Miles Kent

The Church Sequence

The Church sequence $\{C_n\}$ is the recursive sequence defined by:

$$C_1 = 0,\ C_2 = 1,\ C_n = C_{n-2} + 2C_{n-1}, \quad \text{for } n \geq 3$$

The first few terms of the sequence are:

$$0, 1, 2, 5, 12, 29, 70, 169, \ldots$$

Generating $\{C_n\}$ recursively

```
chu (generic function with 1 method)
```

```
function chu(n)
    if n == 1
        return 0
    elseif n == 2
        return 1
    else
        chu(n-2) + 2*chu(n-1)
    end
end
```

```
chu2 (generic function with 1 method)
```

```
function chu2(n)
    lst = [0, 1]
    if n == 1
        return [0]
    elseif n == 2
        return [0, 1]
    else
        for i in 1:(n-2)
            append!(lst, 2*lst[end] + lst[end-1])
        end
        return lst
    end
end
```

```
[0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782]
```

```
chu2(15)
```

```
[0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782]
```

```
[chu(n) for n in 1:15]
```

Generating $\{C_n\}$ using matrix exponentials

Find a matrix A so that:

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a + 2b \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

```
A = 2×2 Matrix{Int64}:
 0  1
 1  2
```

Which Church numbers are in the vector $A^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

```
[5, 12]
```

```
# uncomment the line below, then run the cell. Enter in your answers in the cell below.

A^3 * [0; 1]
```

Answer

The Church numbers in the vector $A^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are: 5 and 12

```
md"""
!!! answer
The Church numbers in the vector ``A^3\begin{bmatrix} 0\\1 \end{bmatrix}`` are:
``5`` and ``12``
"""
# 4th and 5th for n = 3
```

What does the exponent m on the matrix A have to be, so that

$$A^m \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} C_{n-1} \\ C_n \end{bmatrix} ?$$

Answer

$m = n - 2$

Answer

Use a list comprehension using powers of the matrix A to generate the third to the fifteenth Church numbers

```
[2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782]
```

```
# Put the list comprehension here. Verify that the numbers agree with the third to fifteenth Church numbers.

[(A^n * [0;1])[2] for n in 1:13] # n - 2 for nth term in bottom position
```

Diagonalizing the matrix A

Find the eigenvalues and corresponding eigenvectors of the matrix A

```
2×2 Matrix{Int64}:
 0  1
 1  2
```

```
A
```

```
M = 1.0
```

```
M = ((A[1,1] + A[2,2]) / 2)[1]
```

```
P = -1.0
```

```
P = det(A)
```

```
L1 = 2.414213562373095
```

```
L1 = M + sqrt(M^2-P)
```

```
L2 = -0.41421356237309515
```

```
L2 = M - sqrt(M^2-P)
```

Answer

Enter the eigenvalues in $\LaTeX$ below:

$$\lambda_1 = 2.41421$$
$$\lambda_2 = -0.41421$$

```
V1 =  [0.414214, 1.0]
```

```
1 V1 = [-1/(A-L1*I)[1]; 1]
```

```
[0.0, 1.66533e-16]
```

```
1 (A-L1*I)*V1
```

```
V2 =  [-2.41421, 1.0]
```

```
1 V2 = [-1/(A-L2*I)[1]; 1]
```

```
[0.0, 4.44089e-16]
```

```
1 (A-L2*I)*V2
```

Answer

Enter the corresponding eigenvectors in $\LaTeX$ below:

$$v_1 = \begin{bmatrix} 0.414214 \\ 1.0 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} -2.41421 \\ 1.0 \end{bmatrix}$$

```
X = 2×2 Matrix{Float64}:
     0.414214  -2.41421
     1.0        1.0
```

```
1 X = [V1 V2]
```

Answer

Enter the matrix X of eigenvectors in $\LaTeX$ below:

$$X = \begin{bmatrix} 0.414214 & -2.41421 \\ 1.0 & 1.0 \end{bmatrix}$$

```
2×2 Matrix{Float64}:
  0.353553  0.853553
 -0.353553  0.146447
```

```
1 inv(X)
```

Answer

Enter the inverse of matrix X in $\LaTeX$ below:

$$X^{-1} = \begin{bmatrix} 0.353553 & 0.853553 \\ -0.353553 & 0.146447 \end{bmatrix}$$

```
D = 2×2 Matrix{Float64}:
     2.41421   0.0
     0.0      -0.414214
```

```
1 D=diagm([L1, L2])
```

Answer

Enter the diagonal matrix Λ of eigenvalues of matrix A in $\LaTeX$ below:

$$\Lambda = \begin{bmatrix} 2.41421 & 0.0 \\ 0.0 & -0.414214 \end{bmatrix}$$

Answer

Using the diagonalization $A = X\Lambda X^{-1}$, find entries in the vector $A^m \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Enter your answer below in $\LaTeX$.

$$A^m \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.414214 & -2.41421 \\ 1.0 & 1.0 \end{bmatrix} \begin{bmatrix} 2.41421^m & 0.0 \\ 0.0 & (-0.414214)^m \end{bmatrix} \begin{bmatrix} 0.353553 & 0.853553 \\ -0.353553 & 0.146447 \end{bmatrix}$$

Answer

Based on your previous work, what is a formula for the nth Church number? Enter your answer in $\LaTeX$ below:

$$C_n = 0.853553 * (2.41421^{n-2}) - 0.146447 * ((-0.414214)^{n-2})$$

Answer

Using the formula you discovered for C_n, make a julia function C(n) for the nth Church number. Enter it in the cell below.

```
C (generic function with 1 method)
```

```
# uncomment the line below and run the cell after you enter in your formula for C(n)

C(n) = 0.853553*(2.41421^(n-2))-0.146447*((-0.414214)^(n-2))
```

Answer

Verify that your function C(n) produces the correct results by making a list comprehension for C(3) to C(15) and checking that the results agree with the third to fifteenth Church numbers. Enter the list comprehension in the cell below.

```
[2.0, 5.0, 12.0, 29.0, 70.0, 169.0, 408.0, 985.0, 2378.0, 5741.0, 13860.0, 33460.0, 80780
```

```
# Put your list comprehension here, then run the cell

[round(C(n)) for n in 3:15] # round due to using floats for sqrts
```

```
[0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, -1.0, -2.0]
```

```
[round(C(n))-chu(n) for n in 3:15] # slightly off as the numbers get really big due
to floating point inaccuracy
```