Category Theory 2022-23

Lecture 6

11th November 2022

Reformulation of limit

The category of D-cones in $\mathcal{C}$ has

- Objects: D-cones
- Morphisms from $(Z', (Z' \xrightarrow{r'_u} Du)_{u \in |G|})$
  to $(Z, (Z \xrightarrow{r_u} Du)_{u \in |G|})$

  are maps $Z' \xrightarrow{h} Z$ in $\mathcal{C}$ s.t. $r_u \circ h = r'_u \ \forall u \in |G|$

A <u>limit D-cone</u> is then just a terminal object in the category of D-cones.

Limit D-cones are determined up to isomorphism, because terminal objects are determined up to isomorphism!

Projective limits

The <u>projective limit</u> of an infinite sequence of maps $(X_{n+1} \xrightarrow{f_n} X_n)_{n \geq 0}$ in C is given by $P \in |C|$ and $(P \xrightarrow{p_n} X_n)_{n \geq 0}$ such that:

- for all n, $p_n = f_n \circ p_{n+1}$; and
- for any family $(Z \xrightarrow{x_n} X_n)_{n \geq 0}$ such that $x_n = f_n \circ x_{n+1}$ $\forall n$, there exists a unique $Z \xrightarrow{z} P$ such that $p_n \circ z = x_n$ $\forall n$.

$$
\begin{tikzcd}
\cdots \arrow[r, "f_3"'] & X_3 \arrow[r, "f_2"'] & X_2 \arrow[r, "f_1"'] & X_1 \arrow[r, "f_0"'] & X_0 \\
\cdots & Z \arrow[u, "x_3"'] \arrow[ur, "x_2"'] \arrow[urr, "x_1"'] \arrow[urrr, "x_0"'] \arrow[rr, dashed, "z"'] & & P \arrow[ull, "p_3"] \arrow[ul, "p_2"] \arrow[u, "p_1"'] \arrow[ur, "p_0"'] & \cdots
\end{tikzcd}
$$

<u>Set</u> has projective limits

$$P := \{ (x_n)_{n \geq 0} \mid \forall n \; f_n(x_{n+1}) = x_n \}$$

$$p_n := (x_n)_{n \geq 0} \mapsto x_n \quad : P \to X_n$$

A category $\mathcal{C}$ is said to be <u>(small-) complete</u> if every small diagram in $\mathcal{C}$ has a limit cone.

<u>Theorem</u> The following are equivalent.

(1) $\mathcal{C}$ is complete

(2) $\mathcal{C}$ has products and equalisers

<u>Proof idea</u>

(1) $\Rightarrow$ (2) is trivial.

For (2) $\Rightarrow$ (1) let $D : G \to \mathcal{C}$ be a diagram

The limit cone is the components of the equaliser of

$$\prod_{u \in |G|} Du \overset{l_1}{\underset{l_2}{\rightrightarrows}} \prod_{(u,v,e) \in \{(u,v,e) \mid u,v \in |G|,\, e \in G(u,v)\}} Dv$$

where $l_1 = (De \circ \pi_u)_{(u,v,e)}$ and $l_2 = (\pi_v)_{(u,v,e)}$ $\square$

A category is said to be <u>finitely complete</u> if every finite diagram has a limit cone.

<u>Theorem</u> The following are equivalent

(1) $\mathcal{C}$ is finitely complete

(2) $\mathcal{C}$ has finite products and equalisers

(3) $\mathcal{C}$ has terminal object and pullbacks

<u>Proof idea</u> $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$ are immediate.
$(2) \Rightarrow (1)$ is proved in same way as last theorem.

For $(3) \Rightarrow (2)$, first construct the binary product $X \times Y$ as the pullback of $X \xrightarrow{!} 1 \xleftarrow{!} Y$.
Then construct the equaliser of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ as the pullback of $X \xrightarrow{(f,g)} Y \times Y \xleftarrow{(1_Y, 1_Y)} Y$.

$\square$

A <u>colimit</u> in C of a diagram $D : G \to C$

is a limit in C^{op} of $D : G^{op} \to C^{op}$

Type of colimit	Diagram shape
I-indexed coproducts	$\overbrace{\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet}^{I}$
Initial object	$\emptyset$
Coequaliser	$\bullet \rightrightarrows \bullet$
Pushout	$\bullet \leftarrow \bullet \rightarrow \bullet$
Direct limit e.g.	$\bullet \to \bullet \to \bullet \to \bullet \to \cdots$

$\mathcal{C}$ is finitely cocomplete if every finite diagram has a colimit.

$\mathcal{C}$ is small cocomplete (or just cocomplete) if every small diagram has a colimit.

Theorem The following are equivalent

- $\mathcal{C}$ is finitely cocomplete
- $\mathcal{C}$ has finite coproducts and coequalisers
- $\mathcal{C}$ has initial object and pushouts

Theorem The following are equivalent

- $\mathcal{C}$ is cocomplete
- $\mathcal{C}$ has coproducts and coequalisers

Both theorems are immediate from the corresponding theorems about limits by duality

Coproducts in Set

The coproduct of $(X_i)_{i \in I}$ is $\sum_{i \in I} X_i := \{ (i,x) \mid i \in I, x \in X_i \}$

together with the cocone

$$\left(X_i \xrightarrow{in_i} \sum_{i \in I} X_i \right)_{i \in I}$$

$$in_i := x \mapsto (i,x)$$

Universal property

For any cocone $\left(X_i \xrightarrow{f_i} Z \right)_{i \in I}$

there exists a unique $\sum_{i \in I} X_i \xrightarrow{h} Z$ such that

$$\begin{tikzcd} \sum_{i \in I} X_i \arrow[r, "h"] & Z \\ X_i \arrow[u, "in_i"] \arrow[ur, "f_i"'] & \end{tikzcd}$$ commutes $\forall i \in I$

namely $h := (i,x) \mapsto f_i(x)$.

In $\underline{\text{Vect}}_k$ finite coproducts coincide with finite products. ($\underline{\text{Vect}}_k$ has $\underset{\sim}{b}i\underline{\text{products}}$.)

Given vector spaces V, W, the coproduct is

$$V \xrightarrow{v \mapsto (v,0)} V \times W \xleftarrow{(0,w) \mapsfrom w} W$$

Indeed given linear $V \xrightarrow{f} U$ and $W \xrightarrow{g} U$ the map $h := (v,w) \longmapsto f(v) + g(w)$ is the unique linear map such that the diagram below commutes

$$\begin{tikzcd}
V \arrow[r, "(-{,}0)"] \arrow[dr, "f"'] & V \times W \arrow[d, "h"] & W \arrow[l, "(0{,}-)"'] \arrow[dl, "g"] \\
& U &
\end{tikzcd}$$

A <u>coequaliser</u> is a colimit for

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

i.e., an object Q together with $Y \xrightarrow{q} Q$ s.t. $q \circ f = q \circ g$ satisfying:

for any $Y \xrightarrow{z} Z$ s.t. $z \circ f = z \circ g$

there exists a unique $Q \xrightarrow{w} Z$ s.t. $z = w \circ q$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, "w"] \\ & & Z \end{tikzcd}$$

Every q arising as a coequaliser is epi.

Morphisms that arise as coequalisers are called <u>regular epis</u>.

Coequalisers in <u>Set</u>

The coequaliser in <u>Set</u> of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$

is $Y \xrightarrow{q} Q$ defined as follows

$$Q := Y/\sim$$

where $\sim$ is the smallest equivalence relation such that $f(x) \sim g(x)$, for every $x \in X$.

$$q := y \mapsto [y]_\sim$$

(every element is mapped to its equivalence class)

Relations in a category

A representative for a <u>relation</u> between X and Y in $\mathcal{C}$ is a jointly monic pair

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[dr, "r_2"'] & X \\ & Y \end{tikzcd}$$

Recall <u>joint monicity</u> means, for any $Z \overset{x}{\underset{y}{\rightrightarrows}} R$

if $r_1 \circ x = r_1 \circ y$ and $r_2 \circ x = r_2 \circ y$ then $x = y$.

(If $\mathcal{C}$ has products this is $\equiv$ to $R \xrightarrow{(r_1, r_2)} X \times Y$ is mono.)

Given relations $\begin{tikzcd} R \arrow[r, "r_1"] \arrow[dr, "r_2"'] & X \\ & Y \end{tikzcd}$ and $\begin{tikzcd} R' \arrow[r, "r_1'"] \arrow[dr, "r_2'"'] & X \\ & Y \end{tikzcd}$

$R \sqsubseteq R'$ means there exists a (necessarily unique) diagonal making the two triangles commute

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, dashed] & X \\ Y & R' \arrow[u, "r_1'"'] \arrow[l, "r_2'"] \end{tikzcd}$$

$R \equiv R'$ means $R \sqsubseteq R'$ and $R' \sqsubseteq R$

Given $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} \begin{matrix} X \\ Y \end{matrix}$

we say $Z \overset{x}{\underset{y}{\rightrightarrows}} \begin{matrix} X \\ Y \end{matrix}$ are related by R (xRy)

if there exists a (necessarily unique) commuting diagonal

$$\begin{tikzcd} Z \arrow[r, "x"] \arrow[d, "y"'] \arrow[dr, dashed] & X \\ Y & R \arrow[l, "r_2"] \arrow[u, "r_1"'] \end{tikzcd}$$

$R \overset{r_1}{\underset{r_2}{\rightrightarrows}} \begin{matrix} X \\ X \end{matrix}$ is an equivalence relation if

$\forall Z \quad \forall Z \xrightarrow{x} X \qquad xRx$

$\forall Z \ \forall x, y : Z \to X \qquad xRy \Rightarrow yRx$

$\forall Z \ \forall x, y, z : Z \to X \qquad xRy \wedge yRz \Rightarrow xRz$

The <u>kernel pair</u> of $X \xrightarrow[f]{} Y$ is the pullback $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X$ of f along itself

$$\begin{tikzcd} R \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

<u>Exercise</u>

1) f is mono $\Leftrightarrow$ it has $X \overset{1_X}{\underset{1_X}{\rightrightarrows}} X$ as its kernel pair.

2) Any kernel pair is an equivalence relation

The symbiotic relationship between coequalisers and kernel pairs

1) If $Y \xrightarrow{q} Q$ is regular epi and q has a kernel pair $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} Q$ then q is the coequaliser of r_1, r_2

2) If $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} Y$ is a kernel pair and it has a coequaliser $Y \xrightarrow{q} Q$ then $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} Y$ is the kernel pair of q

A diagram

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} Y \xrightarrow{q} Q$$

in which r_1, r_2 is the kernel pair of q and q is the coequaliser of r_1, r_2 is called an exact fork.