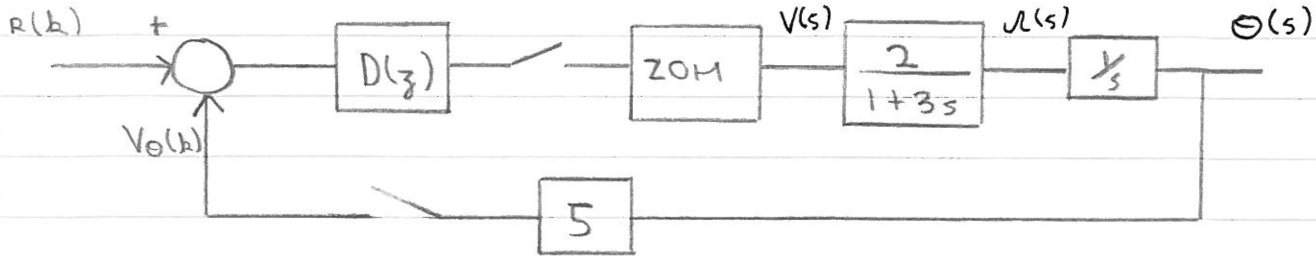
Summer 2004

Q2(c).



$$G(z) = \mathcal{Z}\left\{\frac{1-e^{-sT}}{s}\cdot\frac{2}{1+3s}\cdot\frac{1}{s}\cdot 5\right\}$$

$$= 3{\cdot}33(1-z^{-1})\,\mathcal{Z}\left\{\frac{1}{s^2(s+0{\cdot}33)}\right\} \qquad T = 0{\cdot}1$$

$$= 3{\cdot}33(1-z^{-1})\frac{1}{0{\cdot}33}\left[\frac{0{\cdot}1z^{-1}}{(1-z^{-1})^2} - \frac{(1-e^{-0{\cdot}033})z^{-1}}{0{\cdot}33(1-z^{-1})(1-e^{-0{\cdot}033}z^{-1})}\right]$$

$$= 10(1-z^{-1})\left[\frac{0{\cdot}1z^{-1}}{(1-z^{-1})^2} - \frac{0{\cdot}032z^{-1}}{0{\cdot}33(1-z^{-1})(1-0{\cdot}97z^{-1})}\right]$$

$$= 10\cancel{(1-z^{-1})}\left[\frac{0{\cdot}1z^{-1}(0{\cdot}33)(1-0{\cdot}97z^{-1}) - 0{\cdot}032z^{-1}(1-z^{-1})}{0{\cdot}33(1-z^{-1})^{\cancel{2}}(1-0{\cdot}97z^{-1})}\right]$$

$$= 10\left[\frac{0{\cdot}033z^{-1} - 0{\cdot}032z^{-2} - 0{\cdot}032z^{-1} + 0{\cdot}032z^{-2}}{0{\cdot}33(1-z^{-1})(1-0{\cdot}97z^{-1})}\right]$$

$$= \frac{0{\cdot}003z^{-1}}{(1-z^{-1})(1-0{\cdot}97z^{-1})}$$

$$= \frac{0{\cdot}003z}{(z-1)(z-0{\cdot}97)}$$

23/4/09 Summer 04

Q4 (a). $\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t)$

$s\underline{X}(s) - \underline{x}(0) = A\underline{X}(s) + B\underline{U}(s)$

$(sI - A)\underline{X}(s) = \underline{x}(0) + B\underline{U}(s)$

$\underline{X}(s) = (sI - A)^{-1}(B\underline{U}(s) + \underline{x}(0))$

Zero input response $\Rightarrow \underline{U}(s) = 0$

$\Rightarrow \underline{X}(s) = (sI - A)^{-1}\underline{x}(0)$

Define $\phi(s) = (sI - A)^{-1}$

$\underline{X}(s) = \phi(s)\underline{x}(0)$

$\Rightarrow \underline{x}(t) = \phi(t)\underline{x}(0)$

$\dot{\underline{x}}(t) = A\underline{x}(t) = \frac{d}{dt}(\phi(t)\underline{x}(0)) = \frac{d\phi}{dt}\underline{x}(0)$

$\ddot{\underline{x}}(t) = A^2\underline{x}(t) = \frac{d}{dt}\left(\frac{d\phi}{dt}\underline{x}(0)\right) = \frac{d^2\phi}{dt^2}\underline{x}(0)$

$\dddot{\underline{x}}(t) = A^3\underline{x}(t) = \frac{d}{dt}\left(\frac{d^2\phi}{dt^2}\underline{x}(0)\right) = \frac{d^3\phi}{dt^3}\underline{x}(0)$

$\Rightarrow \frac{d^i}{dt^i}\phi(t) = A^i\phi(t)$

This is true if:

$\phi(t) = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$

Define the matrix exponential as:

$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$

$\Rightarrow \underline{x}(t) = e^{At}\underline{x}(0)$

(ii) The controllability matrix is:

$C_x = [B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{N-1}B]$

Now consider the transformation $\underline{z} = T\underline{x}$

$\underline{z} = T\underline{x} \Rightarrow \underline{x} = T^{-1}\underline{z}$

$\dot{\underline{z}} = T\dot{\underline{x}} \Rightarrow \dot{\underline{x}} = T^{-1}\dot{\underline{z}}$

$\dot{\underline{x}} = A\underline{x} + B\underline{u}$

$\Rightarrow T^{-1}\dot{\underline{z}} = AT^{-1}\underline{z} + B\underline{u}$

$\Rightarrow \dot{\underline{z}} = \underbrace{TAT^{-1}}_{A_2}\underline{z} + \underbrace{TB}_{B_2}\underline{u}$

The controllability matrix of the transformed system is
$C_z = [B_z \,\vdots\, A_z B_z \,\vdots\, A_z^2 B_z \,\vdots\, \cdots \, A_z^{N-1} B_z]$

$B_z = TB$
$A_z B_z = TAT^{-1}TB = TAB$
$A_z^2 B_z = TAT^{-1}TAT^{-1}TB = TA^2B$
$A_z^3 B_z = TAT^{-1}TAT^{-1}TAT^{-1}TB = TA^3B$

$\Rightarrow C_z = [TB \,\vdots\, TAB \,\vdots\, TA^2B \,\vdots\, \cdots \,\vdots\, TA^{N-1}B] = TC_x$

$C_z = TC_x$
$C_z C_x^{-1} = TC_x C_x^{-1}$
$\Rightarrow T = C_z C_x^{-1}$