Chapter 5. State Estimation for Control

5.1 State Estimation (Observers)

The state-space control-law requires the state-vector $\underline{x}(t)$ of the process. However some or all of the states may be unavailable due to:

1) Expense:

2) Requires off-line analysis:

3) Impossible to measure:

4) Process model is mathematical, eg. Derived from step test, system identification, frequency response etc..

An estimator (sometimes called observer) will use what measurements are available to provide the estimates of the unmeasured states.

5.1.1 Direct State Estimation

Consider the MIMO process:

$$\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B\underline{u}\\ \underline{y} &= C\underline{x}\end{aligned}$$

Then if the rank(C)=N, the states can be estimated directly from output measurements:

Of course in the special case, when C is square and of full rank:

5.1.2 Open-Loop Estimator

We have an underlying model of the process, which describes how the states depend on the input – hence propose the estimator:

$$\frac{d}{dt}\underline{\hat{x}} = A\underline{\hat{x}} + B\underline{u}$$

However: i) The model is only approximate

ii) Difficult to determine exactly the initial state

iii) Unmeasured disturbances are not included

5.1.3 The Closed-Loop Estimator (Luenberger Observer)

This is a full-state estimator, which will make use of the output measurement vector $\underline{y}(t)$ to close the loop and to correct for model errors, disturbances and incorrect initial conditions.

Consider the MIMO process model:

$$\begin{aligned}\underline{\dot{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x}\end{aligned}$$

The full-state estimator (Luenberger Observer) is:

$$\begin{aligned}&\frac{d}{dt}\underline{\hat{x}} = A\underline{\hat{x}} + B\underline{u} + G\left(\underline{y}(t) - \underline{\hat{y}}(t)\right) \\ &\underline{\hat{y}}(t) = C\underline{\hat{x}}(t)\end{aligned}$$

Where G is the estimator gain matrix:

Define the state-estimation error vector:

Then we can develop an expression for the estimation error dynamics as follows:

$$\frac{d}{dt}\underline{e}(t) = \frac{d}{dt}\underline{x}(t) - \frac{d}{dt}\underline{\hat{x}}(t)$$

Hence we can write:

Which can be rearranged to yield:

$$\underline{\dot{e}}(t) = A\underline{e}(t) - G\left(\underline{y}(t) - \underline{\hat{y}}(t)\right)$$

But: $\underline{y}(t) = C\underline{x}(t)$ and $\underline{\hat{y}}(t) = C\underline{\hat{x}}(t)$, hence the error dynamics are:

$$\underline{\dot{e}}(t) = A\underline{e}(t) - G\left(C\underline{x}(t) - C\underline{\hat{x}}(t)\right)$$

Now assign: $F = A - GC$

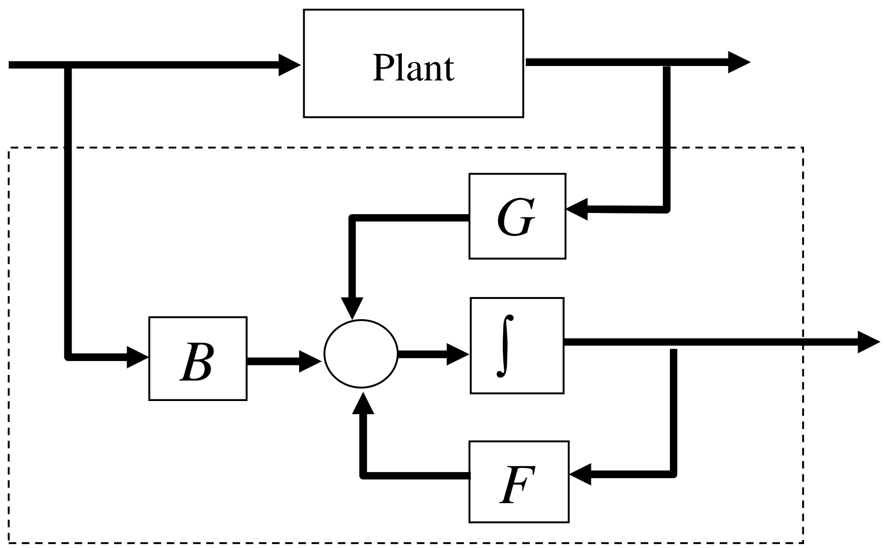
We can now specify the error dynamics by choice of the eigenvalues of F – ie. the N roots of:

$$\det\left(\lambda I - F\right) = 0$$

The estimator could be realised as:

$$\frac{d}{dt}\underline{\hat{x}} = A\underline{\hat{x}} + B\underline{u} + G\left(\underline{y}(t) - \underline{\hat{y}}(t)\right)$$

Which could be built as follows:



Choice of the Estimator Poles

The estimator poles are equivalent to the eigenvalues of F

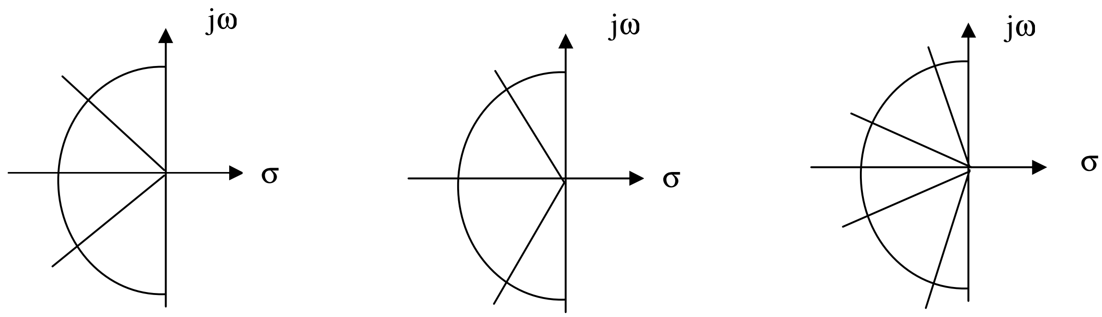
We require: i) A stable estimator
ii) Estimator error dynamics to be much faster than the dominant state dynamics

The closed-loop response of a system will be dominated by a slow dominant pole of pole-pair. It is common to choose the N observer poles so that they are:

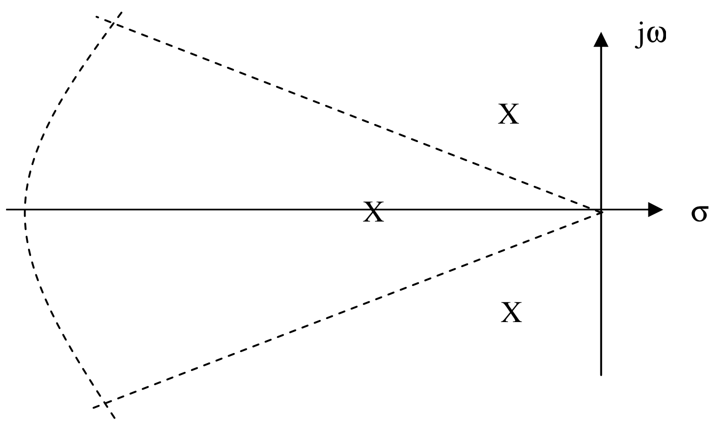
A common configuration for the observer poles are based on the N left hand plane roots of:

$$\left(\frac{s}{\omega_N}\right)^{2N} = (-1)^{N+1}$$

eg:



Consider a third order process under closed-loop control using a full state estimator:



<u>EXAMPLE:</u> Consider the model of a chemical reactor, where Ca and Cb are chemical concentrations, q(t) is a flowrate and T the reactor temperature.

$$\frac{d}{dt}\begin{bmatrix} Ca \\ Cb \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} Ca \\ Cb \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} q(t)$$

$$T(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} Ca \\ Cb \end{bmatrix}$$

Develop a *soft-sensor* to provide estimates of the concentrations from flowrate and temperature measurements.

The open-loop poles at: s=-2, s=-1

The full-state estimator is:

$$\frac{d}{dt}\begin{bmatrix}\hat{C}a\\ \hat{C}b\end{bmatrix} = (A - GC)\begin{bmatrix}\hat{C}a\\ \hat{C}b\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix} q(t) + GT(t)$$

$$G = \begin{bmatrix}g1\\ g2\end{bmatrix}$$

$$A - GC = \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix} - \begin{bmatrix}g1\\ g2\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}-2-g1 & -g1\\ -g2 & -1-g2\end{bmatrix}$$

Poles of the estimator are given by the roots of:

$$\det\left[\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix} - \begin{pmatrix}-2-g1 & -g1\\ -g2 & -1-g2\end{pmatrix}\right] = 0$$

Yields the characteristic equation:

$$s^2 + (3 + g1 + g2)s + (2 + 2g2 + g1) = 0$$

Now this process is open-loop- with fastest pole at s=-2

Choose the two estimator poles at s=-10 twice, yields the desired characteristic equation:

$$s^2 + 20s + 100 = 0$$

Hence: $\begin{matrix} 3+g1+g2=20 \\ 2+2g2+g1=100 \end{matrix}$ which yields, g1=-64, g2=81

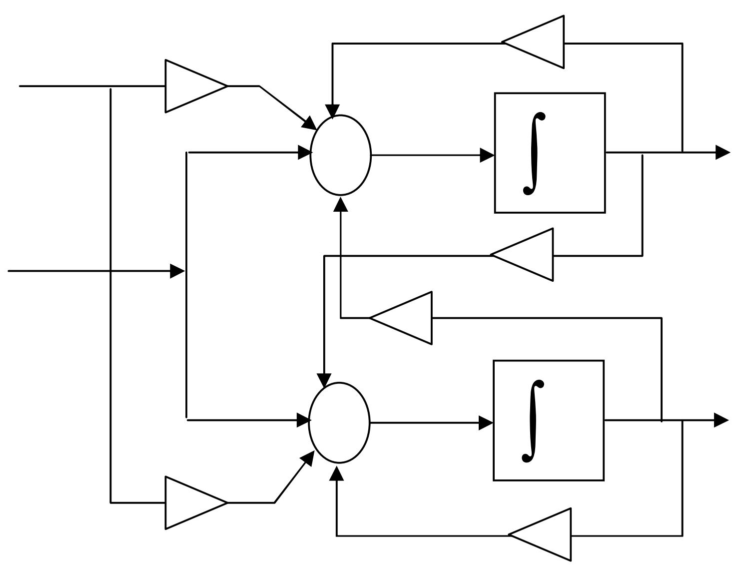
Then:

$$F = A - GC = \begin{bmatrix} -2-g1 & -g1 \\ -g2 & -1-g2 \end{bmatrix} = \begin{bmatrix} 62 & 64 \\ -81 & -82 \end{bmatrix}$$

The soft-sensor equations are then:

$$\frac{d}{dt}\begin{bmatrix} \hat{C}a \\ \hat{C}b \end{bmatrix} = \begin{bmatrix} 62 & 64 \\ -81 & -82 \end{bmatrix}\begin{bmatrix} \hat{C}a \\ \hat{C}b \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} q(t) + \begin{bmatrix} -64 \\ 81 \end{bmatrix} T(t)$$

Which could be constructed using two integrators:



5.2 Observability

Can the state-vector $\underline{x}(t)$ be estimated from input $\underline{u}(t)$ and output $\underline{y}(t)$signals.

The N^th order MIMO process representation:

$$\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + B\,\underline{u} \\ \underline{y} &= C\,\underline{x} \end{aligned}$$

is observable if : $$rank(O_x) = N$$

where:

$$O_x = \begin{bmatrix} C \\ \hline CA \\ \hline CA^2 \\ \hline \vdots \\ \hline CA^{N-1} \end{bmatrix}$$

5.3 The Observer Canonical Form

Consider the transfer function for an N^th order SISO process:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{f_0 + f_1 s + \cdots f_{N-1} s^{N-1}}{s^N + e_{N-1} s^{N-1} + \cdots e_0}$$

Could be rewritten as:

$$\frac{Y(s)}{U(s)} = \frac{\dfrac{f_o}{s^N} + \dfrac{f_1}{s^{N-1}} + \dfrac{f_2}{s^{N-2}} + \cdots \dfrac{f_{N-1}}{s}}{1 + \dfrac{e_{N-1}}{s} + \dfrac{e_{N-2}}{s^2} \cdots \dfrac{e_0}{s^N}}$$

Cross-multiplying yields:

$$\left(1+\frac{e_{N-1}}{s}+\frac{e_{N-2}}{s^2}\cdots\frac{e_0}{s^N}\right)Y(s)=\left(\frac{f_o}{s^N}+\frac{f_1}{s^{N-1}}+\frac{f_2}{s^{N-2}}+\cdots\frac{f_{N-1}}{s}\right)U(s)$$

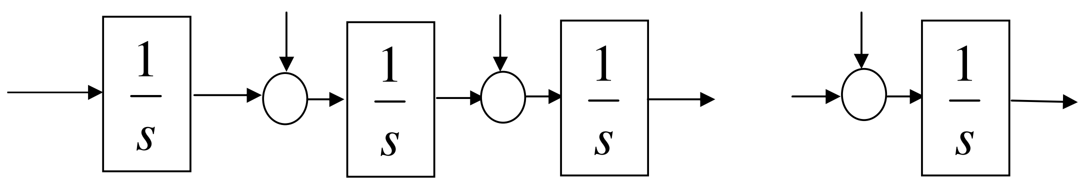
Solving for Y(s):

$$Y(s)=\frac{1}{s}\left(f_{N-1}U-e_{N-1}Y\right)+\frac{1}{s^2}\left(f_{N-2}U-e_{N-2}Y\right)+\cdots\frac{1}{s^N}\left(f_0U-e_0Y\right)$$

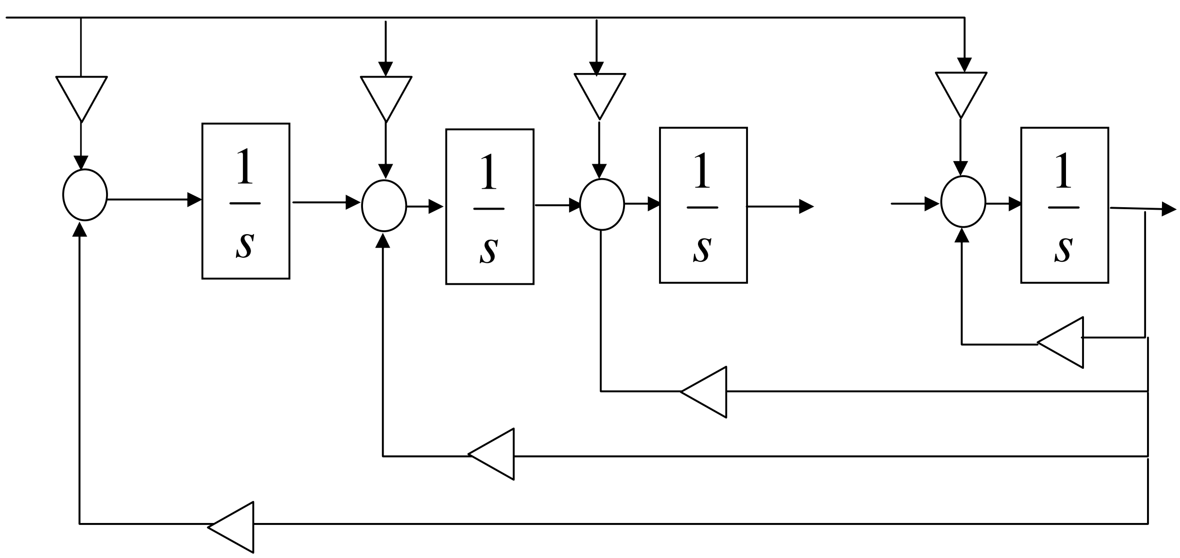
which could be written as:

$$Y(s)=\frac{1}{s}\left(z_1+\frac{1}{s}\left(z_2+\frac{1}{s}\left(z_3+\frac{1}{s}\left(z_4+\frac{1}{s}\left(\cdots\cdots z_{N-1}+\frac{1}{s}\left(z_N\right)\right)\right)\right)\right)\right)$$

which could represented as:



This yields the observer canonical format:



The observer canonical state-space equations are:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} -e_{N-1} & 1 & 0 & 0 & \cdots & 0 \\ -e_{N-2} & 0 & 1 & 0 & \cdots & 0 \\ -e_{N-3} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ -e_1 & 0 & 0 & 0 & \cdots & 1 \\ -e_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} f_{N-1} \\ f_{N-2} \\ f_{N-3} \\ \vdots \\ f_0 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \underline{x}(t)$$

A process model in this form is always observable.

Consider the design equation for the full state estimator:

$$\det\left(sI - \begin{bmatrix} -e_{N-1} & 1 & 0 & 0 & \cdots & 0 \\ -e_{N-2} & 0 & 1 & 0 & \cdots & 0 \\ -e_{N-3} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ -e_1 & 0 & 0 & 0 & \cdots & 1 \\ -e_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_N \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \right)$$
$$= s^N + c_{N-1}s^{N-1} + c_{N-2}s^{N-2} + \cdots c_1 s + c_0$$

Or:

$$\det\left(sI - \begin{bmatrix} g_1 - e_{N-1} & 1 & 0 & 0 & \cdots & 0 \\ g_2 - e_{N-2} & 0 & 1 & 0 & \cdots & 0 \\ g_3 - e_{N-3} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ g_{N-1} - e_1 & 0 & 0 & 0 & \cdots & 1 \\ g_N - e_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right)$$
$$= s^N + c_{N-1}s^{N-1} + c_{N-2}s^{N-2} + \cdots c_1 s + c_0$$

Which of course could be solved by the simple choice:

5.4 Combining Estimators with Controllers

An alternative representation for the *Luenberger Observer* can be obtained as follows:

$$\frac{d}{dt}\underline{\hat{x}} = A\underline{\hat{x}} + B\underline{u} + G\left(\underline{y} - \underline{\hat{y}}\right)$$
$$\frac{d}{dt}\underline{e} = (A - GC)\underline{e}$$

which could be represented as:

$$\frac{d}{dt}\begin{bmatrix} \underline{\hat{x}}(t) \\ \hline \underline{e}(t) \end{bmatrix} = \left[\begin{array}{c|c} A & GC \\ \hline 0 & A-GC \end{array}\right] + \begin{bmatrix} \underline{\hat{x}}(t) \\ \hline \underline{e}(t) \end{bmatrix} + \begin{bmatrix} B \\ \hline 0 \end{bmatrix}\underline{u}(t)$$

5.4.1 The Separation Principle

Consider now a regulator uses the estimated state rather than the actual state measurement:

The closed-loop state equation becomes:

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) - BK\underline{\hat{x}}(t)$$

But we have defined the estimation error as: $\underline{e}(t) = \underline{x}(t) - \underline{\hat{x}}(t)$

Hence:

The closed-loop state equation can be written as:

The combined dynamics of the estimator error and the process state are given in more compact form as:

$$\frac{d}{dt}\left[\begin{array}{c}\underline{x}(t) \\ \hline \underline{e}(t)\end{array}\right]=\left[\begin{array}{c|c}A-BK & BK \\ \hline 0 & A-GC\end{array}\right]\left[\begin{array}{c}\underline{x}(t) \\ \hline \underline{e}(t)\end{array}\right]$$

The poles of the closed-loop process are then given by the roots of:

$$\det\left(sI-\left[\begin{array}{c|c}A-BK & BK \\ \hline 0 & A-GC\end{array}\right]\right)=0$$

which could be rearranged as:

$$\det\left(\left[\begin{array}{c|c}sI_N & 0 \\ \hline 0 & sI_N\end{array}\right]-\left[\begin{array}{c|c}A-BK & BK \\ \hline 0 & A-GC\end{array}\right]\right)=0$$

or:

$$\det\left(\left[\begin{array}{c|c}sI_N-A+BK & -BK \\ \hline 0 & SI_N-A+GC\end{array}\right]\right)=0$$

A little revision:

$$\det\left[\begin{array}{c|c}A & B \\ \hline C & D\end{array}\right]=|A||D|-|B||C|$$

Hence the characteristic equation for the closed-loop system with estimator is:

$$|sI_N-A+BK||sI_N-A+GC|=0$$

This leads us to the “Separation Principle”:

5.4.2 The Equivalent Transfer Function

Consider the estimator:

$$\frac{d}{dt}\underline{\hat{x}} = (A - GC)\underline{\hat{x}} + B\underline{u} + G\underline{y}(t)$$

If the following state regulator is used:

$$\underline{u}(t) = -K\underline{\hat{x}}(t)$$

Then the estimator equations become

$$\frac{d}{dt}\underline{\hat{x}} = (A - GC)\underline{\hat{x}} - BK\underline{\hat{x}} + G\underline{y}(t)$$

Taking Laplace transforms:

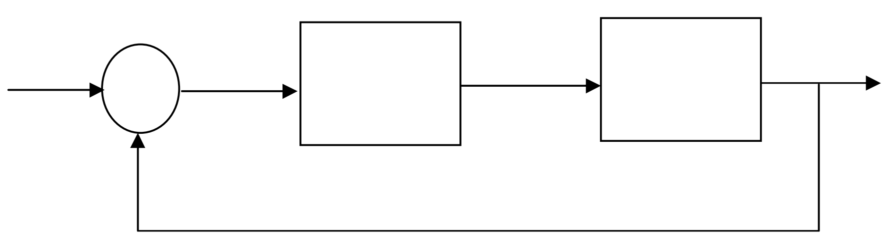
$$s\underline{\hat{x}}(s) = (A - GC - BK)\underline{\hat{x}}(s) + G\underline{Y}(s)$$

which could be rearranged to yield:

$$\underline{\hat{x}}(s) = (sI - A + GC + BK)^{-1} G\underline{Y}(s)$$

the controller can then be easily determined as:

Which yields the following classical regulator



Appendix A: Basics of Digital State Space Control

A.1 Discrete-Time Control

Consider the SISO discrete time process:

$$\underline{x}(k+1) = A_d\,\underline{x}(k) + B_d u(k) + E_d d(k)$$
$$y(k) = C\underline{x}(k)$$

The controllability matrix for this process model is:

$$C_x = \left[B_d \;\vdots\; A_d B_d \;\vdots\; A_d{}^2 B_d \;\vdots\; \cdots \;\vdots\; A_d{}^{N-1} B_d\right]$$

This process is controllable if:

The discrete-time regulator is simply:

$$u(k) = -K\,\underline{x}(k)$$

This yields the closed-loop state equation:

$$\underline{x}(k+1) = \left(A_d - B_d K\right)\underline{x}(k) + E_d d(k)$$

The poles of the closed-loop system are given by the roots of:

$$\det\left(zI - A_d + B_d K\right) = 0$$

Specify the desired characteristic equation:

$$C_{des}(z) = z^N + C_{N-1} z^{N-1} + \cdots C_1 z + C_0 = 0$$

By selecting the N desired closed-loop poles on the z plane (Use the template):

A.2 Discrete time Estimators

If the sample time is very small then it is possible to design the estimator as a continuous time estimator:

$$\frac{d}{dt}\underline{\hat{x}} = (A - GC)\underline{\hat{x}} + B\underline{u} + G\,\underline{y}(t)$$

the dynamics of the continuous design are given by roots of:

$$\det(sI - A + GC) = 0$$

Now simply discretise the design:

$$\underline{\hat{x}}(k+1) = F_d\,\underline{\hat{x}}(k) + B_d\,\underline{u}(k) + G_d\,\underline{y}(k)$$

where:

$$\Phi_{obs}(t) = L^{-1}\{sI - A + GC\}$$
$$F_d = \Phi_{obs}(T)$$
$$B_d = \int_0^T \Phi_{obs}(\eta)B\,d\eta$$
$$G_d = \int_0^T \Phi_{obs}(\eta)G\,d\eta$$

Alternatively if T is relatively large then design completely in the discrete domain using the discrete model:

$$\underline{x}(k+1) = A_d\,\underline{x}(k) + B_d\,\underline{u}(k)$$
$$\underline{y}(k) = C\,\underline{x}(k)$$

Specify the estimator as:

$$\underline{\hat{x}}(k+1) = A_d\,\underline{\hat{x}}(k) + B_d\,\underline{u}(k) + G\left(\underline{y}(k) - C\,\underline{\hat{x}}(k)\right)$$

The error dynamics are determined by the roots of:

$$\det(zI - A_d + GC) = 0$$

Design G to place poles on the Z plane.