An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 14: The Bolzano-Weierstrass Theorem

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1 Introduction: A Cornerstone of Analysis

The Bolzano-Weierstrass Theorem is one of the most celebrated results in real analysis. It states that every bounded sequence has a convergent subsequence. More precisely, in the framework we've developed:

Every bounded sequence has a Cauchy subsequence.

This theorem is remarkable because it guarantees the existence of convergent behavior even in sequences that may appear chaotic or oscillatory. It is the foundation for:

- Proving that continuous functions on closed intervals attain their extrema
- Establishing compactness of closed and bounded subsets of $\mathbb{R}^n$
- Sequential compactness arguments throughout analysis
- Existence proofs in optimization and differential equations

In this lecture, we'll prove the Bolzano-Weierstrass Theorem by combining all the tools we've developed: the monotone subsequence theorem from Lecture 13, the result that bounded monotone sequences are Cauchy from Lecture 12, and the corresponding result for antitone sequences from Problem Sets 12 and 13.

2 New Theorems

2.1 abs_le: The Companion to abs_lt

We've previously seen `abs_lt`, which characterizes when $|x| < y$. Now we need its companion for non-strict inequalities.

Theorem (abs_le): For real or rational numbers x, y with $y \geq 0$:

$$|x| \leq y \iff -y \leq x \leq y$$

This biconditional allows us to split absolute value inequalities into simultaneous bounds. In the forward direction, $|x| \leq y$ gives us both $x \leq y$ and $-x \leq y$, which is equivalent to $-y \leq x$. In the reverse direction, if $-y \leq x \leq y$, then both x and $-x$ are bounded by y, so $|x| \leq y$.

2.2 IsCauchyOfAntitoneBdd: The Dual of Our Earlier Result

In Lecture 12, we proved that bounded monotone (non-decreasing) sequences are Cauchy. The dual result handles non-increasing sequences.

Theorem (IsCauchyOfAntitoneBdd): If a sequence $a : \mathbb{N} \to X$ (where X is $\mathbb{Q}$ or $\mathbb{R}$) is antitone (non-increasing) and bounded below by some M, then a is Cauchy.

The proof technique is elegant: negate the sequence to convert the antitone problem into a monotone one. If a is antitone and bounded below by M, then $b = -a$ is monotone and bounded above by $-M$. By our earlier theorem, b is Cauchy. Then $a = -b$ is also Cauchy (Cauchy sequences are closed under negation).

This was the content of Problem Set 12, and we'll use it as a black box here.

2.3 AntitoneSubseq_of_UnBddPeaks: The Missing Half

In Lecture 13, we proved that sequences without unbounded peaks have monotone subsequences. The complementary result handles the opposite case.

Theorem (AntitoneSubseq_of_UnBddPeaks): If a sequence $a : \mathbb{N} \to X$ has unbounded peaks, then there exists a subsequence $\sigma : \mathbb{N} \to \mathbb{N}$ such that:

- σ is strictly increasing (a `Subseq`)
- $a \circ \sigma$ is antitone (non-increasing)

The intuition is simple: if there are arbitrarily large peaks, we can choose a subsequence consisting entirely of peaks. Since each peak looks down on all future values, and our subsequence consists of peaks ordered by their indices, the sequence values along this subsequence must be non-increasing.

This is the content of Problem Set 13, and we'll use it as a black box in our proof.

3 The Main Theorem: Bolzano-Weierstrass

Now we're ready to state and prove the main result.

Theorem (BolzanoWeierstrass): If a sequence $a : \mathbb{N} \to X$ (where X is $\mathbb{Q}$ or $\mathbb{R}$) is bounded, then there exists a subsequence $\sigma : \mathbb{N} \to \mathbb{N}$ such that:

- σ is strictly increasing (a `Subseq`)
- $a \circ \sigma$ is Cauchy

3.1 Strategic Overview

The proof relies on the dichotomy established in Lecture 13: every sequence either has unbounded peaks or doesn't. We'll handle each case separately:

Case 1: The sequence has unbounded peaks.

1. By `AntitoneSubseq_of_UnBddPeaks`, extract an antitone subsequence σ
2. The boundedness of a gives us bounds on $a \circ \sigma$
3. By `IsCauchyOfAntitoneBdd`, conclude that $a \circ \sigma$ is Cauchy

Case 2: The sequence does not have unbounded peaks.

1. By `MonotoneSubseq_of_BddPeaks` from Lecture 13, extract a monotone subsequence σ

2. The boundedness of a gives us bounds on $a \circ \sigma$

3. By `IsCauchyOfMonotoneBdd` from Lecture 12, conclude that $a \circ \sigma$ is Cauchy

The key insight is that in either case, we obtain a monotone (or antitone) bounded subsequence, and both types are guaranteed to be Cauchy by our earlier theorems.

3.2 Understanding Boundedness

When we say a sequence a is bounded (`SeqBdd a`), we mean that there exists some $M > 0$ such that $|a(n)| \leq M$ for all n.

Using `abs_le`, we can decompose this into two separate bounds:

- **Upper bound:** $a(n) \leq M$ for all n

- **Lower bound:** $-M \leq a(n)$ for all n

These bounds are preserved by passing to subsequences: if $|a(n)| \leq M$ for all n, then $|a(\sigma(n))| \leq M$ for all n and any subsequence σ.

3.3 Lean Solution

```
Statement BolzanoWeierstrass (a : ℕ → X)
    (ha : SeqBdd a)
  : ∃ σ, Subseq σ ∧ IsCauchy (a ∘ σ) := by
choose M Mpos hM using ha
have aBddAbove : ∀ n, a n ≤ M := by
  intro n;
  specialize hM n;
  rewrite [abs_le] at hM;
  apply hM.2
have aBddBelow : ∀ n, -M ≤ a n := by
  intro n;
  specialize hM n;
```

```
  rewrite [abs_le] at hM;
  apply hM.1
by_cases hPeaks : UnBddPeaks a
choose σ σsubseq σanti using
  AntitoneSubseq_of_UnBddPeaks a hPeaks
use σ
split_ands
exact σsubseq
apply IsCauchyOfAntitoneBdd (a ∘ σ) (-M)
intro n
change -M ≤ a (σ n)
apply aBddBelow
apply σanti
choose σ σsubseq σmono using
  MonotoneSubseq_of_BddPeaks a hPeaks
use σ
split_ands
apply σsubseq
apply IsCauchyOfMonotoneBdd (a ∘ σ) M
intro n
change a (σ n) ≤ M
apply aBddAbove
apply σmono
```

3.4 Natural Language Proof

Proof: Let $a : \mathbb{N} \to X$ be a bounded sequence. By the definition of `SeqBdd`, there exists $M > 0$ such that $|a(n)| \leq M$ for all n.

Extracting the bounds. Using `abs_le`, we can rewrite $|a(n)| \leq M$ as the conjunction:

$$-M \leq a(n) \leq M$$

This gives us:

- **Upper bound:** $a(n) \leq M$ for all n
- **Lower bound:** $-M \leq a(n)$ for all n

Case analysis. By the law of excluded middle, either UnBddPeaks(a) holds or it doesn't. We'll handle each case separately.

Case 1: a has unbounded peaks. Suppose UnBddPeaks(a) holds.

By the theorem `AntitoneSubseq_of_UnBddPeaks`, there exists a subsequence $\sigma : \mathbb{N} \to \mathbb{N}$ such that:

- σ is strictly increasing (a `Subseq`)
- $a \circ \sigma$ is antitone (non-increasing)

The subsequence $a \circ \sigma$ inherits the lower bound from a: for all n,

$$-M \leq a(n) \leq a(\sigma(n))$$

Therefore, $a \circ \sigma$ is antitone and bounded below by $-M$.

By the theorem `IsCauchyOfAntitoneBdd`, we conclude that $a \circ \sigma$ is Cauchy.

Thus σ satisfies both required properties: it is a `Subseq` and $a \circ \sigma$ is Cauchy.

Case 2: a does not have unbounded peaks. Suppose $\neg$UnBddPeaks(a).

By the theorem `MonotoneSubseq_of_BddPeaks` from Lecture 13, there exists a subsequence $\sigma : \mathbb{N} \to \mathbb{N}$ such that:

- σ is strictly increasing (a `Subseq`)
- $a \circ \sigma$ is monotone (non-decreasing)

The subsequence $a \circ \sigma$ inherits the upper bound from a: for all n,

$$a(\sigma(n)) \leq a(n) \leq M$$

Therefore, $a \circ \sigma$ is monotone and bounded above by M.

By the theorem `IsCauchyOfMonotoneBdd` from Lecture 12, we conclude that $a \circ \sigma$ is Cauchy.

Thus σ satisfies both required properties: it is a `Subseq` and $a \circ \sigma$ is Cauchy.

Conclusion. In both cases, we have constructed a subsequence σ that is strictly increasing and such that $a \circ \sigma$ is Cauchy. This completes the proof. **QED**

4 The Architecture of the Proof

Let's step back and appreciate the elegant structure of this argument. The Bolzano-Weierstrass Theorem is the capstone of a carefully constructed edifice:

4.1 The Foundation: Monotone Sequences

At the base, we have the fundamental fact that *bounded monotone sequences are Cauchy*. This was proved in Lecture 12 through a clever contradiction argument: if a bounded monotone sequence were not Cauchy, we could construct an infinite sequence of gaps that would eventually force the sequence to exceed its bound.

The dual result for antitone sequences follows by a simple negation argument.

4.2 The Dichotomy: Peaks and Valleys

The next layer introduces the concept of peaks and establishes a dichotomy: every sequence either has unbounded peaks or doesn't.

- **Unbounded peaks:** The sequence keeps returning to new local maxima from which it never recovers
- **Bounded peaks:** After some point, the sequence has no more peaks; it must eventually start climbing

4.3 The Extraction: From Dichotomy to Monotonicity

Lecture 13 showed that sequences without unbounded peaks have monotone subsequences. The proof used the beautiful orbit construction: by iterating a carefully chosen function, we extracted a subsequence that automatically avoided all peaks and hence was forced to be monotone.

Problem Set 13 established the dual result: sequences with unbounded peaks have antitone subsequences. The proof is simpler here – just choose the peaks themselves as the subsequence.

4.4 The Synthesis: Bolzano-Weierstrass

Finally, we combine all these results. Any bounded sequence falls into one of the two cases of our dichotomy. In either case, we can extract a monotone (or antitone) subsequence, and this subsequence remains bounded. By our foundational results, any bounded monotone sequence is Cauchy.

The theorem emerges not from a single clever argument, but from the careful assembly of multiple results, each building on the last.

5 Why This Matters

The Bolzano-Weierstrass Theorem is fundamental to analysis for several reasons:

5.1 Compactness

In topology, a space is called *sequentially compact* if every sequence has a convergent subsequence. The Bolzano-Weierstrass Theorem shows that closed and bounded subsets of $\mathbb{R}$ are sequentially compact. This is a key step in proving the Heine-Borel Theorem, which characterizes compact subsets of Euclidean space.

5.2 Existence Proofs

Many existence results in analysis rely on the Bolzano-Weierstrass Theorem. For example:

- **Extreme Value Theorem:** A continuous function on a closed interval attains its maximum and minimum
- **Bolzano's Theorem:** A continuous function that changes sign must have a zero
- **Optimization:** Continuous functions on compact sets attain optimal values

5.3 Approximation

In numerical analysis and applied mathematics, the Bolzano-Weierstrass Theorem guarantees that bounded sequences of approximations have convergent subsequences, providing a foundation for iterative methods.

5.4 Functional Analysis

The theorem extends to infinite-dimensional spaces (with appropriate modifications), forming the basis for understanding weak and weak* convergence in Banach spaces.

6 Historical Context

The theorem is named after Bernard Bolzano (1781-1848) and Karl Weierstrass (1815-1897), two pioneers of rigorous analysis.

Bolzano's work in the early 19th century was remarkably modern, anticipating many ideas that would only become mainstream later. He understood the importance of proving existence results rigorously and recognized that intuition about continuity and limits needed careful formalization.

Weierstrass, working several decades later, is often credited with putting analysis on a rigorous foundation. He insisted on ε-δ definitions and constructed famous counterexamples (like a continuous nowhere-differentiable function) that showed the limitations of geometric intuition.

The Bolzano-Weierstrass Theorem represents their shared insight: *boundedness plus infinite choice forces accumulation.* If you have infinitely many values confined to a bounded region, they must cluster somewhere.

7 Extensions and Variants

The Bolzano-Weierstrass Theorem has many generalizations:

7.1 Higher Dimensions

In $\mathbb{R}^n$, every bounded sequence has a convergent subsequence. The proof follows the same pattern: extract monotone subsequences in each coordinate, using a diagonal argument to ensure they converge simultaneously.

7.2 Metric Spaces

In complete metric spaces, a set K is compact if and only if every sequence in K has a convergent subsequence (whose limit is in K). This characterization of compactness is one of the most useful in analysis.

7.3 Weak Convergence

In infinite-dimensional Banach spaces, bounded sequences may not have norm-convergent subsequences (the unit ball might not be compact). However, by the Banach-Alaoglu theorem, bounded sequences in the dual space have weak* convergent subsequences.

7.4 Topological Spaces

The most general setting is arbitrary topological spaces, where compactness can be defined without reference to sequences. The relationship between sequential compactness and compactness depends on the topology (they're equivalent for metric spaces).

8 Proof Techniques: What We Learned

This proof showcases several important mathematical techniques:

8.1 Case Analysis with by_cases

The `by_cases` tactic lets us split a proof based on whether a proposition holds or not. This is the formalization of proof by cases, allowing us to leverage the law of excluded middle.

8.2 Decomposing Absolute Values

The `abs_le` theorem shows how to work with absolute value inequalities by splitting them into simultaneous one-sided bounds. This technique is ubiquitous in analysis.

8.3 Duality Arguments

The proof of `IsCauchyOfAntitoneBdd` exemplifies a powerful technique: convert a problem about antitone sequences to one about monotone sequences by negation. This reduces the number of theorems we need to prove from scratch.

8.4 Leveraging Prior Results

The proof of Bolzano-Weierstrass is short precisely because we've built up the necessary machinery in earlier lectures. This demonstrates the importance of:

- Breaking complex proofs into lemmas
- Identifying and proving useful general results

- Structuring a theory to make the main theorems natural consequences

9 Looking Forward

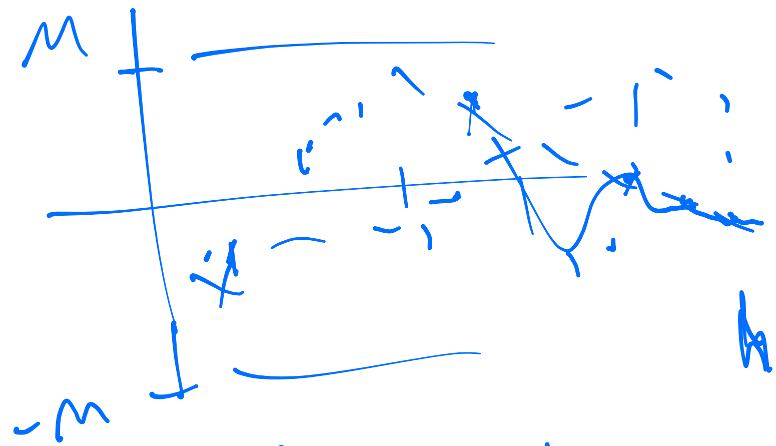
With the Bolzano-Weierstrass Theorem in hand, we're now equipped to prove many classical results in analysis:

- The Extreme Value Theorem: continuous functions on closed intervals are bounded and attain their bounds
- The Intermediate Value Theorem: continuous functions on intervals take on all intermediate values
- The Heine-Borel Theorem: characterizing compact subsets of Euclidean space
- Uniform continuity results on compact sets
- Fixed point theorems

Each of these results relies on the fundamental insight captured by Bolzano-Weierstrass: in the realm of the bounded, there is always convergence hiding somewhere, waiting to be extracted.

Thm (Bolzano-Weierstrass): $a : \mathbb{N} \to X$ ($X = \mathbb{Q}/\mathbb{R}$),

ha : SeqBdd a. <u>Goal</u>: $\exists \sigma$, Subseq σ $\wedge$ IsCauchy $(a \circ \sigma)$.



choose M Mpos ha using ha

hUp : $\forall n$, $a_n \le M$:= by intro n; bound

hLo : $\forall n$, $-M \le a_n$.

m : X
Mpos : M > 0
ha : $\forall n$, $|a_n| \le M$.

by_cases h : UnBddPeaks a.

choose σ SubσAnti using AntiofUnBddPeaks a h.

Use σ

split_ands

apply Subσ

have hLo' : $\forall n$, $-M \le (a \circ \sigma)(n)$:= by intro n; change $-M \le a(\sigma n)$; apply hLo (σ n)

apply IsCauchyofAntiToneBdd $(a \circ \sigma)$ $(-M)$ hLo' Anti

h : ¬ UnBddPeaks a.

choose σ SubσMono using MonotoneSubseqofBddPeaks a h

Use σ

split_ands

apply Subσ

have hN' : ∀n, $a \circ \sigma (n) \le M$:= — —
apply IsCauchy_of_MonoBdd $(a \circ \sigma)$ M hN' mono.

Thm (Monotone Subseq_of_BddPeaks)

have $a : \mathbb{N} \to X$,

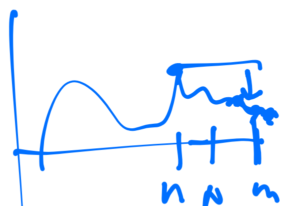
ha : ¬ UnBdd Peaks a

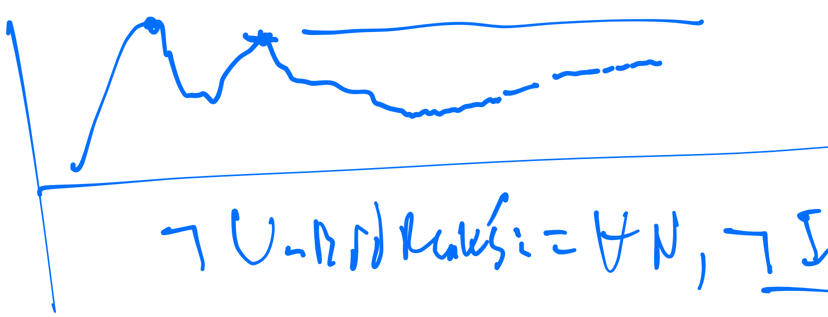
Goal: $\exists \sigma$, Subseq σ ∧ Monotone $(a \circ \sigma)$

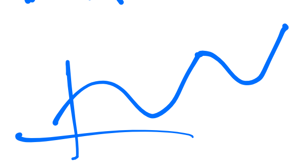
Change $\neg(\forall N, \exists n > N, \forall m > n, a_m \le a_n)$ at ha.

Recall: IsAPeak a n := $\forall m \ge n$, $a_m \le a_n$

UnBdd Peaks a := $\forall N, \exists n > N$, IsAPeak a n.

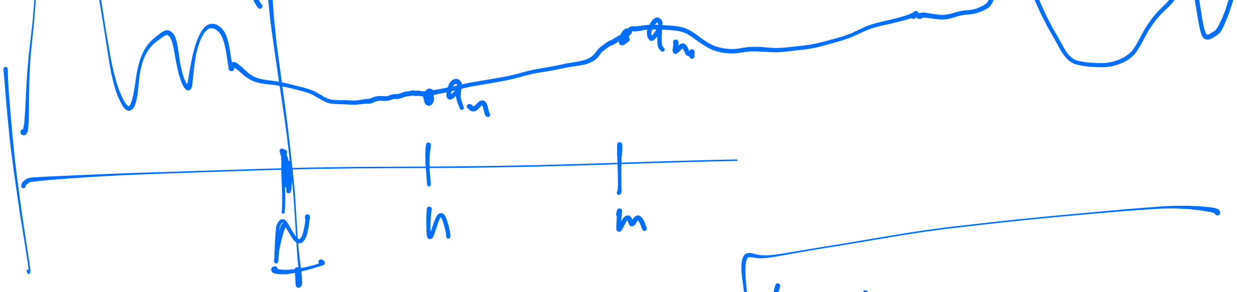






¬UnBddPeaks := $\forall N$, ¬ IsAPeak a N.

push_neg at ha | ha : $\exists N, \forall n > N, \exists m > n, a_n < a_m$



Choose N hN using ha.

$hN : \forall n > N, \exists m > n, a_n < a_m$.

$hN : \forall n, n > N \to \exists m > n, a_n < a_m$.   ($m = \tau(n)$)

Idea: we want to apply

$\boxed{\text{Subseq\_of\_Iterate } (\tau)\ (h\tau : \tau(n) > n)\ (n_0) \Rightarrow \text{Subseq } \tau^{[n]}\ (n_0).}$

Choose τ tbnd hat using hN.

$P \wedge Q \to R$
$\langle n, n > N \rangle$
$P \to (Q \to R)$

$\tau : \mathbb{N} \to (\_ : n > N) \to \mathbb{N}$
$\text{tbnd} : \forall n, (\_ : n > N) \to \tau\ n\ \_ > n$
$\text{hat} : \forall n, (\_ : n > N) \to a_n < a_{\tau\ n\ \_}$

Idea: let $\tau' : \mathbb{N} \to \mathbb{N} :=$ fun $n \mapsto$
if $h : n > N$ then $\tau\ n\ h$ else $N+1$

Alt: let $\tau' :=$ fun $n \mapsto \tau\ (n+N+1)$ (by bound)

have τ'bnd : $\forall n, \tau'\ n > n$:= by
intro n; rewrite [τ'];

specialize τbnd (n+N+1) (by omega);

τbnd   let σ : ℕ → ℕ := fun n ↦ τ'^[n] (N+1)

Subseq_of_Iterate τ'

τ'bnd (N+1)