23/4/09

Summer 2005

Q1 (b). $s = \frac{2}{T}\frac{z-1}{z+1}$

$$C(s) = \frac{M(s)}{E(s)} = \frac{1}{s+a} = \frac{1}{\frac{2}{T}\frac{z-1}{z+1} + a} = \frac{T(z+1)}{2(z-1) + aT(z+1)}$$

$$= \frac{T(z+1)}{(aT+2)z + (aT-2)}$$

Matched pole zero

$C(s) = \frac{1}{s+a}$

$\Rightarrow s = -a$ is a pole

$\Rightarrow z = e^{-aT}$

$$\frac{T(z+1)}{(aT+2)z + (aT-2)} = \frac{1}{z - e^{-aT}}$$

$$\frac{1}{z - e^{-aT}} = \frac{\frac{T(z+1)}{aT+2}}{z + \frac{aT-2}{aT+2}} = \frac{\frac{T(z+1)}{aT+2}}{z + \frac{\frac{aT}{2} - 1}{\frac{aT}{2} + 1}}$$

$$\Rightarrow -e^{-aT} = \frac{aT/2 - 1}{aT/2 + 1} \qquad \Rightarrow e^{-aT} = \frac{1 - \frac{Ta}{2}}{1 + Ta/2}$$

(c). $Y(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots$

$U(z) = \frac{1}{1 - z^{-1}}$

$Y(z) = G(z)U(z) = (g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots)\frac{1}{1-z^{-1}}$

$\Rightarrow (1 - z^{-1})(h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots) = (g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots$

$\Rightarrow h_0 + (h_1 - h_0)z^{-1} + (h_2 - h_1)z^{-2} + \ldots = g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots$

$\Rightarrow G(z) = h_0 + (h_1 - h_0)z^{-1} + (h_2 - h_1)z^{-2} + \ldots$

$Y(z) = (h_0 + (h_1 - h_0)z^{-1} + (h_2 - h_1)z^{-2} + \ldots)\, U(z)$

$y(k) = (h_1 - h_0)u(k-1) + (h_2 - h_1)u(k-2)$

$y(k+1) = (h_1 - h_0)u(k) + (h_2 - h_1)u(k-1) + (h_3 - h_2)u(k-2)$

$y(k+2) = (h_1 - h_0)u(k+1) + (h_2 - h_1)u(k) + (h_3 - h_2)u(k-1) + (h_4 - h_3)u(k-2)$

$$\begin{bmatrix} y(k+1) \\ y(k+2) \end{bmatrix} = \begin{bmatrix} h_1 - h_0 & 0 \\ h_2 - h_1 & h_1 - h_0 \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} + \underline{y}_f(k)$$

$$\underline{y}_f(k) = \begin{bmatrix} h_2 - h_1 & h_3 - h_2 \\ h_3 - h_2 & h_4 - h_3 \end{bmatrix} \begin{bmatrix} u(k-1) \\ u(k-2) \end{bmatrix} = \begin{bmatrix} -0.2 & 0.5 \\ 0.5 & -0.2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.4 \end{bmatrix}$$

$$\begin{bmatrix} y(k+1) \\ y(k+2) \end{bmatrix} = \begin{bmatrix} 0 & 0.5 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -0.2 & 0.5 \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} + \begin{bmatrix} 0.05 \\ 0.4 \end{bmatrix}$$

$$\begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -0.2 & 0.5 \end{bmatrix}^{-1} \begin{bmatrix} -0.05 \\ 0.1 \end{bmatrix}$$

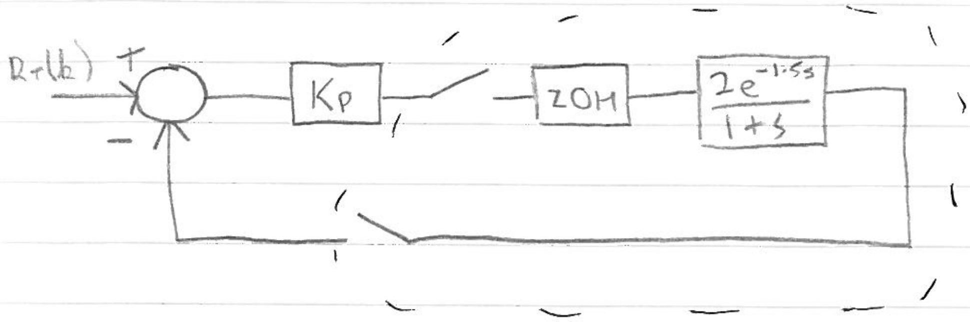
$$\begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} = 4 \begin{bmatrix} 0.5 & 0 \\ 0.2 & 0.5 \end{bmatrix} \begin{bmatrix} -0.05 \\ 0.1 \end{bmatrix}$$

$$\begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} = 4 \begin{bmatrix} -0.025 \\ 0.04 \end{bmatrix}$$

$$\begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0.16 \end{bmatrix}$$

Summer 05

Q2(c).



$$G(z) = \mathcal{Z}\left\{ \frac{1-e^{-sT}}{s} \cdot \frac{2e^{-1.5s}}{1+s} \right\}$$

$$= 2(1-z^{-1})z^{-1}\,\mathcal{Z}_m\left\{ \frac{1}{s(s+1)} \right\}\Big|_{m=0.5}$$

$$m = 1 - \frac{\theta}{T} = 1 - \frac{0.5}{1} = 0.5$$

Using modified $\mathcal{Z}$ transform tables

$$\frac{1}{s(s+a)} \rightarrow \frac{z^{-1}}{a}\left(\frac{1}{1-z^{-1}} - \frac{e^{-amT}}{1-e^{-aT}z^{-1}} \right)$$

$$\frac{1}{s(s+1)} \rightarrow \frac{z^{-1}}{1}\left(\frac{1}{1-z^{-1}} - \frac{e^{-0.5}}{1-e^{-1}z^{-1}} \right)$$

$$= z^{-1}\left(\frac{1}{1-z^{-1}} - \frac{0.61}{1-0.37z^{-1}} \right)$$

$$= z^{-1}\left(\frac{1-0.37z^{-1}-0.61+0.61z^{-1}}{(1-z^{-1})(1-0.37z^{-1})} \right)$$

$$= z^{-1}\left(\frac{0.39+0.24z^{-1}}{(1-z^{-1})(1-0.37z^{-1})} \right)$$

$$= G(z) = \frac{2z^{-2}(0.39+0.24z^{-1})}{1-0.37z^{-1}}$$

$$\Rightarrow K_p G(z) = \frac{2K_p(0.39 + 0.24z^{-1})}{z^2 - 0.37z}$$

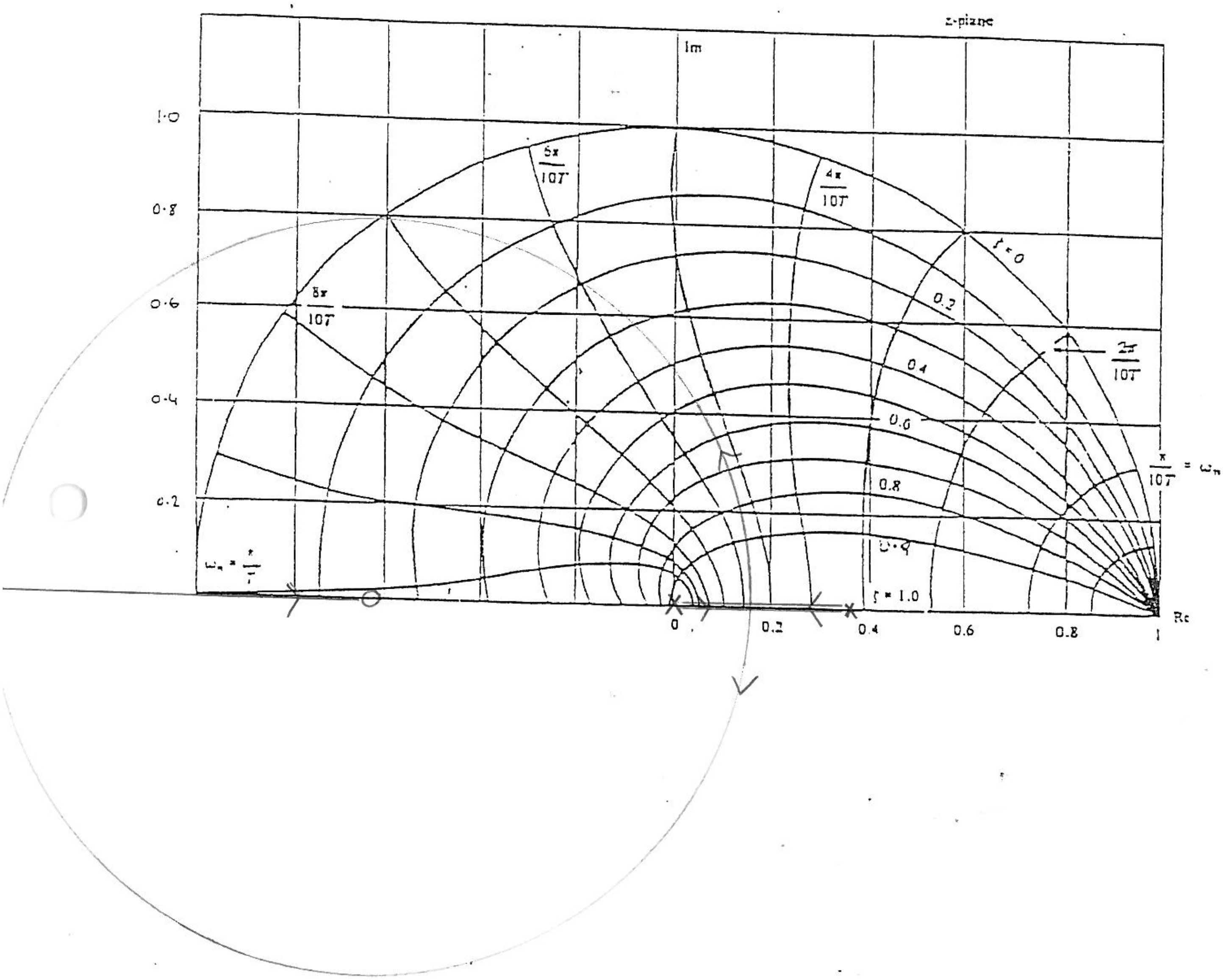
$$z^2 - 0.37z = 0$$
$$z(z - 0.37) = 0$$
$$z = 0 \quad z = 0.37$$ poles

$$0.39 + 0.24z^{-1} = 0$$
$$z = -0.615$$ zero

$$\sum_{i=1}^{n} \frac{1}{\sigma - p_i} = \sum_{j=1}^{m} \frac{1}{\sigma - z_j}$$

$$\frac{1}{\sigma} + \frac{1}{\sigma - 0.37} = \frac{1}{\sigma + 0.615}$$
$$(\sigma - 0.37)(\sigma + 0.615) + \sigma(\sigma + 0.615) = \sigma(\sigma - 0.37)$$
$$\cancel{\sigma^2} + 0.615\sigma \cancel{- 0.37\sigma} - 0.228 + \sigma^2 + 0.615\sigma = \cancel{\sigma^2} \cancel{- 0.37\sigma}$$
$$\sigma^2 + 1.23\sigma - 0.228 = 0$$
$$\sigma = \frac{-1.23 \pm \sqrt{1.23^2 + 4(0.228)}}{2}$$
$$= \frac{-1.23 \pm 1.56}{2}$$
$$= -1.395 \quad 0.165$$

As $K_p \uparrow$ $\xi \downarrow$ $\omega_N \uparrow$
Eventually closed loop system becomes unstable



Z Plane Design Template

Please submit with your script

24/4/09

Summer 2005

Q 3 (b).

$y(k+1) = a_1 y(k) + a_2 y(k-1) + b_1 u(k) + b_2 u(k-1)$

$z Y(z) = a_1 Y(z) + a_2 z^{-1} Y(z) + b_1 U(z) + b_2 z^{-1} U(z)$

$Y(z)(z - a_1 - a_2 z^{-1}) = U(z)(b_1 + b_2 z^{-1})$

$$\Rightarrow \frac{Y(z)}{U(z)} = \frac{b_1 + b_2 z^{-1}}{z - a_1 - a_2 z^{-1}} = \frac{b_1 z + b_2}{z^2 - a_1 z - a_2} = \frac{B(z)}{A(z)}$$

$\hat{\underline{\theta}}(k) = [\hat{a}_1(k)\ \hat{a}_2(k)\ \hat{b}_1(k)\ \hat{b}_2(k)]^T = [2\ -1\ 0\ 0.5]^T$

$a_1 = 2 \quad b_1 = 0$

$a_2 = -1 \quad b_2 = 0.5$

$n = 2 \Rightarrow n_q = n_s = n - 1 = 1$

$\Rightarrow Q(z) = z + q_1$

$\quad S(z) = s_0 z + s_1$

The Diophantine equation is:

$A_d(z) = A(z)Q(z) + S(z)B(z)$

$A_d(z) = (z^2 - a_1 z - a_2)(z + q_1) + (s_0 z + s_1)(b_1 z + b_2)$

$A_d(z) = z^3 + q_1 z^2 - a_1 z^2 - a_1 q_1 z - a_2 z - a_2 q_1 + b_1 s_0 z^2 + b_2 s_0 z + b_1 s_1 z + b_2 s_1$

$\quad = z^3 + (q_1 - a_1 + b_1 s_0) z^2 + (b_1 s_1 + b_2 s_0 - a_1 q_1 - a_2) z + b_2 s_1 - a_2 q_1$

$A_d(z) = z^3 + c_1 z^2 + c_2 z + c_3$

$q_1 - a_1 + b_1 s_0 = c_1 \Rightarrow q_1 + b_1 s_0 = c_1 + a_1$

$b_1 s_1 + b_2 s_0 - a_1 q_1 - a_2 = c_2 \Rightarrow b_1 s_1 + b_2 s_0 - a_1 q_1 = c_2 + a_2$

$b_2 s_1 - a_2 q_1 = c_3$

$$\begin{bmatrix} 1 & b_1 & 0 \\ -a_1 & b_2 & b_1 \\ -a_2 & 0 & b_2 \end{bmatrix} \begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} c_1 + a_1 \\ c_2 + a_2 \\ c_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0.5 & 0 \\ 1 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} c_1 + 2 \\ c_2 - 1 \\ c_3 \end{bmatrix}$$

Poles at $z = 0.8$ twice

$\Rightarrow$ place fast pole at $0.8^5 = 0.33$

$$A_{cl}(z) = (z-0.8)^2(z-0.33)$$
$$= (z^2 - 1.6z + 0.64)(z-0.33)$$
$$= z^3 - 1.93z^2 + 1.168z - 0.2112$$

$\Rightarrow c_1 = -1.93$

$c_2 = 1.168$

$c_3 = -0.2112$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 0.5 & 0 \\ 1 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0.07 \\ 0.168 \\ -0.2112 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} q_1 \\ s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0.5 & 0 \\ 1 & 0 & 0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0.07 \\ 0.168 \\ -0.2112 \end{bmatrix} = \begin{bmatrix} 0.07 \\ 0.616 \\ -0.5624 \end{bmatrix}$$

$\Rightarrow Q(z) = z + 0.07$

$S(z) = 0.616 \quad -0.5624$

$T(z) = t_0 A_0 = t_0(z - 0.33)$

$$t_0 = \frac{A_c(z)}{B(z)}$$

For unity DC gain

$$\lim_{z \to 1} \frac{t_0 B(z)}{A_c(z)} = 1$$

$$\Rightarrow t_0 = \frac{A_c(1)}{B(1)} = 0.02$$

21/4/09 Summer 05

Q4 (a). $\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t)$

Taking Laplace transform yields

$\underline{X}(s) = (sI - A)^{-1}(B\underline{U}(s) + \underline{x}(0))$

Now solve for the state trajectory using the inverse Laplace transform
$\underline{x}(t) = L^{-1}\{\underline{X}(s)\}$

$L^{-1}\{W(s)V(s)\} = w(t) \otimes v(t) = \int_0^t w(t-\tau)v(\tau)\,d\tau$

Define $\phi(s) = (sI - A)^{-1}$ Transition matrix

$\underline{X}(s) = \phi(s)\underline{x}(0) + \underbrace{\phi(s)}_{W(s)}\underbrace{B\underline{U}(s)}_{V(s)}$

Taking inverse Laplace transform yields

$\underline{x}(t) = \phi(t)\underline{x}(0) + L^{-1}\{W(s)V(s)\}$

$\underline{x}(t) = \phi(t)\underline{x}(0) + w(t) \otimes v(t)$

$\underline{x}(t) = \phi(t)\underline{x}(0) + \int_0^t \underbrace{\phi(t-\tau)}_{w(t-\tau)}\underbrace{B\underline{u}(\tau)}_{v(\tau)}\,d\tau$

Consider the zero-input response
$\underline{x}(t) = \phi(t)\underline{x}(0)$
which is the solution to $\dot{\underline{x}} = A\underline{x}$

$\dot{\underline{x}}(t) = \frac{d}{dt}(\phi(t)\underline{x}(0)) = \frac{d\phi}{dt}\underline{x}(0)$ $\dot{\underline{x}} = A\underline{x} = A\phi(t)\underline{x}(0)$

$\ddot{\underline{x}}(t) = \frac{d^2}{dt^2}(\phi(t)\underline{x}(0)) = \frac{d^2\phi}{dt^2}\underline{x}(0)$ $\ddot{\underline{x}} = A\dot{\underline{x}} = A^2\phi(t)\underline{x}(0)$

$\dddot{\underline{x}}(t) = \frac{d^3}{dt^3}(\phi(t)\underline{x}(0)) = \frac{d^3\phi}{dt^3}\underline{x}(0)$ $\dddot{\underline{x}} = A\ddot{\underline{x}} = A^3\phi(t)\underline{x}(0)$

$\frac{d\phi}{dt} = A\phi(t)$, $\frac{d^2\phi}{dt^2} = A^2\phi(t)$, $\frac{d^3\phi}{dt^3} = A^3\phi(t)$

$\Rightarrow \frac{d^i\phi}{dt^i} = A^i\phi(t)$

This will be true if $\phi(t) = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$

Define the matrix exponential function as:

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A_3t^3}{3!} + \ldots$$

$$\Rightarrow \phi(t) = e^{At}$$

$$\phi(t-\tau) = e^{A(t-\tau)}$$

$$= e^{At} \circ e^{-A\tau}$$

$$\Rightarrow \underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{At} \cdot e^{-A\tau} B\underline{u}(\tau)\,d\tau$$

$$\underline{x}(t) = e^{At}\left(\underline{x}(0) + \int_0^t e^{-A\tau} B\underline{u}(\tau)\,d\tau\right)$$

(ii) The state trajectory is given by

$$\underline{x}(t) = \phi(t)\underline{x}(0) + \int_0^t \phi(t-\tau) B\underline{u}(\tau)\,d\tau$$

Consider that the initial time is t_0 with initial state $\underline{x}(t_0)$

$$\underline{x}(t) = \phi(t-t_0)\underline{x}(t_0) + \int_{t_0}^t \phi(t-\tau) B\underline{u}(\tau)\,d\tau$$

Consider what happens to the state vector over a time step T

$t_0 = kT$; $t = (k+1)T$

$$\underline{x}((k+1)T) = \phi((k+1)T - kT)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T - \tau) B\underline{u}(\tau)\,d\tau$$

$$\underline{x}((k+1)T) = \phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T - \tau) B\underline{u}(\tau)\,d\tau$$

Assume a zero order hold is utilized

$\Rightarrow \underline{u}(t) = \underline{u}(kT)$ for $kT \leq t \leq (k+1)T$

$$\underline{x}((k+1)T) = \phi(t)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi((k+1)T - \tau) B\,d\tau \underbrace{\underline{u}(kT)}_{\text{CONSTANT OVER INTEGRAL}}$$

Now make the following substitution

$\eta = (k+1)T - \tau$

$d\eta = -d\tau$

$$\underline{x}((k+1)T) = \phi(T)\underline{x}(kT) - \int_T^0 \phi(\eta) B\,d\eta\, \underline{u}(kT)$$

$(k+1)T \to (k+1)$
$kT \to k$

$$\underline{x}(k+1) = \phi(T)\underline{x}(k) + \int_0^T \phi(\eta) B \, d\eta \, \underline{u}(k)$$

$$\underline{x}(k+1) = A_d \underline{x}(k) + B_d \underline{u}(k)$$

$$A_d = \phi(T) = e^{AT} = I + AT/1! + A^2T^2/2! + \dots$$

T is small
$\Rightarrow A_d = I + AT$

$$B_d = \int_0^T \phi(\eta) B \, d\eta = \int_0^T e^{A\eta} B \, d\eta = \tfrac{1}{A} e^{A\eta} B \Big|_0^T = \tfrac{1}{A}(e^{AT} - I)B$$
$$= A^{-1}(A_d - I)B$$
$$= A^{-1}(I + AT - I)B$$
$$= A^{-1}ATB$$
$$= TB$$

$$\underline{x}(k+1) = (I + AT)\underline{x}(k) + TB\underline{u}(k)$$

$$\underline{x}(k+1) - \underline{x}(k) = \underline{x}(k) + AT\underline{x}(k) + TB\underline{u}(k) - \underline{x}(k)$$

$$\Rightarrow \frac{\Delta \underline{x}(k+1)}{T} = A\underline{x}(k) + B\underline{u}(k)$$

(ii) $$\frac{d}{dt}\underline{x}(t) = \overbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}^{A} \underline{x}(t)$$

The eigenvalues of A are roots of:
$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & 0 \\ 0 & \lambda - b \end{vmatrix} = 0$$

$(\lambda - a)(\lambda - b) = 0$
$\lambda_1 = a \quad \lambda_2 = b$

Since $N = 2$ we can write:
$e^{At} = \alpha_0(t) I + \alpha_1(t) A$

$e^{at} = \alpha_0(t) I + \alpha_1(t) a$
$e^{bt} = \alpha_0(t) I + \alpha_1(t) b$
$\Rightarrow e^{at} - e^{bt} = \alpha_1(t)(a - b)$
$$\Rightarrow \alpha_1(t) = \frac{e^{at} - e^{bt}}{a - b}$$

$$e^{at} = \alpha_0(t) I + \frac{a(e^{at} - e^{bt})}{a-b}$$

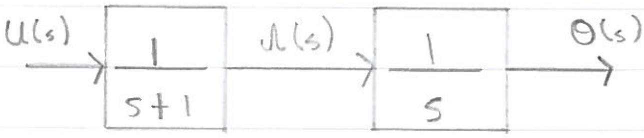
$$\Rightarrow \alpha_0(t) I = e^{at} - \frac{a(e^{at} - e^{bt})}{a-b}$$

$$\Rightarrow e^{At} = e^{at} - \frac{a(e^{at} - e^{bt})}{a-b} + \frac{e^{at} - e^{bt}}{a-b} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

25/4/09

Summer 2005

Q 4 (c).



$\Omega(s) = \frac{1}{s+1}U(s)$
$s\Omega(s) + \Omega(s) = U(s)$
$\frac{d\omega}{dt} + \omega(t) = u(t)$
$\frac{d\omega}{dt} = -\omega + u$

$\Theta(s) = \frac{\Omega(s)}{s}$
$\Omega(s) = s\Theta(s)$
$\omega = \frac{d\Theta}{dt}$
OR $\frac{d\Theta}{dt} = \omega$

$$\frac{d}{dt}\begin{bmatrix}\Theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & -1\end{bmatrix}\begin{bmatrix}\Theta \\ \omega\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}u$$

$U(s) = C(s)E(s)$
$E(s) = \Theta_{des}(s) - \Theta(s)$
$\Rightarrow U(s) = C(s)(\Theta_{des}(s) - \Theta(s))$
$= (Ks + K_z)(\Theta_{des}(s) - \Theta(s))$
$= Ks\Theta_{des}(s) - Ks\Theta(s) + K_z\Theta_{des}(s) - K_z\Theta(s)$

$\Rightarrow u(t) = K\frac{d\Theta_{des}(t)}{dt} - K\frac{d\Theta}{dt} + K_z\Theta_{des}(t) - K_z\Theta(t)$

Setpoint constant
$\Rightarrow \frac{d\Theta_{des}(t)}{dt} = 0$
$\Rightarrow u(t) = K_z\Theta_{des}(t) - K\frac{d\Theta}{dt} - K_z\Theta(t)$
$= K_1\Theta_{des}(t) - K_2\omega(t) - K_1\Theta(t)$
$= K_1\Theta_{des}(t) - [K_1 \;\; K_2]\begin{bmatrix}\Theta \\ \omega\end{bmatrix}$

$C_{des}(s) = \det(sI - A + BK)$
$= \det\left(\begin{pmatrix}s & 0 \\ 0 & s\end{pmatrix} - \begin{pmatrix}0 & 1 \\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix}(K_1 \;\; K_2)\right)$

$= \det\left(\begin{pmatrix}s & -1 \\ 0 & s+1\end{pmatrix} + \begin{pmatrix}0 & 0 \\ K_1 & K_2\end{pmatrix}\right)$

$$= \det \begin{pmatrix} s & -1 \\ K_1 & s+1+K_2 \end{pmatrix}$$

$$= s(s+1+K_2) + K_1$$
$$= s^2 + (K_2+1)s + K_1$$

$$C_{des}(s) = (s+(2+2j))(s+(2-2j))$$
$$= s^2 + 2s - \cancel{2j}s + 2s + \cancel{2j}s + 4 - \cancel{4j} + \cancel{4j} - 4j^2$$
$$= s^2 + 4s + 8$$

$K_1 = 8$ $\quad$ $K_2 + 1 = 4$

$\Rightarrow K_2 = 3$

$K = K_2 = 3$ $\quad$ $Kz = K_1$

$3z = 8$

$z = 2.67$

$$\Rightarrow C(s) = 3(s + 2.67)$$

30/4/09 Summer 2005

Q 5(a). $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{f_0 + f_1 s + \ldots f_{N-1} s^{N-1}}{s^N + e_{N-1} s^{N-1} + \ldots e_0}$

$$\frac{Y(s)}{U(s)} = \frac{f_0/s^N + f_1/s^{N-1} + \ldots f_{N-1}/s}{1 + e_{N-1}/s + e_{N-2}/s^2 + \ldots e_0/s^N}$$

Cross multiplying yields:

$$\left(1 + e_{N-1}/s + e_{N-2}/s^2 \ldots e_0/s^N\right) Y(s) = \left(f_0/s^N + f_1/s^{N-1} + f_2/s^{N-2} + \ldots f_{N-1}/s\right) U(s)$$

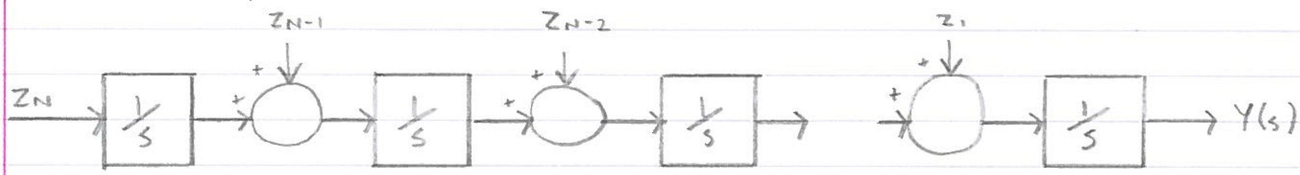
Solving for $Y(s)$

$$Y(s) = \tfrac{1}{s}(f_{N-1}U - e_{N-1}Y) + \tfrac{1}{s^2}(f_{N-2}U - e_{N-2}Y) + \ldots \tfrac{1}{s^N}(f_0 U - e_0 Y)$$

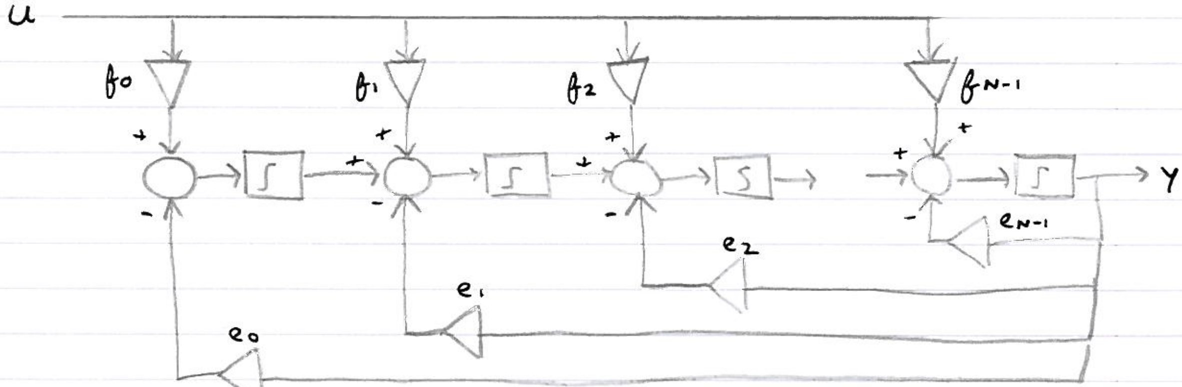
which could be written as $z_i = f_{N-i}U - e_{N-i}Y$

$$Y(s) = \tfrac{1}{s}\left(z_1 + \tfrac{1}{s}\left(z_2 + \tfrac{1}{s}\left(z_3 + \tfrac{1}{s}\left(\ldots z_{N-1} + \tfrac{1}{s}(z_N)\right)\right)\right)\right)$$

Could be represented as



This yields the observer canonical format



$$\frac{Y(s)}{U(s)} = \frac{K\overset{f_0}{a} + \overset{Kf_1}{Ks}}{s^2 + \underset{e_1}{(b+c)}s + \underset{e_0}{bc}} = \frac{Ks + Ka}{s^2 + (b+c)s + bc}$$

$$Y(s) = \tfrac{1}{s}(KU - (b+c)Y) + \tfrac{1}{s^2}(KaU - bcY)$$