Chapter 2. State-Space Theory

2.1 Relationship between the State-Space and Transfer Function Representations

Consider the linear N^{th} order process, with M inputs and P outputs, (for simplicity lets first assume no disturbances):

$$\frac{d}{dt}\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t)$$
$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t)$$

Now apply the Laplace transform:

First some revision:

$$L\{\dot{\underline{x}}(t)\} = \begin{bmatrix} L\{\dot{x}_1(t)\} \\ L\{\dot{x}_2(t)\} \\ \vdots \\ L\{\dot{x}_N(t)\} \end{bmatrix} = \begin{bmatrix} sX_1(s) \\ sX_2(s) \\ \vdots \\ sX_N(s) \end{bmatrix} - \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_N(0) \end{bmatrix}$$

Hence the Laplace transform of equation (1) yields:

Similarly the Laplace transform of equation (2) yields:

$$\underline{Y}(s) = C\underline{X}(s) + D\underline{U}(s)$$

Rearranging equation (3):

$$s\underline{X}(s) - A\underline{X}(s) = B\underline{U}(s) + \underline{x}(0)$$

Can now solve for the state by premulyiplying both sides of equation (4) by $(sI\text{-}A)^{-1}$:

$$(sI - A)^{-1}(sI - A)\underline{X}(s) = (sI - A)^{-1}\left(B\underline{U}(s) + \underline{x}(0)\right)$$

Of course we know that: $\underline{Y}(s) = C\underline{X}(s) + D\underline{U}(s)$

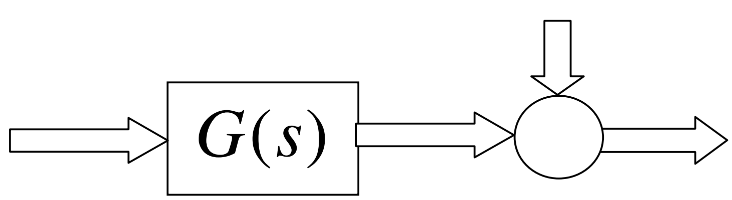
Then we get an expression for the output in terms of the input and the initial state vector:

$$\underline{Y}(s) = C(sI - A)^{-1}\left(B\underline{U}(s) + \underline{x}(0)\right) + D\underline{U}(s)$$

which could be written as:

$$\underline{Y}(s) = \left(C(sI - A)^{-1}B + D\right)\underline{U}(s) + C(sI - A)^{-1}\underline{x}(0)$$

This could be represented by the following block diagram:



Hence the MIMO transfer function of the process is:

$$G(s) = C(sI - A)^{-1} B + D$$

Now if it is assumed that: $\underline{x}(0) = 0$

Or:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_P(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1M}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2M}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{P1}(s) & G_{P2}(s) & \cdots & G_{PM}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_M(s) \end{bmatrix}$$

Consider now an expression for the ith output:

$$Y_i(s) = G_{i1}(s)U_1(s) + G_{i2}(s)U_2(s) + \cdots G_{iM}(s)U_M(s)$$

<u>**Note:**</u> the efficiency of the state-space model:

The state-space model of an Nth order process with P outputs and M inputs requires:

A transfer function matrix model of the same process requires PxM separate transfer functions:

2.1.1 Determining the Poles of a State-Space Model

First some matrix revision:

Hence we can write:

$$(sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)}$$

and the transfer function matrix is:

$$\begin{aligned} G(s) = C(sI - A)^{-1} B + D &= \frac{C\ adj(sI - A)\, B}{\det(sI - A)} + D \\ &= \frac{C\ adj(sI - A)\, B + \det(sI - A) D}{\det(sI - A)} \end{aligned}$$

Hence G(s) could be written as:

$$G(s) = \begin{bmatrix} \frac{F_{11}(s)}{E(s)} & \frac{F_{12}(s)}{E(s)} & \cdots & \frac{F_{1M}(s)}{E(s)} \\ \frac{F_{21}(s)}{E(s)} & \frac{F_{22}(s)}{E(s)} & \cdots & \frac{F_{2M}(s)}{E(s)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F_{P1}(s)}{E(s)} & \frac{F_{P2}(s)}{E(s)} & \cdots & \frac{F_{PM}(s)}{E(s)} \end{bmatrix}$$

Hence the N poles of the N^{th} order process are given by the N roots of the characteristic polynomial:

$$\det(sI - A) = 0$$

<u>Tutorial:</u> Determine the transfer function and the poles of :

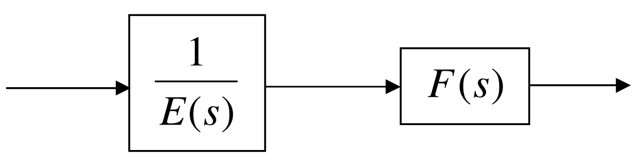
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$\underline{y}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2.2 How to Obtain the State-Space Representation from the Transfer Function

Consider for simplicity the SISO Nth order process:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{f_r s^r + f_{r-1} s^{r-1} + \cdots + f_0}{s^N + e_{N-1} s^{N-1} + \cdots + e_0} =$$



Therefore we can write:

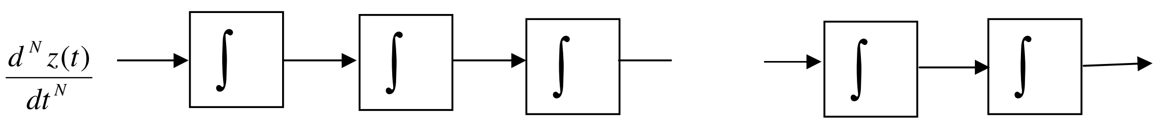
$$\frac{Z(s)}{U(s)} = \frac{1}{E(s)} = \frac{1}{s^N + e_{N-1} s^{N-1} + \cdots + e_0}$$

Which could be written as:

$$s^N Z(s) = U(s) - e_{N-1} s^{N-1} Z(s) - \cdots - e_1 s Z(s) - e_0 Z(s)$$

Taking the inverse Laplace transform yields the N$^{\text{th}}$ order differential equation:

Transform this into N first order equations:



Now assign the state vector:

$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ \dot{z}(t) \\ \vdots \\ z^{(N-1)}(t) \end{bmatrix}$$

Then:

$$\frac{d}{dt}\underline{x}(t) = \frac{d}{dt}\begin{bmatrix} z(t) \\ \dot{z}(t) \\ \vdots \\ z^{(N-1)}(t) \end{bmatrix} = \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \\ \vdots \\ z^{N}(t) \end{bmatrix} =$$

But we know that:

$$\frac{d^N z(t)}{dt^N} = u(t) - e_{N-1}\frac{d^{N-1} z(t)}{dt^{N-1}} - \cdots - e_1 \frac{dz(t)}{dt} - e_0 z(t)$$

Could be written in matrix form as:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} u(t)$$

The output equation was given by:

$$Y(s) = F(s)Z(s) = \left(f_r s^r + f_{r-1}s^{r-1} + \cdots + f_0\right)Z(s)$$

Taking inverse Laplace transforms yields:

But:

Hence the output equation is now:

$$y(t) = \begin{bmatrix} & & & & & & \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{r+1}(t) \\ \hdashline x_{r+2}(t) \\ \vdots \\ x_N(t) \end{bmatrix}$$

2.2.1 The Control Canonical Form

Hence the SISO Nth order process:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{f_r s^r + f_{r-1}s^{r-1} + \cdots + f_0}{s^N + e_{N-1}s^{N-1} + \cdots + e_0} =$$

can be represented by the SISO state-space equations:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + Bu(t) \\ y(t) &= C\underline{x}(t) \end{aligned}$$

where:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -e_0 & -e_1 & -e_2 & -e_3 & \cdots & -e_{N-1} \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \qquad \end{bmatrix}$$

$$C = \begin{bmatrix} f_0 & f_1 & \cdots & f_r & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad \end{bmatrix}$$

*This is called the CONTROL CANONICAL FORM

Example: $\qquad G(s) = \dfrac{s+2}{s(s+1)(s+3)} =$

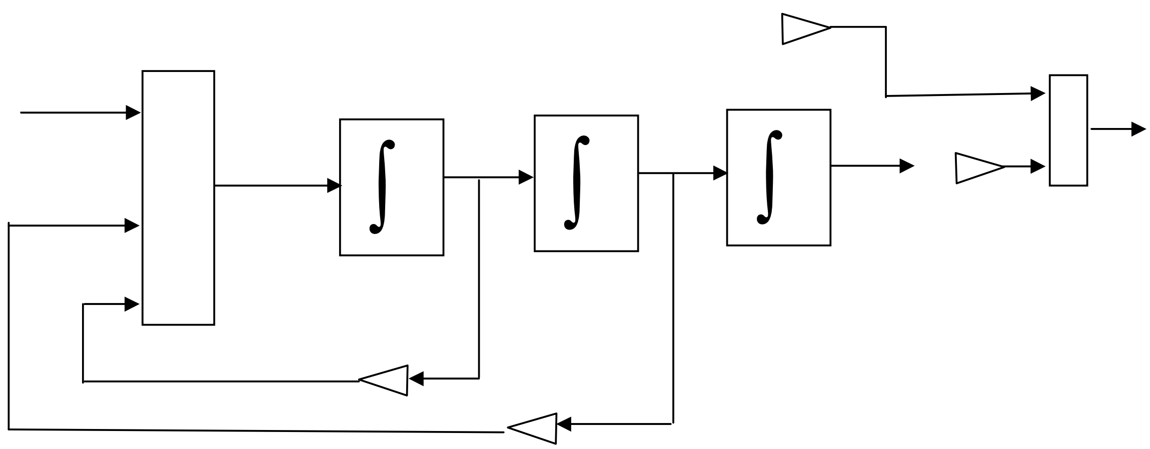
Modelled as the following control canonical form:

$$\dot{\underline{x}}(t) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\underline{x}(t) + \begin{bmatrix} \\ \\ \\ \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} & & \end{bmatrix}\underline{x}(t)$$

where: $\underline{x}(t) = [x_1(t) \quad x_2(t) \quad x_3(t)]^T$

Could be represented as the following simulation diagram:



BUT: Remember the tutorial question of section 2.1.1:

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\underline{x}(t)$$

2.3 State Transformation Theory

Consider the linear MIMO process:

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$
$$\underline{y}(t) = C\underline{x}(t)$$

Consider an arbitrary transformation of the state vector $\underline{x}$(t) to a new state vector $\underline{z}$(t):

Rewrite the state equations:

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$

$$\underline{y}(t) = C\underline{x}(t)$$

The transformed state-equations are then:

$$\underline{\dot{z}}(t) = TAT^{-1}\underline{z}(t) + TB\underline{u}(t)$$
$$\underline{y}(t) = CT^{-1}\underline{z}(t)$$

Just to prove that this is the same system, only represented differently internally, consider the transfer functions:

Orig.: $G(s) = C(sI - A)^{-1}B$ Trans.: $G_2(s) = C_2(sI - A_2)^{-1}B_2$

First some revision: $(QR)^{-1} = R^{-1}Q^{-1}$

The transfer function : $G_2(s) = CT^{-1}\left(sI - TAT^{-1}\right)^{-1}TB$

$$G_2(s) = C\left(sIT - TAT^{-1}T\right)^{-1}TB =$$

which then could be written as:

$$G_2(s) = C\left(T^{-1}sIT - T^{-1}TA\right)^{-1}B =$$

<u>Tutorial:</u>

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\underline{x}(t)$$

i) Determine the transfer function: G(s)=Y(s)/U(s)
ii) Transform the state equations using :

$$T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

iii) Determine the transfer function of the transformed process
iv) Represent each system as a simulation diagram

2.4 Solution of the State-Space Equations

Consider the state-equation:

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$

Taking Laplace transforms yields:

$$\underline{X}(s) = (sI - A)^{-1}(B\underline{U}(s) + \underline{x}(0))$$

Now solve for the state trajectory using the inverse Laplace transform:

Revision: $L^{-1}\{W(s)V(s)\} = w(t) \otimes v(t)$

Now if we define : $\Phi(s) = (sI - A)^{-1}$

Then:

We can now write:

$$\underline{X}(s) = \Phi(s)\underline{x}(0) + \Phi(s)B\underline{U}(s)$$

Taking inverse Laplace transforms yields:
Which yields the solution of the state-trajetory in the time–domain as:

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \int_0^t \Phi(t-\tau)B\underline{u}(\tau)d\tau$$

Now if we assume that D=0 then: $\underline{y}(t) = C\underline{x}(t)$

But since: $\Phi(s) = (sI - A)^{-1}$

Then: $C\Phi(s)B = C(sI - A)^{-1}B = G(s)$

The output equation could be rewritten then as:

$$\underline{y}(t) = C\Phi(t)\underline{x}(0) + \int_0^t G(t-\tau)\underline{u}(\tau)d\tau$$

If we just concentrate on the zero-state response:

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_i(t) \\ \vdots \\ y_p(t) \end{bmatrix} = \int_0^t \begin{bmatrix} g_{11}(t-\tau) & g_{12}(t-\tau) & g_{13}(t-\tau) & \cdots & g_{1M}(t-\tau) \\ \vdots & & & & \vdots \\ g_{i1}(t-\tau) & g_{i2}(t-\tau) & g_{i3}(t-\tau) & \cdots & g_{iM}(t-\tau) \\ \vdots & & & & \vdots \\ g_{P1}(t-\tau) & g_{P2}(t-\tau) & g_{P3}(t-\tau) & \cdots & g_{PM}(t-\tau) \end{bmatrix} \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \\ \vdots \\ u_M(\tau) \end{bmatrix} d\tau$$

An expression for the i^{th} output $y_i(t)$ is:

$$y_i(t) = \int_0^t g_{i1}(t-\tau)u_1(\tau)d\tau + \int_0^t g_{i2}(t-\tau)u_2(\tau)d\tau + \cdots + \int_0^t g_{iM}(t-\tau)u_M(\tau)d\tau$$

2.5 How to Calculate the Transition Matrix Φ(t)

2.5.1 The Laplace-Transform Method

$$\Phi(t) = L^{-1}\left\{(sI - A)^{-1}\right\}$$

Example: $$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\Phi(s) = (sI - A)^{-1} = \left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\right]^{-1}$$

$$\Phi(t) = L^{-1}\{\Phi(s)\} = \begin{pmatrix} L^{-1}\left\{\dfrac{s+3}{(s+2)(s+1)}\right\} & L^{-1}\left\{\dfrac{1}{(s+2)(s+1)}\right\} \\ L^{-1}\left\{\dfrac{-2}{(s+2)(s+1)}\right\} & L^{-1}\left\{\dfrac{s}{(s+2)(s+1)}\right\} \end{pmatrix}$$

Take each term in turn and use partial fractions:

Eg. $$L^{-1}\left\{\frac{s+3}{(s+2)(s+1)}\right\} =$$

Repeating for the other three terms yields:

$$\Phi(t) = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

2.5.2 Matrix Exponential Method

Consider first the zero-input response:

$$\underline{x}(t) = \Phi(t)\underline{x}(0)$$

which is of course the solution to:

Revision: Consider first the scalar differential equation:

$$\dot{x}(t) = ax(t)$$

If the initial condition is x(0), then this is solved to yield:

Now considering: $\dot{\underline{x}}(t) = A\underline{x}(t)$

Propose the solution:

Proof: We know the solution is $\underline{x}(t) = \Phi(t)\underline{x}(0)$

Hence we can write:

$$\dot{\underline{x}}(t) = \frac{d}{dt}\left(\Phi(t)\underline{x}(0)\right) = \frac{d\Phi}{dt}\underline{x}(0)$$

$$\ddot{\underline{x}}(t) = \frac{d^2}{dt^2}\left(\Phi(t)\underline{x}(0)\right) = \frac{d^2\Phi}{dt^2}\underline{x}(0)$$

$$\dddot{\underline{x}}(t) = \frac{d^3}{dt^3}\left(\Phi(t)\underline{x}(0)\right) = \frac{d^3\Phi}{dt^3}\underline{x}(0)$$

In fact we could write:

$$\frac{d^i}{dt^i}\Phi(t) = A^i\Phi(t)$$

This will be true if:

$$\Phi(t) = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

We will now define the matrix exponential function as:

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

<u>Example:</u> $\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$

Approximate Φ(t) to four terms using the matrix exponential method:

$$e^{At} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\frac{t}{1!} + \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}\frac{t^2}{2!} + \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}\frac{t^3}{3!}$$

2.5.3 Using Cayley Hamilton Theory

The Cayley Hamilton theory states that any NxN matrix A obeys:

$$A^{N+i} = \beta_{i0} I + \beta_{i1} A + \beta_{i2} A^2 + \cdots$$

Example: $$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Then: $$A^3 = \beta_{10} I + \beta_{11} A$$

$$A^3 = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} \beta_{10} & 0 \\ 0 & \beta_{10} \end{bmatrix} + \begin{bmatrix} 0 & \beta_{11} \\ -\beta_{11} & -2\beta_{11} \end{bmatrix}$$

Now consider the matrix exponential function:

$$\Phi(t) = e^{At} = I + \frac{At}{1!} + \frac{A^2 t^2}{2!} + \cdots$$

By Cayley Hamilton theory:

$$\begin{aligned}
A^N &= \beta_{00} I + \beta_{01} A + \beta_{02} A^2 + \cdots + \beta_{0(N-1)} A^{N-1} \\
A^{N+1} &= \beta_{10} I + \beta_{11} A + \beta_{12} A^2 + \cdots + \beta_{1(N-1)} A^{N-1} \\
A^{N+2} &= \beta_{20} I + \beta_{21} A + \beta_{22} A^2 + \cdots + \beta_{2(N-1)} A^{N-1} \\
&\vdots \\
A^{N+\infty} &= \beta_{\infty 0} I + \beta_{\infty 1} A + \beta_{\infty 2} A^2 + \cdots + \beta_{\infty (N-1)} A^{N-1}
\end{aligned}$$

$$
\begin{aligned}
e^{At} &= I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots \frac{A^{N-1}t^{N-1}}{(N-1)!} \\
&+ (\beta_{00}I + \beta_{01}A + \beta_{02}A^2 + \cdots + \beta_{0(N-1)}A^{N-1})\frac{t^N}{N!} \\
&+ (\beta_{10}I + \beta_{11}A + \beta_{12}A^2 + \cdots + \beta_{1(N-1)}A^{N-1})\frac{t^{N+1}}{(N+1)!} \\
&+ (\beta_{20}I + \beta_{21}A + \beta_{22}A^2 + \cdots + \beta_{2(N-1)}A^{N-1})\frac{t^{N+2}}{(N+2)!} \\
&\vdots \\
&+ (\beta_{\infty 0}I + \beta_{\infty 1}A + \beta_{\infty 2}A^2 + \cdots + \beta_{\infty(N-1)}A^{N-1})\frac{t^{N+\infty}}{(N+\infty)!}
\end{aligned}
$$

Bringing all similar terms in A together:

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2 + \cdots \alpha_{N-1}(t)A^{N-1}$$

<u>How to find the scalar functions of time:</u>

From a singular value decomposition of A, then every eigenvalue λ_i of A must also satisfy:

$$e^{\lambda_i t} = \alpha_0(t) + \alpha_1(t)\lambda_i + \alpha_2(t)\lambda_i^2 + \cdots \alpha_{N-1}(t)\lambda_i^{N-1}$$

<u>Revision</u>: The eigenvalues of A are the roots of:

$$\det(\lambda I - A) = 0$$

If we have N distinct eigenvalues of A: λ_1, λ_2, ….λ_N

$$
\begin{aligned}
e^{\lambda_1 t} &= \alpha_0(t) + \alpha_1(t)\lambda_1 + \alpha_2(t)\lambda_1^2 + \cdots \alpha_{N-1}(t)\lambda_1^{N-1} \\
&\vdots \\
e^{\lambda_N t} &= \alpha_0(t) + \alpha_1(t)\lambda_N + \alpha_2(t)\lambda_N^2 + \cdots \alpha_{N-1}(t)\lambda_N^{N-1}
\end{aligned}
$$

Example:

$$
\underline{\dot{x}}(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \underline{x}(t)
$$

The eigenvalues of A are roots of:

$$
\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = 0
$$

Since N=2 then we can write: $e^{At} = \alpha_0(t)I + \alpha_1(t)A$

This equation must be obeyed by the eigenvalues of A:

$$
\begin{aligned}
e^{3t} &= \alpha_0(t) + \alpha_1(t)(3) \\
e^{-t} &= \alpha_0(t) + \alpha_1(t)(-1)
\end{aligned}
$$

Now: $$e^{At} = \frac{1}{4}\left(e^{3t} + 3e^{-t}\right)I + \frac{1}{4}\left(e^{3t} - e^{-t}\right)\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

2.6 The Discrete State Space Equations

Consider the MIMO process:

$$\begin{aligned}\underline{\dot{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t)\end{aligned}$$

The state trajectory is given by:

$$\underline{x}(t) = \Phi(t)\underline{x}(0) + \int_0^t \Phi(t-\tau)B\underline{u}(\tau)d\tau$$

Where the initial state at time 0 is $\underline{x}(0)$

Now consider that the initial time is t_0, with initial state:

Then the trajectory could be rewritten as:

Consider what happens to the state vector over a time step T:

$$\underline{x}\big((k+1)T\big) = \Phi\big((k+1)T - kT\big)\underline{x}(kT) + \int_{kT}^{(k+1)T} \Phi\big((k+1)T-\tau\big)B\underline{u}(\tau)d\tau$$

$$\underline{x}\big((k+1)T\big) = \Phi\big(T\big)\underline{x}(kT) + \int_{kT}^{(k+1)T} \Phi\big((k+1)T-\tau\big)B\underline{u}(\tau)d\tau$$

If we assume that a zero-order hold (ZOH) is utilised:

Then:

$$\underline{x}\big((k+1)T\big) = \Phi(T)\underline{x}(kT) + \int_{kT}^{(k+1)T} \Phi\big((k+1)T - \tau\big) B\, d\tau\, \underline{u}(kT)$$

Now make the following substitution:

$$\underline{x}\big((k+1)T\big) = \Phi(T)\underline{x}(kT) - \int_{T}^{0} \Phi(\eta) B\, d\eta\ \underline{u}(kT)$$

Remembering that:

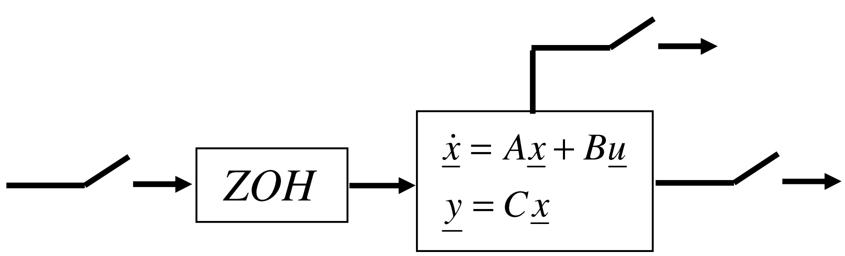
And simplifying the notation:

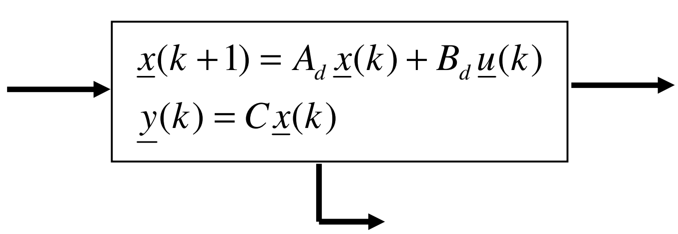
$$\underline{x}(k+1) = \Phi(T)\underline{x}(k) + \int_{0}^{T} \Phi(\eta) B\, d\eta\ \underline{u}(k)$$

This yields the discrete-time state-space equations:

$$\begin{aligned} \underline{x}(k+1) &= A_d\, \underline{x}(k) + B_d\ \underline{u}(k) \\ \underline{y}(k) &= C\underline{x}(k) \end{aligned}$$

where:





Tutorial:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

i) Determine the discrete-time state-space representation, assuming a ZOH, and sample time T=0.2 seconds.
ii) If $u(t) = 1 \quad \forall t > 0^+, \quad x_1(0) = x_2(0) = 1$, determine and sketch the responses for the states x_1, x_2, and the output y(t).
iii) Use Simulink to compare the responses of both the continuous and discrete process models.

2.6.1 Discrete Transfer Function Matrix

Consider the MIMO discrete time system:

$$\underline{x}(k+1) = A_d\,\underline{x}(k) + B_d\,\underline{u}(k)$$
$$\underline{y}(k) = C\underline{x}(k)$$

Taking Z transforms:

$$z\underline{X}(z) = A_d\,\underline{X}(z) + B_d\underline{U}(z)$$
$$\underline{Y}(z) = C\underline{X}(z)$$

Hence we can write:

Which yields the following:

$$\underline{Y}(z) = C(zI - A_d)^{-1}B_d\underline{U}(Z)$$

The poles of the discrete time process on the Z plane are given by the roots of the characteristic equation:

$$\det(zI - A_d) = 0$$

2.6.2 Realisation of Digital Filters

Consider the discrete-time transfer function for a SISO process:

$$\frac{Y(z)}{U(z)} = G(z) = \frac{z^{-d}\left(b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}\right)}{1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}} = \frac{B(z)}{A(z)}$$

This could be constructed as:

$$U(z) \longrightarrow \boxed{\frac{1}{A(z)}} \longrightarrow \boxed{B(z)} \longrightarrow Y(z)$$

Hence:

$$\left(1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n}\right)Q(z) = U(z)$$

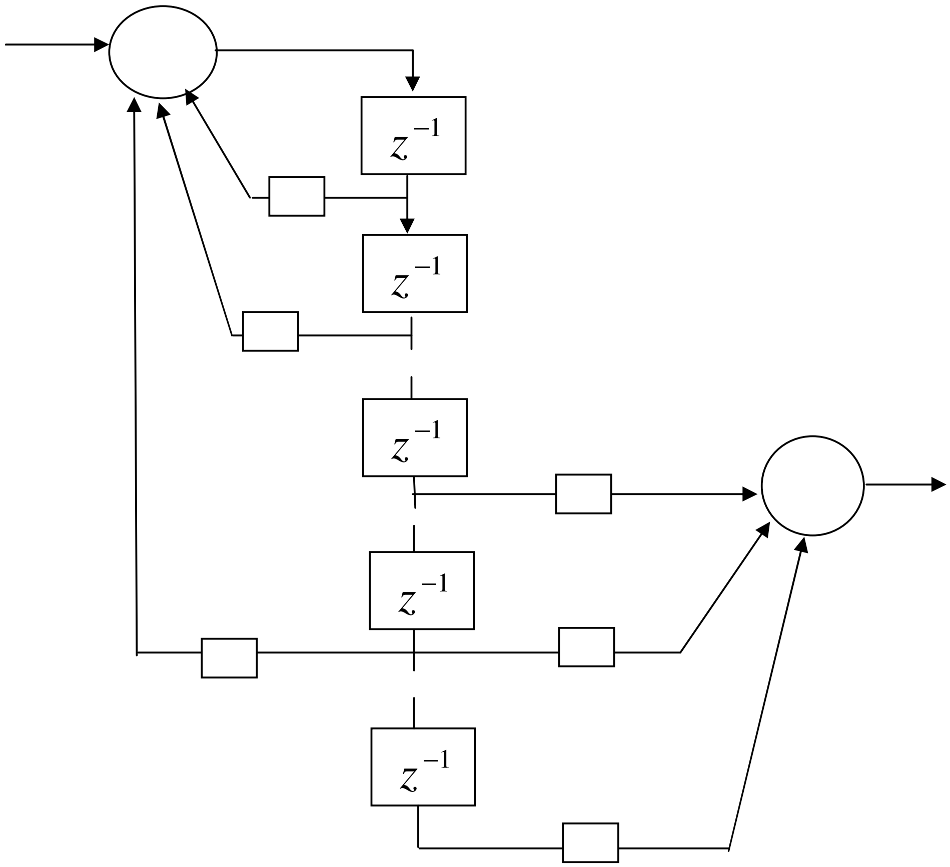
Taking inverse Z transforms yields:

$$q(k) = a_1 q(k-1) + a_2 q(k-2) + \cdots + a_n q(k-n) + u(k)$$

For the output equation :

$$z^{-d}\left(b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}\right)Q(z) = Y(z)$$

Which could be constructed as:



<u>Tutorial:</u>

Use this diagram to develop the following control-canonical state-space representation from the transfer function given above:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \\ \hdashline \vdots \\ x_{m+d}(k+1) \end{bmatrix} = \left[\begin{array}{cccc:ccc} a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ \hdashline \multicolumn{6}{c:}{I_{m+d-1}} & \underline{0}_{m+d-1} \end{array}\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \\ \vdots \\ x_{m+d}(k) \end{bmatrix} + \left[\begin{array}{c} 1 \\ \hdashline \underline{0}_{m+d-1} \end{array}\right] u(k)$$

$$y(k) = \left[\begin{array}{c:cccc} \underline{0}_d^T & b_1 & b_2 & \cdots & b_m \end{array}\right] \underline{x}(k)$$