# Notes on curved-sky quadratic estimation

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JC: Document to be included with the pipeline release after submission of the revised L08.

## 1 Notes on curved-sky quadratic estimation

Lensing and others quadratic estimators used in Ref. [?] are all built multiplying in position space spin transforms of spin-weighted fields. The purpose of this document is to collect the relevant formulae in the spin-weight formalism. The numerical implementation of the estimator responses and noise covariances in the released lensing estimation pipeline follows this document.

The gradient (g) and curl (c) modes of definite parity of a spin-$r$ field ${}_r\alpha(\hat{n})$ with ${}_{-r}\alpha(\hat{n}) = {}_r\alpha^*(\hat{n})$ are defined through

$$\begin{aligned} g^r_{LM} &= -\tfrac{1}{2}\left({}_{|r|}\alpha_{LM} + (-1)^r\ {}_{-|r|}\alpha_{LM}\right) \\ c^r_{LM} &= -\tfrac{1}{2i}\left({}_{|r|}\alpha_{LM} - (-1)^r\ {}_{-|r|}\alpha_{LM}\right). \end{aligned}$$

where ${}_{\pm r}\alpha_{LM} \equiv \int d^2n\ {}_{\pm r}\alpha(\hat{n})\ {}_{\pm r}Y^*_{LM}(\hat{n})$. Prior to projection onto gradient and curl modes, and prior to proper normalization, the quadratic estimators can all be written in the form

$${}_r\hat{\alpha}(\hat{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_\mathrm{o}s_\mathrm{i}}\ {}_{s_\mathrm{i}}\bar{X}_{\ell m}\ {}_{s_\mathrm{o}}Y_{\ell m}(\hat{n})\right)\left(\sum_{\ell m} w_\ell^{t_\mathrm{o}t_\mathrm{i}}\ {}_{t_\mathrm{i}}\bar{X}_{\ell m}\ {}_{t_\mathrm{o}}Y_{\ell m}(\hat{n})\right) \tag{1}$$

where $s_\mathrm{i}, t_\mathrm{i}$ are input spins, $s_\mathrm{o}, t_\mathrm{o}$ outputs spins with $t_\mathrm{o} + s_\mathrm{o} = r$, and $w_\ell^{s_\mathrm{o}s_\mathrm{i}}, w_\ell^{t_\mathrm{i}t_\mathrm{o}}$ associated weights.

The maps ${}_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$${}_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right), \tag{2}$$

and (for the purposes of the analytical calculations in this document) are isotropically related to the data maps ${}_sX$ through a matrix $F$,

$${}_s\bar{X}_{\ell m} \equiv \sum_{s_2=0,2,-2} F_\ell^{ss_2}\ {}_{s_2}X_{\ell m} \quad \text{(isotropic approximation of } \bar{X} = \mathcal{B}^\dagger \mathrm{Cov}^{-1} X^{\mathrm{dat}} \text{ in the notation of Ref. [?].} \tag{3}$$

For independently filtered temperature and polarization such as the Planck 2018 baseline analysis, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to $T, E$ and $B$ respectively, with spin-space matrix $F$ in Eq. (3)

$$F = \begin{pmatrix} F^T_\ell & 0 & 0 \\ 0 & \frac{1}{2}\left(F^E_\ell + F^B_\ell\right) & \frac{1}{2}\left(F^E_\ell - F^B_\ell\right) \\ 0 & \frac{1}{2}\left(F^E_\ell - F^B_\ell\right) & \frac{1}{2}\left(F^E_\ell + F^B_\ell\right) \end{pmatrix} \tag{4}$$

where

$$F^X_\ell = \frac{1}{C^{XX,\mathrm{fid}}_\ell + N^{\mathrm{X}}_\ell / b^2_\ell}, \qquad X = T, E, B. \tag{5}$$

In Ref [?], $F^X_\ell$ is set to zero outside $100 \leq \ell \leq 2048$, $N^T_\ell$ is 35$\mu$K-amin, $N^P_\ell$ is 55$\mu$K-amin, $b_\ell$ is Gaussian beam of FWHM 5-amin, and $F^X_\ell$ contains further an additional small rescaling. For joint temperature and polarization filtering, the $F$ matrix becomes: JC: Why minus sign again?

$$F = \begin{pmatrix} F^{TT}_\ell & -\frac{1}{2}F^{TE} & -\frac{1}{2}F^{TE} \\ -F^{TE} & \frac{1}{2}\left(F^{EE}_\ell + F^B_\ell\right) & \frac{1}{2}\left(F^{EE}_\ell - F^B_\ell\right) \\ -F^{TE} & \frac{1}{2}\left(F^{EE}_\ell - F^B_\ell\right) & \frac{1}{2}\left(F^{EE}_\ell + F^B_\ell\right) \end{pmatrix} \tag{6}$$

where the entries $F^{T,E,B}$ are the elements of

$$\begin{pmatrix} C^{TT}_\ell + N^T_\ell & C^{TE}_\ell & 0 \\ C^{TE}_\ell & C^{EE}_\ell + N^E_\ell & 0 \\ 0 & 0 & C^{BB}_\ell + N^B_\ell \end{pmatrix}^{-1} \tag{7}$$

The formulae exposed in this document can be derived through simple application of this formal relation,

$$\begin{aligned} &\sum_{m_1, m_2} \int d^2n \; {}_{s_1}Y_{\ell_1 m_1}(\hat{n}) \; {}_{s_2}Y_{\ell_2 m_2}(\hat{n}) \; {}_{r_1}Y_{LM}(\hat{n}) \int d^2n' \; {}_{t_1}Y_{\ell_1 m_1}(\hat{n}') \; {}_{t_2}Y_{\ell_2 m_2}(\hat{n}') \; {}_{r_2}Y_{L'M'}(\hat{n}') \\ &= \delta_{LL'}\delta_{MM'} \frac{2\ell_1 + 1}{4\pi} \frac{2\ell_2 + 1}{4\pi} 2\pi \int_{-1}^{1} d\mu \; d^{\ell_1}_{s_1, t_1}(\mu) d^{\ell_2}_{s_2 t_2}(\mu) d^L_{r_1 r_2}(\mu) \quad (s_1 + s_2 + r_1 = 0 = t_1 + t_2 + r_2). \end{aligned} \tag{8}$$

where $d^\ell_{mm'}$ are Wigner small d-matrices.

### 1.1 Gaussian noise covariance

Q.E. noise covariance can be evaluated with a series of one-dimensional integrals as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1), we now obtain their gradient (g) and curl (c) covariances with four integrals as follows.

For an isotropy estimator ${}_r\hat{\alpha}$ let $s = (s_{\mathrm{i}}, s_{\mathrm{o}}, w^{s_{\mathrm{i}} s_{\mathrm{o}}})$ collectively describes the in and out spins and weight function of the left leg, and similarly with $t$ for the right leg (with $s_{\mathrm{o}} + t_{\mathrm{o}} = r$). In the same way, let $u$ and $v$ describes another esimator ${}_{r'}\hat{\alpha}$ (with $u_{\mathrm{o}} + v_{\mathrm{o}} = r'$). Then, their Gaussian covariance may be written $\langle {}_r\hat{\alpha}_{LM} \, {}_{r'}\hat{\alpha}^*_{L'M'} \rangle|_{\mathrm{Gauss}} \equiv \delta_{LL'}\delta_{MM'} n^{stuv}_L$ with

$$\boxed{n^{stuv}_L = (-1)^{r+r'} 2\pi \int_{-1}^{1} d\mu \; d^L_{-r-r'}(\mu) \left[\xi^{su}(\mu)\,\xi^{tv}(\mu) + \xi^{sv}(\mu)\,\xi^{tu}(\mu)\right]} \tag{9}$$

where $\xi$ are position-space correlation functions

$$\boxed{\xi^{st}(\mu) \equiv \sum_\ell \left(\frac{2\ell + 1}{4\pi}\right) w^{s_{\mathrm{o}} s_{\mathrm{i}}}_\ell w^{t_{\mathrm{o}} t_{\mathrm{i}}}_\ell \bar{C}^{s_{\mathrm{i}} t_{\mathrm{i}}}_\ell d^\ell_{s_{\mathrm{o}}, t_{\mathrm{o}}}(\mu) \text{ with } \bar{C}^{s_{\mathrm{i}} t_{\mathrm{i}}}_\ell \equiv \left\langle {}_{s_{\mathrm{i}}}\bar{X}_{\ell m} \, {}_{t_{\mathrm{i}}}\bar{X}^*_{\ell m} \right\rangle}. \tag{10}$$

Projecting onto gradient and curl modes results in

$$\boxed{\begin{aligned}\left\langle \hat{g}^r_{LM}\hat{g}^{*,r'}_{L'M'}\right\rangle\Big|_{\text{Gauss.}} &= \delta_{LL'}\delta_{MM'}\frac{1}{2}\left[n^{stuv}_L + (-1)^r n^{-s-tuv}_L\right]\\ \left\langle \hat{c}^r_{LM}\hat{c}^{*,r'}_{L'M'}\right\rangle\Big|_{\text{Gauss.}} &= \delta_{LL'}\delta_{MM'}\frac{1}{2}\left[n^{stuv}_L - (-1)^r n^{-s-tuv}_L\right]\\ \left\langle \hat{g}^r_{LM}\hat{c}^{*,r'}_{L'M'}\right\rangle\Big|_{\text{Gauss.}} &= 0 = \left\langle \hat{c}^r_{LM}\hat{g}^{*,r'}_{L'M'}\right\rangle\Big|_{\text{Gauss.}}\end{aligned}} \tag{11}$$

Ref. [?] calculates the covariance matrix based on these equations using the empirical, realisation dependent power spectra $\bar{C}^{s_i,t_i}_\ell$.

## 1.2 Responses

We now turn to the calculation of the response of the estimator to a source of anisotropy. Anisotropy can sometimes be parametrized at the level of the CMB maps, (for example for lensing), with

$${}_s\delta X(\hat{n}) = \sum_{a=\pm r} {}_a\alpha(\hat{n})\left(\sum_{\ell m} R^{a,s}_\ell \, {}_sX_{\ell m}\, {}_{s-a}Y_{\ell m}(\hat{n})\right) \tag{12}$$

for response kernel functions $R^{r,s}_\ell$. More generally, let the covariance of the CMB data respond as follows to a spin-$r$ anisotropy source $\alpha$:

$$\delta\left\langle {}_sX(\hat{n})\, {}_tX^*(\hat{n}')\right\rangle = \sum_{\ell m, a=\pm r} {}_a\alpha(\hat{n})W^{a,st}_\ell \, {}_{s-a}Y_{\ell m}(\hat{n})\, {}_tY^*_{\ell m}(\hat{n}') + W^{a,ts}_\ell \, {}_sY_{\ell m}(\hat{n})\, {}_{t-a}Y^*_{\ell m}(\hat{n}')\, {}_{-a}\alpha(\hat{n}') \tag{13}$$

for some weights functions $W^{a,st}_\ell$. For map-level descriptions in Eq. (12) then holds

$$W^{a,st}_\ell = R^{a,s}C^{st}_\ell. \tag{14}$$

However, Eq. (13) is more general. Examples include:

1. Lensing: The source of anisotropy is the spin-1 field ${}_1\alpha(\hat{n})$, with linear response (see Ref. [?])

$$\delta_sX(\hat{n}) = -\frac{1}{2}\alpha_1(\hat{n})\bar{\eth}_sX(\hat{n}) - \frac{1}{2}\alpha_{-1}(\hat{n})\eth\, {}_sX(\hat{n}) \tag{15}$$

where $\eth$ and $\bar{\eth}$ are the spin raising and spin lowering operator respectively. Hence

$$R^{-1,s}_\ell = -\frac{1}{2}\sqrt{(l-s)(l+s+1)} \text{ and } R^{1,s}_\ell = +\frac{1}{2}\sqrt{(l+s)(l-s+1)} \tag{16}$$

2. CMB modulation: The source is spin 0, with response

$$\delta_sX(\hat{n}) = {}_0\alpha(\hat{n})_sX(\hat{n}) \text{ hence } R^{st}_\ell = \delta_{st} \tag{17}$$

3. Point sources in temperature ($S^2$, from Ref. [?]): here anisotropy is sought of the form

$$\delta\left\langle T(\hat{n})\, T(\hat{n}')\right\rangle = \delta_{\hat{n}\hat{n}'}S^2(\hat{n}) \text{ hence } W^{r,st}_\ell = \frac{1}{4}\delta_{r0}\delta_{s0}\delta_{t0} \tag{18}$$

4. Noise variance map anisotropies (basically the same as point sources but acting on beam-convolved maps)

$$W^{r,st}_\ell = \frac{1}{4}\delta_{r0}\delta_{s0}\delta_{t0}\frac{1}{b^2_\ell} \tag{19}$$

Let as before $s, t$ denote collectively the QE spins and weight functions for an estimator ${}_r\hat{\alpha}(\hat{n})$ of spin $r = s_o + t_o$, and let $r'$ be the spin of anisotropy source ${}_{r'}\beta(\hat{n})$ with covariance response kernel $W^{r'}$ as above. Let $\mathcal{R}_L^{g_r g_{r'}}\delta_{LL'}\delta_{MM'}$ be defined as the response of the gradient mode of $\alpha_{LM}$ to the gradient mode of $\beta_{L'M'}$, and similarly for the curl. It holds:

$$\boxed{\begin{aligned} \mathcal{R}_L^{g_r g_{r'}} &= R_L^{st,r'} + (-1)^{r'} R_L^{st,-r'} \\ \mathcal{R}_L^{c_r c_{r'}} &= R_L^{st,r'} - (-1)^{r'} R_L^{st,-r'} \\ \mathcal{R}_L^{g_r c_{r'}} &= 0 = \mathcal{R}_L^{c_r g_{r'}}, \end{aligned}} \tag{20}$$

where

$$\boxed{R_L^{st,r'} = (-1)^r 2\pi \int_{-1}^{1} d\mu\, d^L_{-r-r'}(\mu) \sum_{\tilde{s}_i,\tilde{t}_i=0,2,-2} \left[\xi^{s_o s_i \tilde{s}_i}(\mu)\psi^{t_o t_i \tilde{t}_i \tilde{s}_i, r'}(\mu) + \xi^{t_o t_i \tilde{t}_i}(\mu)\psi^{s_o s_i \tilde{s}_i \tilde{t}_i, r'}(\mu)\right]} \tag{21}$$

and

$$\boxed{\begin{aligned} \xi^{s_o s_i \tilde{s}_i}(\mu) &\equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_o s_i} F_\ell^{s_i \tilde{s}_i} d^\ell_{s_o,\tilde{s}_i}(\mu) \\ \psi^{s_o s_i \tilde{s}_i \tilde{t}_i, r'}(\mu) &\equiv (-1)^{r'} \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w^{s_o s_i} F_\ell^{s_i \tilde{s}_i} W_\ell^{-r', -\tilde{t}_i \tilde{s}_i} d^\ell_{s_o, -\tilde{t}_i + r'}(\mu) \end{aligned}} \tag{22}$$

If there is a unique source of anisotropy, properly normalized gradient and curl estimators are then given by $\hat{g}^r_{LM}/\mathcal{R}_L^{g_r g_r}$ and $\hat{c}^r_{LM}/\mathcal{R}_L^{c_r c_r}$.

## 1.3 Optimal QE weights

Optimal (in the sense of minimal Gaussian variance) QE weights are easily gained from the representation in Eq. 13 of the anisotropy. Let the CMB likelihood gradients be

$$_{\pm r}\hat{\alpha}(\hat{n}) = \frac{\delta}{\delta_{\mp r}\alpha(\hat{n})} - \frac{1}{2}\, {}_{s_1}X \mathrm{Cov}^{-1}_{s_1 s_2}\, {}_{s_2}X \Big|_{\alpha \equiv 0} \tag{23}$$

where $\mathrm{Cov}_{s_1 s_2}(\hat{n}, \hat{n}') \equiv \langle {}_{s_1}X(\hat{n})\, {}_{s_2}X(\hat{n}')\rangle$, and where ${}_r\alpha(\hat{n})$ and ${}_{-r}\alpha(\hat{n})$ are treated as independent variables. Using Eq. (13) and comparing to Eq. (1), we find

$$\boxed{w_\ell^{st} = \delta_{st} \text{ (1st leg)} \quad w_\ell^{-s+r,t} = 2W_\ell^{-r,-st} \text{ (2nd leg)}} \tag{24}$$

JC: why 2 again?