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Reduced Row Echelon Form

When we reduce an $n \times m$ matrix A to reduced row echelon (rref) form, no only do we learn about the null space of A, but we also learn how the dependent columns of A are linear combinations of the independent (pivot) columns and how to factor A as CR, where C contains the independent columns of A and $R = \text{rref}(A)$.

In this assignment you can use the function "rref" on a matrix to reduce it to reduced row echelon form. For example, if

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 & 1 \\ -1 & 4 & -3 & 1 & 1 \\ 3 & -1 & 2 & -2 & 4 \end{bmatrix}$$

then

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \frac{11}{17} & \frac{19}{17} \\ 0 & 1 & 0 & \frac{-29}{17} & \frac{21}{17} \\ 0 & 0 & 1 & \frac{-48}{17} & \frac{16}{17} \end{bmatrix}$$

Helper Functions

I've created several helper functions to aid in completing this assignment. They are:

- randmat(): returns a matrix A and a vector $\boldsymbol{b}$ in the equation $A\boldsymbol{x} = \boldsymbol{b}$ for your problem.
- id(n): Returns the $n \times n$ identity matrix
- testsoln(A,X,xp,b): Here A is the matrix returned by randmat(), X is the matrix of null space basis vectors, xp is a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$, and $\boldsymbol{b}$ is the vector returned by the randmat() function. This function tests five random solution vectors $\boldsymbol{x} = \boldsymbol{x}_p + (\textbf{linear combination of nullspace vectors})$ in the equation $A\boldsymbol{x} = \boldsymbol{b}$. If the vector solves the equation, it returns "true", otherwise it returns "false".

For each problem below, you must:

- Find matrices C and R, then show that $A = CR$
- Find a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$
- Find the matrix X of null space basis vectors
- Use the testsoln function to check your answers. You should get five "true" results.

Example

Consider the matrix $A0$ below:

```
A0, b0 = randmat();
```

$$\begin{bmatrix} -8 & 8 & -16 & 32 \\ 5 & -7 & 14 & -30 \\ 3 & -4 & 8 & -17 \end{bmatrix}$$

```
A0 = [-8//1   8//1  -16//1   32//1;
       5//1  -7//1   14//1  -30//1;
       3//1  -4//1    8//1  -17//1]; latexify(A0)
```

$$\begin{bmatrix} -16 \\ 20 \\ 11 \end{bmatrix}$$

```
b0 = [-16//1; 20//1; 11//1]; latexify(b0)
```

$$\begin{bmatrix} -8 & 8 & -16 & 32 & -16 \\ 5 & -7 & 14 & -30 & 20 \\ 3 & -4 & 8 & -17 & 11 \end{bmatrix}$$

```
A0b0 = [A0 b0]; latexify(A0b0) # Augmented matrix
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -3 \\ 0 & 1 & -2 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
A0b0rr = rref(A0b0); latexify(A0b0rr) # Reduced row echelon form of augmented matrix
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
A0rr = A0b0rr[:,1:4]; latexify(A0rr) #
```

The matric C

$$\begin{bmatrix} -8 & 8 \\ 5 & -7 \\ 3 & -4 \end{bmatrix}$$

```
C = A0[:,[1,2]]; latexify(C) # Columns are basis of column space of A0
```

The matrix R

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 5 \end{bmatrix}$$

```
Arr = A0rr[1:2,:]; latexify(Arr) # Reduced row echelon form of A0
```

Checking that $A = CR$

```
true
```

```
A0==C*Arr # A0 factored as C * Arr
```

A particular solution x_p to $Ax = b$

```
xp =  [-3//1, -5//1, 0//1, 0//1]
```

```
xp = [-3//1; -5//1; 0; 0] # particular solution to Ax = b
```

$$\begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$$

```
F = Arr[:,[3,4]]; latexify(F)
```

The matrix of null space basis vectors; each column is a basis vector of $N(A)$

$$\begin{bmatrix} 0 & -1 \\ 2 & -5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
X = [-F; id(2)]; latexify(X) # Columns are basis of null space
```

Using the "testsoln" function to check our answers

```
[true, true, true, true, true]
```

```
testsoln(A0,X,xp,b0) # Check that general solution satisfies Ax = b
```

The Problems

For each problem, copy the A matrices and the b vectors into the same variables after disabling the cell containing the randmat() function.

Problem 1

```
A1, b1 = randmat();
```

Get A and b

```
A1 = 3×4 Matrix{Rational{Int64}}:
      -6//1  -3//1   6//1  21//1
     -32//1  -7//1  -4//1  31//1
     -26//1  -7//1   2//1  37//1

b1 =  [3//1, 61//1, 43//1]
```

Matrix C

$$\begin{bmatrix} -6 & -3 \\ -32 & -7 \\ -26 & -7 \end{bmatrix}$$

```
1 C1 = A1[:,1:2]; latexify(C1)
```

Matrix R

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -4 & -9 \end{bmatrix}$$

```
1 R1 = rref(A1)[1:2,:]; latexify(R1)
```

Verify $A = CR$

```
true
```

```
1 A1==C1*R1
```

Particular Solution x_p

$$\begin{bmatrix} -6 & -3 & 6 & 21 & 3 \\ -32 & -7 & -4 & 31 & 61 \\ -26 & -7 & 2 & 37 & 43 \end{bmatrix}$$

```
1 Aug1 = [A1 b1]; latexify(Aug1)
```

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -3 \\ 0 & 1 & -4 & -9 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
1 rrefAug1 = rref(Aug1); latexify(rrefAug1)
```

$$\begin{bmatrix} -3 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

```
1 xp1 = [-3//1; 5//1; 0//1; 0//1]; latexify(xp1)
```

Verify solution vector

```
true
```

```
A1*xp1==b1
```

$$\begin{bmatrix} 1 & 1 \\ -4 & -9 \end{bmatrix}$$

```
F1 = R1[:,[3,4]]; latexify(F1)
```

Null Space $null(A1) = X1$

$$\begin{bmatrix} 0 & -1 \\ 2 & -5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
X1 = [-F1; id(2)]; latexify(X)
```

Check answers

```
[true, true, true, true, true]
```

```
testsoln(A1,X1,xp1,b1)
```

Problem 2

```
A2, b2 = randmat();
```

A2

$$\begin{bmatrix} 4 & 6 & -2 & 36 & 24 & 4 & -18 \\ -3 & 3 & -2 & 3 & 27 & 0 & -9 \\ -2 & -2 & 6 & -14 & -12 & 1 & 6 \\ 9 & -9 & 4 & -9 & -87 & -3 & 27 \end{bmatrix}$$

```
A2 =[
  4//1   6//1  -2//1   36//1   24//1   4//1  -18//1;
 -3//1   3//1  -2//1    3//1   27//1   0//1   -9//1;
 -2//1  -2//1   6//1  -14//1  -12//1   1//1    6//1;
  9//1  -9//1   4//1   -9//1  -87//1  -3//1   27//1
]; latexify(A2) # (4x7)
```

b2

$$\begin{bmatrix} 102 \\ 13 \\ -43 \\ -56 \end{bmatrix}$$

```
b2 =[
 102//1;
  13//1;
 -43//1;
 -56//1
]; latexify(b2)
```

C2

$$\begin{bmatrix} 4 & 6 & -2 & 24 \\ -3 & 3 & -2 & 27 \\ -2 & -2 & 6 & -12 \\ 9 & -9 & 4 & -87 \end{bmatrix}$$

```
C2 = A2[:,[1,2,3,5]]; latexify(C2) # (4x4)
```

R2

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 & \frac{-3}{4} & -3 \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{4} & 0 \end{bmatrix}$$

```
R2 = rref(A2); latexify(R2) # (4x7)
```

Verify $A2 = C2 \cdot R2$

```
true
```

```
A2==C2*R2
```

Particular solution for x_{p_2} given $Ax_{p_2} = b$

$$\begin{bmatrix} 4 & 6 & -2 & 36 & 24 & 4 & -18 & 102 \\ -3 & 3 & -2 & 3 & 27 & 0 & -9 & 13 \\ -2 & -2 & 6 & -14 & -12 & 1 & 6 & -43 \\ 9 & -9 & 4 & -9 & -87 & -3 & 27 & -56 \end{bmatrix}$$

```
Aug2 = [A2 b2]; latexify(Aug2)
```

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 1 & 0 & 14 \\ 0 & 1 & 0 & 4 & 0 & \frac{-3}{4} & -3 & \frac{-3}{4} \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{4} & 0 & \frac{7}{4} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{4} & 0 & \frac{9}{4} \end{bmatrix}$$

```
rrefAug2 = rref(Aug2); latexify(rrefAug2)
```

Pivot: $x_1,\ x_2,\ x_3,\ x_5$

Free: $x_4,\ x_6,\ x_7$

$$x_1 + 3x_4 + x_6 = 14$$

$$x_2 + 4x_4 + \frac{-3}{4}x_6 + -3x_7 = \frac{-3}{4}$$

$$x_3 + \frac{3}{4}x_6 = \frac{7}{4}$$

$$x_5 + \frac{1}{4}x_6 = \frac{9}{4}$$

Let $x_4, x_6, x_7 = 0$

$$x_1 = 14$$

$$x_2 = \frac{-3}{4}$$

$$x_3 = \frac{7}{4}$$

$$x_5 = \frac{9}{4}$$

Therefore, x_{p_2}

$$\begin{bmatrix} 14 \\ \frac{-3}{4} \\ \frac{7}{4} \\ 0 \\ \frac{9}{4} \\ 0 \\ 0 \end{bmatrix}$$

```
xp2 = [
14//1;
-3//4;
7//4;
0//1;
9//4;
0//1;
0//1
];latexify(xp2)
```

Verify that x_{p_2} is a solution

```
true
```

```
A2*xp2==b2 # (4x7) x (7x1) == (4x1)
```

Get $X2\ null(A2)$

Get I and F

```
P2 = 7×7 Matrix{Rational{Int64}}:
 1//1  0//1  0//1  0//1  0//1  0//1  0//1
 0//1  1//1  0//1  0//1  0//1  0//1  0//1
 0//1  0//1  1//1  0//1  0//1  0//1  0//1
 0//1  0//1  0//1  0//1  1//1  0//1  0//1
 0//1  0//1  0//1  1//1  0//1  0//1  0//1
 0//1  0//1  0//1  0//1  0//1  1//1  0//1
 0//1  0//1  0//1  0//1  0//1  0//1  1//1
```

Need to swap columns

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 4 & \frac{-3}{4} & -3 \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{4} & 0 \end{bmatrix}$$

```
IF_2 = R2*inv(P2); latexify(IF_2)
```

F2

$$\begin{bmatrix} 3 & 1 & 0 \\ 4 & \frac{-3}{4} & -3 \\ 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

```
F2 = IF_2[:,5:7]; latexify(F2)
```

X2

$$\begin{bmatrix} -3 & -1 & 0 \\ -4 & \frac{3}{4} & 3 \\ 0 & \frac{-3}{4} & 0 \\ 1 & 0 & 0 \\ 0 & \frac{-1}{4} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
X2 = P2*[-F2; id(3)]; latexify(X2)
```

Verify answers

```
[true, true, true, true, true]
```

```
testsoln(A2,X2,xp2,b2)
```

Problem 3

$$\begin{bmatrix} 0 & 8 & 8 & -7 & -22 & -9 \\ -1 & 6 & 5 & 9 & -21 & 0 \\ 2 & -2 & 0 & -7 & -16 & -9 \end{bmatrix}$$

```
A3, b3 = randmat(); latexify(A3)
```

A3

$$\begin{bmatrix} 0 & 8 & 8 & -7 & -22 & -9 \\ -1 & 6 & 5 & 9 & -21 & 0 \\ 2 & -2 & 0 & -7 & -16 & -9 \end{bmatrix}$$

```
A3 = [
  0//1   8//1  8//1  -7//1  -22//1  -9//1;
 -1//1   6//1  5//1   9//1  -21//1   0//1;
  2//1  -2//1  0//1  -7//1  -16//1  -9//1
];latexify(A3)
```

b3

$$\begin{bmatrix} 68 \\ -25 \\ 82 \end{bmatrix}$$

```
b3 =
[
  68//1;
 -25//1;
  82//1
];latexify(b3)
```

Find matrices C and R

Get R3

$$\begin{bmatrix} 1 & 0 & 1 & 0 & \frac{-1383}{79} & \frac{-405}{79} \\ 0 & 1 & 1 & 0 & \frac{-324}{79} & \frac{-81}{79} \\ 0 & 0 & 0 & 1 & \frac{-122}{79} & \frac{9}{79} \end{bmatrix}$$

```
R3 = rref(A3); latexify(R3) # C3 has 3 columns
```

Get C3

$$\begin{bmatrix} 0 & 8 & -7 \\ -1 & 6 & 9 \\ 2 & -2 & -7 \end{bmatrix}$$

```
C3 = A3[:,[1,2,4]]; latexify(C3)
```

Show that $A = CR$

```
true
```

```
1 A3==C3*R3
```

Find a particular solution to $Ax = b$

$$\begin{bmatrix} 0 & 8 & 8 & -7 & -22 & -9 & 68 \\ -1 & 6 & 5 & 9 & -21 & 0 & -25 \\ 2 & -2 & 0 & -7 & -16 & -9 & 82 \end{bmatrix}$$

```
1 Aug3 = [A3 b3]; latexify(Aug3)
```

$$\begin{bmatrix} 1 & 0 & 1 & 0 & \frac{-1383}{79} & \frac{-405}{79} & \frac{2983}{79} \\ 0 & 1 & 1 & 0 & \frac{-324}{79} & \frac{-81}{79} & \frac{486}{79} \\ 0 & 0 & 0 & 1 & \frac{-122}{79} & \frac{9}{79} & \frac{-212}{79} \end{bmatrix}$$

```
1 rrefAug3 = rref(Aug3); latexify(rrefAug3)
```

Pivot: $x_1,\ x_2,\ x_4$

Free: $x_3,\ x_5,\ x_6,\ x_7$

$$x_1 = \frac{2983}{79}$$

$$x_2 = \frac{486}{79}$$

$$x_4 = \frac{-212}{79}$$

$$\begin{bmatrix} \frac{2983}{79} \\ \frac{486}{79} \\ 0 \\ \frac{-212}{79} \\ 0 \\ 0 \end{bmatrix}$$

```
xp3 = [
2983//79;
486//79;
0//1;
-212//79;
0//1;
0//1;
]; latexify(xp3)
```

Verify the x vector solves the equation

```
true
```

```
A3*xp3==b3
```

Find the matrix X of null space basis vectors

$$\begin{bmatrix} 1 & 0 & 1 & 0 & \frac{-1383}{79} & \frac{-405}{79} \\ 0 & 1 & 1 & 0 & \frac{-324}{79} & \frac{-81}{79} \\ 0 & 0 & 0 & 1 & \frac{-122}{79} & \frac{9}{79} \end{bmatrix}$$

```
latexify(R3)
```

```
P3 = 6×6 Matrix{Rational{Int64}}:
 1//1  0//1  0//1  0//1  0//1  0//1
 0//1  1//1  0//1  0//1  0//1  0//1
 0//1  0//1  0//1  1//1  0//1  0//1
 0//1  0//1  1//1  0//1  0//1  0//1
 0//1  0//1  0//1  0//1  1//1  0//1
 0//1  0//1  0//1  0//1  0//1  1//1
```

```
P3 =
[
 1//1  0//1  0//1  0//1  0//1  0//1;
 0//1  1//1  0//1  0//1  0//1  0//1;
 0//1  0//1  0//1  1//1  0//1  0//1;
 0//1  0//1  1//1  0//1  0//1  0//1;
 0//1  0//1  0//1  0//1  1//1  0//1;
 0//1  0//1  0//1  0//1  0//1  1//1
]
```

$$\begin{bmatrix} 1 & 0 & 0 & 1 & \frac{-1383}{79} & \frac{-405}{79} \\ 0 & 1 & 0 & 1 & \frac{-324}{79} & \frac{-81}{79} \\ 0 & 0 & 1 & 0 & \frac{-122}{79} & \frac{9}{79} \end{bmatrix}$$

```
IF_3 = R3*inv(P3); latexify(IF_3)
```

$$\begin{bmatrix} 1 & \frac{-1383}{79} & \frac{-405}{79} \\ 1 & \frac{-324}{79} & \frac{-81}{79} \\ 0 & \frac{-122}{79} & \frac{9}{79} \end{bmatrix}$$

```
F3 = IF_3[:,4:6]; latexify(F3)
```

$$\begin{bmatrix} -1 & \frac{1383}{79} & \frac{405}{79} \\ -1 & \frac{324}{79} & \frac{81}{79} \\ 1 & 0 & 0 \\ 0 & \frac{122}{79} & \frac{-9}{79} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
X3 = P3*[-F3; id(3)]; latexify(X3)
```

Use the testsoln function to check your answers.

```
[true, true, true, true, true]
```

```
testsoln(A3,X3,xp3,b3)
```