Category Theory 2022-23

Lecture 9

2nd December 2022

Monoids in Set

A monoid is a structure $(X, \bullet, e)$ where X is a set and $\bullet : X \times X \to X$ and $e \in X$ are such that

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z \qquad \forall x y z \in X$$

$$x \bullet e = x = e \bullet x \qquad \forall x \in X$$

(N.B. given X and $\bullet$, e is uniquely detemined.)

A homomorphism of monoids from $(X, \bullet, e)$ to $(X', \bullet', e')$ is a function $h : X \to X'$ such that

$$h(x \bullet y) = h(x) \bullet' h(y)$$

$$h(e) = e'$$

The category Mon has monoids as objects and homomorphisms as morphisms.

Monoids in a category $\mathcal{C}$ with finite products

A *monoid* is a structure $(X, \bullet, e)$ where $X \in |\mathcal{C}|$

and $X \times X \xrightarrow{\bullet} X$ and $1 \xrightarrow{e} X$ are such that

$$\begin{tikzcd}
X \times (X \times X) \arrow[r, "1 \times \bullet"] \arrow[d, "\alpha_X"'] & X \times X \arrow[dr, "\bullet"] & \\
 & & X \\
(X \times X) \times X \arrow[r, "\bullet \times 1_X"] & X \times X \arrow[ur, "\bullet"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
X \times 1 \arrow[d, "1_X \times e"'] & X \arrow[l, "\rho_X^{-1}"'] \arrow[r, "\lambda_X^{-1}"] \arrow[d, "1_X"] & 1 \times X \arrow[d, "e \times 1_X"] \\
X \times X \arrow[r, "\bullet"'] & X & X \times X \arrow[l, "\bullet"]
\end{tikzcd}$$

(Exercise: $1 \xrightarrow{e} X$ is uniquely determined by $X \times X \xrightarrow{\bullet} X$.)

A *homomorphism* from $(X, \bullet, e)$ to $(X', \bullet', e')$

is a map $X \xrightarrow{h} X'$ s.t.

$$\begin{tikzcd}
X \times X \arrow[r, "h \times h"] \arrow[d, "\bullet"'] & X' \times X' \arrow[d, "\bullet'"] \\
X \arrow[r, "h"] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[dl, "e"'] \arrow[dr, "e'"] & \\
X \arrow[rr, "h"] & & X'
\end{tikzcd}$$

$\mathbf{Mon}_{\mathcal{C}}$: The category of monoids in $\mathcal{C}$ and homomorphisms

Groups in <u>Set</u>

A <u>group</u> is a monoid $(X, \cdot, e)$ such that,
for every $x \in X$, there exists $x^{-1} \in X$ with $x \cdot x^{-1} = e = x^{-1} \cdot e$

(N.B. x^{-1} is uniquely determined.)

A <u>homomorphism</u> of groups from $(X, \cdot, e)$ to $(X', \cdot', e')$
is simply a monoid homomorphism $h : X \to X'$

(N.B. It follows that $h(x^{-1}) = h(x)^{-1}$

inverse in X
inverse in X'

Also, the fact that $(X', \cdot', e')$ is a group means that
the equation $h(e) = e'$ follows from $h(x \cdot y) = h(x) \cdot' h(y)$ alone.)

<u>Grp</u> is the category of groups and homomorphisms

Groups in a category C with finite products

A *group* is a monoid $(X, X\times X \xrightarrow{\bullet} X, 1 \xrightarrow{e} X)$ for which there exists $X \xrightarrow{(\cdot)^{-1}} X$ such that

$$\begin{tikzcd}
X\times X \arrow[d, "1\times(-)^{-1}"'] & X \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "!"] & X\times X \arrow[d, "(-)^{-1}\times 1"] \\
 & 1 \arrow[d, "e"] & \\
X\times X \arrow[r, "\bullet"'] & X & X\times X \arrow[l, "\bullet"]
\end{tikzcd}
\qquad \Delta := X \xrightarrow{\langle 1_X, 1_X\rangle} X\times X$$

(Exercise $X \xrightarrow{(-)^{-1}} X$ is uniquely determined by $X\times X \xrightarrow{\bullet} X$.)

A *homomorphism* of groups is just a homomorphism of monoids.

(Exercise For any monoid homomorphism $(X,\bullet,e) \xrightarrow{h} (X,\bullet,e)$ between groups it follows that

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "(-)^{-1}"'] & X \arrow[d, "(-)^{-1}"] \\
X \arrow[r, "h"] & X
\end{tikzcd}$$

Moreover preservation of units is a consequence of preservation of multiplication.)

Grp_C is the category of groups in C and homomorphisms.

Grp_C is a full subcategory of Mon_C

C is a full subcategory of D if

$|C| \subseteq |D|$

and $C(X,Y) = D(X,Y) \quad \forall X,Y \in |C|$

Examples

In **Set**: ordinary groups and their homomorphisms

In **Top**: topological groups and their continuous homomorphisms

(In a topological group, group multiplication

$$\cdot : X \times X \longrightarrow X$$

is jointly continuous; i.e continuous w.r.t the product topology, and the inverse function

$$(-)^{-1} : X \to X$$

is continuous. These properties imply that X is Hausdorff (T_2).)

In **Man** (the category of smooth maps between differentiable manifolds): Lie groups and smooth homomorphisms.

Monoids in a category $\mathcal{C}$ with monoidal structure

A *monoid* is a structure $(X, \bullet, e)$ where $X \in |\mathcal{C}|$ and $X \otimes X \xrightarrow{\bullet} X$ and $I \xrightarrow{e} X$ are such that

$$\begin{tikzcd}
X \otimes (X \otimes X) \arrow[r, "1_X \otimes \bullet"] \arrow[d, "\alpha_X"'] & X \otimes X \arrow[dr, "\bullet"] & \\
(X \otimes X) \otimes X \arrow[r, "\bullet \otimes 1_X"] & X \otimes X \arrow[r, "\bullet"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes I \arrow[d, "1_X \otimes e"'] & X \arrow[l, "\rho_X^{-1}"'] \arrow[r, "\lambda_X^{-1}"] \arrow[d, "1_X"] & I \otimes X \arrow[d, "e \otimes 1_X"] \\
X \otimes X \arrow[r, "\bullet"'] & X & X \otimes X \arrow[l, "\bullet"]
\end{tikzcd}$$

(Exercise $I \xrightarrow{e} X$ is uniquely determined by $X \otimes X \xrightarrow{\bullet} X$.)

A *homomorphism* from $(X, \bullet, e)$ to $(X', \bullet', e')$ is a map $X \xrightarrow{h} X'$ s.t.

$$\begin{tikzcd}
X \otimes X \arrow[r, "h \otimes h"] \arrow[d, "\bullet"'] & X' \otimes X' \arrow[d, "\bullet'"] \\
X \arrow[r, "h"] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dl, "e"'] \arrow[dr, "e'"] & \\
X \arrow[rr, "h"] & & X'
\end{tikzcd}$$

$\mathrm{Mon}_{\mathcal{C}}$: The category of monoids in $\mathcal{C}$ and homomorphisms

Examples

A monoid in $\underline{\mathrm{Vect}}_K$ is a vector space V

with $V \otimes V \xrightarrow{\bullet} V$ and $K \xrightarrow{e} V$ corresponding to

a function $V \times V \xrightarrow{\bullet} V$ and element $\underline{1} \in V$

satisfying

bilinearity
$$k\underline{u} \bullet \underline{w} = k(\underline{u} \bullet \underline{w}) \qquad (\underline{u} + \underline{v}) \bullet \underline{w} = \underline{u} \bullet \underline{w} + \underline{v} \bullet \underline{w}$$
$$\underline{u} \bullet k\underline{w} = k(\underline{u} \bullet \underline{w}) \qquad \underline{u} \bullet (\underline{v} + \underline{w}) = \underline{u} \bullet \underline{v} + \underline{u} \bullet \underline{w}$$

monoid laws
$$\underline{u} \bullet (\underline{v} \bullet \underline{w}) = (\underline{u} \bullet \underline{v}) \bullet \underline{w}$$
$$\underline{u} \bullet \underline{1} = \underline{u} = \underline{1} \bullet \underline{u}$$

I.e. a monoid in $\underline{\mathrm{Vect}}_K$ is exactly

an <u>associative K-algebra</u>

A monoid in $\underline{\mathrm{Cat}}$ (w.r.t. product $\times$)

is exactly a (small) strict monoidal category

It is often useful to define varieties of mathematical structure (e.g. algebraic structures) <u>internally</u> in the context of an ambient category C. What we can define in this way depends on the category-theoretic structure of C we use.

<u>Monoidal structure</u>: algebraic structures defined using equations in which the same variables appear in the same order on each side of the equation, and each variable appears only once on each side. (E.g. monoids: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ and $x \cdot e = x = e \cdot x$.)

<u>Symmetric monoidal structure</u>: as above, except that the variables are not required to appear in the same order on both sides of an equation. (E.g. commutative monoides. $x \cdot y = y \cdot x$.)

<u>Finite products</u>: general algebraic structures defined using equations (e.g. groups, Abelian groups, rings).

<u>Finite limits</u>: more exotic structures with, for example, partially defined operations whose domains have 'positive' descriptions (e.g. categories – composition is partially defined in the sense that composable arrows have to have matching domain/codomain).

Rings and modules in $\mathcal{C}$ with finite products.

A ring (with unit) is given by an object R and maps

$$1 \xrightarrow{0} R \qquad R \times R \xrightarrow{+} R \qquad 1 \xrightarrow{1} R \qquad R \times R \xrightarrow{\cdot} R$$

satisfying diagrams expressing the usual laws. Exercise work these out.

Example In Top the rings are exactly the topological rings.

Exercise Define the category $\mathrm{Ring}_{\mathcal{C}}$ of rings (with unit) and ring homomorphisms.

We can also formulate the derived algebraic notion of R-module

An R-module in $\mathcal{C}$ is $(X, \underline{0}, \underline{+}, \underline{\cdot})$ where

$$1 \xrightarrow{\underline{0}} X \qquad X \times X \xrightarrow{\underline{+}} X \qquad R \times X \xrightarrow{\underline{\cdot}} X$$

satisfy diagrams expressing the usual laws; e.g.

$$\begin{tikzcd}
R \times (R \times X) \arrow[r, "1_R \times \underline{\cdot}"] \arrow[d, "\alpha"'] & R \times X \arrow[dr, "\underline{\cdot}"] & \\
& & X \\
(R \times R) \times X \arrow[r, "\cdot \times 1_X"] & R \times X \arrow[ur, "\underline{\cdot}"'] &
\end{tikzcd}$$

$\left(r_1 \underline{\cdot} (r_2 \underline{\cdot} x) = (r_1 \cdot r_2) \underline{\cdot} x \right)$

Similarly a homomorphism of modules is ... (Exercise!)

If R is a field in Set then we obtain the category Vect_R

If R is $\mathbb{R}$ or $\mathbb{C}$ in Top then we obtain topological vector spaces etc.

Internal categories in a category C with finite limits

An *internal category* is a tuple $\mathbb{D} = (D_0, D_1, d_0, d_1, \underline{id}, \underline{\circ})$

where: $D_0, D_1 \in |C|$

D_0 ~ object of objects

D_1 ~ object of morphisms

$$\begin{tikzcd} D_0 \arrow[r, "\underline{id}"] & D_1 \arrow[l, bend right=60, "d_1"'] \arrow[l, bend left=60, "d_0"] \end{tikzcd}$$

are maps in C with $d_0 \circ \underline{id} = 1_{D_0} = d_1 \circ \underline{id}$

d_0 & d_1 ~ domain and codomain of a map

$\underline{id}$ ~ identity map on an object

$$D_1 \times_{D_0} D_1 \xrightarrow{\underline{\circ}} D_1$$

where

$$\begin{tikzcd} D_1 \times_{D_0} D_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D_1 \arrow[d, "d_1"] \\ D_1 \arrow[r, "d_0"'] & D_0 \end{tikzcd}$$

$\underline{\circ}$ ~ composition of maps

Such that

$$\begin{tikzcd} D_1 \times_{D_0} D_0 \arrow[d, "1_{D_1} \times_{D_0} \underline{id}"'] & D_1 \arrow[l, "\langle 1_{D_1}, d_0 \rangle"'] \arrow[r, "\langle d_1, 1_{D_1} \rangle"] \arrow[d, "1_{D_1}"] & D_0 \times_{D_0} D_1 \arrow[d, "\underline{id} \times_{D_0} 1_{D_1}"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\underline{\circ}"] & D_1 & D_1 \times_{D_0} D_1 \arrow[l, "\underline{\circ}"'] \end{tikzcd}$$

$\underline{id}$ gives right identities w.r.t. $\underline{\circ}$

$\underline{id}$ gives left identities w.r.t. $\underline{\circ}$

and

$$\begin{tikzcd} D_1 \times_{D_0} D_1 \times_{D_0} D_1 \arrow[r, "1_{D_1} \times_{D_0} \underline{\circ}"] \arrow[d, "\underline{\circ} \times_{D_0} 1_{D_1}"'] & D_1 \times_{D_0} D_1 \arrow[d, "\underline{\circ}"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\underline{\circ}"'] & D_1 \end{tikzcd}$$

composition is associative

The top left vertex is constructed as the pullback

$$\begin{tikzcd} D_1 \times_{D_0} D_1 \times_{D_0} D_1 \arrow[r, "\langle \pi_2, \pi_3 \rangle"] \arrow[d, "\langle \pi_1, \pi_2 \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_1 \times_{D_0} D_1 \arrow[d, "\pi_1"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\pi_2"'] & D_1 \end{tikzcd}$$

The object of composable triples of morphisms.

(The notation $D_1 \times_{D_0} D_1 \times_{D_0} D_1$ is not very precise! Also my verbal description of this construction in the lecture was a bit misleading.)

Exercise

- Give precise constructions of the maps $\circ \times_{D_0} 1_{D_1}$ and $1_{D_1} \times_{D_0} \circ$ that appear in the associativity diagram.

- Define internal functor between internal categories.

- Define internal natural transformation between internal functors

Week 9 puzzle

① What is a monoid in the strict monoidal category $[\mathcal{C},\mathcal{C}]$ of endofunctors on a category $\mathcal{C}$?

I am looking for an answer of the form: a monoid is an endofunctor together with certain natural transformations satisfying certain properties.

② Find natural examples of such monoids.

This puzzle will be answered in the week 10 lecture