EE4011: RF IC Design

Transmission Lines – A Reminder

Wavelengths and Frequencies

A transmission line is a physical medium which is used to carry power from a source to a load. The most familiar transmission line used to carry high frequency signals is a co-axial cable. When the wavelength of the ac signal is comparable to or less than the dimensions of the components or interconnections in a system then the wave-like propagation properties of the high-frequency signal cannot be ignored.

$$\lambda = \frac{v}{f} = \frac{c}{f\sqrt{\varepsilon_r}} = \frac{3\times10^8}{f\sqrt{\varepsilon_r}}$$

c= velocity of em wave in vacuum
v = velocity of em wave in other dielectric
f = frequency λ = wavelength ε_r = dielectric constant

Wavelengths of signals at various frequencies (in vacuum):

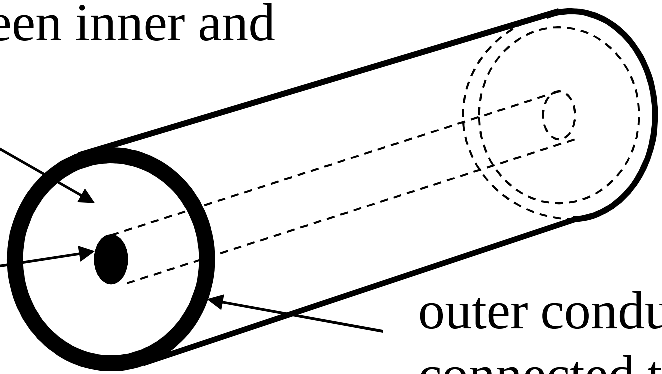
					microwaves				sub-mm
f:	<300kHz	300kHz	3MHz	30MHz	300MHz	3GHz	30GHz	300GHz	>300GHz
λ:	>1km	1km	100m	10m	1m	10cm	1cm	1mm	<1mm

Frequencies between 300MHz and 300GHz are designated “microwave” and have associated wavelengths of between 1mm and 1m which are comparable to component and interconnect dimensions. Frequencies above 300GHz are designated “sub-millimetre” because their wavelengths are less than 1mm.

Sample Transmission Line: A Coaxial Cable

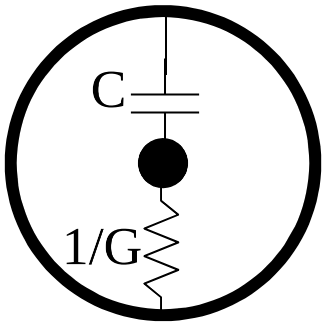
insulating material between inner and outer conductors

centre conductor connected to signal

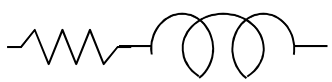


outer conductor usually connected to ground

Cross-section:



Along Length:



There are 4 important quantities determining the propagation of a signal on a co-ax cable. They are all specified *per unit length* of the line:

C: The capacitance between the inner and outer conductors

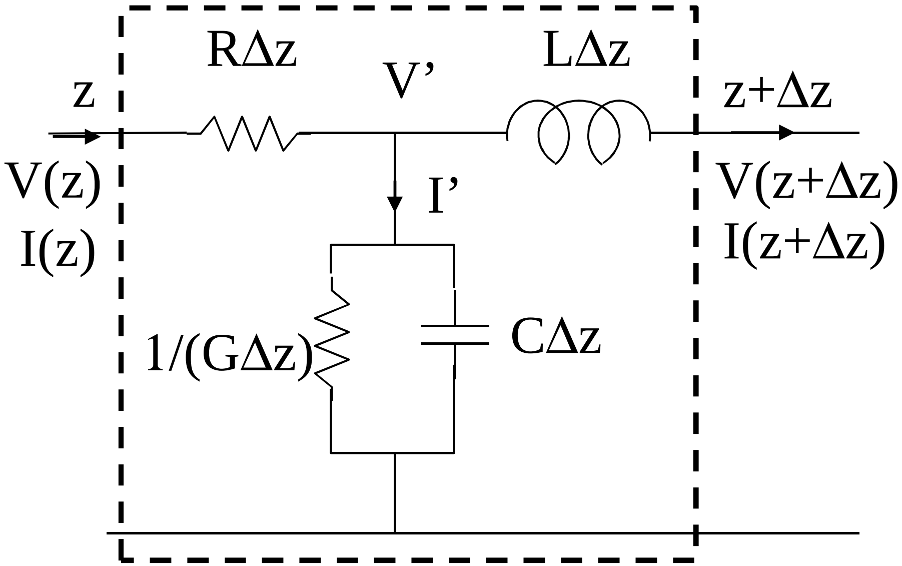
G: The DC conductance (leakage) between the inner and outer conductors

R: The resistance along the line

L: The inductance along the line

Transmission Line Equivalent Circuit

An infinitesimal length of transmission line (length Δz) can be represented by the following equivalent circuit and the full transmission line can be considered to be made up of an infinite set of these sections cascaded together.



A more symmetrical equivalent circuit with half of the resistance and inductance placed at each side of the midpoint would be a better physical representation but this circuit is good enough to derive the transmission line equations.

The complex impedance per unit length of the transmission line is: $Z = R + j\varpi L$

The complex admittance per unit length of the transmission line is: $Y = G + j\varpi C$

The characteristic impedance of the transmission line is Z_0: $Z_0 = \sqrt{\dfrac{Z}{Y}} = \sqrt{\dfrac{R + j\varpi L}{G + j\varpi C}}\ \Omega$

Transmission Line Equations - 1

$$V' = V(z) - I(z)R\Delta z$$

$$V(z+\Delta z) = V' - I(z+\Delta z)\, j\varpi L \Delta z$$

$$\Rightarrow V(z+\Delta z) = V(z) - I(z)R\Delta z - I(z+\Delta z)\, j\varpi L\Delta z$$

$$\Rightarrow \frac{V(z+\Delta z) - V(z)}{\Delta z} = -RI(z) - j\varpi L\, I(z+\Delta z)$$

$$\Rightarrow \frac{V(z+\Delta z) - V(z)}{\Delta z} = -RI(z) - j\varpi L\left[I(z) + \frac{dI}{dz}\Delta z \right]$$

Taking first two terms of the Taylor expansion of I(z+Δz)

$$\Rightarrow \frac{V(z+\Delta z) - V(z)}{\Delta z} = -\left(R + j\varpi L\right)I(z) - j\varpi L \frac{dI}{dz}\Delta z$$

as $\Delta z \to 0$:

$$\frac{dV}{dz} = -\left(R + j\varpi L\right)I(z) = -Z\, I(z)$$

Transmission line equations - 2

$$V' = V(z) - I(z)R\Delta z$$

$$I' = V'(G + j\varpi C)\Delta z = \left(V(z) - I(z)R\Delta z\right)(G + j\varpi C)\Delta z$$

$$I(z+\Delta z) = I(z) - I'$$

$$\Rightarrow I(z+\Delta z) = I(z) - \left(V(z) - I(z)R\Delta z\right)(G + j\varpi C)\Delta z$$

$$\Rightarrow \frac{I(z+\Delta z) - I(z)}{\Delta z} = -(G + j\varpi C)V(z) + (G + j\varpi C)I(z)R\Delta z$$

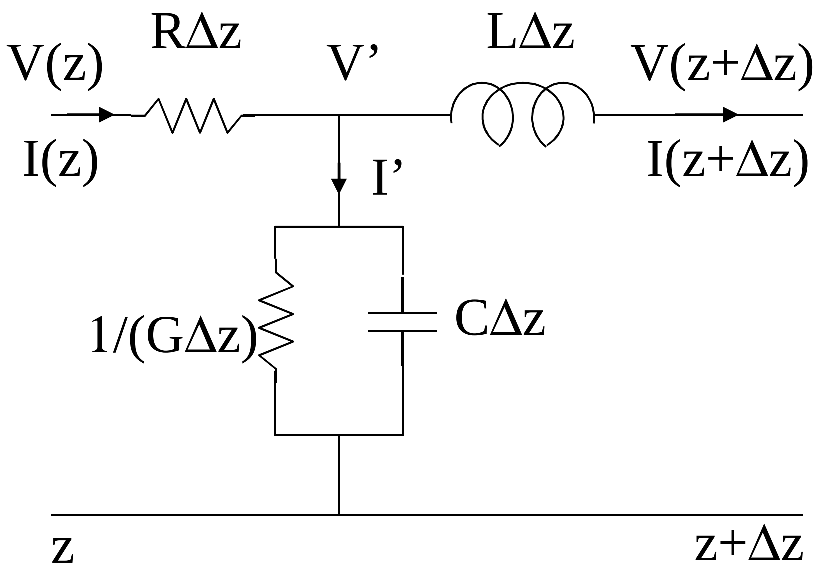
$$\text{as} \quad \Delta z \to 0:$$

$$\frac{dI}{dz} = -\left(G + j\varpi C\right)V(z) = -YV(z)$$

Transmission line equations - 3

$$\frac{dV}{dz} = -ZI$$

$$\frac{dI}{dz} = -YV \qquad \text{(just derived)}$$



taking derivatives again w.r.t. z :

$$\frac{d^2V}{dz^2} = -Z\frac{dI}{dz} = -Z(-YV) = ZYV = \gamma^2 V$$

$$\frac{d^2I}{dz^2} = -Y\frac{dV}{dz} = -Y(-ZI) = ZYI = \gamma^2 I$$

where

$$\gamma = \sqrt{ZY} = \sqrt{(R + j\varpi L)(G + j\varpi C)}$$

These equations show that the second derivatives of voltage and current are directly proportional to voltage and current respectively. The proportionality constant is γ^2. γ is the propagation constant of the transmission line. These equations are often referred to as the Telegrapher's equations and were developed in their original form by Oliver Heaviside, an English scientist who was a telegraph operator in his early years.

Solution to Transmission Line Equations

A general solution to the transmission line equations is given by the sum of exponentials:

$$V = V_i e^{-\gamma z} + V_r e^{\gamma z} \qquad I = I_i e^{-\gamma z} + I_r e^{\gamma z}$$

$$I = I_i e^{-\gamma z} + I_r e^{\gamma z} \Rightarrow \frac{dI}{dz} = -\gamma I_i e^{-\gamma z} + \gamma I_r e^{\gamma z} \qquad \text{But}: \frac{dI}{dz} = -YV = -YV_i e^{-\gamma z} - YV_r e^{\gamma z}$$

Comparing coefficients in the two expressions for dI/dz :

$$-YV_i = -\gamma I_i \Rightarrow I_i = V_i \frac{Y}{\gamma} = V_i \frac{G + j\varpi C}{\sqrt{(R + j\varpi L)(G + j\varpi C)}} = V_i \sqrt{\frac{G + j\varpi C}{R + j\varpi L}} = \frac{V_i}{Z_0}$$

$$-YV_r = \gamma I_r \Rightarrow I_r = -V_r \frac{Y}{\gamma} = -V_r \frac{G + j\varpi C}{\sqrt{(R + j\varpi L)(G + j\varpi C)}} = -V_r \sqrt{\frac{G + j\varpi C}{R + j\varpi L}} = -\frac{V_r}{Z_0}$$

So:

$$V = V_i e^{-\gamma z} + V_r e^{\gamma z} \qquad I = \frac{V_i}{Z_0} e^{-\gamma z} - \frac{V_r}{Z_0} e^{\gamma z}$$

Incident and Reflected Waves

z $\longrightarrow$

Transmission Line

$$V = V_i e^{-\gamma z} + V_r e^{\gamma z} \qquad Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\varpi L}{G + j\varpi C}}\ \Omega$$

$$I = \frac{V_i}{Z_0} e^{-\gamma z} - \frac{V_r}{Z_0} e^{\gamma z} \qquad \gamma = \sqrt{ZY} = \sqrt{(R + j\varpi L)(G + j\varpi C)} = \alpha + j\beta$$

α is the attenuation constant in nepers/m and β is the phase constant in radians/m

This shows that on a transmission line the voltage can be considered to be the sum of two voltage waveforms – one travelling in the positive z direction (the incident wave) and another travelling in the negative z direction (the reflected wave). This also applies to the current waveforms. Additionally, the current waveforms are related to the voltage waveforms through the characteristic impedance as shown. The quantities V_i, I_i etc. are *phasors* i.e. they are implicitly a function of time:

$$V_i = V_{mi} \cos(\varpi t + \theta_i)$$

Therefore, the voltage and current at any point on a transmission line are functions of *both* time and distance.

Transmission Line Attenuation

Simplified case where there is no reflected wave:

$$V = V_i e^{-\gamma z} = V_i e^{-(\alpha + j\beta)z}$$

V_i is the phasor voltage at z = 0 i.e. $V_i = V_{mi} e^{j\varphi}$

$$\Rightarrow V = V_{mi} e^{j\varphi} e^{-(\alpha + j\beta)z} = V_{mi} e^{-\alpha z} e^{j(\varphi - \beta z)}$$

$$\Rightarrow V = V_{mi} e^{-\alpha z} \angle(\varphi - \beta z)$$

This shows that the magnitude of the wave as it travels in the z direction is reduced by a factor $e^{-\alpha z}$ from its value at z=0 and the phase difference between z=0 and any other z is $-\beta z$. The signal attenuation factor over a distance of 1m (z=1) is:

$$f = e^{-\alpha}$$

This attenuation per unit length is often expressed in dB/m i.e.

$$\alpha_{dB/m} = 20 \log_{10}(e^{-\alpha}) = -\alpha 20 \log_{10} e \Rightarrow \left|\alpha_{dB/m}\right| = 8.686\alpha$$

Common Simplifications & Assumptions

$$V = V_i e^{-\gamma z} + V_r e^{\gamma z}$$

$$\gamma = \sqrt{ZY} = \sqrt{(R + j\varpi L)(G + j\varpi C)} = \alpha + j\beta \qquad Z_0 = \sqrt{\frac{R + j\varpi L}{G + j\varpi C}}$$

At high frequencies $\varpi L >> R$ and $\varpi C >> G$

So

$$\gamma \approx \sqrt{(j\varpi L)(j\varpi C)} = j\varpi\sqrt{LC} \Rightarrow \alpha \approx 0 \quad \text{and} \quad \beta \approx \varpi\sqrt{LC}$$

and

$$Z_0 \approx \sqrt{\frac{j\varpi L}{j\varpi C}} = \sqrt{\frac{L}{C}}$$

i.e. at sufficiently high frequencies γ is imaginary and Z_0 is real.

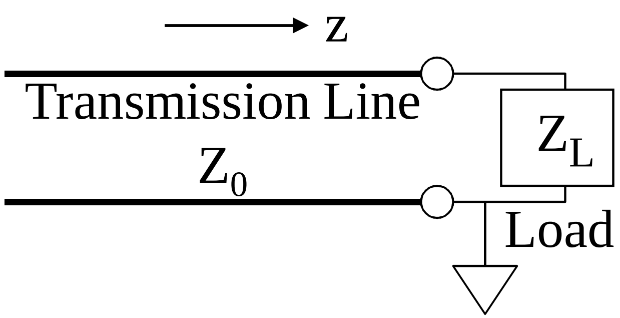
$$Z_0 = \sqrt{\frac{L}{C}} \Rightarrow \sqrt{L} = Z_0\sqrt{C} \Rightarrow \beta = \varpi\sqrt{LC} = \varpi Z_0 C \qquad \text{rad/m}$$

The phase velocity is defined as:

$$v = \frac{\varpi}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{Z_0 C} \qquad \text{m/s}$$

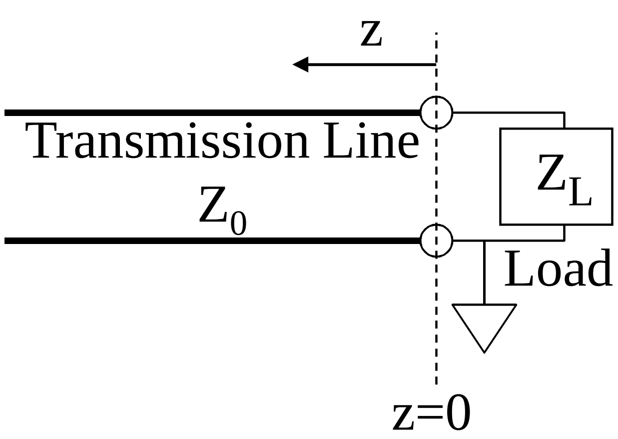
Having the various quantities defined in terms of Z_0 and C is useful because Z_0 and C are often specified.

Shifting the point of reference to the load



$$V = V_i e^{-\gamma z} + V_r e^{\gamma z} \qquad I = \frac{V_i}{Z_0} e^{-\gamma z} - \frac{V_r}{Z_0} e^{\gamma z}$$

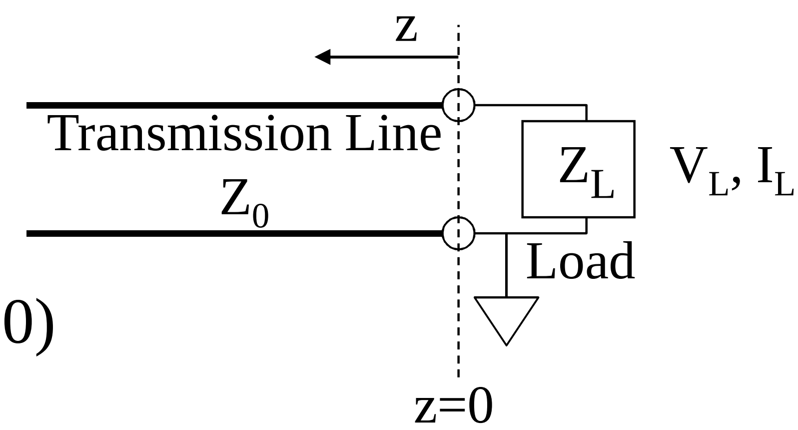
In many cases it is convenient to define the origin (z=0) at the load and consider distance z to be measured *away from* the load. In this case the signs in the exponentials have to be reversed:



$$V = V_i(0) e^{\gamma z} + V_r(0) e^{-\gamma z}$$

$$I = \frac{V_i(0)}{Z_0} e^{\gamma z} - \frac{V_r(0)}{Z_0} e^{-\gamma z}$$

Reflection coefficient at the load



Using Ohm's law at the load: $V_L = Z_L I_L$

Using the transmission line equations at the load (z=0):

$$V_L = V_i(0) + V_r(0) \Rightarrow Z_L I_L = V_i(0) + V_r(0)$$

$$I_L = \frac{V_i(0)}{Z_0} - \frac{V_r(0)}{Z_0} \Rightarrow Z_0 I_L = V_i(0) - V_r(0)$$

Dividing the last two equations into each other: $\dfrac{Z_L}{Z_0} = \dfrac{V_i(0) + V_r(0)}{V_i(0) - V_r(0)}$

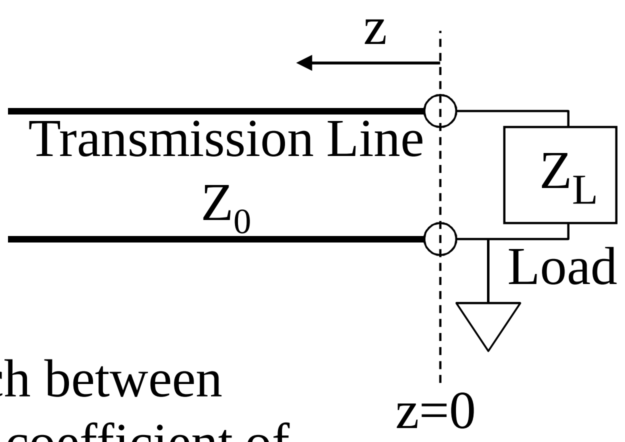
The reflection coefficient at the load is defined as: $\Gamma = \dfrac{V_r(0)}{V_i(0)} \Rightarrow V_r(0) = \Gamma V_i(0)$

Putting this into the previous equation:

$$\frac{Z_L}{Z_0} = \frac{V_i(0) + V_r(0)}{V_i(0) - V_r(0)} = \frac{V_i(0) + \Gamma V_i(0)}{V_i(0) - \Gamma V_i(0)} = \frac{1+\Gamma}{1-\Gamma} \Rightarrow \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Impedance Match and Line Voltage

At the load $\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$



Γ is a complex number which can be written as:

$$\Gamma = |\Gamma| e^{j\theta}$$

If $Z_L = Z_0$, $\Gamma=0$ and there is a perfect impedance match between the transmission line and the load giving a reflection coefficient of zero and no reflected wave. If Z_L is not the same as Z_0, there is a reflected wave and the voltage along the transmission line is:

$$V(z) = V_i(0)e^{\gamma z} + \Gamma V_i(0)e^{-\gamma z}$$

$$\Rightarrow V(z) = V_i(0)e^{\gamma z}\left(1 + \Gamma e^{-2\gamma z}\right) = V_i(0)e^{\gamma z}\left(1 + |\Gamma| e^{j\theta} e^{-2\gamma z}\right)$$

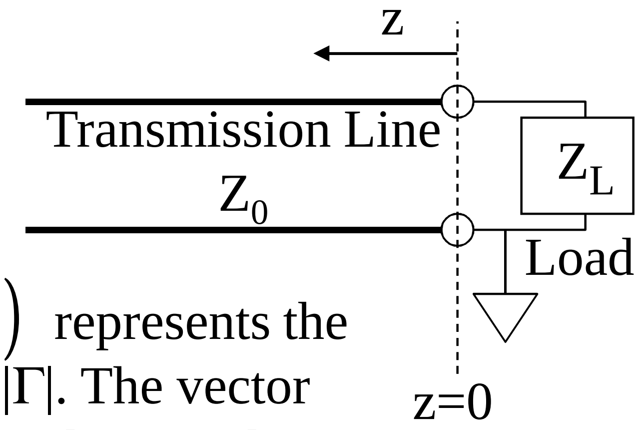
For a lossless line $\gamma = j\beta$ so

$$V(z) = V_i(0)e^{j\beta z}\left(1 + |\Gamma| e^{j\theta} e^{-2j\beta z}\right) = V_i(0)e^{j\beta z}\left(1 + |\Gamma| e^{-j(2\beta z - \theta)}\right)$$

Voltage Standing Wave Ratio (VSWR)

The voltage at any point along the transmission line is:

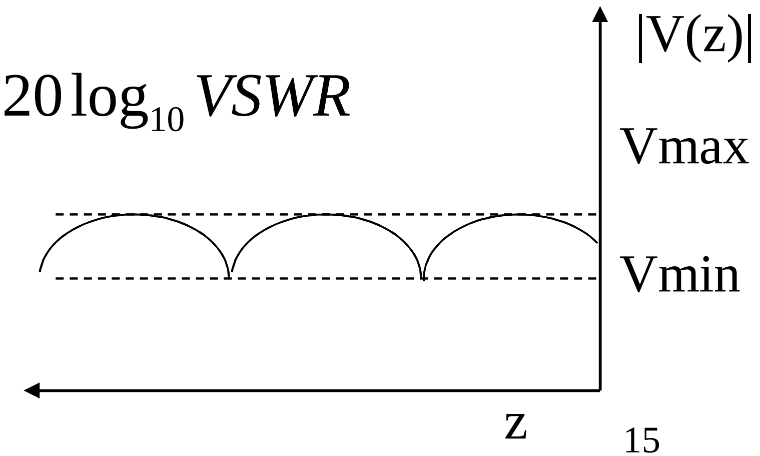
$$V(z) = V_i(0)e^{j\beta z}\left(1+|\Gamma|e^{-j(2\beta z-\theta)}\right)$$



On the complex plane the quantity $\left(1+|\Gamma|e^{-j(2\beta z-\theta)}\right)$ represents the set of points on a circle with centre (1,0) and radius $|\Gamma|$. The vector length from the origin to these points will have min and max values when $e^{-j(2\beta z-\theta)}$ is –1 or +1 respectively giving rise to min and max $|V(z)|$:

$$|V(z)|_{\min} = |V_i(0)|\left(1-|\Gamma|\right) \quad |V(z)|_{\max} = |V_i(0)|\left(1+|\Gamma|\right)$$

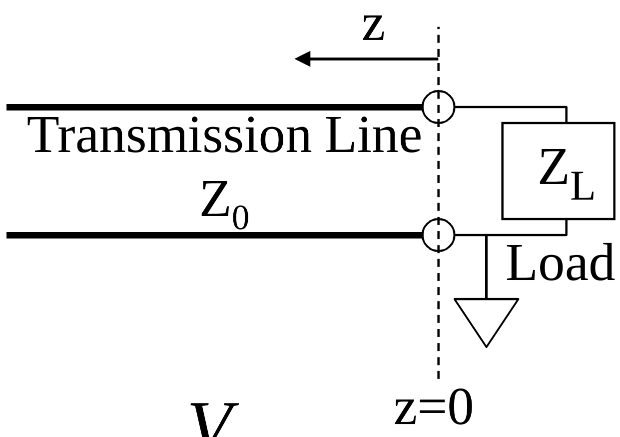
$$VSWR = \frac{|V(z)|_{\max}}{|V(z)|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad VSWR_{dB} = 20\log_{10} VSWR$$



The ratio of the max and min values is called the Voltage Standing Wave Ratio (VSWR).
If Γ=0, VSWR=1 i.e. the amplitude is constant along the line.

Power Transfer

For a lossless line the power carried by the incident and reflected waves is related to the root mean square (r.m.s.) voltages and currents and the power flowing towards the load is the difference between the incident and reflected powers:



$$P = P_i - P_r = V_{i_{rms}} I_{i_{rms}} - V_{r_{rms}} I_{r_{rms}} = V_{i_{rms}} \frac{V_{i_{rms}}}{Z_0} - V_{r_{rms}} \frac{V_{r_{rms}}}{Z_0}$$

$$P = \frac{1}{Z_0}\left(V_{i_{rms}}^2 - V_{r_{rms}}^2\right) = \frac{V_{i_{rms}}^2}{Z_0}\left(1 - \frac{V_{r_{rms}}^2}{V_{i_{rms}}^2}\right)$$

This shows that for maximum power transfer to the load $\Gamma=0$ i.e. $Z_L=Z_0$

$$V_{i_{rms}} = \frac{V_i}{\sqrt{2}} \quad V_{r_{rms}} = \frac{V_r}{\sqrt{2}} \Rightarrow \frac{V_{r_{rms}}^2}{V_{i_{rms}}^2} = \frac{V_r^2}{V_i^2} = |\Gamma|^2 \Rightarrow P = \frac{V_{i_{rms}}^2}{Z_0}\left(1 - |\Gamma|^2\right)$$

Wave Variables

$$P_i = \frac{V_{i_{rms}}^2}{Z_0} = \frac{1}{Z_0}\left(\frac{V_i}{\sqrt{2}}\right)^2 = \frac{V_i^2}{2Z_0} \qquad P_r = \frac{V_{r_{rms}}^2}{Z_0} = \frac{1}{Z_0}\left(\frac{V_r}{\sqrt{2}}\right)^2 = \frac{V_r^2}{2Z_0}$$

Wave variables a and b can be defined such that the power in the incident and reflected waves can be calculated from the square of these wave variables:

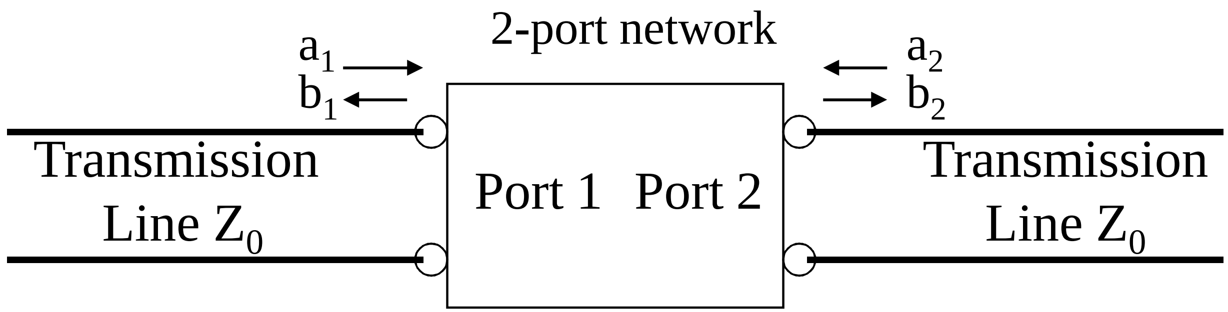
$$a = \frac{V_i}{\sqrt{Z_0}} \Rightarrow P_i = \frac{1}{2}|a|^2 \qquad b = \frac{V_r}{\sqrt{Z_0}} \Rightarrow P_r = \frac{1}{2}|b|^2$$

The net power flow towards the load is then:

$$P = P_i - P_r = \frac{1}{2}\left(|a|^2 - |b|^2\right)$$

The wave variable a is associated with the incident wave and the wave variable b is associated with the reflected wave. For given incident and reflected voltages the wave variables depend on the characteristic impedance Z_0.

Wave variables for a two-port network



$$a_1 = \frac{V_{i1}}{\sqrt{Z_0}} \qquad b_1 = \frac{V_{r1}}{\sqrt{Z_0}}$$

$$a_2 = \frac{V_{i2}}{\sqrt{Z_0}} \qquad b_2 = \frac{V_{r2}}{\sqrt{Z_0}}$$

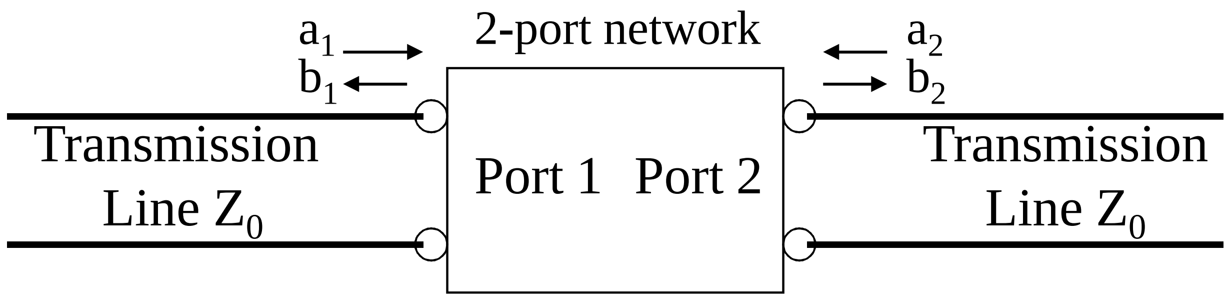
Note the directions of a and b on port 1 and port 2

V_{i1} and V_{i2} are the voltage waves *travelling towards* ports 1 and 2 respectively and V_{r1} and V_{r2} are the voltage waves *travelling away* from ports 1 and 2 respectively.

The net power *flowing into* ports 1 and 2 is given by:

$$P_1 = \frac{1}{2}\left(|a_1|^2 - |b_1|^2\right) \qquad P_2 = \frac{1}{2}\left(|a_2|^2 - |b_2|^2\right)$$

Scattering (s-) parameters for a two-port



The scattering parameters for a two-port network give the relationship between the reflected and the incident wave variables:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

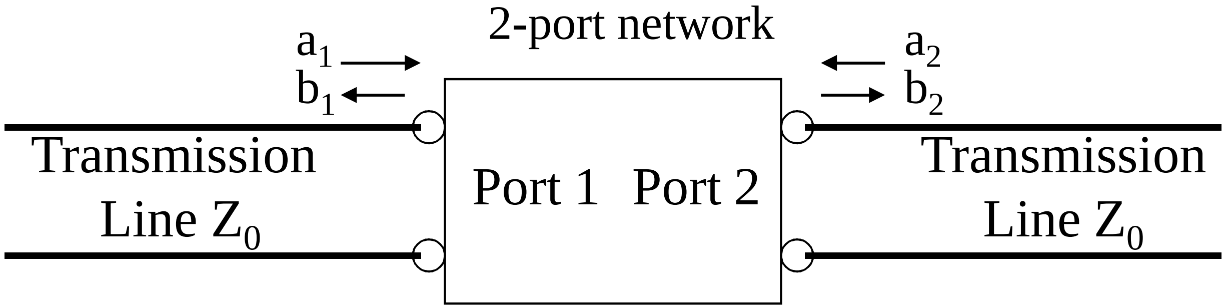
$$b_1 = s_{11} a_1 + s_{12} a_2$$

$$b_2 = s_{21} a_1 + s_{22} a_2$$

$$s_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \qquad s_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \qquad s_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \qquad s_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0}$$

s_{11} and s_{21} are determined when a_2=0. s_{12} and s_{22} are determined when a_1=0

Interpretation of s-parameters



$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

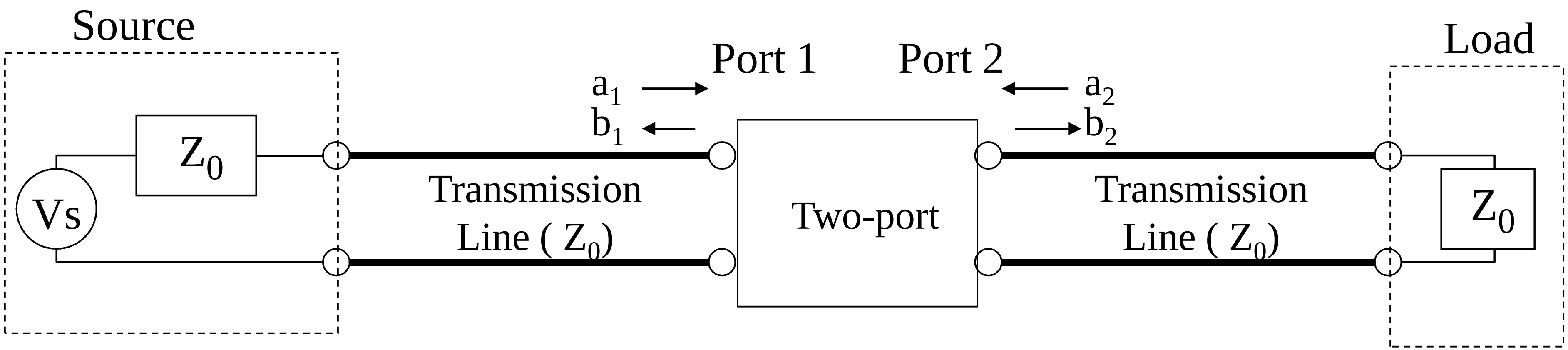
S_{11} is the input reflection coefficient of the two-port i.e. the fraction of the incident wave which is reflected back into the transmission line connected to port 1. If the input is perfectly matched this will be zero. S_{11} is usually plotted on a Smith chart, like other reflection coefficients.

S_{21} is related to the power gain of the circuit and should at least be greater than 1 so that the power flowing out of port 2 is higher than the power flowing in to port 1

S_{22} is the output reflection coefficient of the two-port i.e. the fraction of the incident wave on port 2 which is reflected back into the transmission line. It is also plotted on a Smith chart.

S_{12} is related to the reverse power gain of the two-port i.e. the fraction of the wave incident at port 2 which is transferred to the input circuit, port 1. Ideally this should be zero so that the output circuit has no influence on the input circuit. In most cases it is non-zero but much less than 1.

S-parameter measurement setup - 1



$$b_1 = s_{11}a_1 + s_{12}a_2$$

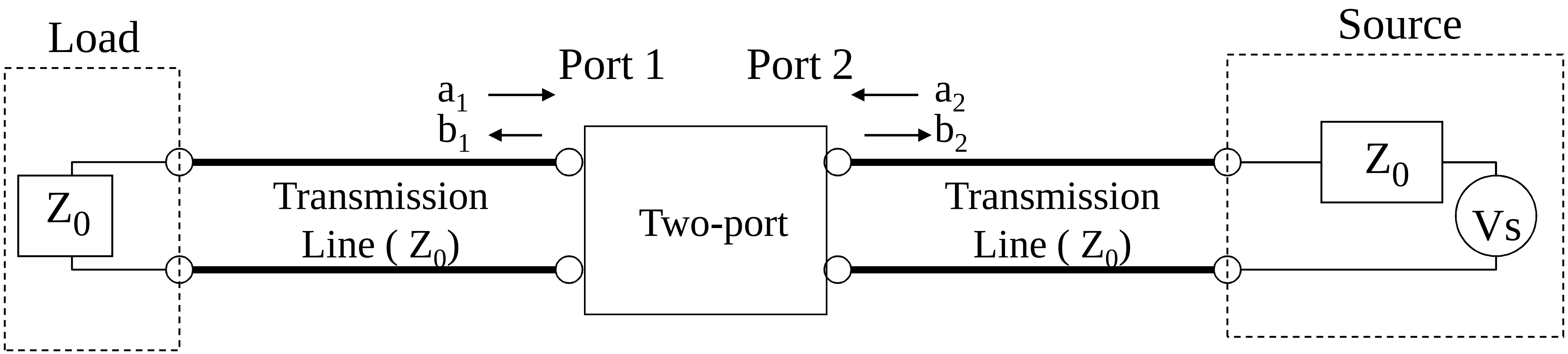
$$b_2 = s_{21}a_1 + s_{22}a_2$$

$$s_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0}$$

$$s_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

With port 1 connected to a source and port 2 connected to a load, no power is being supplied externally to flow into port 2. Therefore the only power flowing into port 2 could be reflected power from the load. By ensuring that the load impedance is set to the characteristic impedance of the transmission line, there will be no power reflected at the load and thus a_2 will be zero.

S-parameter measurement setup - 2



$$b_1 = s_{11}a_1 + s_{12}a_2$$

$$b_2 = s_{21}a_1 + s_{22}a_2$$

$$s_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0}$$

$$s_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0}$$

With port 2 connected to a source and port 1 connected to a load, no power is being supplied externally to flow into port 1. Therefore the only power flowing into port 1 could be reflected power from the load. By ensuring that the load impedance is set to the characteristic impedance of the transmission line, there will be no power reflected at the load and thus a_1 will be zero.