Category Theory 2022-23

Lecture 1

7th October 2022

A *category* $\mathcal{C}$ is given by:

- A collection $|\mathcal{C}|$ of *objects*.
- For every pair X, Y of objects, a collection
  $$\mathcal{C}(X,Y) \text{ of } \textit{morphisms} \ (\textit{maps})$$
- For every object X an *identity morphism*
  $$1_X \in \mathcal{C}(X,X)$$
- For every triple X, Y, Z of objects a *composition function*
  $$(-)\circ(-) : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

Such that:

this means $f \in \mathcal{C}(X,Y)$

- For every X, Y and $X \xrightarrow{f} Y$, $\quad f \circ 1_X = f = 1_Y \circ f \quad$ (identity laws)
- For every $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, $\quad h \circ (g \circ f) = (h \circ g) \circ f \quad$ (associativity)

Exercise The identities are determined uniquely by the composition function.

By the associativity law, chains of morphisms such as

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

yield a unique composite map $X \xrightarrow{h \circ g \circ f} W$.

Equalities between such composites can be expressed as *commutative diagrams*, e.g.,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1_X"'] \arrow[dr, "f"] & Y \arrow[d, "1_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

expresses the identity laws.

Examples

Set The category of sets (and functions).

Objects: sets

Morphisms: **Set** (X,Y) = the set of all functions from X to Y (which makes sense as X, Y are sets).

Identities: 1_X = the identity function $x \mapsto x : X \to X$

Composition: $g \circ f$ = the composite function $x \mapsto g(f(x))$

In the category **Set** the collection of objects $|\mathbf{Set}|$ is the collection of all sets, which is not itself a set, rather a proper class.

However, for every X, Y, **Set** (X,Y) **is** a set.

A category $\mathcal{C}$ is **locally small** if, for every $X, Y \in |\mathcal{C}|$, the **"hom set"** $\mathcal{C}(X,Y)$ is a set.

$\mathcal{C}$ is **small** if it is locally small and $|\mathcal{C}|$ is also a set.

The category **Set** is thus locally small but not small.

Grp The category of groups (and homomorphisms)

Objects: groups

Morphims: $\mathbf{Grp}(G,H) =$ set of homomorphisms from G to H

Identities and composition: identity functions and function composition (as in **Set**)

Top The category of topological spaces (and continuous functions)

Objects: topological spaces

Morphisms: $\mathbf{Top}(S,T) =$ set of continuous functions from S to T

Identities and composition: function identities & composition

a field

$\mathbf{Vect}_K$ The category of vector spaces (and linear tranformations) over K

Objects: Vector spaces over K

Morphisms: $\mathbf{Vect}_K(U,V) =$ set of linear transformations from U to V

Identities and composition: Function identities & composition.

Commonalities

All examples so far are locally small but not small.

In all, morphisms are classes of functions, with identities and composition given by function identities and composition

The following examples are of a different character

Rel The category of (sets and) relations.

Objects: sets

Morphisms: $\mathbf{Rel}(X,Y) := $ set of relations between X and Y

(Recall a relation between X and Y is a function $R : X \times Y \to \{\text{true}, \text{false}\}$.)

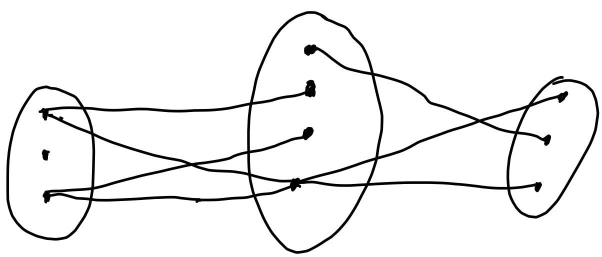
Identities: $1_X = $ the identity relation $x \, 1_X \, x' \Leftrightarrow x = x'$

Composition: $S \circ R = $ relation composition $R ; S$

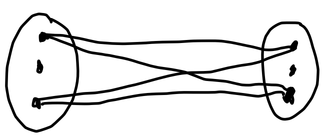
$$x (R;S) z \iff \exists y \;\; xRy \text{ and } ySz$$

This is again a locally small but not small category. However composition is not function composition.

E.g. A composition $3 \to 4 \to 3$



$=$



Let $(G, \cdot, e)$ be any group.

$\underline{G}$ The group G as a category

<u>Objects</u>: Just one object, $*$

<u>Morphisms</u>: $\underline{G}(*,*) := G$

<u>Identities</u>: $1_* := e$

<u>Composition</u>: $y \circ x := y \cdot x$

More generally, the same construction gives a category $\underline{M}$ for any monoid $(M, \cdot, e)$.

$\underline{G}$ (and $\underline{M}$) is a small category.

It has only one object,

Categorification:

Monoid := one object category

Let $(P, \leq)$ be any poset (partially ordered set)

$\underline{P}$ The poset P as a category

Objects: $|\underline{P}| := P$

Morphisms: $\underline{P}(x,y) := \begin{cases} \{*_{x,y}\} & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$

a singleton set

Identities: $1_x := *_{x,x}$

Composition: $*_{y,z} \circ *_{x,y} := *_{x,z}$

More generally, the same construction gives a category for any preorder $(P, \leq)$.

$\underline{P}$ is a small category.

Moreover, there is at most one morphism between any two objects.

Categorification:

preorder := category in which hom sets have at most one element

Special morphisms in a category $\mathcal{C}$

the inverse of f

$X \xrightarrow{f} Y$ is an <u>isomorphism</u> (<u>iso</u>) if there exists $\boxed{Y \xrightarrow{f^{-1}} X}$ such that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$.

Trivially every identity is an isomorphism. It is its own inverse.

<u>Exercises</u>
- If f is an isomorphism then f^{-1} is determined uniquely by f.
- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two isomorphisms then $g \circ f$ is also an iso.

In <u>Set</u> the isomorphisms are the bijections

<u>Grp</u>	bijective homomorphisms
<u>Top</u>	homeomorphisms
<u>Vect</u>$_{\mathbb{K}}$	linear isomorphisms
<u>Rel</u>	graphs of bijections

<u>G</u> every morphism is an isomorphism

<u>P</u> (a poset) the only isomorphisms are the identities

Categorification:

group := 1-object category in which every morphism is an isomorphism

poset := category in which hom-sets have at most one morphism and the only isomorphisms are identities.

$X \xrightarrow{f} Y$ is a <u>Monomorphism</u> (<u>mono</u>) if, for every parallel pair $Z \overset{x}{\underset{y}{\rightrightarrows}} X$, $f \circ x = f \circ y \Rightarrow x = y$

<u>Exercises</u>
- Every isomorphism is a monomorphism
- If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two monos then $g \circ f$ is also mono.
- If $g \circ f$ is mono then so is f.
- If $g \circ f$ is iso and g is mono then f and g are both isos.

In <u>Set</u> the monomorphisms are the injections

<u>Top</u>	continuous injections
<u>Grp</u>	injective homomorphisms
<u>Vect$_K$</u>	injective linear transformations

<u>P</u> (a poset) every morphism is a monomorphism

<u>Weekly puzzle</u> In <u>Rel</u>:

- Is there a mono from 2 (a 2-element set) to 1 (a singleton)?
- Is there a mono from 3 to 2?
- Characterise the monomorphisms in <u>Rel</u>.