Category Theory 2022-23

Lecture 10

9th December 2022

$\underline{\mathrm{Grp}}$ is a typical category of algebras and their homomorphisms.

The forgetful functor $U : \underline{\mathrm{Grp}} \to \underline{\mathrm{Set}}$ has a left adjoint, the free group functor $F : \underline{\mathrm{Set}} \to \underline{\mathrm{Grp}}$

For any set X, group G and function $f : X \to G$ there is a unique homomorphism $h : FX \to G$ s.t.

$$\begin{tikzcd}
FX \arrow[r, "h"] & G \\
X \arrow[u, "i_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

where $i_X(x)$ = the generator in FX associated to x.

The free group FX can be constructed explicitly as equivalence classes of certain expressions

This is an example of a general method for constructing free algebras explicitly

Such constructions are "fussy" Mac Lane [CWM]

CSL (Complete semilattices)

Another category of algebra-like structures

A *complete semilattice* is a partially ordered set $(X, \leq)$ in which every subset $A \subseteq X$ *supremum* (least upper bound) $\bigvee A \in X$.

Every complete semilattice has a *least element* $(\bigvee \emptyset)$.

It also has all *infima* (greatest lower bounds) $\bigwedge A$

(I.e a complete semilattice is always a complete lattice.)

A homomorphism of CSLs is a function $h : X \to Y$ that preserves suprema

$$\forall A \subseteq X \qquad h\left(\bigvee_X A\right) = \bigvee_{X'} \boxed{h(A)} \ \text{direct image}$$

It follows that homomorphisms are order preserving

$$x \leq_X x' \implies h(x) \leq_{X'} h(x')$$

(Homorphisms preserve least element, but need not preserve $\bigwedge$.)

Alternative Formulation

A complete semilattice is a structure $(X, \bigvee)$ where $\bigvee : \wp X \to X$ satisfies

$$\bigvee \{x\} = x \qquad \forall x \in X$$

$$\bigvee \{ \bigvee B \mid B \in \mathcal{A} \} = \bigvee (\bigcup \mathcal{A}) \qquad \forall \mathcal{A} \subseteq \wp B$$

Prop For any set X, csl $(Y, \bigvee)$ and function $f : X \to Y$, there exists a unique csl homomorphism $h : (\wp X, \bigcup) \to (Y, \bigvee)$ s.t.

$$\begin{tikzcd}
\wp X \arrow[r, "h"] & Y \\
X \arrow[u, "\{\cdot\}"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Equivalently The left adjoint to the forgetful functor $U : \underline{CSL} \to \underline{Set}$ is

$$\begin{tikzcd}
X \arrow[d, "f"'] & & (\wp X, \bigcup) \arrow[d, "A \mapsto f(A)"] \\
Y \arrow[r, phantom, "\longmapsto"] & & (\wp Y, \bigcup)
\end{tikzcd}$$

One of the two covariant powerset functors from Lec. 2.

KHaus Objects: compact Hausdorff spaces
Morphisms: Continuous function

Surprising facts

- The forgetful U: **KHaus** $\to$ **Set** has a left adjoint
- **KHaus** is a category of algebras & homomorphisms.

Neither of these points are easy to show.

Today we address

- How to find adjoints without constructing them explicitly (using adjoint functor theorems).
- When an adjunction can be viewed as a free-algebra construction for a category of algebras and homomorphism.

We shall use **Grp**, **CSL** and **KHaus** as our running examples.

Rather than constructing the left adjoint directly we prove that it exists using Freyd's <u>adjoint functor theorem</u>

<u>FAFT</u> Suppose $G : \mathcal{D} \to \mathcal{C}$ is a functor from a complete category $\mathcal{D}$ and G preserves limits. Then G has a left adjoint if and only if it enjoys the following <u>Solution set condition</u>: for any $X \in |\mathcal{C}|$, there exists a family $(X \xrightarrow{g_i} GY_i)_{i \in I}$ of maps in $\mathcal{C}$ indexed by a set I, such that, for every map $X \xrightarrow{f} GY$ in $\mathcal{C}$ there exist $i \in I$ and $Y_i \xrightarrow{h} Y$ in $\mathcal{D}$ such that $f = Gh \circ g_i$.

Recommended:
- Read the proof
- Read about the Special Adjoint Functor Theorem

See [CWM] (or other textbook)

We prove that $\underline{\mathrm{KHaus}}$ is complete and $U : \underline{\mathrm{KHaus}} \to \underline{\mathrm{Set}}$ preserves limits simultaneously by proving a stronger property: U creates limits.

A functor $U : \mathcal{A} \to \mathcal{C}$ **creates limits** if, for any diagram $D : G \to \mathcal{A}$ (G a graph), and any $\{L \xrightarrow{q_x} UD_x\}_{x \in G}$ limit cone in $\mathcal{C}$ for the diagram UD, we have:

- there exists a unique D-cone $\{K \xrightarrow{p_x} D_x\}_{x \in G}$ in $\mathcal{A}$ that is mapped by U to $\{L \xrightarrow{q_x} UD_x\}_{x \in G}$; and
- $\{K \xrightarrow{p_x} D_x\}_{x \in G}$ is a limit cone for D in $\mathcal{A}$.

(The above formulation is standard and useful. However, it breaks a category-theoretic taboo, it is not stable under equivalence of categories.)

U : KHaus → Set creates products (& equalisers)

Let $(X_i)_{i\in I}$ be a family of comp. Haus. space.

By Tychonoff's theorem (requires AC!) the product topological space $\prod_{i\in I} X_i$ is itself a compact Hausdorff space.

Since $\left(\prod_{i\in I} X_i \xrightarrow{\pi_i} X_i\right)_{i\in I}$ is a cone in KHaus that is also a product cone in Top it is a fortiori a limit cone in KHaus.

Now let $(P \xrightarrow{p_i} UX_i)$ be any product cone in Set.

Topologise P with the unique topology that turns the product isomorphism $P \xrightarrow{i} \prod_{i\in I} X_i$ in Set into a homeomorphism.

This is again compact Hausdorff since $\prod_{i\in I} X_i$ is.

Moreover, any other topology on P for which the cone $(P \xrightarrow{p_i} X_i)$ remains continuous would have to be finer than the given one since the product topology is the coarsest such topology.

But no strictly finer topology can be compact Hausdorff.

So U creates products. A similar argument shows it creates equalisers, hence arbitrary limits.

The solution set condition for $U : \underline{\text{kHaus}} \to \underline{\text{Set}}$

Given a set X, consider any compact Hausdorff space Y and function $f: X \to Y$ with dense image. Then the function $y \mapsto \{A \subseteq X \mid y \in \overline{f(A)}\}$ (where $\overline{f(A)}$ is the closure of $f(A)$ in Y) is injective. So Y is in bijective correspondence with a subset of $\mathcal{P}\mathcal{P}X$.

Consider the family $\left(g : X \to Z\right)_{(g, Z, \tau)}$

indexed by the set

$$\{(g, Z, \tau) \mid Z \subseteq \mathcal{P}\mathcal{P}X,\ \tau \text{ is a compact Hausdorff topology on } Z,\ g: X \to Z \text{ has dense image}\}$$

This satisfies the required condition. Consider any function $f: X \to Y$ where Y is a compact Haus. space.

Let $Y' := \overline{\text{image}(f)}$. Since $Y' \subseteq Y$ is closed it carries a compact Hausdorff topology. Clearly $f_{Y'}: X \to Y'$ has dense image. By the remarks at the start, Y' is homeomorphic to some compact Hausdorff space (Z, τ) with $Z \subseteq \mathcal{P}\mathcal{P}X$ via $i: Z \to Y'$. The function $g := i^{-1} \circ f_{Y'}$ indexed by (g, Z, τ) and continuous function $i: Z \to Y' \subseteq Y$ provide the required factorisation of f. □

Revisiting CSLs.

The endofunctor $\mathcal{P} : \underline{Set} \to \underline{Set}$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{P}X \arrow[d, "\mathcal{P}f"] \\ \mathcal{P}Y \end{tikzcd} \mathcal{P}f := A \mapsto f(A)$$

carries the structure of a monoid in $[\underline{Set}, \underline{Set}]$

$\{\cdot\}_X := x \mapsto \{x\} : X \to \mathcal{P}X$ $\quad$ $\{\ \} : 1_{\underline{Set}} \Rightarrow \mathcal{P}$

$\bigcup_X := A \mapsto \bigcup A : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$ $\quad$ $\bigcup : \mathcal{P}^2 \Rightarrow \mathcal{P}$

$$\begin{tikzcd} & \mathcal{P}X \arrow[dl, "\{\cdot\}_{\mathcal{P}X}"'] \arrow[d, "1_{\mathcal{P}X}"] \arrow[dr, "\mathcal{P}\{\cdot\}_X"] & \\ \mathcal{P}^2X \arrow[r, "\bigcup_X"'] & \mathcal{P}X & \mathcal{P}^2X \arrow[l, "\bigcup_X"] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{P} \arrow[dl, Rightarrow, "\{\cdot\}\mathcal{P}"'] \arrow[d, Rightarrow, "1_{\mathcal{P}}"] \arrow[dr, Rightarrow, "\mathcal{P}\{\cdot\}"] & \\ \mathcal{P}^2 \arrow[r, Rightarrow, "\bigcup"'] & \mathcal{P} & \mathcal{P}^2 \arrow[l, Rightarrow, "\bigcup"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{P}^3X \arrow[r, "\mathcal{P}\bigcup_X"] \arrow[d, "\bigcup_{\mathcal{P}X}"'] & \mathcal{P}^2X \arrow[d, "\bigcup_X"] \\ \mathcal{P}^2X \arrow[r, "\bigcup_X"'] & \mathcal{P}X \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}^3 \arrow[r, Rightarrow, "\mathcal{P}\bigcup"] \arrow[d, Rightarrow, "\bigcup\mathcal{P}"'] & \mathcal{P}^2 \arrow[d, Rightarrow, "\bigcup"] \\ \mathcal{P}^2 \arrow[r, Rightarrow, "\bigcup"'] & \mathcal{P} \end{tikzcd}$$

The algebraic redefinition of csls has an elegent definiton involving the above structure

A <u>csl</u> (X, V) is $PX \xrightarrow{V} X$ such that

$$\begin{tikzcd}
 & X \arrow[dl, "\{\}_X"'] \arrow[d, "1_X"] \\
PX \arrow[r, "V"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
P^2X \arrow[r, "PV"] \arrow[d, "U"'] & PX \arrow[d, "V"] \\
PX \arrow[r, "V"'] & X
\end{tikzcd}$$

A <u>homomorphism</u> of csls from (X, V) to (X', V') is $X \xrightarrow{h} X'$ s.t.

$$\begin{tikzcd}
PX \arrow[r, "Ph"] \arrow[d, "V"'] & PX' \arrow[d, "V"] \\
X \arrow[r, "h"'] & X'
\end{tikzcd}$$

Definition A *monad* on $\mathcal{C}$ is a monoid in the strict monoidal category $[\mathcal{C}, \mathcal{C}]$.

I.e. A monad is $(T, \underbrace{\eta}_{\text{the unit}}, \underbrace{\mu}_{\text{the multiplication}})$

where $T : \mathcal{C} \to \mathcal{C}$ is a functor

$\eta : 1_{\mathcal{C}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are nat transfs.

Such that

$$\begin{tikzcd}
& T \arrow[dl, Rightarrow, "\eta T"'] \arrow[d, Rightarrow, "1_T"] \arrow[dr, Rightarrow, "T\eta"] & \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T & T^2 \arrow[l, Rightarrow, "\mu"]
\end{tikzcd}$$

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}$$

An (Eilenberg-Moore) *algebra* for a monad T is (A, a) where $A \in |\mathcal{C}|$ and $TA \xrightarrow{a} A$ is such that

$$\begin{tikzcd}
 & A \arrow[dl, "\eta_A"'] \arrow[dr, "1_A"] & \\
TA \arrow[rr, "a"'] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2A \arrow[r, "Ta"] \arrow[d, "\mu"'] & TA \arrow[d, "a"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}$$

A *(homo)morphism* of algebras from (A, a) to (B, a) is $A \xrightarrow{h} B$ in $\mathcal{C}$ such that

$$\begin{tikzcd}
TA \arrow[r, "Th"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

N.B. The definition of a monad implies that (TX, μ_X) is always an algebra. By naturality of μ, for any $X \xrightarrow{f} Y$, we have Tf is a homomorphism from (TX, μ_X) to (TY, μ_Y)

The definition of algebra implies that for any algebra (A, a) the map $TA \xrightarrow{a} A$ is a homomorphism from (TA, μ) to (TA, a).

The Eilenberg-Moore category of algebras

If (T, η, μ) is a monad on $\mathcal{C}$, we write $\mathcal{C}^T$ for the category where objects are algebras for the monad T and whose maps are homomorphisms.

Let $U : \mathcal{C}^T \to \mathcal{C}$ be the forgetful functor

Prop U has a left adjoint $F : \mathcal{C} \to \mathcal{C}^T$

$$F : X \mapsto (TX, \mu_X)$$

$$\begin{tikzcd} X \arrow[d, "f"'] & & (TX, \mu_X) \arrow[d, "Tf"] \\ Y \arrow[r, phantom, "\longmapsto"] & & (TY, \mu_Y) \end{tikzcd}$$

Prop U creates limits.

Exercise Prove the 2 propositions.

Adjunctions give rise to monads

Prop Any adjunction $F \dashv G : D \to C$ gives rise to a monad (T, η, μ) on C defined by

$$T := GF$$

$\eta : 1_C \Rightarrow GF$ is the unit of the adjunction

$\mu := G\varepsilon F : GFGF \Rightarrow GF$

where $\varepsilon : FG \Rightarrow 1_D$ is the adjunction counit

Given an adjunction and associated monad as above define the comparison functor $K : D \longrightarrow C^T$ by

$$K(Y) := \left(GY,\ GFGY \xrightarrow{G\varepsilon_Y} GY\right)$$

$$\begin{tikzcd} Y \arrow[d, "g"] \\ Y' \end{tikzcd} \longmapsto \begin{tikzcd} GY \arrow[d, "Gg"] \\ GY' \end{tikzcd}$$

In the case of Grp, CSL and kHaus the comparison functor is an isomorphism of categories. !

The notion of algebra for a monad encompasses the familiar examples (groups, rings, modules, csls) of algebras (for which free-algebras exist).

It also explicates in what sense kHaus is a category of algebras.

Beck's theorem gives a beautiful method for proving that the comparison functor is an isomorphism.

Beck's (monadicity theorem)

For an adjunction $F \dashv G : D \to C$ the following are equivalent.

- G creates absolute coequalisers.
- The comparison functor $K : D \to C^{GF}$ is an isomorphism of categories.

(One says that the functor G is monadic.)

There are many variants of this theorem (see [CWM]) Some establishing conditions for showing that K is an equivalence of categories.

An absolute coequaliser in C is a coequaliser diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "q"] & Q \end{tikzcd}$$

that satisfies: for all categories E and functors $F: C \to E$, it holds that F preserves the above coequaliser.

$G: D \to C$ <u>creates absolute coequalisers</u> if: for any $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$ in D and absolute coequaliser $\begin{tikzcd} GX \arrow[r, shift left, "Gf"] \arrow[r, shift right, "Gg"'] & GY \arrow[r, "q"] & Q \end{tikzcd}$ in C, there exists a unique $Y \xrightarrow{r} Z$ in D s.t. $GZ = Q$ and $Gr = q$; furthermore this unique r is a coequaliser for $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$.

<u>Outline proof</u> that $U: \underline{Grp} \to \underline{Set}$ creates absolute coequalisers.

Let $\begin{tikzcd} G \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & H \end{tikzcd}$ be group homomorphisms such that

$$\begin{tikzcd} G \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & H \arrow[r, "q"'] & Q \end{tikzcd}$$ is an absolute coequaliser in <u>Set</u>

Since absolute, the coequaliser is preserved by $X \mapsto X \times X : \underline{Set} \to \underline{Set}$

$$\begin{tikzcd} G^2 \arrow[r, shift left, "g^2"] \arrow[r, shift right, "h^2"'] \arrow[d, "\bullet_G"'] & H^2 \arrow[r, "q^2"] \arrow[d, "\bullet_H"] & Q^2 \arrow[d, dashed, "\bullet_Q"] \\ G \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & H \arrow[r, "q"'] & Q \end{tikzcd}$$

We thus obtain a map $Q^2 \xrightarrow{\bullet_Q} Q$ as in the diagram, giving Q a group structure in the unique way making q a homomorphism. $\square$