An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 12: Bounded Monotone Sequences are Cauchy

Prof. Alex Kontorovich

This text is automatically generated by LLM from "Real Analysis, The Game", Lecture 12

1 Iterated Subsequences and Orbits

Let's warm up to the topics of this lecture with a foundational exercise that will illuminate the deep connection between iteration and monotonic growth.

Suppose you have a sequence of natural numbers, $\sigma : \mathbb{N} \to \mathbb{N}$, and all you know about it is that it always exceeds the identity:

$$h_\sigma : \forall n, n < \sigma(n)$$

Now, this of course doesn't mean that $\sigma(n)$ is itself strictly increasing (what we call a `Subseq`). The sequence could jump around all over the place, as long as its graph stays above that of $y = x$.

But hopefully it's "intuitively clear" from h_σ that σ eventually blows up, gets larger and larger over time, just not monotonically so. That is, there should be *some* way to "accelerate" σ so that it becomes a `Subseq`. The only problem is: how do you *actually* do this?

1.1 The Key Idea: Orbits

The key idea is that of an **orbit**. In astronomy, you can imagine looking up at the sky night after night and trying to track the location of, say, Jupiter against the "fixed" stars (celestial sphere). You start your observations with Jupiter having some "phase-space" (position, velocity) x_0; let T be the function that runs Newtonian dynamics for one day, so that $T(x_0)$ is the new phase-space of Jupiter tomorrow, moving as it does according to Newton's laws and gravity. Then $T(T(x_0))$ is the phase space after two days, and so on. The whole **orbit** of Jupiter over time is then the sequence:

$$x_0, T(x_0), T(T(x_0)), T(T(T(x_0))), \ldots$$

In mathematics, if you have any function $f : X \to X$ that takes an abstract space X to itself, and you start with some base point $x_0 : X$, then we will write $f^{[n]}(x_0)$ for f iterated n times applied to x_0. The sequence $n \mapsto f^{[n]}(x_0)$ is called the "orbit of x_0 under the action of f".

1.2 Application to Our Problem

How does that help us in our present situation? We could start with any base point $n_0 : \mathbb{N}$, and we know from h_σ specialized to $n = n_0$ that $n_0 < \sigma(n_0)$, but we have no idea how big $\sigma(n_0)$ is; it could be huge. So how do we ensure that the next term exceeds $\sigma(n_0)$? (Want to think about it for a minute before reading on?)

Given our previous discussion, hopefully you see right away that: if we were to specialize h_σ to $n = \sigma(n_0)$, we would get: $\sigma(n_0) < \sigma(\sigma(n_0))$. So now it's clear: the way to get larger and larger terms from the sequence σ is to take the orbit!

1.3 New Tools

1.3.1 Function Iteration: succ_iterate

While $\sigma^{[k]}(\sigma(n)) = \sigma^{[k+1]}(n)$ is true by definition, it takes an argument by induction to show that if instead of adding a σ on the right, we add it on the left:

$$\sigma(\sigma^{[k]}(n)) = \sigma^{[k+1]}(n)$$

We'll spare you the proof of that argument, and give you the theorem `succ_iterate`.

1.3.2 Subsequence from Successor: subseq_of_succ

To prove that σ is a `Subseq`, the definition of which speaks of all $i < j$, it's enough to do it one step at a time. The theorem `subseq_of_succ` says that it's enough to show that $\sigma(n) < \sigma(n+1)$ holds for all n to conclude `Subseq` σ. You can `apply` this fact to reduce showing `Subseq` σ to just showing that σ increases from n to $n+1$.

1.3.3 Tactic: show

The `show` tactic has syntax `show fact by proof`. For example, if you want to rewrite by the fact that $\sigma(\sigma^{[n]}(n_0)) = \sigma^{[n+1]}(n_0)$ without a separate `have` declaration, you can write:

```
rewrite [show σ (σ^[n] n₀) = σ^[n+1] n₀ by apply
   succ_iterate]
```

1.4 The Mathematical Statement

Theorem (Subseq_of_Iterate): If a sequence $\sigma : \mathbb{N} \to \mathbb{N}$ grows faster than the identity, $n < \sigma(n)$, then the orbit of any base point $n_0 : \mathbb{N}$ under σ – this means the sequence $n_0, \sigma(n_0), \sigma^{[2]}(n_0), \ldots$ – is a `Subseq`, that is, is strictly increasing.

1.5 Strategic Approach

The proof uses the `subseq_of_succ` theorem to reduce the problem to showing that consecutive terms in the orbit are strictly ordered. Specifically, we need to show:

$$\sigma^{[n]}(n_0) < \sigma^{[n+1]}(n_0)$$

The key insight is that $\sigma^{[n+1]}(n_0) = \sigma(\sigma^{[n]}(n_0))$ by the `succ_iterate` theorem, and we can apply the hypothesis h_σ to the point $\sigma^{[n]}(n_0)$ to get the desired inequality.

1.6 Lean Solution

```
Statement Subseq_of_Iterate (σ : ℕ → ℕ) (hσ : ∀ n, n < σ
    n) (n₀ : ℕ) :
```

```
Subseq (fun n ↦ σ^[n] n₀) := by
apply subseq_of_succ
intro n
specialize hσ (σ^[n] n₀)
have f : σ (σ^[n] n₀) = σ^[n+1] n₀ := by apply
   succ_iterate
rewrite [f] at hσ
apply hσ
```

1.7 Natural Language Proof

Proof: We use the theorem `subseq_of_succ`, which reduces proving that a sequence is strictly increasing to showing that consecutive terms satisfy $f(n) < f(n+1)$.

For our orbit sequence $n \mapsto \sigma^{[n]}(n_0)$, we need to show:

$$\sigma^{[n]}(n_0) < \sigma^{[n+1]}(n_0)$$

By the `succ_iterate` theorem:

$$\sigma^{[n+1]}(n_0) = \sigma(\sigma^{[n]}(n_0))$$

Now we apply the hypothesis h_σ to the point $\sigma^{[n]}(n_0)$:

$$\sigma^{[n]}(n_0) < \sigma(\sigma^{[n]}(n_0)) = \sigma^{[n+1]}(n_0)$$

This completes the proof. **QED**

1.8 Why This Matters

This result establishes that even non-monotonic sequences can have monotonic subsequences when they satisfy certain growth conditions. The orbit construction provides a canonical way to extract monotonic behavior from sequences that grow faster than the identity. This technique will be crucial for our main theorem about bounded monotonic sequences.

2 Monotone and Bounded Sequences are Cauchy

The fundamental theorem connecting monotonicity, boundedness, and convergence states that every bounded monotone sequence is Cauchy (and hence convergent). This result is central to real analysis and provides a powerful tool for establishing convergence without explicitly finding the limit.

The intuition may seem clear: a monotone sequence that is bounded cannot have persistent gaps, because such gaps would eventually cause the sequence to exceed its bound. But getting the details just right will take some work. We postpone to the next section the proof of a "helper lemma" and jump right in to the main argument.

2.1 New Tools

2.1.1 Definition: Monotone

A sequence $a : \mathbb{N} \to X$ (where X is some ordered type) is said to be **monotone** if $a(n) \leq a(m)$ whenever $n \leq m$.

2.1.2 Theorem: Monotone_of_succ

To prove monotonicity, it is enough to prove it one step at a time; that is, if $a(m) \leq a(m+1)$ holds for all m, then a is `Monotone`.

2.1.3 Tactic: push_neg

The negation of $\forall$ is $\exists$, and vice-versa. To push a chain of negations through, write `push_neg`.

2.2 Abstraction and Generality

Let $a : \mathbb{N} \to X$ be a sequence. Wait, what is X here? Well, we're trying to work up to the construction of the real numbers, so we'd better not presuppose their existence; so maybe X should be the rationals. On the other hand, once we prove the fact that monotone bounded sequences are Cauchy, maybe we'll want to use that fact for real-valued sequences. So we'll need to prove the theorem *twice*? No, of course not; that's the beauty of abstraction!

We'll set X to be an abstract "Type", but assume things about it like there's a linear order (so we can say $x \leq y$), and a norm (so we can say

$|x|$), and that we can add/subtract/multiply/divide elements of X and get elements of X (that X is a "field"). We'll just make enough of these assumptions for the proof to work. And then if we want to apply this general theorem to a rational sequence $a : \mathbb{N} \to \mathbb{Q}$ or a real sequence $a : \mathbb{N} \to \mathbb{R}$, Lean will *automatically* infer that all of the necessary conditions on X are satisfied in these cases.

2.3 The Mathematical Statement

Theorem (IsCauchyOfMonotoneBdd): Let $a : \mathbb{N} \to X$ be a monotone sequence that is bounded above by some $M : X$. Then a is Cauchy.

2.4 Strategic Approach

What could go wrong? Suppose (by contradiction) that the sequence a is not Cauchy. This will mean that there are arbitrarily late "gaps" of size ε in the sequence. The iterated gap theorem (see next section/level) then shows these gaps accumulate without bound, contradicting the boundedness assumption.

The initial "obvious" steps are these:

1. Assume by contradiction the hypothesis h that a is not Cauchy. So there exists some $\varepsilon > 0$ such that for *any* N, we can find $m \geq n \geq N$ with $|a(m) - a(n)| \geq \varepsilon$. (The `push_neg` tactic will come in handy here, and you might want to `change` at h so Lean sees through the definition of `IsCauchy`.)

2. Once you choose this ε, you have the statement: `h : ∀N, ∃n ≥N, m ≥ N, |a m - a n| ≥ε`. Think hard about what this means: n is actually a function of N, that is, $n = n(N)$; same with $m = m(N)$. And when we're given an N, we also get a proof that $n(N) \geq N$, and a proof that $m(N) \geq n(N)$, and a proof that $|a(m(N)) - a(n(N))| \geq \varepsilon$. Do you see why that's the same as this statement `h`?

3. Now we can `choose using h` to extract these maps $n(N)$ and $m(N)$, except I think the situation is clearer if we rename them to $\tau(N)$ and $\sigma(N)$. The fact that $\tau(N) \geq N$ can be called `hτ` and the fact that $\sigma(N) \geq \tau(N)$ is `hσ`. The fact that `|a (σ(N)) - a (τ(N))| ≥ε` we can call `hgap`.

Now what? I hope the intuition is at least somewhat clear: the sequence a grows by at least ε from time to time, and is increasing, so after enough of those "bumps" up by ε, we'll exceed the claimed upper bound M. But how do we actually implement this in practice?

Well, start with $N = 0$. Then we get some value $0 \leq \tau(0) \leq \sigma(0)$ for which $|a(\sigma(0)) - a(\tau(0))| \geq \varepsilon$. Of course we can drop the absolute values, since a is monotonic. How can we continue?

2.5 Postponed Helper Lemma

Once we get our hands on some ε amount of growth, we need to iterate it to get $k \cdot \varepsilon$ growth, for any k:

Theorem (IterateGap): Given $(a : \mathbb{N} \to X)$ $(ha : \text{Monotone } a)$ $(\varepsilon : X)$ $(\varepsilon_{pos} : \varepsilon > 0)$ $(\tau : \mathbb{N} \to \mathbb{N})$ $(h_\tau : \forall n, \tau(n) \geq n)$ $(\sigma : \mathbb{N} \to \mathbb{N})$ $(h_\sigma : \forall n, \sigma(n) \geq \tau(n))$ $(h_{gap} : \forall n, \varepsilon \leq |a(\sigma(n)) - a(\tau(n))|)$:

$$\forall k, k \cdot \varepsilon \leq a(\sigma^{[k]}(0)) - a(0)$$

That is, the orbit $\sigma^{[k]}(0)$ is *exactly* the subsequence along which a is guaranteed to grow by at least $k \times \varepsilon$.

2.6 Lean Solution

```
Statement IsCauchyOfMonotoneBdd {X : Type*}
 [NormedField X] [LinearOrder X] [IsStrictOrderedRing X]
 [FloorSemiring X] (a : ℕ → X) (M : X)
 (hM : ∀ n, a n ≤ M) (ha : Monotone a)
 : IsCauchy a := by
intro ε hε
by_contra h
push_neg at h
choose τ hτ σ hσ hgap using h
have f1 : ∀ k, k * ε ≤ a (σ^[k] 0) - a 0 := by apply
   IterateGap a ha ε hε τ hτ σ hσ hgap
let k : ℕ := ⌈(M - a 0) / ε⌉₊ + 1
have hk' : (M - a 0) / ε ≤ ⌈(M - a 0) / ε⌉₊ := by bound
have hk : (M - a 0) / ε < k := by
 change (M - a 0) / ε < (⌈(M - a 0) / ε⌉₊ + 1 : ℕ);
    push_cast; linarith [hk']
```

```
specialize f1 k
specialize hM (σ^[k] 0)
have f2 : (M - a 0) < k * ε := by field_simp at hk;
   rewrite [show k * ε = ε * k by ring_nf]; apply hk
linarith [f1, f2, hM]
```

2.7 Natural Language Proof

Proof: We proceed by contradiction. Suppose a is not Cauchy. Then there exists some $\varepsilon > 0$ such that for every $N \in \mathbb{N}$, we can find indices $m \geq n \geq N$ with $|a(m) - a(n)| \geq \varepsilon$.

That means that we can construct functions $\tau, \sigma : \mathbb{N} \to \mathbb{N}$ such that:

- $\tau(k) \geq k$ for all k
- $\sigma(k) \geq \tau(k)$ for all k
- $|a(\sigma(k)) - a(\tau(k))| \geq \varepsilon$ for all k

By the monotonicity of a and the ordering $\tau(k) \leq \sigma(k)$, we have $a(\tau(k)) \leq a(\sigma(k))$, so:

$$\varepsilon \leq |a(\sigma(k)) - a(\tau(k))| = a(\sigma(k)) - a(\tau(k))$$

Now we can apply the IterateGap theorem to conclude:

$$\forall k \in \mathbb{N}, \quad k \cdot \varepsilon \leq a(\sigma^{[k]}(0)) - a(0)$$

Choose $k = \lceil (M - a(0))/\varepsilon \rceil_+ + 1$. Then:

$$k > \frac{M - a(0)}{\varepsilon}$$

which implies:

$$k \cdot \varepsilon > M - a(0)$$

From our gap accumulation result:

$$k \cdot \varepsilon \leq a(\sigma^{[k]}(0)) - a(0)$$

Therefore:

$$M - a(0) < k \cdot \varepsilon \leq a(\sigma^{[k]}(0)) - a(0)$$

This gives us $M < a(\sigma^{[k]}(0))$, contradicting our assumption that $a(n) \leq M$ for all n.

Therefore, a must be Cauchy. **QED**

3 Iterated Gaps: The Helper Lemma

Now we prove the leftover result from the previous section. This technical lemma captures the precise mechanism by which persistent gaps in a monotone sequence accumulate under iteration.

Recall our setup: we have a monotone sequence $a : \mathbb{N} \to X$ and two subsequences $\sigma, \tau : \mathbb{N} \to \mathbb{N}$ with the properties:

- σ grows faster than τ: $\sigma(n) \geq \tau(n)$ for all n
- τ grows faster than the identity function: $\tau(n) \geq n$ for all n
- There is some positive ε so that: $\varepsilon \leq |a(\sigma(n)) - a(\tau(n))|$

Our goal is to show that if we iterate σ exactly k times – written in Lean as $\sigma^{[k]}$ – then we'll accumulate at least $k \cdot \varepsilon$ growth from the initial value.

3.1 The Mathematical Statement

Theorem (IterateGap): Given a monotone sequence $a : \mathbb{N} \to X$ and subsequences $\tau, \sigma : \mathbb{N} \to \mathbb{N}$ satisfying the conditions above, we have:

$$\forall k \in \mathbb{N}, \quad k \cdot \varepsilon \leq a(\sigma^{[k]}(0)) - a(0)$$

3.2 Strategic Approach

The proof proceeds by induction on k. The key insight is that each iteration of σ introduces at least an additional gap of size ε, and monotonicity ensures these gaps accumulate additively.

For the base case $k = 0$, the statement becomes $0 \leq a(0) - a(0) = 0$, which is trivial.

For the inductive step, we use the gap condition at the point $\sigma^{[k]}(0)$ to establish that moving from $\sigma^{[k]}(0)$ to $\sigma^{[k+1]}(0)$ creates an additional gap of at least ε, which combines with the inductive hypothesis.

3.3 Lean Solution

```
Statement {X : Type*} [NormedField X] [LinearOrder X]
 [IsStrictOrderedRing X] (a : ℕ → X) (ha : Monotone a)
 (ε : X) (εpos : ε > 0) (τ : ℕ → ℕ) (hτ : ∀ n, τ n ≥ n)
 (σ : ℕ → ℕ) (hσ : ∀ n, σ n ≥ τ n)
 (hgap : ∀ n, ε ≤ |a (σ n) - a (τ n)|)
  : ∀ (k : ℕ), k * ε ≤ a (σ^[k] 0) - a 0 := by
intro k
induction' k with k hk
norm_num
specialize hgap (σ^[k] 0)
rewrite [
 show |a (σ (σ^[k] 0)) - a (τ (σ^[k] 0))|
  = a (σ (σ^[k] 0)) - a (τ (σ^[k] 0))
 by apply abs_of_nonneg (by bound)] at hgap
rewrite [show σ (σ^[k] 0) = σ^[k + 1] 0 by apply
   succ_iterate] at hgap
have f1 : 0 ≤ a (τ (σ^[k] 0)) - a (σ^[k] 0) := by bound
push_cast
linarith [f1, hk, hgap]
```

3.4 Natural Language Proof

Proof: We proceed by induction on k.

Base case: When $k = 0$, we need to show $0 \cdot \varepsilon \leq a(\sigma^{[0]}(0)) - a(0)$. Since $\sigma^{[0]}(0) = 0$, this becomes $0 \leq a(0) - a(0) = 0$, which is immediate.

Inductive step: Assume the result holds for some $k \geq 0$, so $k \cdot \varepsilon \leq a(\sigma^{[k]}(0)) - a(0)$. We must show $(k+1) \cdot \varepsilon \leq a(\sigma^{[k+1]}(0)) - a(0)$.

By the gap hypothesis applied at $n = \sigma^{[k]}(0)$:

$$\varepsilon \leq |a(\sigma(\sigma^{[k]}(0))) - a(\tau(\sigma^{[k]}(0)))|$$

Since $\sigma^{[k]}(0) \geq 0$ and h_τ implies $\tau(\sigma^{[k]}(0)) \geq \sigma^{[k]}(0)$, and h_σ implies $\sigma(\sigma^{[k]}(0)) \geq \tau(\sigma^{[k]}(0))$, monotonicity gives us:

$$a(\sigma^{[k]}(0)) \leq a(\tau(\sigma^{[k]}(0))) \leq a(\sigma(\sigma^{[k]}(0)))$$

Therefore the absolute value equals the difference:

$$\varepsilon \leq a(\sigma(\sigma^{[k]}(0))) - a(\tau(\sigma^{[k]}(0)))$$

Using `succ_iterate`, we have $\sigma(\sigma^{[k]}(0)) = \sigma^{[k+1]}(0)$. Also, by monotonicity:

$$a(\tau(\sigma^{[k]}(0))) - a(\sigma^{[k]}(0)) \geq 0$$

Combining these inequalities:

$$\begin{align}
a(\sigma^{[k+1]}(0)) - a(0) &= a(\sigma^{[k+1]}(0)) - a(\sigma^{[k]}(0)) + a(\sigma^{[k]}(0)) - a(0) \tag{1}\\
&\geq a(\sigma^{[k+1]}(0)) - a(\tau(\sigma^{[k]}(0))) + a(\sigma^{[k]}(0)) - a(0) \tag{2}\\
&\geq \varepsilon + k \cdot \varepsilon \tag{3}\\
&= (k+1) \cdot \varepsilon \tag{4}
\end{align}$$

This completes the induction. **QED**

3.5 Why This Technical Lemma Matters

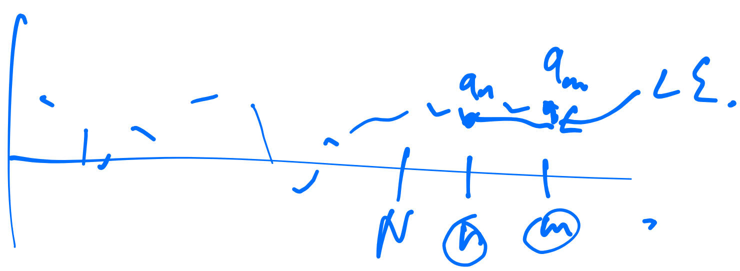
This theorem captures the precise mechanism by which monotone sequences with persistent gaps eventually violate any proposed upper bound. The iteration of subsequences, combined with the additive accumulation of gaps, provides the quantitative tool needed to make the intuitive argument rigorous.

The result demonstrates a fundamental principle: in monotone systems, local growth patterns (represented by the gap condition) scale linearly under iteration. This scaling property is what allows us to construct a contradiction with boundedness in the main theorem.

Where are we? What's a Cauchy Sequence?

Defn $a : \mathbb{N} \to X$ ← satisfies enough axioms fulfilled by $\mathbb{Q}/\mathbb{R}$, Need $|\cdot|$, $x \leq y$.

is Cauchy if: $\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall m \geq n, |a_m - a_n| < \varepsilon$



Aside: Why not $|a_n - a_{n+1}| < \varepsilon$? Think about it!

Prop'd: ① If $a : \mathbb{N} \to \mathbb{R}$ & converges (to some L), then a is Cauchy.

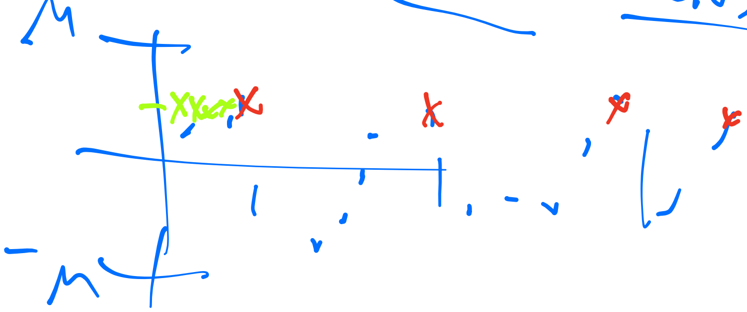
→ ② a & b are Cauchy ⇒ so is a+b.

→ ③ If a Cauchy ⇒ a is bdd

Compare: ① $a \to L$, $b \to M$, $a+b \to L+M$

③ $a \to L \Rightarrow$ bdd

Working up to Bolzano Weierstrass:



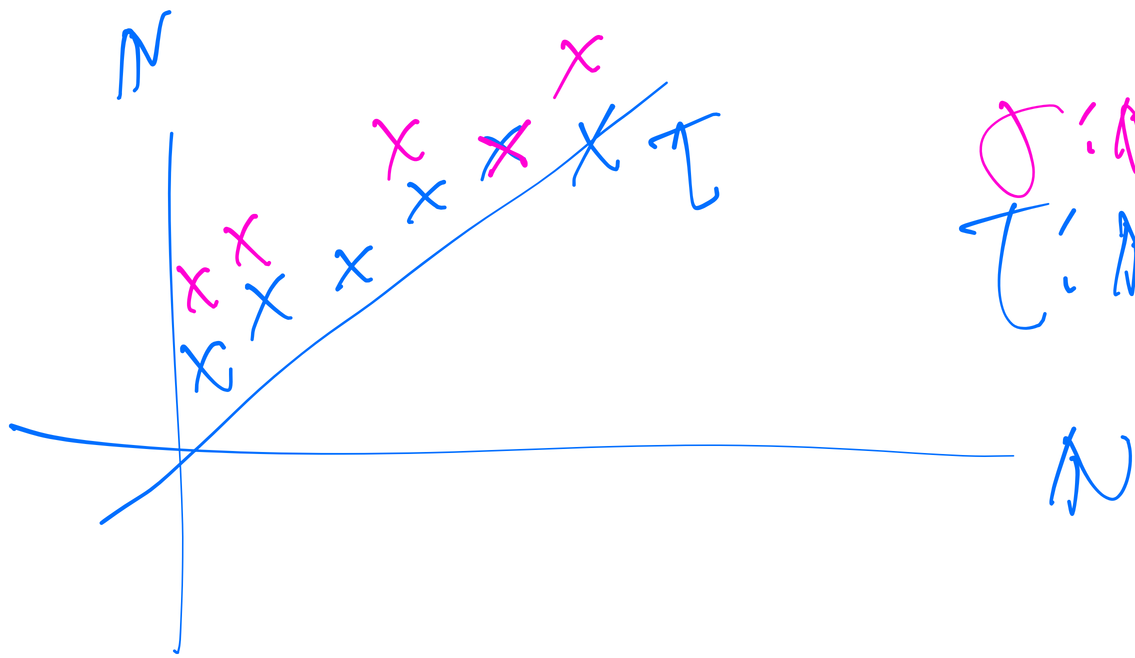
$\forall$ bdd seq $a : \mathbb{N} \to \mathbb{R}$, $\exists$ subseq $\sigma : \mathbb{N} \to \mathbb{N}$. s.t. $a \circ \sigma$ converges

B-W Not true for $a : \mathbb{N} \to \mathbb{Q}$, So at then

Bolzano -Weierstass NOT true for sequences in the rationals!!! So we're forced to contend with the definition the reals.
Example: 1, 1.4, 1.41, 1.414, ... sequence lies between [1, 2], and converges to sqrt 2 which is not rational.

Claim: If a sequence `a : N -> X` is Monotone (non-decreasing, if i <= j, then a i <= a j) and bounded, then it is Cauchy.

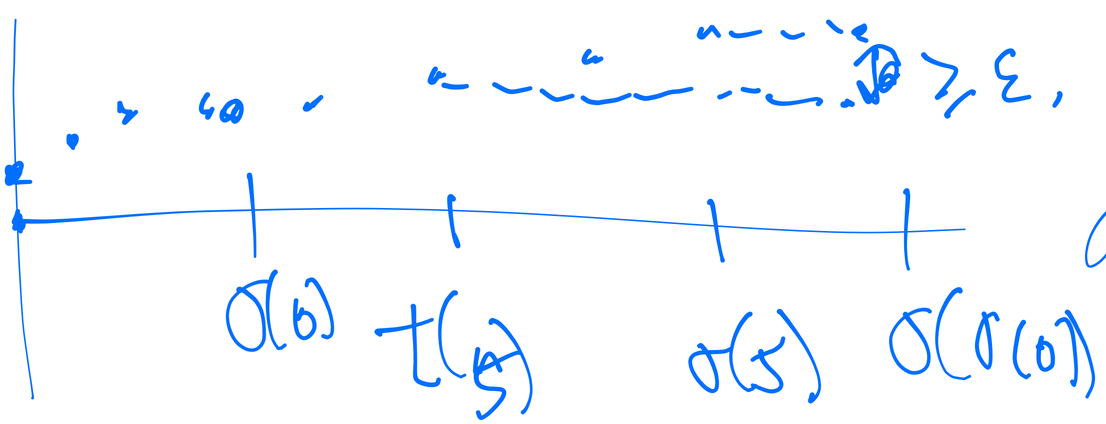
Warmup: We have a sequence `a : N -> X` and epsilon > 0, (ha : Monotone a) with two sequences into naturals, sigma, tau : N -> N, with the following properties: tau (n) >= n, and sigma(n) >= tau (n).



$\sigma : \mathbb{N} \to \mathbb{N}$

$\tau : \mathbb{N} \to \mathbb{N}$

We also assume that `| a (sigma n) - a (tau n)| >= epsilon`. Claim: take the orbit of 0 under sigma, the values of `a` on this orbit grow by epsilon * k.

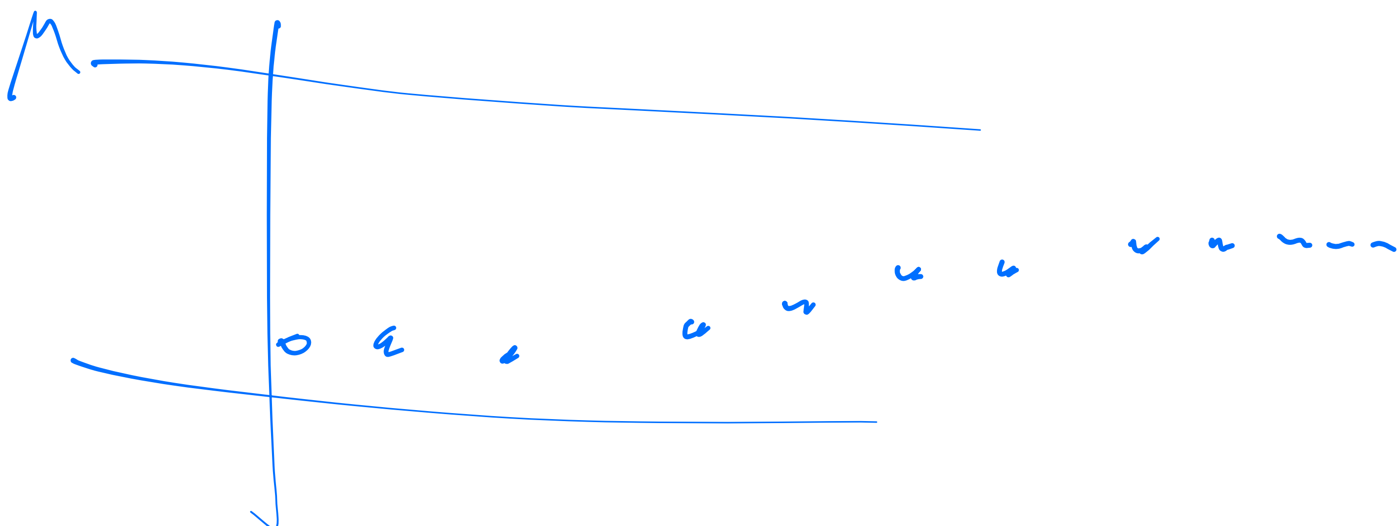


$a : \mathbb{N} \to \mathbb{R}$

Orbit: $0, \sigma(0), \sigma(\sigma(0)), \sigma^{(3)}(0)$

Come back in a min...

Claim: If "a" is Monotone & bdd $\Rightarrow$ a is Cauchy!



Why "should" this be true?
What could go wrong?

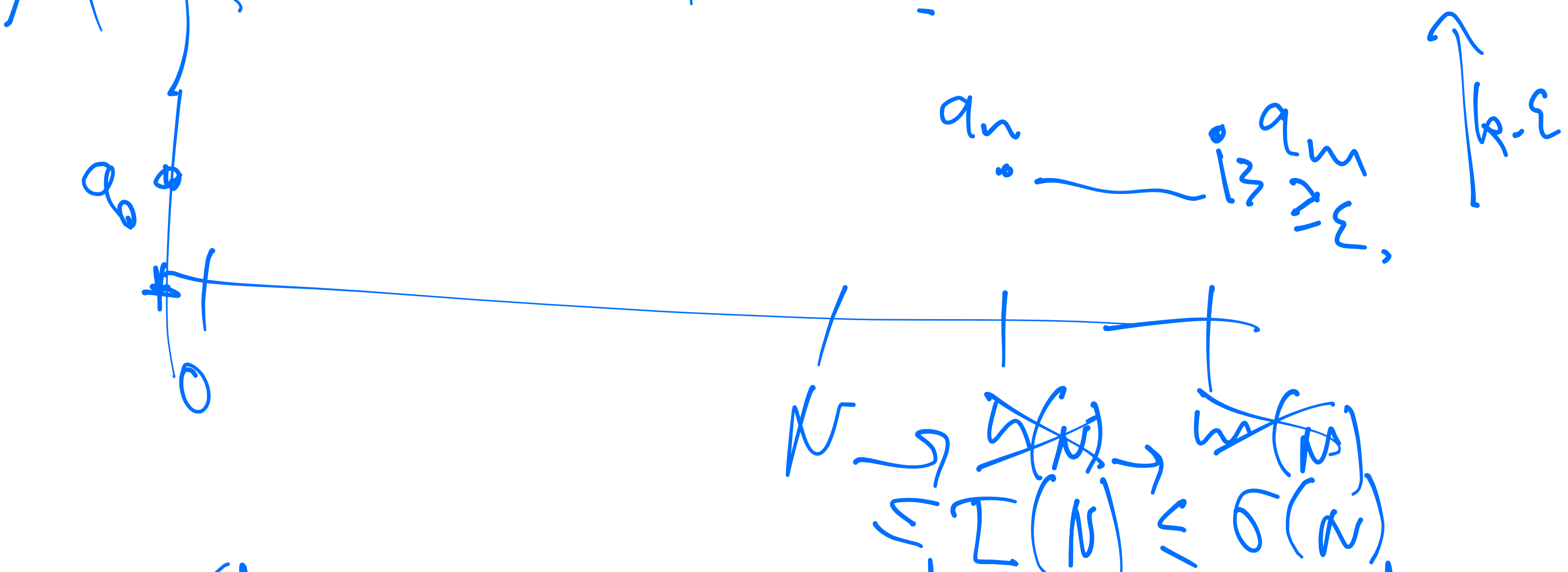
By contra h.

Change $\neg(\forall \varepsilon>0, \exists N, \forall n \geq N, \forall m \geq n, |a_m - a_n| < \varepsilon)$ at h,

$\boxed{\begin{array}{l} h : \neg \text{IsCauchy } a. \\ \text{Goal: False} \end{array}}$

`push_neg`: push negation through.

h: $\exists\, \varepsilon > 0,\ \forall N,\ \exists\, n \geq N,\ \exists\, m \geq n,\ |a_m - a_n| \geq \varepsilon.$



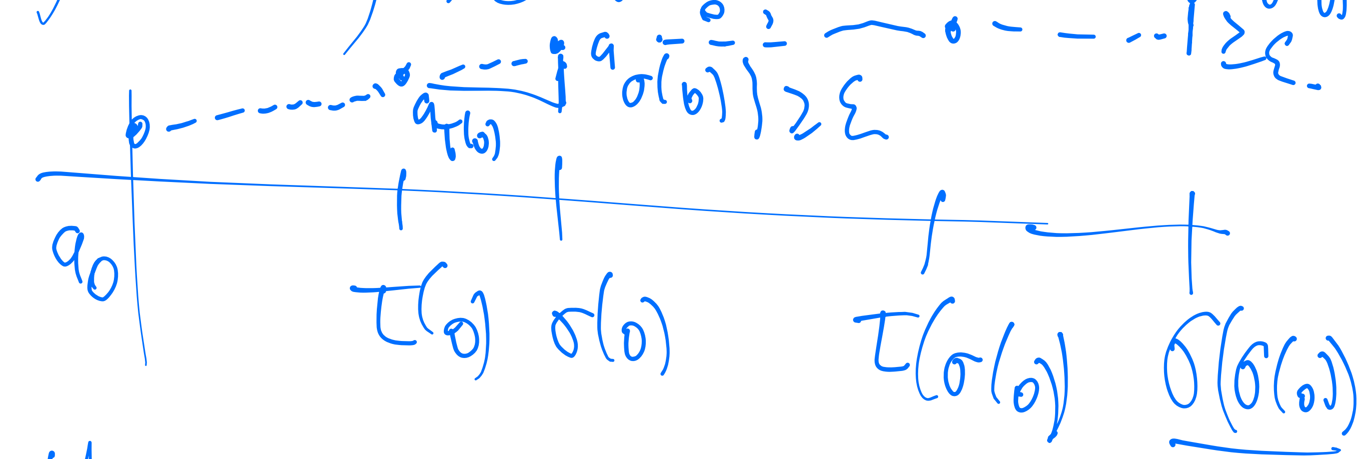
Choose ε ε_{pos} $h\varepsilon$ using h,

Now: $\varepsilon_{pos} : \varepsilon > 0$

$\cancel{n(N)}$ $\tau(k)$, $h\varepsilon : \forall N,\ \exists\, n \geq N,\ \exists\, m \geq n,\ |a_m - a_n| \geq \varepsilon,$

Choose τ $h\tau$ σ $h\sigma$ hgap using $h\varepsilon$.

Next idea: How to chain growing by ε's?

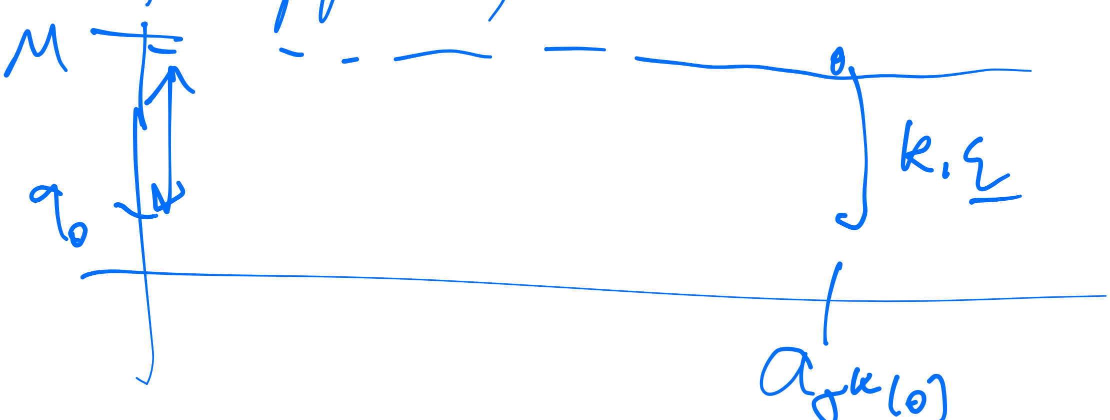


Next guaranteed ε growth happens between $\tau(\sigma^{(2)}(0))$ and $\sigma^{(3)}(0)$

Claim 1: $\forall k,\ k \cdot \varepsilon \le a_{\sigma^{[k]}(0)} - a_0$

(Iterate Gap)

by applying Iterate Gap, set ,



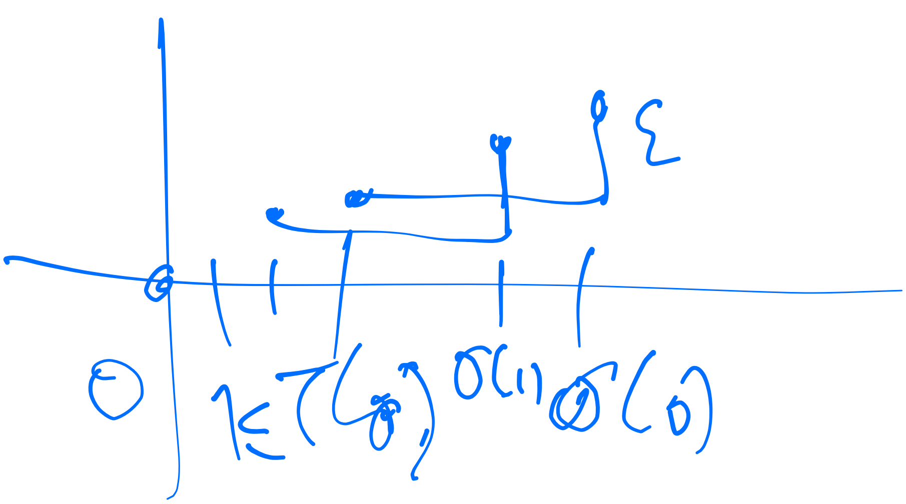
Let $k := \left\lceil \frac{M - a_0}{\varepsilon} \right\rceil_+ + 1$.

<u>Homework:</u> prove $\frac{M - a_0}{\varepsilon} < k$;

Hence $M - a_0 < k \cdot \varepsilon \le a_{\sigma^{(k)}(0)} - a_0$

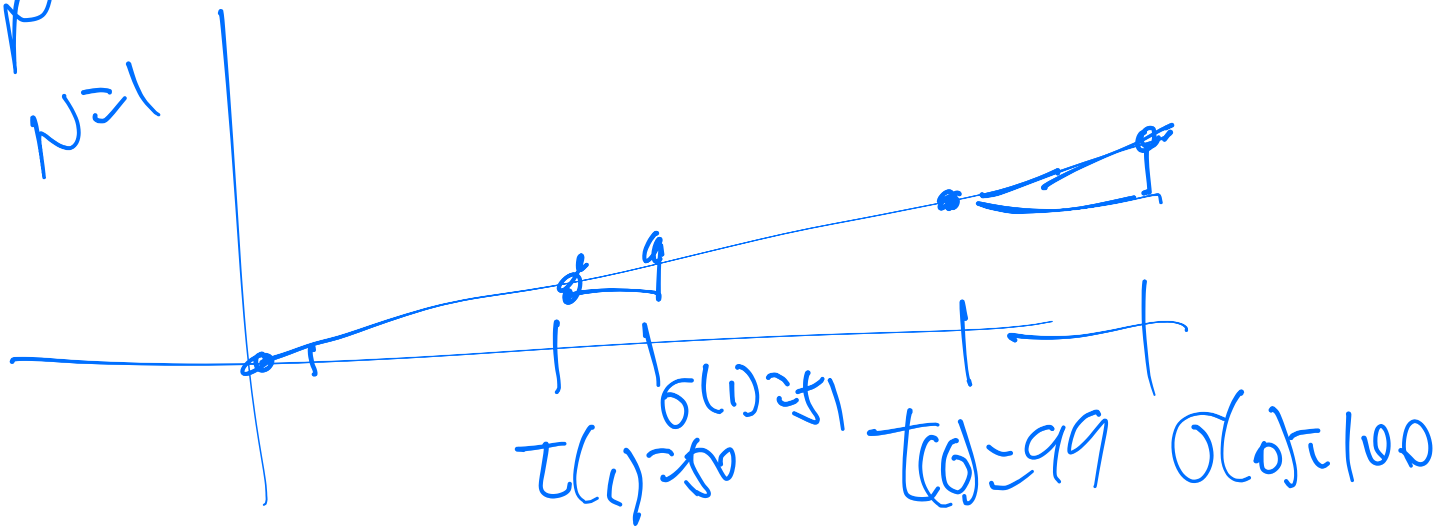
$\Rightarrow a_{\sigma^{(k)}(0)} > M$, $\cancel{\xi}$.

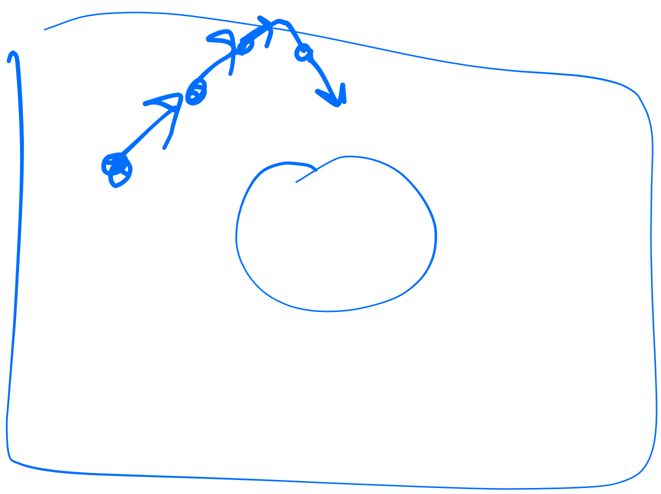
Look again



$n=0,$

$N=1$





$I^{(a)}(4),$

$\sigma((\sigma\ n_0))$

$n+0=n$ by def

$0+n=n$ by pf.

$\sigma^{k+1}=\sigma^{k}(\sigma):=$ by def

$$\sigma(\sigma^k) = \sigma^{k+1} := \text{by succ\_iterate}$$