CONTENTS

I. NOTES ON CURVED-SKY QE RESPONSES ETC.

JC: Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [?] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$_{s_o+t_o}\hat{d}(\boldsymbol{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_i}\ {}_{s_i}\bar{X}_{\ell m}\ {}_{s_o}Y_{\ell m}(\boldsymbol{n})\right)\left(\sum_{\ell m} w_\ell^{t_i}\ {}_{t_i}\bar{X}_{\ell m}\ {}_{t_o}Y_{\ell m}(\boldsymbol{n})\right) \tag{1.1}$$

where s_i, t_i are input spins, $w_\ell^{s_i}, w_\ell^{t_i}$ associated weights, and s_o, t_o outputs spins. The maps ${}_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$${}_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{1.2}$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields $\bar{X}$ of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G_{LM}^s &= -\tfrac{1}{2}\left({}_{|s|}d_{LM} + (-1)^s\ {}_{-|s|}d_{LM}\right) \\ C_{LM}^s &= -\tfrac{1}{2i}\left({}_{|s|}d_{LM} - (-1)^s\ {}_{-|s|}d_{LM}\right). \end{aligned}$$

A. Semi-analytical QE $N_L^{(0)}$ calculation

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1.1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_i, t_i, w^{s_i})$ collectively describes the in and out spins and weight function, and similarly for t, u and v. Let the response function $\mathcal{R}_L^{st,uv}$ be defined as JC: Change $\mathcal{R}$ to N or similarJC: The exact def of $\bar{C}$ is not super clear, I think there are signs lying around, lets see

$$(-1)^{t_o+v_o}\mathcal{R}_L^{st,uv} \equiv 2\pi\int_{-1}^{1} d\mu\, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_o-v_o, s_o+u_o}(\mu) \tag{1.3}$$

where ξ are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_i} w_\ell^{t_i} \bar{C}_\ell^{s_i t_i} d^\ell_{-t_o, s_o}(\mu) \text{ with } \bar{C}_\ell^{s_i t_i} \equiv \left\langle {}_{s_i}\bar{X}_{\ell m}\ {}_{t_i}\bar{X}^*_{\ell m}\right\rangle \tag{1.4}$$

and $d^\ell_{mm'}$ are Wigner small d-matrices. Then

$$\left\langle G_{LM}^{s_o+t_o} G_{LM}^{*,u_o+v_o}\right\rangle = \frac{1}{2}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right) + (-1)^{s_o+t_o}\mathcal{R}_L^{-su,tv} + (-1)^{u_o+v_o}\mathcal{R}_L^{s-v,t-u}\right] \tag{1.5}$$

$$\left\langle C_{LM}^{s_o+t_o} C_{LM}^{*,u_o+v_o}\right\rangle = -\frac{1}{2}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right) - (-1)^{s_o+t_o}\mathcal{R}_L^{-su,tv} - (-1)^{u_o+v_o}\mathcal{R}_L^{s-v,t-u}\right] \tag{1.6}$$

$$\left\langle G_{LM}^{s_o+t_o} C_{LM}^{*,u_o+v_o}\right\rangle = 0 \tag{1.7}$$

JC: TTTT checked OK. Second term sign looks a bit not symmetric but is factually correct I believe, after sign-weight of the weights functions and $\bar{C}$

a. Sketchy derivation to cleanup For this we need a result using the spin-weight spherical harmonic theorem. Define $\mathcal{R}_L^{st,uv}$ through

$$\begin{aligned}\mathcal{R}^{st,uv}(\boldsymbol{n},\boldsymbol{n}') \equiv& (-1)^{t_o+v_o+t_i+v_i}\left(\sum_{\ell m} g_\ell^{s_i} g_\ell^{t_i} C_\ell^{s_i t_i}\ {}_{s_o}Y_{\ell m}(\boldsymbol{n})\ {}_{-t_o}Y^*_{\ell m}(\boldsymbol{n}')\right)\left(\sum_{\ell m} g_\ell^{u_i} g_\ell^{v_i} C_\ell^{u_i v_i}\ {}_{u_o}Y_{\ell m}(\boldsymbol{n})\ {}_{-v_o}Y^*_{\ell m}(\boldsymbol{n}')\right)\\ &\equiv (-1)^{t_o+v_o+t_i+v_i}\sum_{LM}\mathcal{R}_L^{stuv}\ {}_{s_o+u_o}Y_{LM}(\boldsymbol{n})\ {}_{-t_o-v_o}Y^*_{LM}(\boldsymbol{n}')\end{aligned} \tag{1.8}$$

Then we can write

$$\left\langle {}_{s_o+t_o}\hat{d}(\boldsymbol{n})\ {}_{u_o+v_o}\hat{d}(\boldsymbol{n}')\right\rangle = \mathcal{R}^{su,tv}(\boldsymbol{n},\boldsymbol{n}') + \mathcal{R}^{sv,tu}(\boldsymbol{n},\boldsymbol{n}') \tag{1.9}$$

Taking the harmonic transform, we get

$$\left\langle {}_{s_o+t_o}\hat{d}_{LM}\ {}_{u_o+v_o}\hat{d}_{L'M'}\right\rangle = (-1)^M\delta_{M,-M'}\delta_{L,L'}\left(\mathcal{R}_L^{su,tv}+\mathcal{R}_L^{sv,tu}\right)(-1)^{u_o+v_o} \tag{1.10}$$

or

$$\left\langle {}_{s_o+t_o}\hat{d}_{LM}\ {}_{u_o+v_o}\hat{d}^*_{L'M'}\right\rangle = \delta_{MM'}\delta_{LL'}\left(\mathcal{R}_L^{su,tv}+\mathcal{R}_L^{sv,tu}\right) \tag{1.11}$$

In general we have

$$\begin{aligned} G^s_{LM} &= -\tfrac{1}{2}\left({}_sd_{LM} + (-1)^s\ {}_{-s}d_{LM}\right) \quad (s\geq 0)\\ C^s_{LM} &= -\tfrac{1}{2i}\left({}_sd_{LM} - (-1)^s\ {}_{-s}d_{LM}\right) \quad (s\geq 0).\end{aligned}$$

The estimator for ${}_{-s_o-t_o}\hat{d}$ is the same as ${}_{s_o+t_o}\hat{d}$ with all spin signs flipped, and with an overall sign $(-1)^{s_o+s_i+t_o+t_i}$. The out-spins part gets canceled by the sign $(-1)^s$ in the above equation. JC: No. better to request $w_\ell^{-s_i,-s_o} = (-1)^{s_o+si}w_\ell^{s_i,s_o}$ so that ${}_{-s_o-t_o}\hat{d} = {}_{s_o+t_o}\hat{d}^*$? Thats what we do nowHence,

$$\begin{aligned}&\delta_{MM'}\delta_{LL'}\left\langle G^{s_o+t_o}_{LM} G^{*,u_o+v_o}_{L'M'}\right\rangle\cdot 4 = \mathcal{R}_L^{su,tv}+\mathcal{R}_L^{sv,tu} + (-1)^{s_i+t_i}\left(\mathcal{R}_L^{-su,-tv}+\mathcal{R}_L^{-sv,-tu}\right)\\ &+(-1)^{u_i+v_i}\left(\mathcal{R}_L^{s-u,t-v}+\mathcal{R}_L^{s-v,t-u}\right)+(-1)^{s_i+t_i+u_i+v_i}\left(\mathcal{R}_L^{-s-u,-t-v}+\mathcal{R}_L^{-s-v,-t-u}\right)\quad (s_o+t_o>=0, u_o+v_o>=0)\end{aligned} \tag{1.12}$$

Since $\mathcal{R}$ is invariant under the simultaneous sign-flip of all spins JC: Wrong, also with sign-weighted weights: $\bar{C}^{-s,-t} = (-1)^{s+t}C^{t,s} = (-1)^{s+t}C^{s,t}(s,t\geq 0$, so $\mathcal{R}^{st,uv}$ takes a sign $(-1)^{s_i+t_i+u_i+v_i}$ and this solves the problem, we can also write this as:

$$\left\langle G^{s_o+t_o}_{LM} G^{*,u_o+v_o}_{LM}\right\rangle = \frac{1}{2}\left[\left(\mathcal{R}_L^{su,tv}+R_L^{sv,tu}\right)+(-1)^{s_o+t_o}\left(\mathcal{R}_L^{-su,tv}+\mathcal{R}_L^{-sv,tu}\right)\right] \tag{1.13}$$

$$\left\langle C^{s_o+t_o}_{LM} C^{*,u_o+v_o}_{LM}\right\rangle = -\frac{1}{2}\left[\left(\mathcal{R}_L^{su,tv}+R_L^{sv,tu}\right)-(-1)^{s_o+t_o}\left(\mathcal{R}_L^{-su,tv}+\mathcal{R}_L^{-sv,tu}\right)\right] \tag{1.14}$$

$$\left\langle G^{s_o+t_o}_{LM} C^{*,u_o+v_o}_{LM}\right\rangle = 0 \tag{1.15}$$

Pfeew

More details of $\mathcal{R}_L$

Recall the spin-weight addition theorem:

$$\sum_m {}_sY^*_{\ell m}(\boldsymbol{n}')\ {}_tY_{\ell m}(\boldsymbol{n}) = \sqrt{\frac{2\ell+1}{4\pi}}e^{-it\gamma}\ {}_tY_{\ell,-s}(\beta,\alpha). \tag{1.16}$$

Hence

$$\mathcal{R}^{st,uv}(\boldsymbol{n},\boldsymbol{n}') = (-1)^{t_\text{o}+v_\text{o}+t_\text{i}+v_\text{i}} e^{-is_\text{o}\gamma - iu_\text{o}\gamma} \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_\text{i}} g^{t_\text{i}} C_\ell^{s_\text{i}t_\text{i}}\; {}_{s_\text{o}}Y_{\ell t_\text{o}}(\beta,\alpha)\right)\left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_\text{i}} g^{v_\text{i}} C_\ell^{u_\text{i}v_\text{i}}\; {}_{u_\text{o}}Y_{\ell v_\text{o}}(\beta,\alpha)\right) \tag{1.17}$$

The product of the brackets is a spin $s_\text{o}+u_\text{o}$ function. Defining its spin weight coefficients as $\mathcal{R}_L$ we get the relation claimed above. What are these coefficients?

$$\mathcal{R}_L^{st,uv} \equiv (-1)^{t_\text{o}+v_\text{o}+t_\text{i}+v_\text{i}} \sqrt{\frac{4\pi}{2\ell+1}} \int d^2n \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_\text{i}} g^{t_\text{i}} C_\ell^{s_\text{i}t_\text{i}}\; {}_{s_\text{o}}Y_{\ell t_\text{o}}(\boldsymbol{n})\right)\left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_\text{i}} g^{v_\text{i}} C_\ell^{u_\text{i}v_\text{i}}\; {}_{u_\text{o}}Y_{\ell v_\text{o}}(\boldsymbol{n})\right) {}_{s_\text{o}+u_\text{o}}Y^*_{L,t_\text{o}+v_\text{o}}(\tag{1.18}$$

Using

$$\boxed{{}_sY_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}(-1)^m e^{im\phi} d^\ell_{-ms}(\theta)} \tag{1.19}$$

The above thing is invariant if all signs are flipped at the same time. we get

$$\boxed{(-1)^{t_\text{o}+v_\text{o}+t_\text{i}+v_\text{i}}\mathcal{R}_L^{st,uv} \equiv 2\pi \int_{-1}^{1} d\mu\, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_\text{o}-v_\text{o},s_\text{o}+u_\text{o}}(\mu),\ \text{with}\ \xi^{st}(\mu) \equiv \sum_\ell \frac{2\ell+1}{4\pi} g_\ell^{s_\text{i}} g_\ell^{t_\text{i}} C_\ell^{s_\text{i}t_\text{i}} d^\ell_{-t_\text{o},s_\text{o}}(\mu)} \tag{1.20}$$

JC: Remember now the weights w (or g...) changes $(-1)^{s_\text{i}+s_\text{o}}$ under sign flip of the two spins, and $\bar{C}^{st}$ takes a $(-1)^{s+t}$. So $\mathcal{R}^{-s-t,-u-v} = (-1)^{s_\text{i}+t_\text{i}+u_\text{i}+v_\text{i}+s_\text{i}+s_\text{o}+t_\text{i}+t_\text{o}+u_\text{i}+u_\text{o}+v_\text{i}+v_\text{o}} = (-1)^{s_\text{o}+t_\text{o}+u_\text{o}+v_\text{o}}\mathcal{R}^{st,uv}$

B. QE responses calculation

JC: should define the response at the covariance matrix level...The covariance matrices is

$$\langle {}_sX(\boldsymbol{n})\, {}_tX^*(\boldsymbol{n}')\rangle = \langle {}_sX(\boldsymbol{n})\, {}_{-t}X(\boldsymbol{n}')\rangle = \sum_{\ell m} C_\ell^{st}\, {}_sY_{\ell m}(\boldsymbol{n})\, {}_tY^*_{\ell m}(\boldsymbol{n}') \tag{1.21}$$

How does this responds to a source of anisotropy (with spin r), ${}_r\alpha(\boldsymbol{n})$? For all cases in this work, we can parametrize this as follows

$$\delta\,\langle {}_sX(\boldsymbol{n})\, {}_tX(\boldsymbol{n}')\rangle = \,{}_r\alpha(\boldsymbol{n}') \sum_{\ell m} {}_rR_\ell^{st}\, {}_sY_{\ell m}(\boldsymbol{n})\, {}_{r-t}Y^*_{\ell m}(\boldsymbol{n}') + ((-t,\boldsymbol{n}') \leftrightarrow (s,\boldsymbol{n})) \tag{1.22}$$

for some set of isotropic response functions R_ℓ. What is the response to the estimator Eq. (1.1)?

Examples include:

- Lensing: The source of anisotropy is the spin-1 field ${}_1\alpha(\boldsymbol{n})$, with response

$$\delta_sX(\boldsymbol{n}) = \frac{1}{2}\alpha_1 \eth_sX(\boldsymbol{n}) + \frac{1}{2}\alpha_{-1}\bar{\eth}\,{}_sX(\boldsymbol{n}) \tag{1.23}$$

  where $\eth$ and $\bar{\eth}$ are the spin lowering and spin raising operator JC: check notation respectively. From their action on the spin-weighted harmonics, the harmonic space responses areJC: ...

$$s_r = s-1, s+1 \quad R^{s,s-1} = JC: ..., R^{s,s+1} = JC: ... \tag{1.24}$$

- Modulation estimator: The source is spin 0, with response

$$\delta_sX(\boldsymbol{n}) =_0 \alpha(\boldsymbol{n})_sX(\boldsymbol{n}) \tag{1.25}$$

  Hence,

$$s_r = s \quad, R_\ell^{ss} = 1 \tag{1.26}$$

- Point source:

$$\delta_sX(\boldsymbol{n}) = \,{}_0\alpha(\boldsymbol{n})\delta^D(\boldsymbol{n}) \tag{1.27}$$

- Noise anisotropies: