Category Theory 2022-23

Lecture 2

14th October 2022

A key philosophy behind category theory is that it is helpful to consider mathematical structures (objects) in combination with a notion of morphism between them.

Categories themselves are a form of mathematical structure.

So what are the morphisms between categories?

A functor $F: \mathcal{C} \to \mathcal{D}$ ($\mathcal{C}, \mathcal{D}$ categories) is given by:

- a function $F_0 : |\mathcal{C}| \to |\mathcal{D}|$
- for every $X, Y \in |\mathcal{C}|$ a function $F_1 : \mathcal{C}(X,Y) \to \mathcal{D}(F_0 X, F_0 Y)$

Such that

(The blue annotations are normally omitted)

- $F_1(1_X) = 1_{F_0 X} \quad \forall X \in |\mathcal{C}|$
- $F_1(g \circ_{\mathcal{C}} f) = (F_1 g) \circ_{\mathcal{D}} (F_1 f) \quad \forall X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$

There is an obvious identity functor from any category to itself. There is also an obvious composite functor $\mathcal{C} \xrightarrow{G \circ F} \mathcal{E}$ for any two functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$. These satisfy the identity and associativity laws.

Exercise Work out the details of the above.

Thus we can form a category whose objects are categories and whose morphisms are functors.

But there are set-theoretic size issues. Categories are in general large structures. Does it make sense to consider a category of all categories and if so is it an object of itself?

We avoid such issues by circumventing them. Define

Cat The category with small categories as objects and functors as morphisms.

Forward pointer: Cat should really be defined as a 2-category

This gives us a locally small category of categories.

In spite of the restriction to small categories in Cat it still makes sense to consider functors between arbitrary categories.

Example functors

$U : \underline{Grp} \to \underline{Set}$ the forgetful functor

$U(G, m, e) := G$ object action

$U((G,m,e) \xrightarrow{h} (G',m',e')) := G \xrightarrow{h} G'$

morphism action

$$\left.\begin{array}{l} U\ \underline{Top} \to \underline{Set} \\ U : \underline{Vectk} \to \underline{Set} \end{array}\right\} \text{forgetful functors analogous to the above}$$

If G, H are groups then

functors from $\underline{G}$ to $\underline{H}$ $\cong$ homomorphisms from G to H

(and ditto for monoids)

If P, P' are preorders / posets then

functors from $\underline{P}$ to $\underline{P}'$ $\cong$ monotone (i.e. order-preserving) functions from P to P'.

We have so far seen 2 main kinds of examples of categories

1 - Categories whose objects are mathematical structures, and whose morphisms are transformations / relations between structures

E.g., <u>Set</u>, <u>Grp</u>, <u>Top</u>, $\underline{\mathbf{Vect}}_K$, <u>Rel</u>, <u>Cat</u>

Such a category is a single mathematical metastructure that encompasses a whole area of mathematics via its structures (objects) and transformations (morphisms)

Category theory is the "mathematics of mathematics." [E. Cheng]

Watch the video: "What is category theory?"

2 – Individual mathematical structures recast as categories.

E.g. Monoids $\underline{M}$, groups $\underline{G}$

posets / preorders $\underline{P}$

The notion of category axiomatises a very general kind of mathematical structure, of which many familiar mathematical structures arise as natural special cases.

Categorification: derive (standard) mathematics as instances of (sometimes more general) category-theoretic mathematics.

We now start to explore a third rich source of categories

3 — Categories obtained from other categories by category-theoretic constructions

A major part of the power of category theory is that it provides a powerful toolbox of constructions on categories

Opposite (or dual) categories

If $\mathcal{C}$ is a category its <u>opposite</u> $\mathcal{C}^{op}$ is defined by:

$|\mathcal{C}^{op}| := |\mathcal{C}|$

$\mathcal{C}^{op}(X,Y) := \mathcal{C}(Y,X)$

1_X in $\mathcal{C}^{op} := 1_X$ in $\mathcal{C}$

Idea: turn the morphisms around

Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}^{op}$ (i.e. $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathcal{C}$)

$g \circ f$ in $\mathcal{C}^{op} := f \circ g$ in $\mathcal{C}$

Examples

For a group G, $(\underline{G})^{op} = \underline{(G^{op})}$ where G^{op} is the <u>opposite group</u>.

If G is abelian then $\underline{G} = \underline{G}^{op}$

For a poset/preorder P, $(\underline{P})^{op} = \underline{(P^{op})}$

where P^{op} is the <u>dual order</u>

Thus opposite categories generalise standard constructions of dual/opposite structures

Observe that $(\mathcal{C}^{op})^{op} = \mathcal{C}$. (Taking opposites is an <u>involution</u>.)

Insert → Observe functors $C \to D$ in 1-1 correspondence with functors $C^{op} \to D^{op}$

Contravariant Functors

A functor $F : C^{op} \to D$ is called a **contravariant functor** from C to D.

(Ordinary functors $F : C \to D$ are said to be **covariant**.)

Example: Dual vector space

If V is a vector space over K, recall its **dual space**

$$V^* := \{ f : V \to K \mid f \text{ is linear} \}$$

The dual space construction is the object action of a contravariant functor from $\underline{Vect}_K$ to itself.

$$V \mapsto V^* \qquad \text{Object action}$$

$$\begin{array}{c} V \\ \uparrow \lambda \\ W \end{array} \longmapsto \begin{array}{c} V^* \\ \downarrow \\ W^* \end{array} \quad f \mapsto f \circ \lambda \qquad \text{Morphism action}$$

We write this map in its orientation in $\underline{Vect}_K$.

Example : The Self-duality of Rel

If $R : X \times Y \to 2$ is a relation from X to Y

then its transpose $R^T : Y \times X \to 2$ is defined by

$$y R^T x \iff x R y$$

Using this we define a contravariant functor from $\underline{\mathrm{Rel}}$ to itself $T : \underline{\mathrm{Rel}}^{op} \to \underline{\mathrm{Rel}}$

$$X \longmapsto X$$

Object action

$$\begin{tikzcd} X \\ Y \arrow[u, "R"] \end{tikzcd} \longmapsto \begin{tikzcd} X \arrow[d, "R^T"] \\ Y \end{tikzcd}$$

Morphism action

orientation in $\underline{\mathrm{Rel}}$

The same definition gives a functor $T^{op} : \underline{\mathrm{Rel}} \to \underline{\mathrm{Rel}}^{op}$ (a contravariant functor from $\underline{\mathrm{Rel}}^{op}$ to itself)

Notice that $T \circ T^{op} = 1_{\underline{\mathrm{Rel}}}$ and $T^{op} \circ T = 1_{\underline{\mathrm{Rel}}^{op}}$. So $\underline{\mathrm{Rel}} \cong \underline{\mathrm{Rel}}^{op}$.

Exercise (Powerset functors)

1. Find a contravariant functor

$$\underline{\mathrm{Set}}^{op} \longrightarrow \underline{\mathrm{Set}}$$

whose object action is $X \longmapsto \mathcal{P}X$ — the powerset of X

2. Find a covariant functor

$$\underline{\mathrm{Set}} \longrightarrow \underline{\mathrm{Set}}$$

whose object action is $X \longmapsto \mathcal{P}X$

3. Find a second (i.e., different) solution to question 2.

Product categories

The product $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}$ and $\mathcal{D}$ is defined by

$$|\mathcal{C} \times \mathcal{D}| := |\mathcal{C}| \times |\mathcal{D}|$$

$$(\mathcal{C} \times \mathcal{D})((X, Y), (X', Y')) := \mathcal{C}(X, X') \times \mathcal{D}(Y, Y').$$

$$1_{(X,Y)} := (1_X, 1_Y)$$

$$(f, g) \circ (f', g') := (f \circ f', g \circ g')$$

There are evident projection functors

$$\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$$

$$\pi_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

Exercises – Fill in the details

– Generalise to arbitrary finite products $\mathcal{C}_1 \times \dots \times \mathcal{C}_n$ and general (indexed) product categories $\prod_{i \in I} \mathcal{C}_i$

It is now clear what is meant by a multi-argument functor

$$F : \mathcal{C}_1 \times \dots \times \mathcal{C}_n \to \mathcal{D}$$

The hom functor

(This plays a fundamental role in category theory)

If $\mathcal{C}$ is a locally small category then the <u>hom functor</u>

$$\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathrm{Set}}$$

is defined by

$$\mathcal{C}(X,Y) := \text{the hom set } \mathcal{C}(X,Y)$$ object action

$$\begin{tikzcd} X & Y \arrow[d, "h"] \\ X' \arrow[u, "g"] & Y' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathcal{C}(X,Y) \arrow[d] \\ \mathcal{C}(X',Y') \end{tikzcd} \quad f \mapsto h \circ f \circ g$$ Morphism action

<u>Exercise</u> Verify that this is a functor.

The hom functor is contravariant in its first argument and covariant in its second.

Coda: Exploiting duality

Every category-theoretic concept gives rise to a dual concept obtained by interpreting the original concept in the opposite category

E.g. epimorphism (epi) the dual of monomorphism

Quick definition $X \xrightarrow{f} Y$ is an epimorphism in $\mathcal{C}$ if f is a monomorphism in $\mathcal{C}^{op}$.

Expanded definition $X \xrightarrow{f} Y$ is an epimorphism if, for all $Y \overset{g}{\underset{h}{\rightrightarrows}} Z$, $g \circ f = h \circ f \Rightarrow g = h$.

From the quick definition, for any property of monomorphisms there is a corresponding dual property of epimorphisms

In Set the epimorphisms are the surjections.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to preserve monomorphisms (respectively epimorphisms) if, for every mono (resp. epi) f, it holds that Ff is also mono (resp. epi)

Week 2 puzzle

- Does every functor $F : \underline{Set} \to \underline{Set}$ preserve monos ?
- " " " " epis ?