

An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 16: Completeness of the Real Numbers

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“Real Analysis, The Game”, Lecture 16*

1 Introduction: The Completeness Property

We have now arrived at one of the most fundamental properties of the real numbers: **completeness**. This is the property that distinguishes the real numbers from the rational numbers and makes analysis possible.

We know that Cauchy sequences of rational numbers converge to real numbers. Indeed, this is how we *constructed* the real numbers – as equivalence classes of Cauchy sequences of rationals. Let’s record this as a theorem:

Theorem (Conv_of_IsCauchy): If a sequence $a : \mathbb{N} \rightarrow \mathbb{Q}$ satisfies `IsCauchy a`, then `SeqConv a` holds – that is, the sequence converges.

Given that such a limit exists, we can define it explicitly:

Definition (Real_of_CauSeq): This takes `IsCauchy a` (for $a : \mathbb{N} \rightarrow \mathbb{Q}$) and returns the real number that a converges to.

This real number does what we want:

Theorem (SeqLim_of_Real_of_Cau): If `ha : IsCauchy (a : N → Q)`, then `SeqLim a (Real_of_CauSeq ha)` holds – that is, the sequence a converges to the real number defined by `Real_of_CauSeq`.

But now we must ask: what about Cauchy sequences of *real* numbers? Do we need yet another number system (the “hyperreals” or perhaps “surreals”, for the Conway fans out there...) to capture limits of reals?

No! And this is a general phenomenon: when you “complete” a space by replacing it with equivalence classes of Cauchy sequences from that space, the space itself is automatically “complete”. That is, all Cauchy sequences in the completed space already converge within the same space. Let’s prove it for the reals!

2 The Question: Are the Reals Complete?

We want to show that any Cauchy sequence of *real* numbers converges to a real number.

But we just learned that real numbers are themselves Cauchy sequences of rationals (or rather, equivalence classes thereof). Now that real numbers are represented by Cauchy sequences of rationals, what does it mean for (x_n) to be a Cauchy sequence of reals? A Cauchy sequence of Cauchy sequences?? Let’s unpack that.

2.1 The Double Array Picture

Let’s say that we have an infinite sequence of real numbers, $x : \mathbb{N} \rightarrow \mathbb{R}$, thought of as $x = (x_0, x_1, x_2, \dots)$.

Each real x_n is secretly represented by a Cauchy sequence $(q_{n,0}, q_{n,1}, q_{n,2}, \dots)$ of rationals. So each x_n is a map $\mathbb{N} \rightarrow \mathbb{Q}$, and the original x is actually a map:

$$q : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{Q}$$

That is, x is really a *double-indexed array* of rationals:

	$j = 0$	$j = 1$	$j = 2$	$j = 3$	\dots	
$i = 0:$	$q_{0,0}$	$q_{0,1}$	$q_{0,2}$	$q_{0,3}$	\dots	\leftarrow represents x_0
$i = 1:$	$q_{1,0}$	$q_{1,1}$	$q_{1,2}$	$q_{1,3}$	\dots	\leftarrow represents x_1
$i = 2:$	$q_{2,0}$	$q_{2,1}$	$q_{2,2}$	$q_{2,3}$	\dots	\leftarrow represents x_2
$i = 3:$	$q_{3,0}$	$q_{3,1}$	$q_{3,2}$	$q_{3,3}$	\dots	\leftarrow represents x_3
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	

Each **row** is a Cauchy sequence of rationals $q_{n,m}$ converging to the corresponding real x_n . Wild! Let’s call **hq** the condition that each of these rows is a Cauchy sequence:

$$\text{hq} : \forall n, \text{IsCauchy } (q \ n)$$

Then to say that this sequence (of sequences) x_n is itself “Cauchy” is to say (as we already know) that:

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall m \geq n, |\text{Real_of_CauSeq}(\text{hq } m) - \text{Real_of_CauSeq}(\text{hq } n)| < \varepsilon$$

2.2 The Goal

Now, let’s assume this condition on our Cauchy sequence (of sequences) x . Our goal is to show that this sequence has a real limit, that is, that there is a single sequence $y : \mathbb{N} \rightarrow \mathbb{Q}$, satisfying

$$\text{hy} : \text{IsCauchy } y$$

and so that $x_n \rightarrow y$, which means:

$$\text{SeqLim}(\lambda n \mapsto \text{Real_of_CauSeq}(\text{hq } n)) (\text{Real_of_CauSeq } \text{hy})$$

How are we supposed to construct such a thing?!?!

3 The Key Insight: Cantor’s Diagonalization

The key insight is reminiscent of Cantor’s **diagonalization** arguments, very prevalent in real analysis: we need to extract a single sequence of rationals from this array!

3.1 The Naive Attempt

The naive attempt: Take the diagonal sequence $(q_{0,0}, q_{1,1}, q_{2,2}, q_{3,3}, \dots)$.

The problem: Each row converges at **its own rate!** Row k might need to go out to index $1000k$ before it’s within ε of its limit, and row $k+1$ might need to go to index k^2 , etc. But the diagonal only samples row k at position k , which might be way too early!

3.2 The Solution: Choosing Convergence Points

The solution: For each row k , we need to pick an index $N(k)$ that’s far enough out that row k has “converged well enough” by that point. Then we take:

$$y(k) = q_{k,N(k)}$$

This sequence y will be Cauchy, and it will represent the limit of our sequence of reals!

How to choose $N(k)$? Use the fact that each x_k (as a real number) is the limit of its Cauchy sequence. So for any tolerance (say $1/k$, or rather, $1/(k+1)$ so it's always strictly positive), there exists an index $N(k)$ so that for all $m \geq N(k)$, $q_{k,m}$ stays within $1/(k+1)$ of x_k (that is, of $q_{k,n}$ for $n \geq N(k)$). That is what we'll choose!

4 The Proof Strategy

Our task is to:

1. Construct the sequence y by choosing appropriate indices $N(k)$ from each row
2. Prove that y is Cauchy
3. Prove that the sequence of reals converges to the real represented by y

Let's walk through the formal proof.

4.1 Step 1: Choosing the Indices

For each $n : \mathbb{N}$, we use `SeqLim_of_Real_of_Cau (hq n)` with tolerance $1/(n+1)$ to obtain an index $N(n)$ such that:

$$\forall m \geq N(n), |q_{n,m} - x_n| < 1/(n+1)$$

We define:

$$y(n) = q_{n,N(n)}$$

This gives us a single sequence of rationals, one from each row, chosen at points where that row has already converged sufficiently close to its limit.

4.2 Step 2: Proving y is Cauchy

To show that y is Cauchy, we need to show that for any $\varepsilon > 0$, there exists N_0 such that for all $n, m \geq N_0$, we have $|y_m - y_n| < \varepsilon$.

The key idea is a triangle inequality argument:

$$|y_m - y_n| = |q_{m,N(m)} - q_{n,N(n)}| \leq |q_{m,N(m)} - x_m| + |x_m - x_n| + |x_n - q_{n,N(n)}|$$

Let's bound each term:

Bounding the first and third terms: We need $|q_{m,N(m)} - x_m|$ and $|q_{n,N(n)} - x_n|$ to be small. By our choice of $N(m)$ and $N(n)$, we have:

$$|q_{m,N(m)} - x_m| < 1/(m+1)$$

$$|q_{n,N(n)} - x_n| < 1/(n+1)$$

For these to be less than $\varepsilon/3$, we need $1/(m+1) < \varepsilon/3$ and $1/(n+1) < \varepsilon/3$. By the Archimedean property, there exists N_2 such that $1/(N_2 + 1) < \varepsilon/3$. For all $n, m \geq N_2$, these bounds hold.

Bounding the middle term: We need $|x_m - x_n| < \varepsilon/3$. Since x is a Cauchy sequence of reals, there exists N_1 such that for all $m, n \geq N_1$, we have $|x_m - x_n| < \varepsilon/3$.

Combining: Let $N_0 = N_1 + N_2$. Then for all $m, n \geq N_0$, all three terms are bounded by $\varepsilon/3$, so:

$$|y_m - y_n| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Thus y is Cauchy.

4.3 Step 3: Proving $x_n \rightarrow \text{Real_of_CauSeq} \text{ hy}$

Let $L = \text{Real_of_CauSeq} \text{ hy}$ be the real number represented by y . We need to show that $x_n \rightarrow L$, that is:

$$\forall \varepsilon > 0, \exists N_0, \forall n \geq N_0, |x_n - L| < \varepsilon$$

Again, we use a triangle inequality:

$$|x_n - L| = |x_n - y_n + y_n - L| \leq |x_n - y_n| + |y_n - L|$$

Bounding the first term: We have $y_n = q_{n,N(n)}$, so:

$$|x_n - y_n| = |x_n - q_{n,N(n)}| < 1/(n+1)$$

For this to be less than $\varepsilon/2$, we need $1/(n+1) < \varepsilon/2$. By the Archimedean property, there exists N_2 such that for all $n \geq N_2$, this holds.

Bounding the second term: Since y is Cauchy with limit L , by $\text{SeqLim_of_Real_of_Cau hy}$, there exists N_3 such that for all $n \geq N_3$, we have:

$$|y_n - L| < \varepsilon/2$$

One more ingredient: We also need to use the fact that x itself is Cauchy. There exists N_1 such that for all $n, m \geq N_1$, $|x_n - x_m| < \varepsilon/2$.

Combining: Let $N_0 = N_1 + N_2 + N_3$. Then for all $n \geq N_0$:

$$|x_n - L| \leq |x_n - y_n| + |y_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus $x_n \rightarrow L$, completing the proof.

5 The Lean Proof

```
Statement Reals_are_Complete (q : ℕ → ℕ → ℚ) (x : ℕ → ℝ)
(hq : ∀ n, IsCauchy (q n))
(hx : ∀ n, x n = Real_of_CauSeq (hq n))
(hxCau : IsCauchy x) :
  ∃ (y : ℕ → ℚ) (hy : IsCauchy y),
  SeqLim x (Real_of_CauSeq hy) := by
choose N hN using fun (n : ℕ) ↞
  SeqLim_of_Real_of_Cau (hq n) (1 / (n + 1)) (by bound
    )
have hN : ∀ n, ∀ m ≥ N n, |(q n m) - x n| < 1 / (n +
  1) := by
intro n m hm
rewrite [hx n]
apply hN n m hm
let y := fun n ↞ q n (N n)
use y
have hy : IsCauchy y := by
intro ε hε
have hε' : (0 : ℝ) < ε := by exact_mod_cast hε
choose N1 hN1 using hxCau (ε / 3) (by norm_num;
  bound)
choose N2 hN2 using ArchProp
(show (0 : ℝ) < ε / 3 by norm_num; bound)
use N1 + N2
intro n hn m hm
change |(q m (N m)) - q n (N n)| < ε
specialize hN1 n (by bound) m hm
have qnBnd := hN n (N n) (by bound)
have qmBnd := hN m (N m) (by bound)
```

```

have f1 : |(q m (N m) : ℝ) - q n (N n)| =
| (q m (N m) - x m) + ((x n - q n (N n)) + (x m - x n)) | := 
  by ring_nf
have f2 : |(q m (N m) - x m) +
((x n - q n (N n)) + (x m - x n))| ≤
| (q m (N m) - x m) | + | (x n - q n (N n)) + (x m - x n) | := 
  by apply abs_add
have f3 : |(x n - q n (N n)) + (x m - x n)| ≤
| (x n - q n (N n)) | + | (x m - x n) | := by apply
abs_add
have f3' : |(x n - q n (N n))| = |q n (N n) - x n| :=
  by apply abs_sub_comm
field_simp at hN2
have hn' : (N1 : ℝ) + N2 ≤ n := by exact_mod_cast hn
have hm' : (n : ℝ) ≤ m := by exact_mod_cast hm
have f4' : (N2 : ℝ) ≤ n := by bound
have f4'' : (N2 : ℝ) * ε ≤ n * ε := by bound
have f4 : (1 : ℝ) / (n + 1) < ε / 3 := by field_simp
; bound
have f5' : (N2 : ℝ) ≤ m := by bound
have f5'' : (N2 : ℝ) * ε ≤ m * ε := by bound
have f5 : (1 : ℝ) / (m + 1) < ε / 3 := by field_simp
; bound
have f6 : |(q m (N m) : ℝ) - q n (N n)| < ε := by
linarith [f1, f2, f3, f4, f5, qnBnd, qmBnd, hn1,
f3']
exact_mod_cast f6
use hy
intro ε hε
choose N1 hn1 using hxCau (ε / 2) (by norm_num; bound)
choose N2 hn2 using ArchProp
(show (0 : ℝ) < ε / 2 by norm_num; bound)
choose N3 hn3 using SeqLim_of_Real_of_Cau hy (ε / 2)
(by norm_num; bound)
use N1 + N2 + N3
intro n hn
let L := Real_of_CauSeq hy

```

```

change |x n - L| < ε
change ∀ n ≥ N3, |y n - L| < ε / 2 at hN3
rewrite [show |x n - L| = |(x n - y n) + (y n - L)| by
    ring_nf]
have f1 : |(x n - y n) + (y n - L)| ≤
    |(x n - y n)| + |(y n - L)| := by apply abs_add
specialize hN n (N n) (by bound)
change |y n - x n| < 1 / (n + 1) at hN
rewrite [show |y n - x n| = |x n - y n| by apply
    abs_sub_comm] at hN
specialize hN3 n (by bound)
field_simp at hN2
have hn' : (N1 : ℝ) + N2 + N3 ≤ n := by norm_cast
have f2' : (N2 : ℝ) ≤ n := by bound
have f2'' : (N2 : ℝ) * ε ≤ n * ε := by bound
have f2 : (1 : ℝ) / (n + 1) < ε / 2 := by field_simp;
    bound
linarith [f1, f2, hN, hN3]

```

6 Understanding the Proof

Let's unpack the key steps in natural language.

6.1 The Construction of y

We use the `choose` tactic to extract, for each row n , an index $N(n)$ such that the n -th row has converged to within $1/(n+1)$ of its limit by index $N(n)$.

Then we define $y(n) = q_{n,N(n)}$ – we take one element from each row, chosen at a point where that row has already converged sufficiently.

6.2 Why y is Cauchy

The proof that y is Cauchy uses a clever three-term decomposition:

$$|y_m - y_n| \leq |y_m - x_m| + |x_m - x_n| + |x_n - y_n|$$

The middle term is small because x is Cauchy (by assumption). The first and third terms are small because we chose y_m and y_n at points where rows m and n had already converged to their limits x_m and x_n .

But there's a subtlety: we need these terms to be small *simultaneously* for large m and n . This is where the Archimedean property comes in. We need both $1/(m+1) < \varepsilon/3$ and $1/(n+1) < \varepsilon/3$. By choosing m, n large enough (specifically, $m, n \geq N_2$ where N_2 is from the Archimedean property), we ensure both inequalities hold.

6.3 Why $x_n \rightarrow L$

Once we know y is Cauchy, it represents some real number $L = \text{Real_of_CauSeq hy}$. To show that $x_n \rightarrow L$, we again use a two-term decomposition:

$$|x_n - L| \leq |x_n - y_n| + |y_n - L|$$

The first term is small because y_n was chosen from row n at a point where row n had converged to x_n . The second term is small because y converges to L (by `SeqLim_of_Real_of_Cau`).

7 The Significance of Completeness

The completeness of the real numbers is what makes real analysis work. Here's why it matters:

7.1 Closure Under Limits

The real numbers are *closed under limits*: if you have a Cauchy sequence of real numbers, its limit is also a real number. You don't need to keep expanding to larger number systems.

This contrasts sharply with the rational numbers. The sequence $a_n = (1 + 1/n)^n$ is Cauchy in \mathbb{Q} , but its limit e is *not* rational. To capture this limit, we needed to construct \mathbb{R} .

7.2 Fixed Point Theorems

Many fixed point theorems rely on completeness. For example, the Banach Fixed Point Theorem states that a contraction mapping on a complete metric space has a unique fixed point, found by iterating from any starting point.

7.3 Series Convergence

Completeness allows us to work with infinite series. A series $\sum a_n$ converges if and only if its partial sums form a Cauchy sequence. Without completeness, we couldn't guarantee that Cauchy sequences converge.

7.4 Construction of Functions

Many important functions are defined as limits of sequences:

- The exponential function: $e^x = \lim_{n \rightarrow \infty} (1 + x/n)^n$
- Trigonometric functions: defined via power series
- Special functions: Bessel functions, gamma function, etc.

Without completeness, these limits might not exist in the space we're working in.

8 The Completion Process

The construction we've just completed illustrates a general principle in mathematics: the **completion** of a space.

8.1 The General Pattern

Start with an incomplete space X (like \mathbb{Q}) equipped with a notion of distance. To complete it:

1. Consider all Cauchy sequences in X
2. Define an equivalence relation: two Cauchy sequences are equivalent if their difference converges to zero
3. The completed space \overline{X} is the set of equivalence classes
4. Define distance in \overline{X} using representatives from the equivalence classes

8.2 Key Properties of Completion

Universality: The completion is unique (up to isomorphism). Any complete space containing X as a dense subset must be isomorphic to \overline{X} .

Preservation: Algebraic operations (addition, multiplication) and topological properties (continuity) extend from X to \overline{X} .

Completeness: The completed space \overline{X} is complete! This is exactly what we proved: Cauchy sequences in \mathbb{R} converge in \mathbb{R} .

8.3 Examples in Mathematics

\mathbb{Q} to \mathbb{R} : The real numbers are the completion of the rationals with respect to the usual absolute value.

p -adic numbers: Using a different notion of distance (the p -adic absolute value), we can complete \mathbb{Q} to obtain the p -adic numbers \mathbb{Q}_p , which have very different properties from \mathbb{R} .

Function spaces: The space $L^2([0, 1])$ of square-integrable functions is the completion of the space of continuous functions under the L^2 norm.

Metric space completion: Any metric space can be completed to a complete metric space.

9 The Diagonal Argument

The construction of y by choosing elements $y_n = q_{n, N(n)}$ from different rows is a form of **diagonal argument**.

10 Philosophical Reflections

The completeness of the real numbers represents a profound mathematical achievement: the construction of a number system where every Cauchy sequence converges.

10.1 The Price of Completeness

To achieve completeness, we had to leave the world of the rationals, where every number can be written as a finite expression p/q , and enter the world

of the reals, where most numbers (like π , e , $\sqrt{2}$) require infinite decimal expansions.

This is the fundamental trade-off: completeness requires accepting infinite processes as completed objects.

10.2 The Benefit of Completeness

In return, we gain a number system where:

- Every bounded monotone sequence converges
- Every Cauchy sequence converges
- Continuous functions on closed intervals attain their extrema
- The Intermediate Value Theorem holds
- Power series define functions

These properties make analysis possible. Without completeness, we would constantly encounter sequences that “should” converge but don’t, functions that “should” have zeroes but don’t, and theorems that “should” hold but fail.

Last time: The reason

Cauchy sequences $\mathbb{N} \rightarrow \mathbb{Q}$

Converge (not in \mathbb{Q} !) in \mathbb{R}

↳ Nat... they are the ends.

What about Cauchy seqs in \mathbb{R} ?

What else do we need to do

So Nat Cauchy sequences in \mathbb{R} converge? Nothing!

Thm: A Cauchy sequence in \mathbb{R}

Converges to \mathbb{R} :
Some L :

def: Real_of_Cauchy $\{a: \mathbb{N} \rightarrow \mathbb{R}\}$

$(\forall \epsilon > 0) (\exists N \in \mathbb{N}) (\forall n, m \geq N) |a_n - a_m| < \epsilon$

Thm: Seqlim_of_Real_of_Cauchy
 $\{a: \mathbb{N} \rightarrow \mathbb{R}\} (\text{Real_of_Cauchy } a) \Rightarrow \text{Seqlim } (\text{fun } n \mapsto (a_n: \mathbb{R}))$

$\Rightarrow (\text{Real_of_Cauchy } a) \vdash \dots$

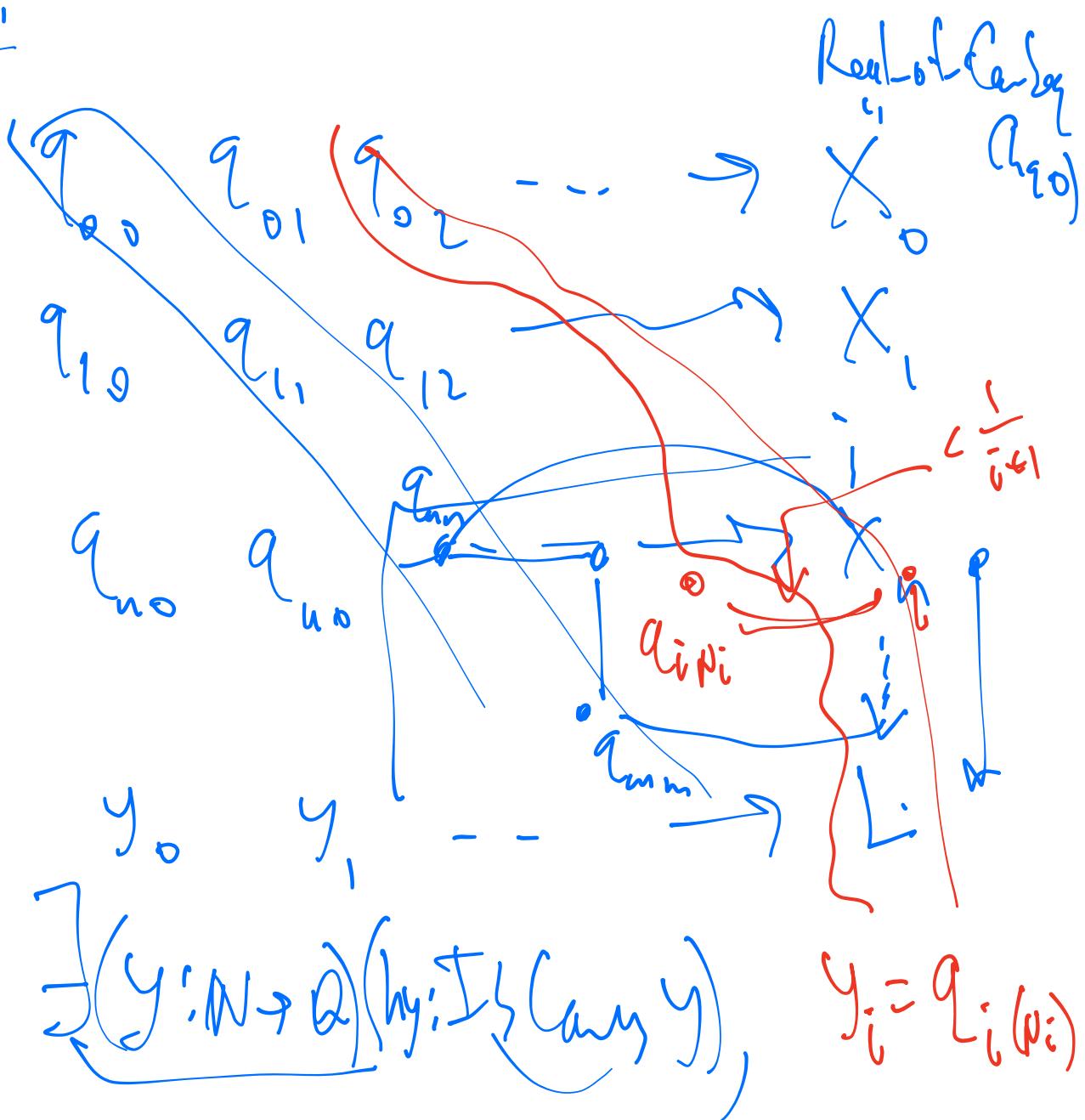
Ques: How to formally state that
Cauchy sequence in \mathbb{R} converges to R ?

Scratch: What $\underline{x_0, \dots, x_n, \dots}$ in \mathbb{R} ,
 $x_0 = [q_{00}, q_{01}, q_{02}, \dots]$ Cauchy seq in \mathbb{Q} ,
 $\downarrow x_0 = (q_{10}, q_{11}, q_{12}, \dots)$,
So $\{x_n\} = q: \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{Q}$.

($h_q: \forall n, \exists \text{Cauchy } (q_n)$)

($h_x: \forall \varepsilon > 0, \exists N, \forall n \geq N, \forall m \geq n$)
 $\left| \text{Real_of_Cauchy}(h_q, n) - \text{Real_of_Cauchy}(h_q, m) \right| < \varepsilon.$
 (h_q)

Attempt 2:



SegLm (fun n \mapsto Real-of-Candy (q_n))

(Real-of-Candy hy)

Attempt 1: Try the Atoms:

$$y_i = q_{ii}.$$

Is this Cauchy?

For $\exists N \forall n \geq N$,

$$|q_{mn} - q_{kn}| < \epsilon?$$

Attempt 2:

Idea: make n th row between

$|q_{nti}|$ of (x_n) by going far enough
(depends on n) into the sequence.

~~for $n \rightarrow \infty$ q_{nti} ($|q_{nti}|$)~~
~~(by limit),~~

~~: $n \rightarrow \infty, i \geq N, j \geq i$~~
 ~~$|q_{n+j} - q_{ni}| < \frac{1}{n+1}$~~

Choose \mathbb{N} wif fun $n \mapsto$ Leg_m of Rekolaw

Want q_{nm} to est.

$$|q_{mNm} - q_{NNn}| < \varepsilon.$$

$$(hq_n) \left(\frac{1}{(n+1)} \right) (by hand)$$

: $n \mapsto \exists N, \forall i \geq N,$

$$|q_{ni} - x_n| < \frac{1}{n+1}.$$

Let $y: \mathbb{N} \rightarrow Q := \lim_n n \mapsto q_n (N_n).$

How can $\{y_n\}$ β Cauchy?

have by: Is Cauchy $y \in y$

① Tho ε $\in \mathbb{E}.$

$$\begin{array}{c} q_{0,0} \dots \\ \vdots \\ N \end{array} \quad \begin{array}{c} \dots \\ \vdots \\ t_{no} \dots \end{array}$$

$$\begin{array}{c} \checkmark \\ \tau_{0i} \rightarrow x_0 \xrightarrow{\frac{1}{n+1}} \varepsilon_1 \\ \downarrow \\ q_{nNn} \rightarrow x_n \xrightarrow{\varepsilon_2} \varepsilon_3 \end{array}$$

$$q_{\text{min}} \leftarrow \frac{1}{m} \sum_{i=1}^m q_i \in \mathcal{E}(3)$$

② choose N_1 $\|hN_1\|$ via Archimedean

(show $(0, R)^2$

$\mathcal{E}(3)$ by

normalization

③ choose N_2 $\|hN_2\|$ using h_x

④ Use $N_1 + N_2$

⑤ into $n h_n$ or h_m

q_{nN_n} q_{mN_m}
Goal: $|y_n - y_m| \leq \epsilon$

⑥ have f): $|y_n - y_m| \leq |y_n - x_n| + |y_m - x_m|$

$+ |x_n - x_m| := \delta_y$

⑦ have f): $|y_n - x_n| < \epsilon_3 := \delta_y$ {easy}

⑧ have f): $|y_m - x_m| < \epsilon_3$ —

⑨ have $f(y_i)$: $|x_n - x_m| < \epsilon_{f(y)}$ \Rightarrow L is ϵ -Cauchy
in L .

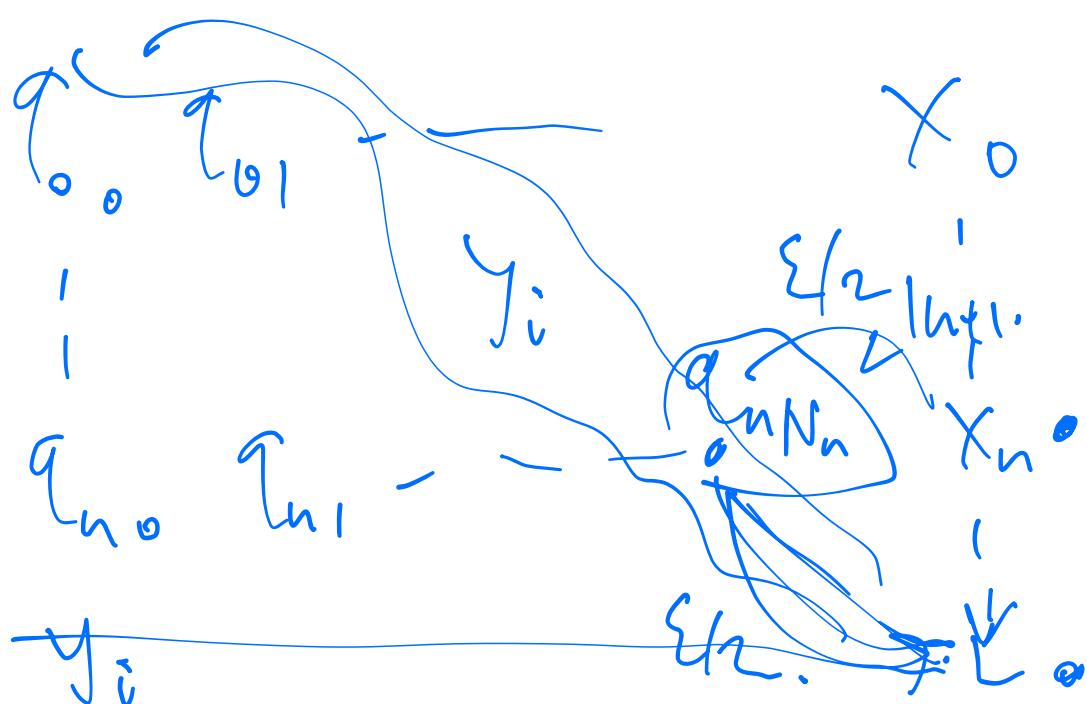
⑩ Invariance $\{f_1, f_L, f_3, f_4\}$.

Next: (y_n) is Cauchy $\Rightarrow \exists \epsilon \subseteq L^R$

Why does $(x_n) \rightarrow L$?

① into ϵ he.

Goal: $|x_n - L| < \epsilon$.



Choose N_1 $\in \mathbb{N}$ ^{very} s.t. $|x_n - x| < \epsilon_2$ by
 $(\exists y_2)$.

Choose N_2 $\in \mathbb{N}_2$ s.t. $\forall n > N_2$ $|y_n - y| < \epsilon_2$
use $N_1 + N_2 + 1$ from $n =$
Then $|x_n - L| \leq |x_n - y_n| + |y_n - L| < \epsilon_2$.