Category Theory 2022-23

Lecture 8

25th November 2022

Monoidal closed structure

Definition (Monoidal closure)

A monoidal category C is <u>(left) closed</u> if, for every $X, Y \in |C|$, there is an object $[X,Y]$ and map

$$[X,Y] \otimes X \xrightarrow{ev_{X,Y}} Y$$

such that, for every map $Z \otimes X \xrightarrow{f} Y$, there exists a unique map $Z \xrightarrow{\Lambda f} [X,Y]$ such that

$$\begin{tikzcd}
{[X,Y] \otimes X} \arrow[r, "ev"] & Y \\
Z \otimes X \arrow[u, "\Lambda f \otimes 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proposition (alternative formulation of monoidal closure)

A monoidal category is (left) closed if and only if, for every object X, there is a functor $[X,-] : C \to C$ together with a natural (in Z and Y) bijection

$$\Lambda : C(Z \otimes X, Y) \xrightarrow{\cong} C(Z, [X,Y])$$

The <u>free vector space</u> $\underline{V}(X)$ over a set X is defined by (Fill in the obvious definitions of addition and scalar multiplication)

$$\underline{V}(X) := \text{functions } X \to K \text{ with finite support}$$

($f : X \to K$ has <u>finite support</u> if $\{x \in X \mid f(x) \neq 0\}$ is finite)

This is characterised up to linear isomorphism by the following <u>universal property</u>

For any set X, vector space W and function $f : X \to W$, there exists a unique linear $h : \underline{V}(X) \to W$ such that

$$\begin{tikzcd} \underline{V}(X) \arrow[r, "h"] & W \\ X \arrow[u, "\delta"] \arrow[ur, "f"'] & \end{tikzcd}$$

(A diagram in <u>Set</u>)

where $\delta_x(y) := \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$

A more pedantic formulation of the same universal property making the role of the forgetful functor $\underline{U} : \underline{\text{Vect}} \to \underline{\text{Set}}$ explicit.

For any set X, vector space W and map $X \xrightarrow{f} \underline{U}W$ in $\underline{\text{Set}}$, there exists a unique map $\underline{V}(X) \xrightarrow{h} W$ in $\underline{\text{Vect}}$ such that

$$\begin{tikzcd} U(\underline{V}(X)) \arrow[r, "\underline{U}h"] & \underline{U}W \\ X \arrow[u, "\delta"] \arrow[ur, "f"'] & \end{tikzcd} \qquad (A)$$

An alternative equivalent statement

There is a functor $\underline{V} : \underline{\text{Set}} \to \underline{\text{Vect}}$ together with natural (in X and W) bijections

$$\psi_{X,W} : \underline{\text{Vect}}(\underline{V}X, W) \xrightarrow{\cong} \underline{\text{Set}}(X, \underline{U}W)$$

Definition of adjunction

An adjunction (F, G, ψ) between $\mathcal{C}$ and $\mathcal{D}$ is given by

- Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$
- Natural (in $X \in |\mathcal{C}|$ and $Y \in |\mathcal{D}|$) bijections

$$\psi_{X,Y} : \mathcal{D}(FX, Y) \xrightarrow{\cong} \mathcal{C}(X, GY)$$

We say that F is left adjoint to G

G right adjoint F

and we write $F \dashv G : \mathcal{D} \to \mathcal{C}$ or $F \dashv G$.

Theorem (Equivalent formulations)

Each of the 3 sets of information below is equivalent to specifying an adjunction (F, G, ψ) between $\mathcal{C}$ and $\mathcal{D}$.

① A functor $G : \mathcal{D} \to \mathcal{C}$, function $F : |\mathcal{C}| \to |\mathcal{D}|$ and family $(X \xrightarrow{\eta_X} GFX)_{X \in |\mathcal{C}|}$ such that, for any $X \in |\mathcal{C}|$, $Y \in |\mathcal{D}|$ and $X \xrightarrow{f} GY$ in $\mathcal{C}$, there exists a unique $FX \xrightarrow{g} Y$ in $\mathcal{D}$ s.t.

$$\begin{tikzcd} GFX \arrow[r, "Gg"] & GY \\ X \arrow[u, "\eta_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes in $\mathcal{C}$.

② A functor $F : \mathcal{C} \to \mathcal{D}$, function $G : |\mathcal{D}| \to |\mathcal{C}|$ and family $(FGY \xrightarrow{\varepsilon_Y} Y)_{Y \in |\mathcal{D}|}$ such that, for any $X \in |\mathcal{C}|$, $Y \in |\mathcal{D}|$ and $FX \xrightarrow{g} Y$ in $\mathcal{D}$, there exists a unique $X \xrightarrow{f} GY$ in $\mathcal{C}$ s.t.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[dr, "g"'] & FGY \arrow[d, "\varepsilon_Y"] \\ & Y \end{tikzcd}$$

commutes in $\mathcal{D}$.

③ Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, and natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ s.t.

$$\begin{tikzcd} FGF \arrow[r, Rightarrow, "\varepsilon F"] & F \\ F \arrow[u, Rightarrow, "F\eta"] \arrow[ur, Rightarrow, "1_F"'] & \end{tikzcd} \qquad \begin{tikzcd} GFG \arrow[r, Rightarrow, "G\varepsilon"] & G \\ G \arrow[u, Rightarrow, "\eta G"] \arrow[ur, Rightarrow, "1_G"'] & \end{tikzcd}$$

Proof outline that ① is equivalent to an adjunction

From adjunction to ①

Let (F, G, ψ) be an adjunction.

Thus $G: \mathcal{D} \to \mathcal{C}$ and $F: |\mathcal{C}| \to |\mathcal{D}|$ are given

Consider $\psi_{X,FX} : \mathcal{D}(FX, FX) \xrightarrow{\cong} \mathcal{C}(X, GFX)$

and define $X \xrightarrow{\eta_X} GFX := \psi(1_{FX})$.

The naturality of $\psi_{X,Y}$ in Y gives us for any map $FX \xrightarrow{g} Y$ in $\mathcal{D}$

$$\begin{tikzcd}
\mathcal{D}(FX, FX) \arrow[r, "\psi_{X,FX}"] \arrow[d, "\mathcal{C}(FX,g)"'] & \mathcal{C}(X, GFX) \arrow[d, "\mathcal{C}(X,Gg)"] \\
\mathcal{D}(FX, Y) \arrow[r, "\psi_{X,Y}"] & \mathcal{C}(X, GY)
\end{tikzcd}$$

$i \mapsto g \circ i$

$h \mapsto Gg \circ h$

In particular $\psi_{X,Y}(g) = \psi_{X,Y}\left(\mathcal{C}(FX,g)(1_{FX})\right)$

$= \mathcal{C}(X, Gg)\left(\psi_{X,FX}(1_{FX})\right) = Gg \circ \eta_X$.

Since $\psi_{X,Y}$ is a bijection, for any $X \xrightarrow{f} GY$ in $\mathcal{C}$,

$\psi^{-1}(f)$ is the unique g s.t. $f = Gg \circ \eta_X$.

From ① to adjunction.

Suppose we have $G : D \to C$, $F : |C| \to |D|$ and $(X \xrightarrow{\eta_X} GFX)_{X \in |C|}$ as in ①.

The functorial action of F is

$$\begin{array}{ccc} X & & \text{the unique } FX \xrightarrow{g} FX' \\ f \downarrow & \longmapsto & \text{such that } \begin{tikzcd} GFX \arrow[r, "Gg"] & GFX' \\ X \arrow[u, "\eta_X"] \arrow[r, "f"'] & X' \arrow[u, "\eta_{X'}"'] \end{tikzcd} \\ X' & & \end{array}$$

The bijection $\psi_{X,Y} : D(FX, Y) \xrightarrow{\cong} C(X, GY)$ is defined by

$$\psi(FX \xrightarrow{g} Y) := X \xrightarrow{\eta_X} GFX \xrightarrow{Gg} GY$$

The diagrammatic property in ① says that this is indeed a bijection.

One then verifies that the functorial action of F preserves identities and composition and the naturality of $\psi_{X,Y}$. $\square$

Examples of adjunctions

The free-vector-space functor $\underline{V} : \underline{Set} \to \underline{Vect}$ is left adjoint to the forgetful $\underline{U} : \underline{Vect} \longrightarrow \underline{Set}$.

(Property (A) is formulation (1) of an adjunction.)

The free-group functor $F : \underline{Set} \to \underline{Grp}$ is left adjoint to the forgetful $\underline{U} : \underline{Grp} \to \underline{Set}$.

(Similar statements hold for other free-algebra functors.)

A monoidal category $\mathcal{C}$ is left closed iff for every object X, the functor $(-)\otimes X : \mathcal{C} \to \mathcal{C}$ has a right adjoint. (The right adjoint is $[X, -]$.)

(The definition of left closure is formulation (2) of adjunction.)

Any equivalence of categories (F, G, α, β) is an adjunction with $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $G \dashv F : \mathcal{C} \to \mathcal{D}$.

(Use formulation (3), exploiting α^{-1} and β^{-1} as appropriate.)

Consider $(-)^* : \underline{\mathrm{Vect}}^{op} \to \underline{\mathrm{Vect}}$, the contravariant functor mapping every vector space to its dual.

Then $(-)^*$ is <u>self adjoint</u> : $(-)^* \dashv (-)^* : \underline{\mathrm{Vect}}^{op} \to \underline{\mathrm{Vect}}$

(There are natural bijections

$$\underline{\mathrm{Vect}}(V, W^*) \cong \underline{\mathrm{Vect}}(V \otimes W, k)$$
$$\cong \mathrm{Vect}(W, V^*) \cong \underline{\mathrm{Vect}}^{op}(V^*, W) \quad .)$$

More generally, if C is symm. mon. closed then, for every $Y \in |C|$, $[-, Y]$ extends to a functor $[-, Y] : C^{op} \to C$ that is self adjoint $[-, Y] \dashv [-, Y] : C^{op} \to C$

<u>Exercise</u> : verify this last point.

- Generalise the statement to categories with (non-symmetric) monoidal structure that are both left and right closed.

Proposition (Adjoints determined up to isomorphism)

Suppose $F \dashv G : D \to C$.

(1) If $F' \dashv G : D \to C$ then $F' \cong F : C \to D$.

(2) If $F \dashv G' : D \to C$ then $G' \cong G : D \to C$.

Proof

(1) We use reformulation (1) of adjunction.

The adjunctions $F \dashv G$ and $F' \dashv G$ give us $X \xrightarrow{\varepsilon_X} GFX$ and $X \xrightarrow{\varepsilon'_X} GF'X$.

Applying reformulation (1) we get $FX \xrightarrow{h_X} F'X$ and $F'X \xrightarrow{h'_X} FX$ unique s.t.

$$\begin{tikzcd} GFX \arrow[r, "Gh_X"] & GF'X \\ X \arrow[u, "\varepsilon_X"] \arrow[ur, "\varepsilon'_X"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} GF'X \arrow[r, "Gh'_X"] & GFX \\ X \arrow[u, "\varepsilon'_X"] \arrow[ur, "\varepsilon_X"'] & \end{tikzcd}$$

It is easy to show that their composites are identities and that $(h_X)_X$ and $(h'_X)_X$ are natural.

(2) Follows from (1) because $F \dashv G : D \to C$ iff $G \dashv F : C^{op} \to D^{op}$.
(This 'iff' is immediate from the definition of adjunction.)

Proposition (Composition of adjunctions)

If $F \dashv G : D \to C$ and $F' \dashv G' : E \to D$ then $F'F \dashv GG' : E \to C$

Proof $C(X, GG'Z) \cong D(FX, G'Z) \cong E(F'FX, Z)$.

Naturality holds because the composition of natural bijections preserves it. $\square$

Definition (Preservation of limits)

A functor $F : C \to C'$ preserves the limit of a diagram $D\ G \to C$ (G a graph) if, for any limit cone $(L, (L \xrightarrow{\beta_x} Dx)_{x \in G})$, it holds that $(FL, (FL \xrightarrow{F\beta_x} FDx)_{x \in G})$ is a limit cone in C' for the diagram $FD : G \to C'$.

$$\left(\begin{tikzcd} & L \arrow[dl, "\beta_x"'] \arrow[d, "\beta_y"] \\ Dx \arrow[r, "De"] & Dy \end{tikzcd} \right)_{x,y,e} \qquad \left(\begin{tikzcd} & FL \arrow[dl, "F\beta_x"'] \arrow[d, "F\beta_y"] \\ FDx \arrow[r, "FDe"] & FDy \end{tikzcd} \right)_{x,y,e}$$

F is said to <u>preserve (existing) limits of shape G</u> if F preserves the limit of D, for every diagram $F : G \to C$ (that has a limit).

Instances of the above: F preserves (finite) products, F preserves pullbacks, F preserves equalisers, F preserves finite limits (a.k.a. is <u>left exact</u>) etc.

F is said to <u>preserve (existing) limits</u> if F preserves (existing) limits of shape G, for every graph G.

There are dual definitions of what it means for F to <u>preserve colimits</u>

Proposition Suppose $F' \dashv F : \mathcal{C} \to \mathcal{C}'$. Then F preserves limits and F' preserves colimits.

(Right adjoints preserve limits, left adjoints preserve colimits.)

Proof Let $(L, (L \xrightarrow{\rho_x} Dx)_{x \in |G|})$ be a limit cone for $D : G \to \mathcal{C}$.

We need to show $(FL, (FL \xrightarrow{F\rho_x} FDx)_x)$ is a limit for $FD : G \to \mathcal{C}'$.

Let $(Z, (Z \xrightarrow{\Gamma_x} FDx)_x)$ be a cone for FD.

Then $(F'Z, (F'Z \xrightarrow{\psi^{-1}\Gamma_x} Dx)_x)$ is a cone for D.

$$\begin{tikzcd}
Z \arrow[r, dashed, "\psi\lambda"] \arrow[d, "\Gamma_x"'] \arrow[dr, "\Gamma_y"' near end] & FL \arrow[d, "F\rho_y"] \arrow[dl, "F\rho_x" near end] \\
\Big(FDx \arrow[r, "FDe"] & FDy \Big)_{x,y,e}
\end{tikzcd}
\qquad
\begin{tikzcd}
F'Z \arrow[r, dashed, "\lambda"] \arrow[d, "\psi^{-1}\Gamma_x"'] \arrow[dr, "\psi^{-1}\Gamma_y"' near end] & L \arrow[d, "\rho_y"] \arrow[dl, "\rho_x" near end] \\
\Big(Dx \arrow[r, "De"] & Dy \Big)_{x,y,e}
\end{tikzcd}$$

Let $F'Z \xrightarrow{\lambda} L$ be the unique cone morphism in $\mathcal{C}$.

Then $Z \xrightarrow{\psi\lambda} FL$ is the unique cone morphism in $\mathcal{C}'$.

The statement for colimits follows by duality because $F' \dashv F : \mathcal{C} \to \mathcal{C}' \iff F' \dashv F : \mathcal{C}'^{op} \to \mathcal{C}^{op}$.

$\square$

Application The 'arithmetic' of exponentiation

Suppose throughout that $\mathcal{C}$ is left monoidal closed.

$$[I, X] \cong X \qquad \text{cf. } x^1 = x$$

$$[Y \otimes Z, X] \cong [Z, [Y, X]] \qquad x^{yz} = (x^y)^z$$

If $\mathcal{C}$ has finite products then

$$[X, 1] \cong 1 \qquad 1^x = 1$$

$$[X, Y \times Z] \cong [X, Y] \times [X, Z] \qquad (yz)^x = y^x z^x$$

If $\mathcal{C}$ has finite coproducts then

$$0 \otimes X \cong 0 \qquad 0 \cdot x = 0$$

$$(Y + Z) \otimes X \cong (Y \otimes X) + (Z \otimes X) \qquad (y + z)x = yx + zx$$

If $\mathcal{C}$ has finite products and coproducts then

$$[0, X] \cong 1 \qquad x^0 = 1$$

$$[Y + Z, X] \cong [Y, X] \times [Z, X] \qquad x^{y+z} = x^y x^z$$

N.B., In the case of a cartesian closed category, $\otimes$ and $\times$ coincide.

Exercise Prove the above!