Category Theory 2022-23

Lecture 3

21st October 2022

Suppose V is a vector space (over k) of dimension n.

Then V^* is also a vector space of dimension n.

If $e_1, \dots, e_n$ is a basis for V

Then $e_1^*, \dots, e_n^*$ is a basis for V^*

where $e_i^* : (\lambda_1 e_1 + \dots + \lambda_n e_n) \mapsto \lambda_i$

The function

$$(\lambda_1 e_1 + \dots + \lambda_n e_n) \mapsto (\lambda_1 e_1^* + \dots + \lambda_n e_n^*)$$

is then a linear isomorphism from V to V^*

The definition of the isomorphism depends crucially on the initial choice of basis. A different choice of basis results in a different isomorphism.

V^{**} is again a vector space of dimension n

The function

$$v \longmapsto (f \in V^* \mapsto f(v)) \ : \ V \longrightarrow V^{**}$$

is a linear isomorphism that is defined independtly of any choice of basis.

In avoiding arbitrary choices, the isomorphism $V \to V^{**}$ is natural in a sense that the isomorphism $V \to V^*$ is not.

The isomorphism $V \to V^{**}$ is defined uniformly in V.

Category theory gives a precise interpretation to this idea of naturality / uniformity : the notion of natural transformation.

Natural transformations are morphisms between functors (between the same two categories).

Given functors $F, G : \mathcal{C} \to \mathcal{D}$

a natural transformation $\alpha : F \Rightarrow G$ (or $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$)

the components of α

is a family $\left(FX \xrightarrow{\alpha_X} GX\right)_{X \in |\mathcal{C}|}$ of

morphisms in $\mathcal{D}$ indexed by objects of $\mathcal{C}$

satisfying :

for every morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$,

the square in $\mathcal{D}$ below commutes

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

(i.e., $\alpha_Y \circ Ff = Gf \circ \alpha_X$)

the naturality condition

For our vector space example, define

$$\varepsilon_V : v \longmapsto (f \in V^* \mapsto f(v))$$

which gives a family $(V \xrightarrow{\alpha_V} V^{**})_{V \in |\underline{\mathrm{Vect}}|}$
indexed by vector spaces (no need to require a finite dimension).

We show that this defines a natural transformation

$$\begin{tikzcd}
\underline{\mathrm{Vect}} \arrow[r, bend left=50, "1_{\underline{\mathrm{Vect}}}"] \arrow[r, bend right=50, "(\cdot)^{**}"'] \arrow[r, phantom, "\Downarrow \varepsilon"] & \underline{\mathrm{Vect}}
\end{tikzcd}$$

defined last week

where $(\cdot)^{**}$ is the composite functor $\underline{\mathrm{Vect}} \xrightarrow{(\cdot)^*} \underline{\mathrm{Vect}}^{op} \xrightarrow{(\cdot)^*} \underline{\mathrm{Vect}}$

The resulting action of $(\cdot)^{**}$ on morphisms is

$$\begin{tikzcd}
V \arrow[d, "l"'] & & V^{**} \arrow[d, "l^{**} := F \in V^{**} \mapsto (g \in W^* \mapsto F(g \circ l))"] \\
W & \longmapsto & W^{**}
\end{tikzcd}$$

Verifying naturality

For any $V \xrightarrow{l} W$ in $\underline{\text{Vect}}$, we need to show that the square below commutes

$$\begin{tikzcd} V \arrow[r, "\varepsilon_V"] \arrow[d, "l"'] & V^{**} \arrow[d, "l^{**}"] \\ W \arrow[r, "\varepsilon_W"'] & W^{**} \end{tikzcd}$$

Indeed, for any $v \in V$, we have

$$\begin{aligned}
(l^{**} \circ \varepsilon_V)(v) &= l^{**}(f \mapsto f(v)) \\
&= g \mapsto (f \mapsto f(v))(g \circ l) \\
&= g \mapsto g(l(v)) \\
&= (\varepsilon_W \circ l)(v)
\end{aligned}$$

For any set X, consider the functions

$$\{\cdot\}_X : x \mapsto \{x\} : X \to \mathcal{P}X$$

union of a set of subsets of X

$$\bigcup_X : \underline{Y} \mapsto \bigcup \underline{Y} : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$$

These are defined in a uniform way for all sets X!

The maps give the components of natural transformations

$$\{\cdot\} : 1_{\underline{set}} \Rightarrow \mathcal{P}$$

$$\bigcup : \mathcal{P}^2 \Rightarrow \mathcal{P} \qquad (\mathcal{P}^2 := \mathcal{P}\mathcal{P})$$

where $\mathcal{P}$ is the <u>covariant powerset functor</u>

$$X \mapsto \mathcal{P}X$$

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array} \longmapsto \begin{array}{c} \mathcal{P}X \\ \downarrow \mathcal{P}_f := X' \mapsto f(X') \\ \mathcal{P}Y \end{array}$$

$(f(X') := \{f(x) \mid x \in X'\}$ the direct image$)$

(This answers part of an exercise from last week.)

Verifying naturality

For any $X \xrightarrow{f} Y$ in $\underline{\mathrm{Set}}$, one needs to check the commutativity of the squares below.

$$
\begin{tikzcd}
X \arrow[r, "\{\cdot\}"] \arrow[d, "f"'] & \mathcal{P}X \arrow[d, "x' \mapsto f(x')"] \\
Y \arrow[r, "\{\cdot\}"'] & \mathcal{P}Y
\end{tikzcd}
$$

$\mathcal{P}_f$

$$
\begin{tikzcd}
\mathcal{P}^2X \arrow[r, "\bigcup"] \arrow[d, "\underline{Y} \mapsto \{f(x') \mid x' \in \underline{Y}\}"'] & \mathcal{P}X \arrow[d, "x' \mapsto f(x')"] \\
\mathcal{P}^2Y \arrow[r, "\bigcup"'] & \mathcal{P}Y
\end{tikzcd}
$$

$\mathcal{P}^2 f$

Given

$$\begin{tikzcd} C \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & D \end{tikzcd}$$

Vertical composition of natural transformations

There is a composite nat. trans.

$$\beta \circ \alpha : F \Rightarrow H$$

defined by

composition in D.

$$(\beta \circ \alpha)_X := \beta_X \circ \alpha_X$$

Exercise Verify the naturality condition.

Functor categories

Given categories $\mathcal{C}, \mathcal{D}$, the <u>functor category</u> $[\mathcal{C}, \mathcal{D}]$ is defined by:

$$|[\mathcal{C}, \mathcal{D}]| := \text{functors } \mathcal{C} \to \mathcal{D}$$

$$[\mathcal{C}, \mathcal{D}](F, G) := \text{nat transformations } F \Rightarrow G$$

$$1_F := \text{identity transformation } \{FX \xrightarrow{1_{FX}} FX\}_{X \in |\mathcal{C}|}$$

$$\beta \circ \alpha := \text{vertical composition}$$

There are size issues! If $\mathcal{C}, \mathcal{D}$ are large then $[\mathcal{C}, \mathcal{D}]$ is very large. However,

- $\mathbb{C}, \mathbb{D}$ small $\Rightarrow$ $[\mathbb{C}, \mathbb{D}]$ small
- $\mathbb{C}$ small, $\mathcal{D}$ locally small $\Rightarrow$ $[\mathbb{C}, \mathcal{D}]$ locally small.

Whiskering

Given
$$\begin{tikzcd} B \arrow[r, "F"] & C \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\alpha\Downarrow"] & D \arrow[r, "H"] & F \end{tikzcd}$$

Define $H\alpha : HG_1 \Rightarrow HG_2$ by $(H\alpha)_X := H\alpha_X$

$\alpha F : G_1F \Rightarrow G_2F$ by $(\alpha F)_X := \alpha_{FX}$

Either way of combing leads to the same

$$H\alpha F : HG_1F \Rightarrow HG_2F$$

Horizontal composition

Given
$$\begin{tikzcd} C \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & E \end{tikzcd}$$

Define $\beta * \alpha : G_1F_1 \Rightarrow G_2F_2$

by $\beta * \alpha := (\beta F_2) \circ (G_1\alpha) \quad \left(= (G_2\alpha) \circ (\beta F_1)\right)$

Exercise Using the above show that functor composition is itself a functor $G, F \mapsto GF : [D, E] \times [C, D] \to [C, E]$.

Proposition

The following are equivalent for a natural transformation
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=L}] & \mathcal{D} \arrow[from=U, to=L, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

(1) Every component $FX \xrightarrow{\alpha_X} GX$ is an isomorphism in $\mathcal{D}$

(2) α is an isomorphism in $[\mathcal{C}, \mathcal{D}]$.

To prove (1) $\Rightarrow$ (2), the main point is that the inverses $\left(FX \xleftarrow{\alpha_X^{-1}} GX\right)_{X \in |\mathcal{C}|}$ satisfy the naturality condition.

Exercise!

A natural transformation enjoying property (1) (or (2)) is called a <u>natural isomorphism</u>

Our vector space example

$$\left(V \xrightarrow{\varepsilon_V} V^{**} \right)_{V \in \underline{\mathrm{FDVect}}}$$

the category of finite dimensional vector spaces

is a natural isomorphism when restricted to finite dimensional vector spaces.

$$\begin{tikzcd} \underline{\mathrm{FDVect}} \arrow[r, bend left, "I"] \arrow[r, bend right, "(\cdot)^{**}"'] \arrow[r, phantom, "\Downarrow \varepsilon"] & \underline{\mathrm{Vect}} \end{tikzcd}$$

The inclusion functor

or $\underline{\mathrm{FDVect}}$

Further constructions of categories

Let C be a category and $I \in |C|$.

<u>The slice category</u> (of C over I) C/I

$$|C/I| := \text{maps in } C \text{ with codomain } I$$

$$C/I\left(\begin{array}{c} X \\ p\downarrow \\ I \end{array}, \begin{array}{c} Y \\ q\downarrow \\ I \end{array}\right) := \left\{ X \xrightarrow{f} Y \;\middle|\; \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & I & \end{tikzcd} \text{ commutes} \right\}$$

Identities and composition inherited from C

<u>The coslice category</u> (of C under I) I/C

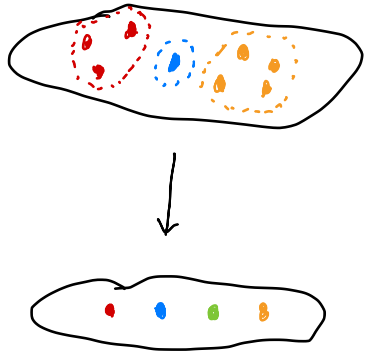
$$|I/C| := \text{maps in } C \text{ with domain } I$$

$$I/C\left(\begin{array}{c} I \\ i\downarrow \\ X \end{array}, \begin{array}{c} I \\ \downarrow j \\ Y \end{array}\right) := \left\{ X \xrightarrow{f} Y \;\middle|\; \begin{tikzcd} & I \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd} \text{ commutes} \right\}$$

Identities and composition inherited from C.

There are forgetful functors $C/I \to C$ and $I/C \to C$

A function
$$\begin{array}{c} X \\ {\scriptstyle p}\big\downarrow \\ I \end{array}$$



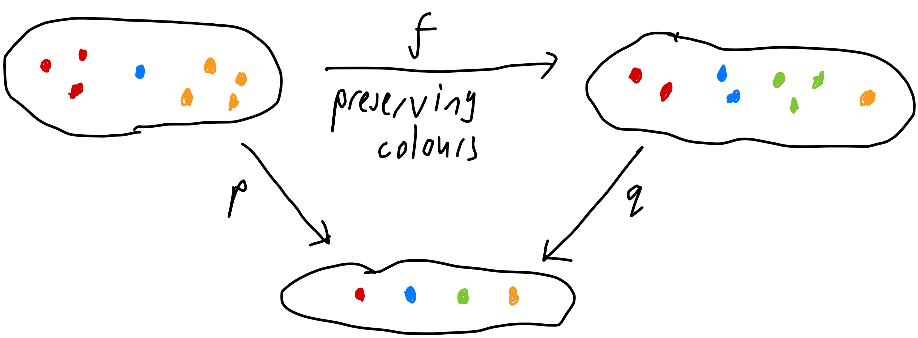
Gives rise to a function

$$i \mapsto p^{-1}(i) : I \to \mathcal{P}X$$

mapping every $i \in I$ to its fibre

Maps in $\underline{\mathbf{Set}}/I$ from $\begin{array}{c} X \\ {\scriptstyle p}\big\downarrow \\ I \end{array}$ to $\begin{array}{c} Y \\ \big\downarrow{\scriptstyle q} \\ I \end{array}$

are functions $X \xrightarrow{f} Y$ that map each $p^{-1}(i)$ to $q^{-1}(i)$



$\equiv$ly families of functions $(f_i : p^{-1}(i) \to q^{-1}(i))_{i \in I}$

The category $\underline{\mathrm{Fam}}_I$ of I-indexed families of sets

<u>Objects</u> $|\underline{\mathrm{Fam}}_I|$:= the collection of I-indexed families of sets $(X_i)_{i\in I}$

$= I \to \underline{\mathrm{Set}}$

<u>Morphisms</u>

$\underline{\mathrm{Fam}}_I\big((X_i)_{i\in I}, (Y_i)_{i\in I}\big) :=$

I-indexed families of functions

$(f_i : X_i \to Y_i)_{i\in I}$

$= \prod_{i\in I} X_i \to Y_i$

<u>Identities</u> $1_{(X_i)_{i\in I}} := (1_{X_i} : X_i \to X_i)_{i\in I}$

<u>Composition</u> $(g_i)_{i\in I} \circ (f_i)_{i\in I} := (g_i \circ f_i)_{i\in I}$.

Shorter description:

$\underline{\mathrm{Fam}}_I := [\underline{I}, \underline{\mathrm{Set}}]$

The <u>discrete</u> category

$|\underline{I}| := I$

only identity morphisms

A functor $F : \underline{Set}/I \to \underline{Fam}_I$

$$\begin{matrix} X \\ p\downarrow \\ I \end{matrix} \longmapsto \left(p^{-1}(i)\right)_{i\in I}$$ object action

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & I & \end{tikzcd} \mapsto \left(f\restriction_{p^{-1}(i)} : p^{-1}(i) \to q^{-1}(i)\right)_{i\in I}$$ morphism action

A functor $S : \underline{Fam}_I \to \underline{Set}/I$

$$(X_i)_{i\in I} \longmapsto \begin{matrix} \sum_{i\in I} X_i \\ \downarrow \pi_1 \\ I \end{matrix} \quad \begin{matrix} := \{(i,x) \mid i\in I, x\in X_i\} \\ := (i,x) \mapsto i \end{matrix}$$ object action

$$(f_i)_{i\in I} \mapsto \left((i,x) \mapsto (i, f_i(x))\right)_{i\in I}$$ morphism action

A natural isomorphism $\phi : 1_{Set/I} \Rightarrow SF$

$$\phi_{\begin{smallmatrix} X \\ \downarrow p \\ I \end{smallmatrix}} := x\in X \longmapsto (p(x), x) : \begin{matrix} X \\ p\downarrow \\ I \end{matrix} \to \begin{matrix} \sum_{i\in I} p^{-1}(x) \\ \downarrow \pi_1 \\ I \end{matrix}$$

A natural isomorphism $\psi : 1_{\underline{Fam}_I} \Rightarrow FS$

$$\psi_{(X_i)_{i\in I}} := \left(x\in X_i \longmapsto (i,x)\right)_{i\in I} : (X_i)_{i\in I} \to (\{i\}\times X_i)_{i\in I}$$

An equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$ is given by:

- Functors $I : \mathcal{C} \to \mathcal{D}$ and $J : \mathcal{D} \to \mathcal{C}$
- Natural isomorphisms $\alpha : 1_{\mathcal{C}} \Rightarrow JI$ and $\beta : 1_{\mathcal{D}} \Rightarrow IJ$

So the categories $\underline{\mathrm{Set}}/I$ and $\underline{\mathrm{Fam}}_I$ are equivalent.

Week 3 puzzle

Find descriptions of the following functor categories in more direct (and perhaps familiar) mathematical terms.

(1) $[\underline{G}, \underline{Set}]$ for any group G.

(2) $[\underline{\Gamma}, \underline{Set}]$

where $\underline{\Gamma}$ is the 2-object category below with only 2 non-identity morphisms

$$E \overset{s}{\underset{t}{\rightrightarrows}} V$$