

Section 3

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Summer '06:

Q5 A binary modulation scheme is described by:-

$$S_i(t) = \begin{cases} S_1(t) = A_1 \cos(\omega_c t) & 0 \leq t \leq T \\ S_2(t) = A_2 \cos(\omega_c t) & 0 \leq t \leq T \end{cases}$$

Where T is an integer times $1/f_c$

Show:-

$$(a) \quad P_e = Q \left[\sqrt{\frac{(A_1 - A_2)^2 T}{4\eta}} \right] \quad [8]$$

(b) Average signal energy per bit, E_b , is a fixed constant. T and η are also fixed. Prove that, to minimise P_e , we require:-

$$A_2 = -A_1 \quad [12]$$

Solution:

$$(a) \quad P_e = Q \left[\sqrt{\frac{E_d}{2\eta}} \right]$$

$$E_d = \int_0^T [A_1 \cos(\omega_c t) - A_2 \cos(\omega_c t)]^2 dt$$

$$\Rightarrow \int_0^T [(A_1 - A_2) \cos(\omega_c t)]^2 dt \rightarrow (A_1 - A_2)^2 \int_0^T \cos^2(\omega_c t) dt$$

$$= (A_1 - A_2)^2 \left(\frac{2\omega_c t + \sin(2\omega_c t)}{4\omega_c} \right) \Big|_0^T$$

$$= (A_1 - A_2)^2 \frac{T}{2} \quad \text{Since } \sin(2\omega_c T) = 0$$

$$\text{Since } T = \frac{1}{f_c}, n \in \mathbb{N}_0$$

$$\therefore P_e = Q \left[\sqrt{\frac{(A_1 - A_2)^2 T}{4B}} \right]$$

(b) Avg. signal energy per bit, E_b , is:-

$$E_b = \frac{1}{2} \left[\int_0^T (S_1^2(t) + S_2^2(t)) dt \right]$$

Hence:

$$E_b = \frac{1}{2} \left[(A_1^2 + A_2^2) \int_0^T \cos^2(\omega_c t) dt \right] = (A_1^2 + A_2^2) \left(\frac{T}{4} \right) = k$$

Constant

k is constant

We require:

$$A_1^2 + A_2^2 = m, \quad m \text{ is a constant}$$

To minimise P_e , we maximise its argument.

$$\Rightarrow \text{Let } y = (A_1 - A_2)^2 \frac{T}{4B}$$

$$\frac{dy}{dA_1} = \left(1 - \frac{dA_2}{dA_1} \right) 2(A_1 - A_2) \frac{T}{4B}$$

$$@ \text{Max/Min } \frac{dy}{dA_1} = 0$$

$$\left(1 - \frac{dA_2}{dA_1}\right) = 0 \quad \text{since } A_1 \neq A_2 \quad \text{because } S_1(z) = A_2(z)$$

↑
Not Allowed

$$2A_1 + \frac{d}{dA_1} (A_2)^2 = 0$$

$$\frac{d}{dA_2} (A_2^2) \frac{dA_2}{dA_1} = 2A_2 \frac{dA_2}{dA_1}$$

$$2A_1 + 2A_2 \frac{dA_2}{dA_1} = 0$$

$$\frac{dA_2}{dA_1} = -\frac{A_1}{A_2}$$

$$\left(1 + \frac{A_1}{A_2}\right) 2(A_1 - A_2) \frac{I}{4R} = 0 \quad \text{if } \frac{dI}{dA_1} = 0$$

$$\Rightarrow 1 + \frac{A_1}{A_2} = 0 \quad \therefore A_2 = -A_1$$

$$\cancel{\% A_2 = A_1}$$

Example:-

Suppose $S_1(t)$ and $S_2(t)$ must have the same energy. Prove that the optimum choice is:-

$$S_2(t) = -S_1(t)$$

and that the resultant S.N.R is:-

$$\left(\frac{S}{N}\right)_{\text{opt}} = \frac{8E}{\eta}$$

where E is the signal energy.

Proof:-

$$(S/N)_{\text{opt}} = \frac{2E_d}{\eta}$$

$$\Rightarrow \frac{2}{\eta} \int_0^T (S_1(t) - S_2(t))^2 dt$$

$$= \frac{2}{\eta} \int_0^T (S_1^2(t) - 2(S_1(t)S_2(t)) + S_2^2(t)) dt$$

$$= \frac{4E}{\eta} - \frac{4}{\eta} \int_0^T S_1(t)S_2(t) dt$$

Using the Schwarz inequality:-

$$\left| \int_0^T S_1(t)S_2(t) dt \right| \leq \underbrace{\sqrt{\int_0^T S_1^2(t) dt}}_E \underbrace{\sqrt{\int_0^T S_2^2(t) dt}}_E$$

Equality holds when $S_2(t) = kS_1(t)$

Since the signals are of equal energy

$$|k| = 1$$

$$\Rightarrow k = \pm 1$$

$k = +1$ cannot be used

$$\therefore k = -1$$

Hence:

$$S_2(t) = -S_1(t)$$

Thus by substituting we get

$$\left(\frac{S}{N}\right)_{o/p} = \frac{4E}{\eta} - \frac{1}{\eta} \int_0^T S_1(t) S_2(t) dt$$

$$= \frac{4E}{\eta} + \frac{4E}{\eta} = \frac{8E}{\eta}$$

This result implies anti-parallel signalling

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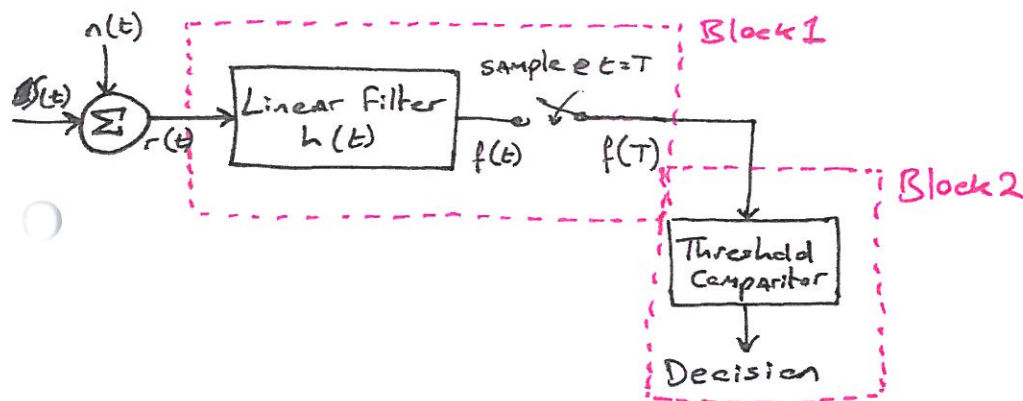
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Dr. C. Murphy

We now turn our attention to the more general question of minimising probability of error for any arbitrary modulation scheme.

Binary Signal Detection & Hypothesis Testing

Once again, we confine our attention to AWGN and a distortionless channel, i.e. no inter-signal interference (I.S.I). The following system diagram summarises the receiver structure:-



The transmitted signal over a symbol interval $(0, T)$ is represented by:-

$$S_i(t) = \begin{cases} S_1(t) & 0 \leq t \leq T \text{ for logic 1} \\ S_2(t) & 0 \leq t \leq T \text{ for logic 0} \end{cases}$$

at the receiver.

The received signal $r(t)$ is represented by:-

$$r(t) = S_i(t) + n(t) \quad i=1,2 \quad 0 \leq t \leq T$$

where $n(t)$ is a zero-mean AWGN (Additive White Gaussian Noise).

As before, there are two steps involved in signal detection. The first step reduces the received signal $r(t)$ to a ~~single number~~ single number $f(T)$. This operation can be performed by a linear filter followed by a sampler, as shown in Block 1 above.

The output of Block 1, sampled at $t=T$, yields:-

$$z(T) = a_i(T) + n_o(T)$$

where $a_i(T)$ is the **signal (ONLY)** component of $z(T)$

$i=1,2$

and $n_o(T)$ is the **noise (ONLY)** component of $z(T)$

(after filtering)

For simplicity, we usually write:-

$$z = a_i + n_0 \quad i = 1, 2$$

Note that the noise component n_0 is also a zero-mean Gaussian random variable (rv) (We will confirm this later). Hence z is also a Gaussian rv with a mean of either a_1 or a_2 , depending on whether $S_1(t)$ or $S_2(t)$ was sent. The sample z is often called the "test statistic".

The second step in the signal detection process consists of comparing z to a threshold level λ in Block 2. The decision is then made according to:-

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \lambda$$

where H_1 & H_2 are the 2 possible hypotheses. Choosing H_1 is equivalent to deciding that signal $S_1(t)$ was sent, and choosing H_2 is equivalent to deciding that $S_2(t)$ was sent. The inequality above indicates that H_1 is chosen if z is larger than λ , and H_2 is chosen if z is less than λ . If $z = \lambda$ an arbitrary decision can be made.

Probability of Error

As before, the overall probability of error, P_e , is given by:

$$P_e = P(S_1) \underbrace{P(H_2|S_1)}_{\text{Probability of deciding } H_2 \text{ when } S_1 \text{ was sent}} + P(S_2) \underbrace{P(H_1|S_2)}_{\text{etc.}}$$

where $P(S_i) : i = 1, 2$ is the probability of sending $S_i(t)$.

In the equi-probable case, $P(S_1) = P(S_2) = \frac{1}{2}$ and:-

$$P_e = \frac{1}{2} (P(H_2|S_1) + P(H_1|S_2))$$

Maximum Likelihood Detector

A popular (~~criterion~~) criterion for choosing λ is based on minimizing P_e .

The calculation for this "minimum error" value of $\lambda = \lambda_0$ starts with this likelihood ratio test:-

$$\Lambda(z) = \frac{f(z|S_1)}{f(z|S_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(S_2)}{P(S_1)}$$

where $f(z|S_i)$ is the conditional pdf, known as the "likelihood of S_i ". Hence we can write:-

$$\Lambda(z) = \frac{f(z|S_1)}{f(z|S_2)} \underset{H_2}{\overset{H_1}{\gtrless}} 1 \quad \text{in the equiprobable case, since } P(S_1) = P(S_2) = \frac{1}{2}$$

If, in addition, the likelihoods $f(z|s_i)$ $i=1,2$ are symmetric, then this further reduces to:-

$$z \underset{H_2}{\overset{H_1}{\geq}} \lambda_0 \quad \text{where } \lambda_0 = \frac{a_1 + a_2}{2}.$$

We will show that $\lambda_0 = \frac{a_1 + a_2}{2}$ is the optimum threshold for minimising P_e in the equiprobable case. (In a) Since this detector minimises P_e , it is known as a "Maximum likelihood detector".

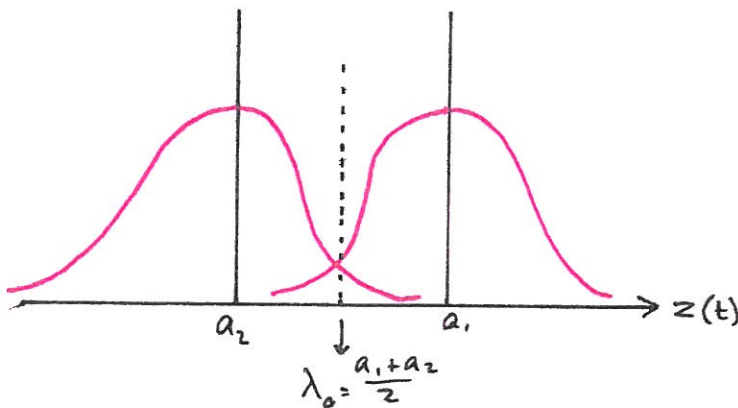
Probability of Error with Gaussian Noise

Assuming Gaussian noise it follows that here we have:-

$$f(z|s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(z-a_1)^2}{2\pi\sigma_n^2}}$$

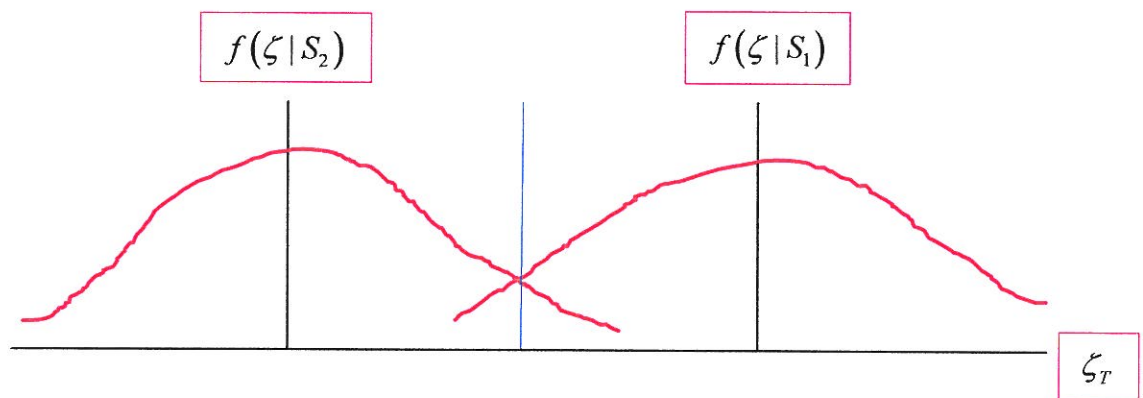
$$f(z|s_2) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(z-a_2)^2}{2\pi\sigma_n^2}}$$

Where σ_n = standard deviation of the filtered noise.



Assuming Gaussian noise it follows that here we have:-

$$\frac{-(\zeta - a_1)^2}{2V_{n0}^2}$$



Now:-

$$P(H_2 | S_1) = \int_{-\infty}^{\infty} f(\zeta | S_1) d\zeta$$

$$P(H_1 | S_2) = \int_{-\infty}^{\infty} f(\zeta | S_2) d\zeta$$

Because of the symmetry of $f(\zeta | S_i)$, P_e reduces to:-

$$P_e = \frac{1}{2} (P(H_2 | S_1) + P(H_1 | S_2))$$

$$P_e = Q\left(\frac{a_1 - a_2}{2\sigma_{n0}}\right)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2} dy$$

Example:

Prove that if $P(S_1) = P(S_2)$ then $\lambda_0 = (a_1 + a_2)/2$ minimises P_e .

Solution:

Assume the threshold is set at λ , and so:-

Optimum Detection

Background Definitions

Wide-Sense Stationary Process

A random process $x(t)$ is “wide-sense stationary” (wss) if its mean is constant, i.e. $E[x(t)] = \mu_x = \int_{-\infty}^{\infty} x f_x(x; t) dx$ and its autocorrelation depends only on the time difference τ . i.e. $E[x(t)x(t+\tau)] = R_{xx}(\tau)$ (see below).

Autocorrelation $R_{xx}(\tau)$

The autocorrelation of $x(t)$ is defined by:-

$$\begin{aligned} R_{xx}(\tau) &= E[x(t)x(t+\tau)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_x(x_1, x_2; t_1, t_2) dx_1 dx_2 \end{aligned}$$

And has the following properties:-

1. Even function:- $R_{xx}(-\tau) = R_{xx}(\tau)$
2. $|R_{xx}(\tau)| \leq E[x^2(t)]$
3. $R_{xx}(0) = E[x^2(t)] = \text{total average normalised power to a } 1\Omega \text{ load.}$

Power Spectrum Density (PSD)

The PSD of $x(t)$ is defined by the Fourier transform of $R_{xx}(\tau)$:-

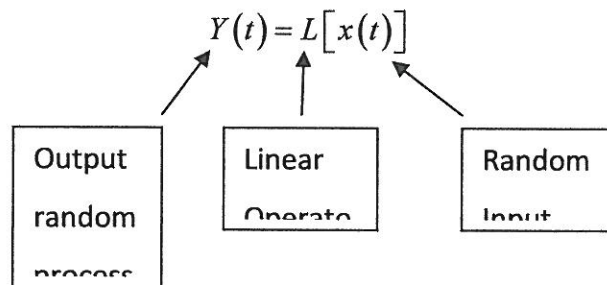
$$\begin{aligned} S_{xx}(\omega) &= \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau \\ \Rightarrow R_{xx}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega \quad \text{Wiener-Khinchin Relations} \end{aligned}$$

PSD $S_{xx}(\omega)$ has the following properties:-

1. $S_{xx}(\omega)$ is real and $s_{xx}(\omega) \geq 0$.
2. $S_{xx}(-\omega) = S_{xx}(\omega)$ (Even also)
3. $R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = R_{xx}(0) = E[x^2(t)]$

Transmission of Random Processes Through Linear Systems

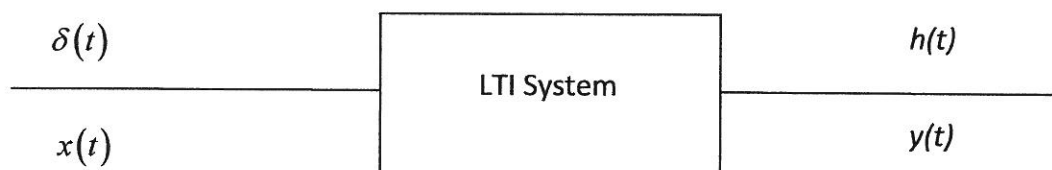
A linear LTI system can be represented by:-



Let $h(t)$ be the impulse response of the LTI. Then:

$$Y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\alpha) x(t - \alpha) d\alpha$$



Mean and Autocorrelation of the Output

The mean of the output

$$\begin{aligned}
\mu_y(t) &= E[y(t)] \\
&= E\left[\int_{-\infty}^{\infty} h(\alpha)x(t-\alpha) d\alpha\right] \\
&= \int_{-\infty}^{\infty} h(\alpha) E[x(t-\alpha)] d\alpha \\
&= \int_{-\infty}^{\infty} h(\alpha)\mu_x(t-\alpha) d\alpha \\
&= h(t) * \mu_x(t)
\end{aligned}$$

If the input is wide-sense stationary we have:

$$\begin{aligned}
E[y(t)] &= \int_{-\infty}^{\infty} h(\alpha)\mu_x d\alpha \\
&= \mu_x \int_{-\infty}^{\infty} h(\alpha) d\alpha \\
&= \mu_x H(0)
\end{aligned}$$

Where $H(0)$ is the frequency response of the linear system at $\omega=0$. Thus, the mean of the output is a constant. The autocorrelation of the output, $R_{yy}(\tau)$, is given by:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta) R_{xx}(\tau + \alpha - \beta) d\alpha d\beta$$

And the output is also wide-sense-stationary.

Power Spectral Density of Output

Taking the Fourier transform of both sides of this last expression yields:

$$\begin{aligned}
S_{YY}(\omega) &= \int_{-\infty}^{\infty} R_{yy}(\tau) e^{-j\omega\tau} d\tau \\
\text{which} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta) R_{xx}(\tau + \alpha - \beta) e^{-j\omega\tau} d\tau d\alpha d\beta \\
\Rightarrow S_{YY}(\omega) &= |H(\omega)|^2 S_{xx}(\omega)
\end{aligned}$$

The autocorrelation of the output, $R_{yy}(\tau)$, is often best evaluated via the inverse Fourier transform of $S_{yy}(\omega)$:-

$$R_{yy}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{xx}(\omega) e^{j\omega\tau} d\omega$$

Hence, the average power in the output $y(t)$ is

$$E[y^2(t)] = R_{yy}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{xx}(\omega) d\omega$$

White Noise

A random process $x(t)$ is called "white noise" if $S_{xx}(\omega) = \frac{\eta}{2}$

It is usually assumed that the mean of white noise is zero. (Note $S_{xx}(\tau)$ relates to $|V|^2$ and the mean relates to V itself.)

Optimum Detection

Recall earlier we had:

$$P_e = Q\left[\frac{a_1 - a_2}{2\sigma_{n_0}}\right].$$

And so, to minimise P_e we search for the filter capable of maximising $\frac{a_1 - a_2}{2\sigma_{n_0}}$

Matched Filter

A matched filter, as we will see, provides the maximum output signal-to-noise ratio for a given transmitted signal. Consider that a known signal $S(t)$ plus AWGN $n(t)$ is the input to a LTI filter, followed by a sampler. Let $a(t)$ be the "signal only" response of the filter. Then, at $t=T$ we have:-

$$\left(\frac{S}{N}\right)_0 = \frac{a^2(T)}{E[n_0^2(T)]}$$

Assuming a 1Ω reference resistor

$$= \frac{a^2(T)}{\sigma_{n_0}^2} \rightarrow \text{since } \sigma_{n_0}^2 = E[(y - \bar{y})^2]$$

But our input noise has zero mean

\Rightarrow so too does our output noise

$\Rightarrow \bar{y} = 0$

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Example:

Find the optimum filter $H(\omega)$ that maximises the output SNR when the input noise is not white (coloured noise).

Solution:

Let $H(\omega)$ be the frequency response of the linear filter
 Let $S_{nn}(\omega)$ be the power spectrum of the input coloured noise. At $t=T$ we have as (follows) before:-

$$a(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) S(\omega) e^{j\omega T} d\omega.$$

and the average output noise power is;-

$$\begin{aligned} N_0 &= E[n_0^2(t)] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{nn}(\omega) |H(\omega)|^2 d\omega \end{aligned}$$

\Rightarrow output SNR is;-

$$\left(\frac{S}{N}\right)_0 = \frac{\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) S(\omega) e^{j\omega T} d\omega \right|^2}{\int_{-\infty}^{\infty} S_{nn}(\omega) |H(\omega)|^2 d\omega}$$

To find the optimum filter set:-

$$\begin{aligned} f_1(\omega) &= \sqrt{S_{nn}(\omega)} H(\omega) \\ f_2(\omega) &= \frac{S(\omega) e^{j\omega T}}{\sqrt{S_{nn}(\omega)}} \end{aligned}$$

we can now write using the Schwarz inequality:-

$$\left| \int_{-\infty}^{\infty} H(\omega) S(\omega) e^{j\omega T} d\omega \right|^2 \leq \dots$$

$$\dots \int_{-\infty}^{\infty} S_{nn}(\omega) |H(\omega)|^2 d\omega > \int_{-\infty}^{\infty} \frac{|S(\omega)|^2}{S_{nn}(\omega)} d\omega \quad \text{since } |e^{j\omega T}| = 1$$

