Notes on curved-sky anisotropy quadratic estimation implementation

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This document supplements the release of the Planck 2018 CMB lensing [1] pipeline, available at `https://github.com/carronj/plancklens`. It collects calculations relevant to curved-sky quadratic estimators in the spin-weight, position-space correlation function formalism, including analytic calculations of estimator responses and Gaussian noise biases between arbitrary pairs of quadratic estimators. It also contains the derivation of optimal, joint gradient and curl mode quadratic estimators for parametrized anisotropy of arbitrary spin.

CONTENTS

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I. SPIN-WEIGHT ESTIMATORS

A complex spin-s field ${}_sf(\hat n)$ is defined with reference to local axes by the condition that it transforms under a clockwise rotation of angle ψ of the local axes according to ${}_sf(\hat n) \rightarrow e^{is\psi}\,{}_sf(\hat n)$. We use further the notation ${}_{-s}f(\hat n) = {}_sf^*(\hat n)$.

The gradient (g) and curl (c) harmonic modes of definite parity of ${}_sf$ are then defined as follows

$$g^s_{\ell m} = -\frac{1}{2}\left({}_{|s|}f_{\ell m} + (-1)^s\,{}_{-|s|}f_{\ell m}\right) \tag{1.1}$$

$$c^s_{\ell m} = -\frac{1}{2i}\left({}_{|s|}f_{\ell m} - (-1)^s\,{}_{-|s|}f_{\ell m}\right) \tag{1.2}$$

where ${}_{\pm s}f_{\ell m} \equiv \int d^2n \,{}_{\pm s}f(\hat n)\,{}_{\pm s}Y^*_{\ell m}(\hat n)$. Spin-0 fields are real and pure gradients. With these conventions, we have in particular for the CMB polarization ${}_{\pm 2}P(\hat n)$

$$ {}_{\pm 2}P_{\ell m} = -(E_{\ell m} \pm iB_{\ell m}). \tag{1.3}$$

where E and B are the polarization gradient and curl modes[1]. Our polarization conventions are such that ${}_{\pm 2}P(\hat n) = Q(\hat n) \pm iU(\hat n)$, with the local x and y axes at each point $\hat n$ pointing south and east (following e.g. the healpix[2] software conventions, or those of Ref. [3], but differing from the IAU standards, see `https://healpix.jpl.nasa.gov/html/intronode12.htm`).
The relation inverse to Eqs. (1.1) and (1.2) is

$$ {}_{\pm|s|}f_{\ell m} = -(\pm)^s\left(g^s_{\ell m} \pm i c^s_{\ell m}\right). \tag{1.4}$$

A. Correlation functions

We use position-space correlation function for fields of arbitrary spins as follows. As is well-known from the case of CMB polarization, in order to undo the dependence on the local axis definition, the fields must first be defined with respect to a common relevant basis [4, 5, e.g.]. For two points on the sphere $\hat n_1$, $\hat n_2$, let γ be the angle at $\hat n_1$ between the local x-axis to the geodesic connecting $\hat n_1$ and $\hat n_2$, and α defined in the same way at $\hat n_2$. See Fig. 1 for the geometry. Clockwise rotations by $\pi-\gamma$ at $\hat n_1$ and by $\pi-\alpha$ at $\hat n_2$ align the local bases, and we may define

$$\begin{aligned}
\xi^{st}_+(\beta) &\equiv \left\langle e^{is(\pi-\gamma)}\,{}_sf(\hat n_1)\,\left({}_tf(\hat n_2)e^{it(\pi-\alpha)}\right)^*\right\rangle\\
\xi^{st}_-(\beta) &\equiv \left\langle \left(e^{is(\pi-\gamma)}\,{}_sf(\hat n_1)\right)^*\left({}_tf(\hat n_2)e^{it(\pi-\alpha)}\right)^*\right\rangle,
\end{aligned}\tag{1.5}$$

In harmonic space, and using relation (1.4) gives the following expression

$$\begin{aligned}
\xi^{st}_+(\beta) &= (+1)^s\sum_\ell\left(\frac{2\ell+1}{4\pi}\right)\left[C_\ell^{g^sg^t} + C_\ell^{c^sc^t} - i\left(C_\ell^{g^sc^t} - C_\ell^{c^sg^t}\right)\right]d^\ell_{st}(\beta)\\
\xi^{st}_-(\beta) &= (-1)^s\sum_\ell\left(\frac{2\ell+1}{4\pi}\right)\left[C_\ell^{g^sg^t} - C_\ell^{c^sc^t} - i\left(C_\ell^{g^sc^t} + C_\ell^{c^sg^t}\right)\right]d^\ell_{-st}(\beta)
\end{aligned}\tag{1.6}$$

where β is the distance between $\hat n_1$ and $\hat n_2$. These two correlation functions carry all of the information on their gradient and curl mode spectra.

B. Quadratic estimators

Prior to projection onto gradient and curl modes, and prior to proper normalization, separable quadratic estimators can be written as a (sum of) products of two position-space maps. Let $\hat q$ be such an unnormalized estimator:

$$ {}_{s_o+t_o}\hat q(\hat n) \equiv \left(\sum_{\ell m} w_\ell^{s_\mathrm{o}s_\mathrm{i}}\,{}_{s_\mathrm{i}}\bar X_{\ell m}\,{}_{s_\mathrm{o}}Y_{\ell m}(\hat n)\right)\left(\sum_{\ell m} w_\ell^{t_\mathrm{o}t_\mathrm{i}}\,{}_{t_\mathrm{i}}\bar X_{\ell m}\,{}_{t_\mathrm{o}}Y_{\ell m}(\hat n)\right) \tag{1.7}$$

where $s_\mathrm{i}, t_\mathrm{i}$ are input spins, $s_\mathrm{o}, t_\mathrm{o}$ outputs spins, and $w_\ell^{s_\mathrm{o}s_\mathrm{i}}, w_\ell^{t_\mathrm{i}t_\mathrm{o}}$ associated weights. For simplicity we use the same symbol w for the first and second leg weights even if they may differ in general for the same spin indices. By consistency with ${}_{-s_o-t_o}\hat q(\hat n) = {}_{s_o+t_o}\hat q^*(\hat n)$ the weights have symmetry $w_\ell^{-s_\mathrm{o}-s_\mathrm{i}} = (-1)^{s_\mathrm{o}+s_\mathrm{i}}w_\ell^{*s_\mathrm{o}s_\mathrm{i}}$.

The maps ${}_s\bar X_{lm}$ are the inverse signal + noise variance filtered CMB maps; the filtered scalar temperature

$$ {}_0\bar X_{\ell m} = \bar T_{\ell m} \tag{1.8}$$

[1] The spin-0 intensity ${}_0T(\hat n)$ gradient mode is $-T_{\ell m}$ and not $T_{\ell m}$

and filtered spin ± 2 Stokes polarization ${}_{\pm 2}\bar{P} = \bar{Q} \pm i\bar{U}$,

$$ {}_{\pm 2}\bar{X}_{\ell m} = {}_{\pm 2}\bar{P}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{1.9}$$

In the notation of Ref. [1], these maps are the result of the filtering step $\bar{X} = \mathcal{B}^\dagger \mathrm{Cov}^{-1} X^{\mathrm{dat}}$, where X^{dat} is the (beam-convolved) data, Cov its covariance matrix, and $\mathcal{B}$ the beam and transfer function mapping the CMB skies to the pixelized data. This inverse-variance filtering operation is trivial (diagonal in harmonic space) for perfectly isotropic data, but in a realistic setting with masked pixels, inhomogeneous noise, etc, this is difficult to perform and several schemes are available with slightly different results.

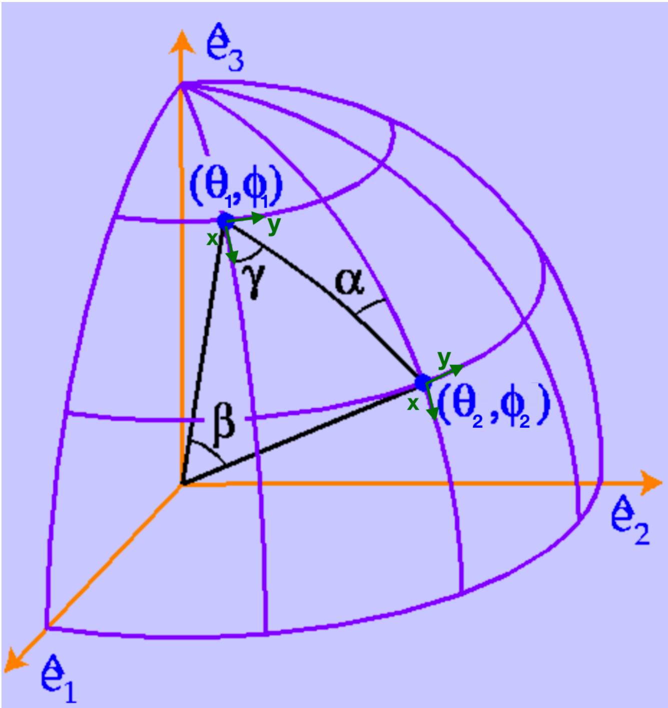

FIG. 1. The geometry and angles in Eq. (1.5), with the local axes in green. Figure originally from Wayne Hu tutorials, `http://background.uchicago.edu/~whu/tamm/webversion/node5.html`.

The formulae exposed in this document can be derived through simple application of this formal relation,

$$\begin{aligned} &\sum_{m_1,m_2} \int d^2 n_1\, {}_{s_1}Y_{\ell_1 m_1}(\hat{n}_1)\, {}_{s_2}Y_{\ell_2 m_2}(\hat{n}_1)\, {}_{r_1}Y_{LM}(\hat{n}_1) \int d^2 n_2\, {}_{t_1}Y_{\ell_1 m_1}(\hat{n}_2)\, {}_{t_2}Y_{\ell_2 m_2}(\hat{n}_2)\, {}_{r_2}Y_{L'M'}(\hat{n}_2) \\ &= \delta_{LL'}\delta_{MM'} \frac{2\ell_1+1}{4\pi}\frac{2\ell_2+1}{4\pi} 2\pi \int_{-1}^{1} d\beta\, d^{\ell_1}_{s_1,t_1}(\beta) d^{\ell_2}_{s_2 t_2}(\beta) d^{L}_{r_1 r_2}(\beta) \quad (\text{whenever } s_1+s_2+r_1 = 0 = t_1+t_2+r_2). \end{aligned} \tag{1.10}$$

where $d^\ell_{mm'}$ are Wigner small d-matrices. we adopt the convention, standard in CMB lensing, to write quadratic estimator multipoles with L, M and use ℓ, m for the CMB fields from which they are built.

II. GAUSSIAN COVARIANCE CALCULATIONS

For two generic estimators as defined in Eq. (1.7), we now obtain the four gradient (g) and curl (c) variances and covariances with two one-dimensional integrals. These terms are often denoted with $N^{(0)}_L$ after proper normalization of the estimators. They act as leading noise biases for estimators of the anisotropy source gradient and curl spectra and cross-spectra.

As is obvious from Eq. (1.7), the result will combine spectra and cross-spectra of the inverse-variance filtered maps $\bar{X}$. Since there are distinct schemes to perform the inverse-variance filtering in a realistic case, we can make here a distinction between the semi-analytical and analytical $N_L^{(0)}$. In the former case, some empirical estimate of these spectra are used (assuming isotropy of the filtered maps). This gives a realization-dependent estimate of the noise. In the latter case, some analytic isotropic approximation of the filtering function is used, giving a realization-independent noise estimate. Neither are as accurate as the much more costly realization-dependent noise debiaser ('RDN0') now routinely used in CMB lensing analyses[1, 6].

For an isotropy estimator ${}_r\hat{q}$ let $s = (s_{\rm i}, s_{\rm o}, w^{s_{\rm i}s_{\rm o}})$ collectively describes the in and out spins and weight function of the left leg, and similarly with t for the right leg (by consistency, $s_{\rm o} + t_{\rm o} = r$). In the same way, let u and v describes another estimator ${}_{r'}\hat{q}'$ (with $u_{\rm o} + v_{\rm o} = r'$). Then, their Gaussian correlation functions are

$$\xi_{\pm}^{rr'}(\beta) = \xi^{\pm s,u}(\beta)\xi^{\pm t,v}(\beta) + \xi^{\pm s,v}(\beta)\xi^{\pm t,u}(\beta), \tag{2.1}$$

where $\xi^{s,t}$ is

$$\xi^{s,t}(\beta) \equiv \sum_{\ell} \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_{\rm o}s_{\rm i}} w_\ell^{*t_{\rm o}t_{\rm i}} \bar{C}_\ell^{s_{\rm i}t_{\rm i}} d^\ell_{s_{\rm o}t_{\rm o}}(\beta) \tag{2.2}$$

and $\bar{C}_\ell^{s_{\rm i}t_{\rm i}} \equiv \left\langle {}_{s_{\rm i}}\bar{X}_{\ell m\ t_{\rm i}}\bar{X}^*_{\ell m}\right\rangle$. Let $N_L^{(0,g^rg^{r'})}$ denote the Gaussian covariance of the gradient modes of the estimators

$$\delta_{LL'}\delta_{MM'}N_L^{(0,g^rg^{r'})} \equiv \left\langle \hat{g}^r_{LM}\hat{g}^{*,r'}_{L'M'}\right\rangle\Big|_{\rm G.} \tag{2.3}$$

and similarly for the curl-curl, gradient-curl and curl-gradient spectra. Projecting the correlation functions in Eq. (2.1) onto gradient and curl modes results in

$$\begin{aligned}
N_L^{(0,g^rg^{r'})} &= \frac{1}{2}\Re\left[C_L^{rr'} + (-1)^r C_L^{-rr'}\right]\\
N_L^{(0,c^rc^{r'})} &= \frac{1}{2}\Re\left[C_L^{rr'} - (-1)^r C_L^{-rr'}\right]\\
N_L^{(0,g^rc^{r'})} &= \frac{1}{2}\Im\left[-C_L^{rr'} - (-1)^r C_L^{-rr'}\right]\\
N_L^{(0,c^rg^{r'})} &= \frac{1}{2}\Im\left[C_L^{rr'} - (-1)^r C_L^{-rr'}\right]
\end{aligned} \tag{2.4}$$

where

$$C_L^{\pm rr'} \equiv 2\pi\int_{-1}^{1} d\mu\, d^L_{\pm rr'}(\mu)\xi_{\pm}^{rr'}(\beta). \tag{2.5}$$

$\Re$ and $\Im$ stands for real and imaginary parts.

A non-zero gradient-curl mode cross-covariance $N_L^{(0,g^rc^{r'})}$ or $N_L^{(0,c^rg^{r'})}$ may be sourced by gradient-curl couplings in the inverse-variance filtered CMB fields (i.e., non-zero $C_\ell^{\bar{T}\bar{B}}$ or $C_\ell^{\bar{E}\bar{B}}$, for example from polarization angle miscalibration or other systematics). This is not the only possibility though.

III. RESPONSE AND CROSS-RESPONSES CALCULATIONS

We now turn to the calculation of the responses of a quadratic estimator given by Eq. (1.7) to a source of anisotropy. In order to do this, we need to parametrize a anisotropy, which we do below in section III A.

Responses are only isotropic in a idealized situation (in the absence of masking and other complications). In this case we can write the inverse-variance filtering step (beam-deconvolved) data maps ${}_sX^{\rm dat}$ with the help of a matrix F, diagonal in harmonic space,

$${}_s\bar{X}_{\ell m} \equiv \sum_{s_2=0,2,-2} F_\ell^{ss_2}\ {}_{s_2}X^{\rm dat}_{\ell m}. \tag{3.1}$$

A. Response parametrization

Anisotropy can sometimes be parametrized at the level of the CMB maps, (for example for lensing), with

$$ {}_s\delta X^{\rm cmb}(\hat n) = \sum_{a=\pm r} {}_a\alpha(\hat n)\left(\sum_{\ell m} R_\ell^{a,s}\; {}_sX^{\rm cmb}_{\ell m}\; {}_{s-a}Y_{\ell m}(\hat n)\right) \tag{3.2}$$

for response kernel functions $R_\ell^{r,s}$. More generally, let the covariance of the CMB (beam-deconvolved) data respond as follows to a spin-r anisotropy source α:

$$ \delta\left\langle {}_sX^{\rm dat}(\hat n_1)\, {}_tX^{*\rm dat}(\hat n_2)\right\rangle = \sum_{\ell m, a=\pm r} {}_a\alpha(\hat n_1) W_\ell^{a,st}\; {}_{s-a}Y_{\ell m}(\hat n_1)\, {}_tY^*_{\ell m}(\hat n_2) + W_\ell^{*a,ts}\; {}_sY_{\ell m}(\hat n_1)\, {}_{t-a}Y^*_{\ell m}(\hat n_2)\, {}_a\alpha^*(\hat n_2) \tag{3.3}$$

for some weights functions $W_\ell^{a,st}$. For map-level descriptions in Eq. (3.2) then holds

$$ W_\ell^{a,st} = R^{a,s} C_\ell^{st}. \tag{3.4}$$

However, Eq. (3.3) is more general. Section V lists weights functions of some relevant cases.

B. Estimator responses calculation

Let as before s,t denote collectively the QE spins and weight functions for an estimator ${}_r\hat q(\hat n)$ of spin $r = s_o + t_o$, and let r' be the spin of anisotropy source ${}_{r'}\alpha(\hat n)$ with covariance response kernel $W^{r'}$ as above. Let $\mathcal{R}_L^{g_r g_{r'}}\delta_{LL'}\delta_{MM'}$ be defined as the response of the gradient mode g^r_{LM} of ${}_r\hat q$ to the gradient mode $g^{r'}_{L'M'}$ of ${}_{r'}\alpha$, and similarly for the curl. It holds:

$$\begin{aligned}
\mathcal{R}_L^{g_r g_{r'}} &= \Re\left[R_L^{st,r'} + (-1)^{r'} R_L^{st,-r'}\right]\\
\mathcal{R}_L^{c_r c_{r'}} &= \Re\left[R_L^{st,r'} - (-1)^{r'} R_L^{st,-r'}\right]\\
\mathcal{R}_L^{g_r c_{r'}} &= \Im\left[-R_L^{st,r'} + (-1)^{r'} R_L^{st,-r'}\right]\\
\mathcal{R}_L^{c_r g_{r'}} &= \Im\left[R_L^{st,r'} + (-1)^{r'} R_L^{st,-r'}\right]
\end{aligned} \tag{3.5}$$

where

$$ R_L^{st,r'} = 2\pi\int_{-1}^{1} d\mu\, d^L_{rr'}(\mu) \sum_{\tilde s_i, \tilde t_i = 0,2,-2}\left[\xi^{s_o s_i \tilde s_i}(\mu)\psi^{t_o t_i \tilde t_i \tilde s_i, r'}(\mu) + \xi^{t_o t_i \tilde t_i}(\mu)\psi^{s_o s_i \tilde s_i \tilde t_i, r'}(\mu)\right] \tag{3.6}$$

and

$$\begin{aligned}
\xi^{s_o s_i \tilde s_i}(\mu) &\equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_o s_i} F_\ell^{s_i \tilde s_i} d^\ell_{s_o, \tilde s_i}(\mu)\\
\psi^{s_o s_i \tilde s_i \tilde t_i, r'}(\mu) &\equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_o s_i} F_\ell^{s_i \tilde s_i} W_\ell^{*-r', -\tilde t_i \tilde s_i} d^\ell_{s_o, -\tilde t_i + r'}(\mu)
\end{aligned} \tag{3.7}$$

Again, in most relevant cases (but not always), the gradient to curl and curl to gradient responses do vanish. If there is a unique source of anisotropy, properly normalized gradient and curl estimators are then given by $\hat g^r_{LM}/\mathcal{R}_L^{g_r g_r}$ and $\hat c^r_{LM}/\mathcal{R}_L^{c_r c_r}$.

IV. DERIVATION OF OPTIMAL QE WEIGHTS

Optimal (defined in the sense of minimal Gaussian variance) QE weights can be easily gained from the representation of the anisotropy. In the presence of the anisotropy (${}_r\alpha(\hat{n})$ in Eq. 3.3), the CMB remains Gaussian. For small anisotropy, a good estimate will be provided by the leading term in a Newton-Raphson estimate of ${}_r\alpha$[7]. This first iteration is given by the gradient of the log-likelihood $\ln p(\alpha, X^{\rm dat})$, normalized by the log-likelihood Hessian (on average equal to the Fisher matrix), all evaluated at zero anisotropy. We can combine un-normalized estimates of the real ($\Re\, {}_r\alpha$) and imaginary ($\Im\, {}_r\alpha$) parts of ${}_r\alpha$ to a spin-r un-normalized estimate with the rule

$$\frac{\delta}{\delta \Re\, {}_r\alpha(\hat{n})} + i \frac{\delta}{\delta \Im\, {}_r\alpha(\hat{n})} = 2 \frac{\delta}{\delta_{-r}\alpha(\hat{n})} \tag{4.1}$$

where the functional derivative with respect to ${}_{-r}\alpha(\hat{n})$ is performed in the usual way, treating ${}_{-r}\alpha(\hat{n})$ and ${}_r\alpha(\hat{n})$ as independent variables. Written in full, this is

$${}_r\hat{\alpha}(\hat{n}) \equiv - \frac{\delta}{\delta_{-r}\alpha(\hat{n})}\, {}_{s_1}X^{\rm dat} {\rm Cov}^{-1}_{s_1 s_2}\, {}_{s_2}X^{\rm dat} \Big|_{\alpha\equiv 0} - \frac{\delta}{\delta\, {}_{-r}\alpha(\hat{n})} \ln \det {\rm Cov} \Big|_{\alpha\equiv 0} \tag{4.2}$$

where ${\rm Cov}_{s_1 s_2}(\hat{n}_1, \hat{n}_2) \equiv \left\langle {}_{s_1}X^{\rm dat}(\hat{n}_1)\, {}_{s_2}X^{\dagger{\rm dat}}(\hat{n}_2) \right\rangle$ is the data covariance matrix. The second term is the likelihood determinant variation. This determinant term is equal to the average of the quadratic estimate just defined, and called for this reason the 'mean-field' (see Ref. [7]). In practice this mean-field is estimated as the average of the quadratic term using a realistic set of simulations.

Performing the derivative using representation (3.3) we find (neglecting the mean-field part)

$${}_r\hat{\alpha}(\hat{n}) = \sum_{s,t} \frac{1}{{}_s2}\, {}_{-s}\bar{X}(\hat{n}) \cdot \left(\sum_{\ell m} 2 W_\ell^{-r,st} \frac{1}{{}_t2}\, {}_t\bar{X}_{\ell m}\, {}_{s+r}Y_{\ell m}(\hat{n}) \right) \tag{4.3}$$

where the symbol ${}_s2$ stands for either $1 (s = 0)$ or $2 (s \neq 0)$. Hence, confronting to definition (1.7),

$$w_\ell^{st} = \frac{\delta_{st}}{{}_s2} \text{ (1st leg)} \quad w_\ell^{-s+r,t} = \frac{2}{{}_t2} W_\ell^{-r,-st} \text{ (2nd leg)} \tag{4.4}$$

Restricting the s, t spins to 0 or ± 2 in Eq. (4.2) provides temperature-only or polarization-only estimators. This derivation recovers for example the minimum-variance (MV), temperature-only and polarization-only lensing estimators as obtained first by Ref. [8](for the gradient term), with identical normalization. Usage of the lensed spectra[9] instead of the unlensed spectra (or even better, the grad-lensed spectra[10, 11]) are well-known slight modifications that improve the estimators. Further restrictions by zeroing maps on the left or right legs gives additional estimators (TE, TB, EB).

V. EXAMPLES

The construction of optimal estimators only requires knowledge of the data covariance response W, Eq. (3.3). In the case that the anisotropy is defined at the levels of the CMB fields rather than the data covariance, W is trivially related to the field response R through Eq. (3.2). In this section, we briefly derive and list a few relevant cases of these responses.

A. Lensing

The source of anisotropy is the spin-1 deflection field ${}_1\alpha(\hat{n})$, with linear response (see Ref. [12]) $\delta_s X^{\rm cmb}(\hat{n}) = -\frac{1}{2}\alpha_1(\hat{n})\bar{\eth}_s X^{\rm cmb}(\hat{n}) - \frac{1}{2}\alpha_{-1}(\hat{n})\eth\, {}_s X^{\rm cmb}(\hat{n})$ where $\eth$ and $\bar{\eth}$ are the spin raising and spin lowering operator respectively. From the action of these operators on the spherical harmonics follow immediately

$$R_\ell^{-1,s} = -\frac{1}{2}\sqrt{(l-s)(l+s+1)}, \quad R_\ell^{1,s} = +\frac{1}{2}\sqrt{(l+s)(l-s+1)} \tag{5.1}$$

B. Modulation

This is relevant for instance searching for a large-scale trispectrum τ_{NL} signature in the CMB[13, 14]. The anisotropy source is a scalar ('f'), with response $\delta_s X^{\rm cmb}(\hat n) = f(\hat n)_s X^{\rm cmb}(\hat n)$, hence

$$R_\ell^{0,s} = 1 \tag{5.2}$$

C. Polarization rotation

This is relevant in the case of systematic polarization angle miscalibration, or within more speculative ideas including cosmic birefringence. The observed polarization is rotated by an angle $\beta(\hat n)$ according to ${}_{\pm 2}P^{\rm cmb} \rightarrow e^{\mp 2i\,\beta(\hat n)}\,{}_{\pm 2}P^{\rm cmb}(\hat n)$. Hence,

$$R_\ell^{0,\pm 2} = \mp 2i \tag{5.3}$$

D. Point sources

Point sources in temperature ('S^2', see Ref. [15]): here anisotropy is sought in the intensity field of the form $\delta\left\langle T^{\rm cmb}(\hat n)\, T^{\rm cmb}(\hat n')\right\rangle \ni \delta^D(\hat n - \hat n')S^2(\hat n)$. Hence,

$$W_\ell^{r,st} = \frac{1}{4}\delta_{r0}\delta_{s0}\delta_{t0} \tag{5.4}$$

E. Noise variance map inhomogeneities

We can look for inhomogeneities in the pixel noise variance $\sigma^2(\hat n)$ across the sky. This leaves an anisotropy signature on the diagonal of the CMB data covariance matrix similar than that of point sources, with the difference that point-sources are convolved with the instrument transfer function but the pixel noise is not. $\delta\left\langle T^{\rm dat}(\hat n)\, T^{\rm dat}(\hat n')\right\rangle \ni \delta_{\hat n \hat n'}\sigma^2(\hat n)$. Hence,

$$W_\ell^{r,st} = \frac{1}{4}\delta_{r0}\delta_{s0}\delta_{t0}\frac{1}{b_\ell^2} \tag{5.5}$$

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