Category Theory 2022-23

Lecture 11

16th December 2022

In $\underline{\mathbf{Set}}$ objects are determined up to isomorphism by their <u>global points</u>

$$X \cong Y \iff \underline{\mathbf{Set}}(1, X) \cong \underline{\mathbf{Set}}(1, Y)$$

The corresponding property does <u>not</u> hold in an arbitrary category with terminal object.

(Exercise: find counterexamples.)

In an arbitrary category $\mathcal{C}$ we need to consider <u>generalised points</u> of X

maps $Z \longrightarrow X$ where Z ranges over all of $|\mathcal{C}|$

Generalised points form a contravariant functor in Z

$$\begin{tikzcd} Z & & \mathcal{C}(Z,X) \arrow[d, "-\circ g"] \\ Z' \arrow[u, "g"] & \mapsto & \mathcal{C}(Z',X) \end{tikzcd}$$

We assume $\mathcal{C}$ is locally small

We write $\underline{y}X := \mathcal{C}(-,X) : \mathcal{C}^{op} \to \underline{Set}$

for this functor, the representable functor from X

(sometimes representable presheaf)

Proposition 1 (Corollary of the Yoneda Lemma to follow)

Objects are determined by generalised points; i.e.,

$$\underline{y}X \cong \underline{y}Y \text{ in } [\mathcal{C}^{op}, \underline{Set}] \Leftrightarrow X \cong Y \text{ in } \mathcal{C}$$

$[\mathcal{C}^{op}, \underline{Set}]$ is well-defined as a locally small category with a class of objects if and only if $\mathcal{C}$ is small. In the case of a locally small $\mathcal{C}$, the collection of objects of $[\mathcal{C}^{op}, \underline{Set}]$ lies outside the realm of sets and classes. Nevertheless one can make sense of $\underline{y}X \cong \underline{y}Y$, which refers to the hom class between 2 objects.

The mapping from object X to representable $\underline{y}X$ is itself a covariant functor

we henceforth assume $\mathcal{C}$ is small

the Yoneda functor $\underline{y} : \mathcal{C} \to [\mathcal{C}^{op}, \underline{Set}]$

$$
\begin{array}{ccc}
X & & \underline{y}X \\
g \downarrow & \longmapsto & \Downarrow \quad \left(Z \xrightarrow{f} X \longmapsto g \circ f \right)_Z \\
Y & & \underline{y}Y
\end{array}
$$

One can equivalently obtain the above functor from our original hom functor

$$\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \underline{Set}$$

using the exponential property of functor categories

$$
\frac{\mathcal{C}^{op} \times \mathcal{C} \to \underline{Set}}{\mathcal{C} \to [\mathcal{C}^{op}, \underline{Set}]}
$$

Proposition 2 (Corollary of the Yoneda lemma to follow)

The Yoneda functor is full and faithful

<u>Application</u> Let G be a group.

$\underline{y} : \underline{G} \to [\underline{G}^{op}, \underline{Set}]$ is (full and) faithful.

Since $\underline{G}$ has only one object,

$U := \alpha \mapsto \alpha_* : [\underline{G}^{op}, \underline{Set}] \to \underline{Set}$ is faithful.

So $U\underline{y} : \underline{G} \to \underline{Set}$ is faithful.

Since functors preserve isos this gives a faithful functor

$$U\underline{y} : \underline{G} \to \underline{Set}_{iso}$$

i.e., we have embedded G in a symmetric group, proving <u>Cayley's theorem</u>.

The argument works in ignorance!

One does not need to know that $[\underline{G}^{op}, \underline{Set}]$ is isomorphic to the category of right G actions (cf. week 3 puzzle).

One does not need to know that $\underline{y}*$ is the transitive right action of G on itself.

The Yoneda lemma

For any $F : \mathcal{C}^{op} \to \underline{\mathrm{Set}}$ and $X \in |\mathcal{C}|$

$$[\mathcal{C}^{op}, \underline{\mathrm{Set}}](\underset{\sim}{y}X, F) \cong FX$$

naturally in X and F.

Proof idea

The required bijections are

$$w \in FX \mapsto \left(Z \xrightarrow{f} X \mapsto F(f)(w) \right)_Z$$

$$\alpha : \underset{\sim}{y}X \Rightarrow F \mapsto \alpha_X(1_X)$$

One must then verify that these are mutual inverses and the naturality property. This is routine. $\square$

<u>Proof of Prop 2</u>: $\underline{y}$ is full & faithful

The Yoneda lemma gives

$$[\mathcal{C}^{op}, \underline{Set}](\underline{y}X, \underline{y}Y) \cong \underline{y}Y(X) = \mathcal{C}(X, Y)$$

where the right-to-left bijection is

$$X \xrightarrow{g} Y \mapsto (Z \xrightarrow{f} X \mapsto g \circ f)$$

which is the morphism action of $\underline{y}$ $\square$

<u>Proof of Prop 1</u>: $\underline{y}X \cong \underline{y}Y \iff X \cong Y$.

$\underline{y}$ is full and faithful. It therefore creates isos. $\square$

$F : \mathcal{C} \to \mathcal{D}$ <u>creates isos</u> if, for any $X, Y \in |\mathcal{C}|$ and iso $g : FX \to FY$ in $\mathcal{D}$, there exists a unique $f : X \to Y$ s.t. $Ff = g$; moreover, this unique f is an iso.

For a <u>small</u> category $\mathbb{C}$, the Yoneda Functor gives a full & faithful embedding of $\mathbb{C}$ into $[\mathbb{C}^{op}, \underline{Set}]$ its category of <u>presheaves</u>.

(other standard notation: $\hat{\mathbb{C}}$, $Psh(\mathbb{C})$)

$$\mathbb{C} \xhookrightarrow{\underline{y}} Psh(\mathbb{C})$$

preserves existing limits

(but <u>not</u> colimits)

complete, cocomplete, cartesian closed, a (Grothendieck) topos

Limits and colimits in $\mathrm{PSh}(\mathbb{C})$ are computed <u>pointwise</u> as in <u>Set</u>

E.g., given presheaves F and G on $\mathbb{C}$.

Define the <u>product presheaf</u> $F \times G$ by:

$$(F \times G)(X) := FX \times GX$$

product in Set

$$F \times G : \begin{array}{c} X \\ {\scriptstyle f}\uparrow \\ X' \end{array} \longmapsto \begin{array}{c} FX \times GX \\ \downarrow {\scriptstyle Ff \times Gf} \\ FX' \times GX' \end{array}$$

<u>Theorem</u> $\mathrm{Psh}(\mathbb{C})$ is cartesian closed.

<u>Proof outline</u> Given presheaves F, G we need to find an exponential presheaf $[F, G]$.

Suppose such a presheaf exists, then it must enjoy the following properties

$$[F,G](X) \cong \mathrm{Psh}(\mathbb{C})\left(\underline{y}X, [F,G]\right) \quad \text{(Yoneda)}$$

$$\cong \mathrm{Psh}(\mathbb{C})(\underline{y}X \times F, G) \quad \text{(defining property of } [F,G])$$

Accordingly, we <u>define</u>

$$[F,G](X) := \mathrm{Psh}(\mathbb{C})(\underline{y}X \times F, G)$$

The corresponding morphism action is determined by the naturality of the above bijections.

Explicitly the morphism action is

$$\begin{tikzcd} X & & \hat{\mathbb{C}}(\underset{\sim}{y}X \times F, G) \arrow[d, "\alpha \mapsto \left((g,w) \in \mathbb{C}(z,Y) \times F(z) \mapsto \alpha_z(f \circ g, y) \right)_z"] \\ Y \arrow[u, "f"] & \longmapsto & \hat{\mathbb{C}}(\underset{\sim}{y}Y \times F, G) \end{tikzcd}$$

A

One now needs to check that the presheaf so defined indeed satisfies the properties required of $[F, G]$.

This is left as an exercise (for the enthusiastic only)

Example $\underline{Psh}(\underline{G})$ (i.e. $[\underline{G}^{op}, \underline{Set}]$ cf. week 7 puzzle)

$\underline{Psh}(\underline{G}) \cong$ category of right G-actions (week 3 puzzle)

Let $\underline{A}, \underline{B}$ be presheaves corresponding to right G-actions $(A, \bullet_A)$, $(B, \bullet_B)$

We calculate $[\underline{A}, \underline{B}]$ in $\underline{Psh}(\underline{G})$.

since $\times$ is symmetric it doesn't matter that we have swapped the order

$$[\underline{A}, \underline{B}](*) \cong \underline{Psh}(\underline{G})(\underline{A} \times \underline{y}(*), \underline{B})$$

$$\cong \underline{G\text{-}Act}_R(\underline{A} \times G, \underline{B})$$

G with its self-right-action

$$\cong B^A \quad \text{(set of } \underline{\text{all}} \text{ functions } A \to B).$$

For the last bijection, any function $f : A \to B$ determines $\tilde{f} : A \times G \to B$ by

$$\tilde{f}(a, g) := f(a \bullet_A g^{-1}) \bullet_B g$$

We show that $\tilde{f}$ is equivariant

$$\tilde{f}((a,g)\cdot h) = \tilde{f}(a\cdot h, g\cdot h)$$
$$= f(a\cdot h\cdot h^{-1}\cdot g^{-1})\cdot g\cdot h$$
$$= f(a\cdot g^{-1})\cdot g\cdot h \quad = \quad \tilde{f}(a,g)\cdot h$$

The mapping $f \mapsto \tilde{f}$ gives the required bijection from B^A to $\underline{G\text{-Act}}_R(A\times G, B)$, with inverse $\phi \mapsto (a \mapsto \phi(a,e))$ for equivariant ϕ (e the group identity)

The presheaf structure on $[\underset{\sim}{A}, \underset{\sim}{B}]$ (see A) corresponds to the following right action on $\underline{G\text{-Act}}_R(A\times G, B)$

For equivariant $\phi : A\times G \longrightarrow B$ and $g\in G$

$$\phi\cdot g : (a,h) \mapsto \phi(a, h\cdot g) \qquad \text{B}$$

Via the bijection $\psi : f \mapsto \tilde{f}$, we obtain the following isomorphic action on B^A

$$(f\cdot g)(a) = (\psi^{-1}(\psi(f)\cdot g))(a) = (\tilde{f}\cdot g)(a,e)$$
$$\overset{\text{B}}{=} \tilde{f}(a,g) = f(a\cdot g^{-1})\cdot g.$$

This explicitly defines the exponential $[A,B]$ in $\underline{G\text{-Act}}_R$.