

Countable and Dependent Choice and Countably Distributive Toposes

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Countable choice (CC):

$$\forall n \in \mathbb{N} \exists x \in X, P(n, x) \Rightarrow \exists f \in X^{\mathbb{N}} \forall n \in \mathbb{N}, P(n, f(n))$$

Dependent choice (DC):

$$\forall x \in X \exists y \in X, R(x, y) \Rightarrow$$

$$\forall x \in X \exists f \in X^{\mathbb{N}}, f(0) = x \wedge \forall n \in \mathbb{N}, R(f(n), f(n+1))$$

- Not difficult to formulate in internal logic of elementary topos with NNO
- $AC \Rightarrow DC \Rightarrow CC$, but converse implications fail.

We begin with logic-free characterisations of CC and DC in elementary toposes with countable limits, which are probably folklore.

Detailed proofs can be found in the masters dissertation of my former student Severin Mejak

S. Mejak

Topos models of set-theoretic principles

Masters dissertation

Faculty of Mathematics and Physics, University of Ljubljana

2019

Logic-free characterisations of countable choice

Proposition T.f.a.e. for an elementary topos \mathcal{E} with NNO

- $\mathcal{E} \models CC$
 - The functor $(-)^{\mathbb{N}} : \mathcal{E} \rightarrow \mathcal{E}$ preserves epis
- A standard formulation of CC in elementary toposes*

Moreover if \mathcal{E} has countable limits, another equivalent statement is

- If $X_1 \xrightarrow{e_1} Y_1, X_2 \xrightarrow{e_2} Y_2, \dots$ are epis then so is

$$\begin{array}{c} X_1 \times X_2 \times X_3 \times \dots \\ \downarrow e_1 \times e_2 \times e_3 \times \dots \\ Y_1 \times Y_2 \times Y_3 \times \dots \end{array}$$

i.e., the product functor

$$\Pi : \mathcal{E}^{\omega} \rightarrow \mathcal{E}$$

preserves epis

A logic-free characterisation of dependent choice

Proposition T.f a.e. for an elementary topos \mathcal{E} with countable limits

- $\mathcal{E} \models DC$
- Every ω^ω -diagram of epis

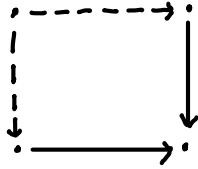
$$\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$$

has a cone of epis

- For every ω^ω -diagram of epis, the limit cone is a cone of epis.

The atomic (Grothendieck) topology is defined on a small category

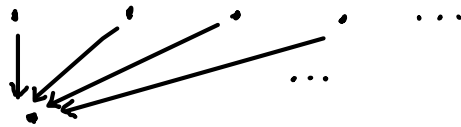
\mathbb{C} if and only if \mathbb{C} is coconfluent: every cospan completes to a commuting square



Atomic topology: a sieve S is a cover $\Leftrightarrow S \neq \emptyset$

The Grothendieck topos $\text{Sh}(\mathbb{C}, \text{at})$ of atomic sheaves is boolean.

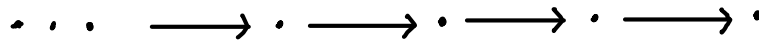
Proposition A sufficient condition for $Sh(\mathbb{C}, at)$ to validate CC is that \mathbb{C} is ω -coconfluent: every ω -cospan in \mathbb{C}



(ω -coconfluent \Leftrightarrow
the atomic topology is closed
under countable intersections)

has a cone.

Proposition A sufficient condition for $Sh(\mathbb{C}, at)$ to validate DC is that every ω^op -chain in \mathbb{C}



has a cone.

Theorem T.f.a.e. for a Grothendieck topos \mathcal{E}

1. $\mathcal{E} \equiv \text{Sh}(\mathcal{C}, \mathcal{J})$ where \mathcal{J} is closed under countable intersections.
2. $\mathcal{E} \equiv \text{Sh}(\mathcal{C}, \mathcal{J})$ for which the associated sheaf functor $\underline{a}: \text{PSh}(\mathcal{C}) \rightarrow \text{Sh}(\mathcal{C}, \mathcal{J})$ preserves countable limits.
3. \mathcal{E} validates CC and Satisfies: for all families $(J_i)_{i \in I}$ of sets and $(A_{i,j})_{i \in I, j \in J_i}$ of objects with I countable, the canonical map

$$\coprod_{\substack{f \in \prod_{i \in I} J_i}} \prod_{i \in I} A_{i, f(i)} \longrightarrow \prod_{i \in I} \coprod_{j \in J_i} A_{i,j}$$

is an isomorphism.

We call a Grothendieck topos satisfying 3 a
Countably distributive topos

Examples of countably distributive toposes

- Presheaf toposes
- (• Grothendieck toposes with finite sites
- Atomic toposes over ω -coconfluent categories
- Subtoposes of completely distributive toposes for which the inverse image of the geometric inclusion preserves countable limits

Any such is \equiv to
a presheaf topos

Examples of
completely distributive
toposes

Do countably geometric morphisms (inverse image functors preserve countable limits) between countably distributive toposes provide a setting for studying classifying toposes for countably geometric theories (extending geometric logic with countable conjunctions and products) ?

Conjecture The topos of probability sheaves is the classifying countably distributive topos for the countably geometric theory of non-separable measure algebras.

Outline proof of theorem

1 \Rightarrow 2 It is standard to define $\underline{a} := (\cdot)^+ \circ (\cdot)^+$ where

$$(\cdot)^+ : \mathcal{E} \rightarrow \mathcal{E}$$

is Grothendieck's $+$ -functor, which preserves finite limits

If \mathcal{I} is closed under countable intersections then it is easily verified that $(\cdot)^+$ preserves countable products, hence \underline{a} does too.

2 \Rightarrow 3 To verify the distributivity law in $\text{Sh}(\mathcal{C}, \mathcal{J})$,

suppose $(A_{i,j})_{i \in I, j \in \mathcal{J}}$ is a family of sheaves

Writing \coprod^s and \coprod^p for sheaf and presheaf coproduct we have

$$\coprod_s \prod_i A_{i,f(i)} \cong \underline{a} \left(\coprod_s \prod_i A_{i,f(i)} \right) \cong \underline{a} \left(\prod_i \coprod_j A_{i,j} \right)$$

$$\cong \prod_i \underline{a} \left(\coprod_j A_{i,j} \right) \cong \prod_i \coprod_j A_{i,j}$$



a preserves countable products

To verify CC in $\text{Sh}(\mathcal{C}, \mathcal{J})$ we use a lemma.

Lemma If $A_1 \xrightarrow{d_1} B_1, A_2 \xrightarrow{d_2} B_2, \dots$ are \mathcal{J} -dense monos in $\text{PSh}(\mathcal{C})$
then so is $\prod_n A_n \xrightarrow{\prod_n d_n} \prod_n B_n$.

Proof of lemma $\underline{a}(\prod_n d_n) \cong \prod_n (\underline{a}(d_n))$ (\underline{a} preserves countable products)

Since each d_n is dense, the r.h.s. is a countable product of isos

Hence the r.h.s. is an iso (CC in meta-theory)

Hence $\prod_n d_n$ is \mathcal{J} -dense.

□

To verify CC in $Sh(\mathcal{C}, \mathcal{J})$, suppose $A_1 \xrightarrow{e_1} B_1, A_2 \xrightarrow{e_2} B_2, \dots$

are epis in $Sh(\mathcal{C}, \mathcal{J})$. We verify that $\prod_i A_i \xrightarrow{\prod_i e_i} \prod_i B_i$

is also epi in $Sh(\mathcal{C}, \mathcal{J})$.

Let $A_n \xrightarrow{q_n} C_n \xrightarrow{m_n} B_n$ be the epi-mono factorisation of e_n in $PSh(\mathcal{C})$.

Then $\prod_i e_i = (\prod_i m_i) \circ (\prod_i q_i)$ is the e-m factorisation of $\prod_i e_i$ in $PSh(\mathcal{C})$.

Since each e_n is epi in $Sh(\mathcal{C}, \mathcal{J})$, each q_n is \mathcal{J} -dense, hence $\prod_i q_i$ is \mathcal{J} -dense by lemma.

Since the mono part of the e-m-factorisation of $\prod_i e_i$ in $PSh(\mathcal{C})$ is \mathcal{J} -dense and $\prod_i e_i$ is a map between sheaves, it is epi in $Sh(\mathcal{C}, \mathcal{J})$.

3 \Rightarrow 1 Suppose \mathcal{E} is a countably distributive Grothendieck topos.

Let \mathcal{C} be a small full subcategory of \mathcal{E} containing a generating set for \mathcal{E} and closed under countable limits in \mathcal{E} .

Let \mathcal{J} be the cover topology on \mathcal{C}

$\{A_i \xrightarrow{c_i} B\}_{i \in J}$ is covering $\iff \{c_i\}_{i \in J}$ is jointly epi in \mathcal{E}

By Giraud theorem / comparison lemma machinery $\mathcal{E} \cong \text{Sh}(\mathcal{C}, \mathcal{J})$.

We show that \mathcal{J} is closed under countable intersections.

Let S_1, S_2, \dots be covering sieves for an object B .

Each $S_n = \{A_{n,i} \xrightarrow{b_{n,i}} C\}_{i \in J_n}$ gives an epi $\coprod_{i \in J_n} A_{n,i} \xrightarrow{c_n := (b_{n,i})_i} C$

We have:

$$\begin{array}{ccc}
 \coprod_{f \in \prod_{i \in \mathbb{N}} J_i} P_f & \xrightarrow{\quad} & \coprod_{f \in \prod_{i \in \mathbb{N}} J_i} \prod_{i \in \mathbb{N}} A_{i, f(i)} \\
 \downarrow (e_f)_f & \searrow & \downarrow \cong \text{ (iso by distributivity)} \\
 & & \prod_{i \in \mathbb{N}} \coprod_{j \in J_i} A_{i,j} \\
 & & \downarrow \prod_{i \in \mathbb{N}} c_i \text{ (epi by CC)} \\
 C & \xrightarrow{\quad \Delta \quad} & \prod_{i \in \mathbb{N}} C
 \end{array}$$

$(d_f)_f := \left\{ \begin{array}{l} \prod_{i \in \mathbb{N}} \coprod_{j \in J_i} A_{i,j} \\ \prod_{i \in \mathbb{N}} c_i \end{array} \right.$

where each $p_f \xrightarrow{e_f} C$ is given by the pullback in \mathcal{C}

$$\begin{array}{ccc}
 p_f & \xrightarrow{\quad} & \prod_{i \in \mathbb{N}} A_{i, f(i)} \\
 \downarrow e_f & \lrcorner & \downarrow d_f \\
 C & \xrightarrow{\Delta} & \prod_{i \in \mathbb{N}} C
 \end{array}$$

The left edge of the top diagram is epi (since pullback of epi) hence defines a cover $\{p_f \xrightarrow{e_f} C\}_{f \in \prod_{i \in \mathbb{N}} J_i}$, which by construction is contained in $\bigcap_{i \in \mathbb{N}} S_n$. By monotonicity of covering property $\bigcap_{i \in \mathbb{N}} S_i \in J$. \square