CONTENTS

I. NOTES ON CURVED-SKY QE RESPONSES ETC.

Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [?] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$_{s_{\rm o}+t_{\rm o}}\hat{d}(\boldsymbol{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_{\rm i}}\ {}_{s_{\rm i}}\bar{X}_{\ell m}\ {}_{s_{\rm o}}Y_{\ell m}(\boldsymbol{n})\right)\left(\sum_{\ell m} w_\ell^{t_{\rm i}}\ {}_{t_{\rm i}}\bar{X}_{\ell m}\ {}_{t_{\rm o}}Y_{\ell m}(\boldsymbol{n})\right) \tag{1.1}$$

where $s_{\rm i}, t_{\rm i}$ are input spins, $w_\ell^{s_{\rm i}}, w_\ell^{t_{\rm i}}$ associated weights, and $s_{\rm o}, t_{\rm o}$ outputs spins. The maps ${}_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$${}_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{1.2}$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields $\bar{X}$ of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G_{LM}^s &= -\tfrac{1}{2}\left({}_{|s|}d_{LM} + (-1)^s\ {}_{-|s|}d_{LM}\right) \\ C_{LM}^s &= -\tfrac{1}{2i}\left({}_{|s|}d_{LM} - (-1)^s\ {}_{-|s|}d_{LM}\right). \end{aligned}$$

A. Semi-analytical QE $N_L^{(0)}$ calculation

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1.1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_{\rm i}, t_{\rm i}, w^{s_{\rm i}})$ collectively describes the in and out spins and weight function, and similarly for t, u and v. Let the response function $\mathcal{R}_L^{st,uv}$ be defined as

$$(-1)^{t_{\rm o}+v_{\rm o}}\mathcal{R}_L^{st,uv} \equiv 2\pi\int_{-1}^{1} d\mu\, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_{\rm o}-v_{\rm o}, s_{\rm o}+u_{\rm o}}(\mu) \tag{1.3}$$

where ξ are position-space correlation functions

$$\xi^{st}(\mu) \equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_{\rm i}} w_\ell^{t_{\rm i}} \bar{C}_\ell^{s_{\rm i}t_{\rm i}} d^\ell_{-t_{\rm o}, s_{\rm o}}(\mu) \text{ with } \bar{C}_\ell^{s_{\rm i}t_{\rm i}} \equiv \left\langle {}_{s_{\rm i}}\bar{X}_{\ell m}\ {}_{t_{\rm i}}\bar{X}^*_{\ell m}\right\rangle \tag{1.4}$$

and $d^\ell_{mm'}$ are Wigner small d-matrices. Then

$$\left\langle G_{LM}^{s_{\rm o}+t_{\rm o}} G_{LM}^{*,u_{\rm o}+v_{\rm o}}\right\rangle = \frac{1}{4}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1 + (-1)^{s_{\rm i}+t_{\rm i}+u_{\rm i}+v_{\rm i}}\right) + (-1)^{s_{\rm i}+t_{\rm i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1 + (-1)^{s_{\rm i}+t_{\rm i}+u_{\rm i}+v_{\rm i}}\right)\right] \tag{1.5}$$

$$\left\langle C_{LM}^{s_{\rm o}+t_{\rm o}} C_{LM}^{*,u_{\rm o}+v_{\rm o}}\right\rangle = -\frac{1}{4}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1 + (-1)^{s_{\rm i}+t_{\rm i}+u_{\rm i}+v_{\rm i}}\right) - (-1)^{s_{\rm i}+t_{\rm i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1 + (-1)^{s_{\rm i}+t_{\rm i}+u_{\rm i}+v_{\rm i}}\right)\right] \tag{1.6}$$

$$\left\langle G_{LM}^{s_{\rm o}+t_{\rm o}} C_{LM}^{*,u_{\rm o}+v_{\rm o}}\right\rangle = \frac{1}{4i}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1 - (-1)^{s_{\rm i}+t_{\rm i}+u_{\rm i}+v_{\rm i}}\right) + (-1)^{s_{\rm i}+t_{\rm i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1 - (-1)^{s_{\rm i}+t_{\rm i}+u_{\rm i}+v_{\rm i}}\right)\right] \tag{1.7}$$

I dont understand the resulting sign and GC spectrum for (the irrelevant case of) odd total input spin. Should nt that always be 1? TTTT checked OK *a. Sketchy derivation to cleanup* For this we need a result using the spin-weight spherical harmonic theorem. Define $\mathcal{R}_L^{st,uv}$ through

$$
\begin{aligned}
\mathcal{R}^{st,uv}(\boldsymbol{n},\boldsymbol{n}') \equiv& (-1)^{t_\mathrm{o}+v_\mathrm{o}} \left(\sum_{\ell m} g_\ell^{s_\mathrm{i}} g_\ell^{t_\mathrm{i}} C_\ell^{s_\mathrm{i}t_\mathrm{i}} \; {}_{s_\mathrm{o}}Y_{\ell m}(\boldsymbol{n}) \; {}_{-t_\mathrm{o}}Y^*_{\ell m}(\boldsymbol{n}')\right) \left(\sum_{\ell m} g_\ell^{u_\mathrm{i}} g_\ell^{v_\mathrm{i}} C_\ell^{u_\mathrm{i}v_\mathrm{i}} \; {}_{u_\mathrm{o}}Y_{\ell m}(\boldsymbol{n}) \; {}_{-v_\mathrm{o}}Y^*_{\ell m}(\boldsymbol{n}')\right) \\
&\equiv (-1)^{t_\mathrm{o}+v_\mathrm{o}} \sum_{LM} \mathcal{R}_L^{stuv} \; {}_{s_\mathrm{o}+u_\mathrm{o}}Y_{LM}(\boldsymbol{n}) \; {}_{-u_\mathrm{o}-v_\mathrm{o}}Y^*_{LM}(\boldsymbol{n}')
\end{aligned}
\tag{1.8}
$$

Then we can write

$$
\left\langle {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}(\boldsymbol{n}) \; {}_{u_\mathrm{o}+v_\mathrm{o}}\hat{d}(\boldsymbol{n}')\right\rangle = \mathcal{R}^{su,tv}(\boldsymbol{n},\boldsymbol{n}') + \mathcal{R}^{sv,tu}(\boldsymbol{n},\boldsymbol{n}') \tag{1.9}
$$

Taking the harmonic transform, we get

$$
\left\langle {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}_{LM} \; {}_{u_\mathrm{o}+v_\mathrm{o}}\hat{d}_{L'M'}\right\rangle = (-1)^M \delta_{M,-M'}\delta_{L,L'} \left(\mathcal{R}_L^{su,tv} + \mathcal{R}_L^{sv,tu}\right) \tag{1.10}
$$

In general we have

$$
\begin{aligned}
G^s_{LM} &= -\tfrac{1}{2}\left({}_sd_{LM} + (-1)^s \; {}_{-s}d_{LM}\right) \quad (s \geq 0) \\
C^s_{LM} &= -\tfrac{1}{2i}\left({}_sd_{LM} - (-1)^s \; {}_{-s}d_{LM}\right) \quad (s \geq 0).
\end{aligned}
$$

The estimator for ${}_{-s_\mathrm{o}-t_\mathrm{o}}\hat{d}$ is the same as ${}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}$ with all spin signs flipped, and with an overall sign $(-1)^{s_\mathrm{o}+s_\mathrm{i}+t_\mathrm{o}+t_\mathrm{i}}$. The out-spins part gets canceled by the sign $(-1)^s$ in the above equation. Hence,

$$
\begin{aligned}
&(-1)^M \delta_{M,-M'}\delta_{L,L'} \left\langle G^{s_\mathrm{o}+t_\mathrm{o}}_{LM} G^{u_\mathrm{o}+v_\mathrm{o}}_{L'M'}\right\rangle \cdot 4 = \mathcal{R}_L^{su,tv} + \mathcal{R}_L^{sv,tu} + (-1)^{s_\mathrm{i}+t_\mathrm{i}}\left(\mathcal{R}_L^{-su,-tv} + \mathcal{R}_L^{-sv,-tu}\right) \\
&+ (-1)^{u_\mathrm{i}+v_\mathrm{i}}\left(\mathcal{R}_L^{s-u,t-v} + \mathcal{R}_L^{s-v,t-u}\right) + (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\left(\mathcal{R}_L^{-s-u,-t-v} + \mathcal{R}_L^{-s-v,-t-u}\right) \quad (s_\mathrm{o}+t_\mathrm{o} >= 0, u_\mathrm{o}+v_\mathrm{o} >= 0)
\end{aligned}
\tag{1.11}
$$

Since $\mathcal{R}$ is invariant under the simultaneous sign-flip of all spins, we can also write this as:

$$
\left\langle G^{s_\mathrm{o}+t_\mathrm{o}}_{LM} G^{*,u_\mathrm{o}+v_\mathrm{o}}_{LM}\right\rangle = \frac{1}{4}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1+(-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right) + (-1)^{s_\mathrm{i}+t_\mathrm{i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1+(-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right)\right] \tag{1.12}
$$

$$
\left\langle C^{s_\mathrm{o}+t_\mathrm{o}}_{LM} C^{*,u_\mathrm{o}+v_\mathrm{o}}_{LM}\right\rangle = -\frac{1}{4}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1+(-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right) - (-1)^{s_\mathrm{i}+t_\mathrm{i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1+(-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right)\right] \tag{1.13}
$$

$$
\left\langle G^{s_\mathrm{o}+t_\mathrm{o}}_{LM} C^{*,u_\mathrm{o}+v_\mathrm{o}}_{LM}\right\rangle = \frac{1}{4i}\left[\left(\mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right)\left(1-(-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right) + (-1)^{s_\mathrm{i}+t_\mathrm{i}}\left(\mathcal{R}_L^{-su,tv} + \mathcal{R}_L^{-sv,tu}\right)\left(1-(-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\right)\right] \tag{1.14}
$$

?

More details of $\mathcal{R}_L$

Recall the spin-weight addition theorem:

$$
\sum_m {}_sY^*_{\ell m}(\boldsymbol{n}') \; {}_tY_{\ell m}(\boldsymbol{n}) = \sqrt{\frac{2\ell+1}{4\pi}} e^{-it\gamma} \; {}_tY_{\ell,-s}(\beta,\alpha). \tag{1.15}
$$

Hence

$$
\mathcal{R}^{st,uv}(\boldsymbol{n},\boldsymbol{n}') = (-1)^{t_\mathrm{o}+v_\mathrm{o}} e^{-is_\mathrm{o}\gamma - iu_\mathrm{o}\gamma}\left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_\mathrm{i}} g^{t_\mathrm{i}} C_\ell^{s_\mathrm{i}t_\mathrm{i}} \; {}_{s_\mathrm{o}}Y_{\ell t_\mathrm{o}}(\beta,\alpha)\right)\left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_\mathrm{i}} g^{v_\mathrm{i}} C_\ell^{u_\mathrm{i}v_\mathrm{i}} \; {}_{u_\mathrm{o}}Y_{\ell v_\mathrm{o}}(\beta,\alpha)\right) \tag{1.16}
$$

The product of the brackets is a spin $s_\mathrm{o} + u_\mathrm{o}$ function. Defining its spin weight coefficients as $\mathcal{R}_L$ we get the relation claimed above. What are these coefficients?

$$\mathcal{R}_L^{st,uv} \equiv (-1)^{t_\mathrm{o}+v_\mathrm{o}} \sqrt{\frac{4\pi}{2\ell+1}} \int d^2n \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_\mathrm{i}} g^{t_\mathrm{i}} C_\ell^{s_\mathrm{i}t_\mathrm{i}} \,{}_{s_\mathrm{o}}Y_{\ell t_\mathrm{o}}(\boldsymbol{n})\right) \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_\mathrm{i}} g^{v_\mathrm{i}} C_\ell^{u_\mathrm{i}v_\mathrm{i}} \,{}_{u_\mathrm{o}}Y_{\ell v_\mathrm{o}}(\boldsymbol{n})\right) {}_{s_\mathrm{o}+u_\mathrm{o}}Y^*_{L,t_\mathrm{o}+v_\mathrm{o}}(\boldsymbol{n}) \tag{1.17}$$

Using

$$\boxed{{}_sY_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}(-1)^m e^{im\phi} d^\ell_{-ms}(\theta)} \tag{1.18}$$

The above thing is invariant if all signs are flipped at the same time. we get

$$\boxed{(-1)^{t_\mathrm{o}+v_\mathrm{o}} \mathcal{R}_L^{st,uv} \equiv 2\pi \int_{-1}^1 d\mu\, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_\mathrm{o}-v_\mathrm{o}, s_\mathrm{o}+u_\mathrm{o}}(\mu), \text{ with } \xi^{st}(\mu) \equiv \sum_\ell \frac{2\ell+1}{4\pi} g_\ell^{s_\mathrm{i}} g_\ell^{t_\mathrm{i}} C_\ell^{s_\mathrm{i}t_\mathrm{i}} d^\ell_{-t_\mathrm{o},s_\mathrm{o}}(\mu)} \tag{1.19}$$

B. QE responses calculation