Notes on curved-sky QE responses etc.

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Contents

JC: Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [?] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$ {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}(\hat{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_\mathrm{o}s_\mathrm{i}}\,{}_{s_\mathrm{i}}\bar{X}_{\ell m}\,{}_{s_\mathrm{o}}Y_{\ell m}(\hat{n})\right)\left(\sum_{\ell m} w_\ell^{t_\mathrm{o}t_\mathrm{i}}\,{}_{t_\mathrm{i}}\bar{X}_{\ell m}\,{}_{t_\mathrm{o}}Y_{\ell m}(\hat{n})\right) \tag{1} $$

where $s_\mathrm{i}, t_\mathrm{i}$ are input spins, $s_\mathrm{o}, t_\mathrm{o}$ outputs spins, and $w_\ell^{s_\mathrm{o}s_\mathrm{i}}, w_\ell^{t_\mathrm{i}t_\mathrm{o}}$ associated weights. The maps ${}_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$$ {}_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{2} $$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields $\bar{X}$ of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$ \begin{aligned} G^s_{LM} &= -\tfrac{1}{2}\left({}_{|s|}d_{LM} + (-1)^s\,{}_{-|s|}d_{LM}\right) \\ C^s_{LM} &= -\tfrac{1}{2i}\left({}_{|s|}d_{LM} - (-1)^s\,{}_{-|s|}d_{LM}\right). \end{aligned} $$

The formulae exposed here can be derived through simple application of this relation,

$$ \sum_{m_1,m_2}\int d^2n \prod_{i=1}^{3} {}_{s_i}Y_{\ell_i m_i}(\hat{n}) \int d^2n' \prod_{i=1}^{3} {}_{t_i}Y_{\ell_i m_i}(\hat{n}') = \frac{2\ell_1+1}{4\pi}\frac{2\ell_2+1}{4\pi}2\pi\int_{-1}^{1} d\mu \prod_{i=1}^{3} d^{\ell_i}_{s_i,t_i}(\mu) \tag{3} $$

0.1 Semi-analytical QE $N_L^{(0)}$ calculations

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_\mathrm{i}, s_\mathrm{o}, w^{s_\mathrm{i}s_\mathrm{o}})$ collectively describes the in and out spins and weight function, and similarly for t, u and v. Let the covariance function $N_L^{st,uv}$ be defined through

$$ \begin{aligned} \delta_{LL'}\delta_{MM'}N_L^{stuv} &\equiv \left\langle {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}_{LM}\;{}_{u_\mathrm{o}+v_\mathrm{o}}\hat{d}^*_{L'M'}\right\rangle\Big|_{\mathrm{Gauss}} \\ &= (-1)^{s_\mathrm{o}+t_\mathrm{o}+u_\mathrm{o}+v_\mathrm{o}}2\pi\int_{-1}^{1} d\mu\, d^L_{-s_\mathrm{o}-t_\mathrm{o},-u_\mathrm{o}-v_\mathrm{o}}(\mu)\left[\xi^{su}(\mu)\,\xi^{tv}(\mu) + \xi^{sv}(\mu)\,\xi^{tu}(\mu)\right] \end{aligned} \tag{4} $$

where ξ are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_{\ell} \left(\frac{2\ell+1}{4\pi}\right) w_{\ell}^{s_o s_i} w_{\ell}^{t_o t_i} \bar{C}_{\ell}^{s_i t_i} d^{\ell}_{s_o, t_o}(\mu) \text{ with } \bar{C}_{\ell}^{s_i t_i} \equiv \left\langle {}_{s_i}\bar{X}_{\ell m}\, {}_{t_i}\bar{X}^{*}_{\ell m} \right\rangle \tag{5}$$

and $d^{\ell}_{mm'}$ are Wigner small d-matrices. Then

$$\boxed{\begin{aligned} \left\langle \hat{G}^{s_o+t_o}_{LM} \hat{G}^{*,u_o+v_o}_{L'M'} \right\rangle\Big|_{\text{Gauss.}} &= \delta_{LL'}\delta_{MM'} \frac{1}{2}\left[N_L^{stuv} + (-1)^{s_o+t_o} N_L^{-s-tuv}\right] \\ \left\langle \hat{C}^{s_o+t_o}_{LM} \hat{C}^{*,u_o+v_o}_{L'M'} \right\rangle\Big|_{\text{Gauss.}} &= \delta_{LL'}\delta_{MM'} \frac{1}{2}\left[N_L^{stuv} - (-1)^{s_o+t_o} N_L^{-s-tuv}\right] \\ \left\langle \hat{G}^{s_o+t_o}_{LM} \hat{C}^{*,u_o+v_o}_{L'M'} \right\rangle\Big|_{\text{Gauss.}} &= 0 \end{aligned}} \tag{6}$$

0.2 QE response calculation

Let the estimator respond to a source with spin-weight $r \geq 0$ as follows

$${}_s\delta X(\hat{n}) = {}_r\alpha(\hat{n}) \left(\sum_{\ell m} R_{\ell}^{rs}\, {}_sX_{\ell m}\, {}_{s-r}Y_{\ell m}(\hat{n})\right) + {}_{-r}\alpha(\hat{n}) \left(\sum_{\ell m} R_{\ell}^{-rs}\, {}_sX_{\ell m}\, {}_{s+r}Y_{\ell m}(\hat{n})\right) \tag{7}$$

for some harmonic space responses R_{ℓ}^{rs}. Let further the isotropic limit of the filtering procedure be the matrix F, defined through

$${}_s\bar{X}_{\ell m} = \sum_{s_2=0,2,-2} F_{\ell}^{ss_2}\, {}_{s_2}X_{\ell m} \quad \text{(isotropic approximation)}. \tag{8}$$

Then the gradient and curl responses of estimator (1) are

$$\boxed{\begin{aligned} \mathcal{R}_L^{gg} &= R_L^{st,r} + (-1)^r R_L^{st,-r} \\ \mathcal{R}_L^{cc} &= R_L^{st,r} - (-1)^r R_L^{st,-r} \\ \mathcal{R}_L^{gc} &= 0 = \mathcal{R}_L^{cg}, \end{aligned}} \tag{9}$$

where $R_L^{st,r}$ is

$$R_L^{st,r} = (-1)^{s_o+t_o} 2\pi \int_{-1}^{1} d\mu\, d^L_{-s_o-t_o,-r}(\mu) \sum_{\tilde{s}_i, \tilde{t}_i = 0,2,-2} \left[\xi^{s_o s_i \tilde{s}_i}(\mu)\psi^{t_o t_i \tilde{t}_i \tilde{s}_i, r}(\mu) + \xi^{t_o t_i \tilde{t}_i}(\mu)\psi^{s_o s_i \tilde{s}_i \tilde{t}_i, r}(\mu)\right] \tag{10}$$

with

$$\begin{aligned} \xi^{s_o s_i \tilde{s}_i}(\mu) &\equiv \sum_{\ell} \left(\frac{2\ell+1}{4\pi}\right) w_{\ell}^{s_o s_i} F_{\ell}^{s_i \tilde{s}_i} d^{\ell}_{s_o, \tilde{s}_i}(\mu) \\ \psi^{s_o s_i \tilde{s}_i \tilde{t}_i, r}(\mu) &\equiv \sum_{\ell} \left(\frac{2\ell+1}{4\pi}\right) w^{s_o s_i} F_{\ell}^{s_i \tilde{s}_i} R_{\ell}^{r t_i} C_{\ell}^{\tilde{s}_i - \tilde{t}_i} d^{\ell}_{s_o, -\tilde{t}_i + r}(\mu) \end{aligned} \tag{11}$$

0.3 QE responses calculation

JC: should define the response at the covariance matrix level...The covariance matrices is

$$\langle {}_sX(\hat{n})\, {}_tX^*(\hat{n}')\rangle = \langle {}_sX(\hat{n})\, {}_{-t}X(\hat{n}')\rangle = \sum_{\ell m} C_\ell^{st}\, {}_sY_{\ell m}(\hat{n})\, {}_tY^*_{\ell m}(\hat{n}') \tag{12}$$

How does this responds to a source of anisotropy (with spin r), ${}_r\alpha(\hat{n})$? For all cases in this work, we can parametrize this as follows

$$\delta \langle {}_sX(\hat{n})\, {}_tX(\hat{n}')\rangle = {}_r\alpha(\hat{n}') \sum_{\ell m} {}_rR_\ell^{st}\, {}_sY_{\ell m}(\hat{n})\, {}_{r-t}Y^*_{\ell m}(\hat{n}') + ((-t,\hat{n}') \leftrightarrow (s,\hat{n})) \tag{13}$$

for some set of isotropic response functions R_ℓ. What is the response to the estimator Eq. (1)?

Examples include:

- Lensing: The source of anisotropy is the spin-1 field ${}_1\alpha(\hat{n})$, with response

$$\delta_s X(\hat{n}) = \frac{1}{2}\alpha_1 \eth_s X(\hat{n}) + \frac{1}{2}\alpha_{-1}\bar{\eth}\, {}_sX(\hat{n}) \tag{14}$$

  where $\eth$ and $\bar{\eth}$ are the spin lowering and spin raising operator JC: check notation respectively. From their action on the spin-weighted harmonics, the harmonic space responses areJC: ...

$$s_r = s-1, s+1 \quad R^{s,s-1} = JC: ..., R^{s,s+1} = JC: ... \tag{15}$$

- Modulation estimator: The source is spin 0, with response

$$\delta_s X(\hat{n}) =_0 \alpha(\hat{n})_s X(\hat{n}) \tag{16}$$

  Hence,

$$s_r = s \quad , R_\ell^{ss} = 1 \tag{17}$$

- Point source:

$$\delta_s X(\hat{n}) = {}_0\alpha(\hat{n})\delta^D(\hat{n}) \tag{18}$$

- Noise anisotropies:

intermediate steps for response calc. Then $\left(R_\ell^{-r,-s} = (-1)^r R_\ell^{r,s}\right)$

$$_{s_o+t_o}\hat{d}_{LM} = \mathcal{R}_L^{st,r}\ {}_{-r}\alpha_{LM} + \mathcal{R}_L^{ts,-r}\ {}_r\alpha_{LM} \text{ with } \mathcal{R}_L^{st,r} = 2\pi \int_{-1}^{1} d\mu\, \Xi^{st,r}(\mu)\, \Xi^t(\mu)\, d^L_{r,s_o+t_o}(\mu) \tag{19}$$

where

$$\Xi^{st,r}(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi}\right) C_\ell^{s_i,-t_i} R_\ell^{r-t_i} w_\ell^{s_i,s_o} d^\ell_{t_i+r,s_o}(\mu) \text{ and } \Xi^t(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{t_i,t_o} d^\ell_{-t_i,t_o}(\mu) \tag{20}$$

1 New derivation of generic response

Let

$$_sX(\hat{n}') = {}_a\alpha(\hat{n}') \left(\sum_{\ell'' m''} R_{\ell''}^{a,s}\ {}_sX_{\ell'' m''}\ {}_{s-a}Y_{\ell'' m''}(\hat{n}')\right) \tag{21}$$

where a sum over spin a is implicit. Let further the spin-weight spectra $C_\ell^{s_1 s_2}$ be defined as

$$\left\langle {}_{s_1}X_{\ell m}\ {}_{s_2}X^\dagger_{\ell' m'}\right\rangle \equiv C_\ell^{s_1 s_2}\delta_{\ell,\ell'}\delta_{m,m'} \tag{22}$$

Let further be the filtering $\mathcal{B}^\dagger \mathrm{Cov}^{-1}$ defined by a matrix

$$
{}_s\bar{X}_{\ell m} = F^{ss'}_{\ell}\;{}_{s'}X_{\ell m} \tag{23}
$$

and the QE be (JC: NB: not exactly as above)

$$
{}_{s_{\mathrm{o}}+t_{\mathrm{o}}}\hat{d}(\hat{n}) \equiv \left(\sum_{\ell m} w_{\ell}^{s_{\mathrm{o}}s_{\mathrm{i}}}\;{}_{s_{\mathrm{i}}}\bar{X}_{\ell m}\;{}_{s_{\mathrm{o}}}Y_{\ell m}(\hat{n})\right)\left(\sum_{\ell' m'} w_{\ell'}^{t_{\mathrm{o}}t_{\mathrm{i}}}\;{}_{t_{\mathrm{i}}}\bar{X}_{\ell' m'}\;{}_{t_{\mathrm{o}}}Y_{\ell' m'}(\hat{n})\right). \tag{24}
$$

Then, with $A \equiv wF$,

$$
\begin{aligned}
{}_{s_0+t_0}\hat{d}_{LM} =&\ {}_a\alpha_{L'M'} R^{a,s_{\mathrm{i}}}_{\ell''} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} \langle\, {}_{s_{\mathrm{i}}}X_{\ell''m''}\;{}_{t_{\mathrm{i}}}X_{\ell'm'} \rangle \int d^2n\;{}_{s_{\mathrm{o}}}Y_{\ell m}\;{}_{t_{\mathrm{o}}}Y_{\ell'm'}\;{}_{s_{\mathrm{o}}+t_{\mathrm{o}}}Y^*_{LM} \int d^2n'\;{}_{s_{\mathrm{i}}}Y^*_{\ell m}\;{}_{s_{\mathrm{i}}-a}Y_{\ell''m''}\;{}_aY_{L'M'} \\
&+ {}_a\alpha_{L'M'} R^{a,t_{\mathrm{i}}}_{\ell''} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} \langle\, {}_{t_{\mathrm{i}}}X_{\ell''m''}\;{}_{s_{\mathrm{i}}}X_{\ell m} \rangle \int d^2n\;{}_{s_{\mathrm{o}}}Y_{\ell m}\;{}_{t_{\mathrm{o}}}Y_{\ell'm'}\;{}_{s_{\mathrm{o}}+t_{\mathrm{o}}}Y^*_{LM} \int d^2n'\;{}_{t_{\mathrm{i}}-a}Y_{\ell''m''}\;{}_{t_{\mathrm{i}}}Y^*_{\ell'm'}\;{}_aY_{L'M'} \\
&= {}_a\alpha_{L'M'} R^{a,s_{\mathrm{i}}}_{\ell''} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} (-1)^{s_{\mathrm{i}}-m'} C^{t_{\mathrm{i}},-s_{\mathrm{i}}}_{\ell'} \int d^2n\;{}_{s_{\mathrm{o}}}Y_{\ell m}\;{}_{t_{\mathrm{o}}}Y_{\ell'm'}\;{}_{s_{\mathrm{o}}+t_{\mathrm{o}}}Y^*_{LM} \int d^2n'\;{}_{s_{\mathrm{i}}}Y^*_{\ell m}\;{}_{s_{\mathrm{i}}-a}Y_{\ell',-m'}\;{}_aY_{L'M'} \\
&+ {}_a\alpha_{L'M'} R^{a,t_{\mathrm{i}}}_{\ell''} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} (-1)^{t_{\mathrm{i}}-m} C^{s_{\mathrm{i}},-t_{\mathrm{i}}}_{\ell} \int d^2n\;{}_{s_{\mathrm{o}}}Y_{\ell m}\;{}_{t_{\mathrm{o}}}Y_{\ell'm'}\;{}_{s_{\mathrm{o}}+t_{\mathrm{o}}}Y^*_{LM} \int d^2n'\;{}_{t_{\mathrm{i}}-a}Y_{\ell,-m}\;{}_{t_{\mathrm{i}}}Y^*_{\ell'm'}\;{}_aY_{L'M'} \\
&= {}_a\alpha_{L'M'} R^{a,s_{\mathrm{i}}}_{\ell'} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} (-1)^{a} C^{t_{\mathrm{i}},-s_{\mathrm{i}}}_{\ell'} \int d^2n\;{}_{s_{\mathrm{o}}}Y_{\ell m}\;{}_{t_{\mathrm{o}}}Y_{\ell'm'}\;{}_{s_{\mathrm{o}}+t_{\mathrm{o}}}Y^*_{LM} \int d^2n'\;{}_{s_{\mathrm{i}}}Y^*_{\ell m}\;{}_{-s_{\mathrm{i}}+a}Y^*_{\ell'm'}\;{}_aY_{L'M'} \\
&+ {}_a\alpha_{L'M'} R^{a,t_{\mathrm{i}}}_{\ell} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} (-1)^{a} C^{s_{\mathrm{i}},-t_{\mathrm{i}}}_{\ell} \int d^2n\;{}_{s_{\mathrm{o}}}Y_{\ell m}\;{}_{t_{\mathrm{o}}}Y_{\ell'm'}\;{}_{s_{\mathrm{o}}+t_{\mathrm{o}}}Y^*_{LM} \int d^2n'\;{}_{-t_{\mathrm{i}}+a}Y^*_{\ell m}\;{}_{t_{\mathrm{i}}}Y^*_{\ell'm'}\;{}_aY_{L'M'} \\
&= (-1)^{M-M'}\;{}_a\alpha_{L'M'} R^{a,s_{\mathrm{i}}}_{\ell'} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} (-1)^{s_{\mathrm{o}}+t_{\mathrm{o}}} C^{t_{\mathrm{i}},-s_{\mathrm{i}}}_{\ell'} \int d^2n\;{}_{s_{\mathrm{o}}}Y_{\ell m}\;{}_{t_{\mathrm{o}}}Y_{\ell'm'}\;{}_{-s_{\mathrm{o}}-t_{\mathrm{o}}}Y_{L-M} \int d^2n'\;{}_{s_{\mathrm{i}}}Y^*_{\ell m}\;{}_{-s_{\mathrm{i}}+a}Y^*_{\ell'm'}\;{}_{-a}Y^*_{L'-M} \\
&+ (-1)^{M-M'}\;{}_a\alpha_{L'M'} R^{a,t_{\mathrm{i}}}_{\ell} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} (-1)^{s_{\mathrm{o}}+t_{\mathrm{o}}} C^{s_{\mathrm{i}},-t_{\mathrm{i}}}_{\ell} \int d^2n\;{}_{s_{\mathrm{o}}}Y_{\ell m}\;{}_{t_{\mathrm{o}}}Y_{\ell'm'}\;{}_{-s_{\mathrm{o}}-t_{\mathrm{o}}}Y_{L-M} \int d^2n'\;{}_{-t_{\mathrm{i}}+a}Y^*_{\ell m}\;{}_{t_{\mathrm{i}}}Y^*_{\ell'm'}\;{}_{-a}Y^*_{L'-M} \\
&= \delta_{LL,MM'}\;{}_a\alpha_{L'M'} R^{a,s_{\mathrm{i}}}_{\ell'} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} (-1)^{s_{\mathrm{o}}+t_{\mathrm{o}}} C^{t_{\mathrm{i}},-s_{\mathrm{i}}}_{\ell'} \left(\frac{2\ell+1}{4\pi}\right)\left(\frac{2\ell'+1}{4\pi}\right) 2\pi \int_{-1}^{1} d\mu\; d^{\ell}_{s_{\mathrm{o}}s_{\mathrm{i}}}(\mu)\; d^{\ell'}_{t_{\mathrm{o}},-s_{\mathrm{i}}+a}(\mu)\; d^{L}_{-s_{\mathrm{o}}-t_{\mathrm{o}},-a}(\mu) \\
&+ \delta_{LL,MM'}\;{}_a\alpha_{L'M'} R^{a,t_{\mathrm{i}}}_{\ell} A^{s_{\mathrm{o}}s_{\mathrm{i}}}_{\ell} A^{t_{\mathrm{o}}t_{\mathrm{i}}}_{\ell'} (-1)^{s_{\mathrm{o}}+t_{\mathrm{o}}} C^{s_{\mathrm{i}},-t_{\mathrm{i}}}_{\ell} \left(\frac{2\ell+1}{4\pi}\right)\left(\frac{2\ell'+1}{4\pi}\right) 2\pi \int_{-1}^{1} d\mu\; d^{\ell}_{s_{\mathrm{o}},-t_{\mathrm{i}}+a}(\mu)\; d^{\ell'}_{t_{\mathrm{o}},t_{\mathrm{i}}}(\mu)\; d^{L}_{-s_{\mathrm{o}}-t_{\mathrm{o}},-a}(\mu)
\end{aligned} \tag{25}
$$