Notes on curved-sky QE responses etc.

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Contents

JC: Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [?] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$ {}_{s_{\rm o}+t_{\rm o}}\hat{d}(\hat{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_{\rm o}s_{\rm i}}\;{}_{s_{\rm i}}\bar{X}_{\ell m}\;{}_{s_{\rm o}}Y_{\ell m}(\hat{n})\right)\left(\sum_{\ell m} w_\ell^{t_{\rm o}t_{\rm i}}\;{}_{t_{\rm i}}\bar{X}_{\ell m}\;{}_{t_{\rm o}}Y_{\ell m}(\hat{n})\right) \tag{1}$$

where $s_{\rm i}, t_{\rm i}$ are input spins, $s_{\rm o}, t_{\rm o}$ outputs spins, and $w_\ell^{s_{\rm o}s_{\rm i}}, w_\ell^{t_{\rm i}t_{\rm o}}$ associated weights. The maps ${}_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$$ {}_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{2}$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields $\bar{X}$ of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G^s_{LM} &= -\tfrac{1}{2}\left({}_{|s|}d_{LM} + (-1)^s\;{}_{-|s|}d_{LM}\right) \\ C^s_{LM} &= -\tfrac{1}{2i}\left({}_{|s|}d_{LM} - (-1)^s\;{}_{-|s|}d_{LM}\right). \end{aligned}$$

The formulae exposed here can be derived through simple application of this relation,

$$ \sum_{m_1,m_2}\int d^2n \prod_{i=1}^{3}{}_{s_i}Y_{\ell_i m_i}(\hat{n})\int d^2n' \prod_{i=1}^{3}{}_{t_i}Y_{\ell_i m_i}(\hat{n}') = \frac{2\ell_1+1}{4\pi}\frac{2\ell_2+1}{4\pi}2\pi\int_{-1}^{1}d\mu\prod_{i=1}^{3}d^{\ell_i}_{s_i,t_i}(\mu) \tag{3}$$

0.1 (Semi-)analytical QE Gaussian noise bias.

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_{\rm i}, s_{\rm o}, w^{s_{\rm i}s_{\rm o}})$ collectively describes the in and out spins and weight function, and similarly for t, u and v. Let the covariance function $N_L^{st,uv}$ be defined through

$$\begin{aligned} \delta_{LL'}\delta_{MM'}N_L^{stuv} &\equiv \left\langle {}_{s_{\rm o}+t_{\rm o}}\hat{d}_{LM}\;{}_{u_{\rm o}+v_{\rm o}}\hat{d}^*_{L'M'}\right\rangle\Big|_{\rm Gauss} \\ &= (-1)^{s_{\rm o}+t_{\rm o}+u_{\rm o}+v_{\rm o}}2\pi\int_{-1}^{1}d\mu\, d^L_{-s_{\rm o}-t_{\rm o},-u_{\rm o}-v_{\rm o}}(\mu)\left[\xi^{su}(\mu)\,\xi^{tv}(\mu)+\xi^{sv}(\mu)\,\xi^{tu}(\mu)\right] \end{aligned} \tag{4}$$

where ξ are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_{\ell} \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_o s_i} w_\ell^{t_o t_i} \bar{C}_\ell^{s_i t_i} d^\ell_{s_o, t_o}(\mu) \text{ with } \bar{C}_\ell^{s_i t_i} \equiv \left\langle {}_{s_i}\bar{X}_{\ell m}\, {}_{t_i}\bar{X}^*_{\ell m} \right\rangle \tag{5}$$

and $d^\ell_{mm'}$ are Wigner small d-matrices. Then

$$\boxed{\begin{aligned} \left\langle \hat{G}^{s_o+t_o}_{LM} \hat{G}^{*,u_o+v_o}_{L'M'} \right\rangle \Big|_{\text{Gauss.}} &= \delta_{LL'}\delta_{MM'} \frac{1}{2}\left[N_L^{stuv} + (-1)^{s_o+t_o} N_L^{-s-tuv}\right] \\ \left\langle \hat{C}^{s_o+t_o}_{LM} \hat{C}^{*,u_o+v_o}_{L'M'} \right\rangle \Big|_{\text{Gauss.}} &= \delta_{LL'}\delta_{MM'} \frac{1}{2}\left[N_L^{stuv} - (-1)^{s_o+t_o} N_L^{-s-tuv}\right] \\ \left\langle \hat{G}^{s_o+t_o}_{LM} \hat{C}^{*,u_o+v_o}_{L'M'} \right\rangle \Big|_{\text{Gauss.}} &= 0 \end{aligned}} \tag{6}$$

0.2 QE responses

Let the covariance of the CMB data respond as follows to a spin-r ($r \geq 0$) anisotropy source α:

$$\delta \left\langle {}_sX(\hat{n})\, {}_tX^*(\hat{n}')\right\rangle = \sum_{\ell m, a=\pm r} {}_a\alpha(\hat{n}) W_\ell^{a,st}\, {}_{s-a}Y_{\ell m}(\hat{n})\, {}_tY^*_{\ell m}(\hat{n}') + W_\ell^{a,ts}\, {}_sY_{\ell m}(\hat{n})\, {}_{t-a}Y^*_{\ell m}(\hat{n}')\, {}_{-a}\alpha(\hat{n}') \tag{7}$$

for some weights functions $W_\ell^{a,st}$. For instance, if the anisotropy can be described at the level of the CMB maps, such as for lensing, with

$${}_s\delta X(\hat{n}) = \sum_{a=\pm r} {}_a\alpha(\hat{n}) \left(\sum_{\ell m} R_\ell^{a,s}\, {}_sX_{\ell m}\, {}_{s-a}Y_{\ell m}(\hat{n})\right) \tag{8}$$

for harmonic responses R, then holds

$$W_\ell^{a,st} = R^{a,s} C_\ell^{st}. \tag{9}$$

However, Eq. (7) is more general. Examples include:

- Lensing: The source of anisotropy is the spin-1 field ${}_1\alpha(\hat{n})$, with response

$$\delta_s X(\hat{n}) = -\frac{1}{2}\alpha_1(\hat{n})\eth_s X(\hat{n}) - \frac{1}{2}\alpha_{-1}(\hat{n})\bar{\eth}\, {}_sX(\hat{n}) \tag{10}$$

  where $\eth$ and $\bar{\eth}$ are the spin lowering and spin raising operator JC: check notation respectively. Hence

$$R_\ell^{1,s} = ... R_\ell^{-1,s} = \tag{11}$$

- Modulation estimator: The source is spin 0, with response

$$\delta_s X(\hat{n}) = {}_0\alpha(\hat{n})_s X(\hat{n}) \tag{12}$$

  Hence,

$$R_\ell^{st} = \delta_{st} \tag{13}$$

- Point sources in temperature:

$$W_\ell^{r,st} = \frac{1}{4}\delta_{r0}\delta_{s0}\delta_{t0} \tag{14}$$

- Noise anisotropies (same as point sources but acting on beam deconvovled maps) JC: does picking a fiducial noise value matter?:

$$W_\ell^{r,st} = \frac{1}{4}\delta_{r0}\delta_{s0}\delta_{t0}\frac{1}{b_\ell^2} \tag{15}$$

Let further the isotropic limit of the filtering procedure be the matrix F, defined through

$$ {}_s\bar{X}_{\ell m} = \sum_{s_2=0,2,-2} F_\ell^{ss_2}\, {}_{s_2}X_{\ell m} \quad \text{(isotropic approximation)}. \tag{16} $$

Then the gradient and curl responses of estimator (1) are

$$ \boxed{\begin{aligned} \mathcal{R}_L^{gg} &= R_L^{st,r} + (-1)^r R_L^{st,-r} \\ \mathcal{R}_L^{cc} &= R_L^{st,r} - (-1)^r R_L^{st,-r} \\ \mathcal{R}_L^{gc} &= 0 = \mathcal{R}_L^{cg}, \end{aligned}} \tag{17} $$

where $R_L^{st,r}$ is

$$ R_L^{st,r} = (-1)^{s_o+t_o} 2\pi \int_{-1}^{1} d\mu\, d^L_{-s_o-t_o,-r}(\mu) \sum_{\tilde{s}_i,\tilde{t}_i=0,2,-2} \left[\xi^{s_o s_i \tilde{s}_i}(\mu)\psi^{t_o t_i \tilde{t}_i \tilde{s}_i,r}(\mu) + \xi^{t_o t_i \tilde{t}_i}(\mu)\psi^{s_o s_i \tilde{s}_i \tilde{t}_i,r}(\mu)\right] \tag{18} $$

with

$$ \begin{aligned} \xi^{s_o s_i \tilde{s}_i}(\mu) &\equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_o s_i} F_\ell^{s_i \tilde{s}_i} d^\ell_{s_o,\tilde{s}_i}(\mu) \\ \psi^{s_o s_i \tilde{s}_i \tilde{t}_i,r}(\mu) &\equiv (-1)^r \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w^{s_o s_i} F_\ell^{s_i \tilde{s}_i} W_\ell^{-r,-\tilde{t}_i \tilde{s}_i} d^\ell_{s_o,-\tilde{t}_i+r}(\mu) \end{aligned} \tag{19} $$

0.3 Optimal QE weights

Optimal QE weights are easily gained from the representation in Eq. 7 of the anisotropy. Let

$$ {}_{\pm r}\hat{g}^\alpha(\hat{n}) = \frac{\delta}{\delta_{\mp r}\alpha(\hat{n})} - \frac{1}{2}\, {}_{s_1}X^{\rm dat} \mathrm{Cov}^{-1}_{s_1 s_2}\, {}_{s_2}X^{\rm dat}. \tag{20} $$

where $\mathrm{Cov}_{s_1 s_2}(\hat{n},\hat{n}') \equiv \left\langle {}_{s_1}X^{\rm dat}(\hat{n})\, {}_{s_2}X^{\rm dat}(\hat{n}')\right\rangle$