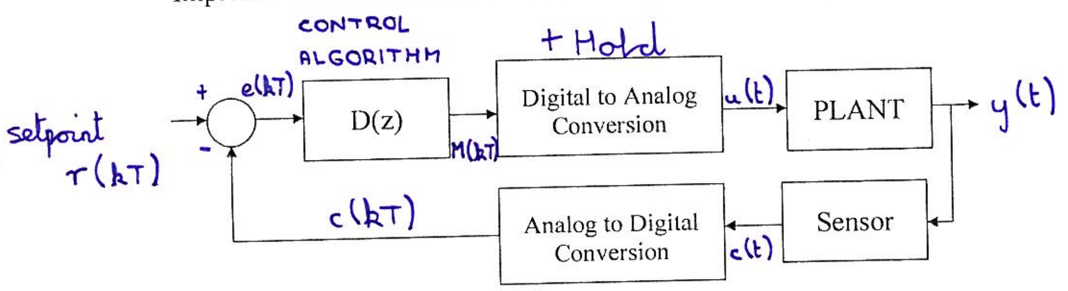
24/9/08 Brian Fitzgibbon

EE4002 Control Engineering

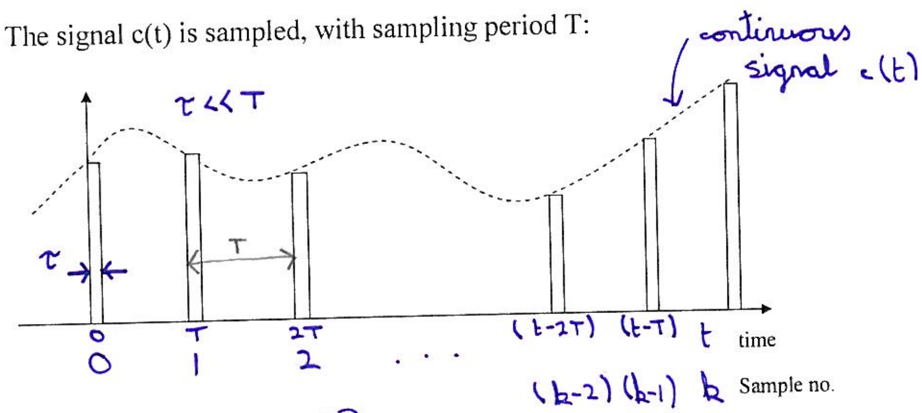
A) Digital Control Systems

Chapter 1. The Basics

Implementation of the control algorithm on a digital computer:



The signal c(t) is sampled, with sampling period T:



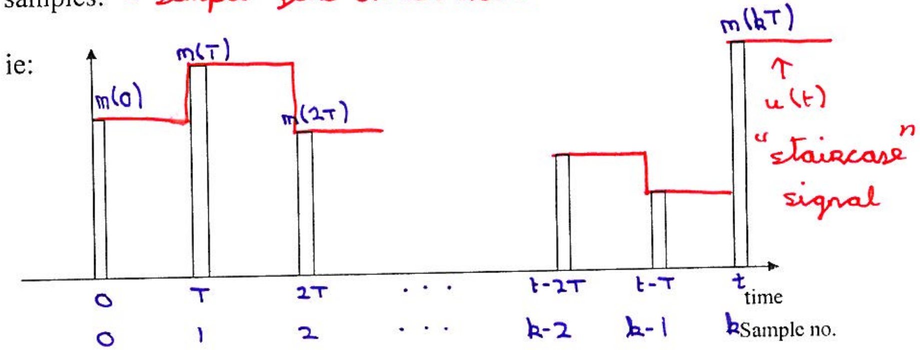
The Setpoint is read every T: sampling period

$$e(kT) = r(kT) - c(kT)$$

Control algorithm processes the error e(kT) to generate m(kT)
Digital to Analog convertor (DAC) converts this binary word representation of m(kT) to an analog voltage. e.g. proportional control $m(kT) = ke(kT)$

It is usual to hold the DAC output voltage constant between samples: - simple zero order hold

ie:



A typical digital proportional control algorithm could be:

```
While True Do
    Increment k
    Sample c(t)
    Read setpoint r(kT)
    Generate error, e(kT)=r(kT)-c(kT)
    Calculate control, m(kT)=Ke(kT)
    Convert to analog+hold
    Wait until period T elapses
End
```

1.1 Basic Approximation of Analog Controllers on a Digital Computer

Design the controller C(s) in the s plane – assuming a continuous system

- assume approximations for integration and differentiation

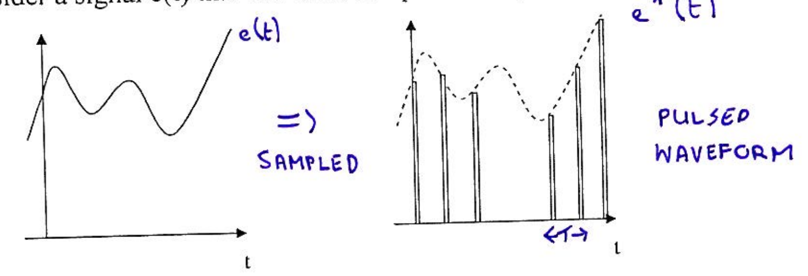
Consider the PID algorithm approximated using the ~~forward~~ backward difference algorithm:

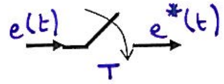
$$e(kT) = r(kT) - c(kT)$$
$$I(kT) = I((k-1)T) + Te(kT)$$
$$D(kT) = \frac{e(kT) - e((k-1)T)}{T}$$
$$m(kT) = K\left(e(kT) + \frac{1}{T_I} I(kT) + T_d D(kT)\right)$$

N.B. Sample time must be small

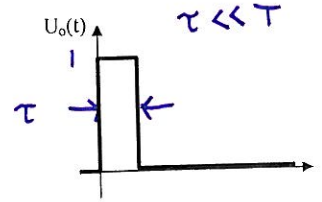
Chapter 2. The Z Transform

Consider a signal e(t) that has been sampled with period T:





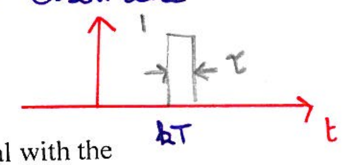
First we will define the unit pulse $U_o(t)$ as:



$\tau << T$

$$U_o(t) = \begin{cases} 1 & \text{for } 0 \le t \le \tau \\ 0 & \text{elsewhere} \end{cases}$$

Also $U_o(t-kT)$



Consider now the product of this unit pulse signal with the continuous signal e(t): if τ is very small

$$U_o(t)\,e(t) = \begin{cases} e(0) & \text{for } 0 \le t \le \tau \\ 0 & \text{elsewhere} \end{cases}$$ extracting the 0^{th} sample

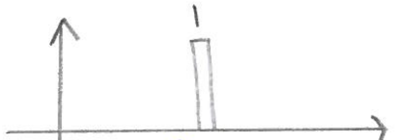
$$U_o(t-kT)\,e(t) = \begin{cases} e(kT) & \text{for } kT \le t \le kT+\tau \\ 0 & \text{elsewhere} \end{cases}$$ extracting the k^{th} sample

2) The unit discrete pulse U_o (kT):

$$U_o(kT)=\begin{cases}1 & for \quad k=0\\ 0 & for \quad k\neq 0\end{cases}$$

$$U_o(z)=Z\{U_o(kT)\}=\sum_{k=0}^{\infty}U_o(kT)z^{-k} = 1z^{-0}+0z^{-1}+0z^{-2}+\dots$$
$$=1$$

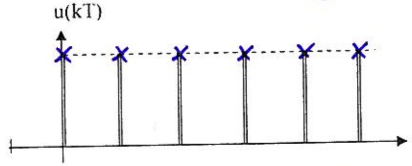
3) The time shifted unit discrete pulse:



$$U_o((k-m)T)=\begin{cases}1 & for \quad k=m\\ 0 & for \quad k\neq m\end{cases}$$

$$Z\{U_o((k-m)T)\}=\sum_{k=0}^{\infty}U_o((k-m)T)z^{-k} = 0z^{-0}+0z^{-1}+\dots+1z^{-m}+0z^{-(m+1)}$$
$$=z^{-m}$$

4) The unit step signal u(kT)



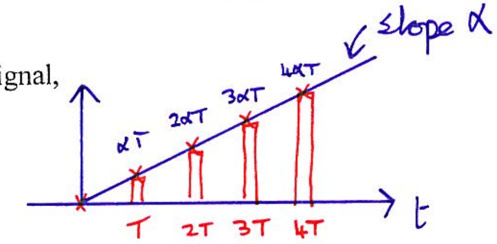
$$u(kT)=\begin{cases}1 & for \quad k\geq 0\\ 0 & for \quad k<0\end{cases}$$

$$U(z)=Z\{u(kT)\}=\sum_{k=0}^{\infty}u(kT)z^{-k} = 1z^{-0}+1z^{-1}+1z^{-2}+\dots$$

=> a geometric progression $\quad U(z)=\frac{z}{z-1}$

5) (Tutorial) Unit ramp signal

Show that the Z transform of the following ramp signal,



$$r(t)=\begin{cases}\alpha t & for \quad t\geq 0\\ 0 & for \quad t<0\end{cases}$$

sampled with sampling time T, is:

$$R(z)=Z\{\underbrace{r(kT)}_{\text{sequence}}\}=\frac{\alpha Tz}{(z-1)^2}$$

6) (Tutorial) Show that the Z transform of the exponential signal,

$$f(t)=\begin{cases}Ke^{-at} & for \quad t\geq 0\\ 0 & for \quad t<0\end{cases}$$

sampled with sampling time T, is:

$$F(z)=Z\{f(kT)\}=\frac{Kz}{z-e^{-aT}}$$

Chapter 3. Spectrum of Sampled Signals

The sampled signal,

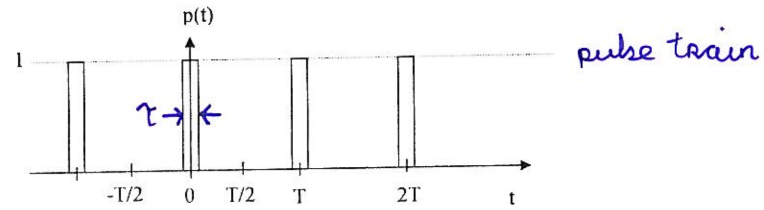
$$e^*(t)=\sum_{k=0}^{\infty}e(kT)\cdot U_o(t-kT)$$

Could be rewritten as:

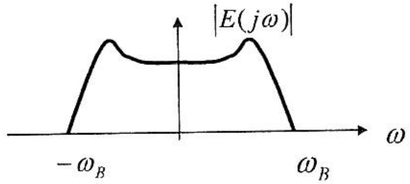
$$e^*(t)=e(t)\sum_{k=0}^{\infty}U_o(t-kT)=e(t)\,p(t)$$

Where:

$$p(t)=\sum_{k=0}^{\infty}U_o(t-kT)$$



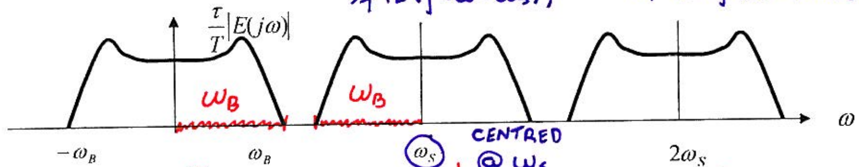
Consider that the spectrum of e(t) is:



where $\omega_s = \frac{2\pi}{T}$

Then the spectrum of the sampled signal e*(t) is:

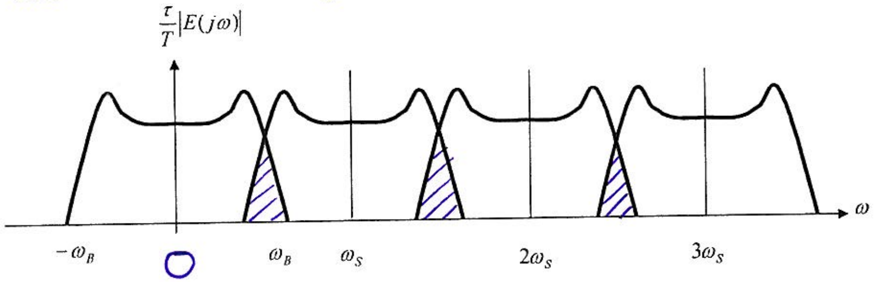
$\tau/T\,|E(j(\omega-\omega_s))|$ $\tau/T\,|E(j(\omega-2\omega_s))|$



N.B. INFINITE NUMBER OF REPLICATIONS OF BASEBAND

3.1 Shannon's Sampling Theorem

if $\omega_s > 2\omega_B$ the spectra are distinct
The baseband can easily be extracted by low pass filtering

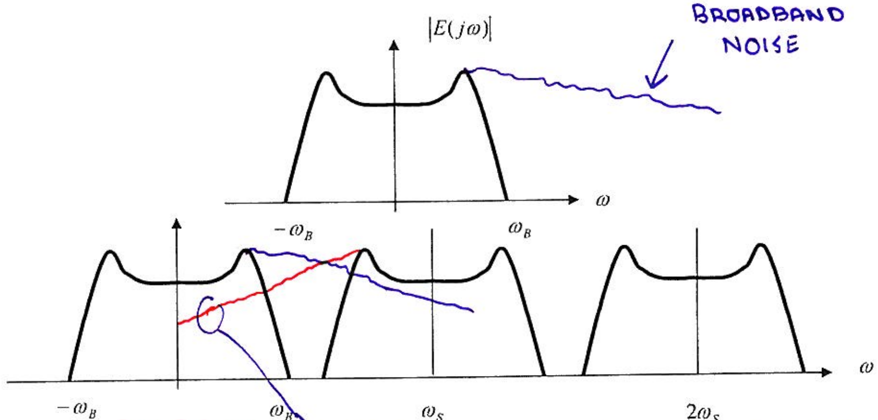


ALIAS DISTORTION - HIGH FREQUENCIES "PRETENDING" TO BE LOW FREQ

Cannot simply reconstruct the base-band from the sampled signal

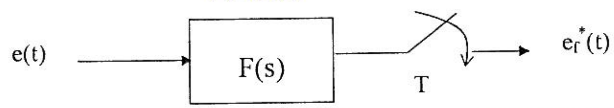
Shannon's Sampling Theorem: For a continuous time signal e(t) with $|E(j\omega)| = 0$ for $|\omega| \geq \omega_B$, then the sampling frequency should be chosen as $\omega_S \geq 2\omega_B$ to ensure that aliasing does not occur.

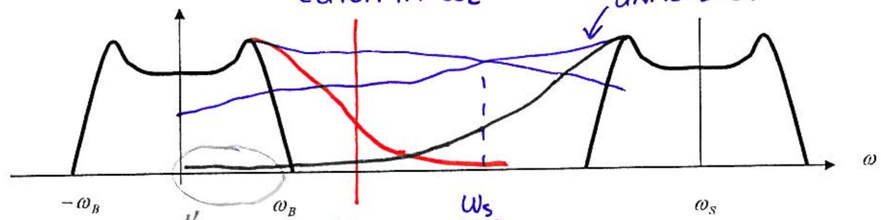
In Practice, there is not a finite spectrum to e(t) due to noise:



high frequency noise appearing in frequency range of interest

Essential to prefilter the signal e(t), before sampling to avoid large aliasing errors:





VERY LITTLE NOISE POWER IS NOW PRESENT IN THE BASEBAND

N.B. we don't want to distort the baseband signal
$\rightarrow \omega_c > \omega_B$

$$|G_{ZOH}(j\omega)| = \frac{\sqrt{2(1-\cos\omega T)}}{\omega\tau}$$

Using the identity:

$$1-\cos\theta = 2\sin^2\left(\theta/2\right)$$

$$1-\cos\omega T = 2\sin^2\left(\frac{\omega T}{2}\right)$$

then:

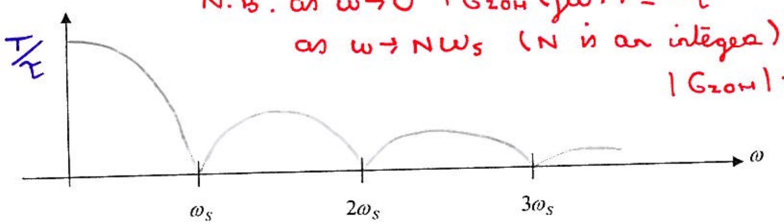
$$|G_{ZOH}(j\omega)| = \frac{\sqrt{2(1-\cos\omega T)}}{\omega\tau} = \frac{\sqrt{4\sin^2\frac{\omega T}{2}}}{\omega\tau} = \frac{2\sin\frac{\omega T}{2}}{\omega\tau} = \frac{T}{\tau}\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}$$

which has the gain frequency response plot:

(Bouncing ball)

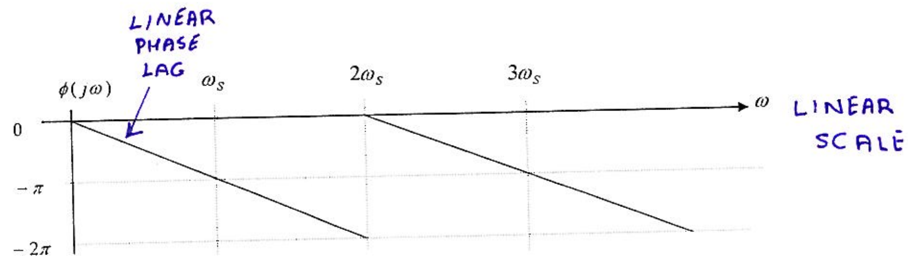
N.B. as $\omega \to 0$ $|G_{ZOH}(j\omega)| = T/\tau$

as $\omega \to N\omega_s$ (N is an integer) $|G_{ZOH}| \to 0$

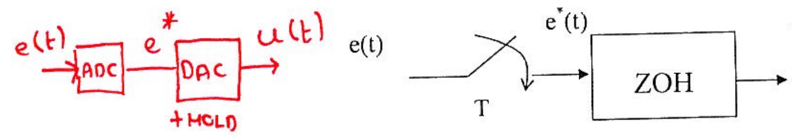


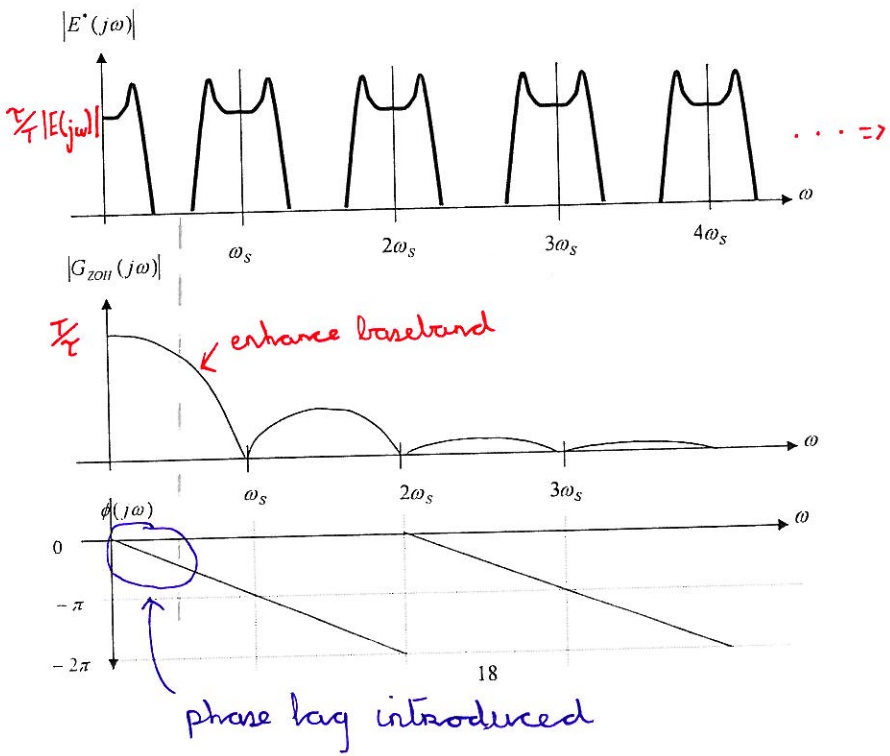
Tutorial: Show that the phase is given by:

$$\phi(j\omega) = \angle G_{ZOH}(j\omega) = -\frac{\omega}{\omega_S}\pi \quad radians$$

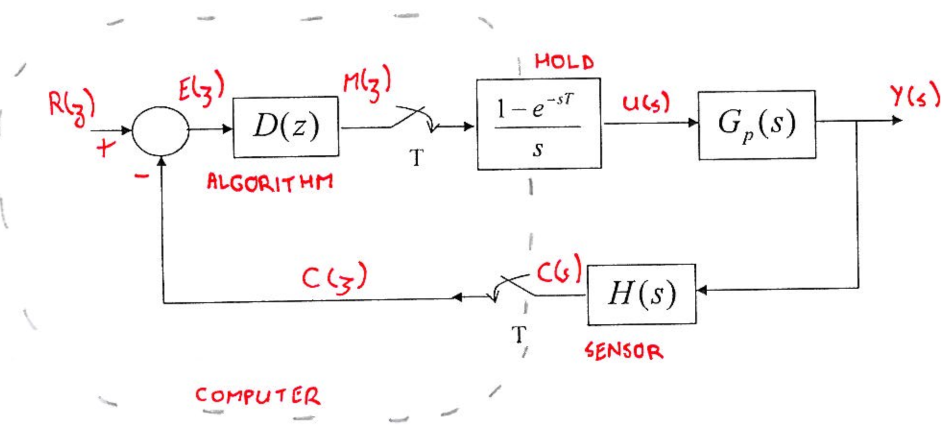


3.3 The Effect of Sampling+Hold on the Spectrum





This will allow the following block diagram to be drawn for a process under digital control:



4.1 The Discrete Time Transfer Function

$$U(z) \longrightarrow \boxed{G(z)} \longrightarrow Y(z)$$

$Y(z) = G(z)U(z)$

Where;

$$\frac{Y(z)}{U(z)} = G(z)$$

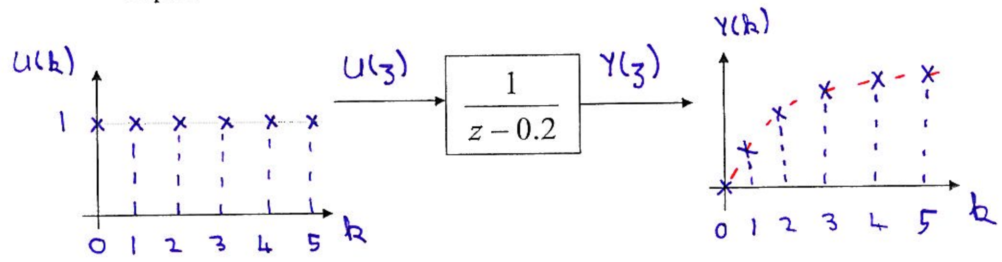
In general for an nth order discrete system we can write:

$$G(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}}$$

$n \geq M$

parameters are constant

Consider the following discrete time system excited by a unit step sequence:



The response can be solved using a number of methods – here we will look at two:

i) Partial Fractions + Tables

The input is a unit step:

$$U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

The output is given in the Z domain as:

$$Y(z) = G(z)U(z) = \frac{1}{z-0.2} \cdot \frac{z}{z-1}$$

Now consider Y(z)/z: → trick to make it easier

$$Y(z)/z = \frac{1}{(z-1)(z-0.2)} = \frac{A}{z-1} + \frac{B}{z-0.2}$$

$$Y(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}} U(z)$$

$$(1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}) U(z)$$

Taking inverse Z transforms, yields:

$$y(k) - a_1 y(k-1) - a_2 y(k-2) \ldots - a_n y(k-n) = b_0 u(k) \ldots + b_m u(k-m)$$

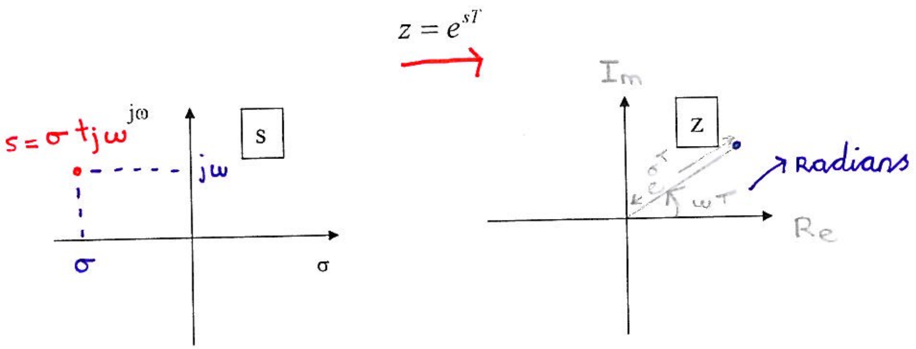
$$y(k) = \sum_{i=1}^{n} a_i y(k-i) + \sum_{j=0}^{m} b_j u(k-j)$$

ARMA MODEL

AUTO REGRESSIVE MOVING AVERAGE

4.2 Stability of Discrete Transfer Functions

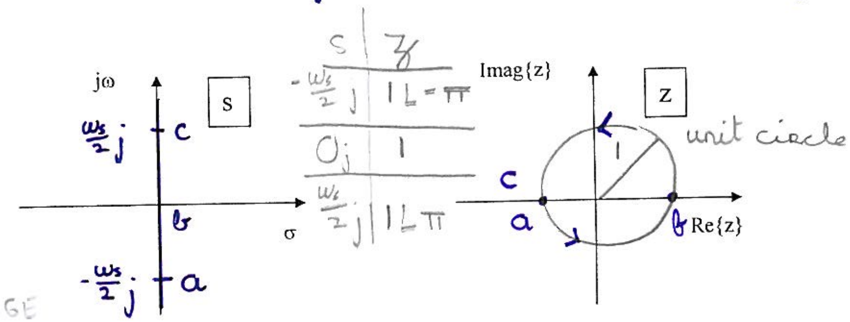
Consider the mapping from the s to the z planes:



$$s = \sigma + j\omega \xrightarrow{e^{sT}} Z = e^{(\sigma + j\omega)T}$$
$$= e^{\sigma T} e^{j\omega T}$$
$$= e^{\sigma T} \angle \omega T$$

N.B. $r \angle \phi = r e^{j\phi}$

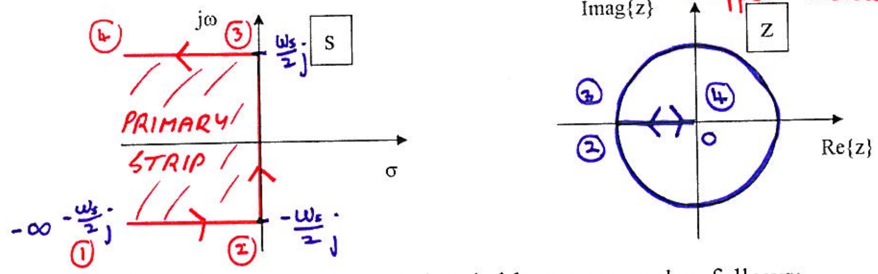
Consider now the mapping of the imaginary axis s=jω from the s plane to the z plane: $s = j\omega \rightarrow e^{j\omega T} = 1 \angle \omega T = 1 \angle \frac{2\pi\omega}{\omega_s}$



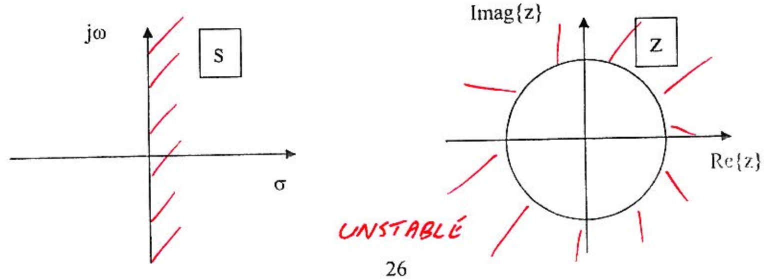
Consider now all points on the left hand side of the s plane:

$\sigma + j\omega \rightarrow e^{\sigma T} \angle \omega T$

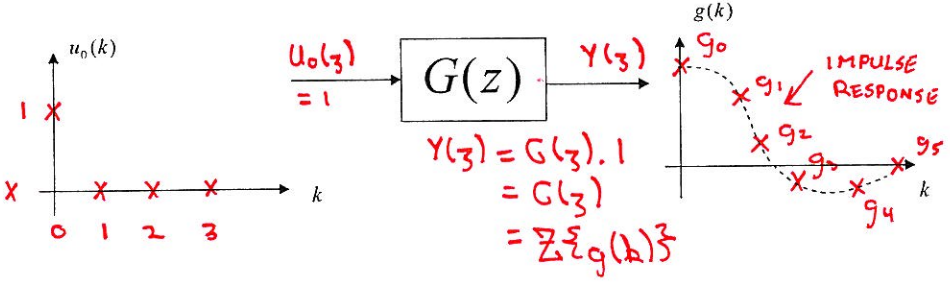
if $\sigma < 0$ $e^{\sigma T} < 1$
all points on LHS of s plane mapped inside unit circle



And all points to the right hand side are mapped as follows:



Consider the following impulse response:



Hence:

$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k}$$

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots \quad \infty \text{ series}$$

Then:

$$Y(z) = G(z)U(z) = \left(g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots\right) U(z)$$

Taking inverse Z transforms yields:

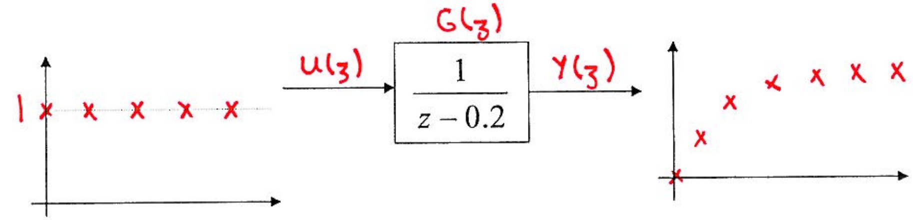
$$y(k) = g_0 u(k) + g_1 u(k-1) + g_2 u(k-2) + \ldots \quad \infty \text{ series}$$

This leads to the discrete-time convolution model:

$$\boxed{y(k) = \sum_{i=0}^{\infty} g_i u(k-i)} \quad \text{where } g_i = g(i)$$

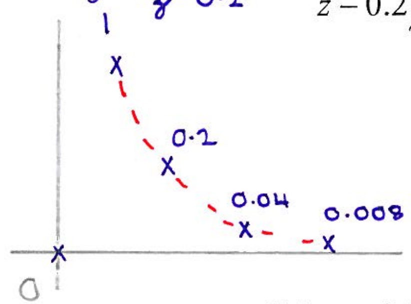
Example:

Consider the following discrete time system excited by a unit step sequence:



First determine the impulse response g(k): - for this example by long division

$$G(z) = \frac{1}{z-0.2}$$



$$z - 0.2 \overline{\big) \; 1 \quad} \quad z^{-1} + 0.2z^{-2} + 0.04z^{-3} + 0.008z^{-4}$$

$$\begin{array}{l} 1 - 0.2z^{-1} \\ \hline 0.2z^{-1} \\ 0.2z^{-1} - 0.04z^{-2} \\ \hline 0.04z^{-2} \\ 0.04z^{-2} - 0.008z^{-3} \\ \hline 0.008z^{-3} \end{array}$$

Hence we get G(z) as an infinite power series

$$G(z) = \overset{+1}{0} + 0.2z^{-1} + 0.04z^{-2} + 0.008z^{-3}$$

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + g_3 z^{-3} + \cdots g_i z^{-i} + \cdots$$

Consider the unit step input:

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Then since:

NON RECURSIVE !!!! (note: requires many past samples