

Notes on curved-sky QE responses etc.

October 26, 2018

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JC: Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [?] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$_{s_o+t_o}\hat{d}(\mathbf{n}) \equiv \left(\sum_{\ell m} w_{\ell}^{s_i s_o} \bar{X}_{\ell m} Y_{\ell m}(\mathbf{n}) \right) \left(\sum_{\ell m} w_{\ell}^{t_i t_o} \bar{X}_{\ell m} Y_{\ell m}(\mathbf{n}) \right) \quad (1)$$

where s_i, t_i are input spins, s_o, t_o outputs spins, and $w_{\ell}^{s_o s_i}, w_{\ell}^{t_i t_o}$ associated weights. The maps ${}_s\bar{X}_{\ell m}$ are the inverse variance filtered CMB maps,

$${}_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}). \quad (2)$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields \bar{X} of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G_{LM}^s &= -\frac{1}{2} (|s| d_{LM} + (-1)^s {}_{-|s|}d_{LM}) \\ C_{LM}^s &= -\frac{1}{2i} (|s| d_{LM} - (-1)^s {}_{-|s|}d_{LM}). \end{aligned}$$

0.1 Semi-analytical QE $N_L^{(0)}$ calculations

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_i, s_o, w^{s_i s_o})$ collectively describes the in and out spins and weight function, and similarly for t, u and v . Let the response function $N_L^{st, uv}$ be defined as

$$(-1)^{t_o+v_o+t_i+v_i} N_L^{st, uv} \equiv 2\pi \int_{-1}^1 d\mu \xi^{st}(\mu) \xi^{uv}(\mu) d_{-t_o-v_o, s_o+u_o}^L(\mu) \quad (3)$$

where ξ are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_{\ell} \left(\frac{2\ell+1}{4\pi} \right) w_{\ell}^{s_i s_o} w_{\ell}^{t_i t_o} \bar{C}_{\ell}^{s_i t_i} d_{-t_o, s_o}^{\ell}(\mu) \text{ with } \bar{C}_{\ell}^{s_i t_i} \equiv \langle {}_{s_i}\bar{X}_{\ell m} {}_{t_i}\bar{X}_{\ell m}^* \rangle \quad (4)$$

and $d_{mm'}^\ell$ are Wigner small d-matrices. Then

$$\boxed{\begin{aligned}\langle G_{LM}^{s_o+t_o} G_{LM}^{*,u_o+v_o} \rangle &= \frac{1}{2} [(-1)^{s_o+t_o} N_L^{-su,tv} + (-1)^{u_o+v_o} N_L^{s-v,t-u} + N_L^{su,tv} + N_L^{sv,tu}] \\ \langle C_{LM}^{s_o+t_o} C_{LM}^{*,u_o+v_o} \rangle &= \frac{1}{2} [(-1)^{s_o+t_o} N_L^{-su,tv} + (-1)^{u_o+v_o} N_L^{s-v,t-u} - N_L^{su,tv} - N_L^{sv,tu}] \\ \langle G_{LM}^{s_o+t_o} C_{LM}^{*,u_o+v_o} \rangle &= 0\end{aligned}} \quad (5)$$

JC: TTTT checked OK. See below for R symmetries and conventions

0.2 QE response calculation

Let the estimator respond to a source (with spin-weight r) as follows

$${}_s X(\mathbf{n}) = {}_r \alpha(\mathbf{n}) \left(\sum_{\ell m} {}_s R_\ell^{rs} {}_s X_{\ell m} {}_{s-r} Y_{\ell m}(\mathbf{n}) \right) + {}_{-r} \alpha(\mathbf{n}) \left(\sum_{\ell m} {}_s R_\ell^{-rs} {}_s X_{\ell m} {}_{s+r} Y_{\ell m}(\mathbf{n}) \right) \quad (6)$$

for some harmonic space responses R_ℓ^{rs} . Then the gradient and curl responses of estimator (1) are

$$\boxed{\begin{aligned}\mathcal{R}_L^{gg} &= R_L^{ts,-r} + R_L^{st,-r} + (-1)^r (R_L^{ts,r} + R_L^{st,r}) \\ \mathcal{R}_L^{cc} &= R_L^{ts,-r} + R_L^{st,-r} - (-1)^r (R_L^{ts,r} + R_L^{st,r}) \\ \mathcal{R}_L^{gc} &= 0 = \mathcal{R}_L^{cg},\end{aligned}} \quad (7)$$

with

$$R_L^{st,r} = 2\pi \int_{-1}^1 d\mu \Xi^{st,r}(\mu) \Xi^t(\mu) d_{r,s_o+t_o}^L(\mu) \quad (8)$$

where

$$\Xi^{st,r}(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi} \right) C_\ell^{s_i,-t_i} R_\ell^{r-t_i} w_\ell^{s_i,s_o} d_{t_i+r,s_o}^\ell(\mu) \text{ and } \Xi^t(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi} \right) w_\ell^{t_i,t_o} d_{-t_i,t_o}^\ell(\mu), \quad C_\ell^{s_i,-t_i} \equiv \langle {}_{s_i} X_{\ell m} {}_{t_i} X_{\ell m}^\dagger \rangle \quad (9)$$

1 Sketchy derivation to cleanup

For this we need a result using the spin-weight spherical harmonic theorem. Define $N_L^{st,uv}$ through

$$\begin{aligned} N^{st,uv}(\mathbf{n}, \mathbf{n}') &\equiv (-1)^{t_o+v_o+\textcolor{red}{t_i+v_i}} \left(\sum_{\ell m} g_\ell^{s_i} g_\ell^{t_i} C_\ell^{s_i t_i} Y_{\ell m}(\mathbf{n})_{-t_o} Y_{\ell m}^*(\mathbf{n}') \right) \left(\sum_{\ell m} g_\ell^{u_i} g_\ell^{v_i} C_\ell^{u_i v_i} Y_{\ell m}(\mathbf{n})_{-v_o} Y_{\ell m}^*(\mathbf{n}') \right) \\ &\equiv (-1)^{t_o+v_o+\textcolor{red}{t_i+v_i}} \sum_{LM} N_L^{stuv} Y_{LM}(\mathbf{n})_{s_o+u_o} Y_{LM}^*(\mathbf{n}')_{-t_o-v_o} \end{aligned} \quad (10)$$

Then we can write

$$\left\langle s_o+t_o \hat{d}(\mathbf{n})_{u_o+v_o} \hat{d}(\mathbf{n}') \right\rangle = N^{su,tv}(\mathbf{n}, \mathbf{n}') + N^{sv,tu}(\mathbf{n}, \mathbf{n}') \quad (11)$$

Taking the harmonic transform, we get

$$\left\langle s_o+t_o \hat{d}_{LM} Y_{LM}(\mathbf{n})_{u_o+v_o} \hat{d}_{L'M'}(\mathbf{n}') \right\rangle = (-1)^M \delta_{M,-M'} \delta_{L,L'} (N_L^{su,tv} + N_L^{sv,tu}) (-1)^{u_o+v_o} \quad (12)$$

or

$$\left\langle s_o+t_o \hat{d}_{LM} Y_{LM}(\mathbf{n})_{u_o+v_o} \hat{d}_{L'M'}^*(\mathbf{n}') \right\rangle = \delta_{MM'} \delta_{LL'} (N_L^{su,tv} + N_L^{sv,tu}) \quad (13)$$

In general we have

$$\begin{aligned} G_{LM}^s &= -\frac{1}{2} ({}_s d_{LM} + (-1)^s {}_{-s} d_{LM}) \quad (s \geq 0) \\ C_{LM}^s &= -\frac{1}{2i} ({}_s d_{LM} - (-1)^s {}_{-s} d_{LM}) \quad (s \geq 0). \end{aligned}$$

The estimator for ${}_{-s_o-t_o} \hat{d}$ is the same as ${}_{s_o+t_o} \hat{d}$ with all spin signs flipped, and with an overall sign $(-1)^{s_o+s_i+t_o+t_i}$. The out-spins part gets canceled by the sign $(-1)^s$ in the above equation. **JC: No. better to request $w_\ell^{-s_i, -s_o} = (-1)^{s_o+s_i} w_\ell^{s_i, s_o}$ so that ${}_{-s_o-t_o} \hat{d} = {}_{s_o+t_o} \hat{d}^*$? That's what we do now** Hence,

$$\begin{aligned} \delta_{MM'} \delta_{LL'} \langle G_{LM}^{s_o+t_o} G_{L'M'}^{*,u_o+v_o} \rangle \cdot 4 &= N_L^{su,tv} + N_L^{sv,tu} + (-1)^{s_i+t_i} (N_L^{-su,-tv} + N_L^{-sv,-tu}) \\ &+ (-1)^{u_i+v_i} (N_L^{s-u,t-v} + N_L^{s-v,t-u}) + (-1)^{s_i+t_i+u_i+v_i} (N_L^{-s-u,-t-v} + N_L^{-s-v,-t-u}) \quad (s_o+t_o \geq 0, u_o+v_o \geq 0) \end{aligned} \quad (14)$$

Since N is invariant under the simultaneous sign-flip of all spins **JC: Wrong, also with sign-weighted weights: $\bar{C}^{-s,-t} = (-1)^{s+t} C^{t,s} = (-1)^{s+t} C^{s,t}$ ($s, t \geq 0$), so $N^{st,uv}$ takes a sign $(-1)^{s_i+t_i+u_i+v_i}$ and this solves the problem, we can also write this as:**

$$\langle G_{LM}^{s_o+t_o} G_{LM}^{*,u_o+v_o} \rangle = \frac{1}{2} [(N_L^{su,tv} + R_L^{sv,tu}) + (-1)^{s_o+t_o} (N_L^{-su,tv} + N_L^{-sv,tu})] \quad (15)$$

$$\langle C_{LM}^{s_o+t_o} C_{LM}^{*,u_o+v_o} \rangle = -\frac{1}{2} [(N_L^{su,tv} + R_L^{sv,tu}) - (-1)^{s_o+t_o} (N_L^{-su,tv} + N_L^{-sv,tu})] \quad (16)$$

$$\langle G_{LM}^{s_o+t_o} C_{LM}^{*,u_o+v_o} \rangle = 0 \quad (17)$$

Pfeew

2 More details

Recall the spin-weight addition theorem:

$$\sum_m {}_s Y_{\ell m}^*(\mathbf{n}') {}_t Y_{\ell m}(\mathbf{n}) = \sqrt{\frac{2\ell+1}{4\pi}} e^{-it\gamma} {}_t Y_{\ell, -s}(\beta, \alpha). \quad (18)$$

Hence

$$N^{st,uv}(\mathbf{n}, \mathbf{n}') = (-1)^{t_o+v_o+t_i+v_i} e^{-is_o\gamma - iu_o\gamma} \left(\sum_{\ell} \sqrt{\frac{2\ell+1}{4\pi}} g^{s_i} g^{t_i} C_{\ell}^{s_i t_i} {}_{s_o} Y_{\ell t_o}(\beta, \alpha) \right) \left(\sum_{\ell} \sqrt{\frac{2\ell+1}{4\pi}} g^{u_i} g^{v_i} C_{\ell}^{u_i v_i} {}_{u_o} Y_{\ell v_o}(\beta, \alpha) \right) \quad (19)$$

The product of the brackets is a spin $s_o + u_o$ function. Defining its spin weight coefficients as N_L we get the relation claimed above. What are these coefficients?

$$N_L^{st,uv} \equiv (-1)^{t_o+v_o+t_i+v_i} \sqrt{\frac{4\pi}{2\ell+1}} \int d^2n \left(\sum_{\ell} \sqrt{\frac{2\ell+1}{4\pi}} g^{s_i} g^{t_i} C_{\ell}^{s_i t_i} {}_{s_o} Y_{\ell t_o}(\mathbf{n}) \right) \left(\sum_{\ell} \sqrt{\frac{2\ell+1}{4\pi}} g^{u_i} g^{v_i} C_{\ell}^{u_i v_i} {}_{u_o} Y_{\ell v_o}(\mathbf{n}) \right) {}_{s_o+u_o} Y_{L, t_o+v_o}^* \quad (20)$$

Using

$$\boxed{{}_s Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} (-1)^m e^{im\phi} d_{-ms}^{\ell}(\theta)} \quad (21)$$

The above thing is invariant if all signs are flipped at the same time. we get

$$\boxed{(-1)^{t_o+v_o+t_i+v_i} N_L^{st,uv} \equiv 2\pi \int_{-1}^1 d\mu \xi^{st}(\mu) \xi^{uv}(\mu) d_{-t_o-v_o, s_o+u_o}^L(\mu), \text{ with } \xi^{st}(\mu) \equiv \sum_{\ell} \frac{2\ell+1}{4\pi} g_{\ell}^{s_i} g_{\ell}^{t_i} C_{\ell}^{s_i t_i} d_{-t_o, s_o}^{\ell}(\mu)} \quad (22)$$

JC: Remember now the weights w (or g ...) changes $(-1)^{s_i+s_o}$ under sign flip of the two spins, and \bar{C}^{st} takes a $(-1)^{s+t}$. So $N^{-s-t, -u-v} = (-1)^{s_i+t_i+u_i+v_i+s_i+s_o+t_i+t_o+u_i+u_o+v_i+v_o} = (-1)^{s_o+t_o+u_o+v_o} N^{st,uv}$

2.1 QE responses calculation

JC: should define the response at the covariance matrix level... The covariance matrices is

$$\langle {}_s X(\mathbf{n}) {}_t X^*(\mathbf{n}') \rangle = \langle {}_s X(\mathbf{n}) {}_{-t} X(\mathbf{n}') \rangle = \sum_{\ell m} C_{\ell}^{st} {}_s Y_{\ell m}(\mathbf{n}) {}_t Y_{\ell m}^*(\mathbf{n}') \quad (23)$$

How does this responds to a source of anisotropy (with spin r), ${}_r \alpha(\mathbf{n})$? For all cases in this work, we can parametrize this as follows

$$\delta \langle {}_s X(\mathbf{n}) {}_t X(\mathbf{n}') \rangle = {}_r \alpha(\mathbf{n}') \sum_{\ell m} {}_r R_{\ell}^{st} {}_s Y_{\ell m}(\mathbf{n}) {}_{r-t} Y_{\ell m}^*(\mathbf{n}') + ((-t, \mathbf{n}') \leftrightarrow (s, \mathbf{n})) \quad (24)$$

for some set of isotropic response functions R_{ℓ} . What is the response to the estimator Eq. (1)?

Examples include:

- Lensing: The source of anisotropy is the spin-1 field ${}_1 \alpha(\mathbf{n})$, with response

$$\delta_s X(\mathbf{n}) = \frac{1}{2} \alpha_1 \bar{\partial}_s X(\mathbf{n}) + \frac{1}{2} \alpha_{-1} \bar{\partial}_{-s} X(\mathbf{n}) \quad (25)$$

where $\bar{\partial}$ and $\bar{\partial}$ are the spin lowering and spin raising operator JC: check notation respectively. From their action on the spin-weighted harmonics, the harmonic space responses are JC: ...

$$s_r = s - 1, s + 1 \quad R^{s, s-1} = \text{JC} : \dots, R^{s, s+1} = \text{JC} : \dots \quad (26)$$

- Modulation estimator: The source is spin 0, with response

$$\delta_s X(\mathbf{n}) = {}_0 \alpha(\mathbf{n}) {}_s X(\mathbf{n}) \quad (27)$$

Hence,

$$s_r = s, R_{\ell}^{ss} = 1 \quad (28)$$

- Point source:

$$\delta_s X(\mathbf{n}) = {}_0 \alpha(\mathbf{n}) \delta^D(\mathbf{n}) \quad (29)$$

- Noise anisotropies:

intermediate steps for response calc. Then $(R_\ell^{-r,-s} = (-1)^r R_\ell^{r,s})$

$$_{s_o+t_o}\hat{d}_{LM} = \mathcal{R}_L^{st,r} {}_{-r}\alpha_{LM} + \mathcal{R}_L^{ts,-r} {}_r\alpha_{LM} \text{ with } \mathcal{R}_L^{st,r} = 2\pi \int_{-1}^1 d\mu \Xi^{st,r}(\mu) \Xi^t(\mu) d_{r,s_o+t_o}^L(\mu) \quad (30)$$

where

$$\Xi^{st,r}(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi} \right) C_\ell^{s_i,-t_i} R_\ell^{r-t_i} w_\ell^{s_i,s_o} d_{t_i+r,s_o}^\ell(\mu) \text{ and } \Xi^t(\mu) = \sum_l \left(\frac{2\ell+1}{4\pi} \right) w_\ell^{t_i,t_o} d_{-t_i,t_o}^\ell(\mu) \quad (31)$$