# Countable and Dependent Choice and Countably Distributive Toposes

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PSSL, Leiden                    16th November 2024

Countable choice (cc):

$$\forall n \in \mathbb{N} \, \exists x \in X, \, P(n,x) \Rightarrow \exists f \in X^{\mathbb{N}} \, \forall n \in \mathbb{N}, \, P(n, f(x))$$

Dependent choice (DC):

$$\forall x \in X \, \exists y \in X, \, R(x,y) \Rightarrow$$
$$\forall x \in X \, \exists f \in X^{\mathbb{N}}, \, f(0) = x \wedge \forall n \in \mathbb{N}, \, R(f(n), f(n+1))$$

- Not difficult to formulate in internal logic of elementary topos with NNO

- $AC \Rightarrow DC \Rightarrow CC$, but converse implications fail.

We begin with logic-free characterisations of CC and DC in elementary toposes with countable limits, which are probably folklore.

Detailed proofs can be found in the masters dissertation of my former student Severin Mejak

S. Mejak
Topos models of set-theoretic principles
Masters dissertation
Faculty of Mathematics and Physics, University of Ljubljana
2019

# Logic-free characterisations of countable choice

## Proposition   T.f.a.e. for an elementary topos $\mathcal{E}$ with NNO

- $\mathcal{E} \vDash CC$
- The functor $(-)^{\mathbb{N}} : \mathcal{E} \to \mathcal{E}$ preserves epis $\Big|$ <span style="color:red">A standard formulation of CC in elementary toposes</span>

Moreover if $\mathcal{E}$ has countable limits, another equivalent statement is

- If $X_1 \xrightarrow{e_1} Y_1$, $X_2 \xrightarrow{e_2} Y_2$, ... are epis then so is

$$X_1 \times X_2 \times X_3 \times \cdots$$
$$\Big\downarrow e_1 \times e_2 \times e_3 \times \cdots$$
$$Y_1 \times Y_2 \times Y_3 \times \cdots$$

<span style="color:purple">i.e., the product functor</span>
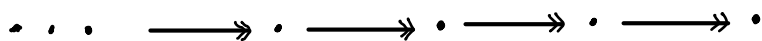$$\textcolor{purple}{\Pi : \mathcal{E}^{\omega} \to \mathcal{E}}$$
<span style="color:purple">preserves epis</span>

# A logic-free characterisation of dependent choice

**Proposition** T.f.a.e. for an elementary topos $\mathcal{E}$ with countable limits

- $\mathcal{E} \vDash DC$

- Every $\omega^{op}$-diagram of epis

$$\cdots \quad \cdot \quad \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot$$
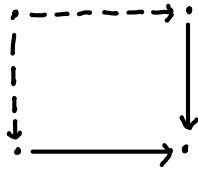
  has a cone of epis

- For every $\omega^{op}$-diagram of epis, the limit cone is a cone of epis.

The <u>atomic (Grothendieck) topology</u> is defined on a small category
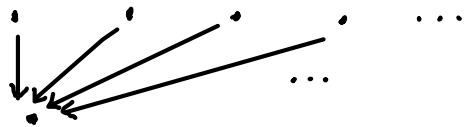$\mathbb{C}$ if and only if $\mathbb{C}$ is <u>coconfluent</u> : every cospan completes
to a commuting square

$$\begin{array}{ccc} \cdot & \dashrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$$

<u>Atomic topology</u> : a sieve $S$ is a cover $\Leftrightarrow$ $S \neq \emptyset$

The Grothendieck topos $Sh(\mathbb{C}, at)$ of <u>atomic sheaves</u> is boolean.

**Proposition** A sufficient condition for $Sh(\mathbb{C}, at)$ to validate CC is that $\mathbb{C}$ is **$\omega$-coconfluent**: every $\omega$-cospan in $\mathbb{C}$



has a cone.

($\omega$-coconfluent $\Leftrightarrow$ the atomic topology is closed under <u>countable</u> intersections)

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**Proposition** A sufficient condition for $Sh(\mathbb{C}, at)$ to validate DC is that every $\omega^{op}$-chain in $\mathbb{C}$



has a cone.

**Theorem** T.f.a.e. for a Grothendieck topos $\mathcal{E}$

1. $\mathcal{E} \equiv Sh(\mathbb{C}, J)$ where $J$ is closed under countable intersections.

2. $\mathcal{E} \equiv Sh(\mathbb{C}, J)$ for which the associated sheaf functor
   $\underline{a}: PSh(\mathbb{C}) \longrightarrow Sh(\mathbb{C}, J)$ preserves countable limits.

3. $\mathcal{E}$ validates CC and satisfies: for all families $(J_i)_{i \in I}$ of sets
   and $(A_{i,j})_{i \in I, j \in J_i}$ of objects with $I$ countable, the canonical map

$$\coprod_{f \in \prod_{i \in I} J_i} \prod_{i \in I} A_{i, f(i)} \longrightarrow \prod_{i \in I} \coprod_{j \in J_i} A_{i,j}$$

   is an isomorphism.

We call a Grothendieck topos satisfying 3 a

**Countably distributive topos**

Examples of countably distributive toposes

- Presheaf toposes

- ( Grothendieck toposes with finite sites  } Any such is $\equiv$ to a presheaf topos

- Atomic toposes over $\omega$-coconfluent categories

- Subtoposes of completely distributive toposes for which the inverse image of the geometric inclusion preserves countable limits

Do _Countably geometric_ morphisms (inverse image functors
preserve countable limits) between countably distributive topose
provide a setting for studying classifying toposes for
_Countably geometric_ theories (extending geometric logic with
countable conjunctions and products) ?

_Conjecture_ The topos of probability sheaves is the classifying
countably distributive toposes for the countably geometric theory
of non-separable measure algebras.

## Outline proof of theorem

**1 ⇒ 2**    It is standard to define $\underline{a} := (\cdot)^{+} \circ (\cdot)^{+}$ where

$$(\cdot)^{+} : Psh(\mathbb{C}) \to Psh(\mathbb{C})$$

is Grothendieck's +-functor, which preserves finite limits

If $J$ is closed under countable intersections then it is easily verified that $(\cdot)^{+}$ preserves countable products, hence $\underline{a}$ does too.

$\underline{2 \Rightarrow 3}$ To verify the distributivity law in $Sh(\mathcal{C}, J)$,

suppose $(A_{i,j})_{i \in I, j \in J_i}$ is a family of sheaves

Writing $\coprod^S$ and $\coprod^P$ for sheaf and presheaf coproduct we have

$$\coprod^S_f \prod_i A_{i, f(i)} \cong a\left(\coprod^P_f \prod_i A_{i, f(i)}\right) \cong a\left(\prod_i \coprod^P_j A_{i,j}\right)$$

$$\cong \prod_i a\left(\coprod^P_j A_{i,j}\right) \cong \prod_i \coprod^S_j A_{i,j}$$

$a$ preserves countable products

To verify CC in $Sh(\mathbb{C}, J)$ we use a lemma.

**Lemma**   If $A_1 \xrightarrow{d_1} B_1$, $A_2 \xrightarrow{d_2} B_2$, ... are $J$-dense monos in $PSh(\mathbb{C})$ then so is $\prod_n A_n \xrightarrow{\prod_n d_n} \prod_n B_n$.

**Proof of lemma**   $\underline{a}\left(\prod_n d_n\right) \cong \prod_n \left(\underline{a}(d_n)\right)$   <span style="color:red">($\underline{a}$ preserves countable products)</span>

Since each $d_n$ is dense, the r.h.s. is a countable product of isos

Hence the r.h.s. is an iso   <span style="color:red">(CC in meta-theory)</span>

Hence $\prod_n d_n$ is $J$-dense.

$\square$

To verify CC in $Sh(\mathbb{C}, J)$, suppose $A_1 \xrightarrow{e_1} B_1, , A_2 \xrightarrow{e_2} B_2, \cdots$

are epis in $Sh(\mathbb{C}, J)$ We verify that $\prod_i A_i \xrightarrow{\prod_i e_i} \prod_i B_i$

is also epi in $Sh(\mathbb{C}, J)$

Let $A_n \xrightarrow{q_n} C_n \xrightarrow{m_n} B_n$ be the epi-mono factorisation of $e_n$ in $Psh(\mathbb{C})$.

Then $\prod_i e_i = (\prod_i m_i) \circ (\prod_i q_i)$ is the e-m factorisation of $\prod_i e_i$ in $Psh(\mathbb{C})$.

Since each $e_n$ is epi in $Sh(\mathbb{C}, J)$, each $q_n$ is J-dense, hence $\prod_i q_i$ is J-dense by lemma

Since the mono part of the e-m-factorisation of $\prod_i e_i$ in $Psh(\mathbb{C})$ is J-dense

and $\prod_i e_i$ is a map between sheaves, it is epi in $Sh(\mathbb{C}, J)$.

## $3 \Rightarrow 1$

Suppose $\mathcal{E}$ is a countably distributive Grothendieck topos.

Let $\mathbb{C}$ be a small full subcategory of $\mathcal{E}$ containing a generating set for $\mathcal{E}$ and closed under countable limits in $\mathcal{E}$.

Let $J$ be the __cover topology__ on $\mathbb{C}$

$$\{A_j \xrightarrow{c_j} B\}_{j \in J} \text{ is covering} \iff \{c_j\}_{j \in J} \text{ is jointly epi in } \mathcal{E}$$
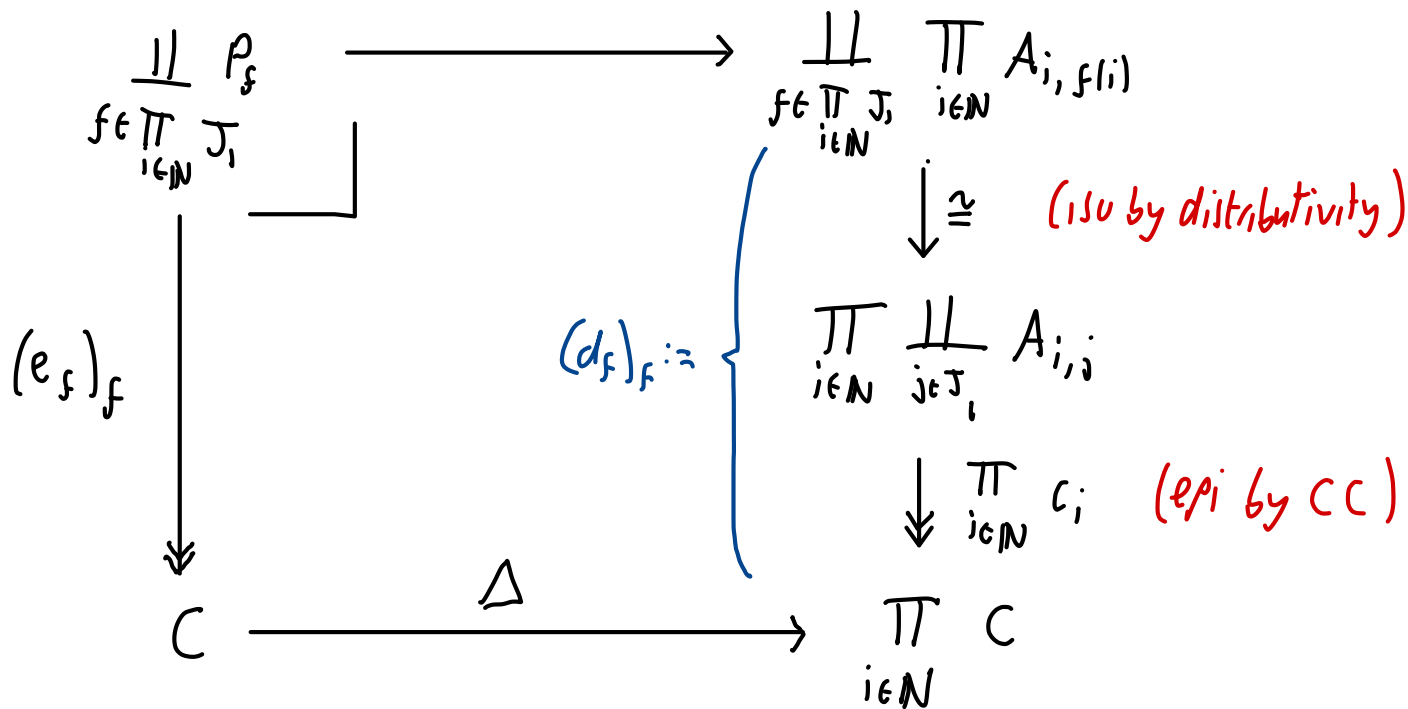
By Giraud theorem / comparison lemma machinery $\mathcal{E} \cong Sh(\mathbb{C}, J)$.

We show that $J$ is closed under countable intersections.

Let $S_1, S_2, \ldots$ be covering sieves for an object $B$.

Each $S_n = \{A_{n,j} \xrightarrow{b_{n,j}} C\}_{j \in J_n}$ gives an epi $\coprod\limits_{j \in J_n} A_{n,j} \xrightarrow{\;c_n := (b_{n,j})_j\;} C$

We have:

$$
\begin{array}{ccc}
\coprod\limits_{f \in \prod\limits_{i \in \mathbb{N}} J_i} P_f & \xrightarrow{\hspace{4cm}} & \coprod\limits_{f \in \prod\limits_{i \in \mathbb{N}} J_i} \;\prod\limits_{i \in \mathbb{N}} A_{i,\,f(i)} \\[2em]
\Big\downarrow {\scriptstyle (e_f)_f} & & \Big\downarrow {\scriptstyle \cong} \quad \text{(iso by distributivity)} \\[2em]
 & (d_f)_f := \Bigg\{ & \prod\limits_{i \in \mathbb{N}} \coprod\limits_{j \in J_i} A_{i,j} \\[2em]
 & & \Big\downarrow {\scriptstyle \prod\limits_{i \in \mathbb{N}} c_i} \quad \text{(epi by CC)} \\[2em]
C & \xrightarrow{\quad \Delta \quad} & \prod\limits_{i \in \mathbb{N}} C
\end{array}
$$

where each $P_f \xrightarrow{e_F} C$ is given by the pullback in $\mathbb{C}$

$$
\begin{array}{ccc}
P_f & \longrightarrow & \prod\limits_{i \in \mathbb{N}} A_{i, F(i)} \\
\downarrow{\scriptstyle e_f} & & \downarrow{\scriptstyle d_f} \\
C & \xrightarrow{\ \Delta\ } & \prod\limits_{i \in \mathbb{N}} C
\end{array}
$$

The left edge of the top diagram is epi (since pullback of epi) hence defines a cover $\{ \hat{P}_f \xrightarrow{e_f} C \}_{f \in \prod\limits_{i \in \mathbb{N}} J_i}$, which by construction is contained in $\bigcap\limits_{i \in \mathbb{N}} S_n$. By monotonicity of covering property $\bigcap\limits_{i \in \mathbb{N}} S_i \in J$. $\qquad \square$