Notes on curved-sky QE responses etc.

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Contents

JC: Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [**?**] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$ {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}(\boldsymbol{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_\mathrm{i} s_\mathrm{o}}\; {}_{s_\mathrm{i}}\bar{X}_{\ell m}\; {}_{s_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\right)\left(\sum_{\ell m} w_\ell^{t_\mathrm{i} t_\mathrm{o}}\; {}_{t_\mathrm{i}}\bar{X}_{\ell m}\; {}_{t_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\right) \tag{1}$$

where $s_\mathrm{i}, t_\mathrm{i}$ are input spins, $s_\mathrm{o}, t_\mathrm{o}$ outputs spins, and $w_\ell^{s_\mathrm{o} s_\mathrm{i}}, w_\ell^{t_\mathrm{i} t_\mathrm{o}}$ associated weights. The maps ${}_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$$ {}_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{2}$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields $\bar{X}$ of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G_{LM}^s &= -\tfrac{1}{2}\left({}_{|s|}d_{LM} + (-1)^s\; {}_{-|s|}d_{LM}\right) \\ C_{LM}^s &= -\tfrac{1}{2i}\left({}_{|s|}d_{LM} - (-1)^s\; {}_{-|s|}d_{LM}\right). \end{aligned}$$

0.1 Semi-analytical QE $N_L^{(0)}$ calculations

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_\mathrm{i}, s_\mathrm{o}, w^{s_\mathrm{i} s_\mathrm{o}})$ collectively describes the in and out spins and weight function, and similarly for t, u and v. Let the response function $N_L^{st,uv}$ be defined as

$$(-1)^{t_\mathrm{o}+v_\mathrm{o}+t_\mathrm{i}+v_\mathrm{i}} N_L^{st,uv} \equiv 2\pi \int_{-1}^{1} d\mu\, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_\mathrm{o}-v_\mathrm{o}, s_\mathrm{o}+u_\mathrm{o}}(\mu) \tag{3}$$

where ξ are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_{\ell} \left(\frac{2\ell+1}{4\pi}\right) w_{\ell}^{s_{\rm i}s_{\rm o}} w_{\ell}^{t_{\rm i}t_{\rm o}} \bar{C}_{\ell}^{s_{\rm i}t_{\rm i}} d^{\ell}_{-t_{\rm o},s_{\rm o}}(\mu) \text{ with } \bar{C}_{\ell}^{s_{\rm i}t_{\rm i}} \equiv \left\langle {}_{s_{\rm i}}\bar{X}_{\ell m}\, {}_{t_{\rm i}}\bar{X}^{*}_{\ell m}\right\rangle \tag{4}$$

and $d^{\ell}_{mm'}$ are Wigner small d-matrices. Then

$$\boxed{\begin{aligned}
\left\langle G_{LM}^{s_{\rm o}+t_{\rm o}} G_{LM}^{*,u_{\rm o}+v_{\rm o}}\right\rangle &= \frac{1}{2}\left[(-1)^{s_{\rm o}+t_{\rm o}} N_L^{-su,-tv} + (-1)^{u_{\rm o}+v_{\rm o}} N_L^{s-v,t-u} + N_L^{su,tv} + N_L^{sv,tu}\right] \\
\left\langle C_{LM}^{s_{\rm o}+t_{\rm o}} C_{LM}^{*,u_{\rm o}+v_{\rm o}}\right\rangle &= \frac{1}{2}\left[(-1)^{s_{\rm o}+t_{\rm o}} N_L^{-su,-tv} + (-1)^{u_{\rm o}+v_{\rm o}} N_L^{s-v,t-u} - N_L^{su,tv} - N_L^{sv,tu}\right] \\
\left\langle G_{LM}^{s_{\rm o}+t_{\rm o}} C_{LM}^{*,u_{\rm o}+v_{\rm o}}\right\rangle &= 0
\end{aligned}} \tag{5}$$

JC: Hmm, implementation still not right. overall minus sign and only O(1) Ok. (typo, two minus signs in first term)!

0.2 QE response calculation

Let the estimator respond to a source (with spin-weight r) as follows

$${}_sX(\boldsymbol{n}) = {}_r\alpha(\boldsymbol{n})\left(\sum_{\ell m} R_{\ell}^{rs}\, {}_sX_{\ell m}\, {}_{s-r}Y_{\ell m}(\boldsymbol{n})\right) + {}_{-r}\alpha(\boldsymbol{n})\left(\sum_{\ell m} R_{\ell}^{-rs}\, {}_sX_{\ell m}\, {}_{s+r}Y_{\ell m}(\boldsymbol{n})\right) \tag{6}$$

for some harmonic space responses R_{ℓ}^{rs}. Then the gradient and curl responses of estimator (1) are

$$\boxed{\begin{aligned}
\mathcal{R}_L^{gg} &= R_L^{ts,-r} + R_L^{st,-r} + (-1)^r\left(R_L^{ts,r} + R_L^{st,r}\right) \\
\mathcal{R}_L^{cc} &= R_L^{ts,-r} + R_L^{st,-r} - (-1)^r\left(R_L^{ts,r} + R_L^{st,r}\right) \\
\mathcal{R}_L^{gc} &= 0 = \mathcal{R}_L^{cg},
\end{aligned}} \tag{7}$$

with JC: filters in this. This is actually $\bar{X}\bar{X}$ spectrum not XX. TT Ok, finish the others.

$$R_L^{st,r} = 2\pi \int_{-1}^{1} d\mu\, \Xi^{st,r}(\mu)\, \Xi^{t}(\mu)\, d^{L}_{-r,s_{\rm o}+t_{\rm o}}(\mu) \tag{8}$$

where

$$\Xi^{st,r}(\mu) = \sum_{l}\left(\frac{2\ell+1}{4\pi}\right) C_{\ell}^{s_{\rm i},-t_{\rm i}} R_{\ell}^{r-t_{\rm i}} w_{\ell}^{s_{\rm i},s_{\rm o}} d^{\ell}_{t_{\rm i}-r,s_{\rm o}}(\mu) \text{ and } \Xi^{t}(\mu) = \sum_{l}\left(\frac{2\ell+1}{4\pi}\right) w_{\ell}^{t_{\rm i},t_{\rm o}} d^{\ell}_{-t_{\rm i},t_{\rm o}}(\mu), \textit{wrong}\quad C_{\ell}^{s_{\rm i},t_{\rm i}} \equiv \left\langle {}_{s_{\rm i}}X_{\ell m}\, {}_{t_{\rm i}}X^{\dagger}_{\ell m}\right\rangle \tag{9}$$

1 Sketchy derivation to cleanup

For this we need a result using the spin-weight spherical harmonic theorem. Define $N_L^{st,uv}$ through

$$
\begin{aligned}
N^{st,uv}(\boldsymbol{n},\boldsymbol{n}') \equiv& (-1)^{t_o+v_o+t_i+v_i}\left(\sum_{\ell m} g_\ell^{s_i} g_\ell^{t_i} C_\ell^{s_i t_i}\ {}_{s_o}Y_{\ell m}(\boldsymbol{n})\ {}_{-t_o}Y^*_{\ell m}(\boldsymbol{n}')\right)\left(\sum_{\ell m} g_\ell^{u_i} g_\ell^{v_i} C_\ell^{u_i v_i}\ {}_{u_o}Y_{\ell m}(\boldsymbol{n})\ {}_{-v_o}Y^*_{\ell m}(\boldsymbol{n}')\right) \\
\equiv&\ (-1)^{t_o+v_o+t_i+v_i}\sum_{LM} N_L^{stuv}\ {}_{s_o+u_o}Y_{LM}(\boldsymbol{n})\ {}_{-t_o-v_o}Y^*_{LM}(\boldsymbol{n}')
\end{aligned}
\tag{10}
$$

Then we can write

$$
\left\langle {}_{s_o+t_o}\hat{d}(\boldsymbol{n})\ {}_{u_o+v_o}\hat{d}(\boldsymbol{n}')\right\rangle = N^{su,tv}(\boldsymbol{n},\boldsymbol{n}') + N^{sv,tu}(\boldsymbol{n},\boldsymbol{n}') \tag{11}
$$

Taking the harmonic transform, we get

$$
\left\langle {}_{s_o+t_o}\hat{d}_{LM}\ {}_{u_o+v_o}\hat{d}_{L'M'}\right\rangle = (-1)^M \delta_{M,-M'}\delta_{L,L'}\left(N_L^{su,tv} + N_L^{sv,tu}\right)(-1)^{u_o+v_o} \tag{12}
$$

or

$$
\left\langle {}_{s_o+t_o}\hat{d}_{LM}\ {}_{u_o+v_o}\hat{d}^*_{L'M'}\right\rangle = \delta_{MM'}\delta_{LL'}\left(N_L^{su,tv} + N_L^{sv,tu}\right) \tag{13}
$$

In general we have

$$
\begin{aligned}
G^s_{LM} &= -\tfrac{1}{2}\left({}_s d_{LM} + (-1)^s\ {}_{-s}d_{LM}\right) \quad (s \geq 0) \\
C^s_{LM} &= -\tfrac{1}{2i}\left({}_s d_{LM} - (-1)^s\ {}_{-s}d_{LM}\right) \quad (s \geq 0).
\end{aligned}
$$

The estimator for ${}_{-s_o-t_o}\hat{d}$ is the same as ${}_{s_o+t_o}\hat{d}$ with all spin signs flipped, and with an overall sign $(-1)^{s_o+s_i+t_o+t_i}$. The out-spins part gets canceled by the sign $(-1)^s$ in the above equation. JC: No. better to request $w_\ell^{-s_i,-s_o} = (-1)^{s_o+si} w_\ell^{s_i,s_o}$ so that ${}_{-s_o-t_o}\hat{d} = {}_{s_o+t_o}\ \hat{d}^*$? Thats what we do nowHence,

$$
\begin{aligned}
&\delta_{MM'}\delta_{LL'}\left\langle G^{s_o+t_o}_{LM} G^{*,u_o+v_o}_{L'M'}\right\rangle \cdot 4 = N_L^{su,tv} + N_L^{sv,tu} + (-1)^{s_i+t_i}\left(N_L^{-su,-tv} + N_L^{-sv,-tu}\right) \\
&+ (-1)^{u_i+v_i}\left(N_L^{s-u,t-v} + N_L^{s-v,t-u}\right) + (-1)^{s_i+t_i+u_i+v_i}\left(N_L^{-s-u,-t-v} + N_L^{-s-v,-t-u}\right) \quad (s_o+t_o >= 0, u_o+v_o >= 0)
\end{aligned}
\tag{14}
$$

Since N is invariant under the simultaneous sign-flip of all spins JC: Wrong, also with sign-weighted weights: $\bar{C}^{-s,-t} = (-1)^{s+t}C^{t,s} = (-1)^{s+t}C^{s,t}(s,t \geq 0)$, so $N^{st,uv}$ takes a sign $(-1)^{s_i+t_i+u_i+v_i}$ and this solves the problem, we can also write this as:

$$
\left\langle G^{s_o+t_o}_{LM} G^{*,u_o+v_o}_{LM}\right\rangle = \frac{1}{2}\left[\left(N_L^{su,tv} + R_L^{sv,tu}\right) + (-1)^{s_o+t_o}\left(N_L^{-su,tv} + N_L^{-sv,tu}\right)\right] \tag{15}
$$

$$
\left\langle C^{s_o+t_o}_{LM} C^{*,u_o+v_o}_{LM}\right\rangle = -\frac{1}{2}\left[\left(N_L^{su,tv} + R_L^{sv,tu}\right) - (-1)^{s_o+t_o}\left(N_L^{-su,tv} + N_L^{-sv,tu}\right)\right] \tag{16}
$$

$$
\left\langle G^{s_o+t_o}_{LM} C^{*,u_o+v_o}_{LM}\right\rangle = 0 \tag{17}
$$

Pfeew

2 More details

Recall the spin-weight addition theorem:

$$
\sum_m {}_sY^*_{\ell m}(\boldsymbol{n}')\ {}_tY_{\ell m}(\boldsymbol{n}) = \sqrt{\frac{2\ell+1}{4\pi}} e^{-it\gamma}\ {}_tY_{\ell,-s}(\beta,\alpha). \tag{18}
$$

Hence

$$N^{st,uv}(\boldsymbol{n},\boldsymbol{n}') = (-1)^{t_o+v_o+t_i+v_i} e^{-is_o\gamma - iu_o\gamma} \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_i} g^{t_i} C_\ell^{s_i t_i} \; {}_{s_o}Y_{\ell t_o}(\beta,\alpha)\right) \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_i} g^{v_i} C_\ell^{u_i v_i} \; {}_{u_o}Y_{\ell v_o}(\beta,\alpha)\right) \tag{19}$$

The product of the brackets is a spin $s_o + u_o$ function. Defining its spin weight coefficients as N_L we get the relation claimed above. What are these coefficients?

$$N_L^{st,uv} \equiv (-1)^{t_o+v_o+t_i+v_i} \sqrt{\frac{4\pi}{2\ell+1}} \int d^2n \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_i} g^{t_i} C_\ell^{s_i t_i} \; {}_{s_o}Y_{\ell t_o}(\boldsymbol{n})\right) \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_i} g^{v_i} C_\ell^{u_i v_i} \; {}_{u_o}Y_{\ell v_o}(\boldsymbol{n})\right) {}_{s_o+u_o}Y^*_{L,t_o+v_o} \tag{20}$$

Using

$$\boxed{{}_sY_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}} (-1)^m e^{im\phi} d^\ell_{-ms}(\theta)} \tag{21}$$

The above thing is invariant if all signs are flipped at the same time. we get

$$\boxed{(-1)^{t_o+v_o+t_i+v_i} N_L^{st,uv} \equiv 2\pi \int_{-1}^{1} d\mu \, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_o-v_o,s_o+u_o}(\mu), \text{ with } \xi^{st}(\mu) \equiv \sum_\ell \frac{2\ell+1}{4\pi} g_\ell^{s_i} g_\ell^{t_i} C_\ell^{s_i t_i} d^\ell_{-t_o,s_o}(\mu)} \tag{22}$$

JC: Remember now the weights w (or g...) changes $(-1)^{s_i+s_o}$ under sign flip of the two spins, and $\bar{C}^{st}$ takes a $(-1)^{s+t}$. So $N^{-s-t,-u-v} = (-1)^{s_i+t_i+u_i+v_i+s_i+s_o+t_i+t_o+u_i+u_o+v_i+v_o} = (-1)^{s_o+t_o+u_o+v_o} N^{st,uv}$

2.1 QE responses calculation

JC: should define the response at the covariance matrix level...The covariance matrices is

$$\langle {}_sX(\boldsymbol{n})\, {}_tX^*(\boldsymbol{n}')\rangle = \langle {}_sX(\boldsymbol{n})\, {}_{-t}X(\boldsymbol{n}')\rangle = \sum_{\ell m} C_\ell^{st}\, {}_sY_{\ell m}(\boldsymbol{n})\, {}_tY^*_{\ell m}(\boldsymbol{n}') \tag{23}$$

How does this responds to a source of anisotropy (with spin r), ${}_r\alpha(\boldsymbol{n})$? For all cases in this work, we can parametrize this as follows

$$\delta \langle {}_sX(\boldsymbol{n})\, {}_tX(\boldsymbol{n}')\rangle = \, {}_r\alpha(\boldsymbol{n}') \sum_{\ell m} {}_rR_\ell^{st}\, {}_sY_{\ell m}(\boldsymbol{n})\, {}_{r-t}Y^*_{\ell m}(\boldsymbol{n}') + ((-t,\boldsymbol{n}') \leftrightarrow (s,\boldsymbol{n})) \tag{24}$$

for some set of isotropic response functions R_ℓ. What is the response to the estimator Eq. (1)?

Examples include:

- Lensing: The source of anisotropy is the spin-1 field ${}_1\alpha(\boldsymbol{n})$, with response

$$\delta_sX(\boldsymbol{n}) = \frac{1}{2}\alpha_1 \eth_sX(\boldsymbol{n}) + \frac{1}{2}\alpha_{-1}\bar{\eth}\, {}_sX(\boldsymbol{n}) \tag{25}$$

  where $\eth$ and $\bar{\eth}$ are the spin lowering and spin raising operator JC: check notation respectively. From their action on the spin-weighted harmonics, the harmonic space responses areJC: ...

$$s_r = s-1, s+1 \quad R^{s,s-1} = JC: ..., R^{s,s+1} = JC: ... \tag{26}$$

- Modulation estimator: The source is spin 0, with response

$$\delta_sX(\boldsymbol{n}) =_0 \alpha(\boldsymbol{n})_sX(\boldsymbol{n}) \tag{27}$$

  Hence,

$$s_r = s \quad , R_\ell^{ss} = 1 \tag{28}$$

- Point source:

$$\delta_sX(\boldsymbol{n}) = \, {}_0\alpha(\boldsymbol{n})\delta^D(\boldsymbol{n}) \tag{29}$$

- Noise anisotropies:

intermediate steps for response calc. Then $\left(R_\ell^{-r,-s} = (-1)^r R_\ell^{r,s}\right)$

$$ {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}_{LM} = \mathcal{R}_L^{st,r}\;{}_{-r}\alpha_{LM} + \mathcal{R}_L^{ts,-r}\;{}_{r}\alpha_{LM} \text{ with } \mathcal{R}_L^{st,r} = 2\pi\int_{-1}^{1} d\mu\,\Xi^{st,r}(\mu)\,\Xi^{t}(\mu)\,d^L_{r,s_\mathrm{o}+t_\mathrm{o}}(\mu) \tag{30}$$

where

$$\Xi^{st,r}(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi}\right) C_\ell^{s_\mathrm{i},-t_\mathrm{i}} R_\ell^{r-t_\mathrm{i}} w_\ell^{s_\mathrm{i},s_\mathrm{o}} d^\ell_{t_\mathrm{i}+r,s_\mathrm{o}}(\mu) \text{ and } \Xi^{t}(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{t_\mathrm{i},t_\mathrm{o}} d^\ell_{-t_\mathrm{i},t_\mathrm{o}}(\mu) \tag{31}$$

3 New derivation of generic response

Let

$$ {}_sX(\boldsymbol{n}') = {}_a\alpha(\boldsymbol{n}') \left(\sum_{\ell''m''} R^{a,s}_{\ell''}\;{}_sX_{\ell''m''}\;{}_{s-a}Y_{\ell''m''}(\boldsymbol{n}')\right) \tag{32}$$

where a sum over spin a is implicit. Let further the spin-weight spectra $C_\ell^{s_1s_2}$ be defined as

$$\left\langle {}_{s_1}X_{\ell m}\;{}_{s_2}X^\dagger_{\ell'm'}\right\rangle \equiv C_\ell^{s_1s_2}\delta_{\ell,\ell'}\delta_{m,m'} \tag{33}$$

Let further be the filtering $\mathcal{B}^\dagger\mathrm{Cov}^{-1}$ defined by a matrix

$$ {}_s\bar{X}_{\ell m} = F_\ell^{ss'}\;{}_{s'}X_{\ell m} \tag{34}$$

and the QE be (JC: NB: not exactly as above)

$$ {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}(\boldsymbol{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_\mathrm{o}s_\mathrm{i}}\;{}_{s_\mathrm{i}}\bar{X}_{\ell m}\;{}_{s_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\right)\left(\sum_{\ell' m'} w_{\ell'}^{t_\mathrm{o}t_\mathrm{i}}\;{}_{t_\mathrm{i}}\bar{X}_{\ell' m'}\;{}_{t_\mathrm{o}}Y_{\ell' m'}(\boldsymbol{n})\right). \tag{35}$$

Then, with $A \equiv wF$,

$$\begin{aligned}
{}_{s_0+t_0}\hat{d}_{LM} =\; & {}_a\alpha_{L'M'}R^{a,s_\mathrm{i}}_{\ell''}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}\left\langle {}_{s_\mathrm{i}}X_{\ell''m''}\;{}_{t_\mathrm{i}}X_{\ell'm'}\right\rangle \int d^2n\;{}_{s_\mathrm{o}}Y_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell'm'}\;{}_{s_\mathrm{o}+t_\mathrm{o}}Y^*_{LM}\int d^2n'\;{}_{s_\mathrm{i}}Y^*_{\ell m}\;{}_{s_\mathrm{i}-a}Y_{\ell''m''}\;{}_aY_{L'M'}\\
&+ {}_a\alpha_{L'M'}R^{a,t_\mathrm{i}}_{\ell''}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}\left\langle {}_{t_\mathrm{i}}X_{\ell''m''}\;{}_{s_\mathrm{i}}X_{\ell m}\right\rangle \int d^2n\;{}_{s_\mathrm{o}}Y_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell'm'}\;{}_{s_\mathrm{o}+t_\mathrm{o}}Y^*_{LM}\int d^2n'\;{}_{t_\mathrm{i}-a}Y_{\ell''m''}\;{}_{t_\mathrm{i}}Y^*_{\ell'm'}\;{}_aY_{L'M'}\\
&= {}_a\alpha_{L'M'}R^{a,s_\mathrm{i}}_{\ell''}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}(-1)^{s_\mathrm{i}-m'}C_\ell^{t_\mathrm{i},-s_\mathrm{i}}\int d^2n\;{}_{s_\mathrm{o}}Y_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell'm'}\;{}_{s_\mathrm{o}+t_\mathrm{o}}Y^*_{LM}\int d^2n'\;{}_{s_\mathrm{i}}Y^*_{\ell m}\;{}_{s_\mathrm{i}-a}Y_{\ell',-m'}\;{}_aY_{L'M'}\\
&+ {}_a\alpha_{L'M'}R^{a,t_\mathrm{i}}_{\ell''}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}(-1)^{t_\mathrm{i}-m}C_\ell^{s_\mathrm{i},-t_\mathrm{i}}\int d^2n\;{}_{s_\mathrm{o}}Y_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell'm'}\;{}_{s_\mathrm{o}+t_\mathrm{o}}Y^*_{LM}\int d^2n'\;{}_{t_\mathrm{i}-a}Y_{\ell,-m}\;{}_{t_\mathrm{i}}Y^*_{\ell'm'}\;{}_aY_{L'M'}\\
&= {}_a\alpha_{L'M'}R^{a,s_\mathrm{i}}_{\ell'}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}(-1)^{a}C_\ell^{t_\mathrm{i},-s_\mathrm{i}}\int d^2n\;{}_{s_\mathrm{o}}Y_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell'm'}\;{}_{s_\mathrm{o}+t_\mathrm{o}}Y^*_{LM}\int d^2n'\;{}_{s_\mathrm{i}}Y^*_{\ell m}\;{}_{-s_\mathrm{i}+a}Y^*_{\ell'm'}\;{}_aY_{L'M'}\\
&+ {}_a\alpha_{L'M'}R^{a,t_\mathrm{i}}_{\ell}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}(-1)^{a}C_\ell^{s_\mathrm{i},-t_\mathrm{i}}\int d^2n\;{}_{s_\mathrm{o}}Y_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell'm'}\;{}_{s_\mathrm{o}+t_\mathrm{o}}Y^*_{LM}\int d^2n'\;{}_{-t_\mathrm{i}+a}Y^*_{\ell m}\;{}_{t_\mathrm{i}}Y^*_{\ell'm'}\;{}_aY_{L'M'}\\
&= (-1)^{M-M'}\;{}_a\alpha_{L'M'}R^{a,s_\mathrm{i}}_{\ell'}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}(-1)^{s_\mathrm{o}+t_\mathrm{o}}C_\ell^{t_\mathrm{i},-s_\mathrm{i}}\int d^2n\;{}_{s_\mathrm{o}}Y_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell'm'}\;{}_{-s_\mathrm{o}-t_\mathrm{o}}Y_{L-M}\int d^2n'\;{}_{s_\mathrm{i}}Y^*_{\ell m}\;{}_{-s_\mathrm{i}+a}Y^*_{\ell'm'}\;{}_{-a}Y^*_{L'-M'}\\
&+ (-1)^{M-M'}\;{}_a\alpha_{L'M'}R^{a,t_\mathrm{i}}_{\ell}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}(-1)^{s_\mathrm{o}+t_\mathrm{o}}C_\ell^{s_\mathrm{i},-t_\mathrm{i}}\int d^2n\;{}_{s_\mathrm{o}}Y_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell'm'}\;{}_{-s_\mathrm{o}-t_\mathrm{o}}Y_{L-M}\int d^2n'\;{}_{-t_\mathrm{i}+a}Y^*_{\ell m}\;{}_{t_\mathrm{i}}Y^*_{\ell'm'}\;{}_{-a}Y^*_{L'-M'}\\
&= \delta_{LL,MM'}\;{}_a\alpha_{L'M'}R^{a,s_\mathrm{i}}_{\ell'}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}(-1)^{s_\mathrm{o}+t_\mathrm{o}}C_\ell^{t_\mathrm{i},-s_\mathrm{i}}\\
&+ \delta_{LL,MM'}\;{}_a\alpha_{L'M'}R^{a,t_\mathrm{i}}_{\ell}A^{s_\mathrm{o}s_\mathrm{i}}_{\ell}A^{t_\mathrm{o}t_\mathrm{i}}_{\ell'}(-1)^{s_\mathrm{o}+t_\mathrm{o}}C_\ell^{s_\mathrm{i},-t_\mathrm{i}}
\end{aligned} \tag{36}$$