Chapter 3. Introduction to State-Space Control

3.1 Continuous Time Regulator Design

What is a regulator?

Consider for simplicity the SISO process:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t) + Ed(t)$$
$$y(t) = C\underline{x}(t)$$

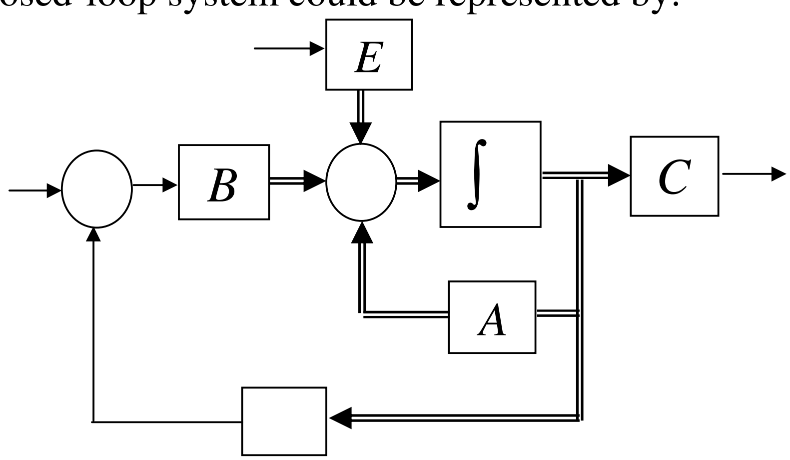
The open-loop dynamic behaviour of the plant to changes in the disturbance d(t) is given by the transfer function:

The poles of this transfer function dictate the dynamics of the open-loop process to changes in d(t):

Assume a regulator control law:

$$u(t) = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \underline{x}(t)$$

The closed-loop system could be represented by:



The closed loop state equation is then:

$$\underline{\dot{x}}(t) = A\underline{x}(t) - BK\,\underline{x}(t) + Ed(t)$$

Which yields the following closed loop transfer function:

$$G_D^{cl}(s) = \frac{Y(s)}{D(s)} = C\left(sI - A_{cl}\right)^{-1} E =$$

The poles of the closed-loop system are given then by the roots of the closed-loop characteristic equation:

For a specified closed-loop performance we will specify the closed-loop poles to be placed at:

This yields the desired characteristic equation:

$$C_{des}(s) = (s - p_1)(s - p_2)\cdots(s - p_N) = 0$$

Hence we choose the gain matrix K so that:

$$\det\left(sI - A + BK\right) = C_{des}(s)$$

Example:

A DC motor is modelled by the following equations:

$$\frac{d\omega}{dt} = \frac{1}{J}\left(K_m i(t) - B\omega(t) - T_L(t)\right)$$
$$\frac{di}{dt} = \frac{1}{L}\left(v(t) - K_m \omega(t) - Ri(t)\right)$$

Where: B=0, J=0.02Kgm^2, K_m=1NmA^{-1}, R=1Ω, L=5mH

The open-loop state-space model is then:

Tutorial:

Show for the open-loop system:

$$\frac{\Omega(s)}{T_L(s)} = \frac{-(5s+1000)}{(s+100)^2}$$

The open-loop poles are obviously s=-100 twice

Suggest the following regulator:

$$u(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\underline{x}(t)$$

Then the closed loop poles are given by the roots of:

$$\det(sI - A + BK) = 0$$

$$\det\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 50 \\ -200 & -200 \end{pmatrix} + \begin{pmatrix} 0 \\ 200 \end{pmatrix}\begin{pmatrix} k_1 & k_2 \end{pmatrix}\right] = 0$$

$$\det\begin{bmatrix} s & -50 \\ 200 + 200k_1 & s + 200 + 200k_2 \end{bmatrix} = 0$$

Which yields the closed-loop characteristic equation:

Now we must specify the desired characteristic equation $C_{des}(s)$:

Assume the following 2nd order structure:

$$C_{des}(s) = s^2 + 2\xi\omega_n s + \omega_n^2$$

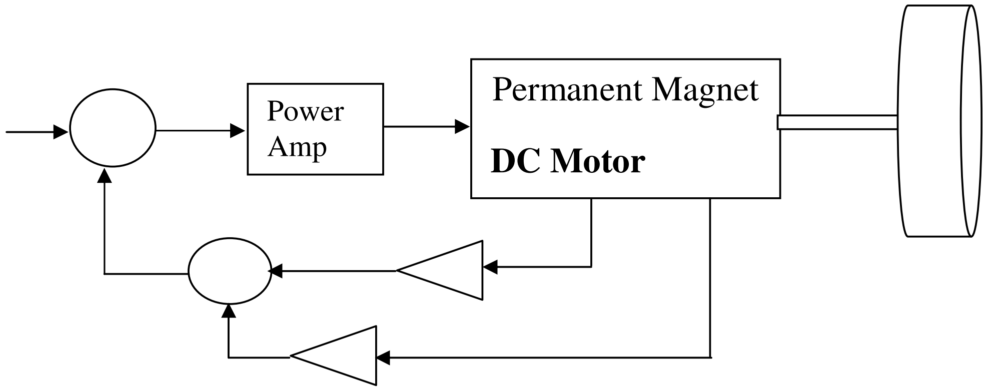
For this example we will choose:

$$C_{des}(s) = s^2 + 282.8s + 40000$$

To achieved the desired pole locations:

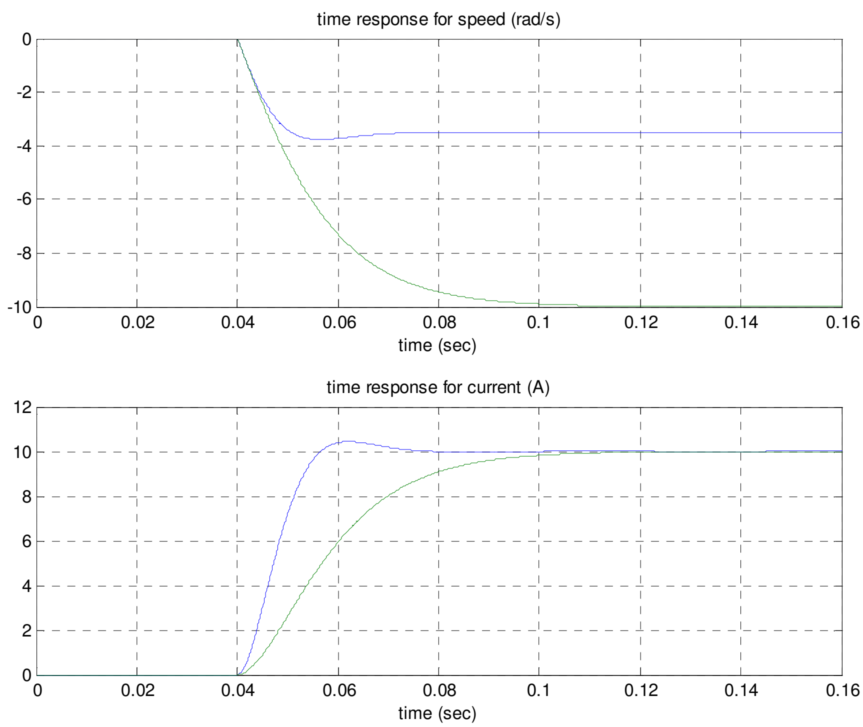
This yields the following regulator:

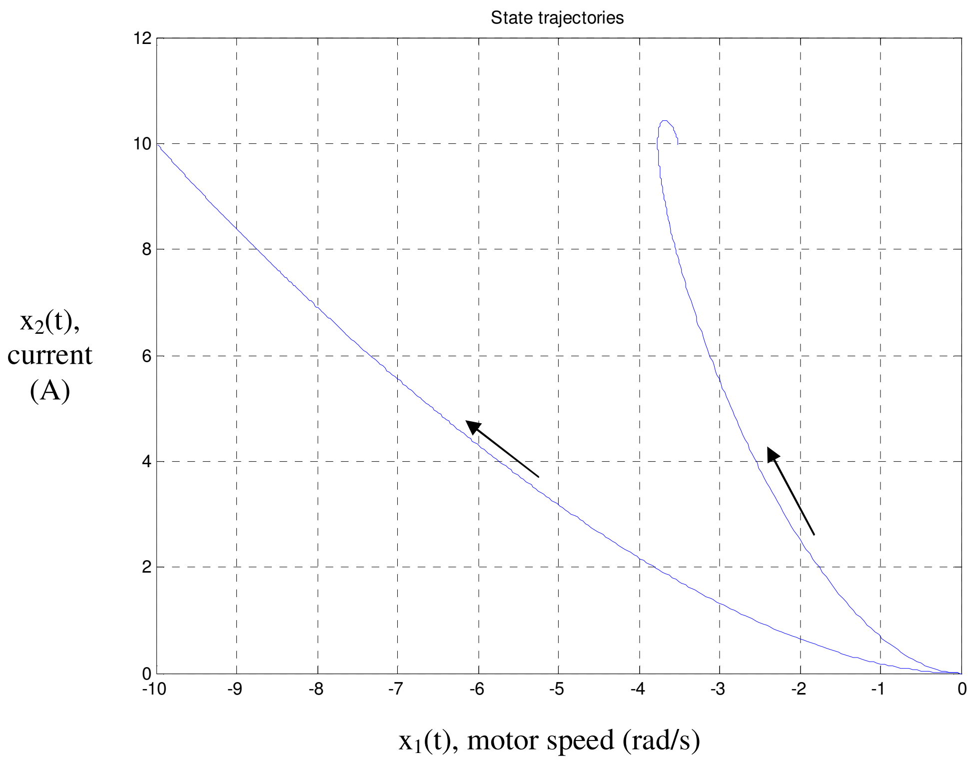
Could be built as follows:



Tutorial: Use the state-space technique to design the following PD speed controller, to achieve the performance highlighted above:

$$m(t) = K\left(e(t) + T_d \frac{de}{dt}\right) \quad \text{where} \quad e(t) = r(t) - \omega(t)$$




State trajectories
12
10
8
6
4
2
0
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
$x_2(t)$, current (A)
$x_1(t)$, motor speed (rad/s)

3.2 Regulator Design for High Order Processes

The state-space pole-placement design method proposed above is difficult to solve for high order processes:

However consider the N^{th} order SISO process in control canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -e_0 & -e_1 & -e_2 & -e_3 & \cdots & -e_{N-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} f_0 & f_1 & \cdots & f_r & 0 & \cdots & 0 \end{bmatrix}$$

If the following regulator is used:

$$u(t) = -K\underline{x}(t) = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}\underline{x}(t)$$

Then the closed-loop state equation becomes:

Lets look at the matrix product BK:

$$BK = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hline 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & \cdots & k_N \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \hline \\ \end{bmatrix} =$$

Hence we can write:

$$(A-BK)=\left[\begin{array}{c:c} \underline{O}_{N-1} & I_{N-1} \\ \hdashline \multicolumn{2}{c}{-\underline{e}^T} \end{array}\right]-\left[\begin{array}{c} \underline{O}_{N-1} \\ \hdashline \underline{K} \end{array}\right]=$$

Hence the characteristic equation of the closed-loop system is:

Now if the desired characteristic equation is:

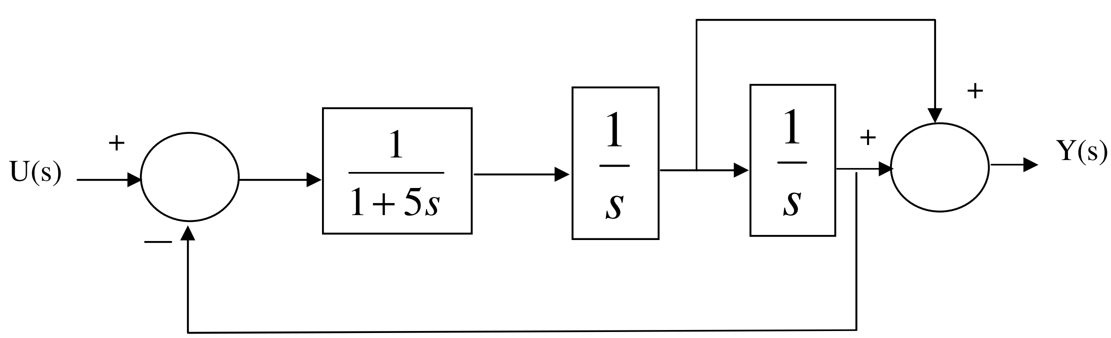
$$C_{des}(s) = s^N + C_{N-1}s^{N-1} + \cdots C_1 s + C_0 = 0$$

It is then easy to choose the gains $k_1, \ldots k_N$, to obtain the desired characteristic equation:

$$\begin{aligned} e_{N-1} + k_N &= C_{N-1} \\ e_{N-2} + k_{N-1} &= C_{N-2} \\ &\vdots \\ e_0 + k_1 &= C_0 \end{aligned}$$

<u>NOTE:</u>

<u>Tutorial:</u> Design a regulator for the following system to place the three closed-loop poles at s=-10.



Even if the process is <u>not</u> even in control canonical form, it would at first glance seem trivial to design a regulator for even a high order process.

Consider the SISO process:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + Bu(t) \\ y(t) &= C\underline{x}(t) \end{aligned}$$

Could easily transform this to a control canonical format:

$$G(s) = C(sI - A)^{-1}C \qquad \begin{aligned} \dot{\underline{x}}_2(t) &= A_2\underline{x}_2(t) + B_2u(t) \\ y(t) &= C_2\underline{x}_2(t) \end{aligned}$$

Then design controller for control canonical format:

<u>BUT!</u>

<u>Tutorial:</u> A certain chemical reactor can be represented by the following state-space equations:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} C(t) \\ T(t) \end{bmatrix} &= \begin{bmatrix} -1 & 5 \\ 2 & -10 \end{bmatrix}\begin{bmatrix} C(t) \\ T(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} q(t) \\ C(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} C(t) \\ T(t) \end{bmatrix} \end{aligned}$$

a) Represent the process in control canonical format

b) Design a state-space regulator to place the closed-loop poles at s=-10 twice using, i) A control canonical form of the model, ii) using the original state-space model.

c) Comment on the practicality and the realisation of each of the controllers.

3.2.1 Design of High Order Regulators Using Transformation Theory

It may be possible to transform the original state-space equations using the transformation,

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bu(t)$$
$$y(t) = C\underline{x}(t)$$

Into control canonical form:

$$\dot{\underline{z}}(t) = A_C\,\underline{z}(t) + B_C u(t) \qquad \dot{\underline{z}}(t) = TAT^{-1}\underline{z}(t) + TBu(t)$$
$$y(t) = C_C\,\underline{z}(t) \qquad y(t) = CT^{-1}\underline{z}(t)$$

Where T is chosen so that :

That is:

$$TAT^{-1} = A_C = \left[\begin{array}{c:c} \underline{O}_{N-1} & I_{N-1} \\ \hdashline \multicolumn{2}{c}{-\underline{e}^T} \end{array}\right]$$

$$TB = B_C = \left[\begin{array}{c} \underline{O}_{N-1} \\ \hdashline 1 \end{array}\right]$$

Now design the regulator:

If the desired characteristic equation is:

$$C_{des}(s) = s^N + C_{N-1}s^{N-1} + \cdots C_1 s + C_0 = 0$$

Then the gain matrix is:

Now we know that: $\underline{z}(t) = T\,\underline{x}(t)$

Hence the control law for the original system is:

<u>Tutorial:</u>

$$\frac{d}{dt}\underline{x}(t) = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t)$$

Determine T to obtain the control canonical form.

3.3 Controllability

There are two common definitions of controllability of the linear MIMO process:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t):$$

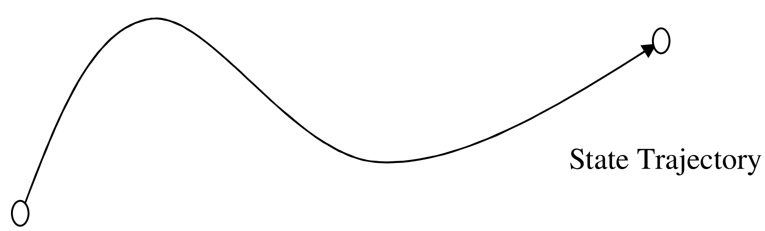
1) A Frequency Domain Definition

This process is controllable using the regulator $\underline{u}(t) = -K\underline{x}(t)$ if the gain matrix K can be selected to place the closed-loop poles anywhere on the s plane.

Important Note:

2) A Time Domain Definition

Consider the possible trajectory through the state-space:



The system is controllable, if for any $\underline{x}_0$ and $\underline{x}_1$, there exists a piecewise continuous control signal $\underline{u}(t)$, that will operate between times t_0 and t_1 to drive the state from any $\underline{x}_0$ at time t_0 to state $\underline{x}_1$ at time t_1

3.3.1 Derivation of the Controllability Matrix

We will derive this test for controllability from the time domain definition and that we know the solution to the state trajectory at time t is given by:

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \int_0^t \Phi(t-\tau)B\underline{u}(\tau)d\tau$$

Without loss of generality, we can express the time domain definition as:

Hence using the state-trajectory equation we can write:

$$\underline{0} = e^{At_1}\underline{x}(0) + \int_0^{t_1} e^{A(t_1-\tau)}B\underline{u}(\tau)d\tau$$

Is there a solution for the control $\underline{u}(t)$over time 0 to t_1 which will ensure that:

Expand the matrix exponential:

$$e^{-A\tau} = I - A\tau + \frac{A^2\tau^2}{2!} - \frac{A^3\tau^3}{3!}\cdots$$

Then:

$$\int_0^{t_1} e^{-A\tau} B\underline{u}(\tau)d\tau = \int_0^{t_1}\left(I - A\tau + \frac{A^2\tau^2}{2!}\cdots\right)B\underline{u}(\tau)d\tau$$

$$= B\int_0^{t_1}\underline{u}(\tau)d\tau + AB\int_0^{t_1} -\tau\underline{u}(\tau)d\tau + A^2B\int_0^{t_1}\frac{\tau^2}{2!}\underline{u}(\tau)d\tau + A^3B\int_0^{t_1} -\frac{\tau^3}{3!}\underline{u}(\tau)d\tau\cdots$$

Hence we could write in matrix form:

$$\underline{x}(0) = -\int_0^{t_1} e^{-A\tau} B\underline{u}(\tau)d\tau = -[\qquad\qquad\qquad]\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Define the controllability matrix as:

$$C_x = \left[B \;\vdots\; AB \;\vdots\; A^2B \;\vdots\; A^3B \;\vdots\; \cdots\right]$$

Now since there are N elements in the initial state: $\underline{x}(0)$:

If C_x was square then of course we could solve for Q as follows:

If however C_x is non square:

Which is solvable if :

Hence a linear MIMO process is controllable if and only if:

Example 1:

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Example 2:

$$\underline{\dot{x}}(t) = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{u}(t)$$

$$C_x = \left[B \;\vdots\; AB \;\vdots\; A^2B \;\vdots\; \cdots\right]$$

$$= \left[\begin{array}{c:c:c} & & \\ & & A^2B \\ & & \end{array}\right]$$

3.3.2 How Controllability is related to the State-Space Model and has nothing to do with the transfer function

Consider the following transfer function, where the zero z is unknown:

$$G(s) = \frac{s-z}{(s+3)(s+4)}$$

This system has the following control-canonical representation;

$$\dot{\underline{x}}_c = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \underline{x}_c + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

With the controllability matrix: $C_c = \left[\begin{array}{c:c} 0 & 1 \\ 1 & -7 \end{array}\right]$

Which is controllable

Now consider the “Observer Canonical Form” of the same system:

$$\dot{\underline{x}}_O = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \underline{x}_O + \begin{bmatrix} 1 \\ -z \end{bmatrix} u$$

The controllability matrix for this realisation is:

$$C_O = \left[\begin{array}{c:c} 1 & -7-z \\ -z & -12 \end{array}\right]$$

So the observer canonical form is controllable if:

3.3.3 Controllability and the State Transformation

Consider the N$^{\text{th}}$ order M input linear process:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \ ;$$

The controllability matrix is:

$$C_x = \left[B \ \vdots \ AB \ \vdots \ A^2B \ \vdots \ \cdots \ \vdots \ A^{N-1}B \right]$$

Now consider the transformation: $\underline{z} = T\underline{x}$

This yields the transformed state-space equations:

$$\dot{\underline{z}} = TAT^{-1}\underline{z} + TB\underline{u}$$

The controllability matrix of the transformed system is:

$$C_z = \left[B_2 \;\vdots\; A_2 B_2 \;\vdots\; A_2{}^2 B_2 \;\vdots\; \cdots \;\vdots\; A_2{}^{N-1} B_2 \right]$$

But:

$$\begin{aligned}
B_2 &= TB \\
A_2 B_2 &= TAT^{-1}TB = \\
A_2{}^2 B_2 &= TAT^{-1}TAT^{-1}TB = \\
A_2{}^3 B_2 &= TAT^{-1}TAT^{-1}TAT^{-1}TB =
\end{aligned}$$

Hence: $$C_z = \left[TB \;\vdots\; TAB \;\vdots\; TA^2 B \;\vdots\; \cdots \;\vdots\; TA^{N-1} B \right] =$$

<u>Note</u>: Since T is non-singular, (Full Rank), then the transformation $\underline{z}=T\underline{x}$ does not contribute to or take away from a process models controllability.

<u>There is another way to look at it</u>: Consider that we wish to transform our system using $\underline{z}=T\underline{x}$ into control canonical form:

3.4 Design of high Order Regulators Using the Controllability Matrix

Consider the design of a state space regulator for the Nth Order SISO process:

$$\dot{\underline{x}} = A\underline{x} + Bu$$
$$y = C\underline{x}$$

First form the controllability matrix based on state vector $\underline{x}$

$$C_x = \left[B \;\vdots\; AB \;\vdots\; A^2B \;\vdots\; \cdots \;\vdots\; A^{N-1}B \right]$$

Next determine the open-loop transfer function:

$$G(s) = C(sI - A)^{-1}B$$

Use G(s) to directly write down the control-canonical state-space format:

$$\dot{\underline{z}} = A_C\,\underline{z} + B_C u$$
$$y = C_C\,\underline{z}$$

Determine the controllability matrix for the CCF:

$$C_z = \left[B_C \;\vdots\; A_C B_C \;\vdots\; A_C{}^2 B_C \;\vdots\; \cdots \;\vdots\; A_C{}^{N-1} B_C \right]$$

Design the regulator for the control canonical form:
Determine the transformation T:

Finally determine the controller gain matrix K:

3.4.1 Ackermann's Gain Formula

Can only be used for single-input systems:

$$\dot{\underline{x}} = A\underline{x} + Bu$$

Assume the control-law:

$$u = -K\underline{x}(t)$$

Form the desired characteristic equation:

$$C_{des}(s) = s^N + C_{N-1}s^{N-1} + \cdots C_1 s + C_0 = 0$$

Form the controllability matrix

$$C_x = \left[B \;\vdots\; AB \;\vdots\; A^2B \;\vdots\; \cdots \;\vdots\; A^{N-1}B\right]$$

Ackermann's gain formula is:

$$K = \begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix} C_x^{-1} C_{des}(A)$$

where:

<u>Tutorial:</u>

$$\dot{\underline{x}} = \begin{bmatrix} -14 & 10 & -22 \\ 13 & 10 & 23 \\ 1 & 0 & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\underline{x}$$

a) Determine the transformation z=Tx which will convert this system into CCF

Design a control-law to place the poles at s=-3, -3±j

b) Repeat the design using Ackermann's formula.

3.5 Regulator Design for Multi-Input Systems

Consider the multi-input system: $\underline{\dot{x}} = A\underline{x} + B\underline{u}$

With the regulator: $\underline{u} = -K\,\underline{x}$

The control gain matrix is:

$$K = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{M1} & k_{M2} & \cdots & k_{MN} \end{bmatrix}$$

With the design equation:

$$\det\left(sI - A + BK\right) = s^{N} + C_{N-1}s^{N-1} + \cdots C_{1}s + C_{0}$$

This can be dealt with in the following ways:

1) Fix some of the gains in K to predefined values:-

2) Instead of pole-placement use the flexibility of having MxN gains to assign the complete eigenstructure of the process.

3) Use an optimisation approach –