

# Notes on curved-sky QE responses etc.

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**JC: Document to be included with the pipeline release after submission of the revised L08.**

Lensing and others quadratic estimators used in [?] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$_{s_o+t_o}\hat{d}(\hat{n}) \equiv \left( \sum_{\ell m} w_{\ell}^{s_o s_i} \bar{X}_{\ell m s_o} Y_{\ell m}(\hat{n}) \right) \left( \sum_{\ell m} w_{\ell}^{t_o t_i} \bar{X}_{\ell m t_o} Y_{\ell m}(\hat{n}) \right) \quad (1)$$

where  $s_i, t_i$  are input spins,  $s_o, t_o$  outputs spins, and  $w_{\ell}^{s_o s_i}, w_{\ell}^{t_o t_i}$  associated weights. The maps  $_{s}\bar{X}_{\ell m}$  are the inverse variance filtered CMB maps,

$$_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad \pm_2\bar{X}_{\ell m} = -(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}). \quad (2)$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered  $\bar{T}, \bar{E}, \bar{B}$  are directly proportional to  $T, E$  and  $B$  respectively. We keep the discussion focussed on generic fields  $\bar{X}$  of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G_{LM}^s &= -\frac{1}{2} (|s| d_{LM} + (-1)^s {}_{-|s|}d_{LM}) \\ C_{LM}^s &= -\frac{1}{2i} (|s| d_{LM} - (-1)^s {}_{-|s|}d_{LM}). \end{aligned}$$

The formulae exposed here can be derived through simple application of this relation,

$$\sum_{m_1, m_2} \int d^2 n \prod_{i=1}^3 {}_{s_i}Y_{\ell_i m_i}(\hat{n}) \int d^2 n' \prod_{i=1}^3 {}_{t_i}Y_{\ell_i m_i}(\hat{n}') = \frac{2\ell_1+1}{4\pi} \frac{2\ell_2+1}{4\pi} 2\pi \int_{-1}^1 d\mu \prod_{i=1}^3 d_{s_i, t_i}^{\ell_i}(\mu) \quad (3)$$

## 0.1 Semi-analytical QE $N_L^{(0)}$ calculations

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let  $s = (s_i, s_o, w^{s_i s_o})$  collectively describes the in and out spins and weight function, and similarly for  $t, u$  and  $v$ . Let the covariance function  $N_L^{st, uv}$  be defined through

$$\begin{aligned} \delta_{LL'} \delta_{MM'} N_L^{stuv} &\equiv \left\langle {}_{s_o+t_o}\hat{d}_{LM} {}_{u_o+v_o}\hat{d}_{L'M'}^* \right\rangle \Big|_{\text{Gauss}} \\ &= (-1)^{s_o+t_o+u_o+v_o} 2\pi \int_{-1}^1 d\mu d_{-s_o-t_o, -u_o-v_o}^L(\mu) [\xi^{su}(\mu) \xi^{tv}(\mu) + \xi^{sv}(\mu) \xi^{tu}(\mu)] \end{aligned} \quad (4)$$

where  $\xi$  are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_{\ell} \left( \frac{2\ell+1}{4\pi} \right) w_{\ell}^{s_o s_i} w_{\ell}^{t_o t_i} \bar{C}_{\ell}^{s_i t_i} d_{s_o, t_o}^{\ell}(\mu) \text{ with } \bar{C}_{\ell}^{s_i t_i} \equiv \langle {}_{s_i} \bar{X}_{\ell m} {}_{t_i} \bar{X}_{\ell m}^* \rangle \quad (5)$$

and  $d_{mm'}^{\ell}$  are Wigner small d-matrices. Then

$$\boxed{\begin{aligned} \left\langle \hat{G}_{LM}^{s_o+t_o} \hat{G}_{L'M'}^{*, u_o+v_o} \right\rangle \Big|_{\text{Gauss.}} &= \delta_{LL'} \delta_{MM'} \frac{1}{2} [N_L^{stuv} + (-1)^{s_o+t_o} N_L^{-s-tuv}] \\ \left\langle \hat{C}_{LM}^{s_o+t_o} \hat{C}_{L'M'}^{*, u_o+v_o} \right\rangle \Big|_{\text{Gauss.}} &= \delta_{LL'} \delta_{MM'} \frac{1}{2} [N_L^{stuv} - (-1)^{s_o+t_o} N_L^{-s-tuv}] \\ \left\langle \hat{G}_{LM}^{s_o+t_o} \hat{C}_{L'M'}^{*, u_o+v_o} \right\rangle \Big|_{\text{Gauss.}} &= 0 \end{aligned}} \quad (6)$$

## 0.2 QE response calculation

Let the covariance of the CMB data respond as follows to a spin- $r$  ( $r \geq 0$ ) anisotropy source  $\alpha$ :

$$\delta \langle {}_s X(\hat{n}) {}_t X^*(\hat{n}') \rangle = \sum_{\ell m, a=\pm r} {}_a \alpha(\hat{n}) W_{\ell}^{a, st} {}_{s-r} Y_{\ell m}(\hat{n}) {}_t Y_{\ell m}^*(\hat{n}') + W_{\ell}^{a, ts} {}_s Y_{\ell m}(\hat{n}) {}_{t-r} Y_{\ell m}^*(\hat{n}') - {}_a \alpha(\hat{n}') \quad (7)$$

for some weights functions  $W_{\ell}^{a, st}$ . For instance, if the anisotropy can be described at the level of the CMB maps, such as for lensing, with

$${}_s \delta X(\hat{n}) = \sum_{a=\pm r} {}_a \alpha(\hat{n}) \left( \sum_{\ell m} R_{\ell}^{a, s} {}_s X_{\ell m} {}_{s-a} Y_{\ell m}(\hat{n}) \right) \quad (8)$$

for harmonic responses  $R$ , then holds

$$W_{\ell}^{a, st} = R_{\ell}^{a, s} C_{\ell}^{st}. \quad (9)$$

However, Eq. (7) is more general. Examples include:

- Lensing: The source of anisotropy is the spin-1 field  ${}_1 \alpha(\hat{n})$ , with response

$$\delta_s X(\hat{n}) = -\frac{1}{2} \alpha_1 \bar{\partial}_s X(\hat{n}) - \frac{1}{2} \alpha_{-1} \bar{\partial}_s X(\hat{n}) \quad (10)$$

where  $\bar{\partial}$  and  $\bar{\partial}$  are the spin lowering and spin raising operator **JC: check notation** respectively. Hence

$$R_{\ell}^{1, s} = \dots R_{\ell}^{-1, s} = \quad (11)$$

- Modulation estimator: The source is spin 0, with response

$$\delta_s X(\hat{n}) = {}_0 \alpha(\hat{n}) {}_s X(\hat{n}) \quad (12)$$

Hence,

$$R_{\ell}^{st} = \delta_{st} \quad (13)$$

- Point sources in temperature:

$$W_{\ell}^{r, st} = \frac{1}{4} \delta_{r0} \delta_{s0} \delta_{t0} \quad (14)$$

- Noise anisotropies (same as point sources but acting on beam deconvolved maps):

$$W_{\ell}^{r, st} = \frac{1}{4} \delta_{r0} \delta_{s0} \delta_{t0} \frac{1}{b_{\ell}^2} \quad (15)$$

Let further the isotropic limit of the filtering procedure be the matrix  $F$ , defined through

$${}_s\bar{X}_{\ell m} = \sum_{s_2=0,2,-2} F_{\ell}^{ss_2} {}_{s_2}X_{\ell m} \quad (\text{isotropic approximation}). \quad (16)$$

Then the gradient and curl responses of estimator (1) are

$$\boxed{\begin{aligned} \mathcal{R}_L^{gg} &= R_L^{st,r} + (-1)^r R_L^{st,-r} \\ \mathcal{R}_L^{cc} &= R_L^{st,r} - (-1)^r R_L^{st,-r} \\ \mathcal{R}_L^{gc} &= 0 = \mathcal{R}_L^{cg}, \end{aligned}} \quad (17)$$

where  $R_L^{st,r}$  is

$$R_L^{st,r} = (-1)^{s_o+t_o} 2\pi \int_{-1}^1 d\mu d_{-s_o-t_o,-r}^L(\mu) \sum_{\tilde{s}_i, \tilde{t}_i=0,2,-2} \left[ \xi^{s_o s_i \tilde{s}_i}(\mu) \psi^{t_o t_i \tilde{t}_i \tilde{s}_i, r}(\mu) + \xi^{t_o t_i \tilde{t}_i}(\mu) \psi^{s_o s_i \tilde{s}_i \tilde{t}_i, r}(\mu) \right] \quad (18)$$

with

$$\begin{aligned} \xi^{s_o s_i \tilde{s}_i}(\mu) &\equiv \sum_{\ell} \left( \frac{2\ell+1}{4\pi} \right) w_{\ell}^{s_o s_i} F_{\ell}^{s_i \tilde{s}_i} d_{s_o, \tilde{s}_i}^{\ell}(\mu) \\ \psi^{s_o s_i \tilde{s}_i \tilde{t}_i, r}(\mu) &\equiv (-1)^r \sum_{\ell} \left( \frac{2\ell+1}{4\pi} \right) w_{\ell}^{s_o s_i} F_{\ell}^{s_i \tilde{s}_i} W_{\ell}^{-r, -\tilde{t}_i \tilde{s}_i} d_{s_o, -\tilde{t}_i+r}^{\ell}(\mu) \end{aligned} \quad (19)$$