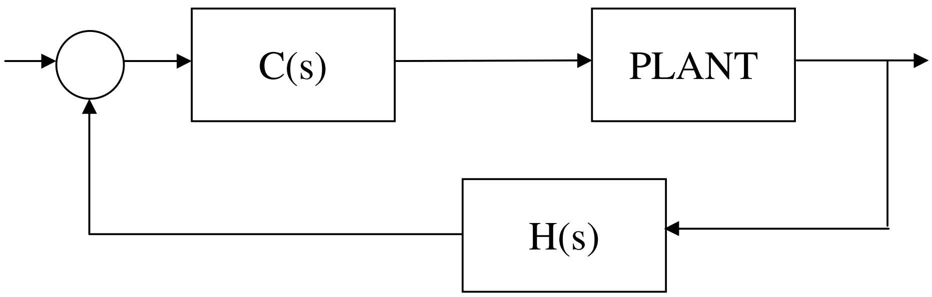
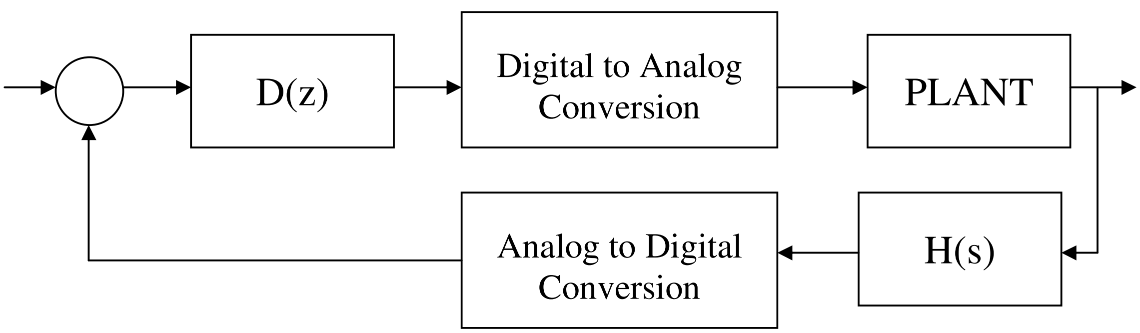
Chapter 5. Digital Emulation of Continuous Time Controllers Using the Z Transform

Given the controller transfer function C(s) –



Implemented as:

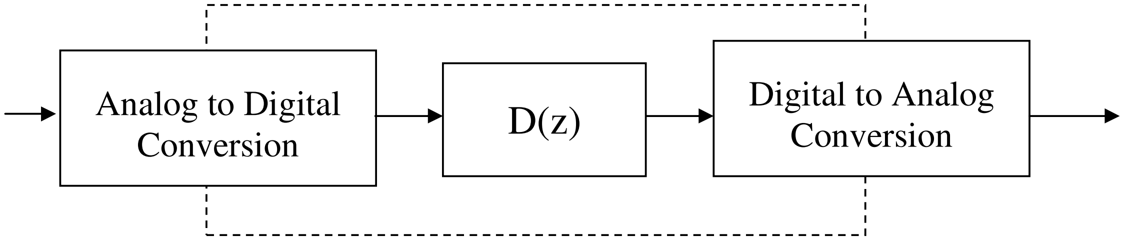


Four common methods:

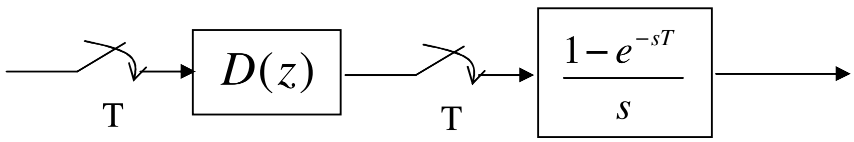
i) Eulers method
ii) Tustins method
iii) Backward difference
iv) Matched pole-zero

All based on the digital approximation of an analog integrator:

This could be approximated on computer by:



Or:



Can’t use tables:

Or the Substitution,

$$s = \frac{1}{T} \ln z$$

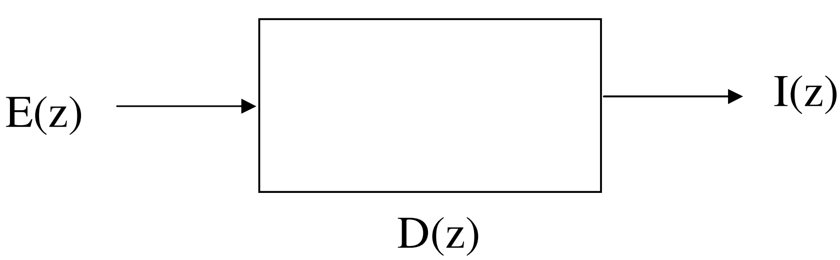
5.1 Eulers Method (Forward Difference)

$$\int_0^t e(\tau)d\tau \approx I(k) =$$

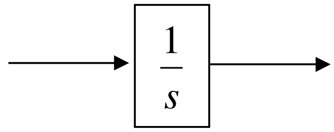
Taking Z transforms of the difference equation:

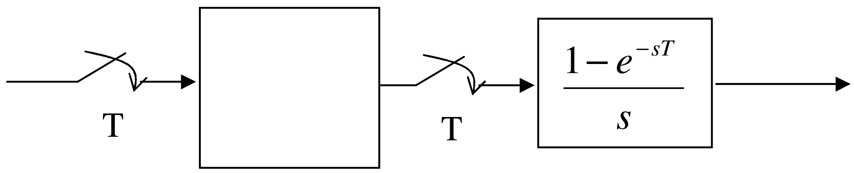
$$I(z) = z^{-1}I(z) + Tz^{-1}E(z)$$

Which yields the discrete-time transfer function:



Hence analog integration could be approximated as:





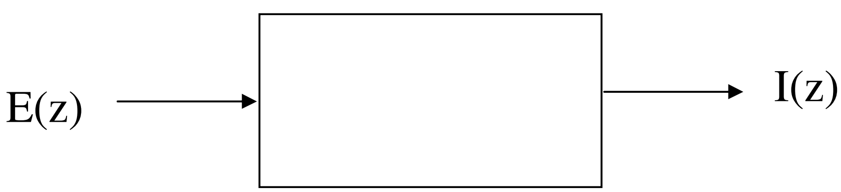
5.2 Tustins Method

$$\int_0^t e(\tau)d\tau \approx I(k) =$$

Taking the Z transforms:

$$I(z) = z^{-1}I(z) + \frac{T}{2}\left(E(z) + z^{-1}E(z)\right)$$

This yields the discrete-time transfer function:



Tutorial: Use Tustin's method to form the difference equations for a discrete approximation of the continuous controller:

$$C(s) = \frac{(s+1)(s+2)}{s}$$

Solution:

Use the substitution:

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

yields:

$$D(z) = \frac{\left(\frac{2}{T}\frac{z-1}{z+1}+1\right)\left(\frac{2}{T}\frac{z-1}{z+1}+2\right)}{\left(\frac{2}{T}\frac{z-1}{z+1}\right)}$$

Tutorial: Show using the backward difference approximation to integration that the following substitution can be obtained:

$$s = \frac{1}{T}\frac{z-1}{z}$$

Show that Euler's Method could possibly yield the transformation of a stable controller C(s) to an unstable algorithm D(z).

We have the following substitutions:

1) Forward Difference (EULERS):

2) Backward Difference:

3) Tustins method:

5.3 Matched-Pole-Zero Method

Unlike these three methods, based on approximations to integration, the matched pole-zero method recognises that the dynamics of the controller C(s) are determined by the position of its poles and zeros

Use the mapping:

Consider the continuous time controller:

$$C(s) = K\frac{(s-\varsigma_1)(s-\varsigma_2)\cdots(s-\varsigma_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- m zeros at $s = \{\varsigma_1, \quad \varsigma_{2,} \quad \cdots \quad \varsigma_m\}$

- p poles at $s = \{p_1, \quad p_{2,} \quad \cdots \quad p_n\}$

Then to map C(s) to D(z):

$$K\frac{(s-\varsigma_1)(s-\varsigma_2)\cdots(s-\varsigma_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \longrightarrow K_D\frac{(z- \qquad)(z- \qquad)\cdots(z- \qquad)}{(z- \qquad)(z- \qquad)\cdots(z- \qquad)}$$

Now choose the gain K_D so that each controller has the same steady–state gain:

$$\lim_{s\to 0} K\frac{(s-\varsigma_1)(s-\varsigma_2)\cdots(s-\varsigma_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} =$$

$$\lim_{z\to 1} K_D\frac{(z-e^{\varsigma_1 T})(z-e^{\varsigma_2 T})\cdots(z-e^{\varsigma_m T})}{(z-e^{p_1 T})(z-e^{p_2 T})\cdots(z-e^{p_n T})}$$

Tutorial: Form the difference equations for the following continuous-time phase-lead compensator, assuming that the sample time is T=0.1 seconds,

$$C(s) = \frac{10(1+0.8s)}{1+0.4s}$$

Chapter 6. Direct Design of Digital Controllers in the Z Domain

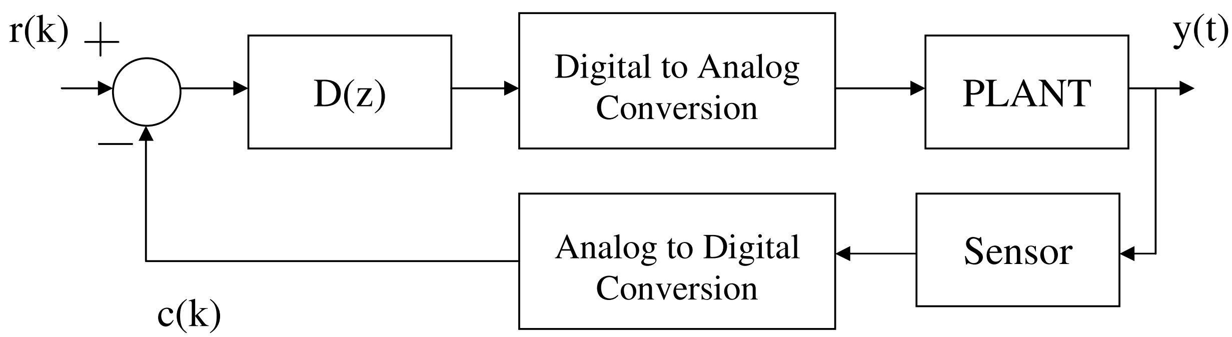
This may be necessary because:

- The sample time is too large

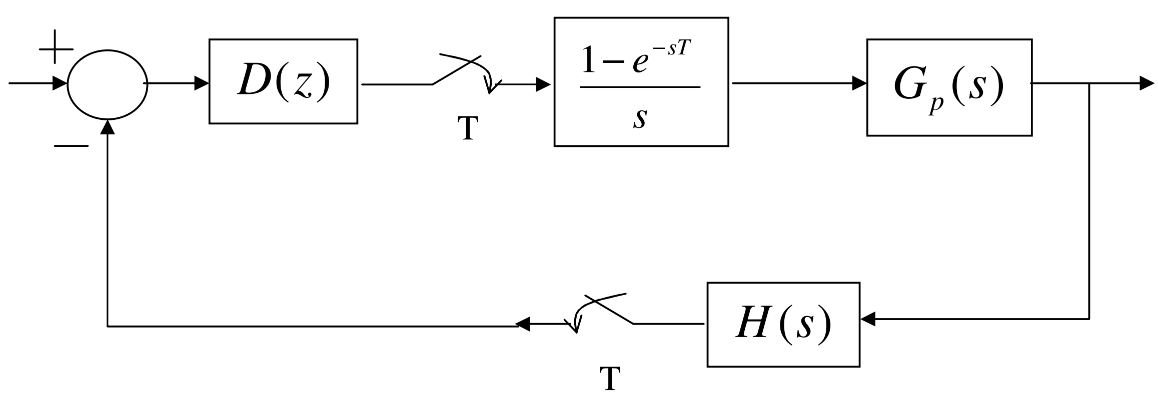
- It is possible to achieve a better performance using a digital controller

6.1 The Plant Transfer Function G(z)

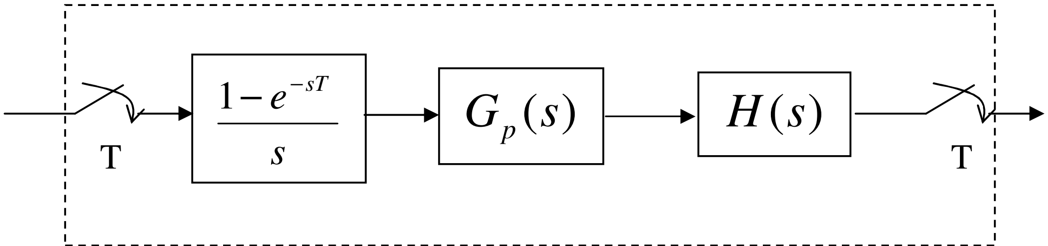
Consider the system under computer control,



Could be represented as :



Consider:



Define:

$$\frac{C(z)}{M(z)} = G(z)$$

where:

$$G(z) = \mathbf{Z}\left\{\frac{1-e^{-sT}}{s} G_p(s) H(s)\right\}$$

6.1.1 Important note on time delays:

Consider that the process contains a pure time delay T_d:

$$G_p(s)H(s) = \frac{B(s)}{A(s)} e^{-sT_d}$$

where:

$$T_d = NT + \theta$$

$$G(z) = \mathbf{Z}\left\{\frac{1-e^{-sT}}{s} \frac{B(s)}{A(s)} e^{-(NT+\theta)s}\right\}$$

But we know by definition that:

$$\mathbf{Z}\left\{e^{-Ts}\right\} = z^{-1}$$
$$\mathbf{Z}\left\{e^{-NTs}\right\} =$$

If we then make the approximation $T_d \approx NT$ then:

$$G(z) \approx \mathbf{Z}\left\{\frac{1-e^{-sT}}{s}\frac{B(s)}{A(s)}e^{-NTs}\right\} =$$

6.1.2 The Modified Z Transform

We can include the frational sample time delay by using the Modified Z Transform to yield a more accurate G(z),

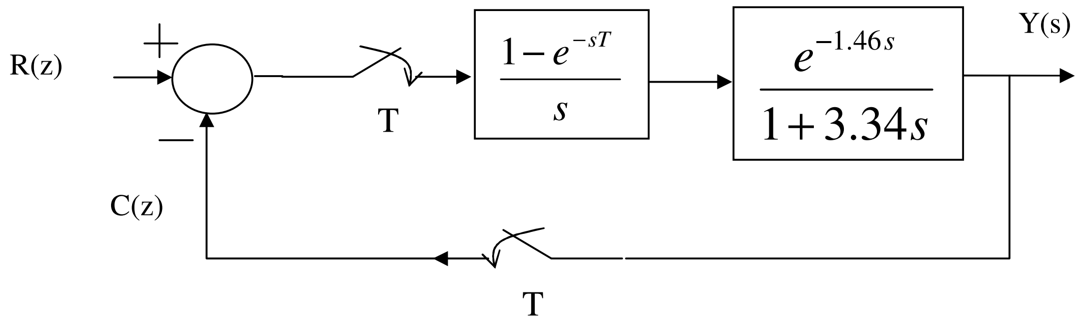
$$G(z) = \left(1 - z^{-1}\right)z^{-N}\mathbf{Z}\left\{\frac{B(s)}{sA(s)}e^{-\theta s}\right\}$$

define a new transform:

$$\mathbf{Z}_m\left\{\frac{B(s)}{sA(s)}\right\} = \mathbf{Z}\left\{\frac{B(s)}{sA(s)}e^{-\theta s}\right\}$$

where:

<u>Example:</u>



Determine the discrete transfer function G(z)=C(z)/M(z) using the modified Z transform, if the sample time is T=1second.

$$G(z) = \left(1 - z^{-1}\right) z^{-N} \mathbf{Z}\left\{ \frac{B(s)}{sA(s)} e^{-\theta s} \right\}$$

$$= \left(1 - z^{-1}\right) z^{-1} \mathbf{Z}\left\{ \frac{1}{s(1+3.34s)} e^{-0.46s} \right\}$$

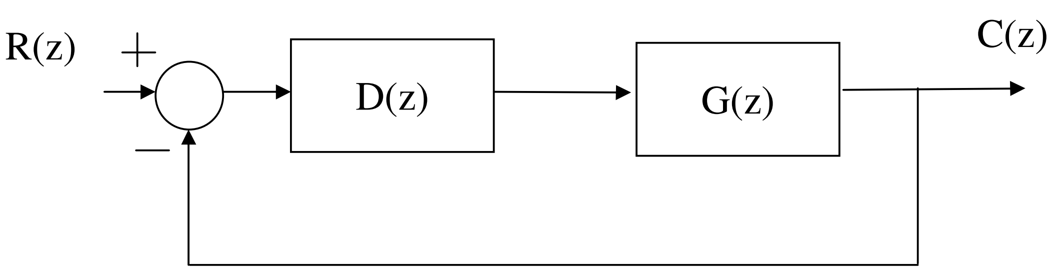
$$= \left(1 - z^{-1}\right) z^{-1} \mathbf{Z}_m\left\{ \frac{1}{s(1+3.34s)} \right\}\Bigg|_{m=}$$

Looking up the modified Z transform tables:

$$\left(1 - z^{-1}\right) z^{-1} \mathbf{Z}_m\left\{ \frac{0.3}{s(s+0.3)} \right\}\Bigg|_{m=0.54} = \frac{z^{-2}(0.15 + 0.11z^{-1})}{1 - 0.74z^{-1}}$$

6.2 Direct Design Equation for D(z)

The closed loop system can now be represented totally in the Z domain as:



Examining the block diagram:

$$C(z) = G(z)D(z)\left(R(z) - C(z)\right)$$

The closed loop transfer function is then:

We can rearrange to yield an expression for the controller transfer function D(z) to achieve some desired closed loop transfer function C(z)/R(z):

If a realistic closed loop response C(z)/R(z) can be specified, then it is possible to determine the controller D(z):

Three methods are commonly used:

1) Deadbeat
2) Dahlins method
3) Kalmans method

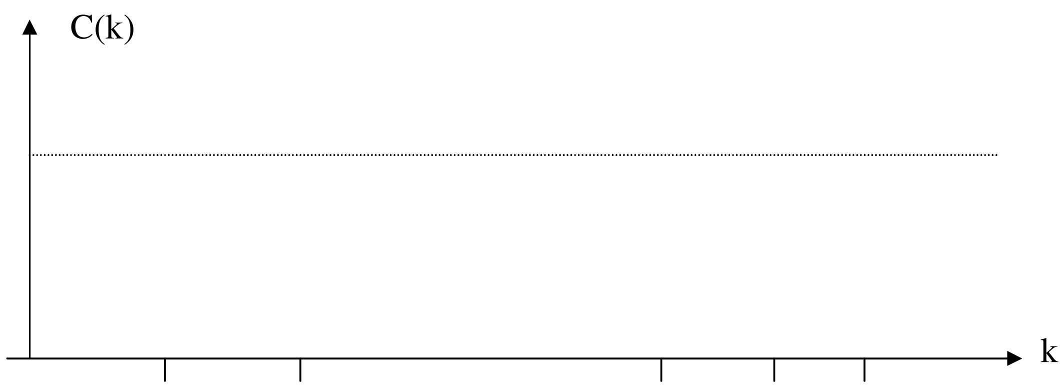
6.3 Deadbeat Control

$$Specification\left\{ \vphantom{\begin{matrix} a \\ b \\ c \\ d \end{matrix}} \right.$$

Consider the process: $$G(s)H(s) = \frac{B(s)}{A(s)} e^{-(nT+\theta)s}$$

$$G(z) = Z_m \left\{ \frac{1-e^{-sT}}{s} e^{-NTs} \frac{B(s)}{A(s)} \right\}_{m=1-\frac{\theta}{T}}$$

If the process is now under deadbeat control, then the response to a unit step in the setpoint is:



For the unit step setpoint: $$r(k) = \begin{cases} 0 & for \quad k < 0 \\ 1 & for \quad k \geq 0 \end{cases}$$

Hence then: $$C(z) = z^{-(N+1)} R(z)$$

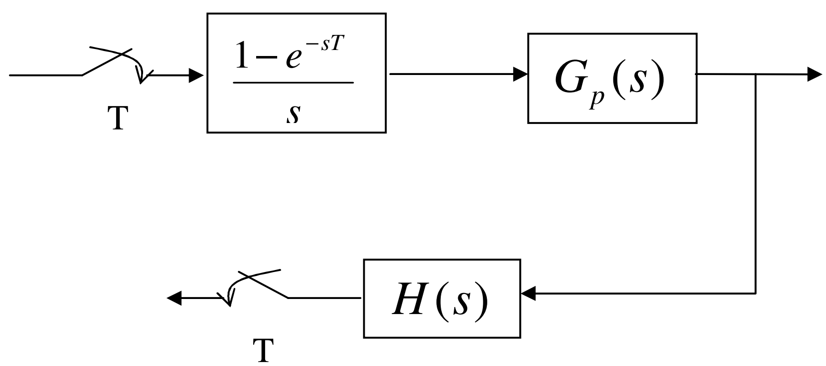
And the desired setpoint response is:

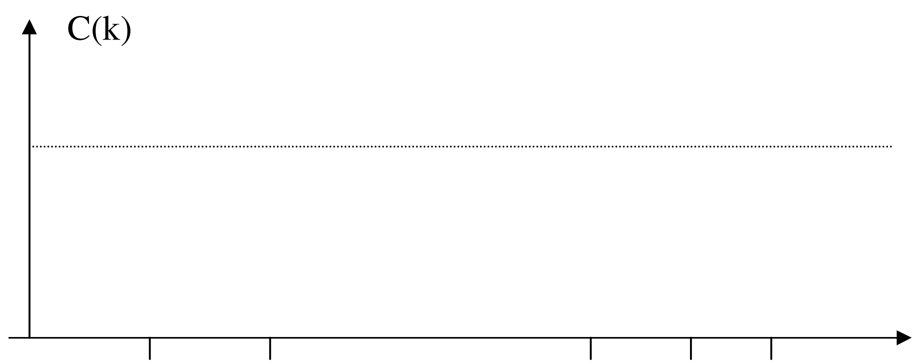
The controller D(z) which will achieve this deadbeat response is:

$$D(z) = \frac{1}{G(z)}\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

6.3.1 Potential Problem of Deadbeat control

We are only specifying what happens to c(k) – in many cases, if we look at the corresponding control signal, we will see an oscillation –



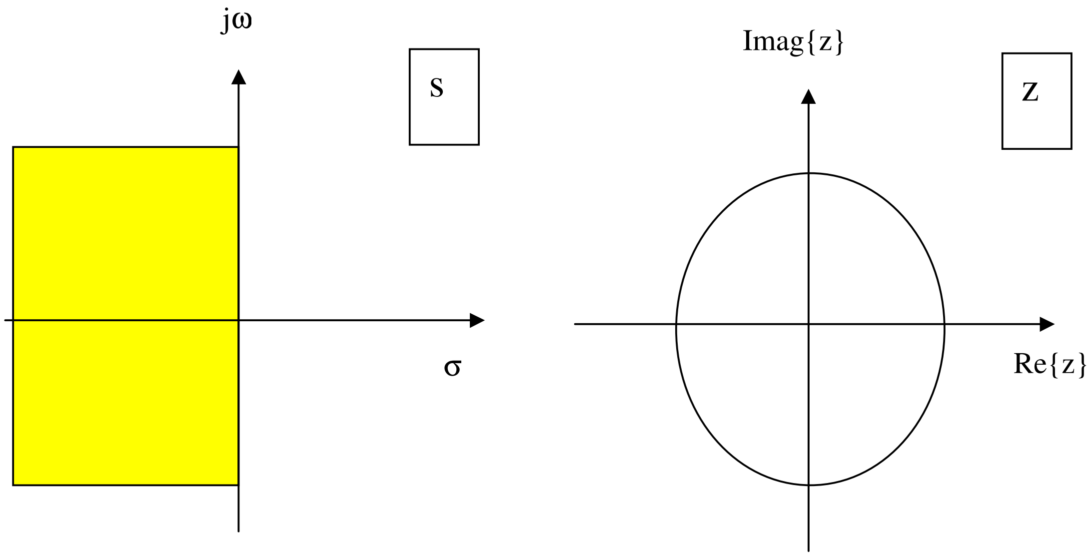


6.3.2 Controller Ringing

Consider that the controller D(z) can be written as:

$$D(z) = K\frac{(z-\varsigma_1)(z-\varsigma_2)\cdots(z-\varsigma_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

Now consider the mapping of the primary strip from the s to the z planes:

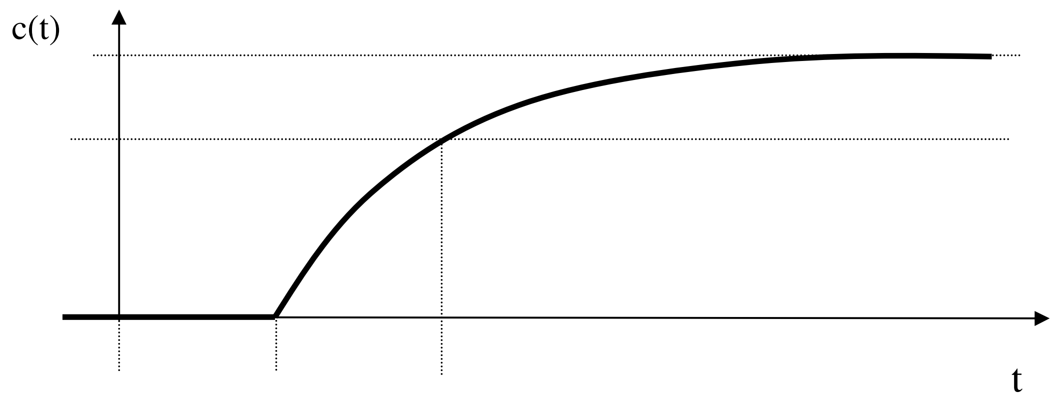


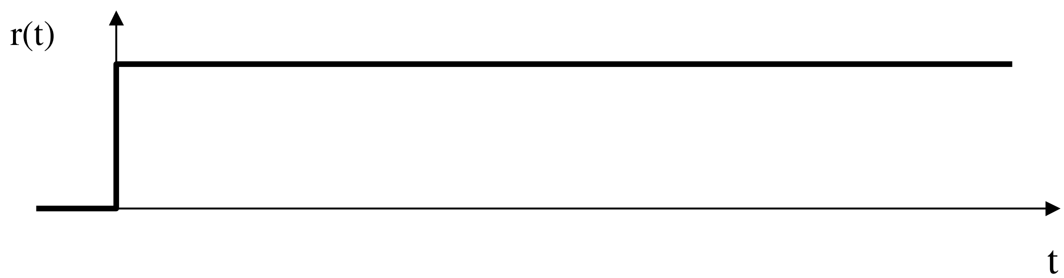
Any poles of D(z) close to the unit circle are underdamped

Any pole of D(z) close to the unit circle and close to z=-1.0,

6.4 Dahlins Method

First specify a desired step-response for the continuous-time signal c(t):





That is :

$$c(t) = \begin{cases} 0 & for \quad t < \theta \\ 1 - e^{-\frac{t-\theta}{\lambda}} & for \quad t \geq \theta \end{cases}$$

Taking Laplace transforms, we could write:

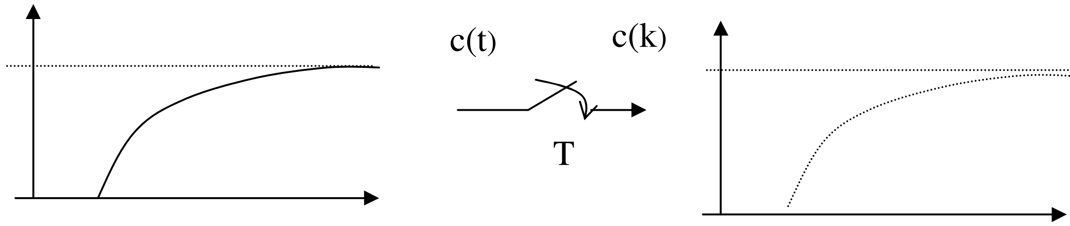
$$C(s) = \boldsymbol{L}\{c(t)\} =$$

Now choose N as:

<u>Then:</u>

$$C(s) =$$

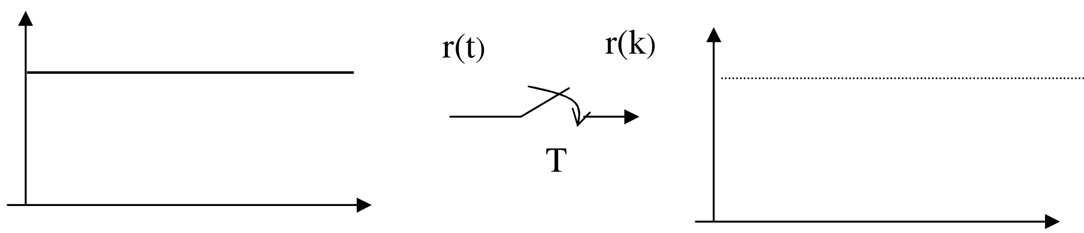
Consider that the signal c(t) was sampled with sample-time T



From the Z transform tables:

$$C(z) = \frac{(1-e^{-T/\lambda})z^{-(N+1)}}{(1-z^{-1})(1-e^{-T/\lambda}z^{-1})}$$

This was the response to a unit step setpoint signal:



Hence:

$$R(z) = \frac{1}{1-z^{-1}}$$

Then the desired closed loop response is:

$$\frac{C(z)}{R(z)} = \frac{(1-e^{-T/\lambda})z^{-(N+1)}}{1-e^{-T/\lambda}z^{-1}}$$

Now use the control design equation:

This yields the controller:

$$D(z) = \frac{1}{G(z)} \frac{(1-e^{-T/\lambda})z^{-(N+1)}}{1-e^{-T/\lambda}z^{-1} - (1-e^{-T/\lambda})z^{-(N+1)}}$$

Which could be factorised as:

$$D(z) = \frac{1}{G(z)} \frac{(1 - e^{-T/\lambda}) z^{-(N+1)}}{(1 - z^{-1}) Q(z)}$$

We can use λ as a tuning parameter:

Note: λ is the time constant of the closed loop transfer function

Example:

Consider again the process:

$$G_p(s)H(s) = \frac{e^{-1.46s}}{1 + 3.34s}$$

The sample time is T=1 seconds. Design a Dahlins controller to achieve a settling time (to within 2% of final value) of 8 seconds.

$$G(z) = \left(1 - z^{-1}\right) z^{-1} \mathbf{Z}_m \left\{ \frac{0.3}{s(s + 0.3)} \right\} \Bigg|_{m=0.54} = \frac{z^{-2}(0.15 + 0.11 z^{-1})}{1 - 0.74 z^{-1}}$$

From the settling time specification :

The pure time delay of the continuous process is:

Use Dahlins method to specify desired closed loop response C/R

$$\frac{C(z)}{R(z)} = \frac{(1-e^{-T/\lambda})z^{-(N+1)}}{1-e^{-T/\lambda}z^{-1}}$$

The controller transfer function D(z) is:

$$D(z) = \frac{1}{G(z)} \frac{(1-e^{-T/\lambda})z^{-(N+1)}}{1-e^{-T/\lambda}z^{-1} - (1-e^{-T/\lambda})z^{-(N+1)}}$$

$$D(z) = \frac{(1-e^{-0.5})z^{-2}}{1-e^{-0.5}z^{-1} - (1-e^{-0.5})z^{-2}} \frac{1-0.74z^{-1}}{z^{-2}(0.15+0.11z^{-1})}$$

Which can be simplified to yield:

$$D(z) = \frac{M(z)}{E(z)} = \frac{2.64(1-0.74z^{-1})}{(1-z^{-1})(1+0.39z^{-1})(1+0.73z^{-1})}$$

Tutorial:

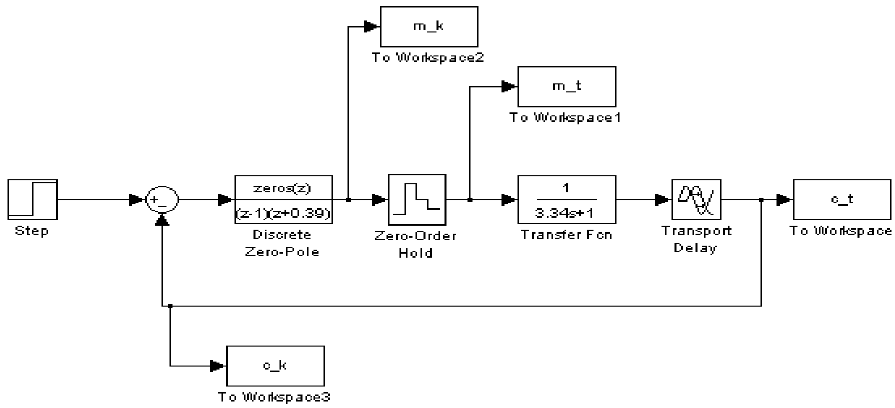
For a unit step in the setpoint signal, determine and plot the sequences for m(k) and c(k) over the first 12 samples, if the initial conditions are defined as:

$$c(k) = 0 \quad \forall \quad k \le 0$$

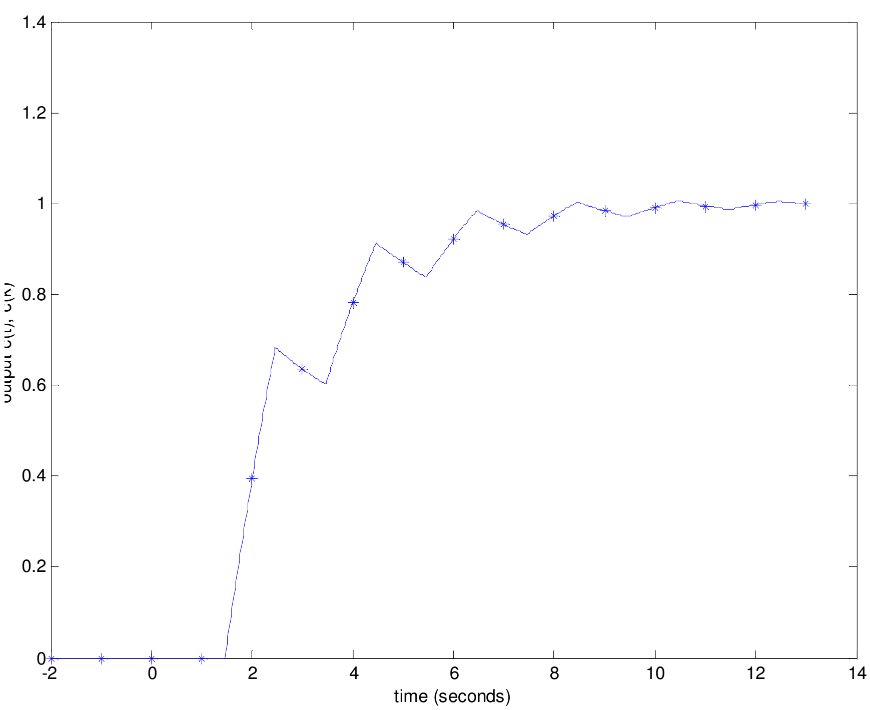
$$m(k) = 0 \quad \forall \quad k < 0$$

<u>Solution:</u>

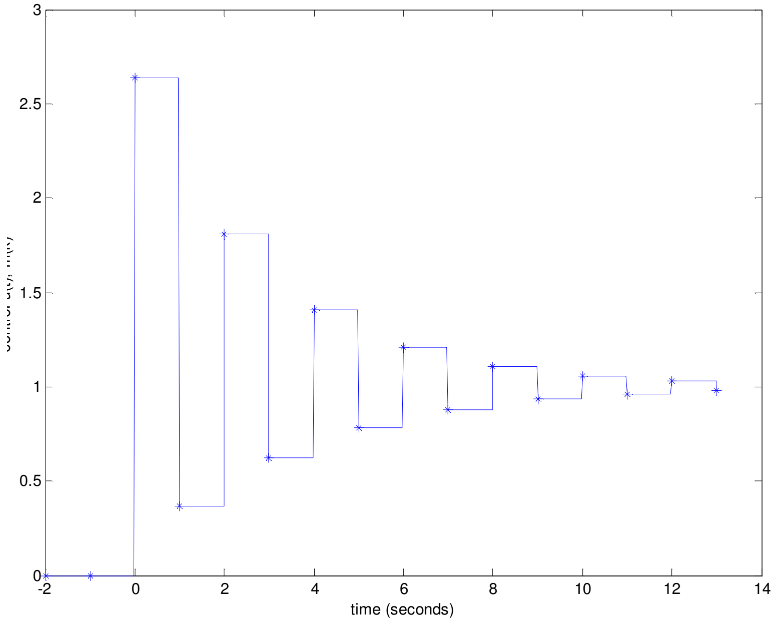
Can be modelled in Simulink as:



The output response is:



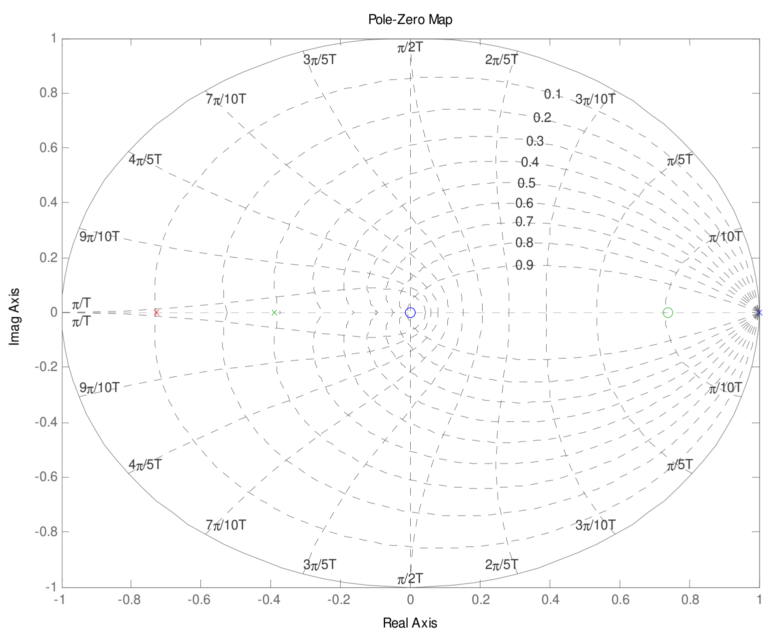
But the control input, u(t) is:



Note: highly underdamped oscillation in the controller output:

Consider controller transfer function:

$$D(z) = \frac{2.64z^2(z-0.74)}{(z-1)(z+0.39)(z+0.73)}$$

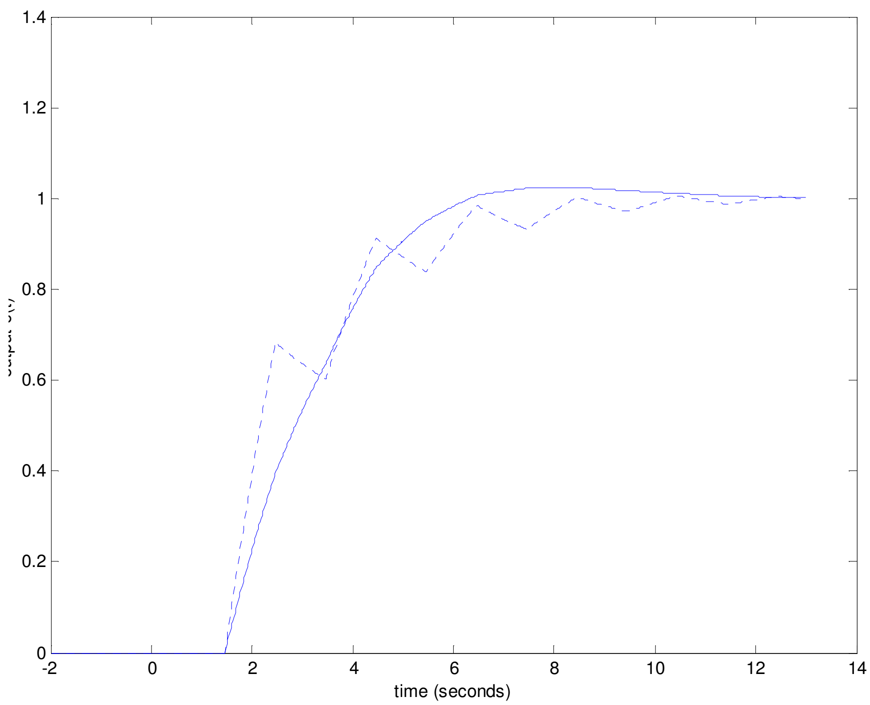


This is easily rectified by replacing the ringing pole by a pole at the origin:

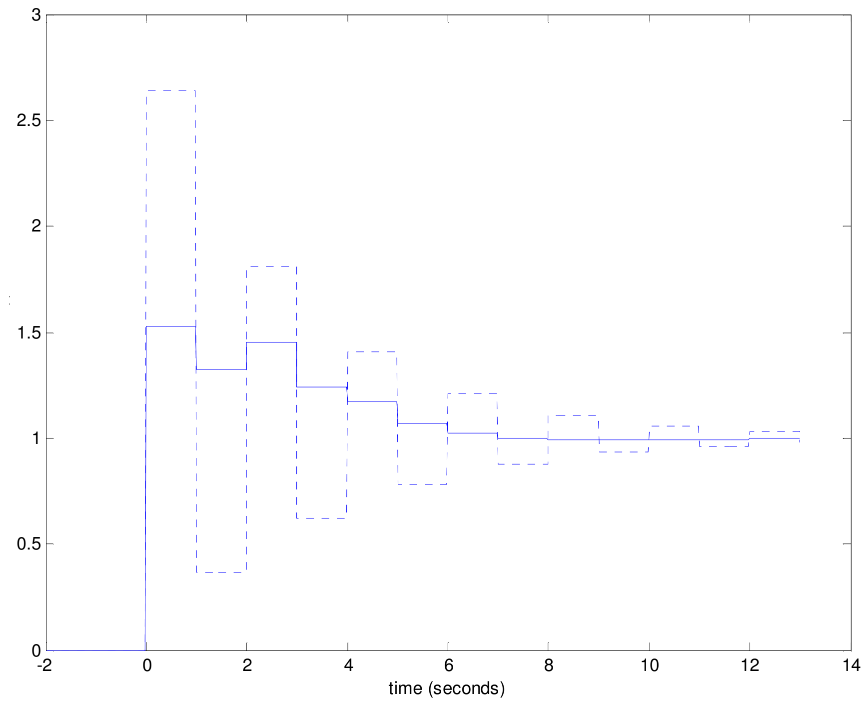
The controller becomes:

$$D'(z) = \frac{2.64z^2(z-0.74)}{(z-1)(z+0.39)(1+0.73)z} =$$

The closed-loop step response for the approximate controller D'(z) compared with the true D(z):



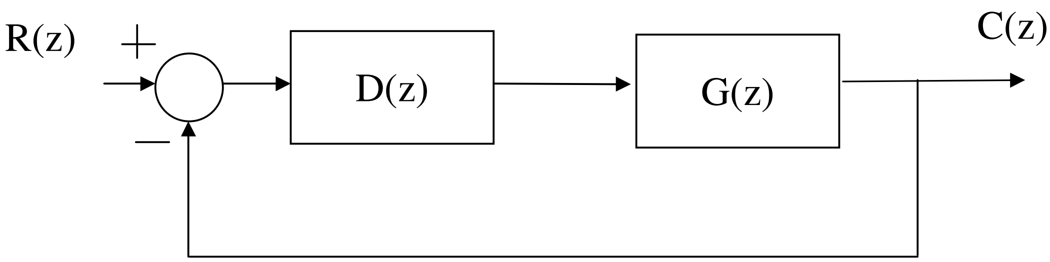
The corresponding controller outputs for both controllers are:



6.5 Kalmans Design Method

The controller D(z) is designed from the start to eliminate ringing –we directly specify the controller output m(k).

Consider the discrete time closed-loop system:

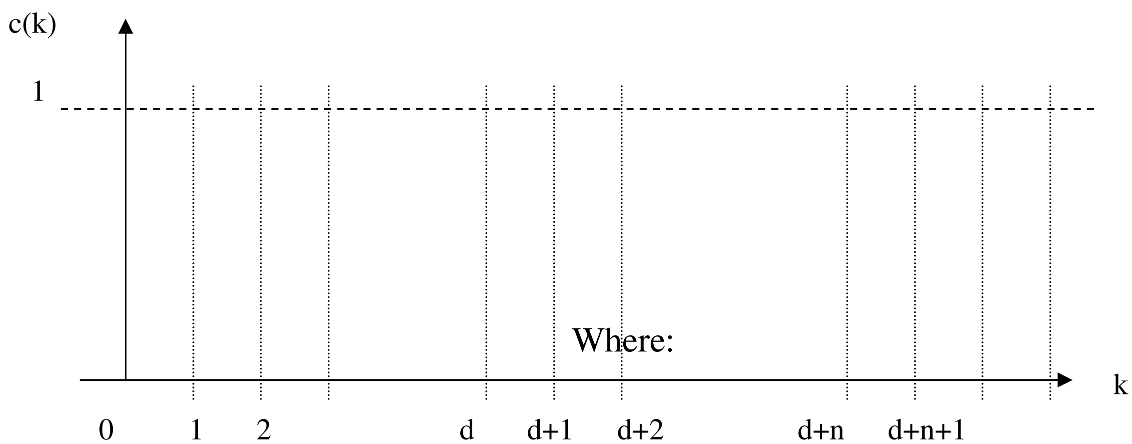


Where the transfer function of the nth order process is:

$$G(z) = \frac{z^{-d}(b_1 z^{-1} + b_2 z^{-2} + \cdots b_m z^{-m})}{1 - a_1 z^{-1} - a_2 z^{-2} \cdots - a_n z^{-n}}$$

Specify D(z) to achieve the following for a unit step in r(k) is:

i) Output c(k) will settle to a steady-state of 1 within (n+d+1) samples:



Where:

Taking Z transform of the sequence c(k):

$$C(z) = \sum_{k=0}^{\infty} c(k) z^{-k}$$

hence:

$$C(z) = z^{-d}\left(c_1 z^{-1} + c_2 z^{-2} + \cdots c_n z^{-n} + z^{-n-1} + z^{-n-2} + z^{-n-3} \cdots\right)$$

Since:

$$R(z) = \frac{1}{1 - z^{-1}}$$

Then we can specify the desired step response C/R as:

$$\frac{C(z)}{R(z)} = (1 - z^{-1})\ z^{-d}\left(c_1 z^{-1} + c_2 z^{-2} + \cdots c_n z^{-n} + z^{-n-1} + z^{-n-2} + z^{-n-3} \cdots\right)$$

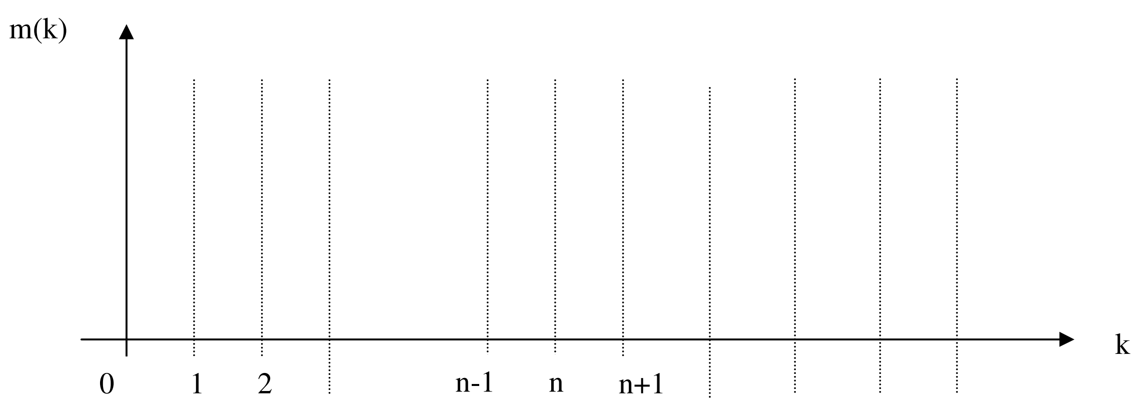
$$= z^{-d}\left(c_1 z^{-1} + (c_2 - c_1) z^{-2} + \cdots (c_n - c_{n-1}) z^{-n} + (1 - c_n) z^{-n-1} + (1-1) z^{-n-2} \cdots\right)$$

We then define :

$$P(z) = z^{-d}\left(p_1 z^{-1} + p_2 z^{-2} + \cdots p_n z^{-n} + p_{n+1} z^{-(n+1)}\right)$$

Note the constraint:

ii) The controller output m(k) will settle to a steady state value within n samples:



Taking Z transform of the sequence m(k):

$$M(z) = \sum_{k=0}^{\infty} m(k) z^{-k}$$

hence:

$$M(z) = m_0 + m_1 z^{-1} + m_2 z^{-2} + \cdots + m_{n-1} z^{-n+1} + m_n z^{-n} + m_n z^{-n-1} \cdots$$

Since:

$$R(z) = \frac{1}{1 - z^{-1}}$$

We can specify the desired controller step response M/R as:

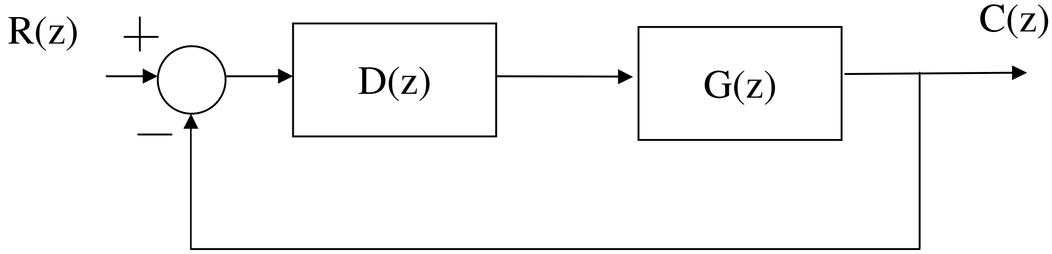
$$\frac{M(z)}{R(z)} = (1 - z^{-1}) \left(m_0 + m_1 z^{-1} + m_2 z^{-2} + \cdots + m_{n-1} z^{-n+1} + m_n z^{-n} + m_n z^{-n-1} \cdots \right)$$

$$= m_0 + (m_1 - m_0) z^{-1} + (m_2 - m_1) z^{-2} + \cdots (m_n - m_{n-1}) z^{-n} + (m_n - m_n) z^{-n-1} + \cdots$$

Now we will define the Q(z) polynomial as:

$$Q(z) = q_0 + q_1 z^{-1} + \cdots q_n z^{-n}$$

Consider the closed-loop diagram:



Then we can say:

$$G(z) = \frac{z^{-d}(b_1 z^{-1} + b_2 z^{-2} + \cdots b_m z^{-m})}{1 - a_1 z^{-1} - a_2 z^{-2} \cdots - a_n z^{-n}} =$$

Since P(z) is arbitrary, then why not specify it as:

$$P(z) = \frac{z^{-d}(b_1 z^{-1} + b_2 z^{-2} + \cdots b_m z^{-m} + \qquad\qquad\qquad)}{\qquad}$$

Then this choice of P(z) forces the following choice of Q(z):

$$Q(z) = \frac{1 - a_1 z^{-1} - a_2 z^{-2} \cdots - a_n z^{-n}}{\qquad}$$

<u>Example:</u>

$$G(z) = \frac{z+1}{2z(z-0.72)^2}$$

6.5.1 Control Design Using Kalmans Method:

$$D(z) = \frac{1}{G(z)} \frac{C(z)/R(z)}{1 - C(z)/R(z)}$$

For the example:

$$G(z) = \frac{z-0.5}{2z(z-0.72)^2}$$

$$D(z) = \frac{Q(z)}{1 - P(z)} =$$

Plotting Responses for m(k) and c(k)

We know: $\frac{C(z)}{R(z)} = P(z)$

$\frac{M(z)}{R(z)} = Q(z)$

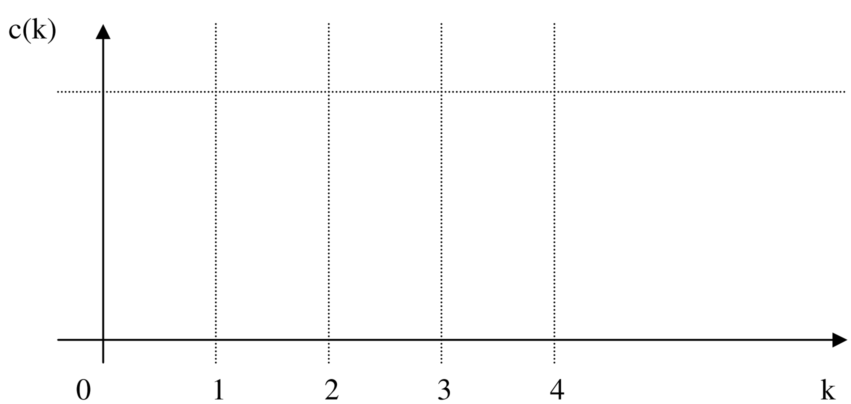
For our example: $P(z) = 2z^{-2} - z^{-3}$

Hence: $C(z) = P(z)R(z) = (2z^{-2} - z^{-3})R(z)$

Taking inverse Z transforms:

$$c(k) = 2r(k-2) - r(k-3)$$

k	0	1	2	3	4	5	……
r(k)							
r(k-2)							
r(k-3)							
c(k)							



Similarly we could easily write:

$$M(z) = Q(z)R(z) = (4 - 5.76z^{-1} + 2.072z^{-2})R(z)$$

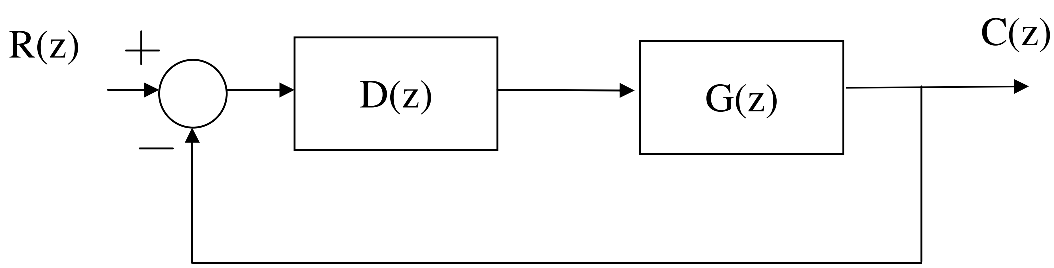
Taking inverse Z transforms:

<u>Tutorial:</u>

Sketch the responses for c(k) and m(k) for a unit ramp setpoint signal.

6.6 A Problem with Direct Digital Design

The closed loop system is:



The characteristic equation is:

$$1 + D(z)G(z) = 0$$

Consider now that the process G(z) could be represented as:

$$G(z) = \frac{B(z)}{A(z)} =$$

Now consider that the controller D(z) is designed based on the following approximate model G'(z):

$$G'(z) = \frac{(z-b')B_1(z)}{(z-a')A_1(z)}$$

The controller algorithm is then:

$$D(z) = \frac{1}{G'(z)} \frac{C(z)/R(z)}{1-C(z)/R(z)} = \frac{(z-a')A_1(z)}{(z-b')B_1(z)} \rule{2cm}{0.4pt}$$

The closed-loop characteristic equation is then:

$$1 + \frac{(z-a')A_1(z)}{(z-b')B_1(z)} \frac{E(z)}{F(z)} \frac{(z-b)B_1(z)}{(z-a)A_1(z)} = 0$$

which could be simplified to yield:

$$(z-b')(z-a)F(z) + (z-b)(z-a')E(z) = 0$$

Now if the open-loop process zero z=b and pole z=a are inside the unit circle:

If however either z=b or z=a are outside the unit circle:

We must therefore never attempt to directly cancel:

- Poles of G(z) outside or on the unit circle

- Zeros of G(z) outside or on the unit circle