

# Category Theory 2022-23

## Lecture 4

28<sup>th</sup> October 2022

Mat<sub>K</sub> - the category of (natural numbers and) matrices over K

$$|\text{Mat}_K| := \mathbb{N}$$

Mat<sub>K</sub> (m,n) := the set of <sup>rows</sup> $n \times$  <sup>columns</sup> $m$  matrices with entries from K

Identities  $1_n := I_n$  ( $n \times n$  identity matrix)

Composition  $B \circ A := BA$  matrix multiplication

A functor  $J : \text{Mat}_K \rightarrow \text{Vect}_K$

$$n \mapsto K^n$$

$$\begin{array}{ccc} m & & K^m \\ A \downarrow & \mapsto & \downarrow JA := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \mapsto A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \\ n & & K^n \end{array}$$

The functor  $J : \text{Mat}_K \rightarrow \text{Vect}_K$  is full and faithful (it is an embedding of categories)

Moreover  $J : \text{Mat}_K \rightarrow \text{FDVect}_K$  is also essentially surjective on objects (it is a weak equivalence of categories)

A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  is:

- full if, for every  $X, Y \in |\mathcal{C}|$ , the morphism action  $F: \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$  is surjective
- faithful if, for every  $X, Y \in |\mathcal{C}|$ , the morphism action  $F: \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$  is injective

A functor that is full and faithful is sometimes called an embedding of categories.

Lemma If  $F$  is an embedding then  $F$  reflects isos (i.e. for any  $X \xrightarrow{f} Y$  in  $\mathcal{C}$ , if  $Ff$  is iso then so is  $f$ ).

(Functors that reflect isos are sometimes called conservative.)

Proof A worthwhile exercise.

A functor  $F : C \rightarrow D$  is :

- essentially surjective on objects if, for every  $Y \in |D|$  there exists  $X \in C$  s.t.  $FX \cong Y$

$FX$  is isomorphic to  $Y$ ; i.e. there exists an iso  $FX \rightarrow Y$  in  $D$

A functor that is full, faithful and essentially surjective on objects is called a weak equivalence

### Theorem

① If  $F$  is part of an equivalence of categories  $(F, G, \alpha, \beta)$  then  $F$  is a weak equivalence.

② If  $F$  is a weak equivalence then assuming a suitable version of the axiom of choice  $F$  arises as part of an equivalence of categories  $(F, G, \alpha, \beta)$ .

## Outline proof of ②

Using essential surjectivity and choice, choose, for any  $Y \in |D|$ , an object  $GY \in |C|$  with iso  $Y \xrightarrow{\beta_Y} FGY$ .

For any  $Y \xrightarrow{g} Y'$  in  $D$ ,  $h = \beta_{Y'} \circ g \circ \beta_Y^{-1}$  is the unique map  $FGY \xrightarrow{h} FG Y'$  s.t.

$$\begin{array}{ccc} Y & \xrightarrow{\beta_Y} & FGY \\ g \downarrow & & \downarrow h \\ Y' & \xrightarrow{\beta_{Y'}} & FG Y' \end{array} \quad \text{Commutative.} \quad \otimes$$

Since  $F: C(GY, GY') \rightarrow D(FGY, FG Y')$  is a bijection ( $F$  an embedding) there is a unique map  $GY \xrightarrow{Gg} GY'$  in  $C$  such that  $h = FGg$ . The assignment  $g \mapsto Gg$  is functorial and  $\otimes$  shows that  $\beta: 1_D \Rightarrow F\beta$  is natural.

To define  $\alpha$ , consider, for any  $X \in |C|$  the function  $F: C(X, GFX) \rightarrow C(FX, FGFX)$ . Since  $F$  is an embedding let  $X \xrightarrow{\alpha_X} GFX$  be the unique map s.t.  $F\alpha_X = \beta_{FX}$ . Since  $F$  reflects isos,  $\alpha_X$  is iso. One then checks that  $\alpha: 1_C \Rightarrow GF$  is natural.  $\square$

The highlighted claims are left as exercises

What is meant by a suitable version of the axiom of choice?

If  $C, D$  are locally small categories, we need the axiom of global choice.

If  $C, D$  are small categories, we need the usual axiom of choice, which has a nice category-theoretic formulation:

(AC) Every epimorphism in Set splits

Suppose  $X \xrightarrow{s} Y \xrightarrow{r} X$  in a category  $C$  are such that  $rs = 1_X$ .

Then  $r$  is epi and  $s$  is mono (exercise)

An epi  $Y \xrightarrow{r} X$  is said to split if  $\exists X \xrightarrow{s} Y$  s.t.  $rs = 1_X$   
A mono  $X \xrightarrow{s} Y$  " split "  $\exists Y \xrightarrow{r} X$  s.t.  $rs = 1_X$

When one has  $X \xrightarrow{s} Y \xrightarrow{r} X$  with  $rs = 1_X$ , then

$Y$  is said to be a retract of  $X$ . The split epi

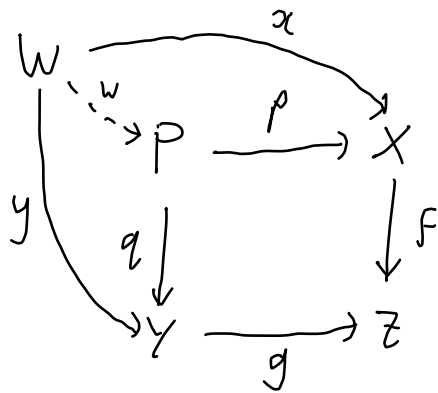
$Y \xrightarrow{r} X$  is called a retraction and the split mono

$X \xrightarrow{s} Y$  is called a section. The composite  $t = sr : Y \rightarrow Y$

is an idempotent:  $t \circ t = t$

# Pullbacks

In a category  $\mathcal{C}$ ,  
a pullback of a cospin  $X \xrightarrow{f} Z \xleftarrow{g} Y$   
is given by a span  $X \xleftarrow{p} P \xrightarrow{q} Y$  for  
which  $f \circ p = g \circ q$  and such that, for any  
span  $X \xleftarrow{x} W \xrightarrow{y} Y$  with  $f \circ x = g \circ y$ ,  
there exists a unique  $W \xrightarrow{w} P$  such that  
 $p \circ w = x$  and  $q \circ w = y$ .



A category  $\mathcal{C}$  is said to have pullbacks  
if every span in  $\mathcal{C}$  has a pullback.

Set has pullbacks.

Given  $X \xrightarrow{f} Z \xleftarrow{g} Y$ , define

$$P := \{ (x, y) \in X \times Y \mid f(x) = g(y) \}$$

$$p := (x, y) \mapsto x \quad : P \rightarrow X$$

$$q := (x, y) \mapsto y \quad : P \rightarrow Y$$

Then

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ q \downarrow & \lrcorner & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

notation for pullback square



# Intuition 1: Fibred Products

Given  $X \xrightarrow{f} I \xleftarrow{g} Y$  in Set

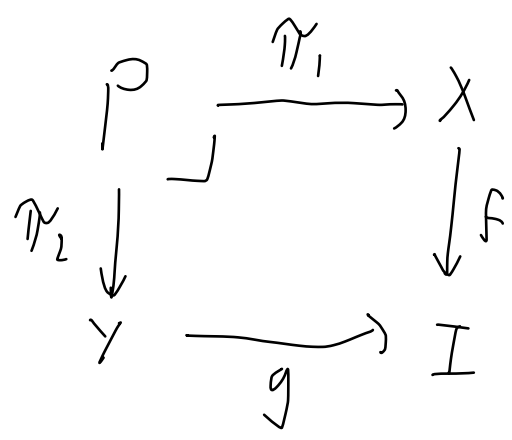
The pullback  $P$  can be defined as a family of products of fibres

$$P = \bigoplus_{i \in I} f^{-1}(i) \times g^{-1}(i)$$

With projections

$$\pi_1 : P \rightarrow X$$

$$\pi_2 : P \rightarrow Y$$



The diagonal map gives  $P$  as the  $I$ -indexed family  $(f^{-1}(i) \times g^{-1}(i))_{i \in I}$ .

## Intuition 2 : reindexing

Given  $J \xrightarrow{r} I \xleftarrow{f} X$  in Set

The pullback can be defined as

$$P := \sum_{j \in J} f^{-1}(r(j))$$

$$f' := (j, \alpha) \mapsto j : P \rightarrow J$$

$$p := (j, \alpha) \mapsto \alpha : P \rightarrow X$$

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ f' \downarrow & \lrcorner & \downarrow f \\ J & \xrightarrow{r} & I \end{array}$$

The left hand edge gives  $P$  as the  $J$ -indexed family

$$(f^{-1}(r(j)))_{j \in J}$$

### Intuition 3 : inverse image

Given  $X \xrightarrow{f} Y$  and  $Y' \subseteq Y$  in Set

the following is a pullback

$$\begin{array}{ccc} f^{-1}(Y') & \xrightarrow{f|_{f^{-1}(Y')}} & Y' \\ i' \downarrow & \lrcorner & \downarrow i \\ X & \xrightarrow{f} & Y \end{array}$$

where the vertical maps are the inclusion functions.

(up to isomorphism, this is a special case of intuition 2.)

## Intuition 4 : Intersection

Given  $X' \subseteq X \supseteq X''$  in Set

The following is a pullback

$$\begin{array}{ccc} X' \cap X'' & \longrightarrow & X' \\ \downarrow & \lrcorner & \downarrow \\ X'' & \longrightarrow & X \end{array}$$

where all maps are inclusion functions.

(This is, up to isomorphism, a special case of intuition 1.)

## Uniqueness up to isomorphism

$$\text{If } \begin{array}{ccc} P & \xrightarrow{p} & X \\ q \downarrow & \lrcorner & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} P' & \xrightarrow{p'} & X \\ q' \downarrow & \lrcorner & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

are both pullback squares then the unique map  $P' \xrightarrow{i} P$  s.t.  $pi = p'$  and  $qi = q'$  is an iso.

## Joint Monomorphism

$$\text{If } \begin{array}{ccc} P & \xrightarrow{p} & X \\ q \downarrow & \lrcorner & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

is a pullback then, for any  $w, v : W \rightarrow P$ ,  
 $(p \circ w = p \circ v \text{ and } q \circ w = q \circ v) \Rightarrow w = v$

## Preservation of Monos

$$\text{If } \begin{array}{ccc} P & \xrightarrow{p} & X \\ q \downarrow & \lrcorner & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

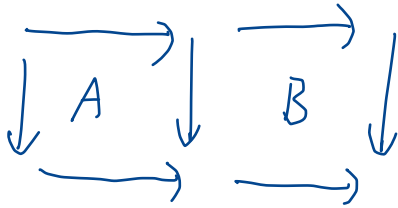
notation for mono

the maps  $p, q$  are jointly mono

is a pullback square and  $f$  is mono then so is  $g$ .

# The pullback lemma

Given a commuting diagram in a category  $\mathcal{C}$  of the form



1. If  $\downarrow \overrightarrow{A}$  and  $\downarrow \overrightarrow{B}$  are both pullbacks then so is  $\downarrow \overrightarrow{AB}$ .
2. If  $\downarrow \overrightarrow{AB}$  and  $\downarrow \overrightarrow{B}$  are both pullbacks then so is  $\downarrow \overrightarrow{A}$ .

We shall prove this next week.  
In the meantime, you can try to prove it yourself

## Subobjects

In any category  $\mathcal{C}$  define a relation  $\leq$  on the set of monomorphism into  $X \in |\mathcal{C}|$  by

$$Y \xrightarrow{m} X \leq Y' \xrightarrow{m'} X \Leftrightarrow \exists \lambda: Y \rightarrow Y' \text{ s.t. } \begin{array}{ccc} Y & \xrightarrow{\lambda} & Y' \\ & \searrow m & \downarrow m' \\ & X & \end{array} \text{ commutes}$$

There exists at most one  $\lambda$  as above and it is always mono.

The relation  $\leq$  is a preorder (reflexive and transitive) on monos into  $X$ .

Define  $m \equiv m' \Leftrightarrow m \leq m'$  and  $m' \leq m$ . This is an equivalence relation, and  $m \equiv m' \Leftrightarrow$  the unique  $\lambda$  as above is iso (Exercise)

A subobject of  $X$  is an equivalence class of monos into  $X$  under  $\equiv$ . The collection  $\text{Sub}(X)$  of subobjects into  $X$  is partially ordered by  $\leq$ .

## Week 4 puzzle

(1) For any set  $X$ , describe the partial order  $\text{Sub}(X)$

(up to isomorphism of partial orders) arising from the category set in direct (and familiar) mathematical terms.

(2) Ditto for any vector space  $V$  w.r.t. the category Vect