Section 3

12:00 — Telecoms — 12/02/10

Summer 06:

Q5 A binary modulation scheme is described by:-

$$S_i(t) = \begin{cases} S_1(t) = A_1 \cos(\omega_c t) & 0 \le t \le T \\ S_2(t) = A_2 \cos(\omega_c t) & 0 \le t \le T \end{cases}$$

Where T is an integer times $1/f_c$

Show:-

(a) $P_e = Q\left[\sqrt{\frac{(A_1 - A_2)^2 T}{4\eta}}\right]$ [8]

(b) Average signal energy per bit, E_b, is a fixed constant. T and η are also fixed.
Prove that, to minimise P_e, we require:-

$A_2 = -A_1$ [12]

Solution:

(a) $P_e = Q\left[\sqrt{\frac{E_d}{2\eta}}\right]$

$$E_d = \int_0^T [A_1 \cos(\omega_c t) - A_2 \cos(\omega_c t)]^2 \, dt$$

$$\Rightarrow \int_0^T [(A_1 - A_2)\cos(\omega_c t)]^2 dt \rightarrow (A_1 - A_2)^2 \int_0^T \cos^2(\omega_c t)\, dt$$

$$= (A_1 - A_2)^2 \left(\frac{2\omega_c t + \sin(2\omega_c t)}{4\omega_c} \Bigg|_0^T \right)$$

$$= (A_1 - A_2)^2 \frac{T}{2} \qquad \text{since } \sin(2\omega_c T) = 0$$

Since $T = n\frac{1}{f_c}$, $n \in N_0$

$$\therefore P_e = Q\left[\sqrt{\frac{(A_1 - A_2)^2 T}{4\eta}}\right]$$

(b) Avg. signal energy per bit, E_b, is:-

$$E_b = \frac{1}{2}\left[\int_0^T \left(S_1^2(t) + S_2^2(t)\right) dt\right]$$

Hence:

Constant.

$$E_b = \frac{1}{2}\left[(A_1^2 + A_2^2)\int_0^T \cos^2(\omega_c t)\, dt\right] = (A_1^2 + A_2^2)\left(\frac{T}{4}\right) = k$$

k is constant

We require:

$$A_1^2 + A_2^2 = m \quad , m \text{ is a constant}$$

To minimise P_e, we maximise its argument.

$\Rightarrow$ Let: $y = (A_1 - A_2)^2 \dfrac{T}{4\eta}$

$$\frac{dy}{dA_1} = \left(1 - \frac{dA_2}{dA_1}\right) 2(A_1 - A_2)\frac{T}{4\eta} \qquad \cancel{@ \text{ Max/Min } \frac{dy}{dA_1} = 0}$$

$$\cancel{\left(1 - \frac{dA_2}{dA_1}\right) = 0 \quad \text{since} \quad A_1 \neq A_2 \quad \text{Because} \quad S_1(t) = S_2(t)}$$

Not Allowed

$$2A_1 + \underbrace{\frac{d}{dA_1}(A_2)^2}_{} = 0$$

$$\frac{d}{dA_2}(A_2^2)\frac{dA_2}{dA_1} = 2A_2\frac{dA_2}{dA_1}$$

$$2A_1 + 2A_2\frac{dA_2}{dA_1} = 0$$

$$\frac{dA_2}{dA_1} = \frac{-A_1}{A_2}$$

$$\left(1 + \frac{A_1}{A_2}\right)2(A_1 - A_2)\frac{I}{4\eta} = 0 \qquad \text{if } \frac{dy}{dA_1} = 0$$

$$\Rightarrow 1 + \frac{A_1}{A_2} = 0 \qquad \therefore \quad A_2 = -A_1$$

$$\cancel{\therefore A_2 = A_1}$$

Example:-

Suppose $S_1(t)$ and $S_2(t)$ must have the same energy. Prove that the optimum choice is:

$$S_2(t) = -S_1(t)$$

and that the resultant S.N.R is:

$$\left(\frac{S}{N}\right)_{o/p} = \frac{8E}{\eta}$$

where E is the signal energy.

Proof:

$$(S/N)_{o/p} = \frac{2E_d}{\eta}$$

$$\Rightarrow \frac{2}{\eta}\int_0^T (S_1(t) - S_2(t))^2\,dt$$

$$= \frac{2}{\eta}\int_0^T \left(S_1^2(t) - 2(S_1(t)S_2(t)) + S_2^2(t)\right) dt$$

$$= \frac{4E}{\eta} - \frac{4}{\eta}\int_0^T S_1(t)S_2(t)\,dt$$

Using the Schwarz inequality:-

$$\left|\int_0^T S_1(t)S_2(t)\,dt\right| \leq \underbrace{\sqrt{\underbrace{\int_0^T S_1^2(t)\,dt}_{E}\;\underbrace{\int_0^T S_2^2(t)\,dt}_{E}}}_{E}$$

Equality holds when: $S_2(t) = kS_1(t)$

Since the signals are of equal energy:

$$|k| = 1$$

$$\Rightarrow \quad k = \pm 1$$

$k = +1$ cannot be used

$\therefore \quad k = -1$

Hence:

$$S_2(t) = -S_1(t)$$

Thus by substituting we get:

$$\left(\frac{S}{N}\right)_{o/p} = \frac{4E}{\eta} - \frac{4}{\eta}\int_0^T S_1(t) S_2(t)\, dt$$

$$= \frac{4E}{\eta} + \frac{4E}{\eta} = \frac{8E}{\eta}$$

This result implied anti-podal signalling

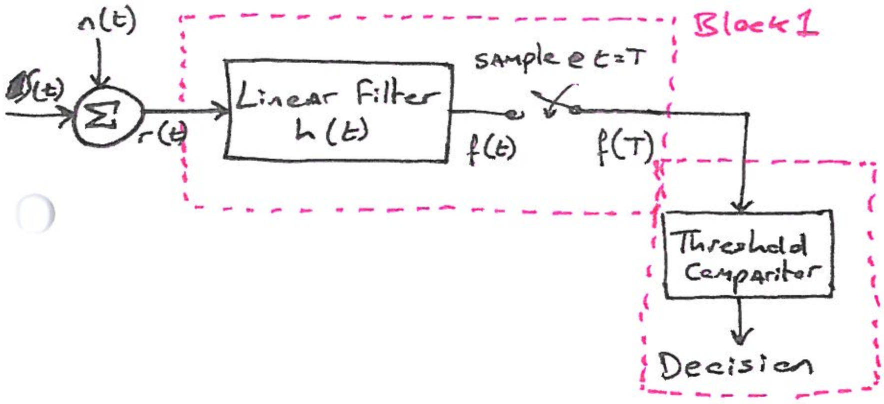
Telecoms

26/01/10 — Dr. C Murphy

We now turn our attention to the more general question of minimising probability of error for any arbitrary modulation scheme.

Binary Signal Detection & Hypothesis Testing

Once again, we confine our attention to AWGN and a distortionless channel, i.e. no inter-signal interference (I.S.I.). The following system diagram summarises the receiver structure:-



The transmitted signal over a symbol interval $(0, T)$ is represented by:-

$$S_i(t) = \begin{cases} S_1(t) & 0 \le t \le T \text{ for logic } 1 \\ S_2(t) & 0 \le t \le T \text{ for logic } 0 \end{cases}$$

at the receiver.

The received signal $r(t)$ is represented by:-

$$r(t) = S_i(t) + n(t) \qquad i = 1, 2 \quad 0 \le t \le T$$

where $n(t)$ is a zero-mean AWGN (Additive White Gaussian Noise).

As before, there are two steps involved in signal detection. The first step reduces the received signal $r(t)$ to a (~~[illegible]~~) single number $f(T)$. This operation can be performed by a linear filter followed by a sampler, as shown in Block 1 above.

The output of Block 1, sampled @ $t=T$, yields:-

$$z(T) = a_i(T) + n_o(T)$$

where $a_i(T)$ is the signal (ONLY) component of $z(T)$ $\qquad i = 1, 2$

and $n_o(T)$ is the noise (ONLY) component of $z(T)$

(after filtering)

For simplicity, we usually write:-

$$z = a_i + n_o \qquad i = 1, 2$$

Note that the noise component n_o is also a zero-mean Gaussian random variable (rv) (We will confirm this later). Hence z is also a Gaussian rv with a mean of either a_1 or a_2, depending on whether $S_1(t)$ or $S_2(t)$ was sent. The sample z is often called the "test statistic".

The second step in the signal detection process consists of comparing z to a threshold level λ in Block 2. The decision is then made according to:-

earlier

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \lambda$$

where H_1 & H_2 are the 2 possible hypotheses. Choosing H_1 is equivalent to deciding that signal $S_1(t)$ was sent, and choosing H_2 is equivalent to deciding that $S_2(t)$ was sent. The inequality above indicates that H_1 is chosen if z is larger than λ, and H_2 is chosen if z is less than λ. If $z = \lambda$ an arbitrary decision can be made.

Probability of Error

As before, the overall probability of error, P_e, is given by:-

$$P_e = P(S_1)\underbrace{P(H_2|S_1)}_{\text{Probability of deciding } H_2 \text{ when } S_1 \text{ was sent}} + P(S_2)\underbrace{P(H_1|S_2)}_{\text{etc.}}$$

where $P(S_i)_{i=1,2}$ is the probability of sending $S_i(t)$.

In the equi-probable case, $P(S_1) = P(S_2) = \frac{1}{2}$ and:-

$$P_e = \frac{1}{2}\left(P(H_2|S_1) + P(H_1|S_2)\right)$$

Maximum Likelihood Detector

A popular ~~(criterion)~~ criterion for choosing λ is based on minimizing P_e. The calculation for this "minimum error" value of $\lambda = \lambda_o$ starts with this likelihood ratio test:-

$$\Lambda(z) = \frac{f(z|S_1)}{f(z|S_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(S_2)}{P(S_1)}$$

where $f(z|S_i)$ is the conditional pdf, known as the "likelihood of S_i". Hence we can write:-

$$\cancel{\Lambda(z) = \frac{f(z|S_1)}{f(z)}}$$

$$\Lambda(z) = \frac{f(z|S_1)}{f(z|S_2)} \underset{H_2}{\overset{H_1}{\gtrless}} 1 \qquad \text{in the equiprobable case, since } P(S_1) = P(S_2) = \tfrac{1}{2}$$

If, in addition, the likelihood, $f(z|s_i)$ $i=1,2$ are symmetric, then this further reduces to:-

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \lambda_0 \quad \text{where } \lambda_0 = \frac{a_1 + a_2}{2}.$$

We will show that $\lambda_0 = \frac{a_1+a_2}{2}$ is the optimum threshold for minimising P_e in the equiprobable case - (th a) Since this detector minimises P_e, it is known as a "maximum likelihood detector"

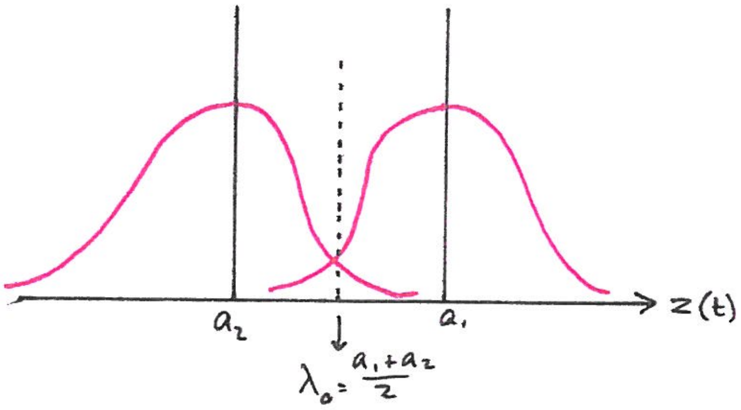
Probability of Error with Gaussian Noise

Assuming Gaussian noise it follows that here we have:-

$$f(z|s_1) = \frac{1}{\sqrt{2\pi\sigma_{n_0}^2}} e^{-(z-a_1)^2/2\pi\sigma_{n_0}^2}$$

$$f(z|s_2) = \frac{1}{\sqrt{2\pi\sigma_{n_0}^2}} e^{-(z-a_2)^2/2\pi\sigma_{n_0}^2}$$

Where σ_{n_0} = standard deviation of the filtered noise.



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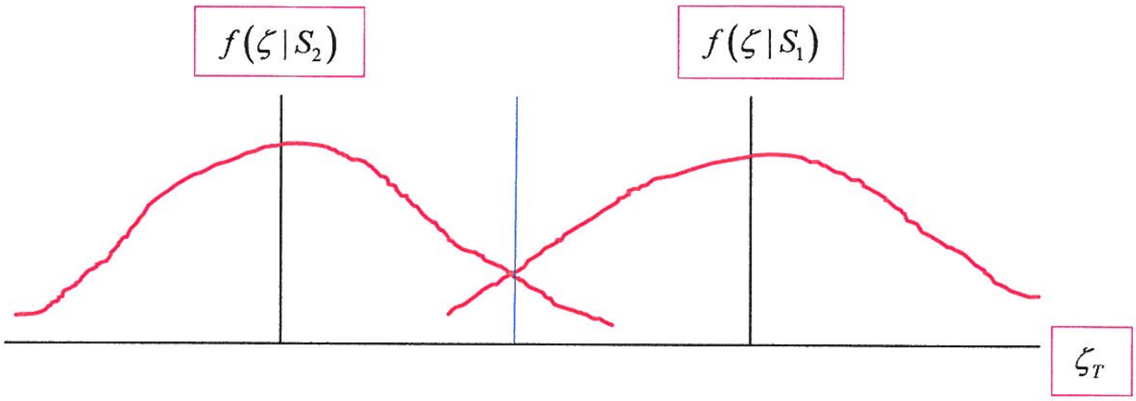
Assuming Gaussian noise it follows that here we have:-

$$\frac{-(\zeta - a_1)^2}{2V_{n0}^2}$$



Now:-

$$P(H_2 \mid S_1) = \int_{-\infty} f(\zeta \mid S_1)\, d\zeta$$

$$P(H_1 \mid S_2) = \int_{\lambda_0}^{\infty} f(\zeta \mid S_2)\, d\zeta$$

Because of the symmetry of $f(\zeta \mid S_i)$, P_e reduces to:-

$$P_e = \tfrac{1}{2}\left(P(H_2 \mid S_1) + P(H_1 \mid S_2)\right)$$

$$P_e = Q\left(\frac{a_1 - a_2}{2\delta n_0}\right)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x e\, dy$$

Example:

Prove that if $P(S_1) = P(S_2)$ then $\lambda_0 = {}^{(a1+a2)}/_2$ minimises P_e.

Solution:

Assume the threshold is set at λ, and so:-

Optimum Detection

Background Definitions

Wide-Sense Stationary Process

A random process x(t) is "wide-sense stationary" (wss) if its mean is constant, i.e. $E\left[x(t)\right] = \mu_x = \int_{-\infty}^{\infty} x f_x(x;t)\,dx$ and its autocorrelation depends only on the time difference τ. i.e. $E\left[x(t)x(t+\tau)\right] = R_{xx}(\tau)$ (see below).

Autocorrelation $R_{xx}(\tau)$

The autocorrelation of *x(t)* is defined by:-

$$R_{xx}(t) = E\left[x(t)x(t+\tau)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_x(x_1,x_2;t_1,t_2)\,dx_1 dx_2$$

And has the following properties:-

1. Even function:- $R_{xx}(-\tau) = R_{xx}(\tau)$
2. $|R_{xx}(\tau)| \leq E[x^2(t)]$
3. $R_{xx}(0) = E[x^2(t)]$ = total average normalised power to a 1Ω load.

Power Spectrum Density (PSD)

The PSD of *x(t)* is defined by the Fourier transform of $R_{xx}(\tau)$:-

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(t)e^{-j\omega t}\,dt$$
$$\Rightarrow R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega t}\,d\omega \qquad \text{Wiener-Khiachin Relations}$$

PSD $S_{xx}(\omega)$ has the following properties:-

1. $S_{xx}(\omega)$ is real and $s_{xx}(\omega) \geq 0$.
2. $S_{xx}(-\omega) = S_{xx}(\omega)$ (Even also)
3. $R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega = R_{xx}(0) = E\left[x^2(t)\right]$

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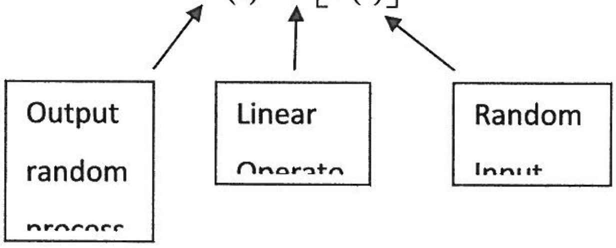
Transmission of Random Processes Through Linear Systems

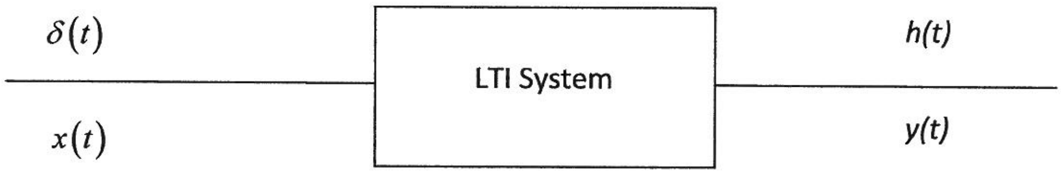
A linear LTI system can be represented by:-

$$Y(t) = L[x(t)]$$



Let *h(t)* be the impulse response of the LTI. Then:

$$\begin{aligned} Y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha \end{aligned}$$



Mean and Autocorrelation of the Output

The mean of the output

$$
\begin{aligned}
\mu_y(t) &= E[y(t)] \\
&= E\left[\int_{-\infty}^{\infty} h(\alpha)x(t-\alpha)\,d\alpha\right] \\
&= \int_{-\infty}^{\infty} h(\alpha)E[x(t-\alpha)]\,d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\mu_x(t-\alpha)\,d\alpha \\
&= h(t) * \mu_x(t)
\end{aligned}
$$

If the input is wide-sense stationary we have:

$$
\begin{aligned}
E[y(t)] &= \int_{-\infty}^{\infty} h(\alpha)\mu_x\,d\alpha \\
&= \mu_x \int_{-\infty}^{\infty} h(\alpha)\,d\alpha \\
&= \mu_x H(0)
\end{aligned}
$$

Where *H(0)* is the frequency response of the linear system at ω=0. Thus, the mean of the output is a constant. The autocorrelation of the output, $R_{yy}(\tau)$, is given by:

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{xx}(\tau+\alpha-\beta)\,d\alpha\,d\beta
$$

And the output is also wide-sense-stationary.

Power Spectral Density of Output

Taking the Fourier transform of both sides of this last expression yields:

$$
\begin{aligned}
&S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{yy}(\tau)e^{-j\omega\tau}\,d\tau \\
&which = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_{xx}(\tau+\alpha-\beta)e^{-j\omega\tau}\,d\tau\,d\alpha\,d\beta \\
&\Rightarrow S_{YY}(\omega) = |H(\omega)|^2 S_{xx}(\omega)
\end{aligned}
$$

The autocorrelation of the output, $R_{yy}(\tau)$, is often best evaluated via the inverse Fourier transform of $S_{yy}(\omega)$:-

$$
R_{yy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_{xx}(\omega)e^{j\omega t}\,d\omega
$$

Hence, the average power in the output *y(t)* is

$$E\left[y^2(t)\right] = R_{yy}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 S_{xx}(\omega)\,d\omega$$

White Noise

A random process *x(t)* is called "white noise" if $S_{xx}(\omega) = \frac{\eta}{2}$

It is usually assumed that the mean of white noise is zero. (Note $S_{xx}(\tau)$ relates to $|V|^2$ and the mean relates to V itself.)

Optimum Detection

Recall earlier we had: $$P_e = Q\left[\frac{a_1 - a_2}{2\sigma_{n_0}}\right].$$

And so, to minimise P_e we search for the filter capable of maximising $\frac{a_1 - a_2}{2\sigma_{n_0}}$

Matched Filter

A matched filter, as we will see, provides the maximum output signal-to-noise ratio for a given transmitted signal. Consider that a known signal *S(t)* plus AWGN *n(t)* is the input to a LTI filter, followed by a sampler. Let *a(t)* be the "signal only" response of the filter. Then, at t=T we have:-

$$\left(\frac{S}{N}\right)_0 = \frac{a^2(T)}{E\left[n_0^2(T)\right]}$$

Assuming a 1Ω reference resistor

$$= \frac{a^2(T)}{\sigma_{n_0}^2} \rightarrow \text{since } \sigma_{n_0}^2 = E\left[(y-\bar{y})^2\right]$$

But our input noise has zero mean

$\Rightarrow$ so too does our output noise

$\Rightarrow \bar{y} = 0$

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Example:

Find the optimum filter $H_o(\omega)$ that maximises the output SNR when the input noise is not white (coloured noise).

Solution:

Let $H(\omega)$ be the frequency response of the linear filter
Let $S_{nn}(\omega)$ be the power spectrum of the input coloured noise. At $t = T$ we have as ~~(follows)~~ before:-

$$a(T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, S(\omega)\, e^{j\omega T}\, d\omega.$$

and the average output noise power is:-

$$N_o = E\left[n_o^2(t)\right]$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{nn}(\omega)\, |H(\omega)|^2\, d\omega$$

=> output SNR is:-

$$\left(\frac{S}{N}\right)_o = \frac{\frac{1}{2\pi}\left|\int_{-\infty}^{\infty} H(\omega)\, S(\omega)\, e^{j\omega T}\, d\omega\right|^2}{\int_{-\infty}^{\infty} S_{nn}(\omega)\, |H(\omega)|^2\, d\omega}$$

To find the optimum filter set:-

$$f_1(\omega) = \sqrt{S_{nn}(\omega)}\, H(\omega)$$
$$f_2(\omega) = \frac{S(\omega)\, e^{j\omega T}}{\sqrt{S_{nn}(\omega)}}$$

we can now write using the Schwarz inequality:-

$$\left|\int_{-\infty}^{\infty} H(\omega)\, S(\omega)\, e^{j\omega T}\, d\omega\right|^2 \leq \ldots$$

$$\ldots \int_{-\infty}^{\infty} S_{nn}(\omega)\, |H(\omega)|^2\, d\omega \times \int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{(S_{nn}(\omega))}\, d\omega$$

since $|e^{j\omega T}| = 1$