Category Theory 2022-23

Lecture 5

4th November 2022

Recall A pullback of $X \xrightarrow{f} Z \xleftarrow{g} Y$ is given by

- $X \xleftarrow{p} P \xrightarrow{q} Y$ s.t. $f \circ p = g \circ q$, and

satisfying

- for any $X \xleftarrow{x} W \xrightarrow{y} Y$ s.t. $f \circ x = g \circ y$, there exists a unique $W \xrightarrow{w} P$ such that $p \circ w = x$ and $q \circ w = y$.

$$\begin{tikzcd}
W \arrow[dr, dashed, "w"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
& P \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

The pullback lemma

Given a commuting diagram in a category $\mathcal{C}$ of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & \cdot \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "B"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

1. If $\downarrow\overset{\rightarrow}{\underset{\rightarrow}{A}}\downarrow$ and $\downarrow\overset{\rightarrow}{\underset{\rightarrow}{B}}\downarrow$ are both pullbacks then so is $\downarrow\overset{\longrightarrow}{\underset{\longrightarrow}{AB}}\downarrow$.

2. If $\downarrow\overset{\longrightarrow}{\underset{\longrightarrow}{AB}}\downarrow$ and $\downarrow\overset{\rightarrow}{\underset{\rightarrow}{B}}\downarrow$ are both pullbacks then so is $\downarrow\overset{\rightarrow}{\underset{\rightarrow}{A}}\downarrow$

(A more general version of 2:

If $\downarrow\overset{\longrightarrow}{\underset{\longrightarrow}{AB}}\downarrow$ is a pullback and p, q are jointly mono then $\downarrow\overset{\rightarrow}{\underset{\rightarrow}{A}}\downarrow$ is a pullback.)

Proof of generalised 2   We have a commuting diagram

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"'] \arrow[dr, phantom, "A"] & X' \arrow[r, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "B"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "g'"'] & Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

where AB is a pullback and f', h are jointly mono. We need to sow that A is a pullback.

Consider any $Y'' \xleftarrow{y''} W \xrightarrow{x'} X'$ s.t. $f'x' = g'y''$.
We must show there is a unique $W \xrightarrow{w} X''$ s.t. $f''w = y''$ and $h'w = x'$.
Note that any such w also satisfies $hh'w = hx'$.

We have $fhx' = gf'x' = gg'y''$, so since AB is a pullback there exists a unique $W \xrightarrow{w} X''$ s.t. $f''w = y''$ and $hh'w = hx'$.
By what we noted above, it is enough to show that this w also satisfies $h'w = x'$.

We now use the property that f' and h are jointly mono.
Since $hh'w = hx'$ and $f'h'w = g'f''w = g'y'' = f'x'$

it follows that $h'w = x'$, as required.

$\square$

Binary product

A *binary product* of $X, Y \in |\mathcal{C}|$ is given by a span $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ in $\mathcal{C}$ such that, for every span $X \xleftarrow{x} Z \xrightarrow{y} Y$ in $\mathcal{C}$ there exists a unique map $Z \xrightarrow{z} P$ s.t. $p_1 \circ z = x$ and $p \circ z = y$.

$$\begin{tikzcd}
 & X \\
Z \arrow[ur, "x"] \arrow[r, dotted, "z"] \arrow[dr, "y"'] & P \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
 & Y
\end{tikzcd}$$

Proposition If $X \xleftarrow{p_1} P \xrightarrow{p_2} Y$ and $X \xleftarrow{p_1'} P' \xrightarrow{p_2'} Y$ are both binary products then the unique map $P' \xrightarrow{i} P$ such that $p_1 \circ i = p_1'$ and $p_2 \circ i = p_2'$ is an isomorphism.

We say that $\mathcal{C}$ *has binary products* if, for every pair $X, Y \in |\mathcal{C}|$, a product $X \xleftarrow{p_1} P \xrightarrow{p_2} Y$ exists in $\mathcal{C}$.

Notation It is common to write $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ for a chosen binary product for X, Y in $\mathcal{C}$, and to write (x, y) for the unique $Z \to X \times Y$ given by $X \xleftarrow{x} Z \xrightarrow{y} Y$ as above.

I- indexed product

The product of a family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ is given by an object $P \in |\mathcal{C}|$ and a family $(P \xrightarrow{p_i} X_i)_{i\in I}$ of maps in $\mathcal{C}$ such that, for every family $(Z \xrightarrow{x_i} X_i)_{i\in I}$, there exists a unique map $Z \xrightarrow{z} P$ s.t $\forall i \in I \quad p_i \circ z = x_i$.

$$\begin{tikzcd} Z \arrow[r, dashed, "z"] \arrow[dr, "x_i"'] & P \arrow[d, "p_i"] \\ & X_i \end{tikzcd} \qquad \forall i$$

I- indexed products are determined up to isomorphism.

- We say $\mathcal{C}$ has (small) products if, for every set I, every I-indexed family of objects $(X_i)_{i\in I}$ has a product.
- We say $\mathcal{C}$ has finite products if, for every finite set I, every I-indexed family of objects $(X_i)_{i\in I}$ has a product.

Notation Common to write $\left(\prod_{i\in I} X_i \xrightarrow{\pi_i} X_i\right)_{i\in I}$ for a chosen product of $(X_i)_{i\in I}$.

We also write $(f_i)_{i\in I}$ for the unique map $Z \longrightarrow \prod_{i\in I} X_i$ such that $\pi_i \circ (f_i)_{i\in I} = f_i \quad \forall i \in I$

Set has products. $\prod_{i\in I} X_i := \{(x_i)_{i\in I} \mid \forall i\in I\ x_i \in X_i\}$

$$\pi_i := (x_i)_{i\in I} \mapsto x_i : \prod_{i\in I} X_i \to X_i \quad \text{(as expected!)}$$

Vect$_k$ has products $\prod_{i\in I} V_i :=$ the cartesian product of vector spaces

Grp has products $\prod_{i\in I} G_i :=$ the direct product of groups

Top has products $\prod_{i\in I} S_i =$ the topological product of spaces.

Verification that topological products are (categorical) products

Recall the topological product $\prod_{i\in I} S_i$ endows the product set with the coarsest topology that makes every projection $\prod_{i\in I} S_i \xrightarrow{\pi_i} S_i$ continuous.

I.e., the topology on $\prod_{i\in I} S_i$ has sub-basis $\{\pi_i^{-1}(U) \mid i\in I,\ U \text{ an open subset of } S_i\}$

Given a top. space Z and continuous $Z \xrightarrow{f_i} S_i$ $\forall i\in I$

There is a unique function $Z \xrightarrow{g} \prod_{i\in I} S_i$ s.t. $f_i = \pi_i \circ g$ $\forall i$

namely $g(z) = (f_i(z))_{i\in I}$.

We need to show g is continuous. This is true because, for any sub-basic open subset $\pi_i^{-1}(U) \subseteq \prod_{i\in I} S_i$, we have

$$g^{-1}(\pi_i^{-1}(U)) = f_i^{-1}(U) \text{ which is open in } Z \text{ because } f_i \text{ is continuous.}$$

Terminal object

A special case of I-indexed product : $I = \emptyset$

There is exactly one empty family of objects.

Its product is an object T such that, for any $Z \in |C|$

there exists a unique morphism from Z to T.

Terminal objects are determined up to isomorphism

Notation Common to write 1 for a terminal object

and $Z \xrightarrow{!_Z} 1$ for the unique map.

Proposition A category has finite products if and only if it has binary products and a terminal object.

Proof $\Rightarrow$ is immediate as binary products and terminal object are special cases of finite products.

$\Leftarrow$ We construct $\prod_{i \in I} (X_i)$ by induction on $n := |I|$.

$n=0$ The product is the terminal object

$n=1$ A singleton product $\prod_{i=1}^{1} X_i$ is $(X_i \xrightarrow{1_{X_i}} X_i)_{i=1}^{1}$

$n>1$ Construct $\prod_{i=1}^{n} X_i$ as $\left(\prod_{i=1}^{n-1} X_i\right) \times X_n$ using the induction hypothesis and a binary product. $\square$

Equalisers

An *equaliser* of a parallel pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathcal{C}$ is a map $E \xrightarrow{e} X$ such that:

- $f \circ e = g \circ e$, and
- for any $Z \xrightarrow{x} X$ such that $f \circ x = g \circ x$, there exists a unique $Z \xrightarrow{z} E$ such that $e \circ z = x$

$$\begin{tikzcd} & E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ Z \arrow[ur, dashed, "z"] \arrow[urr, "x"'] & & & \end{tikzcd}$$

Equalizers are determined up to isomorphism.

We say that $\mathcal{C}$ *has equalisers* if every parallel pair of maps in $\mathcal{C}$ has an equaliser.

Proposition Equaliser maps $E \xrightarrow{e} X$ are always mono.

A morphism in a category is called a *regular mono* if there exists some parallel pair for which it is an equaliser map.

Exercise Split monos are regular.

__Set__ has equalisers

Given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ define $E := \{x \in X \mid f(x) = g(x)\}$

$e := x \mapsto x : E \to X$

(__Vect__$_k$ and __Grp__ have equalisers.)

__Top__ has equalisers.

Given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ continuous functions between topological spaces

define E as above endowed with the __subspace topology__ (from X).

Exercises

- In __Set__ every monomorphism is regular.

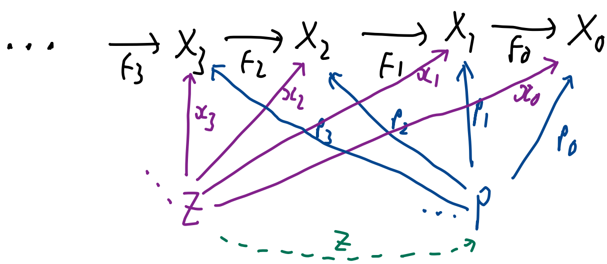
- A continuous function is a regular mono in __Top__ if and only if it is a topological embedding.

(Hint: to show that every topological embedding is a regular mono, make use of the 'indiscrete' topological space $\{0,1\}$ in which only $\varnothing$ and $\{0,1\}$ are open.)

Projective limits

The <u>projective limit</u> of an infinite sequence of maps $(X_{n+1} \xrightarrow{f_n} X_n)_{n \geq 0}$ in C is given by $P \in |C|$ and $(P \xrightarrow{p_n} X_n)_{n \geq 0}$ such that:

- for all n, $p_n = f_n \circ p_{n+1}$; and
- for any family $(Z \xrightarrow{x_n} X_n)_{n \geq 0}$ such that $x_n = f_n \circ x_{n+1}$ $\forall n$, there exists a unique $Z \xrightarrow{z} P$ such that $p_n \circ z = x_n$ $\forall n$.



<u>Set</u> has projective limits

$$P := \{ (x_n)_{n \geq 0} \mid \forall n \; f_n(x_{n+1}) = x_n \}$$

$$p_n := (x_n)_{n \geq 0} \mapsto x_n \quad : P \to X_n$$

All notions we have seen today are instances of a general notion of <u>limit over a diagram</u>

A diagram has a shape given by a <u>graph</u>

Pullback	$\bullet \to \bullet \leftarrow \bullet$
Binary product	$\bullet \quad \bullet$
I-indexed product	$\overbrace{\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet}^{I}$
Terminal object	the empty graph
Equaliser	$\bullet \rightrightarrows \bullet$
Projective limit	$\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$

Note that these graphs are <u>directed</u> and they can have multiple edges between the same two vertices. They are sometimes accordingly called <u>multidigraphs</u> (a.k.a. <u>quivers</u>)

A *graph* (as we shall call it for convenience; more precisely *multidigraph* or *quiver*) G has a collection $|G|$ of *vertices*, and for each $x, y \in |G|$ a collection $G(x,y)$ of *edges* with *source* x and *target* y.

Similarly define *locally finite* & *finite*

G is *locally small* if every $G(x,y)$ is a set.
G is small if it is locally small and $|G|$ is a set.

A *graph (homo)morphism* $H : G \to G'$ is given by:

- a function $H_0 : |G| \to |G'|$
- for every $x, y \in |G|$ a function $H_1 : G(x,y) \to G(H_0 x, H_0 y)$

Graph := category of small graphs and graph morphisms.

N.B. • A category is a graph with additional structure (identities + composition)
• Every functor is a graph morphism.
• These observations give a forgetful functor *Cat* $\to$ *Graph*

Let G be a graph and C a category.

A <u>G-diagram</u> in C is a graph morphism $D : G \to C$

The diagram D is <u>small</u> / <u>finite</u> if G is small / finite

A <u>D-cone</u> $(Z, (\Gamma_u)_{u \in |G|})$

- an object $Z \in |C|$, and
- a family $(Z \xrightarrow{\Gamma_u} Du)_{u \in |G|}$ of maps in C

such that, for every edge $e \in D(u,v)$ in C,

$De \circ \Gamma_u = \Gamma_v$

$$\begin{tikzcd} & Z \arrow[dl, "\Gamma_u"'] \arrow[dr, "\Gamma_v"] & \\ Du \arrow[rr, "De"'] & & Dv \end{tikzcd}$$

A <u>limit D-cone</u> is a D-cone $(L, (L \xrightarrow{\rho_u} Du)_{u \in |G|})$ such that, for every D-cone $(Z, (\Gamma_u)_{u \in |G|})$, there exists a unique $Z \xrightarrow{z} L$ such that $\rho_u \circ z = \Gamma_u$ $\forall u \in |G|$.

A category $\mathcal{C}$ is said to be <u>(small-) complete</u> if every small diagram in $\mathcal{C}$ has a limit cone.

<u>Theorem</u> The following are equivalent.

(1) $\mathcal{C}$ is complete

(2) $\mathcal{C}$ has products and equalisers

A category is said to be <u>finitely complete</u> if every finite diagram has a limit cone.

<u>Theorem</u> The following are equivalent

(1) $\mathcal{C}$ is finitely complete

(2) $\mathcal{C}$ has finite products and equalisers

(3) $\mathcal{C}$ has terminal object and pullbacks

<u>Week 5 puzzle</u>

The category <u>Rel</u> of relations has products. Find a concrete description of products in <u>Rel</u>.