Category Theory 2022-23

Lecture 7

18th November 2022

Monoidal structure on $\mathcal{C}$

is given by

- An object $I \in |\mathcal{C}|$
- A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- Natural isomorphisms

$$X \otimes (Y \otimes Z) \xrightarrow{\alpha_{XYZ}} (X \otimes Y) \otimes Z$$

$$I \otimes X \xrightarrow{\lambda_X} X \qquad X \otimes I \xrightarrow{\rho_X} X$$

Such that the following equalities hold

$$\begin{tikzcd}
X \otimes (Y \otimes (Z \otimes W)) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \alpha"'] & (X \otimes Y) \otimes (Z \otimes W) \arrow[r, "\alpha"] & ((X \otimes Y) \otimes Z) \otimes W \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "\alpha"] & & (X \otimes (Y \otimes Z)) \otimes W \arrow[u, "\alpha \otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (I \otimes Y) \arrow[rr, "\alpha"] \arrow[dr, "1 \otimes \lambda"'] & & (X \otimes I) \otimes Y \arrow[dl, "\rho \otimes 1"] \\
& X \otimes Y &
\end{tikzcd}$$

$$I \otimes I \xrightarrow{\lambda} I \quad = \quad I \otimes I \xrightarrow{\rho} I$$

We say that $\mathcal{C}$ is a <u>monoidal category</u>

If $\mathcal{C}$ is a monoidal category then the same $\otimes, I$ define monoidal structure on $\mathcal{C}^{op}$. (Use $\alpha^{-1}, \rho^{-1}, \lambda^{-1}$.)

If $\mathcal{C}$ is a monoidal category then the same structure exhibits $\mathcal{C}_{iso}$ as a monoidal category. ($\mathcal{C}_{iso}$ is the category with $|\mathcal{C}_{iso}| = |\mathcal{C}|$, whose morphisms are the isos in $\mathcal{C}$.)

Symmetric monoidal structure is given by monoidal structure together with a natural isomorphism

$$X \otimes Y \xrightarrow{\sigma_{XY}} Y \otimes X$$

satisfying

$$\begin{tikzcd} X \otimes Y \arrow[rr, "1"] \arrow[dr, "\sigma"'] & & X \otimes Y \\ & Y \otimes X \arrow[ur, "\sigma"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \sigma"'] & (X \otimes Y) \otimes Z \arrow[r, "\sigma"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha"] \\ X \otimes (Z \otimes Y) \arrow[r, "\alpha"] & (X \otimes Z) \otimes Y \arrow[r, "\sigma \otimes 1"] & (Z \otimes X) \otimes Y \end{tikzcd}$$

$$\begin{tikzcd} X \otimes I \arrow[rr, "\sigma"] \arrow[dr, "\rho"'] & & I \otimes X \arrow[dl, "\lambda"] \\ & X & \end{tikzcd}$$

We say that C is a symmetric monoidal category (smc)

Again, the same structure exhibits C^{op} as an smc.

Finite products

Binary product defines a functor $(-)\times(-) : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$

$$
\begin{array}{ccc}
X & & Y \\
f\downarrow & & g\downarrow \\
X' & & Y'
\end{array}
\quad \longmapsto \quad
\begin{array}{c}
X\times Y \\
\downarrow f\times g := (f\circ\pi_1,\ g\circ\pi_2) \\
X'\times Y'
\end{array}
$$

Define $I := 1$ (terminal object)

$$X\times(Y\times Z) \xrightarrow{\alpha} (X\times Y)\times Z \qquad \text{given by } \alpha := ((\pi_1, \pi_1\circ\pi_2), \pi_2\circ\pi_2)$$

$$1\times X \xrightarrow{\lambda} X \qquad \lambda := \pi_2$$

$$X\times 1 \xrightarrow{\rho} X \qquad \rho := \pi_1$$

This is symmetric monoidal structure

$$X\times Y \xrightarrow{\sigma} Y\times X \qquad \sigma := (\pi_2, \pi_1)$$

Any category with finite products is a symmetric monoidal category

By duality, so is any category with finite coproducts.

A category can carry more than one (symmetric) monoidal structure.

$\underline{\mathrm{Rel}}$

The set-theoretic coproduct $X+Y$ is both coproduct and product in the category $\underline{\mathrm{Rel}}$.

(Cf. Week 5 puzzle.)

So this is one (symmetric) monoidal structure on $\underline{\mathrm{Rel}}$.

Although the set-theoretic product $X \times Y$ is not the product in $\underline{\mathrm{Rel}}$ it is a symmetric monoidal product. E.g., the functorial action is given by

$$\begin{array}{ccccc} X & & X' & & X \times X' \\ R \downarrow & & \downarrow R' & \longmapsto & \downarrow R \times R' \\ Y & & Y' & & Y \times Y' \end{array}$$

$$(x, x') \,(R \times R')\, (y, y') \iff x R y \text{ and } x' R y'.$$

$[\mathcal{C},\mathcal{C}]$ – the category of endofunctors on $\mathcal{C}$.

$$G \otimes F := GF \qquad \text{(composition)}$$
$$I := 1_{\mathcal{C}}$$

This defines strict monoidal structure, all α, λ, ρ maps are identities.

This monoidal structure is not symmetric

$\underline{Mat_K}$

Define $m \otimes n := mn$

Given $n \xrightarrow{A} m$ $n' \xrightarrow{B} m'$ define $n \otimes n' \xrightarrow{A \otimes B} mm'$ to be the $mm' \times nn'$ matrix

$$\begin{bmatrix} (a_{11} B) & \cdots & (a_{1n} B) \\ \vdots & & \vdots \\ (a_{m1} B) & & (a_{mn} B) \end{bmatrix}$$

i.e C where $C_{(i-1)m'+i',\,(j-1)n'+j'} = a_{ij} \cdot b_{i'j'}$ for $\begin{matrix} 1 \leq i \leq m \\ 1 \leq i' \leq m' \\ 1 \leq j \leq n \\ 1 \leq j' \leq n' \end{matrix}$

$I := 1$

ρ and λ are easy.

Have fun working out α!

This is <u>symmetric</u> monoidal structure

$m \otimes n \xrightarrow{\sigma} n \otimes m$ is the $(mn \times mn)$ square matrix:

$$\sigma_{d,e} = \begin{cases} 1 & \text{if } \exists\, 1 \leq i \leq m \;\; 1 \leq j \leq n \text{ s.t. } \begin{matrix} d = (i-1)m + j \\ e = i + (j-1)n \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

More generally $\underline{\mathrm{Vect}}_K$

The <u>tensor product</u> $V \otimes W$ of vector spaces enjoys the following characterising property

There is a bilinear map

$$\psi_{VW} : V \times W \longrightarrow V \otimes W$$

Such that, for any vector space U and bilinear $f : V \times W \to U$, there exists a unique linear map $g : V \otimes W \to U$ satisfying $\circledast$

$$\begin{tikzcd} V \otimes W \arrow[r, "g"] & U \\ V \times W \arrow[u, "\psi"] \arrow[ur, "f"'] & \end{tikzcd}$$

N.B. This is <u>not</u> a diagram in $\underline{\mathrm{Vect}}_K$!

So linear maps $V \otimes W \to U$ are in 1-1-correspondence with bilinear maps $V \times W \to U$

Define $I := K$

Then $\otimes, I$ endow $\underline{\mathrm{Vect}}_K$ with symmetric monoidal structure.

<u>Exercise</u> Work out the details, either abstractly using $\circledast$ or concretely using an explicit construction of $V \otimes W$ (e.g. as a quotient space).

Monoidal closed structure

A monoidal category C is (left) closed if, for every $X, Y \in |C|$, there is an object $[X,Y]$ and map

$$[X,Y] \otimes X \xrightarrow{ev_{X,Y}} Y$$

such that, for every map $Z \otimes X \xrightarrow{f} Y$, there exists a unique map $Z \xrightarrow{\Lambda f} [X,Y]$ such that

$$\begin{tikzcd}
{[X,Y] \otimes X} \arrow[r, "ev"] & Y \\
Z \otimes X \arrow[u, "\Lambda f \otimes 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

There is also a notion of right closed using maps

$$X \otimes [X,Y]^R \longrightarrow Y$$

In the case of a symmetric monoidal category the notions of left and right closed coincide, and one simply says closed

A category is Cartesian closed if it has finite products and the product monoidal structure is closed.

$\underline{\mathbf{Set}}$ is cartesian closed

Define $[X, Y] := Y^X$ (set of all functions $X \to Y$)

$$ev_{X,Y} := (f, x) \mapsto f(x) : Y^X \times X \to Y$$

Given $Z \times X \xrightarrow{f} Y$, we must show there is a unique $Z \xrightarrow{\Lambda f} Y^X$ such that the diagram below commutes

$$\begin{tikzcd}
Y^X \times X \arrow[r, "ev"] & Y \\
Z \times X \arrow[u, "\Lambda_f \times 1_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

A

This diagram says

$$\begin{aligned}
f(x,z) &= (ev \circ (\Lambda_f \times 1_X))(z, x) \\
&= ev(\Lambda_f(z), x) \\
&= \Lambda_f(z)(x)
\end{aligned}$$

So the function

$$\Lambda_f := z \mapsto (x \mapsto f(z,x)) : Z \to Y^X$$

is uniquely determined by the commutivity.

$\underline{\text{Vect}}_k$ is monoidal closed

Define $[V, W] := V \multimap W$ the vector space of all linear functions from V to W

Notice that the evaluation function

$$f, v \mapsto f(v) \; : \; (V \multimap W) \times V \longrightarrow W$$

is bilinear. So it corresponds to a unique map

$$(V \multimap W) \otimes V \xrightarrow{ev} W$$

There is also a 1-1-correspondence

$$\frac{\text{bilinear } [U \times V, W]}{\text{linear } [U, V \multimap W]} \qquad \downarrow \uparrow$$

$f : U \times V \to W \quad \longmapsto \quad (u \mapsto (v \mapsto f(u,v)))$

$g : U \to (V \multimap W) \quad \longmapsto \quad ((u,v) \mapsto g(u)(v))$

This gives us Λ viz:

$$\underline{\text{Vect}}_k(U \times V, W) \underset{\cong}{\xrightarrow{\Lambda}} \underline{\text{Vect}}_k(U, V \multimap W)$$

Week 7 puzzle

Consider the functor categories

$[\underline{G}, \underline{Set}]$

$[\underline{\Delta}, \underline{Set}]$

from Week 3. Both categories are cartesian closed. Find explicit descriptions of the closed structure