Notes on curved-sky quadratic estimation

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This document supplements the release of the Planck 2018 CMB lensing [1] pipeline. It collects the formulae relevant to curved-sky quadratic estimators in the spin-weight, position-space correlation function formalism, including in particular estimator cross-responses and Gaussian noise biases between arbitrary pairs of quadratic estimators. JC: Document to be included with the pipeline release after submission of the revised L08. in progress

CONTENTS

I. SPIN-WEIGHT ESTIMATORS

JC: too many αsA complex spin-s field ${}_sf(\hat{n})$ is defined with reference to local axes by the condition that it transforms under a rotation of angle ψ of the local axes according to ${}_sf(\hat{n}) \rightarrow e^{is\psi}\; {}_sf(\hat{n})$. We use further the notation ${}_{-s}f(\hat{n}) = {}_sf^*(\hat{n})$.

The gradient (g) and curl (c) harmonic modes of definite parity of ${}_sf$ are then defined as follows

$$g^s_{\ell m} = -\frac{1}{2}\left({}_{|s|}f_{\ell m} + (-1)^s\; {}_{-|s|}f_{\ell m}\right) \tag{1.1}$$

$$c^s_{\ell m} = -\frac{1}{2i}\left({}_{|s|}f_{\ell m} - (-1)^s\; {}_{-|s|}f_{\ell m}\right) \tag{1.2}$$

where ${}_{\pm s}f_{\ell m} \equiv \int d^2n\; {}_{\pm s}f(\hat{n})\; {}_{\pm s}Y^*_{\ell m}(\hat{n})$. Spin-0 fields are real and pure gradients. With these conventions, we have in particular for the CMB polarization ${}_{\pm 2}P(\hat{n})$

$${}_{\pm 2}P_{\ell m} = -(E_{\ell m} \pm iB_{\ell m}). \tag{1.3}$$

where E and B are the polarization gradient and curl mode[1]. Our polarization conventions are such that ${}_{\pm 2}P(\hat{n}) = Q(\hat{n}) \pm iU(\hat{n})$, with the local x and y axes at each point $\hat{n}$ pointing south and east (following e.g. the healpix software conventions `https://healpix.jpl.nasa.gov/html/intronode12.htm`).
The relation inverse to Eqs. (1.1) and (1.2) is

$${}_{\pm|s|}f_{\ell m} = -(\pm)^s\left(g^s_{\ell m} \pm ic^s_{\ell m}\right). \tag{1.4}$$

A. Correlation functions

We use position-space correlation function for fields of arbitrary spins as follows. For two points on the sphere $\hat{n}_1$, $\hat{n}_2$, let γ be the angle at $\hat{n}_1$ that aligns the local x-axis to the geodesic connecting $\hat{n}_1$ and $\hat{n}_2$ (with the x-axis pointing towards $\hat{n}_2$), β the angle between $\hat{n}_1$ and $\hat{n}_2$, and α is defined just as γ but at $\hat{n}_2$[2, 3]. See Fig. 1 for the geometry. Then the following two correlators only depend on the distance between the two points and carry all of the information on their gradient and curl mode cross-spectra:

$$\begin{aligned}\xi^{st}_+(\beta) &\equiv \left\langle e^{-is\alpha}\; {}_sf(\hat{n}_1)\; \left({}_tf(\hat{n}_2)e^{-it\gamma}\right)^*\right\rangle \\ \xi^{st}_-(\beta) &\equiv \left\langle \left(e^{-is\alpha}\; {}_sf(\hat{n}_1)\right)^*\; \left({}_tf(\hat{n}_2)e^{-it\gamma}\right)^*\right\rangle\end{aligned} \tag{1.5}$$

JC: Mixups with n1 and n2 defs...fix this!Fourier transforming and using relation (1.4) gives the following expression

[1] Note that the spin-0 intensity ${}_0T(\hat{n})$ gradient mode is $-T_{\ell m}$ and not $T_{\ell m}$

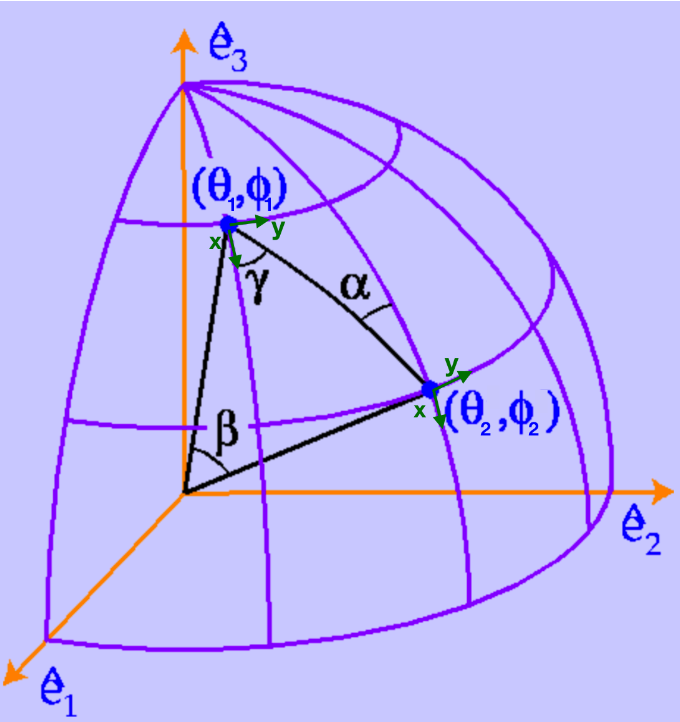

FIG. 1. The geometry and angles in Eq. (1.5), with the local axes in green. It holds $\alpha(\hat{n}_2,\hat{n}_1) = \pi - \gamma(\hat{n}_1,\hat{n}_2)$ and $\gamma(\hat{n}_2,\hat{n}_1) = \pi - \alpha(\hat{n}_1,\hat{n}_2)$. Figure originally from Wayne Hu tutorials, `http://background.uchicago.edu/~whu/tamm/webversion/node5.html`.

$$
\begin{aligned}
\xi^{st}_{+}(\beta) &= (+1)^s \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) \left[C_\ell^{g^s g^t} + C_\ell^{c^s c^t} - i\left(C_\ell^{g^s c^t} + C_\ell^{c^s g^t}\right)\right] d^\ell_{st}(\beta) \\
\xi^{st}_{-}(\beta) &= (-1)^s \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) \left[C_\ell^{g^s g^t} - C_\ell^{c^s c^t} - i\left(C_\ell^{g^s c^t} - C_\ell^{c^s g^t}\right)\right] d^\ell_{-st}(\beta)
\end{aligned}
\tag{1.6}
$$

B. Quadratic estimators

Prior to projection onto gradient and curl modes, and prior to proper normalization, separable quadratic estimators can be written as a (sum of) products of two position-space maps. Let $\hat{q}$ be such an unnormalized estimator:

$$
\begin{aligned}
{}_{s_o+t_o}\hat{q}(\hat{n}) \equiv & \left(\sum_{\ell m} w_\ell^{s_o s_i}\, {}_{s_i}\bar{X}_{\ell m}\, {}_{s_o}Y_{\ell m}(\hat{n})\right) \\
& \cdot \left(\sum_{\ell m} w_\ell^{t_o t_i}\, {}_{t_i}\bar{X}_{\ell m}\, {}_{t_o}Y_{\ell m}(\hat{n})\right)
\end{aligned}
\tag{1.7}
$$

where s_i, t_i are input spins, s_o, t_o outputs spins, and $w_\ell^{s_o s_i}, w_\ell^{t_i t_o}$ associated weights. For simplicity we use the same symbol w for the first and second leg weights even if they may differ in general for the same spin indices. By consistency with ${}_{-s_o-t_o}\hat{q}(\hat{n}) = {}_{s_o+t_o}\hat{q}^*(\hat{n})$ the weights have symmetry $w_\ell^{-s_o-s_i} = (-1)^{s_o+s_i} w_\ell^{*s_o s_i}$.

The maps ${}_s\bar{X}_{lm}$ are the inverse signal + noise variance filtered CMB maps; the filtered scalar temperature

$$
{}_0\bar{X}_{\ell m} = \bar{T}_{\ell m} \tag{1.8}
$$

and filtered spin ± 2 Stokes polarization ${}_{\pm 2}\bar{P} = \bar{Q} \pm i\bar{U}$,

$$
{}_{\pm 2}\bar{X}_{\ell m} = {}_{\pm 2}\bar{P}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{1.9}
$$

In the notation of Ref. [1], these maps are the result of the filtering step $\bar{X} = \mathcal{B}^\dagger \mathrm{Cov}^{-1} X^{\mathrm{dat}}$. This operation is trivial (diagonal in harmonic space) for perfectly isotropic data, but in a realistic setting (masked pixels, inhomogeneous noise, etc) this is difficult to perform and several schemes are available with slightly different results.

The formulae exposed in this document can be derived through simple application of this formal relation,

$$
\sum_{m_1,m_2} \int d^2n_1\, {}_{s_1}Y_{\ell_1 m_1}(\hat{n}_1)\, {}_{s_2}Y_{\ell_2 m_2}(\hat{n}_1)\, {}_{r_1}Y_{LM}(\hat{n}_1) \int d^2n_2\, {}_{t_1}Y_{\ell_1 m_1}(\hat{n}_2)\, {}_{t_2}Y_{\ell_2 m_2}(\hat{n}_2)\, {}_{r_2}Y_{L'M'}(\hat{n}_2)
$$
$$
= \delta_{LL'}\delta_{MM'} \frac{2\ell_1+1}{4\pi}\frac{2\ell_2+1}{4\pi} 2\pi \int_{-1}^{1} d\beta\, d^{\ell_1}_{s_1,t_1}(\beta) d^{\ell_2}_{s_2 t_2}(\beta) d^{L}_{r_1 r_2}(\beta) \quad (\text{whenever } s_1+s_2+r_1 = 0 = t_1+t_2+r_2). \tag{1.10}
$$

where $d^{\ell}_{mm'}$ are Wigner small d-matrices. we adopt the convention, standard in CMB lensing, to write quadratic estimator multipoles with L, M and use ℓ, m for the CMB fields from which they are built.

II. GAUSSIAN COVARIANCE CALCULATIONS

For two generic estimators as defined in Eq. (1.7), we now obtain their gradient (g) and curl (c) variances and covariances (usually denoted with $N_L(0)$) with two one-dimensional integrals.

As is obvious from Eq. (**??**), the result will combine spectra and cross-spectra of the inverse-variance filtered maps $\bar{X}$. Since there are distinct schemes to perform the inverse-variance filtering in a realistic case, we make here a distinction between the semi-analytical and analytical $N^{(0)}$. In the former case, some empirical estimate of these spectra are used (assuming isotropy of the filtered maps). This gives a realization-dependent estimate of the noise. In the latter case, some analytic isotropic approximation of the filtering function is used, giving a realization-independent noise estimate. Neither are as accurate as the much more costly realization-dependent noise debiaser now routinely used in CMB lensing analyses[1, RDN0].

JC: describe here the matrix F

For an isotropy estimator ${}_r\hat{q}$ let $s = (s_{\rm i}, s_{\rm o}, w^{s_{\rm i}s_{\rm o}})$ collectively describes the in and out spins and weight function of the left leg, and similarly with t for the right leg (by consistency, $s_{\rm o} + t_{\rm o} = r$). In the same way, let u and v describes another estimator ${}_{r'}\hat{q}$ (with $u_{\rm o} + v_{\rm o} = r'$). Then, their Gaussian correlation functions are

$$
\xi^{rr'}_{\pm}(\beta) = \xi^{\pm s,u}(\beta)\xi^{\pm t,v}(\beta) + \xi^{\pm s,v}(\beta)\xi^{\pm t,u}(\beta), \tag{2.1}
$$

where $\xi^{s,t}$ is

$$
\xi^{s,t}(\beta) \equiv \sum_{\ell} \left(\frac{2\ell+1}{4\pi}\right) w_{\ell}^{s_{\rm o}s_{\rm i}} w_{\ell}^{*t_{\rm o}t_{\rm i}} \bar{C}_{\ell}^{s_{\rm i}t_{\rm i}} d^{\ell}_{s_{\rm o}t_{\rm o}}(\beta) \tag{2.2}
$$

and $\bar{C}_{\ell}^{s_{\rm i}t_{\rm i}} \equiv \left\langle {}_{s_{\rm i}}\bar{X}_{\ell m}\, {}_{t_{\rm i}}\bar{X}^*_{\ell m}\right\rangle$. Projecting onto gradient and curl modes results in

$$
\begin{aligned}
\left.\left\langle \hat{g}^{r}_{LM}\hat{g}^{*,r'}_{L'M'}\right\rangle\right|_{\rm G.} &= \delta_{LL'}\delta_{MM'}\frac{1}{2}\Re\left[C_L^{rr'} + (-1)^r C_L^{-rr'}\right]\\
\left.\left\langle \hat{c}^{r}_{LM}\hat{c}^{*,r'}_{L'M'}\right\rangle\right|_{\rm G.} &= \delta_{LL'}\delta_{MM'}\frac{1}{2}\Re\left[C_L^{rr'} - (-1)^r C_L^{-rr'}\right]\\
\left.\left\langle \hat{g}^{r}_{LM}\hat{c}^{*,r'}_{L'M'}\right\rangle\right|_{\rm G.} &= \delta_{LL'}\delta_{MM'}\frac{1}{2}\Im\left[-C_L^{rr'} - (-1)^r C_L^{-rr'}\right]\\
\left.\left\langle \hat{c}^{r}_{LM}\hat{g}^{*,r'}_{L'M'}\right\rangle\right|_{\rm G.} &= \delta_{LL'}\delta_{MM'}\frac{1}{2}\Im\left[C_L^{rr'} - (-1)^r C_L^{-rr'}\right]
\end{aligned} \tag{2.3}
$$

where

$$
C_L^{\pm rr'} \equiv 2\pi \int_{-1}^{1} d\mu\, d^L_{\pm rr'}(\mu)\xi^{rr'}_{\pm}(\beta) \tag{2.4}
$$

($\Re$ and $\Im$ stands for real and imaginary parts respectively). Ref. [1] calculates the covariance matrix based on these equations using the empirical, realisation dependent power spectra $\bar{C}_{\ell}^{s_i,t_i}$. A gradient-curl mode cross-covariance may be sourced by gradient-curl couplings in the inverse-variance filtered CMB fields (i.e., non-zero $C_{\ell}^{\bar{T}\bar{B}}$ or $C_{\ell}^{\bar{E}\bar{B}}$). This is not the only possibility though.

III. RESPONSE AND CROSS-RESPONSES CALCULATIONS

JC: (for the purposes of the analytical calculations in this document) are isotropically related to the (beam-deconvolved) data maps ${}_sX^{\rm dat}$ through a matrix F,

$$
{}_s\bar{X}_{\ell m} \equiv \sum_{s_2=0,2,-2} F_{\ell}^{ss_2}\, {}_{s_2}X_{\ell m} \tag{3.1}
$$

We now turn to the calculation of the response of the estimator to a source of anisotropy. Anisotropy can sometimes be parametrized at the level of the CMB maps, (for example for lensing), with

$$
{}_s\delta X(\hat{n}) = \sum_{a=\pm r} {}_a\alpha(\hat{n}) \left(\sum_{\ell m} R_{\ell}^{a,s}\, {}_sX_{\ell m}\, {}_{s-a}Y_{\ell m}(\hat{n})\right) \tag{3.2}
$$

for response kernel functions $R_{\ell}^{r,s}$. More generally, let the covariance of the CMB data respond as follows to a spin-r anisotropy source α:

$$\delta\left\langle {}_sX(\hat n_1)\, {}_tX^*(\hat n_2)\right\rangle = \sum_{\ell m, a=\pm r} {}_a\alpha(\hat n_1) W_\ell^{a,st}\, {}_{s-a}Y_{\ell m}(\hat n_1)\, {}_tY^*_{\ell m}(\hat n_2) + W_\ell^{*a,ts}\, {}_sY_{\ell m}(\hat n_1)\, {}_{t-a}Y^*_{\ell m}(\hat n_2)\, {}_a\alpha^*(\hat n_2) \quad (3.3)$$

for some weights functions $W_\ell^{a,st}$. For map-level descriptions in Eq. (3.2) then holds

$$W_\ell^{a,st} = R^{a,s} C_\ell^{st}. \quad (3.4)$$

However, Eq. (3.3) is more general. Section **??** lists weights functions of some relevant cases.

Let as before s,t denote collectively the QE spins and weight functions for an estimator ${}_r\hat\alpha(\hat n)$ of spin $r = s_o + t_o$, and let r' be the spin of anisotropy source ${}_{r'}\beta(\hat n)$ with covariance response kernel $W^{r'}$ as above. Let $\mathcal{R}_L^{g_r g_{r'}}\delta_{LL'}\delta_{MM'}$ be defined as the response of the gradient mode of α_{LM} to the gradient mode of $\beta_{L'M'}$, and similarly for the curl. It holds:

$$\begin{aligned}
\mathcal{R}_L^{g_r g_{r'}} &= \Re\left[R_L^{st,r'} + (-1)^{r'} R_L^{st,-r'}\right] \\
\mathcal{R}_L^{c_r c_{r'}} &= \Re\left[R_L^{st,r'} - (-1)^{r'} R_L^{st,-r'}\right] \\
\mathcal{R}_L^{g_r c_{r'}} &= \Im\left[-R_L^{st,r'} + (-1)^{r'} R_L^{st,-r'}\right] \\
\mathcal{R}_L^{c_r g_{r'}} &= \Im\left[R_L^{st,r'} + (-1)^{r'} R_L^{st,-r'}\right]
\end{aligned} \quad (3.5)$$

where

$$R_L^{st,r'} = 2\pi \int_{-1}^{1} d\mu\, d^L_{rr'}(\mu) \sum_{\tilde s_i, \tilde t_i = 0,2,-2} \left[\xi^{s_o s_i \tilde s_i}(\mu)\psi^{t_o t_i \tilde t_i \tilde s_i, r'}(\mu) + \xi^{t_o t_i \tilde t_i}(\mu)\psi^{s_o s_i \tilde s_i \tilde t_i, r'}(\mu)\right] \quad (3.6)$$

and

$$\begin{aligned}
\xi^{s_o s_i \tilde s_i}(\mu) &\equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_o s_i} F_\ell^{s_i \tilde s_i} d^\ell_{s_o, \tilde s_i}(\mu) \\
\psi^{s_o s_i \tilde s_i \tilde t_i, r'}(\mu) &\equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w^{s_o s_i} F_\ell^{s_i \tilde s_i} W_\ell^{*-r', -\tilde t_i \tilde s_i} d^\ell_{s_o, -\tilde t_i + r'}(\mu)
\end{aligned} \quad (3.7)$$

Again, in most relevant cases, the gradient to curl and curl to gradient responses do vanish. If there is a unique source of anisotropy, properly normalized gradient and curl estimators are then given by $\hat g^r_{LM}/\mathcal{R}_L^{g_r g_r}$ and $\hat c^r_{LM}/\mathcal{R}_L^{c_r c_r}$.

IV. DERIVATION OF OPTIMAL QE WEIGHTS

Optimal (in the sense of minimal Gaussian variance) QE weights are easily gained from the representation in Eq. 3.3 of the anisotropy. Let the CMB likelihood gradients be

$${}_{\pm r}\hat\alpha(\hat n) = \frac{\delta}{\delta_{\mp r}\alpha(\hat n)} - \frac{1}{2}\, {}_{s_1}X \mathrm{Cov}^{-1}_{s_1 s_2}\, {}_{s_2}X\Big|_{\alpha\equiv 0} \quad (4.1)$$

where $\mathrm{Cov}_{s_1 s_2}(\hat n, \hat n') \equiv \langle {}_{s_1}X(\hat n)\, {}_{s_2}X(\hat n')\rangle$, and where ${}_r\alpha(\hat n)$ and ${}_{-r}\alpha(\hat n)$ are treated as independent variables. Using Eq. (3.3) and comparing to Eq. (1.7), we find

$$\boxed{w_\ell^{st} = \delta_{st}\ \text{(1st leg)} \quad w_\ell^{-s+r,t} = 2W_\ell^{-r,-st}\ \text{(2nd leg)}} \quad (4.2)$$

JC: why 2 again? JC: FIXME: The right expression is

$${}_r\hat g(\hat n) = \sum_s {}_{-s}\bar X(\hat n) \cdot \left(2W_\ell^{-r,st}\, {}_t\bar X_{\ell m}\, {}_{s+r}Y_{\ell m}(\hat n)\right) \quad (4.3)$$

where $\bar X$ has the $(0,2,-2)$ elements (note the additional factor of 2! in pol w.r.t. to naive spin defs.)

$$\begin{pmatrix} \bar T \\ -\frac{1}{2}\left(\bar E + i\bar B\right) \\ -\frac{1}{2}\left(\bar E - i\bar B\right) \end{pmatrix} \quad (4.4)$$

Factor of 2 in front of W comes from 2 $\delta/\delta\ {}_{-r}\alpha(\hat n)$ to get $d/dre + d/dim$ (?).

V. EXAMPLES

The construction of optimal estimators only requires knowledge of the data covariance response W, Eq. 3.3. For perturbation of CMB fields, W is trivially related to the field response R through Eq. (3.2). In this section, we briefly derive and list a few relevant cases of these responses.

A. Lensing

The source of anisotropy is the spin-1 deflection field ${}_1\alpha(\hat{n})$, with linear response (see Ref. [4]) $\delta_s X(\hat{n}) = -\frac{1}{2}\alpha_1(\hat{n})\bar{\eth}_s X(\hat{n}) - \frac{1}{2}\alpha_{-1}(\hat{n})\eth\ {}_s X(\hat{n})$ where $\eth$ and $\bar{\eth}$ are the spin raising and spin lowering operator respectively. From the action of these operators on the spherical harmonics follow immediately

$$\begin{aligned} R_\ell^{-1,s} &= -\frac{1}{2}\sqrt{(l-s)(l+s+1)} \\ R_\ell^{1,s} &= +\frac{1}{2}\sqrt{(l+s)(l-s+1)} \end{aligned} \tag{5.1}$$

B. Modulation

This is relevant for instance in searches for a trispectrum τ_{NL} signature in the CMB[**?**]. The anisotropy source is a scalar ('f'), with response $\delta_s X(\hat{n}) = f(\hat{n})_s X(\hat{n})$, hence

$$R_\ell^{0,s} = 1 \tag{5.2}$$

C. Polarization rotation

This is relevant in the case of systematic polarization angle miscalibration, or within more speculative ideas including cosmic birefringence. The observed polarization is rotated by an angle $\beta(\hat{n})$ according to ${}_{\pm 2}P \rightarrow e^{\mp 2i\,\beta(\hat{n})}\ {}_{\pm 2}P(\hat{n})$. Hence,

$$R_\ell^{0,\pm 2} = \mp 2i \tag{5.3}$$

D. Point sources

Point sources in temperature (S^2, see Ref. [5]): here anisotropy is sought of the form $\delta\left\langle T(\hat{n})\,T(\hat{n}')\right\rangle = \delta_{\hat{n}\hat{n}'} S^2(\hat{n})$. Hence,

$$W_\ell^{r,st} = \frac{1}{4}\delta_{r0}\delta_{s0}\delta_{t0} \tag{5.4}$$

E. Noise variance map anisotropies

This is conceptually the same as point sources but acting on beam-convolved maps

$$W_\ell^{r,st} = \frac{1}{4}\delta_{r0}\delta_{s0}\delta_{t0}\frac{1}{b_\ell^2} \tag{5.5}$$

[1] N. Aghanim *et al.* (Planck), "Planck 2018 results. VIII. Gravitational lensing," (2018), arXiv:1807.06210 [astro-ph.CO].

[2] Gayoung Chon, Anthony Challinor, Simon Prunet, Eric Hivon, and Istvan Szapudi, "Fast estimation of polarization power spectra using correlation functions," Mon. Not. Roy. Astron. Soc. **350**, 914 (2004), arXiv:astro-ph/0303414 [astro-ph].

[3] Anthony Challinor and Antony Lewis, "Lensed CMB power spectra from all-sky correlation functions," Phys. Rev. **D71**, 103010 (2005), arXiv:astro-ph/0502425 [astro-ph].

[4] Anthony Challinor and Gayoung Chon, "Geometry of weak lensing of CMB polarization," Phys. Rev. **D66**, 127301 (2002), arXiv:astro-ph/0301064 [astro-ph].

[5] Stephen J. Osborne, Duncan Hanson, and Olivier Doré, "Extragalactic Foreground Contamination in Temperature-based CMB Lens Reconstruction," JCAP **1403**, 024 (2014), arXiv:1310.7547 [astro-ph.CO].

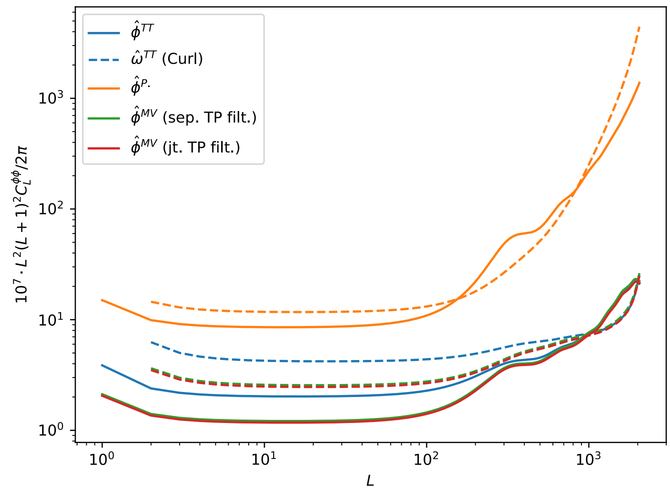

FIG. 2. Lensing gradient and curl reconstruction noise levels for a *Planck*-like experiment.