Category Theory 2022-23

Lecture 13

6th January 2023

Let S be a topological space and $\mathcal{O}(S)$ the collection of <u>open subsets</u> of S partially ordered by subset inclusion ($\subseteq$)

For $U \in \mathcal{O}(S)$ define

$$C(U) := \{ f : U \to \mathbb{R} \mid f \text{ continuous} \}$$

If $V \subseteq U$ then we have

$$f \longmapsto f\restriction_V \ : C(U) \to C(V)$$

The above defines a functor

$$C : \mathcal{O}(S)^{op} \longrightarrow \underline{\mathrm{Set}} \qquad (\mathcal{O}(S) \text{ qua category})$$

i.e., a presheaf.

The presheaf C satisfies a further condition expressing that the continuity of a function $f: U \to \mathbb{R}$ is determined locally within U

The sheaf property for C

Suppose $(U_i)_{i \in I}$ is an open cover of $U \in \mathcal{O}(S)$.

Suppose $(f_i : U_i \to \mathbb{R})_{i \in I}$ is a family of continuous functions such that

$$\forall i, j \in I \quad f_i\restriction_{U_i \cap U_j} = f_j\restriction_{U_i \cap U_j} \; : U_i \cap U_j \to \mathbb{R}$$

then there exists a unique continuous function

$$f : U \to \mathbb{R}$$

such that $\quad f\restriction_{U_i} = f_i \qquad \forall i \in I$.

Sheaf for a topological space

Let $P : \mathcal{O}(S)^{op} \to \underline{\mathrm{Set}}$ be a presheaf for S

A family $(x_i \in P(U_i))_{i \in I}$ is said to be *matching* if

$$\forall i,j \in I \qquad x_i \restriction_{U_i \cap U_j} = x_j \restriction_{U_i \cap U_j} \in P(U_i \cap U_j)$$

(Given $x \in P(U)$ and $V \subseteq U$ we write $x \restriction_V$ for the element $P(\underbrace{U \to V}_{\text{the unique map } U \to V \text{ in } \mathcal{O}(S)})(x) \in P(V)$.)

An element $x \in P(U)$ where $U := \bigcup_{i \in I} U_i$ is an *amalgamation* of $(x_i \in P(U_i))_{i \in I}$ if

$$\forall i \in I \qquad x \restriction_{U_i} = x_i$$

It is easy to show that any family $(x_i \in P(U_i))_{i \in I}$ that has an amalgamation is necessarily matching.
(Exercise.)

Definition (Sheaf)

A presheaf is said to be a sheaf if every matching family has a unique amalgamation.

By our initial discussion

$$C(U) := \{ f : U \to \mathbb{R} \mid f \text{ continuous} \}$$

is a sheaf.

Other example sheaves on S

For any set X

$$F_X(U) := \{ f : U \to X \mid f \text{ any set-theoretic function} \}$$

For any topological space T

$$C_T(U) := \{ f : U \to T \mid f \text{ continuous} \}$$

If S, T are smooth manifolds

$$D_T(U) := \{ f : U \to T \mid f \text{ smooth} \}$$

If S, T are complex manifolds

$$H_T(U) := \{ f : U \to T \mid f \text{ holomorphic} \}$$

Let's fix $S, T := \mathbb{C}$ and look in more detail at the sheaf $H : \mathcal{O}(\mathbb{C})^{op} \to \underline{\text{Set}}$

$$H(U) := \{ h : U \to \mathbb{C} \mid h \text{ holomorphic} \}$$

$H(U)$ is indeed a presheaf

if $h: U \to \mathbb{C}$ is holomorphic and $V \subseteq U$ then $h\restriction_V : V \to \mathbb{C}$ is holomorphic

and moreover a sheaf

if $(h_i : U_i \to \mathbb{C})_{i \in I}$ is a matching family of holomorphic functions then the unique amalgamating function $h: U \to \mathbb{C}$ $(U := \bigcup_{i \in I} U_i)$ is holomorphic.

It further satisfies a more specific property

Given $h: U \to \mathbb{C}$ holomorphic and <u>connected</u> open $U' \supseteq U$ there is at most one holomorphic $h' : U' \to \mathbb{C}$ s.t. $h'\restriction_U = h$.

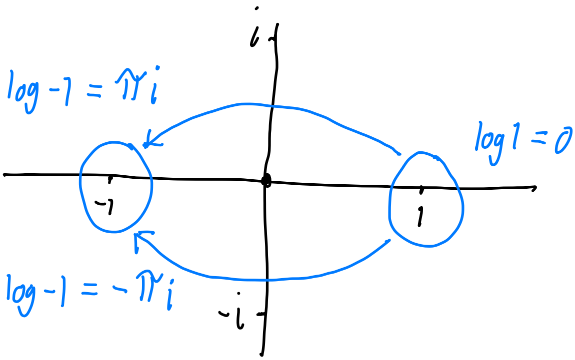
The last property suggests the idea of <u>analytic continuation</u>.

<u>Naively</u>:

Given holomorphic $h: U \to \mathbb{C}$ find as large as possible connected open $U' \supseteq U$ for which there exists a necessarily unique holomorphic $h' : U' \to \mathbb{C}$ with $h'\restriction_U = h$.

This is <u>too</u> naive!

E.g., we cannot extend the complex logarithm function in the above naive way



The full analytic continuation of $\log z$ does not define it on $\mathbb{C} - \{0\}$ but rather on an ascending/descending spiral surface above $\mathbb{C} - \{0\}$

the complex logarithm function

$$\begin{tikzcd}
\mathbb{R} \times \mathbb{R}_+ \arrow[rr, "(\theta, r) \mapsto \ln r + i\theta"] \arrow[dr, "(\theta, r) \mapsto re^{i\theta}"'] & & \mathbb{C} \arrow[dl, "z \mapsto e^z"] \\
& \mathbb{C} - \{0\} \arrow[d] & \\
& \mathbb{C} &
\end{tikzcd}$$

the domain of the full complex logarithm function is a <u>Riemann surface</u> over $\mathbb{C}$

Analytic continuations can be gathered together into a single Riemann surface, the <u>universal analytic function</u>

A <u>germ</u> is a pair $g = (z_0, (a_n)_{n\geq 0})$ such that the power series

$$F_g(z) := a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

Converges on an open subset of $\mathbb{C}$ containing z_0.

Let G be the set of germs with the following topology.
$V \subseteq G$ is <u>open</u> if:

for every $g = (z_0, (a_n)_{n\geq 0}) \in V$
there exists open $U_0 \subseteq \mathbb{C}$ with $z_0 \in U_0$ such that

- for every $z_0' \in U_0$, $F_g(z_0')$ is defined, <u>and</u>
- $g' \in V$ where g' is the unique germ $(z_0', (a_n')_{n\geq 0})$ satisfying

$$F_{g'}(z) = F_g(z) \text{ on some open neighbourhood of } z_0'$$

The above defines a topological space $\mathcal{G}$ together with a projection function

$$p := (z_0, (a_n)_{n\geq 0}) \mapsto z_0 \quad : \mathcal{G} \to \mathbb{C}$$

The function p is continuous. ($\mathcal{G}$ is a bundle over $\mathbb{C}$.)

The function p is also étale:

For every $g \in \mathcal{G}$ there exists open $V \subseteq \mathcal{G}$ with $g \in V$ such that $p(V) \subseteq \mathbb{C}$ is open and $p|_V : V \to p(V)$ is a homeomorphism

The space $\mathcal{G}$ is Hausdorff. (Exercise.)

The space $\mathcal{G}$ is known as the universal holomorphic function as it comprises all analytic functions.

It allows a precise definition of analytic continuation

Analytic continuations

Given a holomorphic function $h : U \to \mathbb{C}$ (for open $U \subseteq \mathbb{C}$) and $z_0 \in U$, let

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

be the power series expansion of h at z_0. Then the <u>analytic continuation</u> of h at z_0 is given by the connected component of $\mathcal{G}$ containing $(z_0, (a_n)_n)$.

The connected component of $\mathcal{G}$ containing $(z_0, (a_n)_n)$

$$\begin{tikzcd} H \arrow[r, "g \mapsto F_g(p(g))"] \arrow[d, "p|_H"'] & \mathbb{C} \\ \mathbb{C} & \end{tikzcd}$$

We have 2 seemingly very different mathematical structures embodying all holomorphic functions

- The sheaf $H : \mathcal{O}(\mathbb{C})^{op} \longrightarrow \underline{Set}$

  Models holomorphic functions <u>locally</u> $(U \rightarrow \mathbb{C})$

- The étale bundle $p : G \longrightarrow \mathbb{C}$

  Models holomorphic functions <u>globally</u>

Category theory shows that these two views of holomorphic functions arise as just one instance of a deep equivalence between sheaves and étale bundles valid for any topological space S.

The category $Sh(S)$ of <u>sheaves on S</u> is the full subcategory of $Psh(\mathcal{O}(S))$ on sheaves.

The category of <u>bundles over S</u> is just the slice category $\underline{Top}/S$.

Define $\Gamma : \underline{Top}/S \to Sh(S)$ by

$$\Gamma\begin{pmatrix} T \\ \downarrow p \\ S \end{pmatrix}(U) = \left\{ s : U \to T \;\middle|\; \begin{array}{l} s \text{ is continuous and} \\ p(s(x)) = x \quad \forall x \in U \end{array} \right\}$$

$\Gamma(p)$ is the <u>sheaf of local sections</u> of p.

<u>Exercise</u> • Verify that $\Gamma(p)$ is indeed a sheaf.

• Define the morphism action of Γ.

A bundle $T \xrightarrow{p} S$ is <u>étale</u> if, for any $y \in T$, there exists open $V \subseteq T$ with $y \in V$ such that $p(V)$ is an open subset of S and $p : V \to p(V)$ is a homeomorphism.

(Étale maps are also known as local homeomorphisms.)

We write $\text{Étale}(S)$ for the full subcategory of $\underline{\text{Top}}/S$ whose objects are étale maps.

<u>Theorem</u>

The functor $\Gamma : \text{Étale}(S) \to Sh(S)$ is (part of) an equivalence of categories.

Over a topological space S, sheaves are equivalent to étale bundles.

Outline proof

We define $\Lambda : Sh(S) \to \acute{E}tale(S)$,
which is the other half of the equivalence

Λ maps a sheaf F to its bundle of germs

Let $F : \mathcal{O}(S)^{op} \to \underline{Set}$ be a sheaf.

For any $x \in S$ consider the set

$$\{ (U, f) \mid U \in \mathcal{O}(S),\ x \in U,\ f \in F(U) \}$$

of elements of F local to x, with the equivalence relation

$$(U, f) \sim (U', f') \Leftrightarrow \exists U'' \subseteq U \cap U' \text{ s.t. } x \in U'' \text{ and } f\restriction_{U''} = f'\restriction_{U''}.$$

Given (U, f) as above the germ of f at x ($\mathrm{germ}_x f$) is the equivalence class $[(U, f)]_\sim$.

We define a bundle $T_F \xrightarrow{p_F} S$

T_F has underlying set

$$T_F = \{(x, \mathrm{germ}_x f) \mid U \in \mathcal{O}(S), x \in U, f \in FU\}$$

A subset $V \subseteq T_F$ is open if it satisfies:

if $(x, \mathrm{germ}_x f) \in V$ where $f \in FU$

then $\exists U' \subseteq U$ with $x \in U'$ s.t.,

$$(x', \mathrm{germ}_{x'}(f|_{U'})) \in V \qquad \forall x' \in U'.$$

(This topology is not in general Hausdorff, even when S is Hausdorff.)

The function $p_F : T_F \to S$ is first projection.

One verifies that p_F is continuous and étale.

The Functor $\Lambda : Sh(S) \to \text{Étale}(S)$

has as its action on objects

$$
\begin{tikzcd}
F \arrow[r, mapsto] & T_F \arrow[d, "p_F"] \\
 & S
\end{tikzcd}
$$

One needs to further define the action on morphisms.

To show we have an equivalence of categories one finds natural isomorphisms

$$
\begin{tikzcd}
T_{\Gamma p} \arrow[rr, "\cong"] \arrow[dr, "p_{\Gamma p}"'] & \arrow[d, phantom, "A"] & T \arrow[dl, "p"] \\
 & S &
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, "\cong", "B"'] & \Gamma \Lambda F
\end{tikzcd}
$$

A is where the property that p is étale is used

B is where the amalgamation property of F is used.