CONTENTS

I. NOTES ON CURVED-SKY QE RESPONSES ETC.

JC: Document to be included with the pipeline release after submission of the revised L08.

Lensing and others quadratic estimators used in [**?**] are all built multiplying in position space spin transforms of spin-weighted fields. We may write all of these in the form

$$ {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}(\boldsymbol{n}) \equiv \left(\sum_{\ell m} w_\ell^{s_\mathrm{i}s_\mathrm{o}}\;{}_{s_\mathrm{i}}\bar{X}_{\ell m}\;{}_{s_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\right)\left(\sum_{\ell m} w_\ell^{t_\mathrm{i}t_\mathrm{o}}\;{}_{t_\mathrm{i}}\bar{X}_{\ell m}\;{}_{t_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\right) \tag{1.1}$$

where $s_\mathrm{i}, t_\mathrm{i}$ are input spins, $w_\ell^{s_\mathrm{o}s_\mathrm{i}}, w_\ell^{t_\mathrm{i}}$ associated weights, and $s_\mathrm{o}, t_\mathrm{o}$ outputs spins. The maps ${}_s\bar{X}_{lm}$ are the inverse variance filtered CMB maps,

$$ {}_0\bar{X}_{\ell m} = -\bar{T}_{\ell m}, \quad {}_{\pm 2}\bar{X}_{\ell m} = -\left(\bar{E}_{\ell m} \pm i\bar{B}_{\ell m}\right). \tag{1.2}$$

For purely analytical calculations, the filtering operation itself can be approximated as isotropic. For independently filtered temperature and polarization, the filtered $\bar{T}, \bar{E}, \bar{B}$ are directly proportional to T, E and B respectively. We keep the discussion focussed on generic fields $\bar{X}$ of arbitrary spins in the following. The gradient (G) and curl (C) modes of definite parity are defined through

$$\begin{aligned} G^s_{LM} &= -\tfrac{1}{2}\left({}_{|s|}d_{LM} + (-1)^s\;{}_{-|s|}d_{LM}\right) \\ C^s_{LM} &= -\tfrac{1}{2i}\left({}_{|s|}d_{LM} - (-1)^s\;{}_{-|s|}d_{LM}\right). \end{aligned}$$

A. Semi-analytical QE $N_L^{(0)}$ calculation

Q.E. noise (co)-variance can be evaluated very easily as was first demonstrated by Ref. []. For two generic estimators as defined in Eq. (1.1), we can jointly obtain their G and C co-variances with 4 one-dimensional integrals as we now describe.

Let $s = (s_\mathrm{i}, s_\mathrm{o}, w^{s_\mathrm{i}s_\mathrm{o}})$ collectively describes the in and out spins and weight function, and similarly for t, u and v. Let the response function $\mathcal{R}_L^{st,uv}$ be defined as JC: Change $\mathcal{R}$ to N or similar

$$ (-1)^{t_\mathrm{o}+v_\mathrm{o}+t_\mathrm{i}+v_\mathrm{i}}\mathcal{R}_L^{st,uv} \equiv 2\pi\int_{-1}^{1} d\mu\,\xi^{st}(\mu)\,\xi^{uv}(\mu)\,d^L_{-t_\mathrm{o}-v_\mathrm{o},s_\mathrm{o}+u_\mathrm{o}}(\mu) \tag{1.3}$$

where ξ are position-space correlation functions

$$ \xi^{st}(\mu) \equiv \sum_\ell \left(\frac{2\ell+1}{4\pi}\right) w_\ell^{s_\mathrm{i}s_\mathrm{o}} w_\ell^{t_\mathrm{i}t_\mathrm{o}} \bar{C}_\ell^{s_\mathrm{i}t_\mathrm{i}} d^\ell_{-t_\mathrm{o},s_\mathrm{o}}(\mu) \text{ with } \bar{C}_\ell^{s_\mathrm{i}t_\mathrm{i}} \equiv \left\langle {}_{s_\mathrm{i}}\bar{X}_{\ell m}\;{}_{t_\mathrm{i}}\bar{X}^*_{\ell m}\right\rangle \tag{1.4}$$

and $d^\ell_{mm'}$ are Wigner small d-matrices. Then

$$ \left\langle G_{LM}^{s_\mathrm{o}+t_\mathrm{o}} G_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}}\right\rangle = \frac{1}{2}\left[(-1)^{s_\mathrm{o}+t_\mathrm{o}}\mathcal{R}_L^{-su,tv} + (-1)^{u_\mathrm{o}+v_\mathrm{o}}\mathcal{R}_L^{s-v,t-u} + \mathcal{R}_L^{su,tv} + R_L^{sv,tu}\right] \tag{1.5}$$

$$ \left\langle C_{LM}^{s_\mathrm{o}+t_\mathrm{o}} C_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}}\right\rangle = \frac{1}{2}\left[(-1)^{s_\mathrm{o}+t_\mathrm{o}}\mathcal{R}_L^{-su,tv} + (-1)^{u_\mathrm{o}+v_\mathrm{o}}\mathcal{R}_L^{s-v,t-u} - \mathcal{R}_L^{su,tv} - R_L^{sv,tu}\right] \tag{1.6}$$

$$ \left\langle G_{LM}^{s_\mathrm{o}+t_\mathrm{o}} C_{LM}^{*,u_\mathrm{o}+v_\mathrm{o}}\right\rangle = 0 \tag{1.7}$$

JC: TTTT checked OK. See below for R symmetries and conventions *a. Sketchy derivation to cleanup* For this we need a result using the spin-weight spherical harmonic theorem. Define $\mathcal{R}_L^{st,uv}$ through

$$\begin{aligned}\mathcal{R}^{st,uv}(\boldsymbol{n},\boldsymbol{n}') \equiv& (-1)^{t_\mathrm{o}+v_\mathrm{o}+t_\mathrm{i}+v_\mathrm{i}}\left(\sum_{\ell m} g_\ell^{s_\mathrm{i}} g_\ell^{t_\mathrm{i}} C_\ell^{s_\mathrm{i}t_\mathrm{i}}\;{}_{s_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\;{}_{-t_\mathrm{o}}Y^*_{\ell m}(\boldsymbol{n}')\right)\left(\sum_{\ell m} g_\ell^{u_\mathrm{i}} g_\ell^{v_\mathrm{i}} C_\ell^{u_\mathrm{i}v_\mathrm{i}}\;{}_{u_\mathrm{o}}Y_{\ell m}(\boldsymbol{n})\;{}_{-v_\mathrm{o}}Y^*_{\ell m}(\boldsymbol{n}')\right)\\ &\equiv (-1)^{t_\mathrm{o}+v_\mathrm{o}+t_\mathrm{i}+v_\mathrm{i}}\sum_{LM}\mathcal{R}_L^{stuv}\;{}_{s_\mathrm{o}+u_\mathrm{o}}Y_{LM}(\boldsymbol{n})\;{}_{-t_\mathrm{o}-v_\mathrm{o}}Y^*_{LM}(\boldsymbol{n}')\end{aligned} \tag{1.8}$$

Then we can write

$$\left\langle {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}(\boldsymbol{n})\;{}_{u_\mathrm{o}+v_\mathrm{o}}\hat{d}(\boldsymbol{n}')\right\rangle = \mathcal{R}^{su,tv}(\boldsymbol{n},\boldsymbol{n}') + \mathcal{R}^{sv,tu}(\boldsymbol{n},\boldsymbol{n}') \tag{1.9}$$

Taking the harmonic transform, we get

$$\left\langle {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}_{LM}\;{}_{u_\mathrm{o}+v_\mathrm{o}}\hat{d}_{L'M'}\right\rangle = (-1)^M\delta_{M,-M'}\delta_{L,L'}\left(\mathcal{R}_L^{su,tv}+\mathcal{R}_L^{sv,tu}\right)(-1)^{u_\mathrm{o}+v_\mathrm{o}} \tag{1.10}$$

or

$$\left\langle {}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}_{LM}\;{}_{u_\mathrm{o}+v_\mathrm{o}}\hat{d}^*_{L'M'}\right\rangle = \delta_{MM'}\delta_{LL'}\left(\mathcal{R}_L^{su,tv}+\mathcal{R}_L^{sv,tu}\right) \tag{1.11}$$

In general we have

$$\begin{aligned} G^s_{LM} &= -\tfrac{1}{2}\left({}_sd_{LM} + (-1)^s\;{}_{-s}d_{LM}\right) \quad (s\geq 0)\\ C^s_{LM} &= -\tfrac{1}{2i}\left({}_sd_{LM} - (-1)^s\;{}_{-s}d_{LM}\right) \quad (s\geq 0).\end{aligned}$$

The estimator for ${}_{-s_\mathrm{o}-t_\mathrm{o}}\hat{d}$ is the same as ${}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}$ with all spin signs flipped, and with an overall sign $(-1)^{s_\mathrm{o}+s_\mathrm{i}+t_\mathrm{o}+t_\mathrm{i}}$. The out-spins part gets canceled by the sign $(-1)^s$ in the above equation. JC: No. better to request $w_\ell^{-s_\mathrm{i},-s_\mathrm{o}} = (-1)^{s_\mathrm{o}+si}w_\ell^{s_\mathrm{i},s_\mathrm{o}}$ so that ${}_{-s_\mathrm{o}-t_\mathrm{o}}\hat{d} ={}_{s_\mathrm{o}+t_\mathrm{o}}\hat{d}^*$? Thats what we do nowHence,

$$\begin{aligned}&\delta_{MM'}\delta_{LL'}\left\langle G^{s_\mathrm{o}+t_\mathrm{o}}_{LM}G^{*,u_\mathrm{o}+v_\mathrm{o}}_{L'M'}\right\rangle\cdot 4 = \mathcal{R}_L^{su,tv}+\mathcal{R}_L^{sv,tu} + (-1)^{s_\mathrm{i}+t_\mathrm{i}}\left(\mathcal{R}_L^{-su,-tv}+\mathcal{R}_L^{-sv,-tu}\right)\\ &+(-1)^{u_\mathrm{i}+v_\mathrm{i}}\left(\mathcal{R}_L^{s-u,t-v}+\mathcal{R}_L^{s-v,t-u}\right) + (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}\left(\mathcal{R}_L^{-s-u,-t-v}+\mathcal{R}_L^{-s-v,-t-u}\right) \quad (s_\mathrm{o}+t_\mathrm{o}>=0, u_\mathrm{o}+v_\mathrm{o}>=0)\end{aligned} \tag{1.12}$$

Since $\mathcal{R}$ is invariant under the simultaneous sign-flip of all spins JC: Wrong, also with sign-weighted weights: $\bar{C}^{-s,-t} = (-1)^{s+t}C^{t,s} = (-1)^{s+t}C^{s,t}(s,t\geq 0$, so $\mathcal{R}^{st,uv}$ takes a sign $(-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}}$ and this solves the problem, we can also write this as:

$$\left\langle G^{s_\mathrm{o}+t_\mathrm{o}}_{LM}G^{*,u_\mathrm{o}+v_\mathrm{o}}_{LM}\right\rangle = \frac{1}{2}\left[\left(\mathcal{R}_L^{su,tv}+R_L^{sv,tu}\right)+(-1)^{s_\mathrm{o}+t_\mathrm{o}}\left(\mathcal{R}_L^{-su,tv}+\mathcal{R}_L^{-sv,tu}\right)\right] \tag{1.13}$$

$$\left\langle C^{s_\mathrm{o}+t_\mathrm{o}}_{LM}C^{*,u_\mathrm{o}+v_\mathrm{o}}_{LM}\right\rangle = -\frac{1}{2}\left[\left(\mathcal{R}_L^{su,tv}+R_L^{sv,tu}\right)-(-1)^{s_\mathrm{o}+t_\mathrm{o}}\left(\mathcal{R}_L^{-su,tv}+\mathcal{R}_L^{-sv,tu}\right)\right] \tag{1.14}$$

$$\left\langle G^{s_\mathrm{o}+t_\mathrm{o}}_{LM}C^{*,u_\mathrm{o}+v_\mathrm{o}}_{LM}\right\rangle = 0 \tag{1.15}$$

Pfeew

More details of $\mathcal{R}_L$

Recall the spin-weight addition theorem:

$$\sum_m {}_sY^*_{\ell m}(\boldsymbol{n}')\;{}_tY_{\ell m}(\boldsymbol{n}) = \sqrt{\frac{2\ell+1}{4\pi}}e^{-it\gamma}\;{}_tY_{\ell,-s}(\beta,\alpha). \tag{1.16}$$

Hence

$$\mathcal{R}^{st,uv}(\boldsymbol{n},\boldsymbol{n}') = (-1)^{t_\mathrm{o}+v_\mathrm{o}+t_\mathrm{i}+v_\mathrm{i}} e^{-is_\mathrm{o}\gamma - iu_\mathrm{o}\gamma}\left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_\mathrm{i}} g^{t_\mathrm{i}} C_\ell^{s_\mathrm{i}t_\mathrm{i}}\ {}_{s_\mathrm{o}}Y_{\ell t_\mathrm{o}}(\beta,\alpha)\right)\left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_\mathrm{i}} g^{v_\mathrm{i}} C_\ell^{u_\mathrm{i}v_\mathrm{i}}\ {}_{u_\mathrm{o}}Y_{\ell v_\mathrm{o}}(\beta,\alpha)\right) \tag{1.17}$$

The product of the brackets is a spin $s_\mathrm{o}+u_\mathrm{o}$ function. Defining its spin weight coefficients as $\mathcal{R}_L$ we get the relation claimed above. What are these coefficients?

$$\mathcal{R}_L^{st,uv} \equiv (-1)^{t_\mathrm{o}+v_\mathrm{o}+t_\mathrm{i}+v_\mathrm{i}} \sqrt{\frac{4\pi}{2\ell+1}} \int d^2n \left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{s_\mathrm{i}} g^{t_\mathrm{i}} C_\ell^{s_\mathrm{i}t_\mathrm{i}}\ {}_{s_\mathrm{o}}Y_{\ell t_\mathrm{o}}(\boldsymbol{n})\right)\left(\sum_\ell \sqrt{\frac{2\ell+1}{4\pi}} g^{u_\mathrm{i}} g^{v_\mathrm{i}} C_\ell^{u_\mathrm{i}v_\mathrm{i}}\ {}_{u_\mathrm{o}}Y_{\ell v_\mathrm{o}}(\boldsymbol{n})\right) {}_{s_\mathrm{o}+u_\mathrm{o}}Y^*_{L,t_\mathrm{o}+v_\mathrm{o}}(\tag{1.18}$$

Using

$$\boxed{{}_sY_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}(-1)^m e^{im\phi} d^\ell_{-ms}(\theta)} \tag{1.19}$$

The above thing is invariant if all signs are flipped at the same time. we get

$$\boxed{(-1)^{t_\mathrm{o}+v_\mathrm{o}+t_\mathrm{i}+v_\mathrm{i}}\mathcal{R}_L^{st,uv} \equiv 2\pi \int_{-1}^{1} d\mu\, \xi^{st}(\mu)\, \xi^{uv}(\mu)\, d^L_{-t_\mathrm{o}-v_\mathrm{o},s_\mathrm{o}+u_\mathrm{o}}(\mu),\ \text{with } \xi^{st}(\mu) \equiv \sum_\ell \frac{2\ell+1}{4\pi} g_\ell^{s_\mathrm{i}} g_\ell^{t_\mathrm{i}} C_\ell^{s_\mathrm{i}t_\mathrm{i}} d^\ell_{-t_\mathrm{o},s_\mathrm{o}}(\mu)} \tag{1.20}$$

JC: Remember now the weights w (or g...) changes $(-1)^{s_\mathrm{i}+s_\mathrm{o}}$ under sign flip of the two spins, and $\bar{C}^{st}$ takes a $(-1)^{s+t}$. So $\mathcal{R}^{-s-t,-u-v} = (-1)^{s_\mathrm{i}+t_\mathrm{i}+u_\mathrm{i}+v_\mathrm{i}+s_\mathrm{i}+s_\mathrm{o}+t_\mathrm{i}+t_\mathrm{o}+u_\mathrm{i}+u_\mathrm{o}+v_\mathrm{i}+v_\mathrm{o}} = (-1)^{s_\mathrm{o}+t_\mathrm{o}+u_\mathrm{o}+v_\mathrm{o}}\mathcal{R}^{st,uv}$

B. QE responses calculation

JC: should define the response at the covariance matrix level...The covariance matrices is

$$\langle {}_sX(\boldsymbol{n})\, {}_tX^*(\boldsymbol{n}')\rangle = \langle {}_sX(\boldsymbol{n})\, {}_{-t}X(\boldsymbol{n}')\rangle = \sum_{\ell m} C_\ell^{st}\ {}_sY_{\ell m}(\boldsymbol{n})\, {}_tY^*_{\ell m}(\boldsymbol{n}') \tag{1.21}$$

How does this responds to a source of anisotropy (with spin r), ${}_r\alpha(\boldsymbol{n})$? For all cases in this work, we can parametrize this as follows

$$\delta \langle {}_sX(\boldsymbol{n})\, {}_tX(\boldsymbol{n}')\rangle = {}_r\alpha(\boldsymbol{n}') \sum_{\ell m} {}_rR_\ell^{st}\ {}_sY_{\ell m}(\boldsymbol{n})\, {}_{r-t}Y^*_{\ell m}(\boldsymbol{n}') + ((-t,\boldsymbol{n}') \leftrightarrow (s,\boldsymbol{n})) \tag{1.22}$$

for some set of isotropic response functions R_ℓ. What is the response to the estimator Eq. (1.1)?

Examples include:

- Lensing: The source of anisotropy is the spin-1 field ${}_1\alpha(\boldsymbol{n})$, with response

$$\delta_sX(\boldsymbol{n}) = \frac{1}{2}\alpha_1 \eth_s X(\boldsymbol{n}) + \frac{1}{2}\alpha_{-1}\bar{\eth}\ {}_sX(\boldsymbol{n}) \tag{1.23}$$

  where $\eth$ and $\bar{\eth}$ are the spin lowering and spin raising operator JC: check notation respectively. From their action on the spin-weighted harmonics, the harmonic space responses areJC: ...

$$s_r = s-1, s+1 \quad R^{s,s-1} = JC: ..., R^{s,s+1} = JC: ... \tag{1.24}$$

- Modulation estimator: The source is spin 0, with response

$$\delta_sX(\boldsymbol{n}) =_0 \alpha(\boldsymbol{n})_sX(\boldsymbol{n}) \tag{1.25}$$

  Hence,

$$s_r = s \quad , R_\ell^{ss} = 1 \tag{1.26}$$

- Point source:

$$\delta_sX(\boldsymbol{n}) = {}_0\alpha(\boldsymbol{n})\delta^D(\boldsymbol{n}) \tag{1.27}$$

- Noise anisotropies: