Category Theory 2022-23

Lecture 14

13th January 2023

A category $\mathcal{E}$ is an <u>elementary topos</u> if:

- it has finite limits
- is cartesian closed
- and has a <u>subobject classifier</u>:

an object Ω and map $1 \xrightarrow{\top} \Omega$ such that, for any mono $X \overset{m}{\rightarrowtail} Y$, there exists a unique map $Y \xrightarrow{\chi_m} \Omega$ such that

$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ Y \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

is a pullback square

(Equivalently, a partial map classifier for 1!)

It follows (with effort!) that every elementary topos $\mathcal{E}$

- has finite colimits
- is <u>locally</u> <u>cartesian closed</u>:

  every slice category $\mathcal{E}/X$ is cartesian closed

- all monos and epis are regular
- every $X \xrightarrow{f} Y$ factors as $f = X \overset{e}{\twoheadrightarrow} Z \overset{m}{\rightarrowtail} Y$ where e is epi and m mono

- in every pullback square with e epi

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "e"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

  - e' is also epi
  - f' mono $\Rightarrow$ f mono

A category $\mathcal{E}$ is a Grothendieck topos if:

- it is an elementary topos
- it has coproducts ($\equiv$ ly is cocomplete)
- it is locally small
- and it has a set of generators:

there exists a set $\mathcal{G} \subseteq |\mathcal{E}|$

such that for any parallel pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{E}$

we have

$$f = g \iff \forall A \in \mathcal{G} \text{ and } A \xrightarrow{x} X \quad f \circ x = g \circ x$$

$\underline{\mathrm{Set}}$ is a Grothendieck topos

The subobject classifier is $1 \xrightarrow{* \mapsto \mathrm{true}} \{\mathrm{true}, \mathrm{false}\}$

Given $X \overset{m}{\rightarrowtail} Y$, the unique $Y \xrightarrow{\chi_m} \{\mathrm{true}, \mathrm{false}\}$

s.t.

$$\begin{tikzcd}
X \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "* \mapsto \mathrm{true}"] \\
Y \arrow[r, "\chi_m"'] & \{\mathrm{true}, \mathrm{false}\}
\end{tikzcd}$$

is $\chi_m(y) = \begin{cases} \mathrm{true} & \text{if } y \in m(X) \\ \mathrm{false} & \text{if } y \notin m(X) \end{cases}$

$\{1\}$ is a (singleton) set of generators

(1 is a generating object.)

$Psh(\mathbb{C})$ is a Grothendieck topos, for any small category $\mathbb{C}$.

A <u>sieve</u> on $X \in |\mathbb{C}|$ is a set S of maps in $\mathbb{C}$ with codomain X s.t.

$$Y \xrightarrow{f} X \in S \text{ and } Z \xrightarrow{g} Y \text{ in } \mathbb{C} \Rightarrow Z \xrightarrow{g \circ f} X \in S$$

$\Omega(X) :=$ the set of all sieves on X

$$\begin{tikzcd} X & & \Omega(X) \arrow[d] \\ Y \arrow[u, "f"] & \longmapsto & \Omega(Y) \end{tikzcd} \qquad S \mapsto \{Z \xrightarrow{g} Y \mid Z \xrightarrow{f \circ g} X \in S\}$$

Given a mono $P \xrightarrow{m} Q$ in $Psh(\mathbb{C})$

$Q \xrightarrow{\chi_m} \Omega$ defined by:

notation for $Q(f)(y)$

$$(\chi_m)_X := y \mapsto \{Z \xrightarrow{f} X \mid \exists x \in P(Z).\ m_Z(x) = y \cdot f\} : Q(X) \to \Omega(X)$$

is unique such that

$$\begin{tikzcd} P \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ Q \arrow[r, "\chi_m"'] & \Omega \end{tikzcd} \qquad \top_X := * \mapsto \{Z \xrightarrow{f} X \mid true\}$$

the maximum sieve

A set of generators is

$$G := \{ \underline{y}(X) \mid X \in |\mathbb{C}| \}$$

the set of all representables

<u>Proof that G is a generating set.</u>

Suppose $P \overset{p}{\underset{q}{\rightrightarrows}} Q$ is a parallel pair in $Psh(\mathbb{C})$

such that

$$\forall X \in |\mathbb{C}|, \forall \underline{y}X \xrightarrow{x} P \text{ in } Psh(\mathbb{C}), \quad p \circ x = q \circ x$$

By Yoneda, this says

$$\forall X \in |\mathbb{C}| \quad \forall x \in P(X) \quad p_X(x) = q_X(x).$$

I.e. $p = q$.

$Sh(S)$ is a Grothendieck topos, for any top. space S.

A key preliminary result is

<u>Theorem</u> The inclusion functor

$$Sh(S) \xhookrightarrow{I} Psh(\mathcal{O}(S))$$

has a left-exact left adjoint

preserves finite limits

$$Psh(\mathcal{O}(S)) \xrightarrow{a} Sh(S)$$

the <u>associated sheaf</u> (or <u>sheafification</u>) functor.

Grothendieck's $(\cdot)^+$ Functor $Psh(\mathcal{O}(S)) \to Psh(\mathcal{O}(S))$

Given a presheaf $P : \mathcal{O}(S)^{op} \to \underline{\text{Set}}$

Define $P^+(U)$ = equivalence classes of matching families covering U

Recall a matching family covering U is

$$(x_i \in P(U_i))_{i \in I} \quad \text{for some } (U_i)_{i \in I} \text{ with } \bigcup_{i \in I} U_i = U$$

such that, for all $i, j \in I$, $x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$

Define

$$(x_i \in P(U_i))_{i \in I} \sim (x'_j \in P(U'_j))_{j \in J}$$

to hold if $\forall i \in I, j \in J$ $x_i|_{U_i \cap U'_j} = x'_j|_{U_i \cap U'_j}$

Exercise: • Work out the morphism action of P^+ and check that it respects the equivalence relation.

• Work out the morphism action of the functor $(\cdot)^+$

The functions

$$x \in P(U) \longmapsto [(x \in P(U))]_\sim$$

indexed by $U \in \mathcal{O}(X)$ define a natural transformation $P \to P^+$

We say P is <u>separated</u> if every matching family has at most one amalgamation.

(Recall a presheaf P is a <u>sheaf</u> if every matching family has a unique amalgamation.)

<u>Lemma</u> (Grothendieck)

1) The functor $(\cdot)^+ : Psh(\mathcal{O}(X)) \to Psh(\mathcal{O}(X))$ preserves finite limits.

2) Given a sheaf Q and presheaf map $P \xrightarrow{f} Q$, there is a unique presheaf map $P^+ \xrightarrow{\bar{f}} Q$ s.t.

$$\begin{tikzcd} P^+ \arrow[r, "\bar{f}"] & Q \\ P \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

3) For every presheaf P, P^+ is separated.

4) If P is separated then P^+ is a sheaf.

It follows immediately from the lemma that $\underline{a} := (\ )^{++}$ defines a left-exact left adjoint to.

$I : Sh(s) \hookrightarrow Psh(\mathcal{O}(s))$

<u>A subtle point</u>. I can be written as a composition of inclusion functors with left adjoints (in black)

separated presheaves

$$
\begin{tikzcd}[column sep=huge]
Sh(s) \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] & Sep(s) \arrow[l, bend right, "(\cdot)^+"'] \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] & Psh(\mathcal{O}(s)) \arrow[l, bend right, "\underline{b}"'] \arrow[l, bend right=70, "(\cdot)^+"'] \arrow[ll, bend left=50, "\underline{a}"] \arrow[ll, phantom, shift right=5, "\top"]
\end{tikzcd}
$$

We also have a functor $(\cdot)^+ : Psh(\mathcal{O}(s)) \to Sep(s)$

Since adjoints compose $\underline{a} \cong (\cdot)^+ \circ \underline{b}$

and by definition $\underline{a} = (\cdot)^+ \circ (\cdot)^+$

However, $\underline{b} \not\cong (\cdot)^+ : Psh(\mathcal{O}(s)) \to Sep(s)$

An advantage of $(\cdot)^+$ is it preserves finite limits ($\underline{b}$ doesn't!)

We are looking at a special type of adjunction

A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be <u>reflective</u> if the inclusion functor $\mathcal{C}' \overset{I}{\hookrightarrow} \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{C}'$.

In such an adjunction, the counit $\varepsilon : FI \Rightarrow 1_{\mathcal{C}'}$ is a natural isomorphism and, for any $Y \in |\mathcal{C}'|$, the component $Y \xrightarrow{\eta_Y} IFY$ of the unit $\eta : 1_{\mathcal{C}} \Rightarrow IF$ is an iso.

These observations are generalised by:

<u>Proposition</u> Given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with counit $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ and unit $\eta : 1_{\mathcal{C}} \Rightarrow GF$.

1) G is faithful iff every component of ε is an epi in $\mathcal{D}$.
2) G is full iff every component of ε is a split mono in $\mathcal{D}$.
3) G is full & faithful iff every ε component is an iso in $\mathcal{D}$.

If G is full and faithful then define $\mathcal{C}'$ to be the full subcategory of $\mathcal{C}$ on objects X such that η_X is an iso. Then $FI : \mathcal{C}' \to \mathcal{D}$ is an equivalence of categories and

$$
\begin{tikzcd}
\mathcal{C}' \arrow[r, hook, "I"] \arrow[d, "FI"'] \arrow[dr, phantom, "\cong\, \Swarrow \eta I"] & \mathcal{C} \\
\mathcal{D} \arrow[ur, "G"'] &
\end{tikzcd}
$$

The full subcategory $\mathcal{C}'$ in the proposition enjoys the property that it is replete:

$$X \in |\mathcal{C}'|,\ Y \in |\mathcal{C}|,\ X \cong Y \Rightarrow Y \in \mathcal{C}'.$$

By the proposition, any adjunction with full and faithful right adjoint (in particular any full reflective subcategory) is equivalent to a replete full reflective subcategory.

Naturally occurring examples of full reflective subcategories are often replete by definition

e.g.
$$\mathrm{Sh}(S) \hookrightarrow \mathrm{Psh}(\mathcal{O}(S))$$

$$\mathrm{Sep}(S) \hookrightarrow \mathrm{Psh}(\mathcal{O}(S))$$

$$\mathrm{Sh}(S) \hookrightarrow \mathrm{Sep}(S).$$

Suppose S is a reflective replete full subcategory of $\mathcal{C}$ with reflection $F \dashv I : S \hookrightarrow \mathcal{C}$.

<u>Proposition</u> I creates limits.

It follows that any reflective full subcategory of a complete category is complete, with limits calculated as in the super-category.

<u>Proposition</u> If a diagram $D : G \to S$ has colimit $\varinjlim D$ in $\mathcal{C}$ then it has colimit $F(\varinjlim D) \circ (\eta ID)$ in S.

So any full reflective subcategory of a cocomplete category is cocomplete, with colimits calculated by reflecting colimits in the supercategory.

(Continuing with the assumptions of the previous page.)

<u>Proposition</u> If $\mathcal{C}$ is cartesian closed and $F: \mathcal{C} \to \mathcal{S}$ preserves finite products then $\mathcal{S}$ is an exponential ideal of $\mathcal{C}$:

$$Y \in |\mathcal{S}|,\ X \in |\mathcal{C}| \Rightarrow [X, Y] \in \mathcal{S}$$

In particular $\mathcal{S}$ is cartesian closed and the inclusion $\mathcal{S} \xhookrightarrow{I} \mathcal{C}$ preserves cartesian closed structure.

Returning to

$$\begin{tikzcd} \mathrm{Sh}(S) \arrow[r, hook, "i", bend right] \arrow[r, phantom, "\perp"] & \mathrm{PSh}(\mathcal{O}(S)) \arrow[l, bend right, "a"'] \end{tikzcd}$$

By Lecture 11, $\mathrm{PSh}(\mathcal{O}(S))$ is complete, cocomplete and cartesian closed.

By the above, $\mathrm{Sh}(S)$ is also complete, cocomplete and cartesian closed.

The subobject classifier in $Sh(S)$

$$\Omega_S(U) := \{U' \in \mathcal{O}(S) \mid U' \subseteq U\}$$

$$\begin{array}{ccc} U & & \Omega_S(U) \\ \cup\!\!\mid & \longmapsto & \downarrow (U' \mapsto U' \cap V) \\ V & & \Omega_S(V) \end{array}$$

This is a sheaf. It is isomorphic to the sheaf C_{S_i} of continuous functions into Sierpinski space $S_i = \{\bot, \top\}$ $\mathcal{O}(S_i) = \{\emptyset, \{\top\}, \{\bot, \top\}\}$

$1 \xrightarrow{\top} \Omega$

$$\top_U := * \mapsto U : 1_U \to \Omega_S(U)$$

Exercise

Verify that the above indeed defines a subobject classifier in $Sh(S)$.

Key Point Consider any subpresheaf P

$$P \subseteq Q$$

of a sheaf Q,

the components of the mono are all subset inclusions

characterise when it holds that P is also a sheaf.

$\boxed{\text{Sh(S) is indeed an elementary topos}}$

Proposition Every representable $\underline{y}(U)$ is a sheaf.

Since the set of representables is a generating set in $Psh(O(S))$, it is also a generating set in $Sh(S)$.

$$\boxed{Sh(S) \text{ is a Grothendieck topos!}}$$

Exercise
- Every representable is isomorphic to a unique subsheaf of 1.
- Every subsheaf of 1 is isomorphic to a unique representable.

Representables coincide with subterminal objects, and $Sh(S)$ has a generating set of subterminals.

Other example families of toposes

- Sheaves on a locale. Grothendieck toposes generalising sheaves on a topological space.
- Sheaves on a site (small category + Grothendieck topology). These are exactly the Grothendieck toposes.
- Realisability toposes. Elementary (but not Grothendieck) toposes related to logic and computability theory.

Higher-dimensional analogues of Grothendieck toposes are important in topology; e.g.,

- Infinity toposes of Jacob Lurie.