Q4.

a(iii) $\underline{x}(k+1) = A_d\, x(k) + B_d\, \underline{u}(k)$.

$A_d = \Phi(T)$ $\qquad\qquad \Phi(t) = e^{At}$.

$= e^{AT}$.

$B_d = \int_0^T \Phi(\tau)\, d\tau\, B$.

$= \int_0^T e^{A\tau}\, d\tau\, B$.

$= \left[A^{-1} e^{A\tau}\right]_0^T B$.

$= A^{-1}\left[e^{AT} - I\right] B$.

$= A^{-1}\left[I + \frac{AT}{1!} + \frac{A^2T^2}{2!} \cdots\cdots - I\right] B$.

$= \left[\frac{TI}{1!} + \frac{AT^2}{2!} + \cdots\cdots\right] B$.

$= T\left[I + \frac{AT}{2!} + \frac{A^2T^2}{3!} \cdots\cdots\right] B$.

$A_d = I + \frac{AT}{1!} + \frac{A^2T^2}{2!} \cdots\cdots$

if T is really small.

$B_d = TB$.

$A_d = I + AT$.

(b) $\dot{\theta} = -u + \phi(t)$.

$\Phi(s) = \frac{k}{1+s\tau}\, u(s)$.

$(1+s\tau)\Phi(s) = k u(s)$.

$s\tau\,\Phi(s)$

$\tau \frac{d\phi}{dt} = k u(t) - \phi(t)$.

$\frac{d\phi}{dt} = -\frac{1}{\tau}\phi + \frac{k}{\tau} u$.

$\frac{d\theta}{dt} = -u + \phi$.

$$x = \begin{bmatrix} \theta \\ \phi \end{bmatrix} \quad \frac{d}{dt}\begin{bmatrix} \theta \\ \phi \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}} \begin{bmatrix} \theta \\ \phi \end{bmatrix} + \overset{B}{\begin{bmatrix} -1 \\ k/\tau \end{bmatrix}} u.$$

$$y = \theta + \phi = \underset{C}{\begin{bmatrix} 1 & 1 \end{bmatrix}} \begin{bmatrix} \theta \\ \phi \end{bmatrix}.$$

S11 Q4b(iii) 25/03/13.

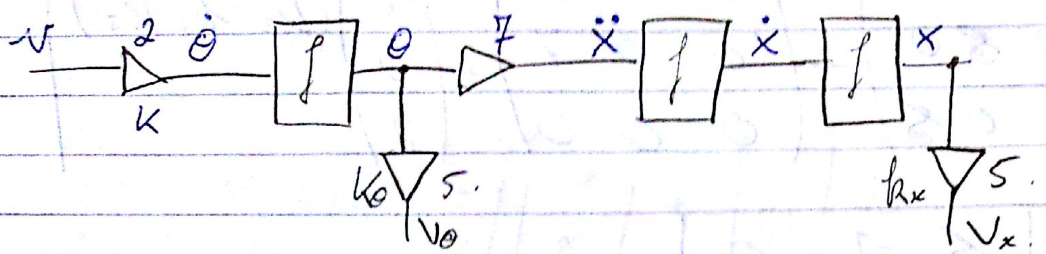
$C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}$

$\Theta = \begin{bmatrix} C \\ CA \end{bmatrix}$ $\Theta = \begin{bmatrix} 1 & 1 \\ 0 & 1 - 1/\tau \end{bmatrix}$

$\det \Theta = 1 - 1/\tau.$

$\text{rank}\, \Theta = 2$ if $\tau \neq 1$.

S11 Q5.



$$\underline{X} = \begin{bmatrix} x \\ \dot{x} \\ \theta \end{bmatrix} \qquad \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} v.$$

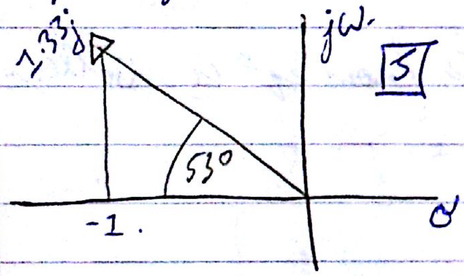
$$\begin{bmatrix} V_x \\ V_\theta \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{C}\begin{bmatrix} x \\ \dot{x} \\ \theta \end{bmatrix}$$

<u>Spec for control</u>

$PO = 10\%$, $T_{s2\%} = 4s$.

$\xi = 0.6$. $\quad \hookrightarrow = \dfrac{4}{\xi\omega_n}$.

$\xi\omega_n = 1$.



Designed dominant pole pair $\Rightarrow -1 \pm 1.33j$.

N.B. There is inherent integral action. $\quad s = -5$

∴ Design regulator.

$A_{cL}: (s+1-1.33j)(s+1+1.33j)(s+5)$
$= s^3 + 7s^2 + 12.8s + 14 = \det(sI - A + Bk)$.

$$\left|\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & K_3 \end{pmatrix}\right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -7 \\ 2k_1 & 2k_2 & s+2k_3 \end{vmatrix}$$

$$= s \begin{vmatrix} s & -7 \\ 2k_2 & s+2k_3 \end{vmatrix} + 1 \begin{vmatrix} 0 & -7 \\ 2k_1 & s+2k_3 \end{vmatrix}.$$

$$s(s^2 + 2k_3 s + 14k_2) + 14k_1.$$

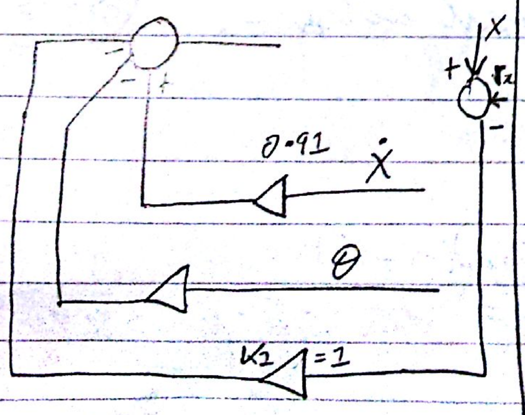
$$s^3 + 2k_3 s^2 + 14k_2 s + 14k_1.$$

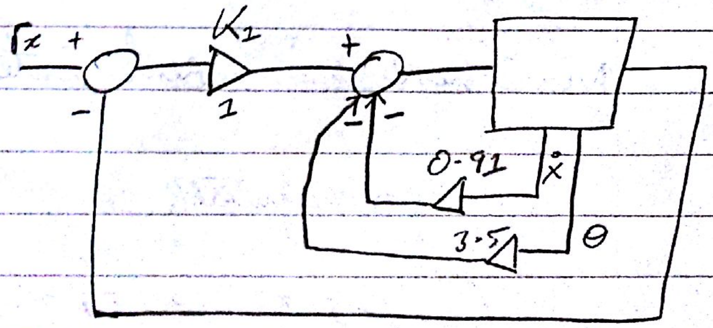
So compare with coefficients from eq^n @ bottom of previous page.

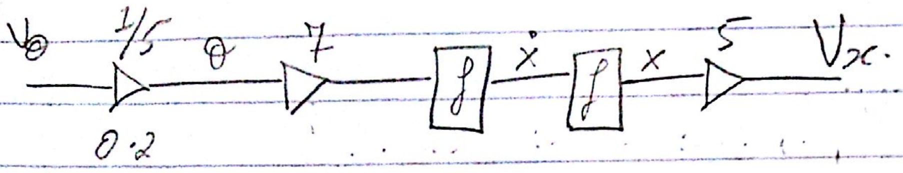
$k_1 = 1.$

$12{\cdot}8 = 14k_2 \rightarrow k_2 = 0{\cdot}91.$

$7 = 2k_3 \rightarrow k_3 = 3{\cdot}5$







$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \overset{B}{\begin{bmatrix} 0 \\ 7/5 \end{bmatrix}} V_\theta$$

$$V_x = \begin{bmatrix} 5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x_1 = x,\ x_2 = \dot{x}.$$

$-1 \pm 1.33j$

$$A_{CL} = (s+5)^2$$
$$= s^2 + 10s + 25 = \det(sI - A + GC) \qquad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \quad C = [5 \ \ 0]$$

$$\det\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}(5 \ \ 0)\right)$$

$$\det\begin{pmatrix} s + 5G_1 & -1. \\ 5G_2 & s \end{pmatrix} = s^2 + 5G_1 s + 5G_2.$$

$$G_2 = 5,\ G_1 = 2.$$

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -25 & 0 \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{\dot{x}} \end{bmatrix} + \begin{bmatrix} 0 \\ 7/5 \end{bmatrix} V_\theta + \begin{bmatrix} 2 \\ 5 \end{bmatrix} V_x$$

$$\frac{d\hat{x}}{dt} = \underbrace{F}_{(A-GC)}\hat{x} + B\underbrace{u}_{V_\theta.} + \underbrace{G}_{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}\overset{V_x.}{Y}$$

See photo of diagram on phone.