Summer 2013

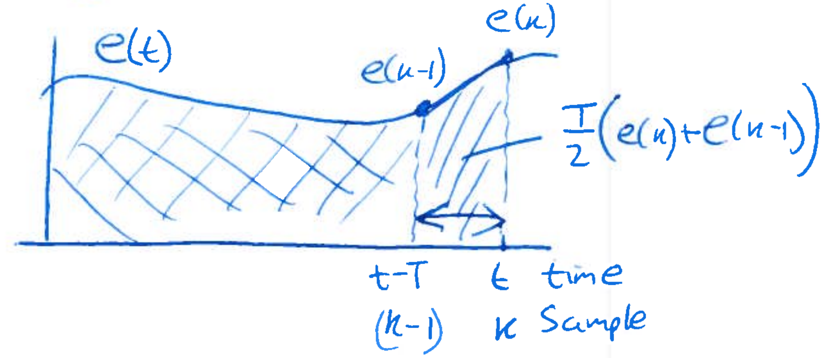
CONTROL EE4002

SOLUTIONS

Q1) a) from notes (8)

b) Approx integration

$\int_0^t e(\tau)\,d\tau \approx I(k)$



$$I(k) = I(k-1) + \frac{T}{2}\left(e(k) + e(k-1)\right)$$

Z transforms

$$(1-z^{-1})I(z) = \frac{T}{2}(1+z^{-1})E(z)$$

$$\frac{I(z)}{E(z)} = \frac{T}{2}\frac{(1+z^{-1})}{1-z^{-1}}$$

Compare $E \rightarrow \boxed{\frac{T}{2}\frac{z+1}{z-1}} \rightarrow I$

with: $E(s) \rightarrow \boxed{\frac{1}{s}} \rightarrow I(s)$

$\therefore$ yields Tustins:

$$S = \frac{2}{T}\frac{z-1}{z+1}$$

(5)

---

c) $m(k) = k\,e(k-1) + 0.7\,m(k-1)$

Z transforms: $(1-0.7z^{-1})M = Kz^{-1}E$

$$\Rightarrow \frac{M}{E} = \frac{Kz^{-1}}{(1-0.7z^{-1})} = \frac{K}{z-0.7}$$

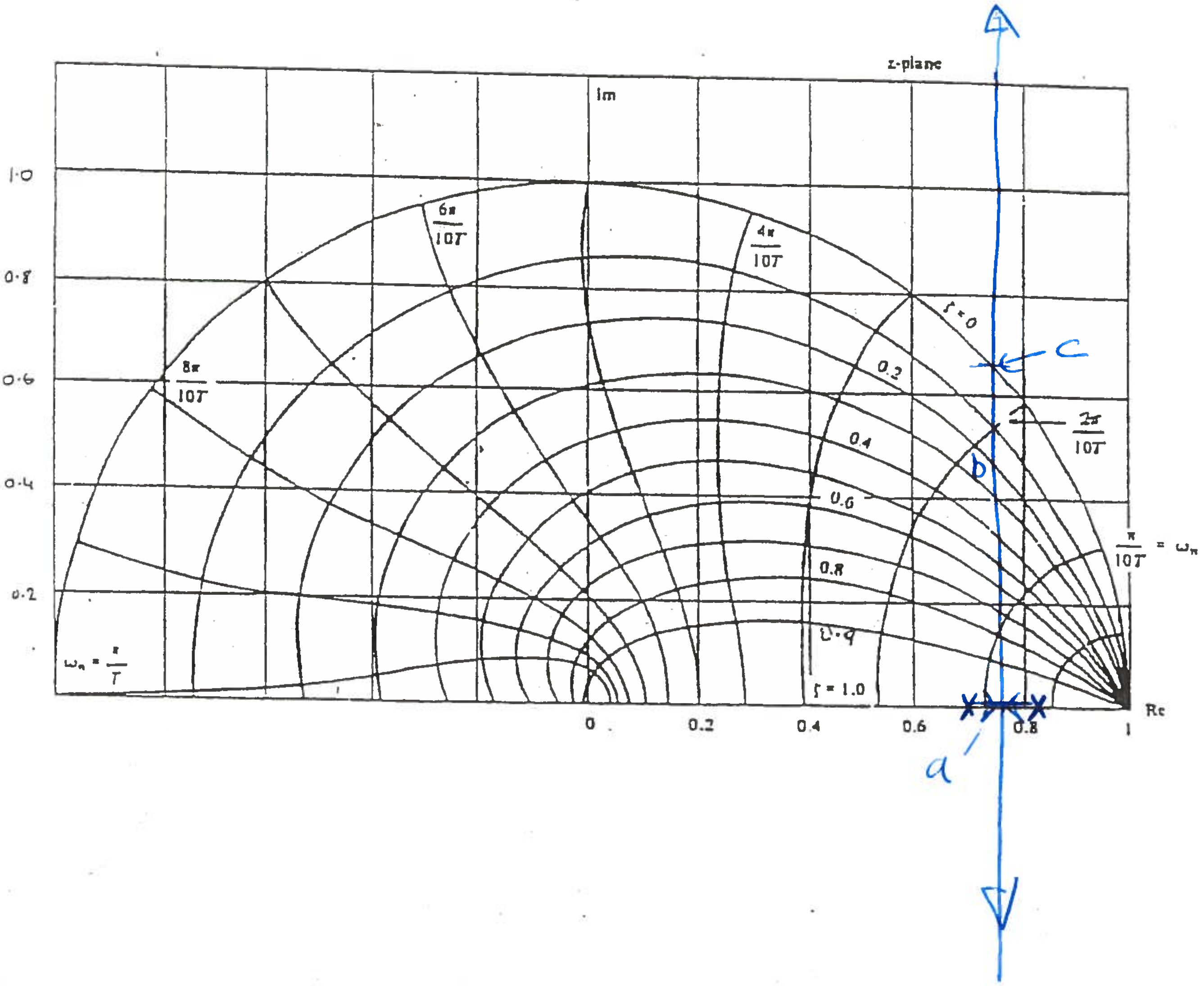
$$\therefore D(z) = \frac{K}{z-0.7}$$

(2)

$$G(z) = Z\left\{\frac{1-e^{-sT}}{s}\cdot\frac{5}{1+5s}\right\}$$

$$= (1-z^{-1})\,Z\left\{\frac{1}{s(s+0.2)}\right\}$$

Tables: $(1-z^{-1})\dfrac{5(1-e^{-0.2T})z^{-1}}{(1-z^{-1})(1-e^{-0.2T}z^{-1})}$



Z Plane Design Template

Please submit with your script

12/12

$$G(z) = \frac{5(0.18)z^{-1}}{1-0.818z^{-1}} = \frac{0.906}{z-0.818}$$

2

look at $D(z)G(z)$

open loop poles @ $z = 0.818$
$z = 0.7$

refer to attached root locus plot

breakaway @ $z = 0.759$

Plots 4 + explain

Second order dynamics

region a) over damped – as $k\uparrow$ response speeds up

region b) underdamped – a $k\uparrow$ damping $\downarrow$ and $\omega_n \uparrow$

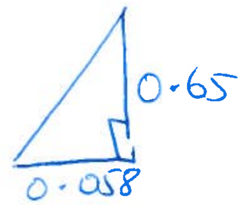
STABILITY — marginal stability at C

$z = 0.76 + 0.65j$

at this point:

$$\left.\frac{0.906}{|z-0.818|} \cdot \frac{k}{|z-0.7|}\right|_{z=0.76+0.65j} = 1$$

$$|z-0.818| = |z-0.7|$$

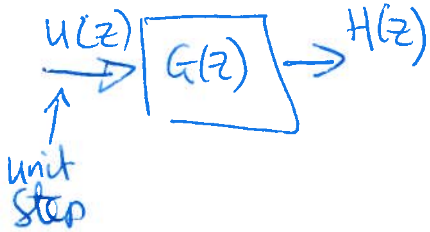
$\Rightarrow$ 

$= 0.652$

$$\therefore \ k = \frac{(0.652)^2}{0.906} = 0.47$$

4

Q2) a)



$H(z) = 0 + 0z^{-1} + 0.5z^{-2} + 1z^{-3} + 1z^{-4} \ldots$

$H(z) = G(z) \cdot \frac{1}{1-z^{-1}}$

$\therefore G(z) = (1-z^{-1})\, H(z)$

$= 0.5z^{-2} + z^{-3} + z^{-4} + z^{-5}$
$\quad - 0.5z^{-3} - z^{-4} - z^{-5} - z^{-6}$

$= 0.5z^{-2} + 0.5z^{-3}$ 2

for input $r(k) = 2k$

$Y(z) = G(z)U(z)$

$= (0.5z^{-2} + 0.5z^{-3})U(z)$

IZT: $y(k) = 0.5u(k-2) + 0.5u(k-3)$

k	0	1	2	3	4	5	
u	0	2	4	6	8	10	ie 2k
y	0	0	0	1	3	5	7

odd number sequence ! 3

b) Consider the closed-loop digital control system:

we want $\frac{C}{R} \Rightarrow P$

$\therefore \frac{DG}{1+DG} = P \Rightarrow DG = (1+DG)P$

$\Rightarrow DG(1-P) = P$

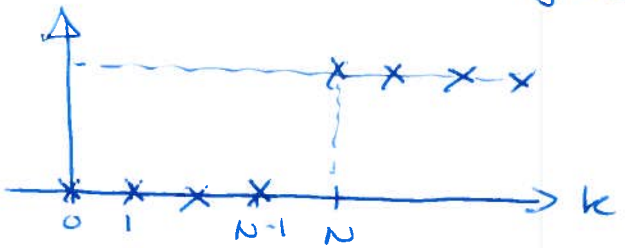
$\therefore D = \frac{P}{G(1-P)}$

we will base our design on a good model

$D = \frac{P}{G_m(1-P)}$

If the actual plant has a pure time delay $T_d = (N-1)T + \theta$

The ~~the~~ "best" response we can achieve is deadbeat in N samples



$\therefore \quad P = z^{-N}$

This yields $D(z) = \dfrac{z^{-N}}{G_M(1-z^{-N})}$

$= \dfrac{1}{G_M(z^N - 1)}$

4

$G_p(z) = \dfrac{k}{(z-0.5)(\underline{z-1.5})}$

Two delays $N=2$

This is the problem

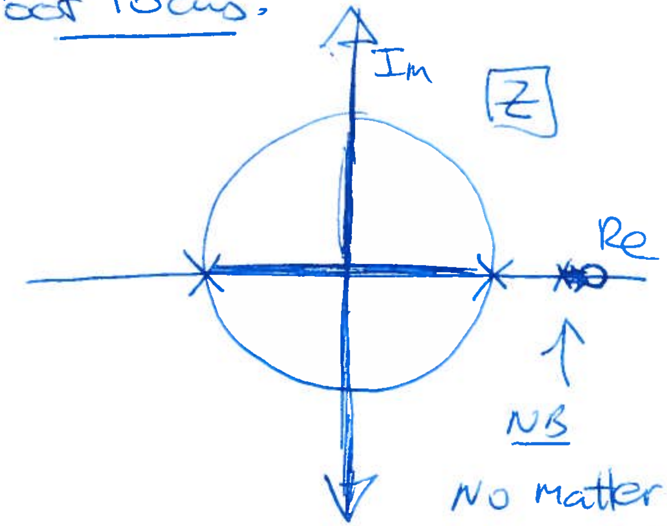
assume the model is nearly perfect

$G_M(z) = \dfrac{k}{(z-0.5)(z-(1.5+\delta))}$

$DG_p = G_p \cdot \dfrac{1}{G_M} \cdot \dfrac{1}{z^2-1}$

$= \dfrac{z-(1.5+\delta)}{z-(1.5)} \cdot \dfrac{1}{(z-1)(z+1)}$

root locus:



6

NB

No matter what value of K
This pole is outside the unit circle
$\therefore$ unstable!

c) Take Laplace Transforms

$$\frac{M(s)}{E(s)} = K_p\left(1 + \frac{1}{sT_I} + sT_d\right)$$

$$= \frac{K_pT_d\left(s^2 + \frac{1}{T_d}s + \frac{1}{T_dT_I}\right)}{s}$$

when equal zeros ; $\frac{1}{T_d^2} = \frac{4}{T_d\,T_I}$

$$\Rightarrow T_d = \frac{T_I}{4}$$

$\therefore$ zeros at $s = -\frac{1}{2T_d}$

$$C(s) = \frac{K_pT_d}{s}\left(s + \frac{1}{2T_d}\right)^2$$

<u>Matched pole zero</u>

zero $s = -\frac{1}{2T_d} \rightarrow e^{-\frac{T}{2T_d}}$

pole $s = 0 \rightarrow e^{-0} = 1$

5

$$\therefore D(z) = \frac{K\left(z - e^{-\frac{T}{2T_d}}\right)^2}{(z-1)\,z}$$ ← included for causality.

To maintain same steady state gain

$$\lim_{s\to 0} s\,C(s) = \frac{K_p}{T_I}$$

$$\lim_{z\to 1} (1-z^{-1})\,D(z) = K\left(1-e^{-T/2T_d}\right)^2$$

$$\therefore K = \frac{K_p}{T_I\left(1-e^{-T/2T_d}\right)^2}$$

$$D(z) = \frac{M(z)}{E(z)} = \frac{K(z-\zeta)^2}{z^2 - z} = \frac{K(z^2 - 2\zeta z + \zeta^2)}{z^2 - z}$$

$$= \frac{K(1 - 2\zeta z^{-1} + \zeta^2 z^{-2})}{1 - z^{-1}}$$

$$\therefore M(z)(1-z^{-1}) = K(1 - 2\zeta z^{-1} + z^{-2})\,E(z)$$

5

<u>IZT</u>

$$M(n) = M(n-1) + K\,e(n) - 2\zeta K e(n-1) + K\zeta^2 e(n-2)$$

$$\therefore \alpha = K = K_p / T_I\left(1-e^{-T/2T_d}\right)^2$$

$$\beta = \frac{-2e^{-T/2T_d}K_p}{T_I\left(1-e^{-T/2T_d}\right)^2} \qquad \gamma = \frac{K_p e^{-\frac{T}{T_d}}}{T_I\left(1-e^{-T/2T_d}\right)^2}$$

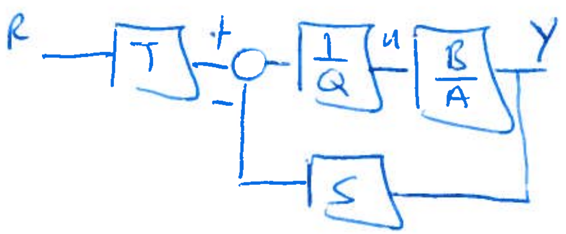
Q3) a)

$$G(z) = \frac{0.5 z^{-2}}{(1-0.8z^{-1})(1-0.7z^{-1})}$$

$$= \frac{0.5}{(z-0.8)(z-0.7)}$$

$$= \frac{0.5}{z^2-1.5z+0.56} = \frac{B}{A}$$

2nd order: propose controller



$Q = z+q_1 \qquad S = s_0 z + s_1$

The ~~Diophantine~~ Closed-loop char eqn is:

$$1 + \frac{B}{A}\cdot\frac{1}{Q}S = 0$$

$$\Rightarrow \quad AQ + BS = 0$$

Closed loop is third order

$\therefore$ need to define 3 closed loop poles

$$\therefore \quad A_{CL} = (z-p_1)(z-p_2)(z-p_3)$$

2 dominant poles at $z = 0.5$

$\Rightarrow$ place fast pole at $(0.5)^5 = 0.03$

$$\therefore \quad A_{CL} = (z-0.5)(z-0.5)(z-0.03)$$

$$= (z^2 - z + 0.25)(z-0.03)$$

$$= z^3 - (1.03)z^2 + (0.28)z - 0.0075$$

Diophantine equation: $AQ + BS \equiv A_{CL}$

$$(z^2-1.5z+0.56)(z+q_1) + 0.5(s_0 z + s_1)$$

$$= z^3 + c_1 z^2 + c_2 z + c_3$$

z^2: $q_1 - 1.5 = c_1 = -1.03$

z: $0.56 - 1.5q_1 + 0.5s_0 = c_2 = 0.28$

z^0: $0.56q_1 + 0.5s_1 = c_3 = -0.0075$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 0.5 & 0 \\ 0.56 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} q_1 \\ \hdashline s_0 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0.47 \\ -0.28 \\ -0.0075 \end{bmatrix}$$

SYLVESTER MATRIX

8

Solve; $q_1 = 0.47$

$$\Rightarrow -1.5 \times 0.47 + 0.5 s_0 = -0.28$$
$$\Rightarrow s_0 = 0.85$$

$$\Rightarrow 0.56 \times 0.47 + 0.5 s_1 = -0.0075$$
$$s_1 \Rightarrow -0.544$$

$$\therefore Q(z) = z + 0.47$$
$$S(z) = 0.85z - 0.544$$

$$T(z) = t_0(z - p_3)$$ ↑ fast pole

$$= t_0(z - 0.03)$$

for $e_{ss} = 0$ for steps

we want
$$\lim_{z \to 1} \frac{BT}{AQ + BS} = 1$$

since $T = t_0(z - 0.03)$

and $(AQ + BS) = (z - 0.03)(z - 0.5)^2$

$$\therefore \text{ we want } \lim_{z \to 1} \frac{t_0\, 0.5\cancel{(z - 0.03)}}{\cancel{(z - 0.03)}(z - 0.5)^2} = 1$$

ie: $t_0 \times 0.5 = (1 - 0.5)^2 = 0.25$

$$\therefore t_0 = 0.5$$

4

$$\therefore T(z) = 0.5(z - 0.03)$$

Q3b) Assume model structure

$$\frac{Y}{u} = G(z) = \frac{b_1 z^{-1} + \cdots b_m z^{-m}}{1 - a_1 z^{-1} + \cdots - a_n z^{-n}}$$

yields candidate diff eqn model:

$$y(k) = a_1 y(k-1) + a_2 y(k-2) + \cdots a_n y(k-n) + b_1 u(k-1) + b_2 u(k-2) + \cdots b_m u(k-m)$$

<u>fit this model to data:</u>

<u>eg for k^{th} sample</u>

$$\hat{y}(k) = \hat{a}_1 y(k-1) + \cdots \hat{a}_n y(k-n) + \hat{b}_1 u(k-1) + \cdots \hat{b}_m u(k-m)$$

$$= \left[y(k-1) \cdots y(k-n) \,\vdots\, u(k-1) \cdots u(k-m) \right] \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \\ \hat{b}_1 \\ \vdots \\ \hat{b}_m \end{bmatrix}$$

$$= \psi^T_{(k)} \underline{\hat{\theta}}$$

Can be repeated for all <u>valid</u> data points

$$N \left\{ \begin{bmatrix} \vdots \\ \hat{y}(k-2) \\ \hat{y}(k-1) \\ \hat{y}(k) \end{bmatrix} \right. = \left[\begin{array}{cccc|cc} & & & & & \vdots \\ y(k-3) & y(k-4) & \cdots & y(k-n-2) & u(k-3) & u(k-m-2) \\ y(k-2) & y(k-3) & \cdots & y(k-n-1) & u(k-2) & u(k-m-1) \\ y(k-1) & y(k-2) & \cdots & y(k-n) & u(k-1) \cdots & u(k-m) \end{array} \right] \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \\ \hat{b}_1 \\ \vdots \\ \hat{b}_m \end{bmatrix}$$

$$\underline{\hat{Y}}(k) = \underline{\Phi}(k) \, \underline{\hat{\theta}}$$ <u>prediction eqn</u>

measurement vector

$$\underline{Y}(k) = \begin{bmatrix} y \\ \vdots \\ y(k-1) \\ y(k) \end{bmatrix}$$ N valid measurements

propose cost: $$J = \sum_{i=0}^{N-1} e^2(k-i)$$

$$e(k-i) = y(k-i) - \hat{y}(k-i)$$

$$J = \underline{E}^T \underline{E}$$

$$= (\underline{Y} - \hat{\underline{Y}})^T (\underline{Y} - \hat{\underline{Y}})$$

using: $\hat{\underline{Y}} = \underline{\Phi}\,\hat{\underline{\Theta}}$

$$J = (\underline{Y} - \underline{\Phi}\hat{\underline{\Theta}})^T (\underline{Y} - \underline{\Phi}\hat{\underline{\Theta}})$$

$$= \hat{\underline{\Theta}}^T \underline{\Phi}^T \underline{\Phi} \hat{\underline{\Theta}} - 2\underline{Y}^T \underline{\Phi} \hat{\underline{\Theta}} + \underline{Y}^T \underline{Y}$$

Completing the square:

$$J = (\hat{\underline{\Theta}} - \underline{\alpha})^T \underline{\Phi}^T \underline{\Phi} (\hat{\underline{\Theta}} - \underline{\alpha}) + J_0 \qquad \text{eqn 1}$$

$$\Rightarrow \hat{\underline{\Theta}}^T \underline{\Phi}^T \underline{\Phi} \hat{\underline{\Theta}} - 2\underline{\alpha}^T \underline{\Phi}^T \underline{\Phi} \hat{\underline{\Theta}} + J_0 + \underline{\alpha}^T \underline{\Phi}^T \underline{\Phi} \underline{\alpha}$$

$\therefore$ choose $\underline{\alpha}^T \underline{\Phi}^T \underline{\Phi} \hat{\underline{\Theta}} = \underline{Y}^T \underline{\Phi} \hat{\underline{\Theta}}$

Transpose $\underbrace{(\underline{\Phi}^T \underline{\Phi})^T}_{\text{symmetric}} \underline{\alpha} = \underline{\Phi}^T \underline{Y}$

$$\therefore (\underline{\Phi}^T \underline{\Phi})\, \underline{\alpha} = \underline{\Phi}^T \underline{Y}$$

yields $\underline{\alpha} = (\underline{\Phi}^T \underline{\Phi})^{-1} \underline{\Phi}^T \underline{Y}$

---

min to eqn 1 occurs when

$$\hat{\underline{\Theta}}_{LS} = \underline{\alpha} = (\underline{\Phi}^T \underline{\Phi})^{-1} \underline{\Phi}^T \underline{Y}$$

7

At the $(k+1)^{st}$ sample

$$\hat{y}(k+1) = \underbrace{[y(k) \cdots \,|\, u(k) \cdots]}_{\underline{\psi}^T(k+1)} \hat{\underline{\Theta}}$$

$$\therefore \hat{\underline{Y}}(k+1) = \begin{bmatrix} \vdots \\ y(k-1) \\ y(k) \\ \hline y(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\Phi}(k) \\ \hline \underline{\psi}^T(k+1) \end{bmatrix}}_{\underline{\Phi}(k+1)} \hat{\underline{\Theta}}$$

$$\therefore \underline{\Phi}(k+1)^T \underline{\Phi}(k+1)$$

$$= \left[\underline{\Phi}(k)^T \,\vdots\, \underline{\psi}(k+1) \right] \begin{bmatrix} \underline{\Phi}(k) \\ \hline \underline{\psi}^T(k+1) \end{bmatrix}$$

$$= \Phi(n)^T \Phi(n) + \underline{\psi}(n+1)\underline{\psi}^T(n+1)$$

By definition

$$P(n+1) = \left[\Phi(n)^T\Phi(n) + \underline{\psi}(n+1)\underline{\psi}^T(n+1)\right]^{-1}$$

$$= \left[P^{-1}(n) + \underline{\psi}(n+1)\underline{\psi}^T(n+1)\right]^{-1}$$

<u>Householders</u> $\Rightarrow$

$A = P$

$B = \underline{\psi}(n+1)$

$C = 1$

$D = \underline{\psi}^T(n+1$

$$P(n+1) = P(n) - P(n)\underline{\psi}(n+1)\underbrace{(1+\psi^T P \psi)^{-1}}_{\text{Scalar}}\psi^T P(n)$$

$$\therefore P(n+1) = P(n) - \frac{P(n)\underline{\psi}(n+1)\underline{\psi}^T(n+1)P(n)}{1 + \underline{\psi}(n+1)^T P(n)\underline{\psi}(n+1)}$$

To finish: $\hat{\underline{\theta}}_{LS}(k+1) = \left[\Phi^T(n+1)\Phi(n+1)\right]^{-1}\Phi^T(n+1)\underline{Y}(n+1)$

$$\hat{\underline{\theta}}_{LS}(n+1) = P(n+1)\left[\Phi(n)^T \vdots \underline{\psi}(n+1)\right]\begin{bmatrix}\underline{Y}(n)\\ \hline y(n+1)\end{bmatrix}$$

$$= P(n+1)\left[\Phi(k)^T\underline{Y}(n) + \underline{\psi}(n+1)y(n+1)\right]$$

$$\therefore \hat{\underline{\theta}}_{LS}(n+1) = \left[P(n) - \frac{P(n)\psi(n+1)\psi^T(n+1)P(n)}{1+\psi^T_{(n+1)}P\psi(n+1)}\right]\left[\Phi^T(n)Y + \underline{\psi}(n+1)y(n+1)\right]$$

6

4) a) $\dot{\underline{x}} = A\underline{x} + B\underline{u}$

Take Laplace transforms

$$sX(s) - \underline{x}(0) = A\underline{X}(s) + B\underline{u}(s)$$

$$\Rightarrow (sI - A)\underline{X}(s) = \underline{x}(0) + B\underline{u}(s)$$

$$\underline{X}(s) = (sI-A)^{-1}\underline{x}(0) + (sI-A)^{-1}B\underline{u}(s)$$

$$\therefore \underline{x}(t) = \mathcal{L}\{\underline{X}(s)\}$$

define $\Phi(s) = (sI-A)^{-1}$

$$\therefore \Phi(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

$$\therefore \underline{x}(t) = \Phi(t)\underline{x}(0) + \mathcal{L}^{-1}\{\Phi(s)B\underline{u}(s)\}$$

$$x(t) = \Phi(t)\underline{x}(0) + \int_0^t \Phi(t-\tau)B\underline{u}(\tau)\,d\tau$$

6

b) $x(n+1) = A_d x(n) + B_d u(n)$

$y(n) = C x(n)$

$$A_d = \Phi(T)$$

$$\Phi(t) = \mathcal{L}^{-1}\left\{\begin{pmatrix} s & 0 \\ 0 & s\end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -2\end{pmatrix}\right\}$$

$$= \mathcal{L}^{-1}\left\{\begin{matrix} s+1 & 0 \\ 0 & s+2\end{matrix}\right\}$$

$$\Phi(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 & 0 \\ 0 & s+1\end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2}\end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t}\end{bmatrix}$$

$$A_d = \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T}\end{bmatrix} = \begin{bmatrix} 0.9048 & 0 \\ 0 & 0.8187\end{bmatrix}$$

$$B_d = \int_0^T \Phi(\eta) B \, d\eta$$

$$= \int_0^T \begin{bmatrix} e^{-\eta} & 0 \\ 0 & e^{-2\eta} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\eta$$

$$= \int_0^T \begin{bmatrix} e^{-\eta} \\ e^{-2\eta} \end{bmatrix} d\eta$$

$$= \begin{bmatrix} -e^{-\eta} \\ -2e^{-2\eta} \end{bmatrix}_0^T$$

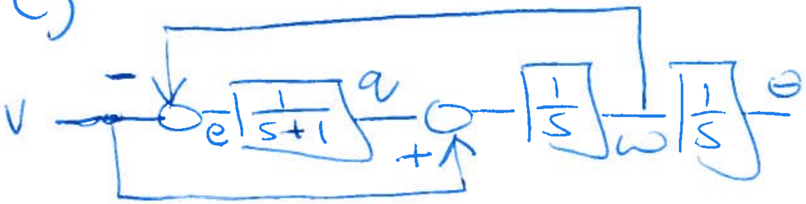
$$= \begin{bmatrix} 1-e^{-0.1} \\ 1-2e^{-0.2} \end{bmatrix} = \begin{bmatrix} 0.0952 \\ -0.637 \end{bmatrix}$$

7

$$\therefore \quad \underline{X}(k+1) = \begin{bmatrix} 0.9048 & 0 \\ 0 & 0.8187 \end{bmatrix} \underline{X}(k) + \begin{bmatrix} 0.0952 \\ -0.637 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{X}(k)$$

c)



$$\dot{\theta} = \omega$$
$$\dot{\omega} = q + v$$
$$e = v - \omega$$
$$\frac{Q}{E} = \frac{1}{s+1} \Rightarrow (s+1)Q = E$$

Take inverse Laplace:

$$\dot{q} + q = e$$
$$\dot{q} = e - q$$
$$= v - \omega - q$$

$$\therefore \quad \dot{\theta} = \omega$$
$$\dot{\omega} = q + v$$
$$\dot{q} = v - \omega - q$$

4

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \\ q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} v$$

poles are roots of:

$$\det(sI - A) = 0$$

$$\det\left(\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 1 & s+1 \end{vmatrix} = s(s^2 + s + 1)$$

$s = 0$

and $s = \dfrac{-1 \pm \sqrt{1-4}}{2}$

$$= -0.5 \pm j\frac{\sqrt{3}}{2}$$

4

<u>Observability</u>: $\Theta = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

$C = [1\ 0\ 0]$

$$CA = [1\ 0\ 0]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} = [0\ 1\ 0]$$

$$CA^2 = [0\ 1\ 0]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} = [0\ 0\ 1]$$

$$\therefore\ \Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is the identity $\rightarrow$ full rank

Since Θ is full rank (3) the states are observable

4

Q5 a) $\dot{\underline{x}} = A\underline{x} + Bu$

$y = C\underline{x}$

let $e(t) = r - y$

$$x_I = \int e(\tau)\,d\tau = \int r - y \, d\tau$$

$$\Rightarrow \quad \dot{x}_I = r - y$$

$$= r - C\underline{x}$$

Augment the state vector

$$\underline{x}_2 = \begin{bmatrix} x \\ \hdashline x_I \end{bmatrix}$$

then $$\frac{d}{dt}\begin{bmatrix} \underline{x} \\ \hdashline x_I \end{bmatrix} = \left[\begin{array}{c:c} A & 0 \\ \hdashline -C & 0 \end{array}\right]\begin{bmatrix} \underline{x} \\ \hdashline x_I \end{bmatrix} + \begin{bmatrix} B \\ \hline 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix} r$$

The control becomes

$$u = -K\underline{x} + K_I x_I$$

$$= \left[\begin{array}{c:c} -K & K_I \end{array}\right]\begin{bmatrix} x_1 \\ \hdashline x_I \end{bmatrix}$$

$$\therefore \frac{d}{dt}\begin{bmatrix} x \\ x_I \end{bmatrix} = \left[\begin{array}{c|c} A & 0 \\ \hline -C & 0 \end{array}\right]\begin{bmatrix} x \\ x_I \end{bmatrix} \cancel{+} \begin{bmatrix} B \\ \hline 0 \end{bmatrix}\begin{bmatrix} -K & K_I \end{bmatrix}\begin{bmatrix} x \\ \hline x_I \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$= \left[\begin{array}{c|c} A - BK & BK_I \\ \hline -C & 0 \end{array}\right]\begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

Char eqn: $$\det\left[\left[\begin{array}{c|c} sI_N & \\ \hline & s \end{array}\right] - \left[\begin{array}{c|c} A - BK & BK_I \\ \hline -C & 0 \end{array}\right]\right] = 0$$

$$\Rightarrow \quad \left|\left[\begin{array}{c|c} sI - A + BK & -BK_I \\ \hline -C & s \end{array}\right]\right| = 0$$

---

7 marks

Q5 b)

$$C(s) = K_c(s+z) = \frac{U}{E}$$

$\therefore \quad U = K_c(s+z)E$

ILT $\quad \therefore \quad u(t) = K_c\frac{de}{dt} + K_c z e$

since $\quad e = r - \theta$

$\therefore \quad u(t) = K_c\frac{d}{dt}(r-\theta) + K_c z(r-\theta)$

$$\approx -K_c\frac{d\theta}{dt} + K_c z r - K_c z\theta$$

$$\Rightarrow K_c z r - K_c z\theta - K_c\omega$$

$$\frac{\theta}{u} = \frac{K}{s^2+as} \quad \Rightarrow \quad \text{control canonical form}$$

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & -a\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}u$$

$$u = k_1 r - \begin{bmatrix}k_1 & k_2\end{bmatrix}\begin{bmatrix}\theta \\ \omega\end{bmatrix}$$

4

poles are roots of:

$$\det(sI - A + BK) = 0$$

$$\det\left[\begin{pmatrix}s & 0 \\ 0 & s\end{pmatrix} - \begin{pmatrix}0 & 1 \\ 0 & -a\end{pmatrix} + \begin{pmatrix}0 & 0 \\ k_1 & k_2\end{pmatrix}\right]$$

$$\Rightarrow \det\begin{bmatrix}s & -1 \\ k_1 & s+a+k_2\end{bmatrix}$$

$$\Rightarrow s^2 + (a+k_2)s + k_1$$

Both poles at $s = -2a$

$$(s+2a)^2 = s^2 + 4as + 4a^2$$

$\therefore \quad a + k_2 = 4a \qquad k_2 = 3a$

and $k_1 = 4a^2$

$\therefore \quad K_c = 3a$

$K_c z = 4a^2$

$3az = 4a^2 \quad \Rightarrow z = \frac{4a}{3}$

4

Q5) c) i) Operating point

EQUIL $\rightarrow$ $\frac{di_d}{dt} = \frac{di_q}{dt} = \frac{dv_{dc}}{dt} = 0$

$\therefore \frac{3}{2}(v_d i_d) = -P_w = -1 \times 10^6$

$v_d i_d = -0.667 \times 10^6$ (1)

We are told $i_q \Rightarrow 0$

$\therefore$ $R i_d = \underset{\uparrow 690}{v_d'} - v_d$

$i_d = \frac{690 - v_d}{R}$ (2)

Next: $\omega L i_d = v_q' - v_q$

---

from (1) and (2)

$v_d(690 - v_d) = -0.667 \times 10^6 \times \underset{0.003}{R}$

$\therefore v_d^2 - 690 v_d - 2000 = 0$

$v_d = \frac{690 \pm \sqrt{690^2 + 8000}}{2}$

$= \frac{690 \pm 695.8}{2}$

$= \underline{\underline{692.9}}$ Volt or -2.9 Volt

$i_d = \frac{690 - 692.9}{R} = -967$ Amp

Linearise the power equation

$C \frac{dv_{dc}}{dt} = f(v_d, i_d, P_w, v_{dc})$

$= \frac{\frac{3}{2}(v_d i_d)}{v_{dc}} + \frac{P_w}{v_{dc}}$

$\frac{d}{dt} \Delta v_{dc} = \left.\frac{\partial f}{\partial v_d}\right|_{op} \Delta v_d + \left.\frac{\partial f}{\partial i_d}\right|_{op} \Delta i_d + \left.\frac{\partial f}{\partial P_w}\right|_{op} \Delta P_w + \left.\frac{\partial f}{\partial v_{dc}}\right| \Delta v_{dc}$

$$\left.\frac{\partial f}{\partial v_d}\right|_{op} = \frac{3}{2}\frac{i_d}{v_{dc}} = \frac{3}{2}\,\frac{-967}{1050} = -1.38$$

$$\left.\frac{\partial f}{\partial i_d}\right|_{op} = \frac{3}{2}\frac{v_d}{v_{dc}} = \frac{3}{2}\,\frac{692.9}{1050} = 0.989$$

$$\left.\frac{\partial f}{\partial P_w}\right|_{op} = \frac{1}{v_{dc}} = \frac{1}{1050} = 9.52\times10^{-4}$$

$$\left.\frac{\partial f}{\partial v_{dc}}\right|_{op} = 0 \Rightarrow -\frac{\frac{3}{2}(v_d i_d)+P_w}{v_{dc}^2} \to 0$$

$$0.2\,\frac{dv_{dc}}{dt} = -1.38\,v_d + 0.989\,i_d + 9.52\times10^{-4}\,P_w$$

$$\frac{dv_{dc}}{dt} = -6.9\,v_d + 4.945\,i_d + 0.0038\,P_w$$

$$\frac{di_d}{dt} = \frac{1}{L}v_d' - \frac{1}{L}v_d - \frac{R}{L}i_d + w i_q$$

$$\frac{di_q}{dt} = \frac{1}{L}v_q' - \frac{1}{L}v_q - \frac{R}{L}i_q - w i_d$$

$w = 314.15$

$\frac{R}{L} = 37.5$

$$\therefore \frac{d}{dt}\begin{bmatrix} i_d \\ i_q \\ v_{dc}\end{bmatrix} = \begin{bmatrix} -37.5 & 314 & 0 \\ -314 & -37.5 & 0 \\ 4.945 & 0 & 0\end{bmatrix}\begin{bmatrix} i_d \\ i_q \\ v_{dc}\end{bmatrix} + \begin{bmatrix} -12500 & 0 \\ 0 & -12500 \\ -6.9 & 0\end{bmatrix}\begin{bmatrix} v_d \\ v_q\end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.0038\end{bmatrix} P_w$$

10

---

ii) With $v_d' = wL i_q + k_1 i_d + k_2 v_{dc} - k_3\int dt$

Sub into state equation for i_d

$L\frac{di_d}{dt} - wL i_q + R i_d = v_d' - wL i_q - k_1 i_d - k_2 v_{dc}$

$\Rightarrow L\frac{di_d}{dt} + R i_d = v_d' - k_1 i_d - k_2 v_{dc}$) dt

Q6)

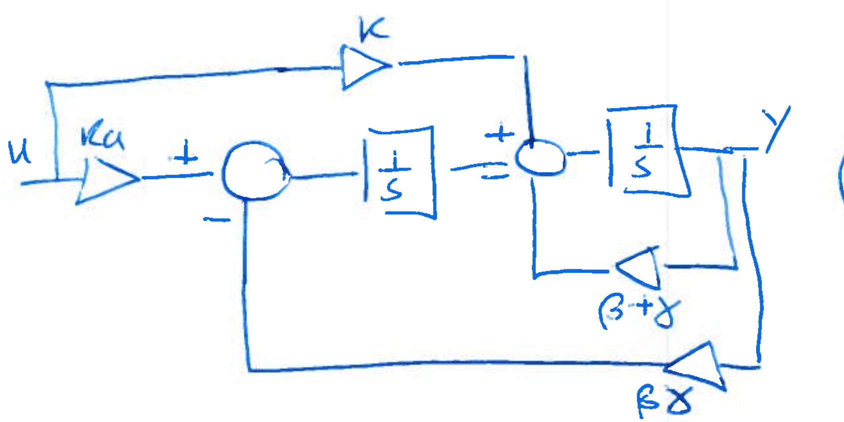
a) $$G(s) = \frac{K(s+a)}{s^2+(\beta+\gamma)s+\beta\gamma}$$

$$= \frac{K\left(\frac{1}{s}+\frac{a}{s^2}\right)}{1+\frac{\beta+\gamma}{s}+\frac{\beta\gamma}{s^2}} = \frac{Y}{U}$$

$$\Rightarrow \left[1+\frac{\beta+\gamma}{s}+\frac{\beta\gamma}{s^2}\right]Y = K\left(\frac{1}{s}+\frac{a}{s^2}\right)U$$

$$Y = \frac{K}{s}U - \frac{\beta+\gamma}{s}Y + \frac{Ka}{s^2}U - \frac{\beta\gamma}{s^2}Y$$

$$= \frac{1}{s}\left[KU - (\beta+\gamma)Y + \frac{1}{s}\left[KaU - \beta\gamma Y\right]\right]$$



5

b) Luenberger observer

$$\frac{d}{dt}\hat{\underline{x}} = (A-GC)\hat{x} + Bu + Gy$$

plant

$$\dot{\underline{x}} = A\underline{x} + Bu \qquad y = C\underline{x}$$

control

$$u = -K\hat{\underline{x}} + Nr$$

---

define estimation error

$$\underline{e} = \underline{x} - \hat{\underline{x}}$$

$$\therefore \hat{\underline{x}} = \underline{x} - \underline{e}$$

$$\therefore \frac{d}{dt}(\underline{x}-\hat{\underline{x}}) = \frac{d}{dt}\underline{e}$$

$$= (A\underline{x}+Bu) - (A\hat{\underline{x}}+Bu+Gy-GC\hat{x})$$

$$= (A\underline{x}+\cancel{Bu}) - (A\hat{x}+\cancel{Bu}+GC(x-\hat{x}))$$

$$= A\underline{x} - A\hat{x} - GC\underline{e}$$

$$= (A-GC)e$$

Plant equations

$$\dot{\underline{x}} = A\underline{x} + Bu$$
$$u = -K\hat{\underline{x}} + Nr$$

$\therefore$ closed loop:

$$\dot{\underline{x}} = A\underline{x} - BK\hat{\underline{x}} + BNr$$

But $\hat{\underline{x}} = \underline{x} - \underline{e}$

$$\therefore \dot{\underline{x}} = A\underline{x} - BK(\underline{x} - \underline{e}) + BNr$$

$$\therefore \dot{\underline{x}} = (A - BK)\underline{x} + BK\underline{e} + BNr$$

Two coupled sets of N 1st order simultaneous diff eqns

$$\frac{d}{dt}\begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} A-BK & BK \\ \hline 0 & A-GC \end{array}\right]}_{A'} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} BN \\ 0 \end{bmatrix} r$$

The poles of this combined $(2N)^{th}$ order process are given by

$$\det(sI - A') = 0$$

ie
$$\left| \left(\begin{array}{c|c} sI_N & 0 \\ \hline 0 & sI_N \end{array}\right) - \left(\begin{array}{c|c} A-BK & BK \\ \hline 0 & A-GC \end{array}\right) \right| = 0$$

$$\Rightarrow \left| \begin{array}{c|c} sI_N - A + BK & -BK \\ \hline 0 & sI_N - A + GC \end{array} \right| = 0$$

$$\Rightarrow \det(sI_N - A + BK)\,\det(sI_N - A + GC) = 0$$

The roots of the 2Nth order process are simply the combination of the N roots of

$$\det(sI - A + BK) = 0$$

and the N roots of

$$\det(sI - A + GC) = 0$$

This means that we can design the regulator using ideal states $\underline{x}$ and design equation $\det(sI - A + BK) = C'_{des}(s)$ to place the N closed loop poles as desired.

We can then place the observer N poles using $\det(sI - A + GC) = C''_{des}(s)$ knowing that they will not affect the already design positions of the N closed loop poles.

5

ii) Luenberger observer

$$\frac{d}{dt}\hat{\underline{x}} = (A - GC)\hat{\underline{x}} + Bu + Gy$$

$$u = -K\hat{\underline{x}} + Nr$$

$$\therefore \frac{d}{dt}\hat{\underline{x}} = (A - GC)\hat{\underline{x}} - BK\hat{\underline{x}} + BNr + Gy$$

$$= (A - GC - BK)\hat{\underline{x}} + BNr + Gy$$

Take Laplace Transforms

$$s\hat{\underline{X}}(s) = (A - GC - BK)\hat{\underline{X}}(s) + BN\underline{R}(s) + G\underline{Y}(s)$$

$$\therefore (sI - A + GC + BK)\hat{\underline{X}}(s) = BN\,R(s) + G\,Y(s)$$

$$\Rightarrow \hat{\underline{X}}(s) = (sI - A + GC + BK)^{-1}\left[BN\underline{R} + GY\right]$$

Back to the control equation:

$$u(s) = -K\hat{\underline{x}}(s) + N R(s)$$