CONTENTS

I. NOTES ON CURVED-SKY QE RESPONSES ETC.

Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [?] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}(\boldsymbol{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_\mathrm{i}}\; {}_{s_\mathrm{i}}\bar{X}_{\ell m}\; {}_{s_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\right)\left(\sum_{\ell m} w_\ell^{t_\mathrm{i}}\; {}_{t_\mathrm{i}}\bar{X}_{\ell m}\; {}_{t_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\right) \tag{1.1}$$

where $s_\mathrm{i}, t_\mathrm{i}$ are input spins, $w_\ell^{s_\mathrm{i}}, w_\ell^{t_\mathrm{i}}$ associated weights, and $s_\mathrm{o}, t_\mathrm{o}$ outputs spins. The maps ${}_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$$_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{1.2}$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields $\bar{X}$ of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G^s_{LM} &= -\tfrac{1}{2}\left({}_{|s|}d_{LM} + (-1)^s\; {}_{-|s|}d_{LM}\right) \\ C^s_{LM} &= -\tfrac{1}{2i}\left({}_{|s|}d_{LM} - (-1)^s\; {}_{-|s|}d_{LM}\right). \end{aligned}$$

A. Semi-analytical QE $N_L^{(0)}$ calculation

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1.1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_\mathrm{i}, t_\mathrm{i}, w^{s_\mathrm{i}})$ collectively describes the in and out spins and weight function, and similarly for t, u and v. Let the response function $\mathcal{R}_L^{st,uv}$ be defined as

$$(-1)^{t_\mathrm{o}+v_\mathrm{o}}\mathcal{R}_L^{st,uv} \equiv 2\pi\int_{-1}^{1} d\mu\, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_\mathrm{o}-v_\mathrm{o}, s_\mathrm{o}+u_\mathrm{o}}(\mu) \tag{1.3}$$

where ξ are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_\mathrm{i}} w_\ell^{t_\mathrm{i}} \bar{C}_\ell^{s_\mathrm{i}t_\mathrm{i}} d^\ell_{-t_\mathrm{o}, s_\mathrm{o}}(\mu) \text{ with } \bar{C}_\ell^{s_\mathrm{i}t_\mathrm{i}} \equiv \left\langle {}_{s_\mathrm{i}}\bar{X}_{\ell m}\; {}_{t_\mathrm{i}}\bar{X}^*_{\ell m}\right\rangle \tag{1.4}$$

and $d^\ell_{mm'}$ are Wigner small d-matrices. Then

$$\left\langle G_{LM}^{s_\mathrm{o}+t_\mathrm{o}} G_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}}\right\rangle = \frac{1}{4}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1 + (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right) + (-1)^{s_\mathrm{i}+t_\mathrm{i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1 + (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right)\right] \tag{1.5}$$

$$\left\langle C_{LM}^{s_\mathrm{o}+t_\mathrm{o}} C_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}}\right\rangle = -\frac{1}{4}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1 + (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right) - (-1)^{s_\mathrm{i}+t_\mathrm{i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1 + (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right)\right] \tag{1.6}$$

$$\left\langle G_{LM}^{s_\mathrm{o}+t_\mathrm{o}} C_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}}\right\rangle = \frac{1}{4i}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1 - (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right) - (-1)^{s_\mathrm{i}+t_\mathrm{i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1 + (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right)\right] \tag{1.7}$$

I dont understand the resulting sign and GC spectrum for (the irrelevant case of) odd total input spin. Should nt that always be 1? TTTT checked OK *a. Sketchy derivation to cleanup* For this we need a result using the spin-weight spherical harmonic theorem. Define $\mathcal{R}_L^{st,uv}$ through

$$\begin{aligned}\mathcal{R}^{st,uv}(\boldsymbol{n},\boldsymbol{n}') \equiv& (-1)^{t_o+v_o}\left(\sum_{\ell m} g_\ell^{s_i} g_\ell^{t_i} C_\ell^{s_i t_i}\ {}_{s_o}Y_{\ell m}(\boldsymbol{n})\ {}_{-t_o}Y^*_{\ell m}(\boldsymbol{n}')\right)\left(\sum_{\ell m} g_\ell^{u_i} g_\ell^{v_i} C_\ell^{u_i v_i}\ {}_{u_o}Y_{\ell m}(\boldsymbol{n})\ {}_{-v_o}Y^*_{\ell m}(\boldsymbol{n}')\right)\\ \equiv& (-1)^{t_o+v_o}\sum_{LM}\mathcal{R}_L^{stuv}\ {}_{s_o+u_o}Y_{LM}(\boldsymbol{n})\ {}_{-u_o-v_o}Y^*_{LM}(\boldsymbol{n}')\end{aligned} \tag{1.8}$$

Then we can write

$$\left\langle {}_{s_o+t_o}\hat{d}(\boldsymbol{n})\ {}_{u_o+v_o}\hat{d}(\boldsymbol{n}')\right\rangle = \mathcal{R}^{su,tv}(\boldsymbol{n},\boldsymbol{n}') + \mathcal{R}^{sv,tu}(\boldsymbol{n},\boldsymbol{n}') \tag{1.9}$$

Taking the harmonic transform, we get

$$\left\langle {}_{s_o+t_o}\hat{d}_{LM}\ {}_{u_o+v_o}\hat{d}_{L'M'}\right\rangle = (-1)^M\delta_{M,-M'}\delta_{L,L'}\left(\mathcal{R}_L^{su,tv}+\mathcal{R}_L^{sv,tu}\right) \tag{1.10}$$

In general we have

$$\begin{aligned} G^s_{LM} &= -\tfrac{1}{2}\left({}_sd_{LM} + (-1)^s\ {}_{-s}d_{LM}\right) \quad (s\geq 0)\\ C^s_{LM} &= -\tfrac{1}{2i}\left({}_sd_{LM} - (-1)^s\ {}_{-s}d_{LM}\right) \quad (s\geq 0).\end{aligned}$$

The estimator for ${}_{-s_o-t_o}\hat{d}$ is the same as ${}_{s_o+t_o}\hat{d}$ with all spin signs flipped, and with an overall sign $(-1)^{s_o+s_i+t_o+t_i}$. The out-spins part gets canceled by the sign $(-1)^s$ in the above equation. Hence,

$$\begin{aligned}(-1)^M\delta_{M,-M'}\delta_{L,L'}\left\langle G^{s_o+t_o}_{LM}G^{u_o+v_o}_{L'M'}\right\rangle\cdot 4 &= \mathcal{R}_L^{su,tv}+\mathcal{R}_L^{sv,tu}+(-1)^{s_i+t_i}\left(\mathcal{R}_L^{-su,-tv}+\mathcal{R}_L^{-sv,-tu}\right)\\ +(-1)^{u_i+v_i}\left(\mathcal{R}_L^{s-u,t-v}+\mathcal{R}_L^{s-v,t-u}\right)&+(-1)^{s_i+t_i+u_i+v_i}\left(\mathcal{R}_L^{-s-u,-t-v}+\mathcal{R}_L^{-s-v,-t-u}\right)\quad (s_o+t_o>=0, u_o+v_o>=0)\end{aligned} \tag{1.11}$$

Since $\mathcal{R}$ is invariant under the simultaneous sign-flip of all spins, we can also write this as:

$$\left\langle G^{s_o+t_o}_{LM}G^{*,u_o+v_o}_{LM}\right\rangle = \frac{1}{4}\left[\left(\mathcal{R}_L^{su,tv}+R_L^{sv,tu}\right)\left(1+(-1)^{s_i+t_i+u_i+v_i}\right)+(-1)^{s_i+t_i}\left(\mathcal{R}_L^{-su,tv}+\mathcal{R}_L^{-sv,tu}\right)\left(1+(-1)^{s_i+t_i+u_i+v_i}\right)\right] \tag{1.12}$$

$$\left\langle C^{s_o+t_o}_{LM}C^{*,u_o+v_o}_{LM}\right\rangle = -\frac{1}{4}\left[\left(\mathcal{R}_L^{su,tv}+R_L^{sv,tu}\right)\left(1+(-1)^{s_i+t_i+u_i+v_i}\right)-(-1)^{s_i+t_i}\left(\mathcal{R}_L^{-su,tv}+\mathcal{R}_L^{-sv,tu}\right)\left(1+(-1)^{s_i+t_i+u_i+v_i}\right)\right] \tag{1.13}$$

$$\left\langle G^{s_o+t_o}_{LM}C^{*,u_o+v_o}_{LM}\right\rangle = \frac{1}{4i}\left[\left(\mathcal{R}_L^{su,tv}+R_L^{sv,tu}\right)\left(1-(-1)^{s_i+t_i+u_i+v_i}\right)+(-1)^{s_i+t_i}\left(\mathcal{R}_L^{-su,tv}+\mathcal{R}_L^{-sv,tu}\right)\left(1-(-1)^{s_i+t_i+u_i+v_i}\right)\right] \tag{1.14}$$

?

More details of $\mathcal{R}_L$

Recall the spin-weight addition theorem:

$$\sum_m {}_sY^*_{\ell m}(\boldsymbol{n}')\ {}_tY_{\ell m}(\boldsymbol{n}) = \sqrt{\frac{2\ell+1}{4\pi}}e^{-it\gamma}\ {}_tY_{\ell,-s}(\beta,\alpha). \tag{1.15}$$

Hence

$$\mathcal{R}^{st,uv}(\boldsymbol{n},\boldsymbol{n}') = (-1)^{t_o+v_o}e^{-is_o\gamma-iu_o\gamma}\left(\sum_\ell\sqrt{\frac{2\ell+1}{4\pi}}g^{s_i}g^{t_i}C_\ell^{s_i t_i}\ {}_{s_o}Y_{\ell t_o}(\beta,\alpha)\right)\left(\sum_\ell\sqrt{\frac{2\ell+1}{4\pi}}g^{u_i}g^{v_i}C_\ell^{u_i v_i}\ {}_{u_o}Y_{\ell v_o}(\beta,\alpha)\right) \tag{1.16}$$

The product of the brackets is a spin $s_{\rm o} + u_{\rm o}$ function. Defining its spin weight coefficients as $\mathcal{R}_L$ we get the relation claimed above. What are these coefficients?

$$\mathcal{R}_L^{st,uv} \equiv (-1)^{t_{\rm o}+v_{\rm o}} \sqrt{\frac{4\pi}{2\ell+1}} \int d^2n \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_{\rm i}} g^{t_{\rm i}} C_\ell^{s_{\rm i} t_{\rm i}}\ {}_{s_{\rm o}}Y_{\ell t_{\rm o}}(\boldsymbol{n})\right) \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_{\rm i}} g^{v_{\rm i}} C_\ell^{u_{\rm i} v_{\rm i}}\ {}_{u_{\rm o}}Y_{\ell v_{\rm o}}(\boldsymbol{n})\right) {}_{s_{\rm o}+u_{\rm o}}Y^*_{L,t_{\rm o}+v_{\rm o}}(\boldsymbol{n}) \tag{1.17}$$

Using

$$\boxed{{}_sY_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}(-1)^m e^{im\phi} d^\ell_{-ms}(\theta)} \tag{1.18}$$

The above thing is invariant if all signs are flipped at the same time. we get

$$\boxed{(-1)^{t_{\rm o}+v_{\rm o}}\mathcal{R}_L^{st,uv} \equiv 2\pi \int_{-1}^{1} d\mu\, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_{\rm o}-v_{\rm o}, s_{\rm o}+u_{\rm o}}(\mu), \text{ with } \xi^{st}(\mu) \equiv \sum_\ell \frac{2\ell+1}{4\pi} g_\ell^{s_{\rm i}} g_\ell^{t_{\rm i}} C_\ell^{s_{\rm i} t_{\rm i}} d^\ell_{-t_{\rm o}, s_{\rm o}}(\mu)} \tag{1.19}$$

B. QE responses calculation