Category Theory 2022-23

Lecture 12

23rd December 2022

The simplicial category Δ

Objects: sets $[n] := \{0, \dots, n\}$ $n \geq 0$

(N.B. $[n]$ has $n+1$ elements)

Morphisms from $[m]$ to $[n]$: order preserving functions $f: [m] \to [n]$

(i.e., $i \leq j \Rightarrow f(i) \leq f(j)$)

A simplicial set is a presheaf $\Delta^{op} \to \underline{Set}$

$\underline{sSet} := Psh(\Delta)$ The category of simplicial sets

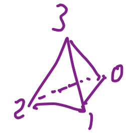
A mathematically important example of a presheaf category

Representable simplicial sets Δ^n = the standard n-simplex

$$\Delta^n := \underset{\sim}{y}[n] = \underline{\Delta}(-,[n])$$

We write Δ^n_m for $\Delta^n[m] = \underline{\Delta}([m],[n])$.

$\Delta^n_m \sim$ the m-simplices within the n-simplex

e.g. the 3-simplex is the tetrahedron 

Δ^3_0 has 4 elements (4 vertices)

Δ^3_1 has 6 non-degenerate elements (6 edges)

(non-degenerate = injective function)

and 10 elements (6 edges + 4 vertices)

Δ^3_2 has 4 non-degenerate elements (4 triangles)

Δ^3_3 has 1 non-degenerate element (1 tetrahedron)

Δ^3_m has 0 non-degenerate elements if $m > 3$.

A functor $\Delta : \underline{\Delta} \to \underline{Top}$

$$[n] \mapsto \Delta_n := \left\{ (x_0, \dots, x_n) \in [0,1]^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \right\}$$

the standard topological n-simplex

$$\begin{tikzcd} {[m]} \arrow[d, "f"'] & & \Delta_m \arrow[d] \\ {[n]} \arrow[rr, phantom, "\longmapsto" near start, shift left=4ex] & & \Delta_n \end{tikzcd}$$

the unique affine function mapping $\hat{i}$ to $\widehat{f(i)}$ for $i \in [m]$

$\hat{i} = (x_0, \dots, x_n)$ where $x_j = \delta_{ij}$

Kronecker delta

A functor $S : \underline{Top} \to \underline{sSet}$

$$T \mapsto \underline{Top}(\Delta -, T)$$

$S(T)[n]$ = all continuous maps from Δ_n to T

$S(T)$ is the <u>total singular complex of T</u>

Any simplicial set encodes a topological space obtained intuitively by gluing together standard top. simplices according to the recipe encoded by the simplicial set

The *geometric realisation* functor $G : \underline{sSet} \to \underline{Top}$ is:

- the unique (up to natural isomorphism) colimit preserving functor such that

$$\begin{tikzcd}
\underline{sSet} \arrow[r, "G"] & \underline{Top} \\
\underline{\Delta} \arrow[u, "\underline{y}"] \arrow[ur, "\Delta"'] \arrow[u, phantom, shift left=4ex, "\cong"] &
\end{tikzcd}$$

- constructed explicitly as a *pointwise left Kan extension* of Δ along $\underline{y}$

- left adjoint to S: $\quad G \dashv S : \underline{Top} \to \underline{sSet}$

The first property above captures the gluing intuition.

The category of elements of a presheaf

Let $P : \mathbb{C}^{op} \to \underline{\mathrm{Set}}$ be a presheaf.

The category $\int P$ of elements has

Objects (X, x) $\quad X \in |\mathbb{C}|$, $x \in PX$

Morphisms from (X, x) to (Y, y):

maps $X \xrightarrow{f} Y$ in $\mathbb{C}$ s.t. $x = P(f)(y)$

($\equiv$ly s.t. $x = y \cdot f$ using 'action notation' for presheaves)

There is an obvious forgetful functor
$U : \int P \to \mathbb{C}$

Theorem (The co-Yoneda lemma!)

Every presheaf P is a colimit in $Psh(\mathbb{C})$ of the diagram $\int P \xrightarrow{u} \mathbb{C} \xrightarrow{\underline{y}} Psh(\mathbb{C})$.

This is often paraphrased: every presheaf is a colimit of representables. It is frequently referred to as "the co-Yoneda lemma", but it is not the only result known by this name.

Proof outline

The colimiting cocone $\left(\underline{y}X \xrightarrow{c_{(X,x)}} P\right)_{(X,x)}$ is given by

$c_{(X,x)} := \psi_X(x)$ where $\psi_X^P : P(X) \xrightarrow{\cong} Psh(\mathbb{C})(\underline{y}X, P)$ is the Yoneda lemma bijection

Given any other cocone $\left(\underline{y}X \xrightarrow{d_{(X,x)}} Q \right)_{(X,x)}$

we need to define the unique cocone morphism $P \xrightarrow{e} Q$ in $Psh(\mathbb{C})$.

The component $e_X : PX \to QX$

is the function

$$x \in PX \longmapsto \left(\psi^Q_X\right)^{-1}\left(d_{(X,x)}\right).$$

One then needs to verify:

- $\left(c_{(X,x)}\right)_{(X,x)}$ is indeed a cocone
- $(e_X)_X$ is natural
- e is a morphism of cocones
- e is the unique cocone morphism

$\square$

An alternative description of $\int P$.

Objects (X, α) where $X \in |\mathbb{C}|$ and $\underline{y}X \xrightarrow{\alpha} P$ in $\mathrm{Psh}(\mathbb{C})$

Morphisms from (X,α) to (Y,β) are maps $X \xrightarrow{f} Y$ in $\mathbb{C}$ such that

$$\begin{tikzcd} \underline{y}X \arrow[r, "\alpha"] \arrow[d, "\underline{y}f"'] & P \\ \underline{y}Y \arrow[ur, "\beta"'] & \end{tikzcd}$$

commutes in $\mathrm{Psh}(\mathbb{C})$.

This is an isomorphic category to $\int P$ by the Yoneda lemma.

It is an instance of a general comma category construction.

Given $F : C \to \mathbb{D}$ and $Z \in |\mathbb{D}|$ the comma category $F \downarrow Z$ has:

Objects (X, g) where $X \in |C|$ and $FX \xrightarrow{g} Z$ in $\mathbb{D}$

Morphisms from (X,g) to (Y,h) are maps $X \xrightarrow{f} Y$ in C s.t. $h \circ Ff = g$ in $\mathbb{D}$.

(More general comma category constructions than this exist too.)

The reformulation of $\int P$ at the top of the page shows that $\int P$ is isomorphic to the comma category $\underline{y} \downarrow P$.

Given $F : \mathbb{C} \to \mathbb{D}$ and $Z \in |\mathbb{D}|$ we have a diagram

$$F \downarrow Z \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathbb{D} \quad \text{in } \mathbb{D} \qquad (A)$$

which has a canonical cocone with vertex Z

$$\left(FX \xrightarrow{g} Z \right)_{(X,g)} \qquad (B)$$

A functor F is said to be <u>dense</u> if, for every $Z \in |\mathbb{D}|$, (B) is a colimit of (A).

<u>Reformulation</u> (The co-Yoneda lemma)

The Yoneda functor $\underline{y} : \mathbb{C} \to \mathrm{Psh}(\mathbb{C})$ is dense.

Theorem ($Psh(\mathbb{C})$ is the free cocompletion of $\mathbb{C}$)

Let $\mathbb{C}$ be a small category.

For any cocomplete category $\mathcal{A}$ and functor $F : \mathbb{C} \to \mathcal{A}$,
there exists a colimit preserving funtor $\overline{F} : \mathbb{C} \to \mathcal{A}$
such that

- $\overline{F}\,\underline{y} \cong F$ in $[\mathbb{C}, \mathcal{A}]$
- For any colimit preserving $\overline{G} : Psh(\mathbb{C}) \to \mathcal{A}$
  for which $\overline{G}\,\underline{y} \cong F$, it holds that $\overline{G} \cong \overline{F}$

$$\begin{tikzcd}
Psh(\mathbb{C}) \arrow[rr, "\overline{F}/\overline{G}", "\text{colimit preserving}"'] & & \mathcal{A} \\
 & \cong & \\
\mathbb{C} \arrow[uu, hook, "\underline{y}"] \arrow[uurr, "F"'] & &
\end{tikzcd}$$

More briefly, for any cocomplete $\mathcal{A}$ and functor $F : \mathbb{C} \to \mathcal{A}$,
there exists a unique (up to natural isomorphism) colimit preserving
functor $\overline{F} : Psh(\mathbb{C}) \to \mathcal{A}$ such that $\overline{F}\,\underline{y} \cong F$.

The above property characterises $Psh(\mathbb{C})$ up to
equivalence of categories.

Given functors $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{A}$ ($\mathcal{A}, \mathcal{C}, \mathcal{D}$ arbitrary categories),

a <u>left Kan extension</u> of F along K ($\mathrm{Lan}_K F$)

is a functor $\underline{F} : \mathcal{D} \to \mathcal{A}$ and natural transformation

$\alpha : F \Rightarrow \underline{F} K$ such that, for any $\underline{G} : \mathcal{D} \to \mathcal{A}$ and

natural transformation $\beta : F \Rightarrow \underline{G} K$, there exists

a unique natural transformation $\gamma : \underline{F} \Rightarrow \underline{G}$ such that

$$\beta = (\gamma K) \circ \alpha$$

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, bend left, "\overline{G}"] \arrow[rr, bend right, "\overline{F}"'] \arrow[rr, phantom, "\Uparrow\gamma"] & & \mathcal{A} \\
& \mathcal{C} \arrow[ul, "K"] \arrow[ur, "F"'] \arrow[u, phantom, "\Uparrow\alpha \ \Uparrow\beta"] &
\end{tikzcd}$$

<u>Proposition</u> Left Kan extensions (if they exist) are uniquely determined up to natural isomorphism.

Theorem Suppose $\mathbb{C}$ is small, $\mathbb{D}$ locally small and $\mathcal{A}$ cocomplete. Then every $F : \mathbb{C} \to \mathcal{A}$ has a left Kan extension along every $K : \mathbb{C} \to \mathbb{D}$, given explicitly by

$$(\mathrm{Lan}_K F)\,Y := \varinjlim \left(K \downarrow Y \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathcal{A} \right)$$

$\varinjlim$ means colimit, $\varprojlim$ means limit

With the morphism action determined by the universal property of colimits

If a left Kan extension is defined in the above way it is said to be a <u>pointwise</u> left Kan extension.

Proposition If $K : \mathbb{C} \to \mathbb{D}$ is full and faithful, and $(\bar{F} : \mathbb{D} \to \mathcal{A},\ \alpha : F \Rightarrow \bar{F}K)$ is a pointwise left Kan extension of F along K then α is a natural isomorphism.

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "\bar{F}"] & \mathcal{A} \\
\mathbb{C} \arrow[u, "K"] \arrow[ur, "F"'] \arrow[u, phantom, ""] &
\arrow[from=2-1, to=1-2, phantom, "\cong", shift left=3]
\end{tikzcd}$$

I.e., $\bar{F}$ really does 'extend' F up to isomorphism

Proposition If $\mathcal{A}$ is cocomplete then for any $F : \mathbb{C} \to \mathcal{A}$ the functor $\mathrm{Lan}_{\underline{y}} F : \mathrm{Psh}(\mathbb{C}) \to \mathcal{A}$ is left adjoint to the functor $G : \mathcal{A} \to \mathrm{Psh}(\mathbb{C})$

$\overline{F}$ below

$$Z \mapsto \mathcal{A}(F-, Z) : \mathcal{A} \to \mathrm{Psh}(\mathbb{C})$$

Proof outline

$$\mathcal{A}(\overline{F}P, Z) \cong \mathcal{A}\left(\varinjlim\left(\underline{y}\downarrow P \xrightarrow{u} \mathbb{C} \xrightarrow{F} \mathcal{A}\right), Z\right) \qquad \text{def. } \overline{F}$$

$$\cong \varprojlim\left(\left(\underline{y}\downarrow P\right)^{op} \xrightarrow{u} \mathbb{C}^{op} \xrightarrow{\mathcal{A}(F-, Z)} \underline{Set}\right) \qquad \text{def } \varinjlim \ \circledast$$

$$\cong \varprojlim\left(\left(\underline{y}\downarrow P\right)^{op} \xrightarrow{u} \mathbb{C}^{op} \xrightarrow{GZ} \underline{Set}\right) \qquad \text{def. } G$$

$$\cong \varprojlim\left(\left(\underline{y}\downarrow P\right)^{op} \xrightarrow{u} \mathbb{C}^{op} \xrightarrow{\mathrm{Psh}(\mathbb{C})(\underline{y}-, GZ)} \underline{Set}\right) \qquad \text{Yoneda lemma}$$

$$\cong \mathrm{Psh}\left(\varinjlim\left(\underline{y}\downarrow P \xrightarrow{u} \mathbb{C} \xrightarrow{\underline{y}} \mathrm{Psh}(\mathbb{C})\right), GZ\right) \qquad \text{def. } \varinjlim \ \circledast$$

$$\cong \mathrm{Psh}(P, GZ) \qquad \text{density of } \underline{y}.$$

$\square$

$\circledast$ Using $\mathbb{C}\left(\varinjlim\left(\mathbb{G} \xrightarrow{D} \mathbb{C}\right), Z\right) \cong \varprojlim\left(\mathbb{G}^{op} \xrightarrow{D} \mathbb{C}^{op} \xrightarrow{\mathbb{C}(-, Z)} \underline{Set}\right)$

Proof that $Psh(\mathbb{C})$ is the free cocompletion of $\mathbb{C}$

Consider any functor $F : \mathbb{C} \to \mathcal{A}$ where $\mathcal{A}$ is cocomplete.

Define $\bar{F} : Psh(\mathbb{C}) \to \mathcal{A}$ to be the pointwise left Kan extension of F along $\underline{y}$, as given by the previous theorem.

$\bar{F}$ has a right adjoint $Z \mapsto \mathcal{A}(F-, Z)$, so $\bar{F}$ preserves colimits.

As a pointwise left K. ext. along a full & faithful functor ($\underline{y}$) we have $\bar{F}\underline{y} \cong F$.

Suppose $\bar{G}$ is another colimit preserving functor with $\bar{G}\underline{y} \cong F$.

Then.

$$\begin{aligned}
\bar{G}P &\cong \bar{G}\left(\varinjlim\left(\underline{y}\downarrow P \xrightarrow{U} \mathbb{C} \xrightarrow{\underline{y}} Psh(\mathbb{C})\right)\right) && \text{density of } \underline{y} \\
&\cong \varinjlim\left(\underline{y}\downarrow P \xrightarrow{U} \mathbb{C} \xrightarrow{\underline{y}} Psh(\mathbb{C}) \xrightarrow{\bar{G}} \mathcal{A}\right) && \bar{G} \text{ preserves colimits} \\
&\cong \varinjlim\left(\underline{y}\downarrow P \xrightarrow{U} \mathbb{C} \xrightarrow{F} \mathcal{A}\right) && \bar{G}\underline{y} \cong F \\
&\cong \bar{F}P && \text{definition of } \bar{F} \text{ as pointwise left Kan extension}
\end{aligned}$$

where every isomorphism is natural in P.

$\square$

The nerve of a category

The *nerve* $N(\mathbb{C})$ of a small category $\mathbb{C}$ is the simplicial set

$$[n] \longmapsto \text{composable sequences } X_0 \xrightarrow{f_0} X_1 \to \cdots \xrightarrow{f_{n-1}} X_n$$

$$\begin{array}{ccc}
[m] & & X_0 \xrightarrow{g_1} X_2 \to \cdots \xrightarrow{g_{m-1}} X_m \\
f \uparrow & \longmapsto & \downarrow \\
[n] & & X_{f(0)} \to X_{f(1)} \to \cdots \to X_{f(n)}
\end{array}$$

where each $X_{f(i)} \longrightarrow X_{f(i+1)}$ is the composite $g_{f(i+1)-1} \circ \cdots \circ g_{f(i)}$ if $f(i+1) > f(i)$ and $1_{X_{f(i)}}$ if $f(i+1) = f(i)$.

The nerve functor $N : \underline{Cat} \to \underline{sSet}$

Define $\underline{K} : \underline{\Delta} \to \underline{Cat}$

$$\underline{K}[n] := \underbrace{\bullet \to \bullet \to \cdots \to \bullet}_{\substack{n+1 \text{ objects} \\ n \text{ morphisms}}} = \underline{[n]}$$

(the partial order $([n], \leq)$ as a category)

$N : \underline{Cat} \to \underline{sSet}$ (cf. $\Delta : \underline{Top} \to \underline{sSet}$)

$$N(\mathbb{C}) = \underline{Cat}(\underline{K}-, \mathbb{C})$$

As before, N has a left adjoint given by the unique (up to isomorphism) colimit preserving functor s.t.

$$\begin{tikzcd} \underline{sSet} \arrow[r] & \underline{Cat} \\ \underline{\Delta} \arrow[u, "\underline{y}"] \arrow[ur, "\underline{K}"'] & \end{tikzcd}$$

A new phenomenon:

The nerve functor $N: \underline{Cat} \to \underline{sSet}$ is full and faithful

Thus (small) categories can be viewed as special simplicial sets.

This viewpoint leads to the notion of quasicategory (a.k.a. $(\infty,1)$-category) which generalises ordinary categories to higher-dimensional 'categories' wrapped up as special simplicial sets.

Such quasicategories are the basis, for example, of Lurie's Higher topos theory

This and related approaches to higher-dimensional categories are a very active research area.