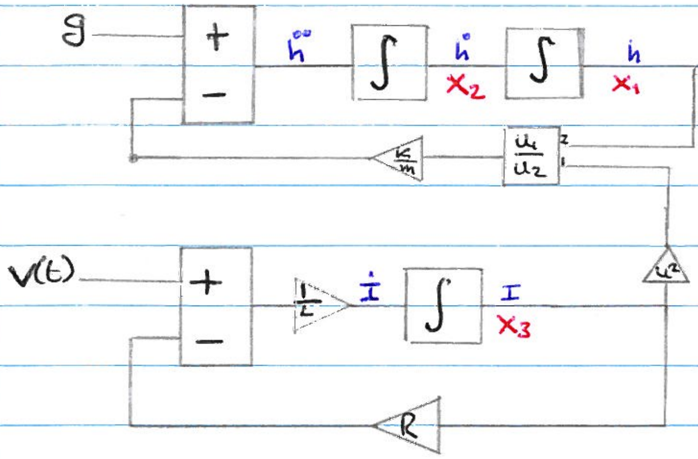
Control Engineering - State Space

Magnetic Suspension System

$$m\frac{d^2h}{dt^2} = mg - \frac{Ki^2(t)}{h(t)} \Rightarrow \frac{d^2h}{dt^2} = g - \frac{Ki^2(t)}{mh(t)}$$

$$L\frac{di(t)}{dt} = v(t) - Ri(t) \Rightarrow \frac{di(t)}{dt} = \frac{1}{L}\left(v(t) - Ri(t)\right)$$



*

$$\dot{x}_1 = x_2(t) = f_1(x_2)$$

$$\dot{x}_2 = g - \frac{K}{m}\frac{x_3^2(t)}{x_1(t)} = f_2(x_3, x_1)$$

$$\dot{x}_3 = \frac{1}{L}\left(u(t) - Rx_3(t)\right) = f_3(x_3, u)$$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ & & \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u(t)$$

By linearisation:

For stable o.p. @ $h = 0.01m$ with $L = 10mH$ $M = 0.05kg$

$g = 10ms^{-2}$ $R = 1\Omega$ $K = 0.01Nm/A$

$$m\cancelto{0}{\frac{d^2h(t)}{dt^2}} = mg - \frac{Ki^2(t)}{h(t)} \Rightarrow mg = \frac{Ki_0^2}{h_0^2}$$

$$L\cancelto{0}{\frac{di(t)}{dt}} = v(t) - Ri(t) \Rightarrow v = Ri_0$$

$$\left.\begin{aligned} i_0 &= 0.7071\,A \\ V &= 0.7071\,V \end{aligned}\right\} \quad x_0 = \begin{bmatrix} h \\ \dot{h} \\ i \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0 \\ 0.7071 \end{bmatrix}$$

Taking state equs. derived previously *

$$\Delta x = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = x - x_0 \qquad \Delta u = u - u_0$$

We have:

$$\frac{d\Delta x_1(t)}{dt} = \left.\frac{df_1}{dx_2}\right|_{op} \Delta x_2(t) \Rightarrow \frac{df_1}{dx_2} = 1 \text{ everywhere}$$

$$\frac{d\Delta x_2(t)}{dt} = \left.\frac{df_2}{dx_1}\right|_{op} \Delta x_1(t) + \left.\frac{df_2}{dx_3}\right|_{op} \Delta x_3(t) \Rightarrow \frac{Kx_3^2}{mx_1^2} = 999.98$$

$$\Downarrow \frac{-2kx_3}{mx_1} = -28.284$$

$$\frac{d\Delta x_3(t)}{dt} = \left.\frac{df_3}{dx_3}\right|_{op} \Delta x_3(t) + \left.\frac{df_3}{du}\right|_{op} \Delta u(t)$$

$$= -\frac{R\Delta x_3(t)}{L} + \frac{1}{L}\Delta u(t)$$

State Space Control

Magnetic Suspension System

$$\Rightarrow \frac{d\Delta x_1(t)}{dt} = 1\Delta x_2(t)$$

$$\Rightarrow \frac{d\Delta x_2(t)}{dt} = 999.98\,\Delta x_1(t) - 28.284\,\Delta x_3(t)$$

$$\Rightarrow \frac{d\Delta x_3(t)}{dt} = -100\,\Delta x_3(t) + 100\,\Delta u(t)$$

$$\Delta y(t) = h(t) - h_0 = \Delta x_1(t)$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 999.98 & 0 & -28.284 \\ 0 & 0 & -100 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}\Delta u(t)$$

$$\Delta y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix}$$

State - Space Control
P.M. DC Motor

2 states $\Rightarrow$ i, ω
1 input $\Rightarrow$ v
1 dist. $\Rightarrow$ T_L

$$\frac{di}{dt} = \frac{1}{L}\left(v(t) - Ri(t) - K_m\omega(t)\right)$$
$$\frac{d\omega}{dt} = \frac{1}{J}\left(K_m i(t) - B\omega(t) - T_L(t)\right)$$

$$\frac{d}{dt}\begin{bmatrix} i(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{K_m}{L} \\ \frac{K_m}{J} & -\frac{B}{J} \end{bmatrix}\begin{bmatrix} i(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v(t) + \begin{bmatrix} 0 \\ -\frac{1}{J} \end{bmatrix} T_L(t)$$

Operating Point: If $T_L(t) = K_F\omega^2(t)$

$$\frac{di}{dt} \to 0 \Rightarrow V_o = Ri_o + K_m\omega_o$$
$$\frac{d\omega}{dt} \to 0 \Rightarrow i_o = \frac{B\omega_o + K_F\omega_o^2}{K_m}$$
$$V_o = \left(K_m + \frac{RB}{K_m}\right)\omega_o + \left(\frac{RK_F}{K_m}\right)\omega_o^2$$

$$\dot{x}_1(t) = \frac{1}{L}\left(u(t) - Rx_1(t) - K_m x_2(t)\right) \qquad f_1(u, x_1, x_2)$$
$$\dot{x}_2(t) = \frac{1}{J}\left(K_m x_1(t) - Bx_2(t) - K_F x_2^2(t)\right) \qquad f_2(x_1, x_2)$$

$$\frac{d\Delta x_1(t)}{dt} = \left.\frac{df_1}{du}\right|_{op}\Delta u(t) + \left.\frac{df_1}{dx_1}\right|_{op}\Delta x_1(t) + \left.\frac{df_1}{dx_2}\right|_{op}\Delta x_2(t)$$

$$\frac{d\Delta x_2(t)}{dt} = \left.\frac{df_2}{dx_1}\right|_{op}\Delta x_1(t) + \left.\frac{df_2}{dx_2}\right|_{op}\Delta x_2(t)$$

$$\frac{d\Delta x_1(t)}{dt} = \frac{1}{L}\Delta u(t) - \frac{R}{L}\Delta x_1(t) - \frac{K_m}{L}\Delta x_2(t)$$
$$\frac{d\Delta x_2(t)}{dt} = \frac{K_m}{J}\Delta x_1(t) - \frac{1}{J}\left(B + 2K_F\omega_o\right)\Delta x_2(t)$$

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{K_m}{L} \\ \frac{K_m}{J} & -\frac{1}{J}(B + 2K_F\omega_o) \end{bmatrix}\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t)$$

State Space Control

Cont. Time Regulator Design

$\underline{\dot{x}}(t) = A\underline{x}(t) + Bu(t) + Ed(t)$

$y(t) = C\underline{x}(t)$

Open Loop TF: $C(sI-A)^{-1}E$

$u(t) = -[k_1 \ k_2 \cdots k_n]\,\underline{x}(t)$

$\Rightarrow \underline{\dot{x}}(t) = (A-Bk)\underline{x}(t) + Ed(t)$

Closed Loop TF: $C(sI-A+Bk)^{-1}E$

C.L.T.F. Poles = Roots of $\det(sI-A+Bk)=0$

Choose: $C_{des}(s) = (s-p_1)(s-p_2)\cdots(s-p_N) = \det(sI-A+Bk)$

For $N=2$: $C_{des}(s) = s^2 + 2\xi\omega_n s + \omega_n^2$

Regs. for high order:

$$(A-Bk) = \left[\begin{array}{c|c} \underline{0}_{N-1} & I_{N-1} \\ \hline \multicolumn{2}{c}{-\underline{e}^T - k} \end{array}\right]$$

Char. Eqn. of C.L. Sys: $-e^T - k = [-e_0-k_1 \ \ -e_1-k_2 \cdots -e_{N-1}-k_N]$

$C_{des}(s) = s^N + C_{N-1}s^{N-1} + \dots C_1 s + C_0 = 0$

Compare: $s^N + (e_{N-1}+k_N)s^{N-1} + \dots + (e_0+k_1) = 0$

High Order Using Controllability Matrix:

- Convert non CCF $\rightarrow G(s) \rightarrow$ CCF
- Find C_z, controllability matrix for CCF.
- Design regulator for CCF, $u(t) = -k_z\underline{z}(t)$

$T = C_z C_x^{-1} \rightarrow k = k_z C_z C_x^{-1}$ for $u = -K\underline{x}(t)$

State-Space Control

Control Canonical Form

$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$

$y(t) = C\underline{x}(t)$

E.G. $G(s) = \dfrac{\overset{f_1}{1}s + \overset{f_0}{2}}{\underset{e_2}{s^3+4s^2}+\underset{e_1}{3s}+\underset{e_0}{0}}$ $N=3$

$$A = \begin{bmatrix} \overbrace{\begin{matrix}0\\0\end{matrix}}^{O_{N-1}} & \overbrace{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}}^{I_{N-1}} \\ -e_0 & -e_1 \quad -e_2 \end{bmatrix} \qquad B = \begin{bmatrix} \overbrace{\begin{matrix}0\\0\end{matrix}}^{O_{N-1}} \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & \boxed{0} \end{bmatrix}$$

N - Highest pwr in numerator - 1

State-Space to TF

$G(s) = C(sI-A)^{-1}B + D$

Transformation Theory

If $\underline{z}(t) = T\underline{x}(t)$ $\qquad T^{-1}\dot{\underline{z}}(t) = AT^{-1}\underline{z}(t) + B\underline{u}(t)$

$\qquad y(t) = CT^{-1}\underline{z}(t)$

$\Rightarrow \dot{\underline{z}}(t) = TAT^{-1}\underline{z}(t) + TB\underline{u}(t)$

$y(t) = CT^{-1}\underline{z}(t)$

Proof:

$G(s) = C(sI-A)^{-1}B + D$ $\qquad G_2(s) = CT^{-1}(sI - TAT^{-1})^{-1}TB$

$G_2(s) = C(sIT - TA)^{-1}TB$ $\qquad = C(T^{-1}sIT - A)^{-1}B$

$G_2(s) = C(sI - A)^{-1}B$

Transition Matrix

$\phi(t) = \mathcal{L}^{-1}\{\phi(s)\}$ $\qquad$ E.G. $\phi(s) = (sI-A)^{-1} = \begin{bmatrix} s+3 & 1 \\ -2 & 5 \end{bmatrix}\dfrac{1}{(s+2)(s+1)}$

Use partial fraction expansion.

$$\phi(t) = \mathcal{L}^{-1}\{\phi(s)\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{s+3}{(s+2)(s+1)}\right\} & \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s+1)}\right\} \\ \mathcal{L}^{-1}\left\{\frac{-2}{(s+2)(s+1)}\right\} & \mathcal{L}^{-1}\left\{\frac{5}{(s+2)(s+1)}\right\} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Matrix Exponential Method

$$\phi(t) = I + \frac{At}{1!} + \frac{At^2}{2!} + \frac{At^3}{3!} + \dots$$