Control Engineering Summer '07 / Autumn '08

4 c) i) $\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t)$

Taking the Laplace transform:

$$\underline{X}(s) = (sI - A)^{-1}(B\underline{U}(s) + \underline{x}(0))$$

Define $\phi(s) = (sI - A)^{-1}$

$$\underline{X}(s) = \phi(s)\underline{x}(0) + \phi(s)B\underline{U}(s)$$

Taking the inverse Laplace transform:

$$\underline{x}(t) = \phi(t)\underline{x}(0) + \int_0^t \phi(t-\tau)B\underline{u}(\tau)\,d\tau$$

ii) $\phi(s) = (sI - A)^{-1}$

$\phi(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

Consider the zero input response:

$\underline{x}(t) = \phi(t)\underline{x}(0)$

This is the solution to:

$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t)$

Which may be solved as:

$\underline{x}(t) = e^{At}\underline{x}(0)$

$\therefore \phi(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = e^{At}$

We know the solution is $\underline{x}(t) = \phi(t)\underline{x}(0)$

$$\frac{d\underline{x}}{dt} = \frac{d\phi}{dt}\underline{x}(0) \qquad \frac{d\underline{x}}{dt} = A\underline{x}(t) = A\phi(t)\underline{x}(0)$$

$$\frac{d^2\underline{x}}{dt^2} = \frac{d^2\phi}{dt^2}\underline{x}(0) \qquad \frac{d^2\underline{x}}{dt^2} = A\frac{d\underline{x}}{dt} = A^2\phi(t)\underline{x}(0)$$

$$\frac{d^3\underline{x}}{dt^3} = \frac{d^3\phi}{dt^3}\underline{x}(0) \qquad \frac{d^3\underline{x}}{dt^3} = A\frac{d^2\underline{x}}{dt^2} = A^3\phi(t)\underline{x}(0)$$

$$\frac{d^i}{dt^i}\phi(t) = A^i\phi(t)$$

$$\phi(t) = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \dots = e^{At}$$

4 c) iii) The state trajectory is given by:

$$\underline{x}(t) = \phi(t)\underline{x}(0) + \int_0^t \phi(t-\tau) B\underline{u}(\tau)\, d\tau$$

For initial time t_0 & state $\underline{x}(t_0)$

$$x(t) = \phi(t-t_0)\underline{x}(t_0) + \int_{t_0}^t \phi(t-\tau) B\underline{u}(\tau)\, d\tau$$

$$\left.\begin{aligned} t &= (k+1)T \\ t_0 &= (kT) \end{aligned}\right\} \text{for a timestep } T$$

$\underline{u}(\tau)$ is constant across integral with ZOH

$$x(k+1)T = \phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi(t-\tau) B\, d\tau\, \underline{u}(kT)$$

$$= \phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \phi[(k+1)T-\tau] B\, d\tau\, \underline{u}(kT)$$

$\eta = (k+1)T - \tau$

$d\eta = -d\tau$

$$x(k+1)T = \phi(T)\underline{x}(kT) - \int_T^0 \phi(\eta)\, d\eta\, B\underline{u}(kT)$$

$$x(k+1)T = \phi(T)\underline{x}(kT) + \int_0^T \phi(\eta)\, d\eta\, B\underline{u}(kT)$$

Simplify: $(k+1)T \to (k+1)$

$(kT) \to k$

$$x(k+1) = e^{AT}\underline{x}(k) + \int_0^T e^{A\eta}\, d\eta\, B\underline{u}(k)$$

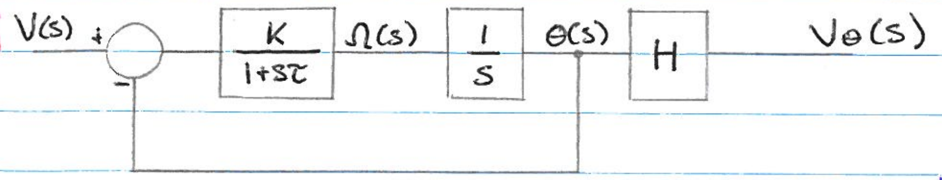
$$x(k+1) = e^{AT}\underline{x}(k) + \tfrac{1}{A}\left(e^{A\eta}\Big|_0^T\right) B\underline{u}(k)$$

$$x(k+1) = e^{AT}\underline{x}(k) + A^{-1}\left(e^{AT} - e^{A0}\right) B\underline{u}(k)$$

$$\therefore \quad x(k+1) = e^{AT}\underline{x}(k) + A^{-1}\left(e^{AT} - 1\right) B\underline{u}(k)$$

Control Engineering Summer '07

4 b) i



$$E(s) = V(s) - \Theta(s) \qquad \Omega(s) = \frac{K E(s)}{1+s\tau}$$

$$e(t) = v(t) - \theta(t) \qquad \Omega(s) + s\tau\Omega(s) = k E(s)$$

$$\Theta(s) = \frac{1}{s}\Omega(s) \qquad \omega(t) + \tau \frac{d\omega(t)}{dt} = k e(t)$$

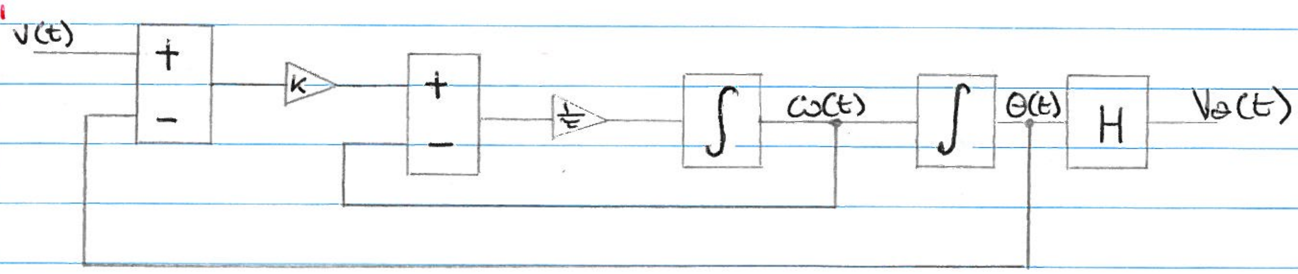
$$s\Theta(s) = \Omega(s)$$

$$\frac{d\theta(t)}{dt} = \omega(t) \qquad \frac{d\omega(t)}{dt} = \frac{k}{\tau}v(t) - \frac{k}{\tau}\theta(t) - \frac{1}{\tau}\omega(t)$$

$$\frac{d}{dt}\begin{bmatrix}\omega(t)\\ \theta(t)\end{bmatrix} = \begin{bmatrix}-\frac{1}{\tau} & -\frac{k}{\tau}\\ 1 & 0\end{bmatrix}\begin{bmatrix}\omega(t)\\ \theta(t)\end{bmatrix} + \begin{bmatrix}\frac{k}{\tau}\\ 0\end{bmatrix} v(t)$$

$$V_\theta(t) = \begin{bmatrix}0 & H\end{bmatrix}\begin{bmatrix}\omega(t)\\ \theta(t)\end{bmatrix}$$

ii



iii) Observability: $O_x = \begin{bmatrix} C \\ CA \end{bmatrix}$

$$CA = \begin{bmatrix}0 & H\end{bmatrix}\begin{bmatrix}-\frac{1}{\tau} & -\frac{k}{\tau}\\ 1 & 0\end{bmatrix} = \begin{bmatrix}H & 0\end{bmatrix}$$

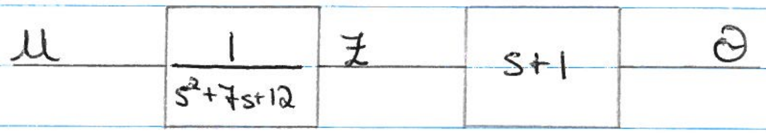
$$O_x = \begin{bmatrix}0 & H\\ H & 0\end{bmatrix} \Rightarrow \det(O_x) = -H^2 \neq 0$$

$\Rightarrow$ States are observable

Control Engineering Summer '06

4 a) i) $$\frac{d^2\Theta(t)}{dt^2} + 7\frac{d\Theta(t)}{dt} + 12\Theta(t) = \frac{du(t)}{dt} + u(t)$$

$$(s^2 + 7s + 12)\Theta_s = (s+1)U(s)$$

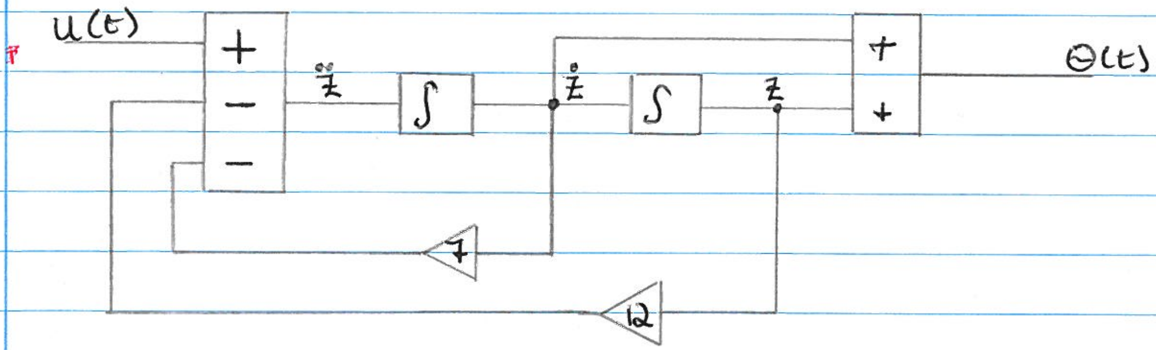


$$\frac{z}{u} = \frac{1}{s^2+7s+12} \qquad\qquad \frac{\Theta}{z} = s+1$$

$$\Rightarrow s^2 z + 7sz + 12z = u \qquad\qquad \Rightarrow \Theta = sz + z$$

$$\Rightarrow \ddot{z} + 7\dot{z} + 12z = u \qquad\qquad \Rightarrow \Theta = \dot{z} + z$$

$$\ddot{z} = u - (7\dot{z} + 12z)$$



ii) $$\frac{d}{dt}\begin{bmatrix} z \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}\begin{bmatrix} z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\Theta(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} z \\ \dot{z} \end{bmatrix}$$

iii) $$\Theta(0) = z_0 + \dot{z}_0 = 1$$

$$\dot{\Theta}(0) = \dot{z}_0 + \ddot{z}_0 = 0$$

$$\Rightarrow \dot{z}_0 + u - 7\dot{z}_0 - 12z_0 = 0$$

Zero I/P $\rightarrow u = 0$ $\Rightarrow -6\dot{z}_0 - 12z_0 = 0$

$$\dot{z}_0 + z_0 = 1$$

$$\Rightarrow -6z_0 = 6 \Rightarrow z_0 = -1$$

$$\dot{z} = 2$$

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4 a iii) $\underline{x}(s) = \phi(s)\,\underline{x}(0) = \phi(s)\begin{bmatrix}-1\\2\end{bmatrix}$

$$\phi(s) = (sI-A)^{-1}$$

$$= \begin{bmatrix}s & -1\\ +12 & s+7\end{bmatrix}^{-1} = \frac{1}{s(s+7)+12}\begin{bmatrix}s+7 & +1\\ -12 & s\end{bmatrix}$$

$$\phi(t) = \mathcal{L}^{-1}\phi_s = \begin{bmatrix} \mathrm{I}\ \mathcal{L}^{-1}\left\{\frac{s+7}{s^2+7s+12}\right\} & \mathrm{II}\ \mathcal{L}^{-1}\left\{\frac{1}{s^2+7s+12}\right\} \\ \mathrm{III}\ \mathcal{L}^{-1}\left\{\frac{-12}{s^2+7s+12}\right\} & \mathrm{IV}\ \mathcal{L}^{-1}\left\{\frac{s}{s^2+7s+12}\right\}\end{bmatrix}$$

$$\frac{A}{s+3} + \frac{B}{s+4}$$

I $sA + 4A + sB + 3B = s+7$

$A+B=1 \quad 4A+3B=7$

$A=4 \quad B=-3$

I $\mathcal{L}^{-1}\left[\frac{4}{s+3} - \frac{3}{s+4}\right]$

$= 4e^{-3t} - 3e^{-4t}$

II $A+B=0 \quad 4A+3B=1$

$A=1 \quad B=-1$

II $\mathcal{L}^{-1}\left[\frac{1}{s+3} - \frac{1}{s+4}\right]$

$= e^{-3t} - e^{-4t}$

III $A+B=0 \quad 4A+3B=-12$

$A=-12 \quad B=12$

III $-12e^{-3t} + 12e^{-4t}$

IV $A+B=1 \quad 4A+3B=0$

$A=-3 \quad B=4$

IV $-3e^{-3t} + 4e^{-4t}$

$$\phi(t) = \begin{bmatrix}4e^{-3t} - 3e^{-4t} & e^{-3t} - e^{-4t}\\ -12e^{-3t} + 12e^{-4t} & -3e^{-3t} + 4e^{-4t}\end{bmatrix}$$

$$\dot{\underline{x}}(t) = \begin{bmatrix}-2e^{-3t} + e^{-4t}\\ 6e^{-3t} - 4e^{-4t}\end{bmatrix}$$

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4 b) i) $$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B$$
$$= \begin{bmatrix} 1 & 2 \end{bmatrix}\left(\begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{1}{(s+1)(s+2)} = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} s+2 \\ 2s+2 \end{bmatrix}\frac{1}{(s+1)(s+2)}$$

$$= \frac{5s+6}{(s+1)(s+2)} = \frac{5s+6}{s^2+3s+2}$$

ii) $$C_x = [B \,\vdots\, AB] \qquad AB = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

$$C_x = \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix} \qquad \det(C_x) = -2 \neq 0$$

$\Rightarrow$ System is controllable

iii) CCF

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 6 & 5 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \qquad C_2 = T C_x \qquad T = C_2 C_x^{-1}$$

$$C_x^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 1 \\ -2 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} +2 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix}$$

Control Engineering Summer '05

4 a) $\frac{d}{dt}\underline{x}(t) = A\,\underline{x}(t) + B\underline{u}(t)$

Taking the Laplace Transform:

$s\underline{X}(s) - \underline{X}(0) = A\underline{X}(s) + B\underline{U}(s)$

$\underline{X}(s) = (sI - A)^{-1}(B\underline{U}(s) + \underline{x}(0))$

Let $\phi(s) = (sI - A)^{-1}$

$\underline{X}(s) = \phi(s)\underline{x}(0) + \phi(s)B\underline{U}(s)$

* Taking the inverse Laplace Transform

$\underline{x}(t) = \phi(t)\underline{x}(0) + \phi(t) \circledast B\underline{u}(t)$

$= \phi(t)\underline{x}(0) + \int_0^t \phi(t-\tau)B\underline{u}(\tau)\,d\tau$

Taking the zero-input response:

$\underline{x}(t) = \phi(t)\underline{x}(0)$

This is the solution to:

$\frac{d}{dt}\underline{x}(t) = A\,\underline{x}(t)$

Which may also be solved:

$x(t) = e^{At}x(0)$

This indicates:

$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) = A\phi(t)x(0) = \frac{d\phi}{dt}x(0)$

$\frac{d^2}{dt^2}\underline{x}(t) = A^2x(t) = A^2\phi(t)x(0) = \frac{d^2\phi}{dt^2}x(0)$

$\frac{d^3}{dt^3}\underline{x}(t) = A^3x(t) = A^3\phi(t)x(0) = \frac{d^3\phi}{dt^3}x(0)$

$\frac{d^i}{dt^i}\phi(t) = A^i\phi(t)$

$\phi(t) = I + \frac{At}{1!} + \frac{At}{2!} + \dots$

$= e^{At}$

$\Rightarrow \underline{x}(t) = e^{At}\underline{x}(0) + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)\,d\tau$

$= e^{At}\left[\int_0^t e^{-A\tau}B\underline{u}(\tau)\,d\tau + \underline{x}(0)\right]$

Control Engineering Summer '05

4 a) The state trajectory is given by:

$$\underline{x}(t) = \phi(t)\underline{x}(0) + \int_0^t \phi(t-\tau) B \underline{u}(\tau) d\tau$$

For initial time t_0 & state $\underline{x}(t_0)$

$$\underline{x}(t) = \phi(t-t_0) x(t_0) + \int_{t_0}^{t} \phi(t-\tau) B \underline{u}(\tau) d\tau$$

let $\left.\begin{aligned} t &= (k+1)T \\ t_0 &= kT \end{aligned}\right\}$ For a sample time T

$$\underline{x}(k+1)T = \phi(T) x(kT) + \int_{kT}^{(k+1)T} \phi[(k+1)T-\tau] B \underline{u}(\tau) d\tau$$

For ZOH, $\underline{u}(\tau) = \underline{u}(kT)$ across the integral

$$\underline{x}(k+1)T = \phi(T) x(kT) + \int_{kT}^{(k+1)T} \phi[(k+1)T-\tau] B \underline{u}(kT) d\tau$$

$\eta = (k+1)T - \tau$

$d\eta = -d\tau$

$$x(k+1)T = \phi(T) x(kT) + \int_0^T \phi(\eta) d\eta \, B \underline{u}(kT)$$

$$A_d = \phi(T) = e^{AT} = I + \frac{AT}{1!} + \frac{A^2T^2}{2!} + \cdots \qquad B_d = \int_0^T e^{A\eta} d\eta \, B = A^{-1}(e^{AT} - I) B$$

$$= A^{-1}(A_d - I) B$$

$$= A^{-1}(I + AT - I) B$$

$$= TB$$

Simplify $(k+1)T \to k+1$

$kT \to k$

$$\underline{x}(k+1) = A_d \underline{x}(k) + TB\underline{u}(k)$$

$$= (I + AT)\underline{x}(k) + TB\underline{u}(k)$$

$$\underline{x}(k+1) - \underline{x}(k) = AT\underline{x}(k) + BT\underline{u}(k)$$

$$\frac{\Delta \underline{x}(k+1)}{T} = A\underline{x}(k) + B\underline{u}(k)$$

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4 b) $\frac{d}{dt}\underline{x}(t) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \underline{x}(t)$

The eigenvalues are the roots of:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a & 0 \\ 0 & \lambda - b \end{vmatrix}$$

$$\Rightarrow (\lambda - a)(\lambda - b)$$

$$\lambda_1 = a \quad \lambda_2 = b$$

Since $N = 2$

$$e^{At} = \alpha_0(t) I + \alpha_1(t) A$$

$$e^{at} = \alpha_0(t) I + \alpha_1(t) a$$

$$e^{bt} = \alpha_0(t) I + \alpha_1(t) b$$

$$e^{at} - e^{bt} = \alpha_1(t)(a - b)$$

$$\alpha_1(t) = \frac{e^{at} - e^{bt}}{a - b} \qquad \alpha_0(t) I = e^{at} - \frac{a(e^{at} - e^{bt})}{a - b}$$

$$e^{At} = e^{at} - \frac{a(e^{at} - e^{bt})}{a - b} + \frac{e^{at} - e^{bt}}{a - b} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

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4 c) $U(s) \rightarrow \boxed{\frac{1}{s+1}} \xrightarrow{\Omega(s)} \boxed{\frac{1}{s}} \rightarrow \Theta(s)$

$$s\Omega(s) + \Omega(s) = U(s) \qquad s\Theta(s) = \Omega(s)$$

$$\frac{d\omega}{dt} = u(t) - \omega(t) \qquad \frac{d\theta}{dt} = \omega(t)$$

$$\frac{d}{dt}\begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$U(s) = C(s)E(s)$$

$$E(s) = \Theta_d(s) - \Theta(s)$$

$$U(s) = C(s)(\Theta_d(s) - \Theta(s))$$

$$= (Ks + Kz)(\Theta_d(s) - \Theta(s))$$

$$= Ks\Theta_d(s) - Ks\Theta(s) + Kz\Theta_d(s) - Kz\Theta(s)$$

$$\Rightarrow u(t) = K\frac{d\theta_d(t)}{dt} - K\frac{d\theta(t)}{dt} + Kz\,\theta_d(t) - Kz\,\theta(t)$$

For a constant setpoint,

$$\frac{d\theta_d(t)}{dt} = 0$$

$$u(t) = Kz\,\theta_d(t) - Kz\,\theta(t) - K\frac{d\theta(t)}{dt}$$

$$= Kz\,\theta_d(t) - \begin{bmatrix} Kz & K \end{bmatrix} \begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix}$$

$$C_{des} \Rightarrow \det(sI - A + BK)$$

$$\det\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ Kz & K \end{pmatrix}\right] = 0$$

$$\det\begin{bmatrix} s & -1 \\ Kz & (s+k)-1 \end{bmatrix} = 0 \qquad s^2 + ks - s + Kz = 0$$

$$s^2 + (k-1)s + Kz = 0$$

$$C_{des} = (s + (2+2j))(s + (2-2j))$$

$$= s^2 + [(2-2j) + (2+2j)]s + 4 - 4j^2$$

$$= s^2 + 4s + 8 \qquad k - 1 = 4 \qquad Kz = 8$$

$$k = 5 \qquad z = 1{\cdot}6, \; Kz = 8$$