An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 13: Monotone Subsequence

Prof. Alex Kontorovich

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1 Introduction: The Monotone Subsequence Theorem

Claim: Every infinite sequence of real (or rational) numbers has either a "monotone" (non-decreasing) or an "antitone" (non-increasing) subsequence. This fundamental result is a cornerstone of real analysis and provides deep insight into the structure of sequences. In this lecture, we'll prove half of this theorem: that sequences without unbounded peaks must have a monotone subsequence.

The strategy revolves around a beautiful dichotomy. We'll define what it means for a sequence to have "unbounded peaks" – intuitively, positions from which the sequence never rises again, and there are arbitrarily many such positions. The key insight is that if a sequence does *not* have this property, then we can construct an increasing subsequence by cleverly avoiding the peaks.

2 New Definitions

2.1 Peaks in a Sequence

Consider a sequence $a : \mathbb{N} \to X$ where X could be $\mathbb{Q}$ or $\mathbb{R}$ (or any linearly ordered set). We say that position n is a **peak** if from that vantage point, we can look forward and see that all future values are no greater than $a(n)$.

Definition (IsAPeak): For a sequence $a : \mathbb{N} \to X$ and index $n : \mathbb{N}$, we say n is a peak of a if:

$$\text{IsAPeak}(a, n) := \forall m > n,\ a(m) \leq a(n)$$

In other words, standing at position n, we can look down at all future values.

2.2 Unbounded Peaks

A sequence might have many peaks, or few peaks, or even no peaks at all. We're particularly interested in sequences whose peaks are "unbounded" – meaning that no matter how far out we look, there's always another peak even further along.

Definition (UnBddPeaks): A sequence $a : \mathbb{N} \to X$ has unbounded peaks if:

$$\text{UnBddPeaks}(a) := \forall k,\ \exists n > k,\ \text{IsAPeak}(a, n)$$

This says that for any bound k, there exists a peak position n beyond k. The set of peaks has no upper bound.

2.3 Conditional Expressions: if-then-else

In Lean, we can define values conditionally. The syntax is:

```
if h : condition then value_if_true else value_if_false
```

The optional `h :` binds a proof of `condition` (when true) or its negation (when false) that can be used in the respective branches. This is a powerful feature that we'll use to construct our subsequence.

3 New Theorem: lt_of_not_ge

One of the fundamental properties of ordered sets is that comparisons are decidable: for any two elements m, n, either $m \leq n$ or $n < m$.

Theorem (lt_of_not_ge): If $\neg(m \leq n)$, then $n < m$.

This seemingly simple fact is crucial for our construction, as it allows us to extract strict inequality from the failure of weak inequality.

4 The Main Theorem: Monotone Subsequence from Bounded Peaks

Now we arrive at our main result. The theorem states that if a sequence does *not* have unbounded peaks, then it must have a monotone increasing subsequence.

Theorem (MonotoneSubseq_of_BddPeaks): Let $a : \mathbb{N} \to X$ be a sequence (where X is $\mathbb{Q}$ or $\mathbb{R}$). If a does not have unbounded peaks, then there exists a subsequence $\sigma : \mathbb{N} \to \mathbb{N}$ such that:

- σ is strictly increasing (a `Subseq`)
- $a \circ \sigma$ is monotone (non-decreasing)

4.1 Strategic Overview

The proof proceeds in several stages:

Stage 1: Unpacking the Hypothesis. The negation of `UnBddPeaks a` means there exists some bound k such that there are no peaks beyond k. Using `choose`, we extract:

- A bound $k : \mathbb{N}$
- For each $n > k$, a witness $\tau(n)$ with $\tau(n) > n$ and $\neg\text{IsAPeak}(a, \tau(n))$

Stage 2: Constructing an Auxiliary Sequence. We define a helper sequence $\tau' : \mathbb{N} \to \mathbb{N}$ that grows faster than the identity:

$$\tau'(n) := \begin{cases} n+1 & \text{if } n \leq k \\ \tau(n) & \text{if } n > k \end{cases}$$

The first case handles indices up to our bound k, while the second case uses our non-peak witnesses for indices beyond k.

Stage 3: Building the Orbit. Using the orbit construction from Lecture 12, we define:

$$\sigma(n) := (\tau')^{[n]}(k+1)$$

This is the n-fold iteration of τ' starting from $k+1$. Since τ' grows faster than the identity, we know from `Subseq_of_Iterate` that σ is strictly increasing.

Stage 4: Proving Monotonicity. The key observation is that for all n, we have $\sigma(n) > k$, so $\sigma(n)$ is never a peak. Since $\sigma(n)$ is not a peak, there must exist some $m > \sigma(n)$ with $a(m) > a(\sigma(n))$. By our construction, $\sigma(n+1) = \tau'(\sigma(n)) > \sigma(n)$ is precisely such an m, giving us $a(\sigma(n)) \leq a(\sigma(n+1))$.

4.2 Lean Solution

```
Statement MonotoneSubseq_of_BddPeaks (a : ℕ → X)
  (ha : ¬ UnBddPeaks a)
  : ∃ σ, Subseq σ ∧ Monotone (a ∘ σ) := by
change ¬ (∀ k, ∃ n > k, ∀ m > n, a m ≤ a n) at ha
push_neg at ha
choose k hk using ha
choose τ τ_gt hτ using hk
let τ' : ℕ → ℕ := fun n => if h : n ≤ k then n + 1 else
   τ n (lt_of_not_ge h)
have τ'_eq : ∀ n, τ' n = if h : n ≤ k then n + 1 else
    τ n (lt_of_not_ge h) := by
  intro n; rfl
have τ'_gt : ∀ n, n < τ' n := by
  intro n;
  by_cases hn : n ≤ k;
  rewrite [τ'_eq];
  bound;
  rewrite [τ'_eq];
  bound
let σ : ℕ → ℕ := fun n ↦ τ'^[n] (k+1)
have σ_eq : ∀ n, σ n = τ'^[n] (k+1) := by intro n; rfl
have hσ : ∀ n, k < σ n := by
```

```
  intro n;
  induction' n with n hn;
  rewrite [σ_eq];
  bound;
  rewrite [σ_eq];
  rewrite [← show τ' (τ'^[n] (k + 1)) =
      τ'^[n + 1] (k + 1)
    by apply succ_iterate];
  rewrite [← σ_eq];
  specialize τ'_gt (σ n);
  linarith [τ'_gt, hn]
use σ
split_ands
apply Subseq_of_Iterate
apply τ'_gt
apply Monotone_of_succ
intro n
specialize hσ n
specialize hτ (σ n) hσ
change a (σ n) ≤ a (σ (n + 1))
rewrite [show σ (n + 1) = τ'^[n+1] (k + 1)
  by apply σ_eq]
rewrite [← show τ' (τ'^[n] (k + 1)) = τ'^[n + 1] (k + 1)
  by apply succ_iterate]
rewrite [← show σ (n) = τ'^[n] (k + 1) by apply σ_eq]
rewrite [τ'_eq]
bound
```

4.3 Natural Language Proof

Proof: Assume $a : \mathbb{N} \to X$ does not have unbounded peaks. We will construct a strictly increasing subsequence along which a is monotone.

Step 1: Extracting the bound. Since $\neg\text{UnBddPeaks}(a)$, the negation of $\forall k, \exists n > k, \text{IsAPeak}(a, n)$ gives us:

$$\exists k, \forall n > k, \neg\text{IsAPeak}(a, n)$$

By the `choose` tactic, we obtain a specific $k : \mathbb{N}$.

Step 2: Extracting witnesses. For each $n > k$, since $\neg \mathrm{IsAPeak}(a, n)$, we have:

$$\neg(\forall m > n, a(m) \leq a(n))$$

which is equivalent to:

$$\exists m > n, a(m) > a(n)$$

Using `choose` again, we obtain a function τ so that:

- $\tau(n) > n$ for all $n > k$
- $a(\tau(n)) > a(n)$ for all $n > k$

Step 3: Defining the auxiliary sequence. We define $\tau' : \mathbb{N} \to \mathbb{N}$ by:

$$\tau'(n) := \begin{cases} n+1 & \text{if } n \leq k \\ \tau(n) & \text{if } n > k \end{cases}$$

where in the second case, we use `lt_of_not_ge` to convert $\neg(n \leq k)$ into $k < n$, allowing us to apply τ.

We verify that τ' grows faster than the identity:

- If $n \leq k$: then $\tau'(n) = n + 1 > n$
- If $n > k$: then $\tau'(n) = \tau(n) > n$

Thus $\forall n, n < \tau'(n)$.

Step 4: Building the orbit. Define $\sigma : \mathbb{N} \to \mathbb{N}$ by:

$$\sigma(n) := (\tau')^{[n]}(k+1)$$

By the theorem `Subseq_of_Iterate` from Lecture 12, since τ' grows faster than the identity, σ is strictly increasing.

Step 5: Verifying the key inequality. We prove by induction that $\sigma(n) > k$ for all n:

- Base case ($n = 0$): $\sigma(0) = (\tau')^{[0]}(k+1) = k+1 > k$

- Inductive step: Assume $\sigma(n) > k$. Then:

$$\begin{aligned}
\sigma(n+1) &= (\tau')^{[n+1]}(k+1) \\
&= \tau'((\tau')^{[n]}(k+1)) \\
&= \tau'(\sigma(n)) \\
&> \sigma(n) \\
&> k
\end{aligned}$$

Step 6: Proving monotonicity. We use the theorem `Monotone_of_succ` which says that it suffices to show $a(\sigma(n)) \leq a(\sigma(n+1))$ for all n.

Fix n. We know $\sigma(n) > k$. By our construction:

$$\sigma(n+1) = \tau'(\sigma(n))$$

Since $\sigma(n) > k$, we have $\tau'(\sigma(n)) = \tau(\sigma(n))$, and by the property of τ:

$$a(\sigma(n)) \leq a(\tau(\sigma(n))) = a(\tau'(\sigma(n))) = a(\sigma(n+1))$$

This establishes that $a \circ \sigma$ is monotone.

Conclusion: We have constructed σ satisfying both required properties: σ is strictly increasing (a `Subseq`) and $a \circ \sigma$ is monotone. **QED**

5 The Bigger Picture

This theorem reveals a fundamental dichotomy in the behavior of sequences. Every sequence falls into one of two camps:

1. **Sequences with unbounded peaks:** These have a non-increasing (antitone) subsequence. (This is the content of the homework exercise.)

2. **Sequences with bounded peaks:** These have a non-decreasing (monotone) subsequence. (This is what we just proved.)

Together, these results will imply the **Bolzano-Weierstrass Theorem for sequences**: every infinite sequence has a subsequence that is Cauchy.

This result is foundational because:

- Combined with boundedness, it guarantees convergent subsequences

- It provides a tool for extracting regular behavior from seemingly chaotic sequences
- It connects to compactness in metric spaces
- It underlies many existence proofs in analysis

5.1 The Role of the if-then-else Construction

The conditional definition of τ' is essential to our proof. We needed a single function that:

- Grows faster than the identity everywhere
- Coincides with our witness function τ for indices beyond k
- Is well-defined for all indices, including those $\leq k$ where τ doesn't exist

The if-then-else construct elegantly handles both cases, allowing us to piece together a globally defined function from our partially defined witness. This technique of "patching together" functions is common in constructive analysis.

5.2 Why Orbits Work Here

The orbit construction $\sigma(n) = (\tau')^{[n]}(k+1)$ is perfect for this problem because:

1. It automatically gives us a strictly increasing sequence (by `Subseq_of_Iterate`)
2. It keeps us in the region where τ' coincides with τ (beyond k)
3. Each iteration moves to a point where the sequence value is strictly larger
4. The starting point $k+1$ is chosen to immediately enter the active region

This demonstrates the power of dynamic systems thinking in analysis: by viewing τ' as a dynamical system and studying its orbits, we extract the desired monotonic behavior.

6 Looking Ahead

In the homework, you'll prove the complementary result: sequences with unbounded peaks have antitone (non-increasing) subsequences. The proof strategy is similar but involves choosing the peaks themselves to form the subsequence.

Together with results about bounded sequences, these theorems form the foundation for understanding when sequences have convergent subsequences – a central question in real analysis.

The techniques developed here – using `choose` to extract witnesses from existence statements, conditional definitions to construct functions, and orbit iterations to build subsequences – are fundamental tools you'll use throughout analysis.

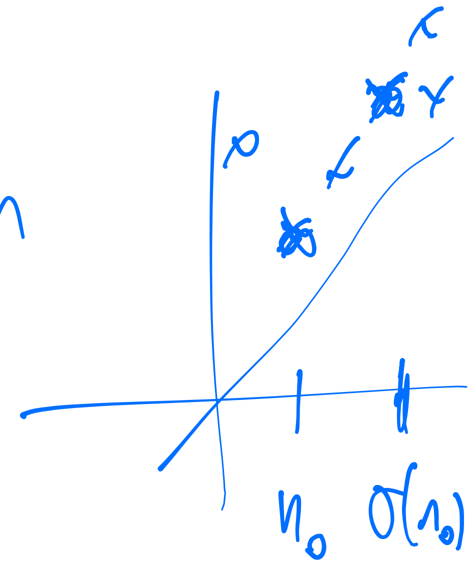
Warmup 1:

Thm Given $\sigma : \mathbb{N} \to \mathbb{N}$,

$n_0 : \mathbb{N}$

$h\sigma : \forall n,\ \sigma(n) > n$

Goal: $\mathrm{Subseq}\left(n \mapsto \sigma^{[n]}(n_0)\right)$



Recall τ is a Subseq : if $\forall i<j,\ \tau_i < \tau_j$,

To know that τ is a Subseq, it's

enough to know $\tau(n) < \tau(n+1)$.

Thm [Subseq_of_succ] $\forall n,\ \tau(n) < \tau(n+1) \to \mathrm{Subseq}\ \tau$

apply Subseq_of_succ

New Goal: $\forall n,\ \sigma^{[n]}(n_0) < \sigma^{[n+1]}(n_0)$.

Intro n

Recall Thm [succ_iterate]: $\sigma(\sigma^{[n]}) = \sigma^{[n+1]}$,

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\Vert$

$\qquad\qquad\qquad\qquad\qquad\qquad$ (By Def, $\sigma^{[n]}(\sigma)$)

rewrite ← show $\sigma(\sigma^{[n]}(1_0)) = \sigma^{[n+1]}(1_0)$ by apply succ_iterate

New goal: $\sigma^{[n]}(1_0) < \sigma(\sigma^{[n]}(1_0))$.

apply h_σ.

$\vdash \frac{m}{n} \le \varepsilon$.

show $m \le \varepsilon \times n$ by field_simp.

$\vdash m \le \varepsilon \times n$.

Next Thm: Given $p : \mathbb{N} \to \mathrm{Prop}$, h: the set of n for which p n holds is unbdd, i.e. $\forall N, \exists n > N, p\ n$.

Goal: produce a subseq σ along which $p(\sigma\ n)$ always holds.

~~h(N)~~

- choose τ hτBnd hτP using h.

defs: $\tau : \mathbb{N} \to \mathbb{N}$

hτBnd : $\forall N, \tau(N) > N$.

hτP : $\forall N, p(\tau(N))$.

- let $\sigma : \mathbb{N} \to \mathbb{N} :=$ fun $n \mapsto \tau^{(n)}(\tau\ 0)$.

- use σ New Goal: Subseq $\sigma \wedge \forall n, p(\sigma(n))$

- split_ands.

↳ apply subseq_of_iterate_τ StBdd (τ0)

↳ • intro n.   Goal: $p(\sigma(n))$.

• change $p(\tau^{[n+1]}(0))$

• rewrite [← show $\tau(\tau^{[n]}(0)) = \tau^{[n+1]}(0)$, by apply succ_iterate]

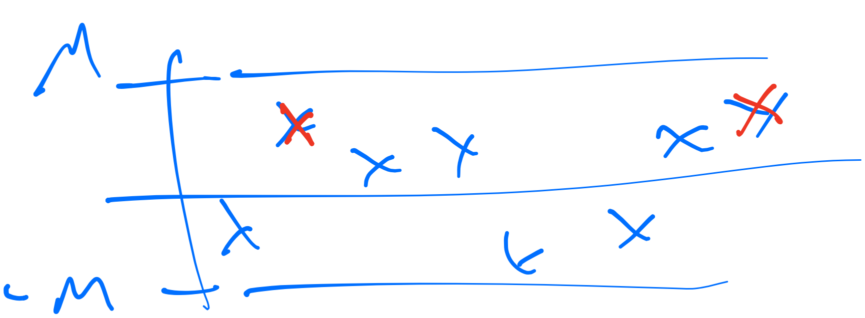
• apply StP $(\underbrace{\tau^{[n]}\ 0})$.

Last time: showed

• Monotone + Bdd ⇒ Cauchy.

What Bolzano–Weierstrass:

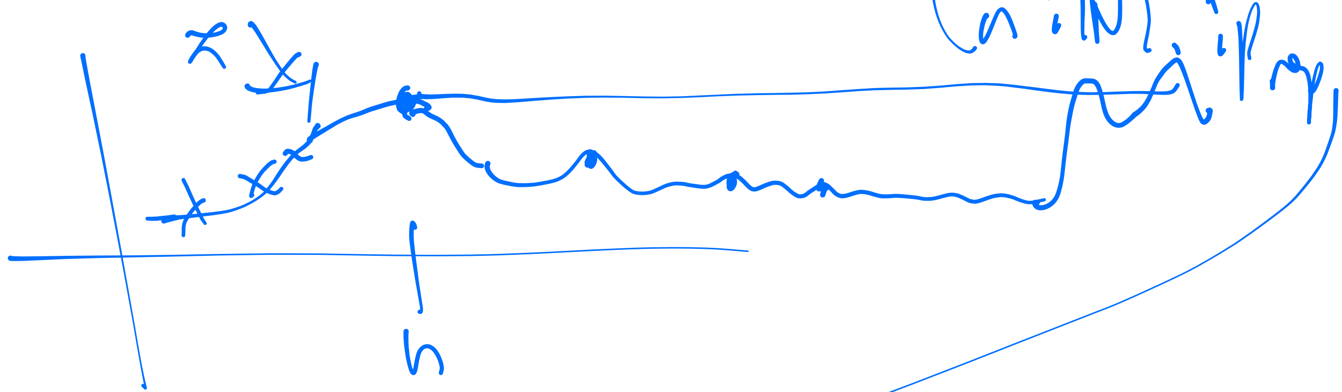
Bdd ⇒ ∃ subseq which is Cauchy.



• Likewise: Antitone (non-increasing) + BddBelow $\Rightarrow$ Cauchy.

<u>Claim:</u> Bdd $\Rightarrow$ $\exists$ subseq which
1) Monotone or Antitone.

Let's think about IsAPeak $(a : \mathbb{N} \to X)$ $(n : \mathbb{N})$, i.e. Prop



$P := \forall m \geq n,\ a_m \leq a_n.$

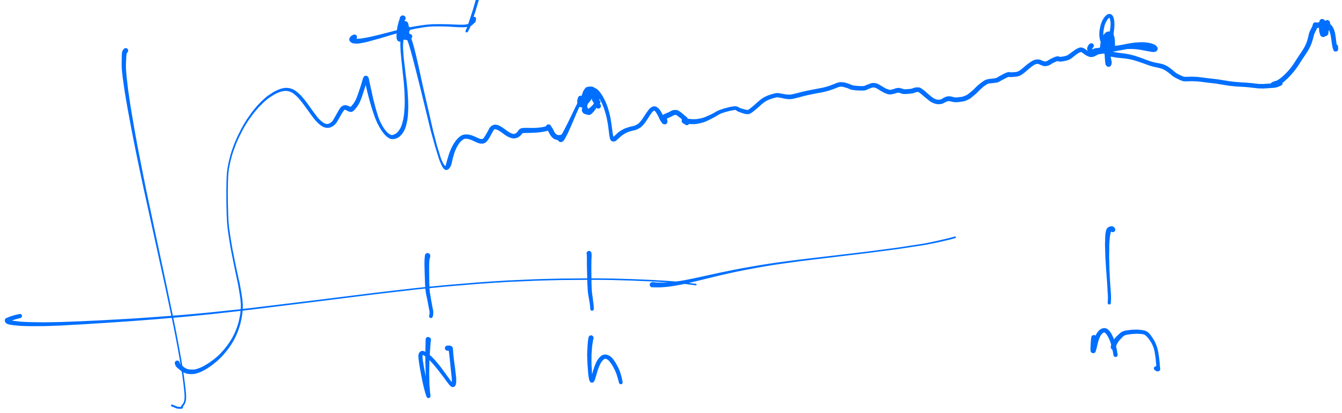
We say $(a : \mathbb{N} \to \mathbb{R})$ has UnBdd Peaks

if: $\forall n,\ \exists m > n,$ IsAPeak $a\ m$.

Thm (Monotone Subseq. of Bdd Peaks):

Given $a:\mathbb{N}\to X$, $h_a: \neg\,\text{HasUnbddPeaks}$,

Goal: $\exists\sigma$, Subseq $\wedge$ Monotone $(a\circ\sigma)$.



- change $\neg(\forall N, \exists n>N, \forall m>n, a_m\le a_n)$ at h_a
- pushing at h_a!

$h_a: \exists N, \forall n>N, \exists m>n, a_n<a_m$

- choose N h_N using h_a
- choose τ τ_{bnd} τ_a using h_N