McGill University
Department of Electrical and Computer Engineering
Patrick Diez, U4 Undergraduate
Sean Lawlor, U4 Undergraduate

Error

ECSE 211: Design Principles and Methods

Overview

McGill University
Department of Electrical and
Computer Engineering
Patrick Diez, U4 Undergraduate
Sean Lawlor, U4 Undergraduate

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- Correctable and Incorrigible Error
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Definition

- In sensors, error is the signed difference between the (known) physical value and the value reported by the sensor
- In control systems, error is the signed difference between the target value of the controlled variable (the setpoint), and its measured value
- In general, error is quantitatively defined as the difference between that which is desired and that which is reported

Correctable and Incorrigible Error

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- To illustrate the nature of error, consider a one-dimensional dartboard on the integer number line, where the score, $s(n)$, is a function of the position on the number line, n, defined as:

$$s(n) = \begin{cases} 17-(5-n)^2 & 0<n<10 \\ 0 & otherwise \end{cases}$$

- Ten (10) darts are shot at the board and hit the following positions on the board (values of n), for a score of 82:

$$5, 6, 10, 5, 8, 4, 10, 8, 8, 6$$

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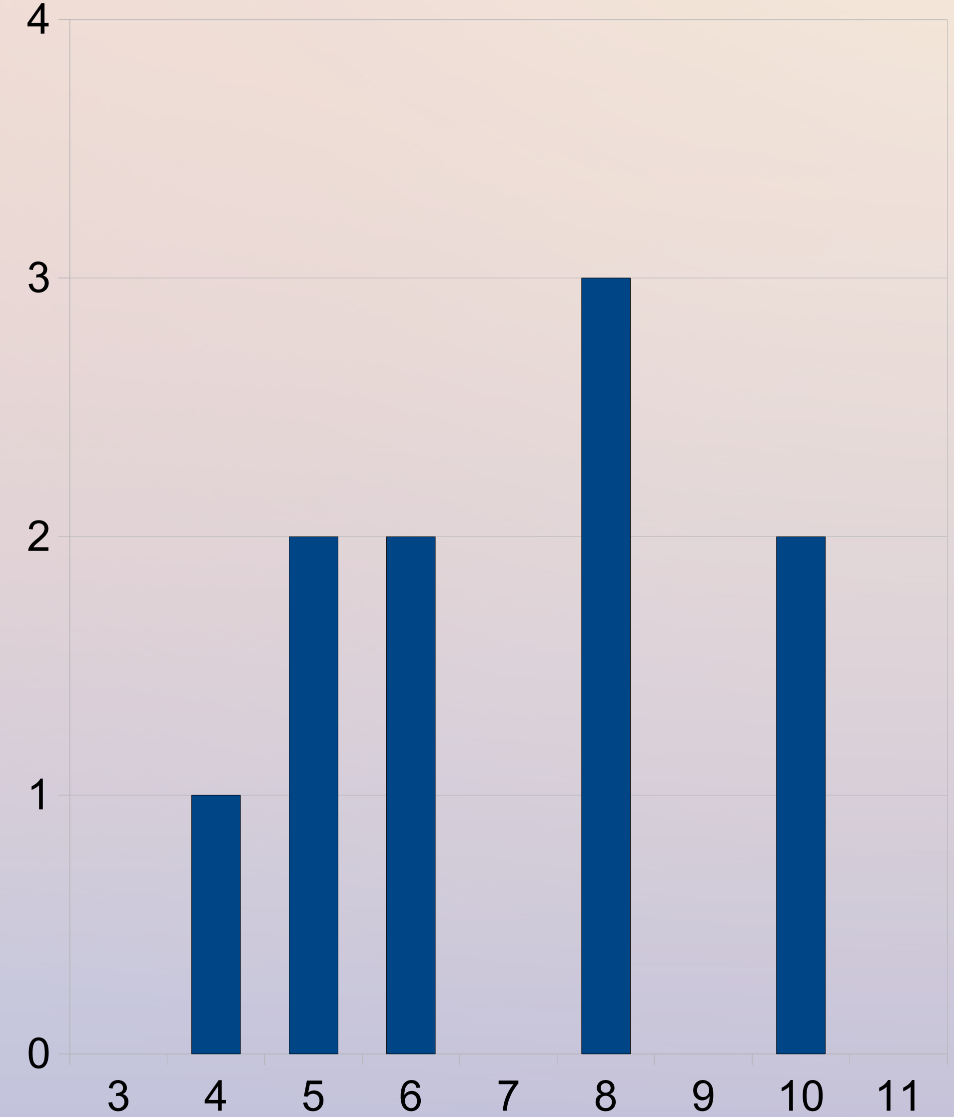
- These results are then plotted (right) on a bar graph whose x-axis is the position on the dartboard, *n*, and whose y-axis is the number of hits
- Using this information, we seek to do two things: improve the score, and predict future hits



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- Clearly, hitting the centre of the dartboard ($n = 5$) yields the highest score ($s(5)=17$), so centring the shots around $n = 5$ will maximize the score
- The **mean** position, μ, determines the location of the centre of the shots, and is equal to the sum of the values of n divided by the number of shots taken

$$\mu = \frac{5+6+10+5+8+4+10+8+8+6}{10} = 7$$

- Thus, were these shots aimed 2 positions to the left, they would yield the highest score, notably 130
- In this sense, the mean represents **correctable error**

Correctable and Incorrigible Error

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- Another means of improving the score is to **cluster** the shots closer to their mean
- The **standard deviation**, σ, is a measure of the shots' clustering, and is equal to the square root of the sum of the squares of the difference between each shot and their mean, divided by one less than the number of shots taken (larger numbers signify less clustering):

$$\sigma=\sqrt{\frac{(5-7)^2+(6-7)^2+\ldots+(6-7)^2}{9}}=2.11$$

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- Unfortunately, unlike the mean, the standard deviation represents an **incorrigible error**, that is, one that cannot be corrected by changing where shots are aimed, but only by improving the accuracy of individual shots
- However, it is possible to predict, using the values μ and σ, the likelihood of a dart hitting a given position on the dartboard
- To do this, a model for the shot distribution, called a **normal distribution**, is used

Normal Distributions

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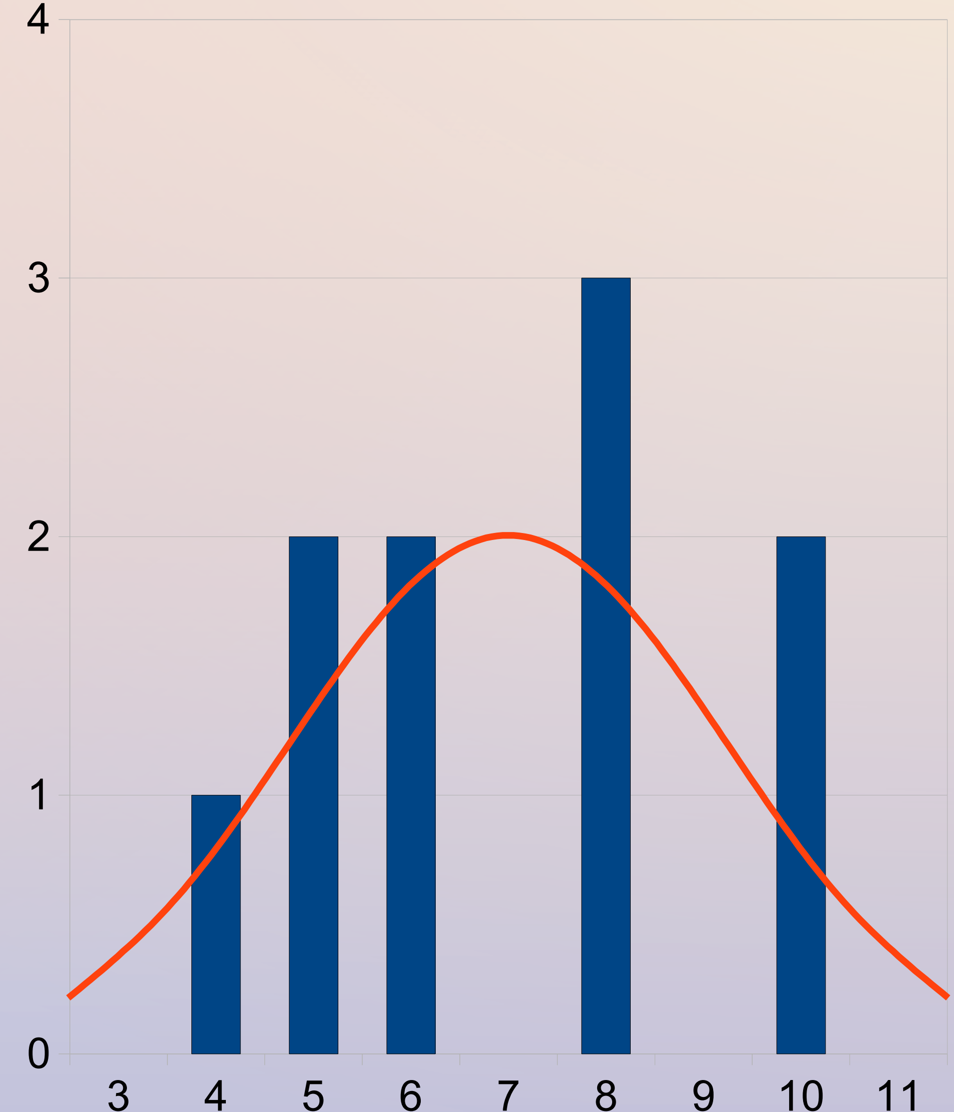
- The normal distribution is a kind of probability distribution well-suited to the representation of test results on the real number line, and takes the form:

$$\phi(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



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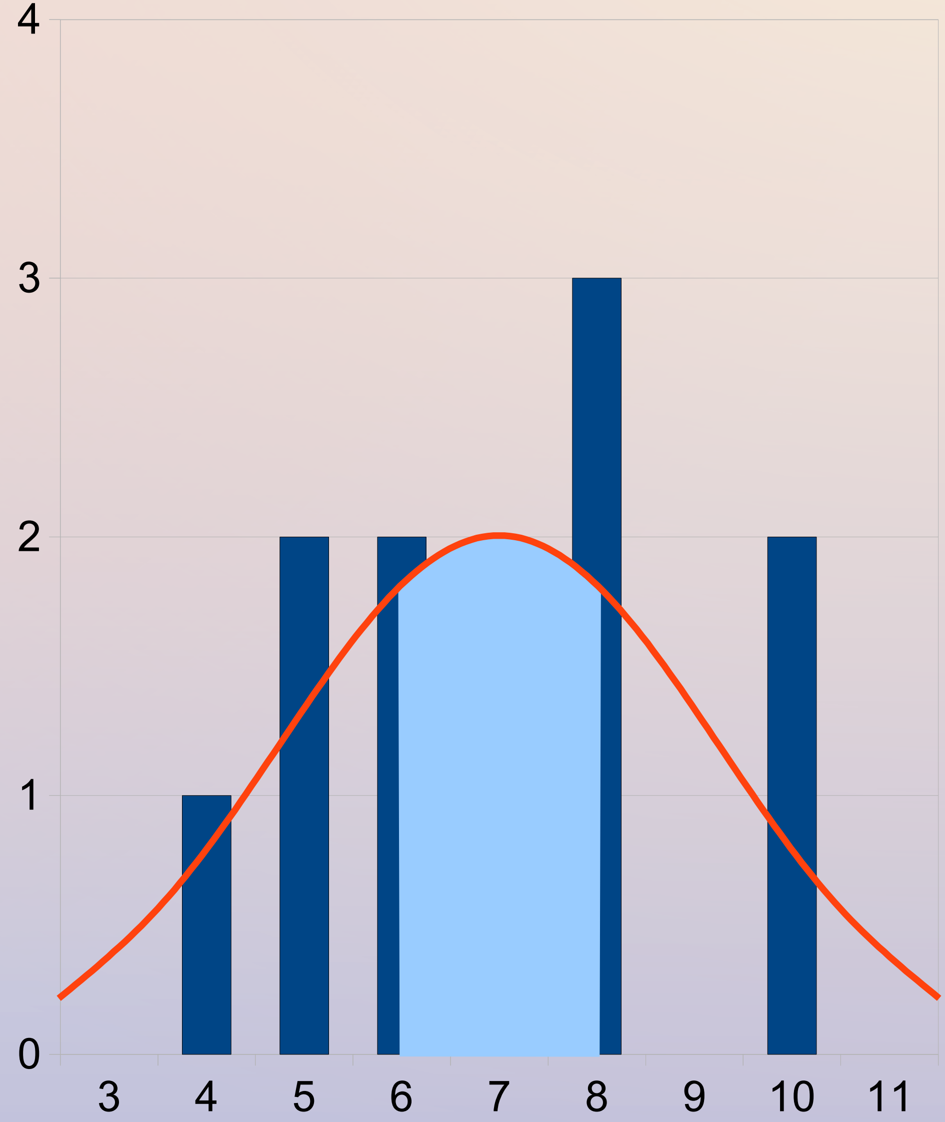
Normal Distributions

- The area under the normal distribution represents the likelihood of a result occurring in that interval
- On the right is highlighted in light blue the area corresponding to the interval (6, 8), with a probability of 38%



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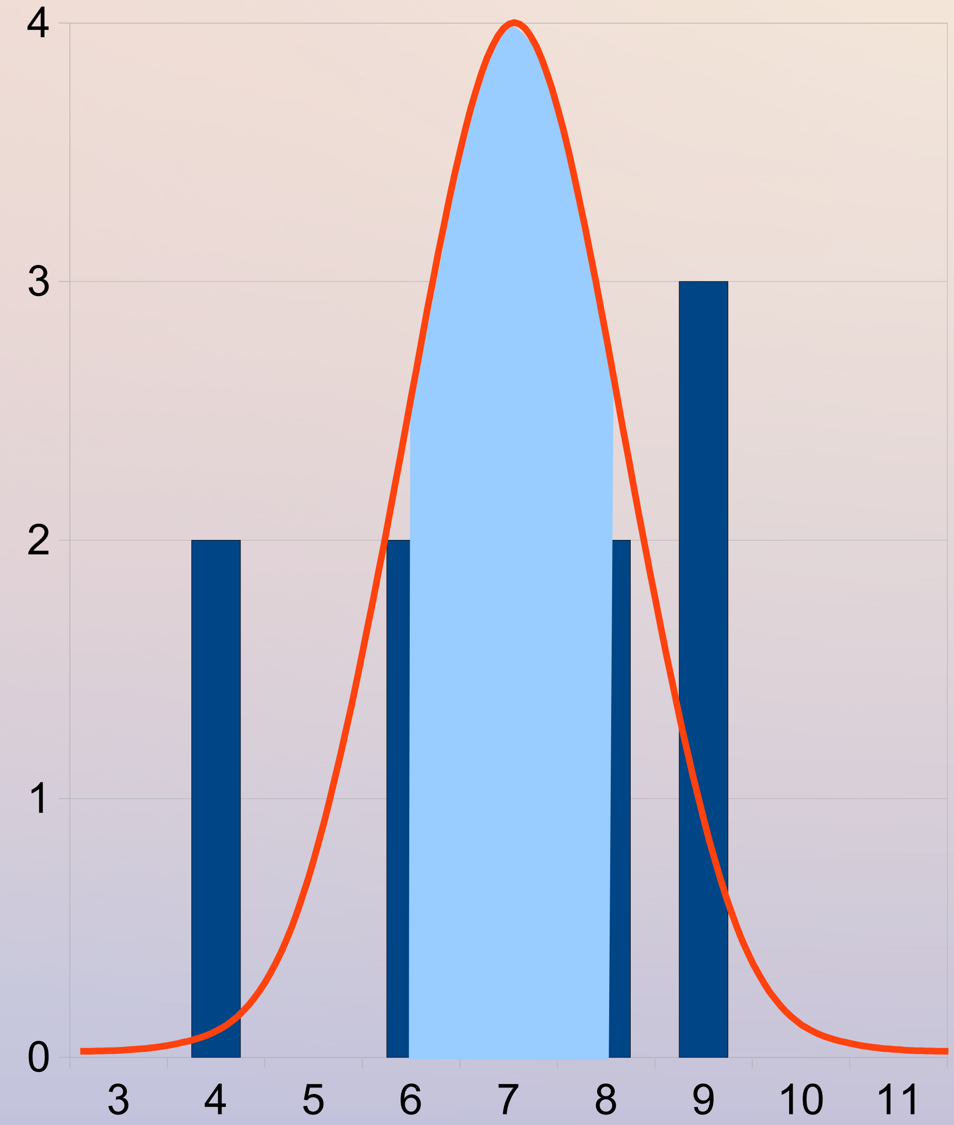
Normal Distributions

- If, in another sample set of 10 shots, σ is decreased, the area under the curve in the same interval will increase (as it is centred around the mean)
- As seen on the right, decreasing σ by a factor of two increased the probability for the same interval to 68%



Normal Distributions

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- To compute these areas, we use the anti-derivative of the normal distribution function, called the **error function**:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

- Since the normal distribution is symmetric about its mean, and the area under it is always 1, we see that by substitution

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(t-\mu)^2}{2\sigma^2}}\,dt = \frac{1}{2} + \frac{1}{2} erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)$$

Normal Distributions

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- Finally, applying the Fundamental Theorem of Calculus,

$$\int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(t-\mu)^2}{2\sigma^2}} dt = \frac{1}{2}\left(erf\left(\frac{b-\mu}{\sigma\sqrt{2}}\right) - erf\left(\frac{a-\mu}{\sigma\sqrt{2}}\right)\right)$$

- Many programs, including Microsoft Excel, OpenOffice.org Calc, Maple, and MATLAB, allow evaluation of the error function, *erf*(x)

Normal Distributions

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- As a rule of thumb, the area under the normal distribution in the interval $(\mu - \sigma, \mu + \sigma)$ is 0.68, representing a 68% probability, and in the interval $(\mu - 2\sigma, \mu + 2\sigma)$ is 0.95, representing a 95% probability
- Thus, if a person aiming at the one-dimensional dartboard's centre hits the dartboard 95% of the time, the standard deviation (σ) of the shots is equal to half the radius of the dartboard
- In this way, it is easy to calculate success rates for given intervals from σ, and vice-versa

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Example: Navigation

- Assume that you have written a controller for your robot's motors that moves it forward a specified distance, in centimetres
- To test it, you perform the following task
  - Make the robot move forward a specified distance
  - Measure and record the **difference** (**error**) between the distance specified and the actual distance moved
  - Repeat using a different distance
- From these tests, you determine that the **mean** of the error is -2 cm, and the **standard deviation** is 0.5 cm

Example: Navigation

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- You can correct for the non-zero mean of your error by making your robot move 2 cm less than the distance specified. This correction can (and should) be done in **software**, and is a simple form of **calibration**
- However, to reduce the standard deviation, **mechanical** (**hardware**) modifications may be required, such as changing the position of the wheels on the robot to improve traction
- Most importantly, knowing $\sigma = 0.5$ cm, you can make the informative statement that: "The robot will stop within 1 cm of the desired position 95% of the time."

Summary

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- The difference between the desired value of a sensor or system and its reported value is called **error**
- Through testing, the **mean** and **standard deviation** of the error can be calculated, and the sensor's or system's **accuracy** can be improved by **compensating** for the mean through the process of **software calibration**
- The standard deviation, however, represents a **hard-to-correct** randomness of the sensor or system, and is used to define the range in which, given a reported value, the actual value will exist 95% of the time