Notes on curved-sky QE responses etc.

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Contents

JC: Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [**?**] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$_{s_o+t_o}\hat{d}(\boldsymbol{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_i s_o} \,_{s_i}\bar{X}_{\ell m} \,_{s_o}Y_{\ell m}(\boldsymbol{n}) \right) \left(\sum_{\ell m} w_\ell^{t_i t_o} \,_{t_i}\bar{X}_{\ell m} \,_{t_o}Y_{\ell m}(\boldsymbol{n}) \right) \tag{1}$$

where s_i, t_i are input spins, s_o, t_o outputs spins, and $w_\ell^{s_o s_i}, w_\ell^{t_i t_o}$ associated weights. The maps $_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$$_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad _{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m} \right). \tag{2}$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields $\bar{X}$ of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G_{LM}^s &= -\tfrac{1}{2} \left(\,_{|s|}d_{LM} + (-1)^s \,_{-|s|}d_{LM} \right) \\ C_{LM}^s &= -\tfrac{1}{2i} \left(\,_{|s|}d_{LM} - (-1)^s \,_{-|s|}d_{LM} \right). \end{aligned}$$

0.1 Semi-analytical QE $N_L^{(0)}$ calculations

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_i, s_o, w^{s_i s_o})$ collectively describes the in and out spins and weight function, and similarly for t, u and v. Let the response function $N_L^{st,uv}$ be defined as

$$(-1)^{t_o+v_o+t_i+v_i} N_L^{st,uv} \equiv 2\pi \int_{-1}^{1} d\mu \, \xi^{st}(\mu) \, \xi^{uv}(\mu) \, d_{-t_o-v_o, s_o+u_o}^L(\mu) \tag{3}$$

where ξ are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_\ell \left(\frac{2\ell+1}{4\pi} \right) w_\ell^{s_i s_o} w_\ell^{t_i t_o} \bar{C}_\ell^{s_i t_i} d_{-t_o, s_o}^\ell(\mu) \text{ with } \bar{C}_\ell^{s_i t_i} \equiv \left\langle \,_{s_i}\bar{X}_{\ell m} \,_{t_i}\bar{X}_{\ell m}^* \right\rangle \tag{4}$$

and $d^\ell_{mm'}$ are Wigner small d-matrices. Then

$$
\boxed{
\begin{aligned}
\left\langle G_{LM}^{s_\mathrm{o}+t_\mathrm{o}} G_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}} \right\rangle &= \frac{1}{2}\left[(-1)^{s_\mathrm{o}+t_\mathrm{o}} N_L^{-su,-tv} + (-1)^{u_\mathrm{o}+v_\mathrm{o}} N_L^{s-v,t-u} + N_L^{su,tv} + N_L^{sv,tu} \right] \\
\left\langle C_{LM}^{s_\mathrm{o}+t_\mathrm{o}} C_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}} \right\rangle &= \frac{1}{2}\left[(-1)^{s_\mathrm{o}+t_\mathrm{o}} N_L^{-su,-tv} + (-1)^{u_\mathrm{o}+v_\mathrm{o}} N_L^{s-v,t-u} - N_L^{su,tv} - N_L^{sv,tu} \right] \\
\left\langle G_{LM}^{s_\mathrm{o}+t_\mathrm{o}} C_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}} \right\rangle &= 0
\end{aligned}
}
\tag{5}
$$

JC: Hmm, implementation still not right. overall minus sign and only O(1) Ok. (typo, two minus signs in first term)!

0.2 QE response calculation

Let the estimator respond to a source (with spin-weight r) as follows

$$
{}_sX(\boldsymbol{n}) = {}_r\alpha(\boldsymbol{n})\left(\sum_{\ell m} R_\ell^{rs}\, {}_sX_{\ell m}\, {}_{s-r}Y_{\ell m}(\boldsymbol{n}) \right) + {}_{-r}\alpha(\boldsymbol{n})\left(\sum_{\ell m} R_\ell^{-rs}\, {}_sX_{\ell m}\, {}_{s+r}Y_{\ell m}(\boldsymbol{n}) \right)
\tag{6}
$$

for some harmonic space responses R_ℓ^{rs}. Then the gradient and curl responses of estimator (1) are

$$
\boxed{
\begin{aligned}
\mathcal{R}_L^{gg} &= R_L^{ts,-r} + R_L^{st,-r} + (-1)^r\left(R_L^{ts,r} + R_L^{st,r} \right) \\
\mathcal{R}_L^{cc} &= R_L^{ts,-r} + R_L^{st,-r} - (-1)^r\left(R_L^{ts,r} + R_L^{st,r} \right) \\
\mathcal{R}_L^{gc} &= 0 = \mathcal{R}_L^{cg},
\end{aligned}
}
\tag{7}
$$

with JC: filters in this. This is actually $\bar{X}\bar{X}$ spectrum not XX. TT Ok, finish the others.

$$
R_L^{st,r} = 2\pi \int_{-1}^{1} d\mu\, \Xi^{st,r}(\mu)\, \Xi^t(\mu)\, d^L_{-r,s_\mathrm{o}+t_\mathrm{o}}(\mu)
\tag{8}
$$

where

$$
\Xi^{st,r}(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi}\right) C_\ell^{s_\mathrm{i},-t_\mathrm{i}} R_\ell^{r-t_\mathrm{i}} w_\ell^{s_\mathrm{i},s_\mathrm{o}} d^\ell_{t_\mathrm{i}-r,s_\mathrm{o}}(\mu) \quad \text{and} \quad \Xi^t(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{t_\mathrm{i},t_\mathrm{o}} d^\ell_{-t_\mathrm{i},t_\mathrm{o}}(\mu), \textit{wrong} \quad C_\ell^{s_\mathrm{i},t_\mathrm{i}} \equiv \left\langle {}_{s_\mathrm{i}}X_{\ell m}\, {}_{t_\mathrm{i}}X^\dagger_{\ell m} \right\rangle
\tag{9}
$$

1 Sketchy derivation to cleanup

For this we need a result using the spin-weight spherical harmonic theorem. Define $N_L^{st,uv}$ through

$$N^{st,uv}(\boldsymbol{n},\boldsymbol{n}') \equiv (-1)^{t_o+v_o+t_i+v_i} \left(\sum_{\ell m} g_\ell^{s_i} g_\ell^{t_i} C_\ell^{s_i t_i} {}_{s_o}Y_{\ell m}(\boldsymbol{n}) {}_{-t_o}Y_{\ell m}^*(\boldsymbol{n}') \right) \left(\sum_{\ell m} g_\ell^{u_i} g_\ell^{v_i} C_\ell^{u_i v_i} {}_{u_o}Y_{\ell m}(\boldsymbol{n}) {}_{-v_o}Y_{\ell m}^*(\boldsymbol{n}') \right)$$

$$\equiv (-1)^{t_o+v_o+t_i+v_i} \sum_{LM} N_L^{stuv} {}_{s_o+u_o}Y_{LM}(\boldsymbol{n}) {}_{-t_o-v_o}Y_{LM}^*(\boldsymbol{n}') \tag{10}$$

Then we can write

$$\left\langle {}_{s_o+t_o}\hat{d}(\boldsymbol{n}) {}_{u_o+v_o}\hat{d}(\boldsymbol{n}') \right\rangle = N^{su,tv}(\boldsymbol{n},\boldsymbol{n}') + N^{sv,tu}(\boldsymbol{n},\boldsymbol{n}') \tag{11}$$

Taking the harmonic transform, we get

$$\left\langle {}_{s_o+t_o}\hat{d}_{LM} {}_{u_o+v_o}\hat{d}_{L'M'} \right\rangle = (-1)^M \delta_{M,-M'} \delta_{L,L'} \left(N_L^{su,tv} + N_L^{sv,tu} \right) (-1)^{u_o+v_o} \tag{12}$$

or

$$\left\langle {}_{s_o+t_o}\hat{d}_{LM} {}_{u_o+v_o}\hat{d}_{L'M'}^* \right\rangle = \delta_{MM'} \delta_{LL'} \left(N_L^{su,tv} + N_L^{sv,tu} \right) \tag{13}$$

In general we have

$$G_{LM}^s = -\tfrac{1}{2} \left({}_s d_{LM} + (-1)^s {}_{-s} d_{LM} \right) \quad (s \geq 0)$$
$$C_{LM}^s = -\tfrac{1}{2i} \left({}_s d_{LM} - (-1)^s {}_{-s} d_{LM} \right) \quad (s \geq 0).$$

The estimator for ${}_{-s_o-t_o}\hat{d}$ is the same as ${}_{s_o+t_o}\hat{d}$ with all spin signs flipped, and with an overall sign $(-1)^{s_o+s_i+t_o+t_i}$. The out-spins part gets canceled by the sign $(-1)^s$ in the above equation. JC: No. better to request $w_\ell^{-s_i,-s_o} = (-1)^{s_o+si} w_\ell^{s_i,s_o}$ so that ${}_{-s_o-t_o}\hat{d} = {}_{s_o+t_o}\hat{d}^*$? Thats what we do nowHence,

$$\delta_{MM'}\delta_{LL'} \left\langle G_{LM}^{s_o+t_o} G_{L'M'}^{*,u_o+v_o} \right\rangle \cdot 4 = N_L^{su,tv} + N_L^{sv,tu} + (-1)^{s_i+t_i} \left(N_L^{-su,-tv} + N_L^{-sv,-tu} \right)$$
$$+ (-1)^{u_i+v_i} \left(N_L^{s-u,t-v} + N_L^{s-v,t-u} \right) + (-1)^{s_i+t_i+u_i+v_i} \left(N_L^{-s-u,-t-v} + N_L^{-s-v,-t-u} \right) \quad (s_o + t_o >= 0, u_o + v_o >= 0) \tag{14}$$

Since N is invariant under the simultaneous sign-flip of all spins JC: Wrong, also with sign-weighted weights: $\bar{C}^{-s,-t} = (-1)^{s+t} C^{t,s} = (-1)^{s+t} C^{s,t}(s,t \geq 0)$, so $N^{st,uv}$ takes a sign $(-1)^{s_i+t_i+u_i+v_i}$ and this solves the problem, we can also write this as:

$$\left\langle G_{LM}^{s_o+t_o} G_{LM}^{*,u_o+v_o} \right\rangle = \frac{1}{2} \left[(N_L^{su,tv} + R_L^{sv,tu}) + (-1)^{s_o+t_o} (N_L^{-su,tv} + N_L^{-sv,tu}) \right] \tag{15}$$

$$\left\langle C_{LM}^{s_o+t_o} C_{LM}^{*,u_o+v_o} \right\rangle = -\frac{1}{2} \left[(N_L^{su,tv} + R_L^{sv,tu}) - (-1)^{s_o+t_o} (N_L^{-su,tv} + N_L^{-sv,tu}) \right] \tag{16}$$

$$\left\langle G_{LM}^{s_o+t_o} C_{LM}^{*,u_o+v_o} \right\rangle = 0 \tag{17}$$

Pfeew

2 More details

Recall the spin-weight addition theorem:

$$\sum_m {}_s Y_{\ell m}^*(\boldsymbol{n}') {}_t Y_{\ell m}(\boldsymbol{n}) = \sqrt{\frac{2\ell+1}{4\pi}} e^{-it\gamma} {}_t Y_{\ell,-s}(\beta,\alpha). \tag{18}$$

Hence

$$N^{st,uv}(\boldsymbol{n}, \boldsymbol{n}') = (-1)^{t_{\mathrm{o}}+v_{\mathrm{o}}+t_{\mathrm{i}}+v_{\mathrm{i}}} e^{-is_{\mathrm{o}}\gamma - iu_{\mathrm{o}}\gamma} \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_{\mathrm{i}}} g^{t_{\mathrm{i}}} C_\ell^{s_{\mathrm{i}}t_{\mathrm{i}}} {}_{s_{\mathrm{o}}}Y_{\ell t_{\mathrm{o}}}(\beta, \alpha) \right) \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_{\mathrm{i}}} g^{v_{\mathrm{i}}} C_\ell^{u_{\mathrm{i}}v_{\mathrm{i}}} {}_{u_{\mathrm{o}}}Y_{\ell v_{\mathrm{o}}}(\beta, \alpha) \right)$$
(19)

The product of the brackets is a spin $s_{\mathrm{o}} + u_{\mathrm{o}}$ function. Defining its spin weight coefficients as N_L we get the relation claimed above. What are these coefficients?

$$N_L^{st,uv} \equiv (-1)^{t_{\mathrm{o}}+v_{\mathrm{o}}+t_{\mathrm{i}}+v_{\mathrm{i}}} \sqrt{\frac{4\pi}{2\ell+1}} \int d^2 n \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_{\mathrm{i}}} g^{t_{\mathrm{i}}} C_\ell^{s_{\mathrm{i}}t_{\mathrm{i}}} {}_{s_{\mathrm{o}}}Y_{\ell t_{\mathrm{o}}}(\boldsymbol{n}) \right) \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_{\mathrm{i}}} g^{v_{\mathrm{i}}} C_\ell^{u_{\mathrm{i}}v_{\mathrm{i}}} {}_{u_{\mathrm{o}}}Y_{\ell v_{\mathrm{o}}}(\boldsymbol{n}) \right) {}_{s_{\mathrm{o}}+u_{\mathrm{o}}}Y^*_{L,t_{\mathrm{o}}+v_{\mathrm{o}}}$$
(20)

Using

$$\boxed{ {}_sY_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} (-1)^m e^{im\phi} d^\ell_{-ms}(\theta) }$$
(21)

The above thing is invariant if all signs are flipped at the same time. we get

$$\boxed{ (-1)^{t_{\mathrm{o}}+v_{\mathrm{o}}+t_{\mathrm{i}}+v_{\mathrm{i}}} N_L^{st,uv} \equiv 2\pi \int_{-1}^1 d\mu \, \xi^{st}(\mu) \, \xi^{uv}(\mu) \, d^L_{-t_{\mathrm{o}}-v_{\mathrm{o}}, s_{\mathrm{o}}+u_{\mathrm{o}}}(\mu), \text{ with } \xi^{st}(\mu) \equiv \sum_\ell \frac{2\ell+1}{4\pi} g^{s_{\mathrm{i}}}_\ell g^{t_{\mathrm{i}}}_\ell C_\ell^{s_{\mathrm{i}}t_{\mathrm{i}}} d^\ell_{-t_{\mathrm{o}}, s_{\mathrm{o}}}(\mu) }$$
(22)

JC: Remember now the weights w (or g...) changes $(-1)^{s_{\mathrm{i}}+s_{\mathrm{o}}}$ under sign flip of the two spins, and $\bar{C}^{st}$ takes a $(-1)^{s+t}$.
So $N^{-s-t,-u-v} = (-1)^{s_{\mathrm{i}}+t_{\mathrm{i}}+u_{\mathrm{i}}+v_{\mathrm{i}}+s_{\mathrm{i}}+s_{\mathrm{o}}+t_{\mathrm{i}}+t_{\mathrm{o}}+u_{\mathrm{i}}+u_{\mathrm{o}}+v_{\mathrm{i}}+v_{\mathrm{o}}} = (-1)^{s_{\mathrm{o}}+t_{\mathrm{o}}+u_{\mathrm{o}}+v_{\mathrm{o}}} N^{st,uv}$

2.1 QE responses calculation

JC: should define the response at the covariance matrix level...The covariance matrices is

$$\langle {}_sX(\boldsymbol{n}) \, {}_tX^*(\boldsymbol{n}') \rangle = \langle {}_sX(\boldsymbol{n}) \, {}_{-t}X(\boldsymbol{n}') \rangle = \sum_{\ell m} C_\ell^{st} \, {}_sY_{\ell m}(\boldsymbol{n}) \, {}_tY^*_{\ell m}(\boldsymbol{n}')$$
(23)

How does this responds to a source of anisotropy (with spin r), ${}_r\alpha(\boldsymbol{n})$? For all cases in this work, we can parametrize this as follows

$$\delta \langle {}_sX(\boldsymbol{n}) \, {}_tX(\boldsymbol{n}') \rangle = {}_r\alpha(\boldsymbol{n}') \sum_{\ell m} {}_rR_\ell^{st} \, {}_sY_{\ell m}(\boldsymbol{n}) \, {}_{r-t}Y^*_{\ell m}(\boldsymbol{n}') + ((-t, \boldsymbol{n}') \leftrightarrow (s, \boldsymbol{n}))$$
(24)

for some set of isotropic response functions R_ℓ. What is the response to the estimator Eq. (1)?
Examples include:

- Lensing: The source of anisotropy is the spin-1 field ${}_1\alpha(\boldsymbol{n})$, with response

$$\delta_sX(\boldsymbol{n}) = \frac{1}{2}\alpha_1 \eth \, {}_sX(\boldsymbol{n}) + \frac{1}{2}\alpha_{-1} \bar{\eth} \, {}_sX(\boldsymbol{n})$$
(25)

where $\eth$ and $\bar{\eth}$ are the spin lowering and spin raising operator JC: check notation respectively. From their action on the spin-weighted harmonics, the harmonic space responses areJC: ...

$$s_r = s - 1, s + 1 \quad R^{s,s-1} = JC: ..., R^{s,s+1} = JC: ...$$
(26)

- Modulation estimator: The source is spin 0, with response

$$\delta_sX(\boldsymbol{n}) = {}_0\alpha(\boldsymbol{n})_sX(\boldsymbol{n})$$
(27)

Hence,

$$s_r = s \quad , R_\ell^{ss} = 1$$
(28)

- Point source:

$$\delta_sX(\boldsymbol{n}) = {}_0\alpha(\boldsymbol{n})\delta^D(\boldsymbol{n})$$
(29)

- Noise anisotropies:

intermediate steps for response calc.   Then $\left(R_\ell^{-r,-s} = (-1)^r R_\ell^{r,s}\right)$

$$_{s_o+t_o}\hat{d}_{LM} = \mathcal{R}_L^{st,r} \ _{-r}\alpha_{LM} + \mathcal{R}_L^{ts,-r} \ _r\alpha_{LM} \ \text{with} \ \mathcal{R}_L^{st,r} = 2\pi \int_{-1}^{1} d\mu \, \Xi^{st,r}(\mu) \, \Xi^t(\mu) \, d_{r,s_o+t_o}^{L}(\mu) \tag{30}$$

where

$$\Xi^{st,r}(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi}\right) C_\ell^{s_i,-t_i} R_\ell^{r-t_i} w_\ell^{s_i,s_o} d_{t_i+r,s_o}^{\ell}(\mu) \ \text{and} \ \Xi^t(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{t_i,t_o} d_{-t_i,t_o}^{\ell}(\mu) \tag{31}$$