An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 7: Uniqueness and Advanced Limit Theorems

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1 Uniqueness of Limits

One of the fundamental properties of convergent sequences is that they converge to a unique limit. This might seem obvious at first glance—after all, how could a sequence be getting arbitrarily close to two different numbers? But as with many intuitive facts in analysis, the rigorous proof requires careful reasoning with our epsilon-N definitions.

The key to proving uniqueness is proof by contradiction. We'll assume a sequence converges to two different limits L and M, and show this leads to an impossibility. The strategy involves choosing epsilon to be half the distance between L and M, then showing the sequence can't simultaneously stay that close to both limits.

1.1 New Tools

1.1.1 Proof by Contradiction: by_contra

The `by_contra` tactic allows us to prove a statement by assuming its negation and deriving a contradiction. The syntax is `by_contra h`, which adds a hypothesis h containing the negation of the current goal and changes the goal to `false`.

1.1.2 Absolute Value Positivity: abs_pos_of_nonzero

The theorem `abs_pos_of_nonzero` states that if $x \neq 0$, then $0 < |x|$. This is essential for working with distances between distinct points.

1.2 The Mathematical Statement

Theorem: If a sequence $a : \mathbb{N} \to \mathbb{R}$ converges to both L and M, then $L = M$.

1.3 Lean Solution

```
LimUnique (a : ℕ → ℝ) (L M : ℝ) (aToL : SeqLim a L) (
   aToM : SeqLim a M) :
    L = M := by
by_contra h
have f0 : L - M ≠ 0 := by bound
have f1 : 0 < |L - M| := by apply abs_pos_of_nonzero f0
have f2 : 0 < |L - M| / 2 := by bound
specialize aToL (|L - M| / 2) f2
specialize aToM (|L - M| / 2) f2
choose N1 hN1 using aToL
choose N2 hN2 using aToM
have f3 : N1 ≤ N1 + N2 := by bound
have f4 : N2 ≤ N1 + N2 := by bound
specialize hN1 (N1 + N2) f3
specialize hN2 (N1 + N2) f4
have f5 : |L - M| = |(L - a (N1+N2)) + (a (N1 + N2) - M)
   | := by ring_nf
have f6 : |(L - a (N1+N2)) + (a (N1 + N2) - M)| ≤
    |(L - a (N1+N2))| + |(a (N1 + N2) - M)| := by apply
       abs_add
have f7 : |(L - a (N1+N2))| = |-(a (N1+N2) - L)| := by
   ring_nf
have f8 : |-(a (N1+N2) - L)| = |(a (N1+N2) - L)| := by
   apply abs_neg
linarith [f5, f6, f7, f8, hN1, hN2]
```

1.4 Natural Language Proof

Proof: Suppose for contradiction that $L \neq M$. Then $L - M \neq 0$, so $|L - M| > 0$, and therefore $\varepsilon := |L - M|/2 > 0$.

Since $a(n) \to L$, there exists N_1 such that $|a(n) - L| < \varepsilon$ for all $n \geq N_1$. Since $a(n) \to M$, there exists N_2 such that $|a(n) - M| < \varepsilon$ for all $n \geq N_2$.

Let $N = N_1 + N_2$. Then for $n = N$, we have both $|a(N) - L| < \varepsilon$ and $|a(N) - M| < \varepsilon$.

Now observe that:

$$|L - M| = |(L - a(N)) + (a(N) - M)|$$

By the triangle inequality:

$$|L-M| \leq |L-a(N)|+|a(N)-M| = |a(N)-L|+|a(N)-M| < \varepsilon+\varepsilon = |L-M|$$

This gives us $|L - M| < |L - M|$, which is a contradiction. Therefore, $L = M$. **QED**

1.5 Why Uniqueness Matters

The uniqueness of limits is fundamental to the entire edifice of analysis. It allows us to speak of "the limit" of a sequence rather than "a limit," and ensures that our notion of convergence is well-defined. Without uniqueness, many standard theorems and techniques would fail or require substantial modification.

2 Eventually: Convergent Sequences Stay Near Their Limits

When a sequence converges to a nonzero limit, it doesn't just get arbitrarily close to that limit—it eventually stays away from zero as well. This "eventually bounded away from zero" property is crucial for many theorems involving quotients and reciprocals.

The intuition is straightforward: if a sequence is converging to some nonzero value L, then eventually the sequence terms must be at least half as large (in absolute value) as L itself. They can't simultaneously be approaching L and shrinking toward zero.

2.1 The Mathematical Statement

Theorem: If $a : \mathbb{N} \to \mathbb{R}$ converges to L with $L \neq 0$, then there exists N such that for all $n \geq N$, we have $|a(n)| \geq |L|/2$.

2.2 Strategic Approach

The key is to use the convergence condition with $\varepsilon = |L|/2$. Since $L \neq 0$, we have $|L| > 0$, so this epsilon is positive. The convergence condition then tells us that eventually $|a(n) - L| < |L|/2$, which by the reverse triangle inequality implies $|a(n)|$ is at least $|L|/2$.

2.3 Lean Solution

```
Statement EventuallyGeHalfLim (a : ℕ → ℝ) (L : ℝ) (aToL
   : SeqLim a L) (LneZero: L ≠ 0) :
    ∃ N, ∀ n ≥ N, |L| / 2 ≤ |a n| := by
specialize aToL (|L| / 2)
have : 0 < |L| := by apply abs_pos_of_nonzero LneZero
have : 0 < |L| / 2 := by bound
specialize aToL this
choose N hN using aToL
use N
intro n hn
specialize hN n hn
have l1 : |L| = |a n + (L - a n)| := by ring_nf
```

```
have l2 : |a n + (L - a n)| ≤ |a n| + |L - a n| := by
   apply abs_add
have l3 : |L - a n| = |-(a n - L)| := by ring_nf
have l4 : |-(a n - L)| = |(a n - L)| := by apply abs_neg
linarith [l1, l2, l3, l4, hN]
```

2.4 Natural Language Proof

Proof: Since $L \neq 0$, we have $|L| > 0$, and therefore $\varepsilon := |L|/2 > 0$.

By convergence of a to L, there exists N such that $|a(n) - L| < |L|/2$ for all $n \geq N$.

For any $n \geq N$, we have:

$$|L| = |a(n)+(L-a(n))| \leq |a(n)|+|L-a(n)| = |a(n)|+|a(n)-L| < |a(n)|+\frac{|L|}{2}$$

Rearranging gives:

$$|L| - \frac{|L|}{2} < |a(n)|$$

Therefore $|a(n)| > |L|/2$, which gives us $|a(n)| \geq |L|/2$. **QED**

2.5 Applications

This result is essential for proving that the reciprocal of a convergent sequence (with nonzero limit) is itself convergent. It ensures the denominators don't approach zero, which would cause the reciprocals to blow up.

3 Continuity of Absolute Value: Sequences of Absolute Values

The absolute value function behaves extremely well with respect to limits—if a sequence converges, then the sequence of absolute values converges to the absolute value of the limit. This is a manifestation of the continuity of the absolute value function.

The key property we'll use is that absolute value is Lipschitz continuous with constant 1, meaning $||x| - |y|| \leq |x - y|$ for all real numbers x and y. This inequality captures the idea that absolute value doesn't increase distances.

3.1 New Tools

3.1.1 Lipschitz Property: abs_Lipschitz

The theorem `abs_Lipschitz` states that for any real numbers x and y, we have $||x| - |y|| \leq |x - y|$. This is sometimes called the reverse triangle inequality for absolute values.

3.2 The Mathematical Statement

Theorem: If $a : \mathbb{N} \to \mathbb{R}$ converges to L, and $b : \mathbb{N} \to \mathbb{R}$ is defined by $b(n) = |a(n)|$ for all n, then b converges to $|L|$.

3.3 Lean Solution

```
Statement AbsLim (a : ℕ → ℝ) (L : ℝ) (aToL : SeqLim a L)
    (b : ℕ → ℝ) (bEqAbsa : ∀ n, b n = |a n|) :
    SeqLim b |L| := by
intro ε hε
specialize aToL ε hε
choose N hN using aToL
use N
intro n hn
specialize hN n hn
specialize bEqAbsa n
rewrite [bEqAbsa]
```

```
have : |(|a n|) - (|L|)| ≤ |a n - L| := by apply
   abs_Lipschitz
bound
```

3.4 Natural Language Proof

Proof: Let $\varepsilon > 0$ be given. Since $a(n) \to L$, there exists N such that $|a(n) - L| < \varepsilon$ for all $n \geq N$.

For any $n \geq N$, we have:

$$|b(n) - |L|| = ||a(n)| - |L|| \leq |a(n) - L| < \varepsilon$$

where we used the Lipschitz property of absolute value in the inequality. Therefore $b(n) \to |L|$. **QED**

3.5 Why This Matters

This theorem is a special case of a much more general principle: continuous functions preserve limits. The absolute value function is continuous everywhere, so it maps convergent sequences to convergent sequences. This principle extends to all continuous functions and is fundamental to mathematical analysis.

4 Reciprocals of Convergent Sequences

One of the most important limit theorems concerns reciprocals: if a sequence converges to a nonzero limit, then the sequence of reciprocals converges to the reciprocal of the limit. This result is crucial for proving theorems about quotients and rational functions.

The proof combines several techniques we've developed: showing the sequence stays bounded away from zero (so reciprocals don't blow up), carefully manipulating algebraic expressions involving fractions, and using the triangle inequality to control error terms.

4.1 New Tools

4.1.1 Absolute Value of Quotients: abs_div

For any real numbers x and y (with $y \neq 0$), we have $|x/y| = |x|/|y|$. This allows us to separate absolute values across division.

4.1.2 Nonzero from Positive Absolute Value: nonzero_of_abs_pos

If $0 < |x|$, then $x \neq 0$. This is useful for verifying that division is valid.

4.2 The Mathematical Statement

Theorem: If $a : \mathbb{N} \to \mathbb{R}$ converges to L with $L \neq 0$, and $b : \mathbb{N} \to \mathbb{R}$ is defined by $b(n) = 1/a(n)$ for all n, then b converges to $1/L$.

4.3 Strategic Approach

The challenge is that the expression $|1/a(n) - 1/L|$ involves reciprocals, which can be difficult to bound directly. The key steps are:

1. Use the EventuallyGeHalfLim theorem to ensure $|a(n)| \geq |L|/2$ eventually

2. Choose epsilon carefully: use $\varepsilon \cdot |L|^2/2$ when applying convergence of a to L

3. Algebraically simplify: $|1/a(n) - 1/L| = |(L - a(n))/(a(n) \cdot L)|$

4. Separate the absolute value using abs_div
5. Bound the denominator using our lower bound on $|a(n)|$
6. Use careful inequalities to show the result is less than ε

4.4 Lean Solution

```
Statement InvLim (a : ℕ → ℝ) (L : ℝ) (aToL : SeqLim a L)
    (LneZero : L ≠ 0) (b : ℕ →
ℝ) (bEqInva : ∀ n, b n = 1 / a n) :
    SeqLim b (1 / L) := by
choose NhalfL hNhalfL using EventuallyGeHalfLim a L aToL
    LneZero
intro ε hε
have : 0 < |L| := by apply abs_pos_of_nonzero LneZero
specialize aToL (ε * |L| * |L| / 2) (by bound)
choose Na hNa using aToL
use Na + NhalfL
intro n hn
specialize bEqInva n
rewrite [bEqInva]
have hnHalfL : NhalfL ≤ n := by bound
have hna : Na ≤ n := by bound
specialize hNhalfL n hnHalfL
specialize hNa n hna
have : 0 < |a n| := by bound
have : a n ≠ 0 := by apply nonzero_of_abs_pos this
have l1 : |1 / a n - 1 / L| = |(L - a n) / (a n * L)| :=
    by field_simp
have l2 :  |(L - a n) / (a n * L)| =  |(L - a n)| / |(a
   n * L)| := by apply abs_div
have l3 : |(L - a n)| / |(a n * L)| = |(L - a n)| / (|a
   n| * |L|) := by bound
have l4 : |L - a n| = |-(a n - L)| := by ring_nf
have l5 : |-(a n - L)| = |(a n - L)| := by apply abs_neg
have l6 : |a n - L| / (|a n| * |L|) < (ε * |L| * |L| /
   2) / (|a n| * |L|) := by field_simp; nlinarith
have l10 : |(L - a n)| / (|a n| * |L|) = |-(a n - L)| /
   (|a n| * |L|) := by rewrite [l4]; rfl
```

```
have l11 : |-(a n - L)| / (|a n| * |L|) = |(a n - L)| /
   (|a n| * |L|) := by rewrite [l5]; rfl
have l13 : ε * |L| * |L| / 2 / (|a n| * |L|) = ε * |L| /
    2 / |a n| := by field_simp
have l14 : ε * |L| / 2 / |a n| ≤ ε := by field_simp;
   bound
linarith [l1, l2, l3, l10, l11, l6, l13, l14]
```

4.5 Natural Language Proof

Proof: Let $\varepsilon > 0$ be given. Since $L \neq 0$, by the EventuallyGeHalfLim theorem, there exists N_1 such that $|a(n)| \geq |L|/2$ for all $n \geq N_1$.

Since $|L| > 0$, we have $\varepsilon \cdot |L|^2/2 > 0$. By convergence of a to L, there exists N_2 such that $|a(n) - L| < \varepsilon \cdot |L|^2/2$ for all $n \geq N_2$.

Let $N = N_1 + N_2$. For any $n \geq N$, we have $|a(n)| \geq |L|/2 > 0$, so $a(n) \neq 0$ and $b(n) = 1/a(n)$ is well-defined.

Now:

$$\begin{aligned}
|b(n) - 1/L| &= |1/a(n) - 1/L| \\
&= \left|\frac{L - a(n)}{a(n) \cdot L}\right| \\
&= \frac{|L - a(n)|}{|a(n)| \cdot |L|} \\
&= \frac{|a(n) - L|}{|a(n)| \cdot |L|} \\
&< \frac{\varepsilon \cdot |L|^2/2}{|a(n)| \cdot |L|} \\
&\leq \frac{\varepsilon \cdot |L|^2/2}{(|L|/2) \cdot |L|} \\
&= \frac{\varepsilon \cdot |L|^2/2}{|L|^2/2} \\
&= \varepsilon
\end{aligned}$$

Therefore $b(n) \to 1/L$. **QED**

4.6 Congratulations, Big Boss Defeated!

You've just completed one of the most challenging proofs in elementary analysis! The reciprocal limit theorem is a major milestone—you've proven that reciprocals preserve convergence (when the limit is nonzero).

4.7 What You Accomplished

This proof required you to orchestrate multiple sophisticated techniques:

- Using `EventuallyGeHalfLim` to ensure denominators stay bounded away from zero
- Choosing a carefully calibrated epsilon ($\varepsilon \cdot |L|^2/2$) to make the algebra work
- Manipulating complex fractional expressions with common denominators
- Applying `abs_div` to separate absolute values across division
- Chaining together a sequence of inequalities to reach the final bound

Each step built on the previous levels, showing how mathematical proofs are constructed from carefully assembled building blocks.

4.8 Applications and Extensions

With this theorem in hand, you now have a complete toolkit for limits of **rational functions**. Combined with earlier results on sums and products, you can now prove:

If $a(n) \to L$ and $c(n) \to M$ with $M \neq 0$, then $a(n)/c(n) \to L/M$.

The proof is straightforward: first show $1/c(n) \to 1/M$ using the reciprocal theorem you just proved, then use the product theorem to show $a(n) \cdot (1/c(n)) \to L \cdot (1/M) = L/M$.

This completes the fundamental arithmetic of limits: sums, products, and quotients. These are the building blocks for analyzing limits of polynomials, rational functions, and much more complex expressions throughout calculus and analysis.

4.9 Mastery of Technique

The reciprocal theorem showcases a crucial lesson in mathematical proof: sometimes the “right” epsilon isn’t the obvious choice. The expression $\varepsilon \cdot |L|^2/2$ might seem mysterious at first, but it’s precisely engineered to make the final inequalities work out. This kind of strategic thinking—working backwards from what you need to figure out what you should assume—is at the heart of mathematical problem-solving.

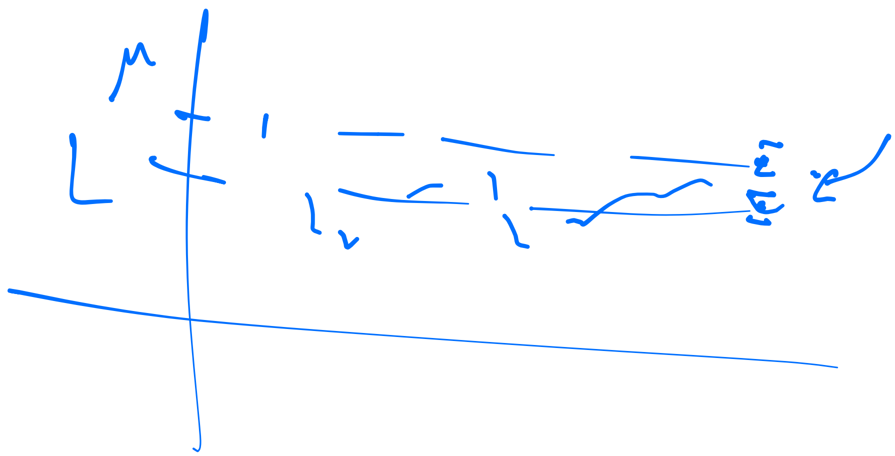
You’ve now mastered the essential techniques for proving limit theorems. Well done!

Thm: Limits Are Unique.

Assume a to L: $a \to L$, a to M: $a \to M$.

Goal: $\underbrace{L = M}$.

Idea: If $L \neq M$, set $\varepsilon = |L-M|/2$.



Idea: argue by contradiction.

In Lean: by_contra LneM. This will add a hypothesis LneM : $\neg\, L = M$, and change Goal to: False.

New theorem name: abs_pos_of_nonzero.

$$x \neq 0 \to 0 < |x|$$

Pf: By contradiction, assume (h) that $L \neq M$.
Then $L - M \neq 0$, and hence (abs_pos_of_nonzero) $0 < |L-M|$.
Clearly also $0 < |L-M|/2$. (f_1).

Specialize aToL ($|L-M|/2$) f_1 | aToL: $\exists N_1, \forall n \geq N_1 \ldots$

" aToM " f_1 | aToM: $\exists N_2, \forall n \geq N_2 \ldots$

Choose N_1 hN_1 using aToL | $hN_1 : \forall n \geq N_1, |a_n - L| < |L-M|/2$

Choose N_2 hN_2 using aToM | hN_2: " N_2 ... M ...

Idea: use $n = N_1 + N_2$. So:

have $f2$: $N_1 \leq N_1 + N_2$:= by linarith

have f_3 : $N_2 \leq$ - - - - -

Specialize hN_1 (N_1+N_2) f_2 | hN_1: $|a_{N_1+N_2} - L| < |L-M|/2$.

" hN_2 ... f_3 | hN_2: $|a_{(N_1+N_2)} - M| <$ ".

Think: f_6: $|L - M| = |(L - a_{(N_1+N_2)}) + (a_{(N_1+N_2)} - M)|$

f_7 — $\leq |L - a_{N_1+N_2}| + |a_{N_1+N_2} - M|$ (abs_add).

have f_4 : $|L - a_{N_1+N_2}| = |-(a_{N_1+N_2} - L)|$:= by ring_nf name f_7 : $|-(a_{N_1+N_2} - L)| = |a_{N_1+N_2} - L|$:= by apply abs_neg.

length $[f_6, f_7, hN_2, f_4, f_5, hN_1]$.

Thm (EventuallyGeHalfLim);

$a : \mathbb{N} \to \mathbb{R}$, $L : \mathbb{R}$,

aToL : $a \to L$, Lnezero : $L \neq 0$.

Goal: $\underbrace{\exists N, \forall n \geq N}_{\text{"eventually"}}, \frac{|L|}{2} \leq |a_n|$.

Note: If $L \neq 0$ and Goal: $\frac{|L|}{2} < |a_n|$, then is FALSE.

(Lnezero is necessary!).

Why: look at: 0, 0, 0, ... ,

pf: have f0: $0 < |L|$:= by apply abs_pos_of_nonzero
Lnezero.

have f_1 : $0 < |L|/2$:= by bound.

specialize aToL $(|L|/2)$ f_1

aToL : $\exists N, \forall n \geq N, |a_n - L| < |L|/2$.

choose N hN using aToL

hN : $\forall n \geq N, |a_n - L| < |L|/2$.
Goal: $\forall n \geq N, \frac{|L|}{2} \leq |a_n|$

use N

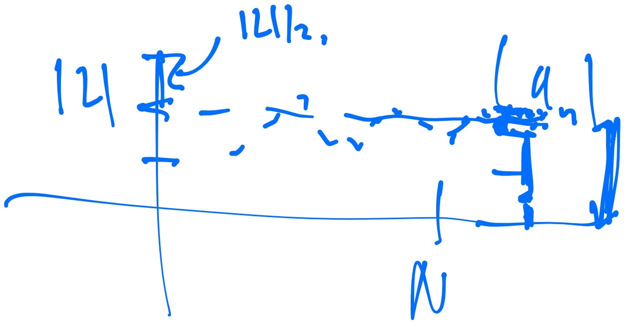
intro n hn

$h_n : n \geq N$.

specialize hN n hn.

hN : $|a_n - L| < |L|/2$.
Goal: $\frac{|L|}{2} \leq |a_n|$

$\frac{|L|}{2} = \frac{1}{2}(|a_n + (L - a_n)|)$.



Idea:

f_2: $|L| = |(L - a_n) + a_n|$

f_3: $\_ \leq |(L - a_n)| + |a_n|$

f_4: $|L - a_n| = |-(a_n - L)|$

f_5: $|-(a_n - L)| = |a_n - L| < \frac{|L|}{2}$.

$|a_n| \geq |L| - |a_n - L|$
$> |L| - \frac{|L|}{2}$.
$= |L|/2$.

Thm (AbsLim): $a \xrightarrow{\text{aToL}} L$ & $b \overset{\text{bEqAbsa}}{=} |a|$

Then $b \to |L|$.

Let $\varepsilon > 0$ be given (intro ε hε). (hε)

Specialize aToL ε hε.

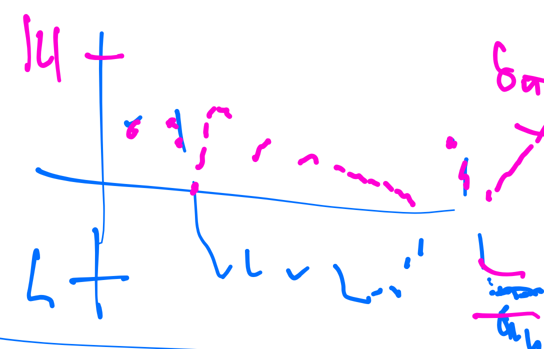
Choose N hN using aToL

$\boxed{\text{hN}: \forall n \geq N, |a_n - L| < \varepsilon.}$

Use N

$\boxed{\text{Goal}: \forall n \geq N, |b_n - |L|| < \varepsilon}$

Let $n \geq N$ be given. (hn)



Specialize hN n hn.

Specialize bEqAbsa n.

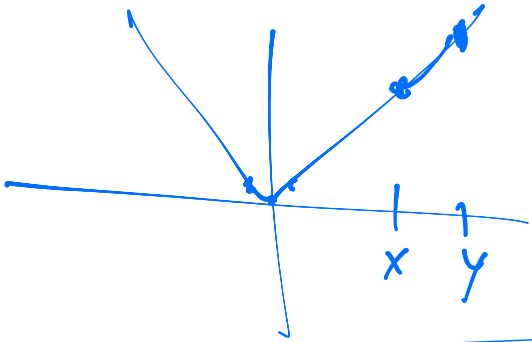
rewrite [bEqAbsa]

$\boxed{\begin{array}{l}\text{hN}: |a_n - L| < \varepsilon \\ \text{Goal}: ||a_n| - |L|| < \varepsilon.\end{array}}$

<u>Aside</u>: A function f is called Lipschitz

if: $|f(x) - f(y)| \leq C \cdot |x - y|$.

Exercise: $|\cdot|$ is Lipschitz with $C = 1$.



abs_Lipschitz:

$$\big||x|-|y|\big| \le |x-y|.$$

have f : $\big||a_n|-|L|\big| \le |a_n - L|$:= by apply abs_Lipschitz

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Thm (InvLim) $a \to L$ & $L \neq 0$ $\Rightarrow$ $1/a \to 1/L$.

Say we know $|a_n - L| < \delta$??.

Goal: $\left|\frac{1}{a_n} - \frac{1}{L}\right| < \varepsilon$

$$\left|\frac{1}{a_n} - \frac{1}{L}\right| = \left|\frac{L - a_n}{a_n L}\right|$$

$$= \frac{|a_n - L|}{|a_n|\,|L|} < \varepsilon,$$

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$,

abs_div

Next time: Idea: make n

so large that $|a_n| \geq |L|/2$.