An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 23: Uniformity II: Continuity

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SOCRATES: Last time, we saw that limits of continuous functions are continuous, but only if the convergence is *uniform.* And we saw how Riemann sums, if they converge to a limit, define the integral of a function on an interval. Let's talk about what conditions might ensure that Riemann sums converge.

SIMPLICIO: Hmm... Is it enough for the function to be continuous?

SOCRATES: Well, that's a good guess. The answer is *no...* and *yes.* Let's think about how we could prove that the sequence of Riemann sums converges. First of all, do we know what it should converge *to*?

SIMPLICIO: Umm... I guess we don't know yet – that's the mystery of the integral! We just want to show that it converges to *something.* Oh! I know what to do in these cases: we need to show that the sequence of Riemann sums is Cauchy.

SOCRATES: Exactly! So let's think about how the difference |`RiemannSum f a b m - RiemannSum f a b n`| can be made small.

SIMPLICIO: Well, if m and n are very large, then the partitions are very fine, so the Riemann sums should be close to each other. Right? I guess if you make the partitions finer, after a certain point the Riemann sums shouldn't change much.

SOCRATES: Yes, that's the right intuition. But we need to be a bit more precise. Is there something we could do to compare Riemann sums with different numbers of subintervals? What if one of the numbers is a multiple of the other?

SIMPLICIO: Oh! If m is a multiple of n, say $m = n \cdot k$, then we can compare `RiemannSum f a b (n * k)` to `RiemannSum f a b n`. Maybe we can show that if the partition with n subintervals is fine enough, then the Riemann sum with $n \cdot k$ subintervals is close to that with n subintervals. I think I see how to do it. But! We'll need a single δ that works for *all* points in $[a, b]$, not a different δ for each point.

SOCRATES: Bingo! You just invented the notion of *uniform continuity*! Tell me what it should mean.

SIMPLICIO: Wait, what's uniform continuity? Does that have anything to do with uniform convergence that we just discussed?

SOCRATES: Oh, sorry! You're right, that can be confusing. We need to make sure to distinguish these two somewhat similar-sounding but actually rather different concepts. I think of uniform convergence as a "vertical" notion: you have a family of different functions f_n converging to F, and the convergence happens uniformly over the entire domain. Uniform continuity, on the other hand, is more of a "horizontal" notion: it applies to a single function f. What should it mean, given our discussion about Riemann sums?

SIMPLICIO: Hmm... So ordinary continuity of a function f at a point x means the following: Given an $\varepsilon > 0$ and a point x, we can find a $\delta > 0$ (which may depend on x!!) such that for all y with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. And so the "uniformity" should say that the $\delta > 0$ can be chosen regardless of what x is. Ah! Something like this? Given an $\varepsilon > 0$, we can find a $\delta > 0$ such that for *all* x and y in the domain, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Again, it's just about reordering the quantifiers?

SOCRATES: Yes, that's exactly right! You've just defined *uniform continuity*. So just like in pointwise convergence where N can depend on the point x but uniform convergence requires a single N that works for all x, here ordinary continuity allows δ to depend on x but uniform continuity requires a single δ that works for all x and y. So that's how they're similar but different!

SIMPLICIO: Ok, I think I get it.

SOCRATES: What we'll prove today is that if a function is uniformly continuous on $[a, b]$, then its Riemann sums converge to a limit, and this limit is what we call the integral of the function on $[a, b]$. But this raises another fundamental question: When is a continuous function *uniformly* continuous? Remember how we proved that $f(x) = x^2 - 1$ is continuous everywhere? Remember what value of δ we chose once ε was given?

SIMPLICIO: For the third time, we chose $\delta = \varepsilon/(2|x|+1)$. Oh, I see! So δ depended on x, which means that $f(x) = x^2 - 1$ is not uniformly continuous on all of $\mathbb{R}$.

SOCRATES: Right. But! When we integrate a function on a finite interval like $[a, b]$, we only care about its behavior on that interval. Let's say we wanted to integrate $f(x) = x^2 - 1$ on the interval $[-10, 10]$. Can we find a δ that works for *all* x, just in that range?

SIMPLICIO: Hmm... Well, on $[-10, 10]$, the maximum value of $|x|$ is 10. Instead of a different value of $\delta = \varepsilon/(2|x|+1)$ for each x, we can just use the largest possible denominator, which is when $|x| = 10$. That is, we can use $\delta = \varepsilon/(2 \cdot 10 + 1) = \varepsilon/21$ for *all* x in $[-10, 10]$.

SOCRATES: Exactly! So on the interval $[-10, 10]$, the function $f(x) = x^2 - 1$ *is* uniformly continuous. In fact, there's a general theorem here: It turns out that any function that is continuous on a "*compact*" interval like $[-10, 10]$ is automatically uniformly continuous on that interval. But this is not a theorem in calculus; it's a result in topology! So to understand some of the next steps of calculus, we need to learn a bit of topology first.

SIMPLICIO: Ok, what's topology?

SOCRATES: Well, of all the subjects in mathematics, topology is among the youngest. Geometry and algebra have been studied for thousands of years, and calculus for a few hundred years. But topology only really started to be developed "officially" in the 19th and early 20th centuries. In fact, once you see just how important topology is for understanding calculus, you'll feel embarrassed for all of the great mathematicians of the past who did calculus without it! The proper notion of compactness was not fully understood until the late 19th century, well after Riemann's fundamental work on integration.

SIMPLICIO: Wow, that's surprising. Ok, so what is compactness?

SOCRATES: Like so many other concepts in this course, I hope you'll come to appreciate how complicated and subtle the idea is, despite the fact that we can now make the definition quite intuitively clear. It took people decades to figure out *exactly* what compactness should mean! Let's go back to uniform continuity. You have continuity at every single point x in your set S. So for every x, there's a little ball $(x - \delta_x, x + \delta_x)$ where continuity holds with δ_x depending on x. And of course all these balls cover your set $S \subseteq \bigcup_x (x - \delta_x, x + \delta_x)$. Here's the key idea for compactness. What if: whenever you can cover S by any number of balls (possibly infinitely many, even uncountably many), you can *always* find a subcover that uses only *finitely many* of those balls? **That** is the key idea and definition of compactness!

SIMPLICIO: Ah! And if I have only finitely many balls covering S, then I can just take the minimum of all the δs for those balls, and that minimum δ will work for the entire set S! I see now how compactness leads to uniform continuity. At least I think I do.

SOCRATES: Good; let's find out for *sure*. :)

Level 1: Riemann Sum Refinement

When working with Riemann sums, a key question is how the accuracy changes when we refine our partition. If a function satisfies uniform continuity on an interval, we can control how much the Riemann sum changes when we increase the number of subintervals.

The Setup

Suppose we have:

- $\varepsilon > 0$ and $\delta > 0$
- A function $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in [a, b]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$
- Natural numbers $n, k \neq 0$
- The partition of $[a, b]$ into n subintervals is fine enough: $(b - a)/n < \delta$

The Result

Theorem (`RiemannSumRefinement`): Under the above conditions, the Riemann sum of f on $[a, b]$ with $n \cdot k$ subintervals differs from that with n subintervals by at most $(b - a) \cdot \varepsilon$.

The Intuition

When we refine a partition from n to $n \cdot k$ subintervals, we're making the intervals smaller. If f is uniformly continuous with tolerance ε and threshold δ, and our partition is already fine enough ($(b-a)/n < \delta$), then the function values at nearby points can't differ by more than ε. Since we're integrating over an interval of length $(b - a)$, the total accumulated error is bounded by $(b - a) \cdot \varepsilon$.

Key Lemma

The proof uses the **sum over product** lemma:

$$\sum_{i=0}^{m \cdot n - 1} f(i) = \sum_{j=0}^{m-1} \sum_{\ell=0}^{n-1} f(j + \ell \cdot m)$$

This allows us to reorganize the refined Riemann sum in a way that makes comparison with the coarser sum possible.

Your Challenge

Prove that if f is uniformly continuous on $[a, b]$ with the given parameters, then:

$$|\mathrm{RiemannSum}(f, a, b, n \cdot k) - \mathrm{RiemannSum}(f, a, b, n)| < (b - a) \cdot \varepsilon$$

The Formal Proof

```
Statement RiemannSumRefinement (f : ℝ → ℝ) {a b : ℝ} (
    hab : a < b) {n k : ℕ}
      (hn : n ≠ 0) (hk : k ≠ 0)
      {ε δ : ℝ} (hε : ε > 0) (hδ : δ > 0)
      (hunif : ∀ x ∈ Icc a b, ∀ y ∈ Icc a b, |y - x| < δ →
          |f y - f x| < ε)
      (hfine : 2 * (b - a) / n < δ) :
   |RiemannSum f a b (n * k) - RiemannSum f a b n| < (b -
        a) * ε   := by
change |(b - a) / (n * k) * ∑ i ∈ range (n * k), f (a +
    (i + 1) * (b - a) / (n * k)) -
        (b - a) / n * ∑ i ∈ range n, f (a + (i + 1) * (b -
            a) / n)| <
   (b - a) * ε
rewrite [sum_of_prod]
rewrite [show (b - a) / (n * k) * ∑ j ∈ Finset.range n,
   ∑ k_1 ∈ Finset.range k, f (a + ((j + k_1 * n) + 1) *
    (b - a) / (n * k)) -
        (b - a) / n * ∑ i ∈ Finset.range n, f (a + (i + 1)
            * (b - a) / n) =
        (b - a) / n * ∑ j ∈ Finset.range n,  ((1 / k) * ∑
           ℓ ∈ Finset.range k, f (a + (ℓ + j * k + 1) * (b
            - a) / (n * k)) -
        f (a + (j + 1) * (b - a) / n)) by  sorry]
have dx : ∀ j ∈ range n, ∀ ℓ ∈ range k,
        |a + (ℓ + j * k + 1) * (b - a) / (n * k) -  (a + (
           j + 1) * (b - a) / n)| < δ := by
```

```
  intro j hj ℓ hℓ
  rewrite [show
    |a + (ℓ + j * k + 1) * (b - a) / (n * k) -  (a + (j
       + 1) * (b - a) / n)|
    = |(ℓ + 1) * (b - a) / (n * k) -  (b - a) / n| by
    field_simp
    ring_nf]
  have h1 : |(ℓ + 1) * (b - a) / (n * k)  -  (b - a) / n
     | ≤ 2 * (b - a) / n := by sorry
  linarith [hfine, hδ, h1]
have dy : ∀ j ∈ range n, ∀ ℓ ∈ range k,
  |f (a + (ℓ + j * k + 1) * (b - a) / (n * k)) -  f (a +
      (j + 1) * (b - a) / n)| < ε := by
  intro j hj ℓ hℓ
  specialize hunif (a + (j + 1) * (b - a) / n) (by sorry
    )
    (a + (ℓ + j * k + 1) * (b - a) / (n * k)) (by sorry)
    (dx j hj ℓ hℓ)
  apply hunif
sorry
```

Understanding the Proof

The proof works by:

Step 1: Using the sum reorganization lemma to express the refined sum in terms of the coarse sum.

Step 2: Showing that corresponding sample points in the two sums are within distance δ of each other.

Step 3: Applying uniform continuity to bound the function value differences by ε.

Step 4: Integrating this pointwise bound over the interval length $(b - a)$ to get the final estimate.

This result is crucial for proving that uniformly continuous functions are integrable, as it shows that Riemann sum sequences become Cauchy when the partitions are fine enough.

Level 2: Integration Converges!

Now we can prove the fundamental result that uniform continuity guarantees integrability. This bridges the gap between the analytical properties of functions and their integrability.

New Definition: Uniform Continuity

Definition (`UnifContOn`): A function $f : \mathbb{R} \to \mathbb{R}$ is *uniformly continuous* on a set S if:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in S, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

```
def UnifContOn (f : ℝ → ℝ) (S : Set ℝ) : Prop :=
  ∀ ε > 0, ∃ δ > 0, ∀ x ∈ S, ∀ y ∈ S, |y - x| < δ → |f y
    - f x| < ε
```

The Key Insight

The crucial difference between ordinary continuity and uniform continuity is the *order of quantifiers*:

Continuity:

$$\forall x \in S, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in S, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Uniform Continuity:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in S, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

In ordinary continuity, δ can depend on both ε and the specific point x. In uniform continuity, δ depends only on ε and works for all points simultaneously.

The Result

Theorem (`HasIntegral_of_UnifContOn`): If f is uniformly continuous on $[a, b]$, then f is integrable on $[a, b]$.

More precisely, the sequence of Riemann sums $\{\text{RiemannSum}(f, a, b, n)\}_{n=1}^{\infty}$ converges to a limit.

The Proof Strategy

We prove integrability by showing that the Riemann sum sequence is Cauchy. For any $\varepsilon > 0$:

Step 1: Use uniform continuity of f with tolerance $\varepsilon/(2(b-a))$ to get $\delta > 0$.

Step 2: Choose N large enough so that for $n, m \geq N$, both partitions are fine enough: $(b-a)/n < \delta$ and $(b-a)/m < \delta$.

Step 3: For any $n, m \geq N$, estimate:

$$|\text{RiemannSum}(f, a, b, m) - \text{RiemannSum}(f, a, b, n)| \tag{1}$$
$$\leq |\text{RiemannSum}(f, a, b, m) - \text{RiemannSum}(f, a, b, mn)| \tag{2}$$
$$+ |\text{RiemannSum}(f, a, b, mn) - \text{RiemannSum}(f, a, b, n)| \tag{3}$$

Step 4: Apply the Riemann Sum Refinement theorem to bound each term by $\varepsilon/2$.

Your Challenge

Prove that uniform continuity implies integrability:

`UnifContOn f (Icc a b) → IntegrableOn f a b`

The Formal Proof

```
Statement HasIntegral_of_UnifContOn (f : ℝ → ℝ) (a b : ℝ
  ) (hab : a < b)
    (hf : UnifContOn f (Icc a b)) : IntegrableOn f a b
      := by
apply SeqConv_of_IsCauchy
intro ε hε
choose δ hδ₁ hδ₂ using hf (ε / (2 * (b - a))) (by bound)
choose N hN using ArchProp (show 0 < δ / (2 * (b - a))
  by bound)
use N
intro n hn m hnm
have pos1 : 0 < 1 / (δ / (2 * (b - a))) := by bound
have NposR : (0 : ℝ) < N := by linarith [hN, pos1]
```

```
have Npos : 0 < N := by exact_mod_cast NposR
have npos : 0 < n := by bound
have mpos : 0 < m := by bound
change |RiemannSum f a b m - RiemannSum f a b n| < ε
have hn_small : 2 * (b - a) / n < δ := by sorry
have hm_small : 2 * (b - a) / m < δ := by sorry
have f1 : |RiemannSum f a b m - RiemannSum f a b n| ≤
    |RiemannSum f a b m - RiemannSum f a b (m * n)| + |
      RiemannSum f a b (n * m) - RiemannSum f a b n| :=
       by
  rewrite [show RiemannSum f a b m - RiemannSum f a b n
     =
      (RiemannSum f a b m - RiemannSum f a b (m * n)) +
      (RiemannSum f a b (n * m) - RiemannSum f a b n) by
          ring_nf]
  apply abs_add
have hfn := RiemannSumRefinement f hab (show n ≠ 0 by
   bound) (show m ≠ 0 by bound) (show 0 < ε / (2 * (b -
   a)) by bound)
    (show 0 < δ by bound) hδ₂ hn_small
have hfm := RiemannSumRefinement f hab (show m ≠ 0 by
   bound) (show n ≠ 0 by bound) (show 0 < ε / (2 * (b -
   a)) by bound)
    (show 0 < δ by bound) hδ₂ hm_small
rewrite [show |RiemannSum f a b (m * n) - RiemannSum f a
    b m| =
    |RiemannSum f a b m - RiemannSum f a b (m * n)| by
      apply abs_sub_comm] at hfm
have bapos : 0 < b - a := by linarith [hab]
rewrite [show (b - a) * (ε / (2 * (b - a))) = ε / 2 by
   field_simp] at hfn hfm
linarith [f1, hfn, hfm]
```

Understanding the Proof

This proof establishes a fundamental connection between topological properties (uniform continuity) and analytical properties (integrability). The key insight is using the common multiple mn as an intermediate step, which allows us to apply the refinement theorem to both terms.

The factor of $1/(2(b-a))$ in the uniform continuity application ensures that after multiplying by the interval length $(b-a)$, each term contributes at most $\varepsilon/2$ to the total error.

Level 3: Compactness

We now address the fundamental question: When is a continuous function uniformly continuous? The answer involves one of the most important concepts in mathematics: compactness.

New Definitions

Ball: An open ball of radius r centered at x is:

$$\mathrm{Ball}(x, r) = (x - r, x + r) = \{y \in \mathbb{R} : |y - x| < r\}$$

```
def Ball (x : ℝ) (r : ℝ) : Set ℝ := Ioo (x - r) (x + r)
```

Compactness: A set $S \subseteq \mathbb{R}$ is *compact* if every open cover has a finite subcover. Formally:

```
def IsCompact (S : Set ℝ) : Prop :=
  ∀ (ι : Type) (xs : ι → ℝ) (δs : ι → ℝ), (∀ i, 0 < δs i
    ) →
    (S ⊆ ⋃ i, Ball (xs i) (δs i)) →
    ∃ (V : Finset ι), S ⊆ ⋃ i ∈ V, Ball (xs i) (δs i)
```

This says: if you can cover S with balls (indexed by any type ι), then you can always find a finite subcollection that still covers S.

The Intuitive Picture

Imagine trying to cover a set S with open balls. Compactness says that no matter how you do this covering (even with uncountably many balls), you can always throw away all but finitely many balls and still cover S.

For intervals, the compact sets are precisely the *closed and bounded* intervals $[a, b]$. Open intervals like (a, b) are not compact, nor are unbounded intervals like $[a, \infty)$.

The Connection to Uniform Continuity

Here's why compactness implies uniform continuity for continuous functions:

Step 1: If f is continuous on S, then for each point $x \in S$, we can find a ball around x where f doesn't vary by more than $\varepsilon/2$.

Step 2: These balls cover S, so by compactness, finitely many of them still cover S.

Step 3: Take the minimum radius of these finitely many balls - this gives us a uniform δ that works for the entire set S.

Key Helpful Lemmas

Union Membership: $(x \in \bigcup_i s_i) \leftrightarrow (\exists i, x \in s_i)$

Finite Minimum: If V is a finite set and $\delta_s : V \to \mathbb{R}^+$, then $\exists \delta > 0, \forall i \in V, \delta \leq \delta_s(i)$

The Result

Theorem (`UnifContOn_of_Compact`): A continuous function on a compact set is uniformly continuous.

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and S is compact, then f is uniformly continuous on S.

Your Challenge

Prove that continuity plus compactness implies uniform continuity:

`FunCont f → IsCompact S → UnifContOn f S`

The Formal Proof

```
Statement UnifContOn_of_Compact (f : ℝ → ℝ) (hf :
   FunCont f) (S : Set ℝ)
    (hS : IsCompact S) : UnifContOn f S := by
intro ε hε
have h1 : ∀ x ∈ S, ∃ δ > 0, ∀ y ∈ S, |y - x| < δ → |f y
   - f x| < ε / 2 := by
  intro x hx
  choose δ δpos hδ using hf x (ε / 2) (by bound)
  use δ, δpos
  intro y hy hxy
  apply hδ y hxy
choose δs δspos hδs using h1
let ι : Type := S
```

```
let xs : ι → ℝ := fun i ↦ i
let δ's : ι → ℝ := fun i ↦ (δs (xs i) i.2 / 2)
have δ'spos : ∀ i, 0 < δ's i := by
  intro i
  change 0 < δs (xs i) i.2 / 2
  linarith [δspos i.1 i.2]
have hScover : S ⊆ ⋃ i, Ball (xs i) (δ's i) := by
  intro x hx
  rewrite [mem_Union]
  use ⟨x, hx⟩
  change x ∈ Ioo (x - (δs x hx) / 2) (x + (δs x hx) / 2)
  rewrite [Set.mem_Ioo]
  split_ands
  linarith [(δspos x hx)]
  linarith [(δspos x hx)]
choose V hV using hS ι xs δ's δ'spos hScover
choose δ δpos hδ using FinMinPos ι V δ's δ'spos
use δ, δpos
intro x hx y hy hxy
have hx1 := hV hx
have hx1' : ∃ i ∈ V, x ∈ Ball (xs i) (δ's i) := by
   sorry
choose i i_in_V x_in_Ball using hx1'
have hxxi : |x - xs i| < (δs (xs i) i.2) / 2 := by
   sorry
have hxxi' : |x - xs i| < (δs (xs i) i.2) := by
   sorry
have hxy' : |y - x| < (δs i i.2) / 2 := by
   sorry
have hyxi : |y - xs i| < (δs i i.2) := by
  rewrite [show y - xs i = (y - x) + (x - xs i) by
     ring_nf]
  have h1 : |(y - x) + (x - xs i)| ≤ |y - x| + |x - xs i
     | := by apply abs_add
  have h2 : |x - xs i| = |xs i - x| := by apply
     abs_sub_comm
  linarith [hxy', hxxi, h1, h2]
have fyfxi := hδs (xs i) i.2 y hy hyxi
have fxix := hδs (xs i) i.2 x hx hxxi'
```

```
rewrite [show f y - f x = (f y - f (xs i)) + (f (xs i) -
    f x) by ring_nf]
have h1 : |(f y - f (xs i)) + (f (xs i) - f x)| ≤ |f y -
    f (xs i)| + |f (xs i) - f x| := by apply abs_add
rewrite [show |f (xs i) - f x| = |f x - f (xs i)| by
   apply abs_sub_comm] at h1
linarith [fyfxi, fxix, h1]
```

Understanding the Proof

This proof is a beautiful example of how abstract topology (compactness) solves concrete analytical problems (uniform continuity).

The local-to-global principle: We start with local information (continuity at each point) and use compactness to extract global information (uniform continuity on the whole set).

The finite extraction: Compactness allows us to replace an infinite covering problem with a finite one, where we can take actual minima.

Applications: Since closed bounded intervals $[a, b]$ are compact, every continuous function on $[a, b]$ is uniformly continuous, and therefore integrable. This explains why elementary calculus can get away with assuming all continuous functions are integrable!

Last time:

uniformly converging

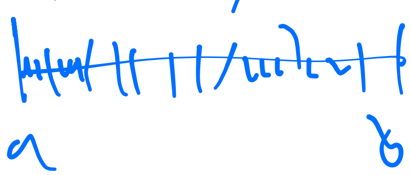
cont.

$$f_n \text{ cont } \& \ f_n(x) \to F(x)$$

Riemann Sum of f a b $: \mathbb{N} \to \mathbb{R} :=$

$$n \mapsto \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a+(i+1)\frac{b-a}{n}\right).$$

(general: any partition, mesh → 0),

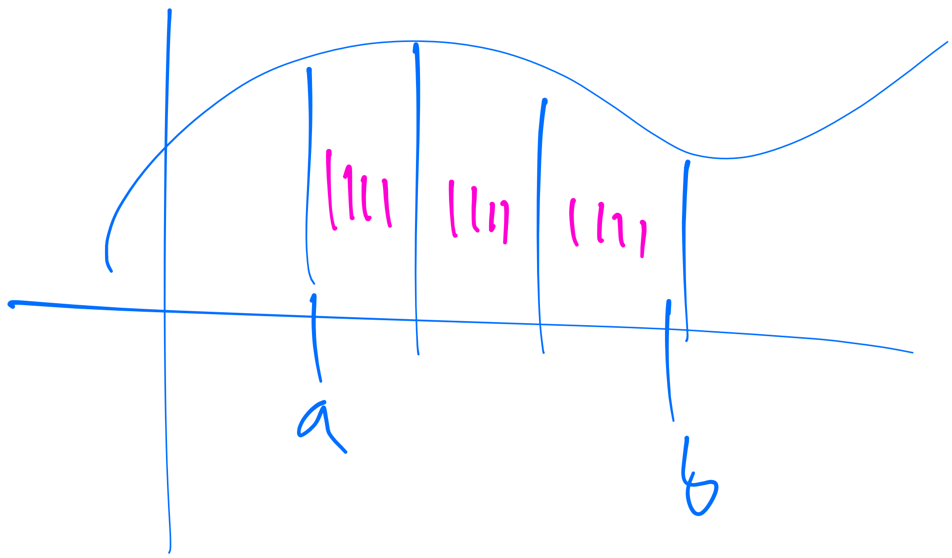


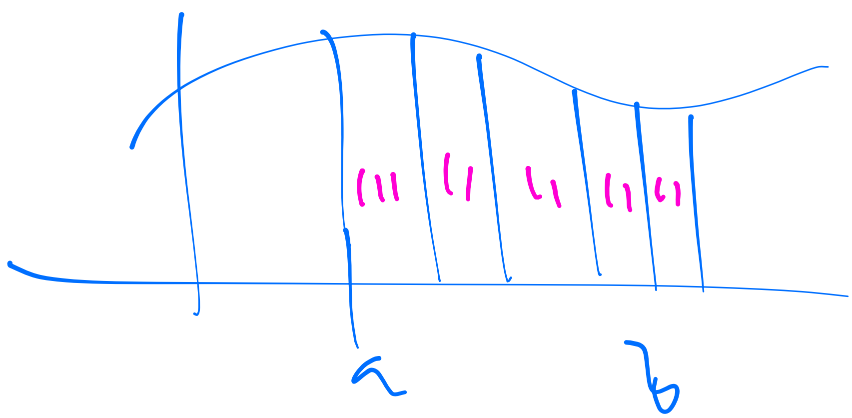
If seq conv, we call that the $\int_a^b f(x)\,dx$.

When does a Riemann Sum converge?

Want to show: {Riemann Sum n} is Cauchy.

$$|RS_n - RS_m| < \text{small}.$$





Is $|RS_n - RS_{n\cdot k}| < \text{small}$?

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f\left(a+(i+1)\frac{b-a}{n}\right).$$

$$\hookrightarrow \frac{b-a}{n\cdot K}\sum_{j=0}^{n\cdot K-1} f\left(a+(j+1)\frac{b-a}{n\cdot K}\right).$$

$j = i + lK$

$$= \frac{b-a}{n\cdot K}\cdot\sum_{l=0}^{K-1}\sum_{i=0}^{n-1} f\left(a+(l+i\cdot K+1)\left(\frac{b-a}{n\cdot K}\right)\right)$$

$$|RS_n - RS_{n\cdot K}| = \left|\frac{b-a}{n}\sum_{i=0}^{n-1}\left(f\left(a+(i+1)\frac{b-a}{n}\right) - \frac{1}{K}\sum_{l=0}^{K-1} \right)\right|$$

$$\le \left|\frac{(b-a)}{n}\sum_{i=0}^{n-1}\left[\frac{1}{K}\sum_{l=0}^{K-1}\left|f\left(a+(l+iK+1)\frac{(b-a)}{n\cdot K}\right) - f\left(a+(i+1)\frac{b-a}{n}\right)\right|\right]\right|.$$

$$||$$

$$\Delta x = \left|\left(\cancel{a}+(l+i\cancel{K}+1)\frac{(b-a)}{nK}\right) - \left(\cancel{a}+(i+1)\frac{b-a}{n}\right)\right|$$

$$= \left| \frac{(\ell+1)\cdot(b-a)}{n\cdot K} \cdot - \frac{b-a}{n} \right| \leq \frac{b-a}{n} < \delta.$$

Continuity: Δf is locally small.

Problem: need uniform control across whole interval!

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f_n $\forall \varepsilon > 0 \; \exists N$,

$\downarrow$ uniform convergence

F

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vs   Uniform Continuity:

Def: f is Unif Cont On S: $S \subseteq \mathbb{R}$

iff: $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x, y \in S$,

$|y-x|<\delta \to |f(y)-f(x)|<\varepsilon,$

Compare: f FunCont On S if,

$\forall \varepsilon>0$ $\forall x \in S$, $\exists \delta_x > 0$ s.t.

$\forall y \in S$, $|y-x|<\delta \to |f(y)-f(x)|<\varepsilon.$

Thm: If f is UnifCont On $[a,b]$ then

RS_n Converges.

Pf: Claim: $\{RS_n\}$ is Cauchy.

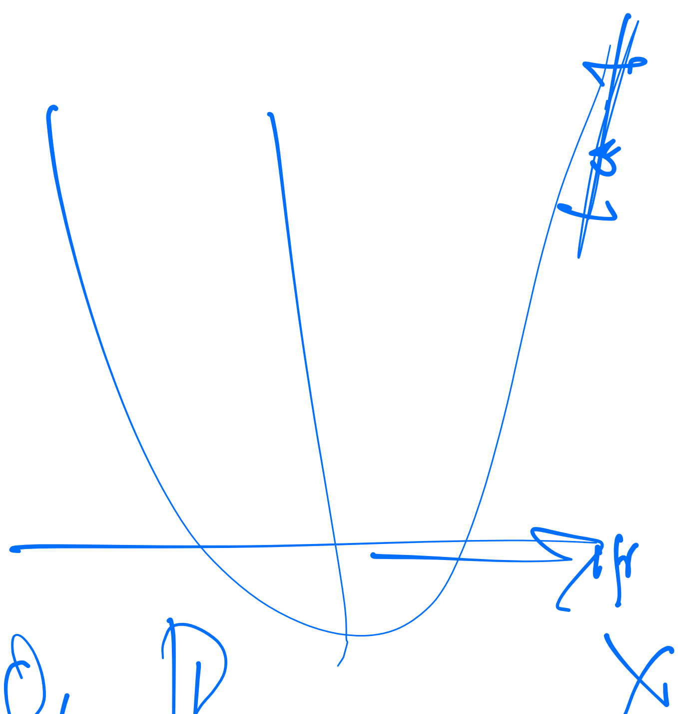
$|RS_n - RS_m| \leq |RS_n - RS_{n\cdot m}| + |RS_{n\cdot m} - RS_{n\cdot m}|$

Thm 1: f unif cont....

$$|RS_n - RS_{n.k}| < \varepsilon \cdot (b-a),$$

Recall, for $f(x) = x^2$,

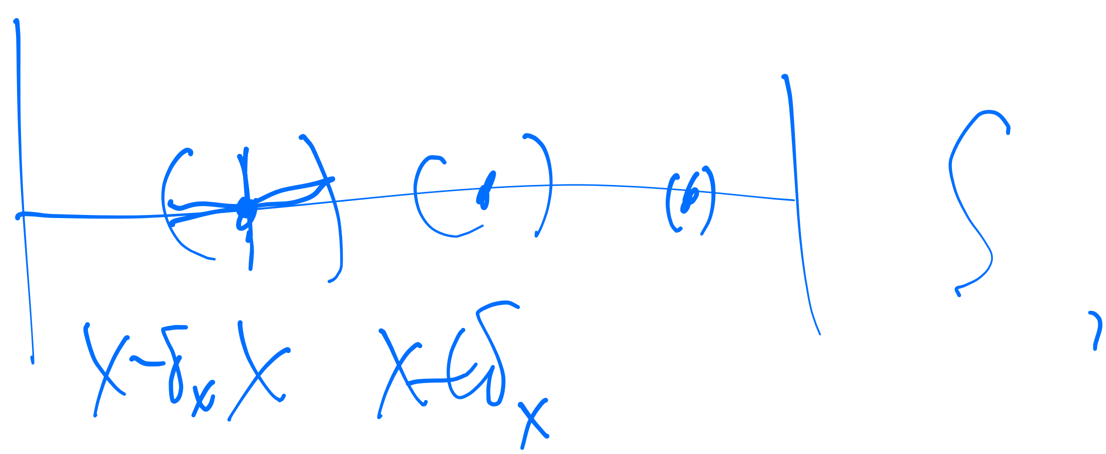
$$\delta = \frac{\varepsilon}{2|x|+1}$$



Not unif cont on $\mathbb{R}$,

But $\int_{-10}^{10}$, take $\delta = \frac{\varepsilon}{21}$.

Thm 3: If f is <u>cont</u> on set S: set $\mathbb{R}$ & S <u>is compact</u> then f is Unif Cont on S.



like to say: take minimal δ that works for all.
But why can't $\delta_x \to 0$?

Key idea: If S is covered by balls, then it is finitely covered by balls.

Def: IsCompact $(S : \text{Set } \mathbb{R})$:

Prop := $\forall (I : \text{Type}) (xs : I \to \mathbb{R}) (\delta s : I \to \mathbb{R}) (\forall i, \delta s\, i > 0) \to$

$$S \subseteq \bigcup_{i \in I} \text{Ball } x_i\ \delta_i$$

$\to \exists V : \text{Finset } I, \ S \subseteq \bigcup_{i \in V}, \text{Ball } x_i\ \delta s_i$

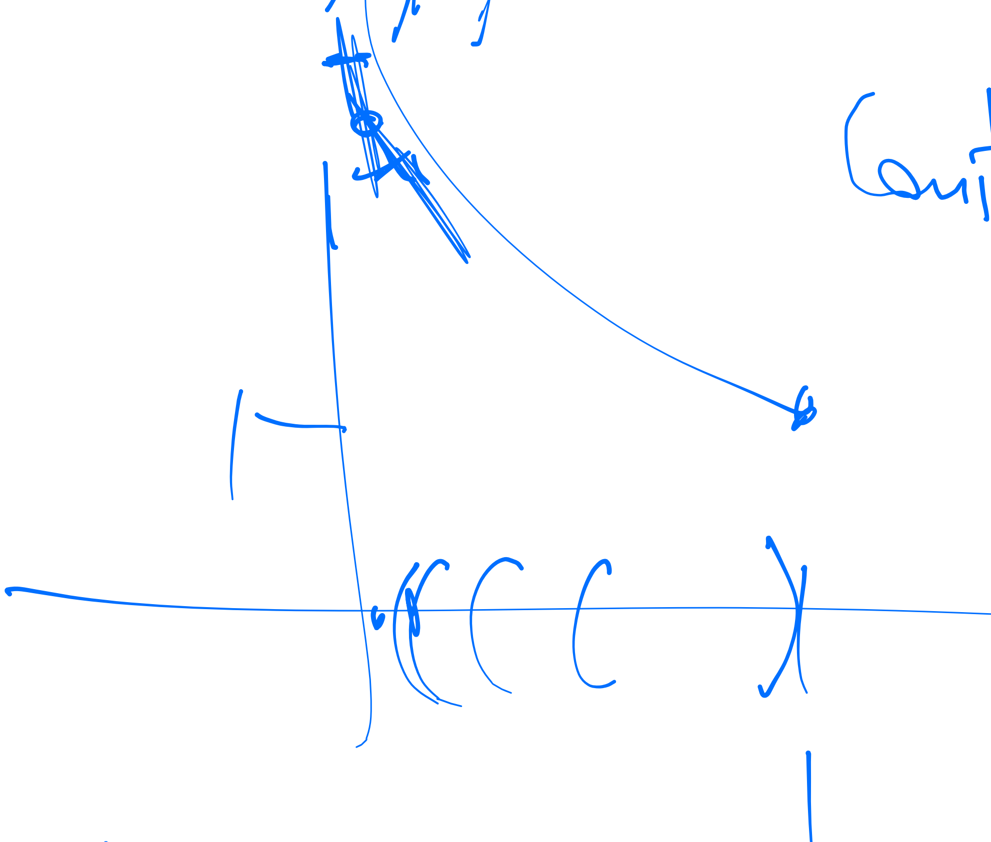
Def: $\text{Ball } x\ r := \text{Ioo } (x-r)(x+r)$

"Compact $\iff$ any cover by balls has finite sub cover"

Is Bounded enough?

$S = (0,1)$ $f(x) = 1/x$.

Cont? V.



Is $1/x$ unif cont on $(0,1)$?

$\mathrm{Obs}: (0,1] \subseteq \bigcup_{n:\mathbb{N}} \left(\frac{1}{n+1}, 1\right)$

But has no finite subcover!

$(0,1)$ is not compact!.

Obs: $[0,1] \subseteq \mathrm{Ball}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$

& has a finite subcover ✓.

(Riemann Sum $\longrightarrow$ Conv $\int$

if f is unif cont.

When is f unif cont?

f is unif cont if S is compact. [-1, 1].

Thm If f is cont on [a, b], then $\int f$ exists.

Thm: $\epsilon > 0$, $\delta > 0$, f

a, b, hab: $a < b$, $n \neq 0$, $k \neq 0$.

Know: $\forall x, y \in [a,b]$, $|y-x| < \delta \Rightarrow$

$|fy - fx| < \varepsilon$.

Know: $\frac{b-a}{n} < \delta$, $\Leftrightarrow \frac{b-a}{\delta} < n$,

Goal: $|RS_{nk} - RS_n| \le (b-a) \cdot \varepsilon$

Thm 2: $a < b$, UnifCont On f $[a,b]$.

Goal: Definable On f a b

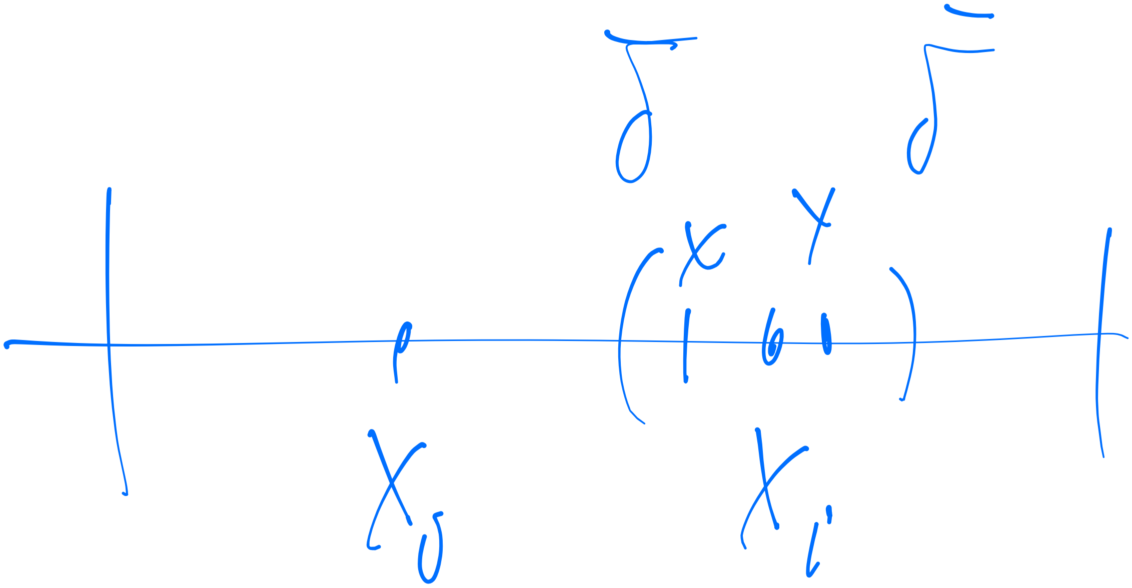
Thm 3: $f : \mathbb{R} \to \mathbb{R}$, $S \subseteq \mathbb{R}$

Hyp 1: FnCont f.

Hyp 2: IsCompact S.

Goal: UnifContOn f S.

Intro $\varepsilon > 0$.



$|x - x_1| < \bar\delta \to |f(x) - f(x_1)| < \varepsilon$,

Find $\delta = \bar\delta/2$.

$$|x - x_i| < \delta/2, \quad |x - y| < \delta/2,$$

$$\Rightarrow |x_i - y| < \delta_i$$

$$|f(x) - f(y)| \leq \underbrace{|f(x) - f(x_i)|}_{< \varepsilon/2} + \underbrace{|f(x_i) - f(y)|}_{< \varepsilon/2}$$