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Ballistics

ECSE 211: Design Principles and Methods

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Overview

- Review
- Analysis: Catapult
- Design Considerations
- Example
- Summary

Review

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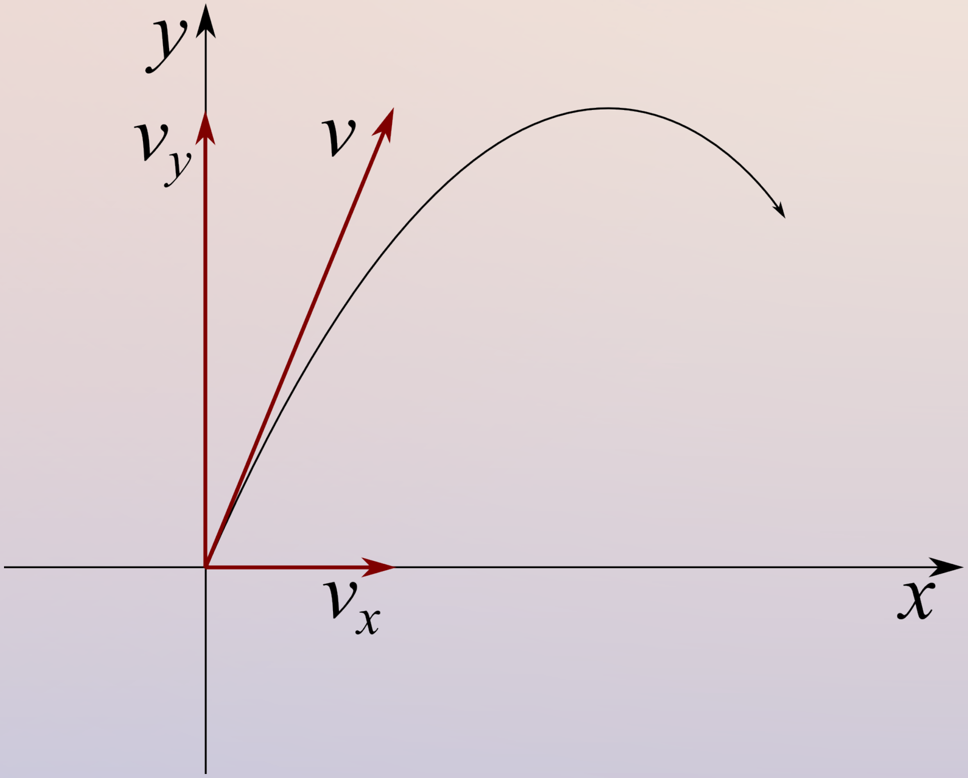
- Recall from high school physics that a projectile's velocity vector can be broken into components
- Each component can then be considered individually
- If a vacuum is assumed, the only force that exists on the projectile is that of gravity



Review

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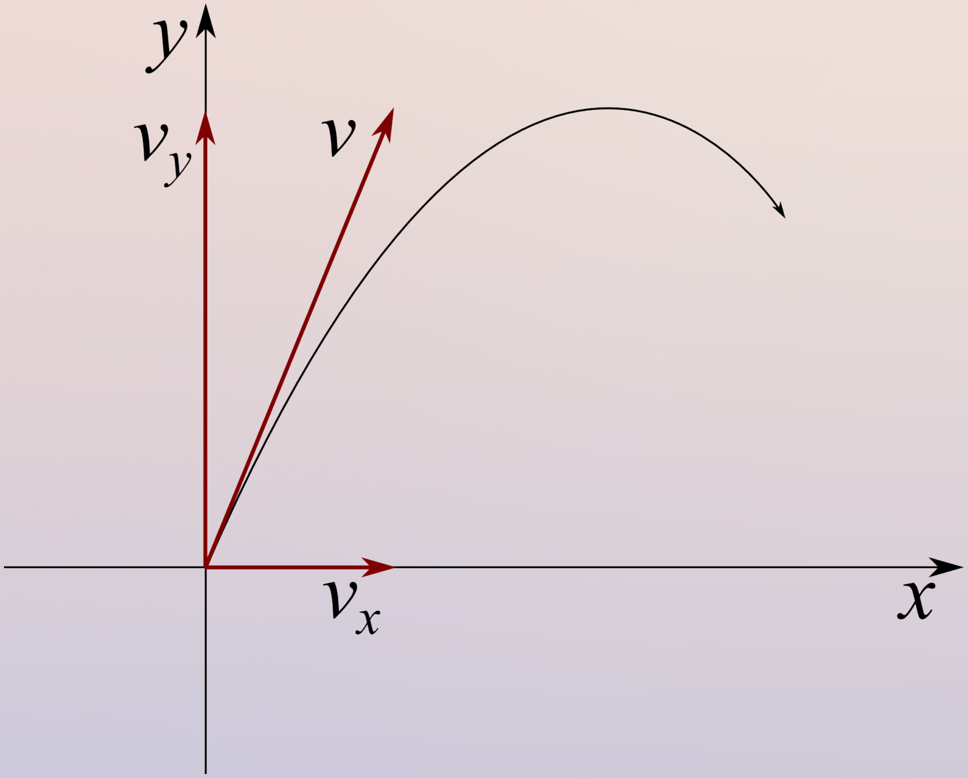
- Since, by Newton's Second Law of Motion,

$$m\frac{d}{dt}v_x = F$$

and there exists no force in the x-direction, v_x must be constant



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- Integrating on both sides of:

$$m\frac{d}{dt}v_y=-mg$$

$$\int_0^t\int_0^\tau \frac{d}{dT}v_y\,dT\,d\tau=\int_0^t\int_0^\tau -g\,dT\,d\tau$$

$$r_y=-\frac{1}{2}gt^2+C_1t+C_0$$

and assuming the projectile is initially at the origin,

$$r_y(t)=-\frac{1}{2}gt^2+v_{y0}t$$

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- Then we have two equations for the position of the projectile as a function of time:

$$r_x(t) = v_{x0} t$$

$$r_y(t) = -\frac{1}{2} g t^2 + v_{y0} t$$

and performing substitution of one parametric equation into the other, we get the trajectory of the projectile:

$$t = \frac{r_x(t)}{v_{x0}} = \frac{x}{v_{x0}}$$

$$y(x) = -\frac{1}{2 v_{x0}^2} g x^2 + \frac{v_{y0}}{v_{x0}} x$$

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- This is a parabola, whose roots are:

$$y(x)=\left(-\frac{1}{2v_{x0}^2}gx+\frac{v_{y0}}{v_{x0}}\right)x=0\Rightarrow x=\left\{0,\frac{2v_{x0}v_{y0}}{g}\right\}$$

Thus if the projectile is launched at a fixed velocity v with a variable angle θ,

$$(v_{x0},v_{y0})=(v\cos(\theta),v\sin(\theta))$$

$$d=\frac{2v_{x0}v_{y0}}{g}=\frac{2v^2\sin(\theta)\cos(\theta)}{g}=\frac{v^2\sin(2\theta)}{g}$$

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- And maximizing d with respect to θ yields the well known result:

$$\frac{d}{d\theta}d=\frac{2v^2\cos(2\theta)}{g}=0\Rightarrow\theta=\frac{\cos^{-1}(0)}{2}=45^{\circ}$$

where d is the distance the projectile travels, and is maximized at:

$$d_{max}=\frac{v^2}{g}$$

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- Finally, if the initial energy of the projectile is known,

$$\frac{2E}{m} = v^2$$

$$d = \frac{2E\sin(2\theta)}{mg}, \quad d_{max} = \frac{2E}{mg}$$

Analysis: Catapult

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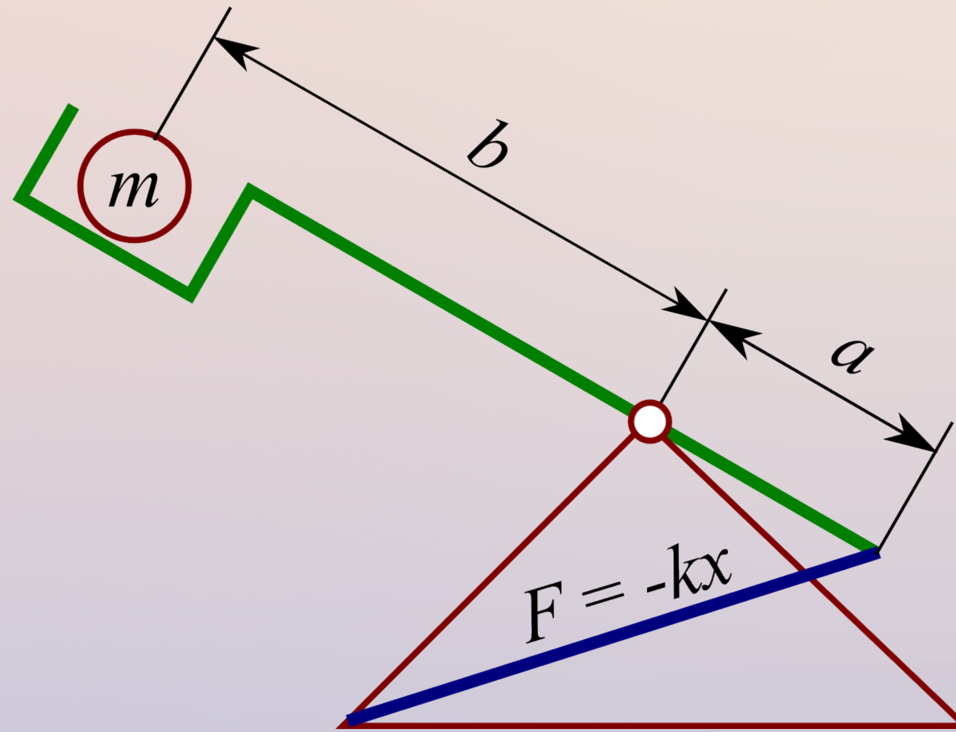
- Consider the (green) catapult on the right, where the blue bar is a spring under tension respecting Hooke's Law
- To aid in the analysis of this catapult, some simplifying assumptions must be made



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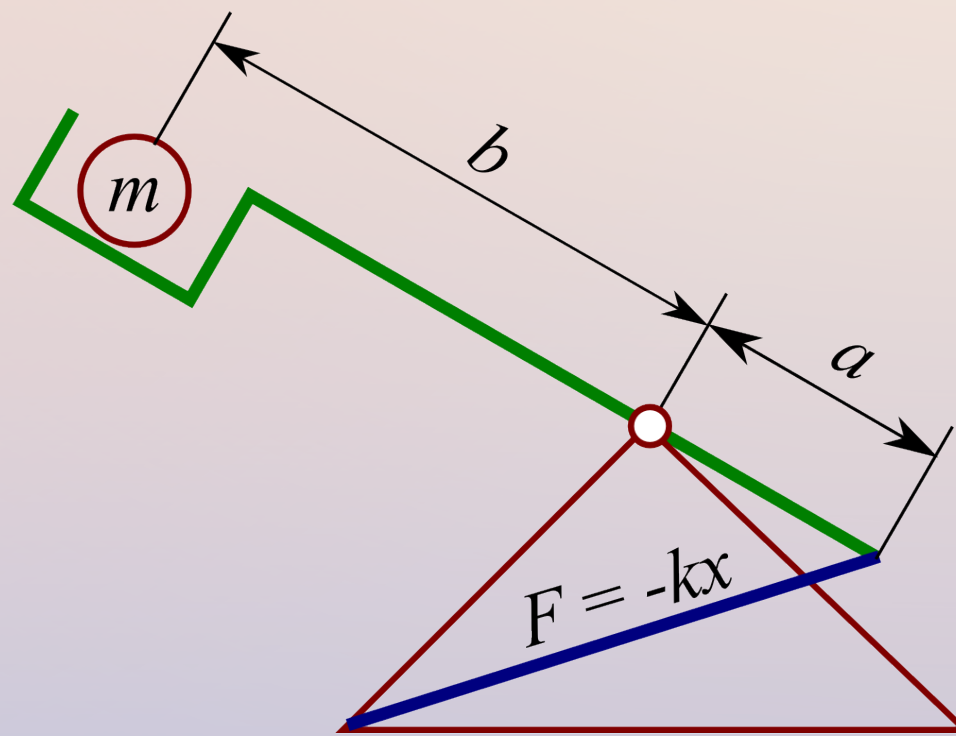
Analysis: Catapult

- First, assume that the force applied on the catapult by the spring is applied entirely tangentially
- Second, assume that gravity is negligible
- Finally, assume that the catapult (without projectile) has a known inertial moment $I_{catapult}$



Analysis: Catapult

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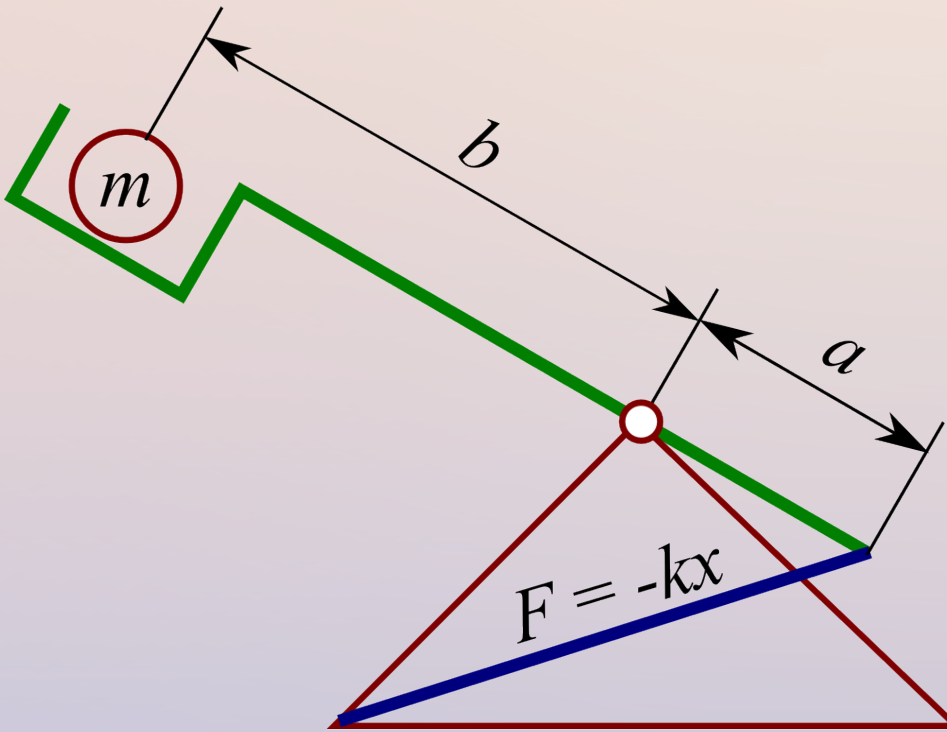
- Then the torque applied by the spring is:

$$\tau_{spring} = (-kx)\,a$$

- Since the force is being applied tangentially, we define θ such that:

$$\theta = \frac{x}{a} \Rightarrow \tau_{spring} = -k\,\theta\,a^2$$



Analysis: Catapult

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- Then applying Newton's Second Law of Motion yields

$$\tau_{spring} = I\alpha$$

$$-k\theta a^2 = I\alpha$$

$$\frac{-ka^2}{I}\theta = \frac{d^2}{dt^2}\theta$$

an ordinary differential equation whose solution is:

$$\theta(t) = C_1 \sin\left(a\sqrt{\frac{k}{I}}t\right) + C_2 \cos\left(a\sqrt{\frac{k}{I}}t\right)$$

Analysis: Catapult

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- Assuming the catapult starts at rest, at a position θ_0, it follows that:

$$\theta(t) = C_1 \sin\left(a\sqrt{\frac{k}{I}}t\right) + C_2 \cos\left(a\sqrt{\frac{k}{I}}t\right)$$

$$\theta(0) = C_1 \sin(0) + C_2 \cos(0) = C_2 = \theta_0$$

$$\left.\frac{d}{dt}\theta(t)\right|_{t=0} = C_1 a\sqrt{\frac{k}{I}}\cos(0) - C_2 a\sqrt{\frac{k}{I}}\sin(0) = C_1 = 0$$

Analysis: Catapult

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- So the position of the catapult as a function of time is:

$$\theta(t) = \theta_0 \cos\left(a\sqrt{\frac{k}{I}}\,t\right)$$

- By taking a derivative, we get its angular velocity:

$$\frac{d}{dt}\theta(t) = -\theta_0 a\sqrt{\frac{k}{I}}\sin\left(a\sqrt{\frac{k}{I}}\,t\right)$$

to conclude that its peak angular velocity is:

$$\omega_{max} = \theta_0 a\sqrt{\frac{k}{I}}$$

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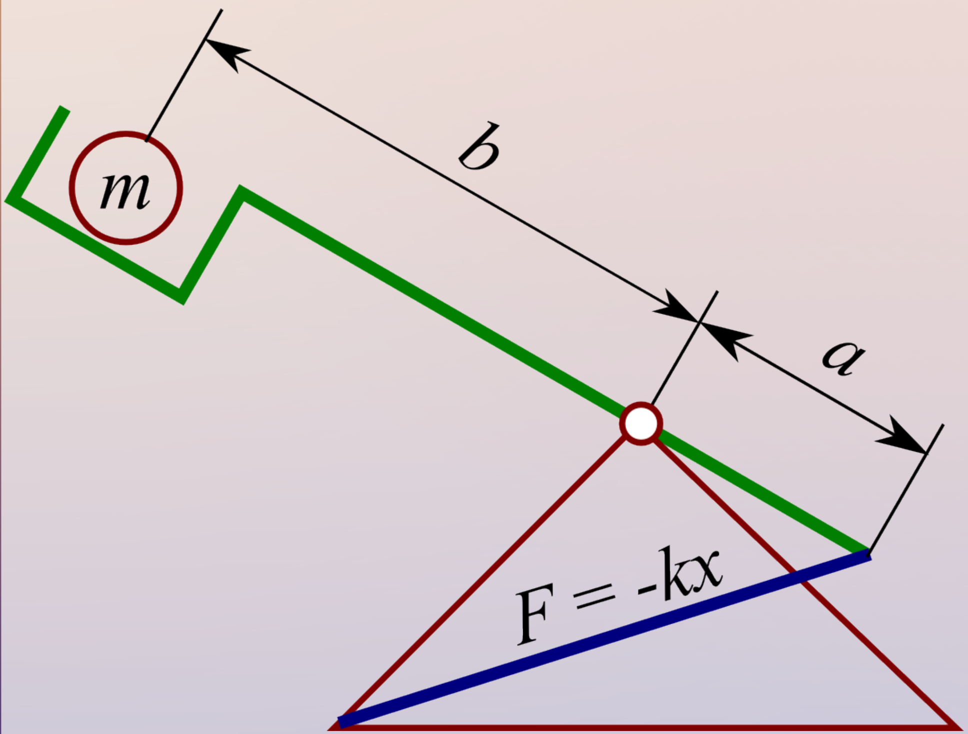

- In this context, I is the inertial moment of the catapult and its projectile:

$$I = I_{catapult} + m_{projectile} b^2$$

- And with some simple geometry, the projectile's launch velocity is:

$$v_0 = \omega_{max} b$$

Analysis: Catapult

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- Its launch velocity can be expressed as a function of the catapult's parameters:

$$v_0 = \theta_0 a b \sqrt{\frac{k}{I}}$$

- And similarly, its initial kinetic energy:

$$E_{projectile} = \frac{1}{2} m v_0^2 = \frac{m k (\theta_0 a b)^2}{2 I}$$

Analysis: Catapult

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- Returning to the earlier equations (on slide 15), it can be shown that ω_{max} occurs when $\theta(t) = 0$. Then the energy released by the spring into the catapult is:

$$E_{spring} = \frac{1}{2} k \theta_0^2 a^2$$

- And finally, the energy efficiency of the catapult is evaluable as:

$$\eta_{catapult} = \frac{E_{projectile}}{E_{spring}} = \frac{m_{projectile} b^2}{I_{catapult} + m_{projectile} b^2}$$

Design Considerations

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- If the catapult is made of a material of constant linear density λ with respect to its length, then its inertial moment is:

$$I_{catapult} = \int_{-a}^{b} \lambda r^2 \, dr = \frac{\lambda \left(a^3 + b^3\right)}{3}$$

- Since this growth is cubic, increasing the length of the catapult (*b*) will, in general, decrease the projectile velocity and the catapult's mechanical efficiency

Design Considerations

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- The distance d_{max} travelled by the projectile can be expressed as a function of the catapult parameters:

$$d_{max} = \frac{v^2}{g} = \frac{k\left(\theta_0 \, a \, b\right)^2}{I\,g}$$

- Thus, increasing the angle rotated by the catapult prior to launching (θ_0), decreasing its inertial moment (I), and increasing the spring constant of its spring (k) are possible means of improving the launch distance

Example

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- A catapult with b = 10 cm launches a 3 g ping-pong ball 2 m, with an initial launch angle of 45°. Its spring was measured to store 1 J of energy prior to launch. Determine the inertial moment of the catapult.

<u>Answer</u>

Working backward:

$$d_{max} = \frac{2E}{mg} \Rightarrow E = \frac{d_{max}\, m\, g}{2} = \frac{2 \times 0.003 \times 9.8}{2} = 29.4\,\text{mJ}$$

Example

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- Using the formula for efficiency:

$$\eta_{catapult} = \frac{E_{projectile}}{E_{spring}} = \frac{m_{projectile} b^2}{I_{catapult} + m_{projectile} b^2}$$

$$I_{catapult} = \frac{E_{spring} - E_{projectile}}{E_{projectile}} m_{projectile} b^2$$

$$I_{catapult} = \frac{1 - 0.0294}{0.0294} \times 0.003 \times 0.1^2 = 0.99 \, \mathrm{g\,m}^2$$

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Summary

- Newton's laws of physics allow the mathematical description of the (parabolic) trajectory of a projectile with constant force on it
- Using a catapult as a sample launching mechanism (naturally, others exist), a detailed analysis of the efficiency of the device can be performed
- This analysis in turn leads to conclusions that influence the design process
- Additionally, experimentation is shown to be usable to calculate otherwise difficult-to-measure values (such as inertial moment)