Category Theory 2022-23

Lecture 4

28th October 2022

$\underline{\mathrm{Mat}_K}$ - the category of (natural numbers and) matrices over K

$$|\underline{\mathrm{Mat}_K}| := \mathbb{N}$$

$$\underline{\mathrm{Mat}_K}(m,n) := \text{the set of } \overset{\text{rows}}{n} \times \overset{\text{columns}}{m} \text{ matrices with entries from } K$$

<u>Identities</u> $1_n := I_n$ ($n \times n$ identity matrix)

<u>Composition</u> $B \circ A := BA$ matrix multiplication

A functor $J : \underline{\mathrm{Mat}_K} \to \underline{\mathrm{Vect}_K}$

$$n \mapsto K^n$$

$$\begin{array}{ccc} m & & K^m \\ A \big\downarrow & \mapsto & \big\downarrow JA := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \mapsto A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \\ n & & K^n \end{array}$$

The functor $J : \underline{\mathrm{Mat}_K} \to \underline{\mathrm{Vect}_K}$ is <u>full</u> and <u>faithful</u> (it is an <u>embedding</u> of categories)

Moreover $J : \underline{\mathrm{Mat}_K} \to \underline{\mathrm{FDVect}_K}$ is also <u>essentially surjective on objects</u> (it is a <u>weak equivalence</u> of categories)

A functor $F : \mathcal{C} \to \mathcal{D}$ is:

- <u>full</u> if, for every $X, Y \in |\mathcal{C}|$, the morphism action $F : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$ is surjective
- <u>faithful</u> if, for every $X, Y \in |\mathcal{C}|$, the morphism action $F : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$ is injective

A functor that is full and faithful is sometimes called an <u>embedding</u> of categories.

<u>Lemma</u> If F is an embedding then F reflects isos (i.e. for any $X \xrightarrow{f} Y$ in $\mathcal{C}$, if Ff is iso then so is f).

(Functors that reflect isos are sometimes called <u>conservative</u>.)

<u>Proof</u> A worthwhile exercise.

A functor $F: C \to D$ is:

- <u>essentially surjective on objects</u> if, for every $Y \in |D|$ there exists $X \in C$ s.t. $FX \cong Y$

FX is isomorphic to Y; i.e. there exists an iso $FX \to Y$ in D

A functor that is full, faithful and essentially surjective on objects is called a <u>weak equivalence</u>

<u>Theorem</u>

1. If F is part of an equivalence of categories (F, G, α, β) then F is a weak equivalence.

2. If F is a weak equivalence then assuming a suitable version of the axiom of choice F arises as part of an equivalence of categories (F, G, α, β).

Outline proof of 2

Using essential surjectivity and choice, choose, for any $Y \in |D|$, an object $GY \in |C|$ with iso $Y \xrightarrow{\beta_Y} FGY$.

For any $Y \xrightarrow{g} Y'$ in D, $h = \beta_{Y'} \circ g \circ \beta_Y^{-1}$ is the unique map $FGY \xrightarrow{h} FGY'$ s.t.

$$\begin{tikzcd} Y \arrow[r, "\beta_Y"] \arrow[d, "g"'] & FGY \arrow[d, "h"] \\ Y' \arrow[r, "\beta_{Y'}"'] & FGY' \end{tikzcd} \quad \text{commutes.} \qquad (*)$$

Since $F : C(GY, GY') \to D(FGY, FGY')$ is a bijection (F an embedding) there is a unique map $GY \xrightarrow{Gg} GY'$ in C such that $h = FGg$. The assignment $g \mapsto Gg$ is functorial and $(*)$ shows that $\beta : 1_D \Rightarrow FG$ is natural.

To define α, consider, for any $X \in |C|$ the function $F : C(X, GFX) \to C(FX, FGFX)$. Since F is an embedding let $X \xrightarrow{\alpha_X} GFX$ be the unique map s.t. $F\alpha_X = \beta_{FX}$. Since F reflects isos, α_X is iso. One then checks that $\alpha : 1_C \Rightarrow GF$ is natural.

$\square$

The highlighted claims are left as exercises

What is meant by a suitable version of the axiom of choice?

If $\mathcal{C}, \mathcal{D}$ are locally small categories, we need the <u>axiom of global choice</u>.

If $\mathcal{C}, \mathcal{D}$ are small categories, we need the usual <u>axiom of choice</u>, which has a nice category-theoretic formulation:

(AC) Every epimorphism in <u>Set</u> splits

Suppose $X \xrightarrow{s} Y \xrightarrow{r} X$ in a category $\mathcal{C}$ are such that $r \circ s = 1_X$.
Then r is epi and s is mono (exercise)

An epi $Y \xrightarrow{r} X$ is said to <u>split</u> if $\exists\, X \xrightarrow{s} Y$ s.t. $r \circ s = 1_X$
A mono $X \xrightarrow{s} Y$ " <u>split</u> " $\exists\, Y \xrightarrow{r} X$ s.t. $r \circ s = 1_X$

When one has $X \xrightarrow{s} Y \xrightarrow{r} X$ with $r \circ s = 1_X$, then Y is said to be a <u>retract</u> of X. The split epi $Y \xrightarrow{r} X$ is called a <u>retraction</u> and the split mono $X \xrightarrow{s} Y$ is called a <u>section</u>. The composite $t = s \circ r : Y \to Y$ is an <u>idempotent</u>: $t \circ t = t$

Pullbacks

In a category $\mathcal{C}$, a <u>pullback</u> of a <u>cospan</u> $X \xrightarrow{f} Z \xleftarrow{g} Y$ is given by a <u>span</u> $X \xleftarrow{p} P \xrightarrow{q} Y$ for which $f \circ p = g \circ q$ and such that, for any span $X \xleftarrow{x} W \xrightarrow{y} Y$ with $f \circ x = g \circ y$, there exists a unique $W \xrightarrow{w} P$ such that $p \circ w = x$ and $q \circ w = y$.

$$
\begin{tikzcd}
W \arrow[dr, dashed, "w"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
& P \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

A category $\mathcal{C}$ is said to <u>have pullbacks</u> if every span in $\mathcal{C}$ has a pullback.

<u>Set</u> has pullbacks.

Given $X \xrightarrow{f} Z \xleftarrow{g} Y$, define

$$P := \{ (x,y) \in X \times Y \mid f(x) = g(y) \}$$

$$p := (x,y) \mapsto x \quad : P \to X$$
$$q := (x,y) \mapsto y \quad : P \to Y$$

Then

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

notation for pullback square

Intuition 1: Fibred Products

Given $X \xrightarrow{f} I \xleftarrow{g} Y$ in $\underline{\mathrm{Set}}$

The pullback P can be defined as a family of products of fibres

$$P = \biguplus_{i \in I} f^{-1}(i) \times g^{-1}(i)$$

with projections

$$\pi_1 : P \to X$$
$$\pi_2 : P \to Y$$

$$\begin{tikzcd}
P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & I
\end{tikzcd}$$

The diagonal map gives P as the I-indexed family $\left(f^{-1}(i) \times g^{-1}(i)\right)_{i \in I}$.

Intuition 2 : reindexing

Given $J \xrightarrow{r} I \xleftarrow{f} X$ in Set

The pullback can be defined as

$$P := \sum_{j \in J} f^{-1}(r(j))$$

$$f' := (j,x) \mapsto j \ : P \to J$$

$$p := (j,x) \mapsto x \ : P \to X$$

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
J \arrow[r, "r"'] & I
\end{tikzcd}$$

The left hand edge gives P as the J-indexed family

$$\left(f^{-1}(r(j))\right)_{j \in J}$$

Intuition 3 : inverse image

Given $X \xrightarrow{f} Y$ and $Y' \subseteq Y$ in <u>Set</u>

the following is a pullback

$$\begin{tikzcd}
f^{-1}(Y') \arrow[r, "f\restriction_{f^{-1}(Y')}"] \arrow[d, "i'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, "i"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where the vertical maps are the inclusion functions.

(up to isomorphism, this is a special case of intuition 2.)

Intuition 4: Intersection

Given $X' \subseteq X \supseteq X''$ in $\underline{Set}$

The following is a pullback

$$\begin{tikzcd} X' \cap X'' \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, hook] \\ X'' \arrow[r, hook] & X \end{tikzcd}$$

where all maps are inclusion functions.

(This is, up to isomorphism, a special case of intuition 1.)

Uniqueness up to isomorphism

If
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
and
$$\begin{tikzcd} P' \arrow[r, "p'"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
are both pullback squares then the unique map $P' \xrightarrow{i} P$ s.t $p \circ i = p'$ and $q \circ i = q'$ is an iso.

Joint Mon(omorph)icity

If
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
is a pullback then, for any $w, v : W \longrightarrow P$,
$(p \circ w = p \circ v$ and $q \circ w = q \circ v) \Rightarrow w = v$

the maps p, q are jointly mono

Preservation of Monos

If:
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, tail, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
notation for mono

is a pullback square and f is mono then so is q.

The pullback lemma

Given a commuting diagram in a category $\mathcal{C}$ of the form

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$$

1. If A and B are both pullbacks then so is AB.

2. If AB and B are both pullbacks then so is A

We shall prove this next week.
In the meantime, you can try to prove it yourself

Subobjects

In any category $\mathcal{C}$ define a relation $\sqsubseteq$ on the set of monomorphism into $X \in |\mathcal{C}|$ by

$$Y \overset{m}{\rightarrowtail} X \sqsubseteq Y' \overset{m'}{\rightarrowtail} X \iff$$

$$\exists\, l : Y \to Y' \text{ s.t. } \begin{tikzcd} Y \arrow[rr, "l"] \arrow[dr, tail, "m"'] & & Y' \arrow[dl, tail, "m'"] \\ & X & \end{tikzcd} \text{ commutes}$$

There exists at most one l as above and it is always mono.
The relation $\sqsubseteq$ is a preorder (reflexive and transitive) on monos into X.
Define $m \equiv m' \iff m \sqsubseteq m'$ and $m' \sqsubseteq m$. This is an equivalence relation, and $m \equiv m' \iff$ the unique l as above is iso (Exercise)

A <u>subobject</u> of X is an equivalence class of monos into X under $\equiv$. The collection $\mathrm{Sub}(X)$ of subobjects into X is partially ordered by $\sqsubseteq$.

Week 4 puzzle

(1) For any set X, describe the partial order $\mathrm{Sub}(X)$ (up to isomorphism of partial orders) arising from the category <u>Set</u> in direct (and familiar) mathematical terms.

(2) Ditto for any vector space V w.r.t. the category <u>Vect$_k$</u>