An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 18: Rearrangements

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SIMPLICIO: Does rearranging the terms of a series change its sum?

SOCRATES: Great question! Here's the way I like to illustrate this idea (that I learned from Walter Rudin's books). Imagine a massive matrix, M, infinite in both directions, to the right, and down. It is upper triangular, has -1s on the diagonal, and at position (i, j) for $i < j$ its entry is $1/2^{(j-i+1)}$. So M looks like this:

$$\begin{pmatrix} -1 & 1/2 & 1/4 & 1/8 & 1/16 & \cdots \\ 0 & -1 & 1/2 & 1/4 & 1/8 & \cdots \\ 0 & 0 & -1 & 1/2 & 1/4 & \cdots \\ 0 & 0 & 0 & -1 & 1/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

What is the sum of the elements in M?

SIMPLICIO: Ok let's see if we can work this out. Hmm this isn't so hard, I'll sum the rows, and then add those up.

The first row sums to $-1 + 1/2 + 1/4 + 1/8 + \cdots = 0$.

The second row sum is $0 + -1 + 1/2 + 1/4 + \cdots = 0$ as well.

In fact, every row obviously sums to 0. So the sum of all the elements in M is 0.

SOCRATES: Very good. What will my next question be?

SIMPLICIO: I bet you're going to ask me to sum the columns instead of the rows...?

SOCRATES: You bet! :)

SIMPLICIO: Alright, let's do that. The first column sums to $-1+0+0+0+\cdots=-1$.

The second column sums to $1/2+-1+0+0+\cdots=-1/2$.

The third column sums to $1/4+1/2+-1+0+\cdots=-1/4$.

Ok, I see the pattern. The nth column sums to $-1/2^{n-1}$. So the sum of all the elements in M is $-1-1/2-1/4-1/8-\cdots=-2$.

Uh oh. So what *is* the sum?

SOCRATES: Well that's just it! There is *no* 'the sum', because it depends on in what order you add the terms up!

Infinite summation is not commutative!

A *lot* of analysis is devoted to the study of this problem. A sequence is called 'conditionally convergent' if it converges, but not when you add absolute values. That is, the matrix $|M|$ would look like this:

$$\begin{pmatrix} 1 & 1/2 & 1/4 & 1/8 & 1/16 & \cdots \\ 0 & 1 & 1/2 & 1/4 & 1/8 & \cdots \\ 0 & 0 & 1 & 1/2 & 1/4 & \cdots \\ 0 & 0 & 0 & 1 & 1/2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and hopefully it's clear (just from the diagonal!) that the sum of all its elements diverges (to infinity).

People like Euler and Ramanujan were absolute wizards with divergent series, getting out of them all kinds of bizarre results, like $1+2+3+4+\cdots$ '=' $-1/12$. As the great Oliver Heaviside supposedly once said, "This series is divergent, therefore we may be able to do something with it!"

SIMPLICIO: I like it! Let's go.

Level 1: Absolute Convergence Implies Convergence

In this level, we introduce the important concept of absolute convergence and prove that it implies regular convergence.

New definition: `AbsSeriesConv`

A series `Series a` is said to converge *absolutely* (`AbsSeriesConv`) if the series of absolute values `Series (fun n ↦ |a n|)` converges.

```
def AbsSeriesConv (a : ℕ → ℝ) : Prop :=
  SeriesConv (fun n ↦ |a n|)
```

The Theorem

Theorem (`Conv_of_AbsSeriesConv`): If `Series (fun n ↦ |a n|)` converges, then `Series a` converges.

In other words: *absolute convergence implies convergence.*

Proof Strategy

The key insight is that if the series of absolute values converges, then it is Cauchy, which means that the tails of the series get arbitrarily small. By the triangle inequality, if $\sum |a_k|$ is small, then $|\sum a_k|$ is also small.

To make this rigorous, we use two helper lemmas (to be proved as homework):

- `DiffOfSeries`: For $n \leq m$, the difference of partial sums equals the sum over the interval:

$$\text{Series}(a, m) - \text{Series}(a, n) = \sum_{k \in [n,m)} a(k)$$

- `Series_abs_add`: For $n \leq m$, we have the triangle inequality:

$$\left| \sum_{k \in [n,m)} a(k) \right| \leq \sum_{k \in [n,m)} |a(k)|$$

The Formal Proof

Proof: We need to show that `Series a` converges. By `SeqConv_of_IsCauchy`, it suffices to prove that `Series a` is Cauchy.

Let $\varepsilon > 0$ be given. Since `Series (fun n ↦ |a n|)` converges, it is Cauchy by `IsCauchyOfSeqConv`.

Using `choose`, we extract N such that for all $n \geq N$ and $m \geq n$:

$$|\text{Series}(|a|, m) - \text{Series}(|a|, n)| < \varepsilon$$

We use the same N for the series of a. Consider $n \geq N$ and $m \geq n$.

By `DiffOfSeries`, we rewrite:

$$\text{Series}(a, m) - \text{Series}(a, n) = \sum_{k \in [n,m)} a(k)$$

and similarly for the absolute value series:

$$\text{Series}(|a|, m) - \text{Series}(|a|, n) = \sum_{k \in [n,m)} |a(k)|$$

Now we chain the inequalities. By `Series_abs_add`:

$$\left| \sum_{k \in [n,m)} a(k) \right| \leq \sum_{k \in [n,m)} |a(k)|$$

By the `bound` tactic (since any real number is bounded by its absolute value):

$$\sum_{k \in [n,m)} |a(k)| \leq \left| \sum_{k \in [n,m)} |a(k)| \right|$$

Combining these with our hypothesis on the convergence of the absolute value series:

$$\left| \sum_{k \in [n,m)} a(k) \right| \leq \sum_{k \in [n,m)} |a(k)| \leq \left| \sum_{k \in [n,m)} |a(k)| \right| < \varepsilon$$

Therefore $|\text{Series}(a, m) - \text{Series}(a, n)| < \varepsilon$, completing the proof by `linarith`. □

The Lean Proof

```
Statement Conv_of_AbsSeriesConv {a : ℕ → ℝ}
    (ha : AbsSeriesConv a) : SeriesConv a := by
  apply SeqConv_of_IsCauchy
  intro ε hε
  apply IsCauchyOfSeqConv at ha
  choose N hN using ha ε hε
  use N
  intro n hn m hnm
  rewrite [DiffOfSeries _ hnm]
  specialize hN n hn m hnm
  rewrite [DiffOfSeries _ hnm] at hN
  have f1 : |∑ k ∈ Ico n m, a k| ≤ ∑ k ∈ Ico n m, |a k|
     := by
    apply Series_abs_add _ hnm
  have f2 : ∑ k ∈ Ico n m, |a k| ≤ |∑ k ∈ Ico n m, (|a k
     |)| := by
    bound
  linarith [f1, f2, hN]
```

Understanding the Theorem

This theorem tells us that absolute convergence is a *stronger* condition than regular convergence. Some important examples:

- The geometric series $\sum_{k=0}^{\infty} r^k$ converges absolutely when $|r| < 1$
- The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges absolutely
- The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges but *not* absolutely (it is **conditionally convergent**)

The distinction between absolute and conditional convergence becomes crucial when studying rearrangements of series, as we will see!

Level 2: Alternating Series Test

In this level, we prove one of the most elegant results in the theory of infinite series: the Alternating Series Test, also known as the Leibniz Test.

The Theorem

Theorem (`AlternatingSeriesTest`): Suppose $a : \mathbb{N} \to \mathbb{R}$ is an `Antitone` sequence that converges to 0. Then the alternating series

$$\sum_{n=0}^{\infty} (-1)^n \cdot a_n$$

converges.

In other words: *if the terms of a series decrease to zero, then the alternating series converges.*

Key Lemmas (to be proved in homework)

The proof relies on several technical lemmas:

- `AntitoneLimitBound`: If $a : \mathbb{N} \to \mathbb{R}$ is `Antitone` and converges to L, then for all n, $L \le a(n)$. This is analogous to `MonotoneLimitBound`.

  Since $a_n \to 0$ and a is antitone, this implies $a_n \ge 0$ for all n.

- `CoherenceOfReals`: If sequences a and b converge to limits L and M respectively, and their difference $a_n - b_n \to 0$, then $L = M$.

- `SeqEvenOdd`: If the even-indexed subsequence $a_{2n} \to L$ and the odd-indexed subsequence $a_{2n+1} \to L$, then $a_n \to L$.

- `MonotoneSeriesEven`: If a is `Antitone`, then the even partial sums

  $$S_{2n} = \sum_{k=0}^{2n-1} (-1)^k a_k$$

  form a monotone increasing sequence.

- `AntitoneSeriesOdd`: If a is `Antitone`, then the odd partial sums

$$S_{2n+1} = \sum_{k=0}^{2n} (-1)^k a_k$$

  form an antitone (decreasing) sequence.

- `BddSeriesEven`: If a is `Antitone`, then for all n,

$$\sum_{k=0}^{2n-1} (-1)^k a_k \leq a_0$$

- `BddSeriesOdd`: If a is `Antitone` and nonnegative, then for all n,

$$0 \leq \sum_{k=0}^{2n} (-1)^k a_k$$

- `DiffGoesToZero`: If a is `Antitone` and $a_n \to 0$, then

$$\sum_{k=0}^{2n} (-1)^k a_k - \sum_{k=0}^{2n-1} (-1)^k a_k \to 0$$

Proof Strategy

The idea is to show that the even and odd partial sums both converge to the same limit. We define:

$$S_{2n} := \sum_{k=0}^{2n-1} (-1)^k a_k$$
$$S_{2n+1} := \sum_{k=0}^{2n} (-1)^k a_k$$

The even partial sums form a monotone increasing sequence bounded above by a_0, so they converge to some limit L. Similarly, the odd partial sums form an antitone decreasing sequence bounded below by 0, so they converge to some limit M.

The key observation is that $S_{2n+1} - S_{2n} = a_{2n} \to 0$, which by `CoherenceOfReals` implies $M = L$. Since both subsequences converge to the same limit, the full sequence of partial sums converges by `SeqEvenOdd`.

The Formal Proof

Proof:

Step 1: By `AntitoneLimitBound`, since a is antitone and $a_n \to 0$, we have $a_n \geq 0$ for all n.

Step 2: Define S_{2n} and S_{2n+1} as above. By `MonotoneSeriesEven`, S_{2n} is monotone increasing.

Step 3: By `AntitoneSeriesOdd`, S_{2n+1} is antitone (decreasing).

Step 4: By `BddSeriesEven`, $S_{2n} \leq a_0$ for all n, so S_{2n} is bounded above.

Step 5: By `BddSeriesOdd`, $0 \leq S_{2n+1}$ for all n, so S_{2n+1} is bounded below.

Step 6: By `IsCauchyOfMonotoneBdd`, the monotone bounded sequence S_{2n} is Cauchy. By `SeqConv_of_IsCauchy`, it converges to some limit L.

Step 7: Similarly, by `IsCauchyOfAntitoneBdd`, the antitone bounded sequence S_{2n+1} is Cauchy, hence converges to some limit M.

Step 8: By `DiffGoesToZero`, we have $S_{2n+1} - S_{2n} \to 0$.

Step 9: By `CoherenceOfReals`, since $S_{2n+1} \to M$, $S_{2n} \to L$, and $S_{2n+1} - S_{2n} \to 0$, we conclude that $M = L$.

Step 10: We observe by reflexivity that $S_{2n} = \text{Series}(\lambda k.(-1)^k \cdot a_k, 2n)$ and $S_{2n+1} = \text{Series}(\lambda k.(-1)^k \cdot a_k, 2n+1)$.

Step 11: Since both the even-indexed partial sums converge to L and the odd-indexed partial sums converge to L, by `SeqEvenOdd`, the full sequence of partial sums `Series (fun n` $\mapsto$ $(-1)^n \cdot a_n$`)` converges to L.

Therefore, the alternating series converges. □

The Lean Proof

```
Statement AlternatingSeriesTest {a : ℕ → ℝ} (ha :
   Antitone a)
    (aLim : SeqLim a 0) : SeriesConv (fun n ↦ (-1)^n * a
        n) := by
  have TermsBddBelow : ∀ n, a n ≥ 0 := by
    apply AntitoneLimitBound ha aLim
  let S2n : ℕ → ℝ := (fun n ↦∑ k ∈ range (2 * n), (-1)^
     k * a k)
  let S2np1 : ℕ → ℝ := (fun n ↦∑ k ∈ range (2 * n + 1),
      (-1)^k * a k)
  have s2nMono : Monotone S2n := by apply
     MonotoneSeriesEven ha
```

```
  have s2np1Anti : Antitone S2np1 := by apply
     AntitoneSeriesOdd ha
  have s2nBdd : ∀ n, S2n n ≤ a 0 := by apply
     BddSeriesEven ha
  have s2np1Bdd : ∀ n, 0 ≤ S2np1 n := by
    apply BddSeriesOdd ha TermsBddBelow
  have s2nCauchy : IsCauchy S2n := by
    apply IsCauchyOfMonotoneBdd s2nMono s2nBdd
  have s2nLim : SeqConv S2n := by
    apply SeqConv_of_IsCauchy s2nCauchy
  have s2np1Cauchy : IsCauchy S2np1 := by
    apply IsCauchyOfAntitoneBdd s2np1Anti s2np1Bdd
  have s2np1Lim : SeqConv S2np1 := by
    apply SeqConv_of_IsCauchy s2np1Cauchy
  choose L hL using s2nLim
  choose M hM using s2np1Lim
  have diffZero : SeqLim (fun n ↦ S2np1 n - S2n n) 0 :=
     by
    apply DiffGoesToZero ha aLim
  have hLM : M = L := CoherenceOfReals hM hL diffZero
  have s2nIs : S2n = fun n ↦ Series (fun k ↦ (-1)^k * a
     k) (2 * n) := by
    rfl
  rewrite [s2nIs] at hL
  have s2np1Is : S2np1 = fun n ↦ Series (fun k ↦ (-1)^k
     * a k) (2 * n + 1) := by
    rfl
  rewrite [s2np1Is] at hM
  rewrite [hLM] at hM
  use L
  apply SeqEvenOdd hL hM
```

Understanding the Theorem

The Alternating Series Test is particularly useful because its hypothesis is easy to verify. Some classic examples:

- The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges to $\ln 2$

- The series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ converges to $\frac{\pi}{4}$
- More generally, any series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k^p}$ converges for $p > 0$

Note that the alternating harmonic series converges but not absolutely (since the harmonic series diverges), making it an example of **conditional convergence**. This will be important in the next levels when we study rearrangements!

Last time: $(\text{Series } a)_n := \sum_{k=0}^{n-1} a_k$.

Def: SeriesConv $(a : \mathbb{N} \to \mathbb{R})$: Prop := SeqConv (Series a)

When does a Series Converge?

Last time: If SeriesConv a $\Rightarrow$ $a_k \to 0$.

Does it matter in what order you add things up?

$$M = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & -1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & 0 & -1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 0 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & & & & & \ddots \end{pmatrix} \begin{array}{l} \to -1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 0 \\ \to 0 + (-1) + \frac{1}{2} + \frac{1}{4} + \cdots = 0 \\ \to \quad = 0 \\ \to \quad = 0 \end{array}$$

$$-1 \quad -\tfrac{1}{2} \quad -\tfrac{1}{4} \quad -\tfrac{1}{8} \quad \cdots \to -2 \qquad\qquad 0$$

Q: $\sum m_{ij} \overset{?}{=} 0$, $= -2$.

Main Observation: Infinite Addition is not necessarily Commutative!!

Def: A ~~sequence~~ series $a : \mathbb{N} \to \mathbb{R}$ ($\sum a_n$) is absolutely convergent 'AbsSeriesConv' if $\sum |a_n|$ converges.

i.e. AbsSeriesConv $(a : \mathbb{N} \to \mathbb{R})$: Prop := SeriesConv $(|a \cdot|)$

Thm 'Conv_of_AbsConv' : $\{a : \mathbb{N} \to \mathbb{R}\}$, (ha : AbsSeriesConv a) : SeriesConv a := by.

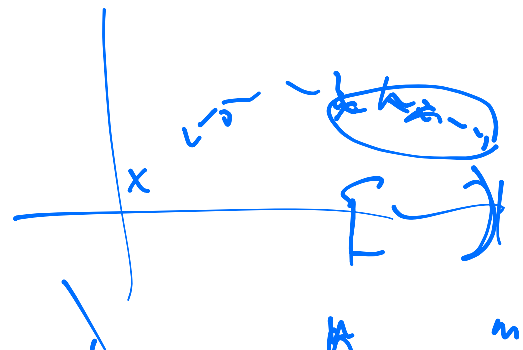
Know: $\sum_n |a_n| \to L$ converges. Want: $\sum_n a_n \xrightarrow{??}$ converges.

Idea: no clue where $\sum a_n$ is supposed to converge

So instead let's try to prove that it's Cauchy!

i.e. Given $\varepsilon > 0$, n, m large, we'll need:

$$\underbrace{\left|\sum_{k=0}^{m-1} a_k - \sum_{k=0}^{n-1} a_k\right|}_{=\left|\sum_{k=n}^{m-1} a_k\right|} < \varepsilon.$$



$$\left|\sum_{k=n}^{m-1} a_k\right| \le \sum_{k=n}^{m-1} |a_k| \quad (\text{`Series\_abs\_add'}).$$

idea: By (hyp), Series $|a_k|$ is Cauchy, so

$$\underbrace{\left|\sum^{m-1} |a_k| - \sum^{n-1} |a_k|\right|}_{=\left|\sum_{n}^{m-1} |a_k|\right|} < \varepsilon.$$

Human sketch pf: let $\varepsilon > 0$ be given. It suffices to produce some N s.t. $\forall n \geq N$, $\forall m \geq n$,

$$\left|\sum_0^{m-1} a_k - \sum_0^{n-1} a_k\right| = \left|\sum_n^{m-1} a_k\right| < \varepsilon.$$

From (ha), using the same value of $\varepsilon > 0$, get N (use it), so that we know that similarly, $\left|\sum_n^{m-1} |a_k|\right| < \varepsilon$. ✓

But from series_abs_add, $\left|\sum a_k\right| \leq \sum |a_k| \leq \left|\sum |a_k|\right|$

To say $\sum_{k=n}^{m-1} b_k$, write: $\sum k \in$ Ico n m, b_k.

Conv_of_IsCauchy ← for $a : \mathbb{N} \to \mathbb{Q}$.

Proved completeness of $\mathbb{R}$.

SeqConv_of_IsCauchy ← for $a : \mathbb{N} \to \mathbb{R}$.

Recall from Calc: Alternating Series Test:

If $(a : \mathbb{N} \to \mathbb{R})$ is Antitone & SeqLim a 0.

then SeriesConv $(n \mapsto (-1)^n \cdot a_n)$.

Why is this true? $\underbrace{(a_0 - a_1)}+\underbrace{(a_2 - a_3)}+\underbrace{(a_4 - a_5)}+\ldots$ ≤ 0 ≤ 0

Need: $\forall n, a_n \geq 0,$ ≥ 0 ≥ 0 ≥ 0

Recall: If $\sum a_k$ converges then $a_n \to 0$.

Not converse. Here: $a_n \searrow 0$ then $\sum (-1)^k a_k$ conv.

Let $S_n = \sum_{k=0}^{n-1} a_k$ Observe: • S_{2n} is Monotone & Bdd $\leq a_0$.

• S_{2n+1} is Antitone, ≥ 0.

• $S_{2n} = a_0 + (-a_1 + a_2) + (-a_3 + a_4) + (-a_5) \cdots \leq a_0$

• $S_{2n+1} = (a_0 - a_1) + (a_2 - a_3) \cdots (-a_{2n-1}) + a_{2n} \geq 0.$

So $S_{2n} \to L$, $S_{2n+1} \searrow M$. S_{2n} & $S_{2n+1} \to L$?: $\Rightarrow$ $S_n \to L$.

Lemma: $a_n \to L$, $b_n \to M$, $a_n - b_n \to 0 \Rightarrow L = M$.

$S_{2n} \to L$ & $S_{2n+1} \to L \overset{?}{\Rightarrow} S_n \to L$.

N_1 N_2 Need $|S_n - L| < \varepsilon$: 2 cases n even, odd.

$n \geq N_1 + N_2$,

Q: If $\exists$ subseq σ, $S_{\sigma(n)} \to L$ then $S_n \to L$?

$\sum_{k=0}^{2n-1} (-1)^k = 0.$ (No!).