Derivation of per-phase magnetizing inductance

With only one phase, for example phase *a* excited by a magnetizing current i_a, the flux distribution in a 2-pole machine is

$$B_a(\theta) = \mu_0 \frac{N_{SP}}{l_g} i_{ma} \cos\theta$$

The energy density at an angle θ is

$$w_{ma}(\theta) = \frac{1}{2}\frac{B_a(\theta)^2}{\mu_0}$$

The differential energy stored at an angle θ (with respect to the *a*-axis) in a differential angle dθ is

$$\mathrm{d}W_{ma}(\theta) = w_{ma}(\theta)\cdot \mathrm{d}(\text{volume})$$

$$= \frac{1}{2}\frac{\left(\mu_0 \frac{N_{SP}}{l_g} i_{ma} \cos\theta\right)^2}{\mu_0} . r\mathrm{d}\theta . l . l_g$$

$$= \frac{\mu_0 N_{SP} r l}{2 l_g} i_{ma}{}^2 . \cos^2\theta \mathrm{d}\theta$$

Integrating both sides with respect to θ from 0 to 2π gives the total energy stored in the airgap.

$$W_{ma} = \frac{\mu_0 N_{SP} r l}{2 l_g} i_{ma}{}^2 . \int_0^{2\pi} \cos^2\theta \,\mathrm{d}\theta$$

$$= \frac{1}{2}\left(\frac{\pi \mu_0 N_{SP}{}^2 r l}{l_g}\right) i_{ma}{}^2$$

The energy stored in an inductor is given by

$$E = \frac{1}{2} L_{a-a}\, i_{ma}{}^2$$

Therefore, the per-phase self inductance is

$$L_{ph} = L_{a-a} = \frac{\pi \mu_0 N_{SP}{}^2 r l}{l_g}$$

The magnetizing inductance is easily derived based on an understanding of the relationships between the self and mutual inductances.

(i) Suppose the winding of phase *b* is aligned with the winding of phase *a*, i.e. they have aligned magnetic axes.

$$\theta_{mb} = 0°$$

$$L_{a-b} = L_{a-a}$$

Thus, any flux generated by phase *a* is perfectly coupled to phase *b*. So the mutual inductance between then is equal to the self inductance of either winding.

(ii) Suppose phase *b* is 180° out of phase with phase *a*.

$$\theta_{mb} = 180°$$

$$L_{a-b} = -L_{a-a}$$

Thus, any flux generated by phase *a* is perfectly coupled to phase *b* but has opposite polarity. So the mutual inductance between then is equal and opposite to the self inductance of either winding.

(iii) Suppose phase b is 90° out of phase with phase a,

$$\theta_{mb} = 90°$$

$$L_{a-b} = 0$$

In this case there is no magnetic flux linkage due to orthogonality of the windings and so the mutual inductance is zero.

The relationship between the mutual and self inductance above is simply a function of $\cos\theta_{mb}$. In the three-phase machine, the phase windings are physically displaced w.r.t. each other by 120° or $\cos\theta_{mb} = -\frac{1}{2}$. Thus the mutual inductances from phases b and c w.r.t. phase a are given by

$$L_{mut} = -\frac{1}{2}L_{ph} = -\frac{1}{2}\frac{\pi\mu_0 N_{SP}{}^2 rl}{l_g} \tag{1}$$

In a three-phase ac machine the back emf of the three phases are given by

$$\begin{bmatrix} e_{ma} \\ e_{mb} \\ e_{mc} \end{bmatrix} = \begin{bmatrix} L_{a-a} & L_{a-b} & L_{a-c} \\ L_{a-b} & L_{b-b} & L_{b-c} \\ L_{a-c} & L_{b-c} & L_{c-c} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} i_{ma} \\ i_{mb} \\ i_{mc} \end{bmatrix}$$

Thus, the back emf for phase a is as follows

$$e_{ma} = L_{a-a}\frac{\mathrm{d}}{\mathrm{d}t}i_{ma} + L_{a-b}\frac{\mathrm{d}}{\mathrm{d}t}i_{mb} + L_{a-c}\frac{\mathrm{d}}{\mathrm{d}t}i_{mc}$$

or

$$e_{ma} = L_{ph}\frac{\mathrm{d}}{\mathrm{d}t}i_{ma} + L_{mut}\frac{\mathrm{d}}{\mathrm{d}t}i_{mb} + L_{mut}\frac{\mathrm{d}}{\mathrm{d}t}i_{mc} \tag{2}$$

Substituting (1) into (2) gives

$$e_{ma} = L_{ph}\left(\frac{\mathrm{d}}{\mathrm{d}t}i_{ma} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}i_{mb} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}i_{mc}\right) \tag{3}$$

We already know that in a balanced three-phase machine

$$i_{ma} + i_{mb} + i_{mc} = 0 \quad \text{or} \quad i_{ma} = -i_{mb} - i_{mc} \tag{4}$$

Substituting (4) into (3) gives

$$e_{ma} = \frac{3}{2}L_{ph}\frac{\mathrm{d}}{\mathrm{d}t}i_{ma} = L_m\frac{\mathrm{d}}{\mathrm{d}t}i_{ma}$$

where

$$L_m = \frac{3}{2}L_{ph} = \frac{3}{2}\frac{\pi\mu_0 N_{SP}{}^2 rl}{l_g}$$

is the per-phase **magnetizing** inductance.