

2

Forward Vehicle Dynamics

Straight motion of an ideal rigid vehicle is the subject of this chapter. We ignore air friction and examine the load variation under the tires to determine the vehicle's limits of acceleration, road grade, and kinematic capabilities.

2.1 Parked Car on a Level Road

When a car is parked on level pavement, the normal force, F_z , under each of the front and rear wheels, F_{z1} , F_{z2} , are

$$F_{z1} = \frac{1}{2}mg\frac{a_2}{l} \quad (2.1)$$

$$F_{z2} = \frac{1}{2}mg\frac{a_1}{l} \quad (2.2)$$

where, a_1 is the distance of the car's mass center, C , from the front axle, a_2 is the distance of C from the rear axle, and l is the wheel base.

$$l = a_1 + a_2 \quad (2.3)$$

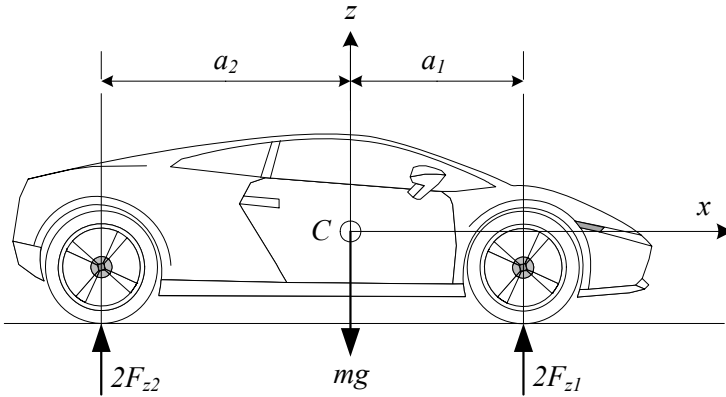


FIGURE 2.1. A parked car on level pavement.

Proof. Consider a longitudinally symmetrical car as shown in Figure 2.1. It can be modeled as a two-axel vehicle. A symmetric two-axel vehicle is equivalent to a rigid beam having two supports. The vertical force under the front and rear wheels can be determined using planar static equilibrium equations.

$$\sum F_z = 0 \quad (2.4)$$

$$\sum M_y = 0 \quad (2.5)$$

Applying the equilibrium equations

$$2F_{z_1} + 2F_{z_2} - mg = 0 \quad (2.6)$$

$$-2F_{z_1}a_1 + 2F_{z_2}a_2 = 0 \quad (2.7)$$

provide the reaction forces under the front and rear tires.

$$\begin{aligned} F_{z_1} &= \frac{1}{2}mg \frac{a_2}{a_1 + a_2} \\ &= \frac{1}{2}mg \frac{a_2}{l} \end{aligned} \quad (2.8)$$

$$\begin{aligned} F_{z_2} &= \frac{1}{2}mg \frac{a_1}{a_1 + a_2} \\ &= \frac{1}{2}mg \frac{a_1}{l} \end{aligned} \quad (2.9)$$

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Example 39 *Reaction forces under wheels.*

A car has 890 kg mass. Its mass center, C , is 78 cm behind the front wheel axis, and it has a 235 cm wheel base.

$$a_1 = 0.78 \text{ m} \quad (2.10)$$

$$l = 2.35 \text{ m} \quad (2.11)$$

$$m = 890 \text{ kg} \quad (2.12)$$

The force under each front wheel is

$$\begin{aligned} F_{z_1} &= \frac{1}{2}mg \frac{a_2}{l} \\ &= \frac{1}{2}890 \times 9.81 \times \frac{2.35 - 0.78}{2.35} = 2916.5 \text{ N} \end{aligned} \quad (2.13)$$

and the force under each rear wheel is

$$\begin{aligned} F_{z_2} &= \frac{1}{2}mg \frac{a_1}{l} \\ &= \frac{1}{2}890 \times 9.81 \times \frac{0.78}{2.35} = 1449 \text{ N.} \end{aligned} \quad (2.14)$$

Example 40 *Mass center position.*

Equations (2.1) and (2.2) can be rearranged to calculate the position of mass center.

$$a_1 = \frac{2l}{mg} F_{z_2} \quad (2.15)$$

$$a_2 = \frac{2l}{mg} F_{z_1} \quad (2.16)$$

Reaction forces under the front and rear wheels of a horizontally parked car, with a wheel base $l = 2.34$ m, are:

$$F_{z_1} = 2000 \text{ N} \quad (2.17)$$

$$F_{z_2} = 1800 \text{ N} \quad (2.18)$$

Therefore, the longitudinal position of the car's mass center is at

$$\begin{aligned} a_1 &= \frac{2l}{mg} F_{z_2} \\ &= 2 \frac{2.34}{2(2000 + 1800)} \times 1800 = 1.1084 \text{ m} \end{aligned} \quad (2.19)$$

$$\begin{aligned} a_2 &= \frac{2l}{mg} F_{z_1} \\ &= 2 \frac{2.34}{2(2000 + 1800)} \times 2000 = 1.2316 \text{ m.} \end{aligned} \quad (2.20)$$

Example 41 *Longitudinal mass center determination.*

The position of mass center C can be determined experimentally. To determine the longitudinal position of C , we should measure the total weight of the car as well as the force under the front or the rear wheels. Figure 2.2 illustrates a situation in which we measure the force under the front wheels.

Assuming the force under the front wheels is $2F_{z_1}$, the position of the mass center is calculated by static equilibrium conditions

$$\sum F_z = 0 \quad (2.21)$$

$$\sum M_y = 0. \quad (2.22)$$

Applying the equilibrium equations

$$2F_{z_1} + 2F_{z_2} - mg = 0 \quad (2.23)$$

$$-2F_{z_1} a_1 + 2F_{z_2} a_2 = 0 \quad (2.24)$$

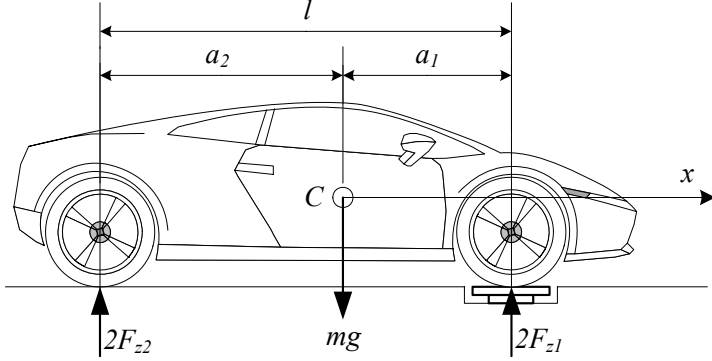


FIGURE 2.2. Measuring the force under the front wheels.

provide the longitudinal position of C and the reaction forces under the rear wheels.

$$\begin{aligned} a_1 &= \frac{2l}{mg} F_{z_2} \\ &= \frac{2l}{mg} (mg - 2F_{z_1}) \end{aligned} \quad (2.25)$$

$$F_{z_2} = \frac{1}{2} (mg - 2F_{z_1}) \quad (2.26)$$

Example 42 *Lateral mass center determination.*

Most cars are approximately symmetrical about the longitudinal center plane passing the middle of the wheels, and therefore, the lateral position of the mass center C is close to the center plane. However, the lateral position of C may be calculated by weighing one side of the car.

Example 43 *Height mass center determination.*

To determine the height of mass center C , we should measure the force under the front or rear wheels while the car is on an inclined surface. Experimentally, we use a device such as is shown in Figure 2.3. The car is parked on a level surface such that the front wheels are on a scale jack. The front wheels will be locked and anchored to the jack, while the rear wheels will be left free to turn. The jack lifts the front wheels and the required vertical force applied by the jacks is measured by a load cell.

Assume that we have the longitudinal position of C and the jack is lifted such that the car makes an angle ϕ with the horizontal plane. The slope angle ϕ is measurable using level meters. Assuming the force under the front wheels is $2F_{z_1}$, the height of the mass center can be calculated by

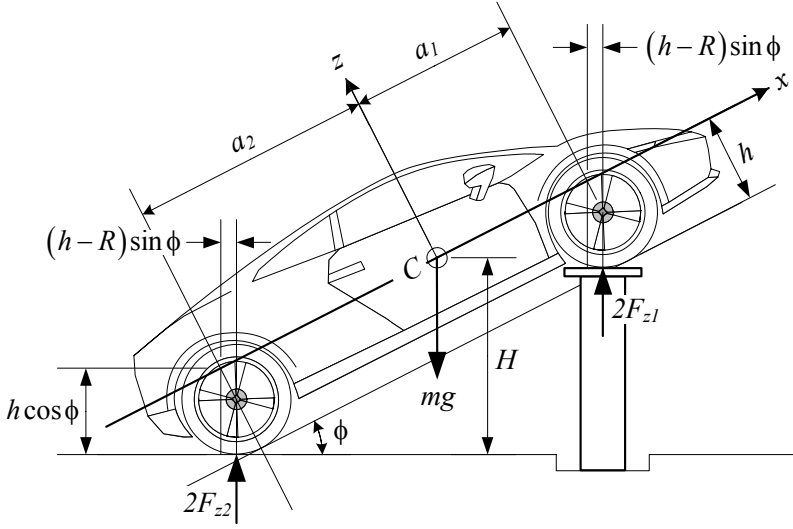


FIGURE 2.3. Measuring the force under the wheels to find the height of the mass center.

static equilibrium conditions

$$\sum F_Z = 0 \quad (2.27)$$

$$\sum M_y = 0. \quad (2.28)$$

Applying the equilibrium equations

$$2F_{z1} + 2F_{z2} - mg = 0 \quad (2.29)$$

$$\begin{aligned} -2F_{z1} (a_1 \cos \phi - (h - R) \sin \phi) \\ + 2F_{z2} (a_2 \cos \phi + (h - R) \sin \phi) &= 0 \end{aligned} \quad (2.30)$$

provides the vertical position of C and the reaction forces under the rear wheels.

$$F_{z2} = \frac{1}{2}mg - F_{z1} \quad (2.31)$$

$$\begin{aligned} h &= \frac{F_{z1} (R \sin \phi + a_1 \cos \phi) + F_{z2} (R \sin \phi - a_2 \cos \phi)}{mg \sin \phi} \\ &= R + \frac{a_1 F_{z1} - a_2 F_{z2}}{mg} \cot \phi \\ &= R + \left(2 \frac{F_{z1}}{mg} l - a_2 \right) \cot \phi \end{aligned} \quad (2.32)$$

A car with the following specifications

$$\begin{aligned}
 m &= 2000 \text{ kg} \\
 2F_{z_1} &= 18000 \text{ N} \\
 \phi &= 30^\circ \approx 0.5236 \text{ rad} \\
 a_1 &= 110 \text{ cm} \\
 l &= 230 \text{ cm} \\
 R &= 30 \text{ cm}
 \end{aligned} \tag{2.33}$$

has a C at height h.

$$h = 34 \text{ cm} \tag{2.34}$$

There are three assumptions in this calculation: 1— the tires are assumed to be rigid disks with radius R, 2— fluid shift, such as fuel, coolant, and oil, are ignored, and 3— suspension deflections are assumed to be zero.

Suspension deflection generates the maximum effect on height determination error. To eliminate the suspension deflection, we should lock the suspension, usually by replacing the shock absorbers with rigid rods to keep the vehicle at its ride height.

Example 44 *Different front and rear tires.*

Depending on the application, it is sometimes necessary to use different type of tires and wheels for front and rear axles. When the longitudinal position of C for a symmetric vehicle is determined, we can find the height of C by measuring the load on only one axle. As an example, consider the motorcycle in Figure 2.4. It has different front and rear tires.

Assume the load under the rear wheel of the motorcycle F_z is known. The height h of C can be found by taking a moment of the forces about the tireprint of the front tire.

$$h = \frac{F_{z_2}(a_1 + a_2)}{mg} - a_1 \cos \left(\sin^{-1} \frac{H}{a_1 + a_2} \right) + \frac{R_f + R_r}{2} \tag{2.35}$$

Example 45 *Statically indeterminate.*

A vehicle with more than three wheels is statically indeterminate. To determine the vertical force under each tire, we need to know the mechanical properties and conditions of the tires, such as the value of deflection at the center of the tire, and its vertical stiffness.

2.2 Parked Car on an Inclined Road

When a car is parked on an inclined pavement as shown in Figure 2.5, the normal force, F_z , under each of the front and rear wheels, F_{z_1} , F_{z_2} , is:

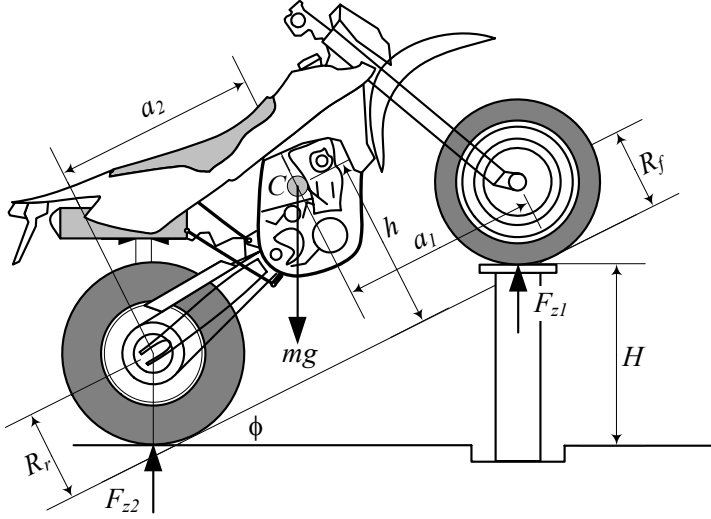


FIGURE 2.4. A motorcycle with different front and rear tires.

$$F_{z1} = \frac{1}{2}mg \frac{a_2}{l} \cos \phi + \frac{1}{2}mg \frac{h}{l} \sin \phi \quad (2.36)$$

$$F_{z2} = \frac{1}{2}mg \frac{a_1}{l} \cos \phi - \frac{1}{2}mg \frac{h}{l} \sin \phi \quad (2.37)$$

$$l = a_1 + a_2$$

where, ϕ is the angle of the road with the horizon. The horizon is perpendicular to the gravitational acceleration \mathbf{g} .

Proof. Consider the car shown in Figure 2.5. Let us assume the parking brake forces are applied on only the rear tires. It means the front tires are free to spin. Applying the planar static equilibrium equations

$$\sum F_x = 0 \quad (2.38)$$

$$\sum F_z = 0 \quad (2.39)$$

$$\sum M_y = 0 \quad (2.40)$$

shows that

$$2F_{x2} - mg \sin \phi = 0 \quad (2.41)$$

$$2F_{z1} + 2F_{z2} - mg \cos \phi = 0 \quad (2.42)$$

$$-2F_{z1}a_1 + 2F_{z2}a_2 - 2F_{x2}h = 0. \quad (2.43)$$

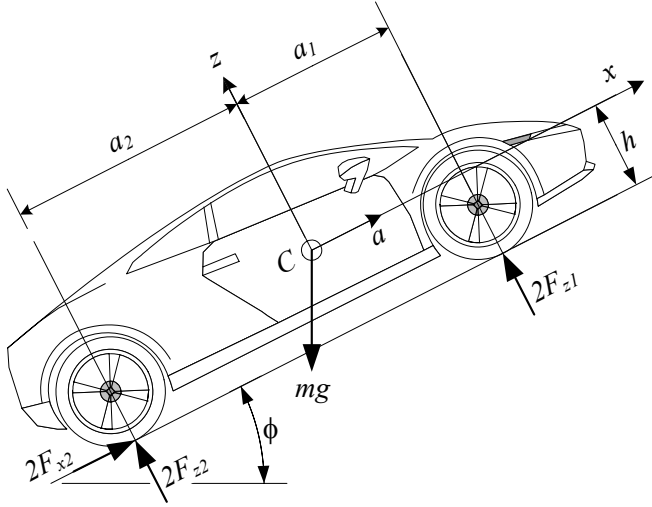


FIGURE 2.5. A parked car on inclined pavement.

These equations provide the brake force and reaction forces under the front and rear tires.

$$F_{z_1} = \frac{1}{2}mg \frac{a_2}{l} \cos \phi - \frac{1}{2}mg \frac{h}{l} \sin \phi \quad (2.44)$$

$$F_{z_2} = \frac{1}{2}mg \frac{a_1}{l} \cos \phi + \frac{1}{2}mg \frac{h}{l} \sin \phi \quad (2.45)$$

$$F_{x_2} = \frac{1}{2}mg \sin \phi \quad (2.46)$$

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Example 46 *Increasing the inclination angle.*

When $\phi = 0$, Equations (2.36) and (2.37) reduce to (2.1) and (2.2). By increasing the inclination angle, the normal force under the front tires of a parked car decreases and the normal force and braking force under the rear tires increase. The limit for increasing ϕ is where the weight vector mg goes through the contact point of the rear tire with the ground. Such an angle is called a **tilting angle**.

Example 47 *Maximum inclination angle.*

The required braking force F_{x_2} increases by the inclination angle. Because F_{x_2} is equal to the friction force between the tire and pavement, its maximum depends on the tire and pavement conditions. There is a specific angle ϕ_M at which the braking force F_{x_2} will saturate and cannot increase any more. At this maximum angle, the braking force is proportional to the normal force F_{z_2}

$$F_{x_2} = \mu_{x_2} F_{z_2} \quad (2.47)$$

where, the coefficient μ_{x_2} is the x -direction friction coefficient for the rear wheel. At $\phi = \phi_M$, the equilibrium equations will reduce to

$$2\mu_{x_2}F_{z_2} - mg \sin \phi_M = 0 \quad (2.48)$$

$$2F_{z_1} + 2F_{z_2} - mg \cos \phi_M = 0 \quad (2.49)$$

$$2F_{z_1}a_1 - 2F_{z_2}a_2 + 2\mu_{x_2}F_{z_2}h = 0. \quad (2.50)$$

These equations provide

$$F_{z_1} = \frac{1}{2}mg\frac{a_2}{l} \cos \phi_M - \frac{1}{2}mg\frac{h}{l} \sin \phi_M \quad (2.51)$$

$$F_{z_2} = \frac{1}{2}mg\frac{a_1}{l} \cos \phi_M + \frac{1}{2}mg\frac{h}{l} \sin \phi_M \quad (2.52)$$

$$\tan \phi_M = \frac{a_1\mu_{x_2}}{l - \mu_{x_2}h} \quad (2.53)$$

showing that there is a relation between the friction coefficient μ_{x_2} , maximum inclination ϕ_M , and the geometrical position of the mass center C . The angle ϕ_M increases by decreasing h .

For a car having the specifications

$$\begin{aligned} \mu_{x_2} &= 1 \\ a_1 &= 110 \text{ cm} \\ l &= 230 \text{ cm} \\ h &= 35 \text{ cm} \end{aligned} \quad (2.54)$$

the tilting angle is

$$\phi_M \approx 0.514 \text{ rad} \approx 29.43^\circ. \quad (2.55)$$

Example 48 Front wheel braking.

When the front wheels are the only braking wheels $F_{x_2} = 0$ and $F_{x_1} \neq 0$. In this case, the equilibrium equations will be

$$2F_{x_1} - mg \sin \phi = 0 \quad (2.56)$$

$$2F_{z_1} + 2F_{z_2} - mg \cos \phi = 0 \quad (2.57)$$

$$-2F_{z_1}a_1 + 2F_{z_2}a_2 - 2F_{x_1}h = 0. \quad (2.58)$$

These equations provide the brake force and reaction forces under the front and rear tires.

$$F_{z_1} = \frac{1}{2}mg\frac{a_2}{l} \cos \phi - \frac{1}{2}mg\frac{h}{l} \sin \phi \quad (2.59)$$

$$F_{z_2} = \frac{1}{2}mg\frac{a_1}{l} \cos \phi + \frac{1}{2}mg\frac{h}{l} \sin \phi \quad (2.60)$$

$$F_{x_1} = \frac{1}{2}mg \sin \phi \quad (2.61)$$

At the ultimate angle $\phi = \phi_M$

$$F_{x_1} = \mu_{x_1} F_{z_1} \quad (2.62)$$

and

$$2\mu_{x_1} F_{z_1} - mg \sin \phi_M = 0 \quad (2.63)$$

$$2F_{z_1} + 2F_{z_2} - mg \cos \phi_M = 0 \quad (2.64)$$

$$2F_{z_1} a_1 - 2F_{z_2} a_2 + 2\mu_{x_1} F_{z_1} h = 0. \quad (2.65)$$

These equations provide

$$F_{z_1} = \frac{1}{2} mg \frac{a_2}{l} \cos \phi_M - \frac{1}{2} mg \frac{h}{l} \sin \phi_M \quad (2.66)$$

$$F_{z_2} = \frac{1}{2} mg \frac{a_1}{l} \cos \phi_M + \frac{1}{2} mg \frac{h}{l} \sin \phi_M \quad (2.67)$$

$$\tan \phi_M = \frac{a_2 \mu_{x_1}}{l - \mu_{x_1} h}. \quad (2.68)$$

Let's name the ultimate angle for the front wheel brake in Equation (2.53) as ϕ_{M_f} , and the ultimate angle for the rear wheel brake in Equation (2.68) as ϕ_{M_r} . Comparing ϕ_{M_f} and ϕ_{M_r} shows that

$$\frac{\phi_{M_f}}{\phi_{M_r}} = \frac{a_1 \mu_{x_2} (l - \mu_{x_1} h)}{a_2 \mu_{x_1} (l - \mu_{x_2} h)}. \quad (2.69)$$

We may assume the front and rear tires are the same and so,

$$\mu_{x_1} = \mu_{x_2} \quad (2.70)$$

therefore,

$$\frac{\phi_{M_f}}{\phi_{M_r}} = \frac{a_1}{a_2}. \quad (2.71)$$

Hence, if $a_1 < a_2$ then $\phi_{M_f} < \phi_{M_r}$ and therefore, a rear brake is more effective than a front brake on uphill parking as long as ϕ_{M_r} is less than the tilting angle, $\phi_{M_r} < \tan^{-1} \frac{a_2}{h}$. At the tilting angle, the weight vector passes through the contact point of the rear wheel with the ground.

Similarly we may conclude that when parked on a downhill road, the front brake is more effective than the rear brake.

Example 49 Four-wheel braking.

Consider a four-wheel brake car, parked uphill as shown in Figure 2.6. In these conditions, there will be two brake forces F_{x_1} on the front wheels and two brake forces F_{x_1} on the rear wheels.

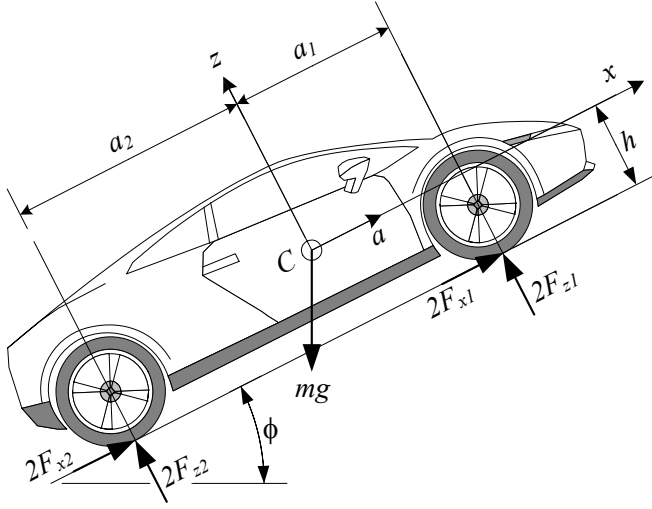


FIGURE 2.6. A four wheel brake car, parked uphill.

The equilibrium equations for this car are

$$2F_{x_1} + 2F_{x_2} - mg \sin \phi = 0 \quad (2.72)$$

$$2F_{z_1} + 2F_{z_2} - mg \cos \phi = 0 \quad (2.73)$$

$$-2F_{z_1}a_1 + 2F_{z_2}a_2 - (2F_{x_1} + 2F_{x_2})h = 0. \quad (2.74)$$

These equations provide the brake force and reaction forces under the front and rear tires.

$$F_{z_1} = \frac{1}{2}mg\frac{a_2}{l}\cos\phi - \frac{1}{2}mg\frac{h}{l}\sin\phi \quad (2.75)$$

$$F_{z_2} = \frac{1}{2}mg\frac{a_1}{l}\cos\phi + \frac{1}{2}mg\frac{h}{l}\sin\phi \quad (2.76)$$

$$F_{x_1} + F_{x_2} = \frac{1}{2}mg\sin\phi \quad (2.77)$$

At the ultimate angle $\phi = \phi_M$, all wheels will begin to slide simultaneously and therefore,

$$F_{x_1} = \mu_{x_1}F_{z_1} \quad (2.78)$$

$$F_{x_2} = \mu_{x_2}F_{z_2}. \quad (2.79)$$

The equilibrium equations show that

$$2\mu_{x_1}F_{z_1} + 2\mu_{x_2}F_{z_2} - mg\sin\phi_M = 0 \quad (2.80)$$

$$2F_{z_1} + 2F_{z_2} - mg\cos\phi_M = 0 \quad (2.81)$$

$$-2F_{z_1}a_1 + 2F_{z_2}a_2 - (2\mu_{x_1}F_{z_1} + 2\mu_{x_2}F_{z_2})h = 0. \quad (2.82)$$

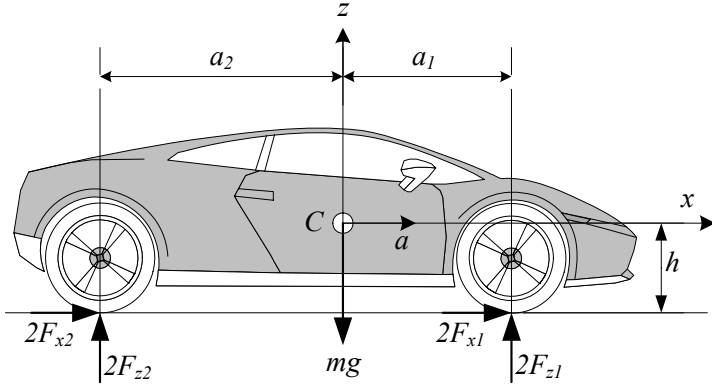


FIGURE 2.7. An accelerating car on a level pavement.

Assuming

$$\mu_{x1} = \mu_{x2} = \mu_x \quad (2.83)$$

will provide

$$F_{z1} = \frac{1}{2}mg \frac{a_2}{l} \cos \phi_M - \frac{1}{2}mg \frac{h}{l} \sin \phi_M \quad (2.84)$$

$$F_{z2} = \frac{1}{2}mg \frac{a_1}{l} \cos \phi_M + \frac{1}{2}mg \frac{h}{l} \sin \phi_M \quad (2.85)$$

$$\tan \phi_M = \mu_x. \quad (2.86)$$

2.3 Accelerating Car on a Level Road

When a car is speeding with acceleration a on a level road as shown in Figure 2.7, the vertical forces under the front and rear wheels are

$$F_{z1} = \frac{1}{2}mg \frac{a_2}{l} - \frac{1}{2}mg \frac{h}{l} \frac{a}{g} \quad (2.87)$$

$$F_{z2} = \frac{1}{2}mg \frac{a_1}{l} + \frac{1}{2}mg \frac{h}{l} \frac{a}{g}. \quad (2.88)$$

The first terms, $\frac{1}{2}mg \frac{a_2}{l}$ and $\frac{1}{2}mg \frac{a_1}{l}$, are called *static parts*, and the second terms $\pm \frac{1}{2}mg \frac{h}{l} \frac{a}{g}$ are called *dynamic parts* of the normal forces.

Proof. The vehicle is considered as a rigid body that moves along a horizontal road. The force at the tireprint of each tire may be decomposed to a normal and a longitudinal force. The equations of motion for the accelerating car come from Newton's equation in x -direction and two static

equilibrium equations.

$$\sum F_x = ma \quad (2.89)$$

$$\sum F_z = 0 \quad (2.90)$$

$$\sum M_y = 0. \quad (2.91)$$

Expanding the equations of motion produces three equations for four unknowns F_{x_1} , F_{x_2} , F_{z_1} , F_{z_2} .

$$2F_{x_1} + 2F_{x_2} = ma \quad (2.92)$$

$$2F_{z_1} + 2F_{z_2} - mg = 0 \quad (2.93)$$

$$-2F_{z_1}a_1 + 2F_{z_2}a_2 - 2(F_{x_1} + F_{x_2})h = 0 \quad (2.94)$$

However, it is possible to eliminate $(F_{x_1} + F_{x_2})$ between the first and third equations, and solve for the normal forces F_{z_1} , F_{z_2} .

$$\begin{aligned} F_{z_1} &= (F_{z_1})_{st} + (F_{z_1})_{dyn} \\ &= \frac{1}{2}mg\frac{a_2}{l} - \frac{1}{2}mg\frac{h}{l}\frac{a}{g} \end{aligned} \quad (2.95)$$

$$\begin{aligned} F_{z_2} &= (F_{z_2})_{st} + (F_{z_2})_{dyn} \\ &= \frac{1}{2}mg\frac{a_1}{l} + \frac{1}{2}mg\frac{h}{l}\frac{a}{g} \end{aligned} \quad (2.96)$$

The static parts

$$(F_{z_1})_{st} = \frac{1}{2}mg\frac{a_2}{l} \quad (2.97)$$

$$(F_{z_2})_{st} = \frac{1}{2}mg\frac{a_1}{l} \quad (2.98)$$

are weight distribution for a stationary car and depend on the horizontal position of the mass center. However, the dynamic parts

$$(F_{z_1})_{dyn} = -\frac{1}{2}mg\frac{h}{l}\frac{a}{g} \quad (2.99)$$

$$(F_{z_2})_{dyn} = \frac{1}{2}mg\frac{h}{l}\frac{a}{g} \quad (2.100)$$

indicate the weight distribution according to horizontal acceleration, and depend on the vertical position of the mass center.

When accelerating $a > 0$, the normal forces under the front tires are less than the static load, and under the rear tires are more than the static load.

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Example 50 *Front-wheel-drive accelerating on a level road.*

When the car is front-wheel-drive, $F_{x_2} = 0$. Equations (2.92) to (2.88) will provide the same vertical tireprint forces as (2.87) and (2.88). However, the required horizontal force to achieve the same acceleration, a , must be provided by solely the front wheels.

Example 51 *Rear-wheel drive accelerating on a level road.*

If a car is rear-wheel drive then, $F_{x_1} = 0$ and the required force to achieve the acceleration, a , must be provided only by the rear wheels. The vertical force under the wheels will still be the same as (2.87) and (2.88).

Example 52 *Maximum acceleration on a level road.*

The maximum acceleration of a car is proportional to the friction under its tires. We assume the friction coefficients at the front and rear tires are equal and all tires reach their maximum tractions at the same time.

$$F_{x_1} = \pm \mu_x F_{z_1} \quad (2.101)$$

$$F_{x_2} = \pm \mu_x F_{z_2} \quad (2.102)$$

Newton's equation (2.92) can now be written as

$$ma = \pm 2\mu_x (F_{z_1} + F_{z_2}). \quad (2.103)$$

Substituting F_{z_1} and F_{z_2} from (2.93) and (2.94) results in

$$a = \pm \mu_x g. \quad (2.104)$$

Therefore, the maximum acceleration and deceleration depend directly on the friction coefficient.

Example 53 *Maximum acceleration for a single-axle drive car.*

The maximum acceleration a_{rwd} for a rear-wheel-drive car is achieved when we substitute $F_{x_1} = 0$, $F_{x_2} = \mu_x F_{z_2}$ in Equation (2.92) and use Equation (2.88)

$$\mu_x mg \left(\frac{a_1}{l} + \frac{h}{l} \frac{a_{rwd}}{g} \right) = ma_{rwd} \quad (2.105)$$

and therefore,

$$\begin{aligned} \frac{a_{rwd}}{g} &= \frac{a_1 \mu_x}{l - h \mu_x} \\ &= \frac{\mu_x}{1 - \mu_x \frac{h}{l}} \frac{a_1}{l}. \end{aligned} \quad (2.106)$$

The front wheels can leave the ground when $F_{z_1} = 0$. Substituting $F_{z_1} = 0$ in Equation (2.88) provides the maximum acceleration at which the front wheels are still on the road.

$$\frac{a_{rwd}}{g} \leq \frac{a_2}{h} \quad (2.107)$$

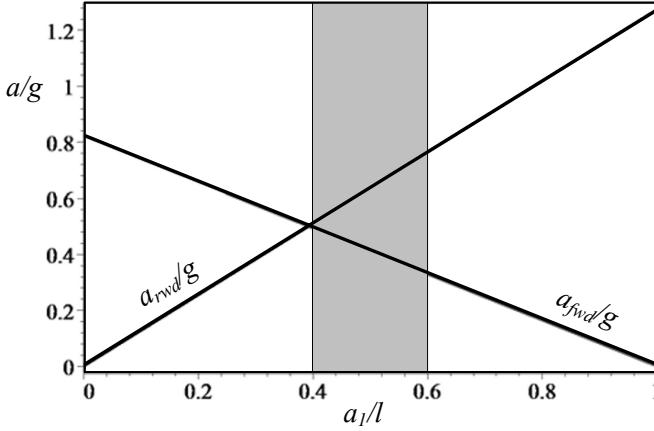


FIGURE 2.8. Effect of mass center position on the maximum achievable acceleration of a front- and a rear-wheel drive car.

Therefore, the maximum attainable acceleration would be the less value of Equation (2.106) or (2.107).

Similarly, the maximum acceleration a_{fwd} for a front-wheel drive car is achieved when we substitute $F_{x_2} = 0$, $F_{x_1} = \mu_x F_{z_1}$ in Equation (2.92) and use Equation (2.87).

$$\begin{aligned} \frac{a_{fwd}}{g} &= \frac{a_2 \mu_x}{l + h \mu_x} \\ &= \frac{\mu_x}{1 + \mu_x \frac{h}{l}} \left(1 - \frac{a_1}{l} \right) \end{aligned} \quad (2.108)$$

To see the effect of changing the position of mass center on the maximum achievable acceleration, we plot Figure 2.8 for a sample car with

$$\begin{aligned} \mu_x &= 1 \\ h &= 0.56 \text{ m} \\ l &= 2.6 \text{ m}. \end{aligned} \quad (2.109)$$

Passenger cars are usually in the range $0.4 < (a_1/g) < 0.6$, with $(a_1/g) \rightarrow 0.4$ for front-wheel-drive cars, and $(a_1/g) \rightarrow 0.6$ for rear-wheel-drive cars. In this range, $(a_{rwd}/g) > (a_{fwd}/g)$ and therefore rear-wheel-drive cars can reach higher forward acceleration than front-wheel-drive cars. It is an important applied fact, especially for race cars.

The maximum acceleration may also be limited by the tilting condition

$$\frac{a_M}{g} \leq \frac{a_2}{h}. \quad (2.110)$$

Example 54 *Minimum time for 0 – 100 km/h on a level road.
Consider a car with the following characteristics:*

$$\begin{aligned}
 \text{length} &= 4245 \text{ mm} \\
 \text{width} &= 1795 \text{ mm} \\
 \text{height} &= 1285 \text{ mm} \\
 \text{wheel base} &= 2272 \text{ mm} \\
 \text{front track} &= 1411 \text{ mm} \\
 \text{rear track} &= 1504 \text{ mm} \\
 \text{net weight} &= 1500 \text{ kg} \\
 h &= 220 \text{ mm} \\
 \mu_x &= 1 \\
 a_1 &= a_2
 \end{aligned} \tag{2.111}$$

Assume the car is rear-wheel-drive and its engine can provide the maximum traction supported by friction. Equation (2.88) determines the load on the rear wheels and therefore, the forward equation of motion is

$$\begin{aligned}
 2F_{x_2} &= 2\mu_x F_{z_2} \\
 &= \mu_x mg \frac{a_1}{l} + \mu_x mg \frac{h}{l} \frac{1}{g} a \\
 &= ma.
 \end{aligned} \tag{2.112}$$

Rearrangement provides the following differential equation to calculate velocity and displacement:

$$\begin{aligned}
 a &= \ddot{x} = \frac{\mu_x g \frac{a_1}{l}}{1 - \mu_x g \frac{h}{l} \frac{1}{g}} \\
 &= g\mu_x \frac{a_1}{l - h\mu_x}
 \end{aligned} \tag{2.113}$$

Taking an integral

$$\int_0^{27.78} dv = \int_0^t a dt \tag{2.114}$$

between $v = 0$ and $v = 100 \text{ km/h} \approx 27.78 \text{ m/s}$ shows that the minimum time for 0 – 100 km/h on a level road is

$$t = \frac{27.78}{g\mu_x \frac{a_1}{l - h\mu_x}} \approx 5.11 \text{ s} \tag{2.115}$$

If the same car was front-wheel-drive, then the traction force would be

$$\begin{aligned}
 2F_{x_1} &= 2\mu_x F_{z_1} \\
 &= \mu_x mg \frac{a_2}{l} - \mu_x mg \frac{h}{l} \frac{1}{g} a \\
 &= m a.
 \end{aligned} \tag{2.116}$$

and the equation of motion would reduce to

$$\begin{aligned}
 a &= \ddot{x} = \frac{\mu_x g \frac{a_2}{l}}{1 + \mu_x g \frac{h}{l} \frac{1}{g}} \\
 &= g \mu_x \frac{a_2}{l + h \mu_x}.
 \end{aligned} \tag{2.117}$$

The minimum time for 0 – 100 km/h on a level road for this front-wheel-drive car is

$$t = \frac{27.78}{g \mu_x \frac{a_2}{l + h \mu_x}} \approx 6.21 \text{ s.} \tag{2.118}$$

Now consider the same car to be four-wheel-drive. Then, the traction force is

$$\begin{aligned}
 2F_{x_1} + 2F_{x_2} &= 2\mu_x (F_{z_1} + F_{z_2}) \\
 &= \frac{g}{l} m (a_1 + a_2) \\
 &= m a.
 \end{aligned} \tag{2.119}$$

and the minimum time for 0 – 100 km/h on a level road for this four-wheel-drive car can theoretically be reduced to

$$t = \frac{27.78}{g} \approx 2.83 \text{ s.} \tag{2.120}$$

2.4 Accelerating Car on an Inclined Road

When a car is accelerating on an inclined pavement with angle ϕ as shown in Figure 2.9, the normal force under each of the front and rear wheels, F_{z_1} , F_{z_2} , would be:

$$F_{z_1} = \frac{1}{2} mg \left(\frac{a_2}{l} \cos \phi - \frac{h}{l} \sin \phi \right) - \frac{1}{2} m a \frac{h}{l} \tag{2.121}$$

$$F_{z_2} = \frac{1}{2} mg \left(\frac{a_1}{l} \cos \phi + \frac{h}{l} \sin \phi \right) + \frac{1}{2} m a \frac{h}{l} \tag{2.122}$$

$$l = a_1 + a_2$$

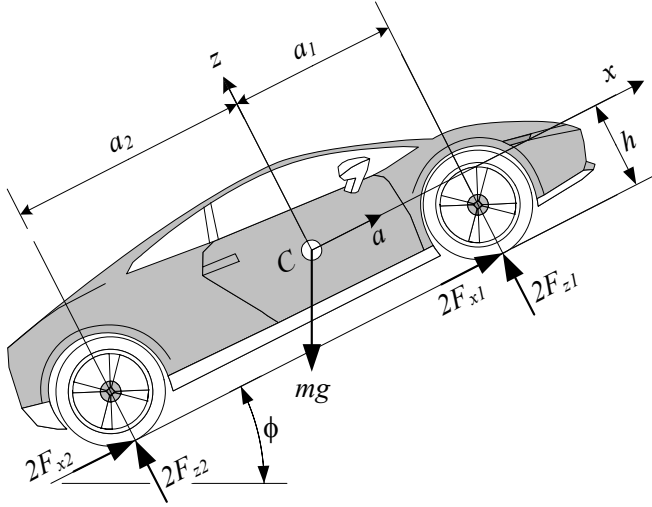


FIGURE 2.9. An accelerating car on inclined pavement.

The dynamic parts, $\pm \frac{1}{2}mg \frac{h}{l} \frac{a}{g}$, depend on acceleration a and height h of mass center C and remain unchanged, while the static parts are influenced by the slope angle ϕ and height h of mass center.

Proof. The Newton's equation in x -direction and two static equilibrium equations, must be examined to find the equation of motion and ground reaction forces.

$$\sum F_x = ma \quad (2.123)$$

$$\sum F_z = 0 \quad (2.124)$$

$$\sum M_y = 0. \quad (2.125)$$

Expanding these equations produces three equations for four unknowns F_{x1} , F_{x2} , F_{z1} , F_{z2} .

$$2F_{x1} + 2F_{x2} - mg \sin \phi = ma \quad (2.126)$$

$$2F_{z1} + 2F_{z2} - mg \cos \phi = 0 \quad (2.127)$$

$$2F_{z1}a_1 - 2F_{z2}a_2 + 2(F_{x1} + F_{x2})h = 0 \quad (2.128)$$

It is possible to eliminate $(F_{x1} + F_{x2})$ between the first and third equations, and solve for the normal forces F_{z1} , F_{z2} .

$$\begin{aligned} F_{z1} &= (F_{z1})_{st} + (F_{z1})_{dyn} \\ &= \frac{1}{2}mg \left(\frac{a_2}{l} \cos \phi - \frac{h}{l} \sin \phi \right) - \frac{1}{2}ma \frac{h}{l} \end{aligned} \quad (2.129)$$

$$\begin{aligned}
F_{z_2} &= (F_{z_2})_{st} + (F_{z_2})_{dyn} \\
&= \frac{1}{2}mg \left(\frac{a_1}{l} \cos \phi + \frac{h}{l} \sin \phi \right) + \frac{1}{2}ma \frac{h}{l}
\end{aligned} \tag{2.130}$$

■

Example 55 *Front-wheel-drive car, accelerating on inclined road.*

For a front-wheel-drive car, we may substitute $F_{x_1} = 0$ in Equations (2.126) and (2.128) to have the governing equations. However, it does not affect the ground reaction forces under the tires (2.129 and 2.130) as long as the car is driven under its limit conditions.

Example 56 *Rear-wheel-drive car, accelerating on inclined road.*

Substituting $F_{x_2} = 0$ in Equations (2.126) and (2.128) and solving for the normal reaction forces under each tire provides the same results as (2.129) and (2.130). Hence, the normal forces applied on the tires do not sense if the car is front-, rear-, or all-wheel drive. As long as we drive in a straight path at low acceleration, the drive wheels can be the front or the rear ones. However, the advantages and disadvantages of front-, rear-, or all-wheel drive cars appear in maneuvering, slippery roads, or when the maximum acceleration is required.

Example 57 *Maximum acceleration on an inclined road.*

The maximum acceleration depends on the friction under the tires. Let's assume the friction coefficients at the front and rear tires are equal. Then, the front and rear traction forces are

$$F_{x_1} \leq \mu_x F_{z_1} \tag{2.131}$$

$$F_{x_2} \leq \mu_x F_{z_2}. \tag{2.132}$$

If we assume the front and rear wheels reach their traction limits at the same time, then

$$F_{x_1} = \pm \mu_x F_{z_1} \tag{2.133}$$

$$F_{x_2} = \pm \mu_x F_{z_2} \tag{2.134}$$

and we may rewrite Newton's equation (2.123) as

$$ma_M = \pm 2\mu_x (F_{z_1} + F_{z_2}) - mg \sin \phi \tag{2.135}$$

where, a_M is the maximum achievable acceleration.

Now substituting F_{z_1} and F_{z_2} from (2.129) and (2.130) results in

$$\frac{a_M}{g} = \pm \mu_x \cos \phi - \sin \phi. \tag{2.136}$$

Accelerating on an uphill road ($a > 0, \phi > 0$) and braking on a downhill road ($a < 0, \phi < 0$) are the extreme cases in which the car can stall. In these cases, the car can move as long as

$$\mu_x \geq |\tan \phi|. \tag{2.137}$$

Example 58 *Limits of acceleration and inclination angle.*

Assuming $F_{z_1} > 0$ and $F_{z_2} > 0$, we can write Equations (2.121) and (2.122) as

$$\frac{a}{g} \leq \frac{a_2}{h} \cos \phi - \sin \phi \quad (2.138)$$

$$\frac{a}{g} \geq -\frac{a_1}{h} \cos \phi - \sin \phi. \quad (2.139)$$

Hence, the maximum achievable acceleration ($a > 0$) is limited by a_2 , h , ϕ ; while the maximum deceleration ($a < 0$) is limited by a_1 , h , ϕ . These two equations can be combined to result in

$$-\frac{a_1}{h} \cos \phi \leq \frac{a}{g} + \sin \phi \leq \frac{a_2}{h} \cos \phi. \quad (2.140)$$

If $a \rightarrow 0$, then the limits of the inclination angle would be

$$-\frac{a_1}{h} \leq \tan \phi \leq \frac{a_2}{h}. \quad (2.141)$$

This is the maximum and minimum road inclination angles that the car can stay on without tilting and falling.

Example 59 *Maximum deceleration for a single-axle-brake car.*

We can find the maximum braking deceleration a_{fwb} of a front-wheel-brake car on a horizontal road by substituting $\phi = 0$, $F_{x_2} = 0$, $F_{x_1} = -\mu_x F_{z_1}$ in Equation (2.126) and using Equation (2.121)

$$-\mu_x mg \left(\frac{a_2}{l} - \frac{h}{l} \frac{a_{fwb}}{g} \right) = ma_{fwb} \quad (2.142)$$

therefore,

$$\frac{a_{fwb}}{g} = -\frac{\mu_x}{1 - \mu_x \frac{h}{l}} \left(1 - \frac{a_1}{l} \right). \quad (2.143)$$

Similarly, the maximum braking deceleration a_{rwb} of a front-wheel-brake car can be achieved when we substitute $F_{x_2} = 0$, $F_{x_1} = \mu_x F_{z_1}$.

$$\frac{a_{rwb}}{g} = -\frac{\mu_x}{1 + \mu_x \frac{h}{l}} \frac{a_1}{l} \quad (2.144)$$

The effect of changing the position of the mass center on the maximum achievable braking deceleration is shown in Figure 2.10 for a sample car with

$$\begin{aligned} \mu_x &= 1 \\ h &= 0.56 \text{ m} \\ l &= 2.6 \text{ m.} \end{aligned} \quad (2.145)$$

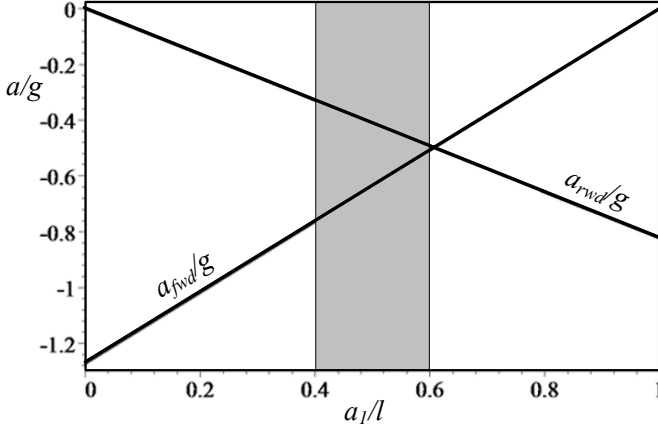


FIGURE 2.10. Effect of mass center position on the maximum achievable deceleration of a front-wheel and a rear-wheel-drive car.

Passenger cars are usually in the range $0.4 < (a_1/l) < 0.6$. In this range, $(a_{fwb}/g) < (a_{rwb}/g)$ and therefore, front-wheel-brake cars can reach better forward deceleration than rear-wheel-brake cars. Hence, front brakes are much more important than the rear brakes.

Example 60 ★ A car with a trailer.

Figure 2.11 depicts a car moving on an inclined road and pulling a trailer. To analyze the car-trailer motion, we need to separate the car and trailer to see the forces at the hinge, as shown in Figure 2.12. We assume the mass center of the trailer C_t is at distance b_3 in front of the only axle of the trailer. If C_t is behind the trailer axle, then b_3 should be negative in the following equations.

For an ideal hinge between a car and a trailer moving in a straight path, there must be a horizontal force F_{x_t} and a vertical force F_{z_t} .

Writing the Newton's equation in x -direction and two static equilibrium equations for both the trailer and the vehicle

$$\sum F_x = m_t a \quad (2.146)$$

$$\sum F_z = 0 \quad (2.147)$$

$$\sum M_y = 0 \quad (2.148)$$

we find the following set of equations:

$$F_{x_t} - m_t g \sin \phi = m_t a \quad (2.149)$$

$$2F_{z_3} - F_{z_t} - m_t g \cos \phi = 0 \quad (2.150)$$

$$2F_{z_3} b_3 - F_{z_t} b_2 - F_{x_t} (h_2 - h_1) = 0 \quad (2.151)$$

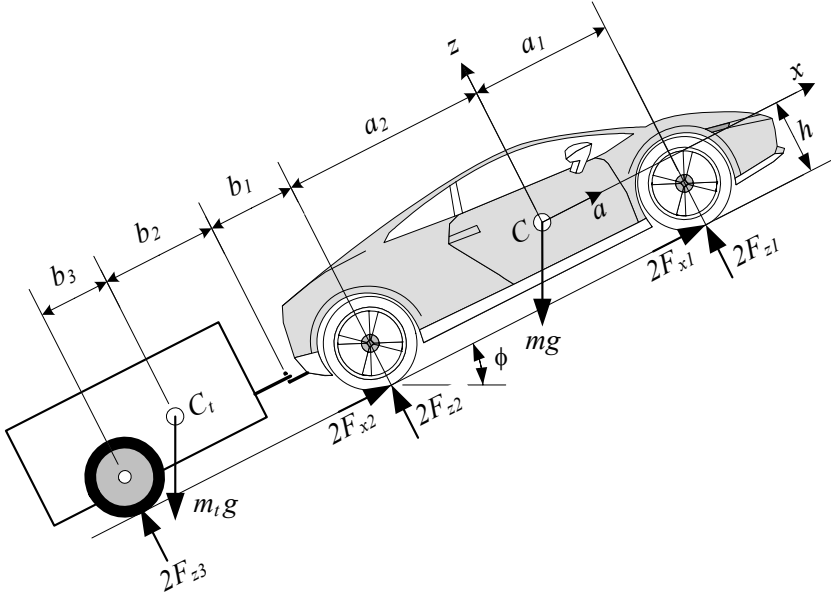


FIGURE 2.11. A car moving on an inclined road and pulling a trailer.

$$2F_{x_1} + 2F_{x_2} - F_{x_t} - mg \sin \phi = ma \quad (2.152)$$

$$2F_{z_1} + 2F_{z_2} - F_{z_t} - mg \cos \phi = 0 \quad (2.153)$$

$$\begin{aligned} 2F_{z_1} a_1 - 2F_{z_2} a_2 + 2(F_{x_1} + F_{x_2}) h \\ - F_{x_t} (h - h_1) + F_{z_t} (b_1 + a_2) = 0 \end{aligned} \quad (2.154)$$

If the value of traction forces F_{x_1} and F_{x_2} are given, then these are six equations for six unknowns: a , F_{x_t} , F_{z_t} , F_{z_1} , F_{z_2} , F_{z_3} . Solving these equations provide the following solutions:

$$a = \frac{2}{m + m_t} (F_{x_1} + F_{x_2}) - g \sin \phi \quad (2.155)$$

$$F_{x_t} = \frac{2m_t}{m + m_t} (F_{x_1} + F_{x_2}) \quad (2.156)$$

$$F_{z_t} = \frac{h_1 - h_2}{b_2 - b_3} \frac{2m_t}{m + m_t} (F_{x_1} + F_{x_2}) + \frac{b_3}{b_2 - b_3} m_t g \cos \phi \quad (2.157)$$

$$\begin{aligned} F_{z_1} = & \frac{b_3}{2l} \left(\frac{2a_2 - b_1}{b_2 - b_3} m_t + \frac{a_2}{b_3} m \right) g \cos \phi \\ & + \left[\frac{2a_2 - b_1}{b_2 - b_3} (h_1 - h_2) m_t - h_1 m_t - h m \right] \frac{F_{x_1} + F_{x_2}}{l(m + m_t)} \end{aligned} \quad (2.158)$$

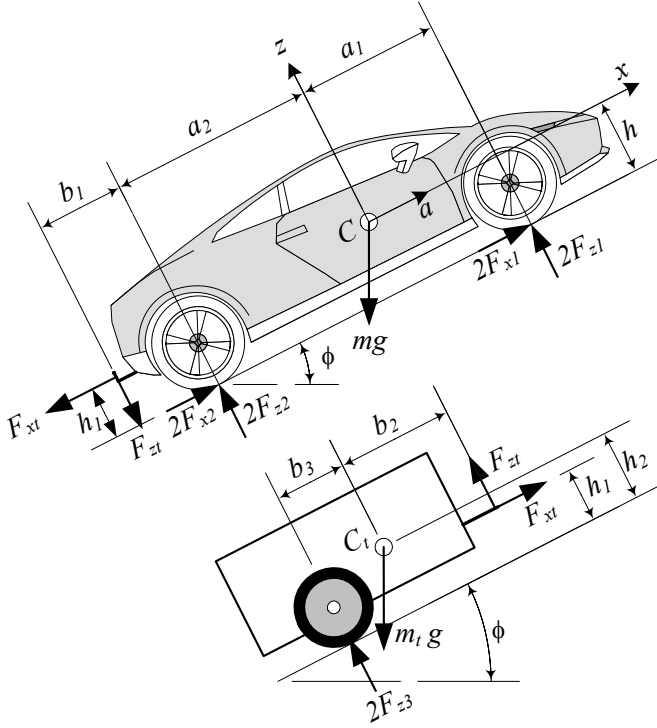


FIGURE 2.12. Free-body-diagram of a car and the trailer when moving on an uphill road.

$$F_{z_2} = \frac{b_3}{2l} \left(\frac{a_1 - a_2 + b_1}{b_2 - b_3} m_t + \frac{a_1}{b_3} m \right) g \cos \phi + \left[\frac{a_1 - a_2 + b_1}{b_2 - b_3} (h_1 - h_2) m_t + h_1 m_t + h m \right] \frac{F_{x_1} + F_{x_2}}{l(m + m_t)} \quad (2.159)$$

$$F_{z_3} = \frac{1}{2} \frac{b_2}{b_2 - b_3} m_t g \cos \phi + \frac{h_1 - h_2}{b_2 - b_3} \frac{m_t}{m + m_t} (F_{x_1} + F_{x_2}) \quad (2.160)$$

$$l = a_1 + a_2. \quad (2.161)$$

However, if the value of acceleration a is known, then unknowns are: $F_{x_1} + F_{x_2}$, F_{x_t} , F_{z_t} , F_{z_1} , F_{z_2} , F_{z_3} .

$$F_{x_1} + F_{x_2} = \frac{1}{2} (m + m_t) (a + g \sin \phi) \quad (2.162)$$

$$F_{x_t} = m_t (a + g \sin \phi) \quad (2.163)$$

$$F_{z_t} = \frac{h_1 - h_2}{b_2 - b_3} m_t (a + g \sin \phi) + \frac{b_3}{b_2 - b_3} m_t g \cos \phi \quad (2.164)$$

$$F_{z_1} = \frac{b_3}{2l} \left(\frac{2a_2 - b_1}{b_2 - b_3} m_t + \frac{a_2}{b_3} m \right) g \cos \phi \quad (2.165)$$

$$+ \frac{1}{2l} \left[\frac{2a_2 - b_1}{b_2 - b_3} (h_1 - h_2) m_t - h_1 m_t - h m \right] (a + g \sin \phi)$$

$$F_{z_2} = \frac{b_3}{2l} \left(\frac{a_1 - a_2 + b_1}{b_2 - b_3} m_t + \frac{a_1}{b_3} m \right) g \cos \phi \quad (2.166)$$

$$+ \frac{1}{2l} \left[\frac{a_1 - a_2 + b_1}{b_2 - b_3} (h_1 - h_2) m_t + h_1 m_t + h m \right] (a + g \sin \phi)$$

$$F_{z_3} = \frac{1}{2} \frac{m_t}{b_2 - b_3} (b_2 g \cos \phi + (h_1 - h_2) (a + g \sin \phi)) \quad (2.167)$$

$$l = a_1 + a_2.$$

Example 61 ★ *Maximum inclination angle for a car with a trailer.*

For a car and trailer as shown in Figure 2.11, the maximum inclination angle ϕ_M is the angle at which the car cannot accelerate the vehicle. Substituting $a = 0$ and $\phi = \phi_M$ in Equation (2.155) shows that

$$\sin \phi_M = \frac{2}{(m + m_t)g} (F_{x_1} + F_{x_2}). \quad (2.168)$$

The value of maximum inclination angle ϕ_M increases by decreasing the total weight of the vehicle and trailer $(m + m_t)g$ or increasing the traction force $F_{x_1} + F_{x_2}$.

The traction force is limited by the maximum torque on the drive wheel and the friction under the drive tire. Let's assume the vehicle is four-wheel-drive and friction coefficients at the front and rear tires are equal. Then, the front and rear traction forces are

$$F_{x_1} \leq \mu_x F_{z_1} \quad (2.169)$$

$$F_{x_2} \leq \mu_x F_{z_2}. \quad (2.170)$$

If we assume the front and rear wheels reach their traction limits at the same time, then

$$F_{x_1} = \mu_x F_{z_1} \quad (2.171)$$

$$F_{x_2} = \mu_x F_{z_2} \quad (2.172)$$

and we may rewrite the Equation (2.168) as

$$\sin \phi_M = \frac{2\mu_x}{(m + m_t)g} (F_{z_1} + F_{z_2}). \quad (2.173)$$

Now substituting F_{z_1} and F_{z_2} from (2.158) and (2.159) results in

$$\begin{aligned} & (mb_3 - mb_2 - m_t b_3) \mu_x \cos \phi_M + (b_2 - b_3) (m + m_t) \sin \phi_M \\ &= 2\mu_x \frac{m_t (h_1 - h_2)}{m + m_t} (F_{x_1} + F_{x_2}). \end{aligned} \quad (2.174)$$

If we arrange Equation (2.174) as

$$A \cos \phi_M + B \sin \phi_M = C \quad (2.175)$$

then

$$\phi_M = \text{atan2}\left(\frac{C}{\sqrt{A^2 + B^2}}, \pm \sqrt{1 - \frac{C^2}{A^2 + B^2}}\right) - \text{atan2}(A, B) \quad (2.176)$$

and

$$\phi_M = \text{atan2}\left(\frac{C}{\sqrt{A^2 + B^2}}, \pm \sqrt{A^2 + B^2 - C^2}\right) - \text{atan2}(A, B) \quad (2.177)$$

where

$$A = (mb_3 - mb_2 - m_t b_3) \mu_x \quad (2.178)$$

$$B = (b_2 - b_3)(m + m_t) \quad (2.179)$$

$$C = 2\mu_x \frac{m_t (h_1 - h_2)}{m + m_t} (F_{x_1} + F_{x_2}). \quad (2.180)$$

For a rear-wheel-drive car pulling a trailer with the following characteristics:

$$\begin{aligned} l &= 2272 \text{ mm} \\ w &= 1457 \text{ mm} \\ h &= 230 \text{ mm} \\ a_1 &= a_2 \\ h_1 &= 310 \text{ mm} \\ b_1 &= 680 \text{ mm} \\ b_2 &= 610 \text{ mm} \\ b_3 &= 120 \text{ mm} \\ h_2 &= 560 \text{ mm} \\ m &= 1500 \text{ kg} \\ m_t &= 150 \text{ kg} \\ \mu_x &= 1 \\ \phi &= 10 \text{ deg} \\ a &= 1 \text{ m/s}^2 \end{aligned} \quad (2.181)$$

we find

$$\begin{aligned} F_{z_1} &= 3441.78 \text{ N} \\ F_{z_2} &= 3877.93 \text{ N} \\ F_{z_3} &= 798.57 \text{ N} \\ F_{z_t} &= 147.99 \text{ N} \\ F_{x_t} &= 405.52 \text{ N} \\ F_{x_2} &= 2230.37 \text{ N.} \end{aligned} \quad (2.182)$$

To check if the required traction force F_{x_2} is applicable, we should compare it to the maximum available friction force μF_{z_2} and it must be

$$F_{x_2} \leq \mu F_{z_2}. \quad (2.183)$$

Example 62 ★ *Solution of equation $a \cos \theta + b \sin \theta = c$.*

The first type of trigonometric equation is

$$a \cos \theta + b \sin \theta = c. \quad (2.184)$$

It can be solved by introducing two new variables r and η such that

$$a = r \sin \eta \quad (2.185)$$

$$b = r \cos \eta \quad (2.186)$$

and therefore,

$$r = \sqrt{a^2 + b^2} \quad (2.187)$$

$$\eta = \text{atan2}(a, b). \quad (2.188)$$

Substituting the new variables show that

$$\sin(\eta + \theta) = \frac{c}{r} \quad (2.189)$$

$$\cos(\eta + \theta) = \pm \sqrt{1 - \frac{c^2}{r^2}}. \quad (2.190)$$

Hence, the solutions of the problem are

$$\theta = \text{atan2}\left(\frac{c}{r}, \pm \sqrt{1 - \frac{c^2}{r^2}}\right) - \text{atan2}(a, b) \quad (2.191)$$

and

$$\theta = \text{atan2}\left(\frac{c}{r}, \pm \sqrt{r^2 - c^2}\right) - \text{atan2}(a, b). \quad (2.192)$$

Therefore, the equation $a \cos \theta + b \sin \theta = c$ has two solutions if $r^2 = a^2 + b^2 > c^2$, one solution if $r^2 = c^2$, and no solution if $r^2 < c^2$.

Example 63 ★ *The function $\tan_2^{-1} \frac{y}{x} = \text{atan2}(y, x)$.*

There are many situations in kinematics calculation in which we need to find an angle based on the sin and cos functions of an angle. However, \tan^{-1} cannot show the effect of the individual sign for the numerator and denominator. It always represents an angle in the first or fourth quadrant. To overcome this problem and determine the angle in the correct quadrant, the atan2 function is introduced as below.

$$\text{atan2}(y, x) = \begin{cases} \tan^{-1} \frac{y}{x} & \text{if } y > 0 \\ \tan^{-1} \frac{y}{x} + \pi \text{ sign } y & \text{if } y < 0 \\ \frac{\pi}{2} \text{ sign } x & \text{if } y = 0 \end{cases} \quad (2.193)$$

In this text, whether it has been mentioned or not, wherever $\tan^{-1} \frac{y}{x}$ is used, it must be calculated based on $\text{atan2}(y, x)$.

Example 64 Zero vertical force at the hinge.

We can make the vertical force at the hinge equal to zero by examining Equation (2.157) for the hinge vertical force F_{z_t} .

$$F_{z_t} = \frac{h_1 - h_2}{b_2 - b_3} \frac{2m_t}{m + m_t} (F_{x_1} + F_{x_2}) + \frac{b_3}{b_2 - b_3} m_t g \cos \phi \quad (2.194)$$

To make $F_{z_t} = 0$, it is enough to adjust the position of trailer mass center C_t exactly on top of the trailer axle and at the same height as the hinge. In these conditions we have

$$h_1 = h_2 \quad (2.195)$$

$$b_3 = 0 \quad (2.196)$$

that makes

$$F_{z_t} = 0. \quad (2.197)$$

However, to increase safety, the load should be distributed evenly throughout the trailer. Heavy items should be loaded as low as possible, mainly over the axle. Bulkier and lighter items should be distributed to give a little positive b_3 . Such a trailer is called nose weight at the towing coupling.

2.5 Parked Car on a Banked Road

Figure 2.13 depicts the effect of a bank angle ϕ on the load distribution of a vehicle. A bank causes the load on the lower tires to increase, and the load on the upper tires to decrease. The tire reaction forces are:

$$F_{z_1} = \frac{1}{2} \frac{mg}{w} (b_2 \cos \phi - h \sin \phi) \quad (2.198)$$

$$F_{z_2} = \frac{1}{2} \frac{mg}{w} (b_1 \cos \phi + h \sin \phi) \quad (2.199)$$

$$w = b_1 + b_2 \quad (2.200)$$

Proof. Starting with equilibrium equations

$$\sum F_y = 0 \quad (2.201)$$

$$\sum F_z = 0 \quad (2.202)$$

$$\sum M_x = 0. \quad (2.203)$$

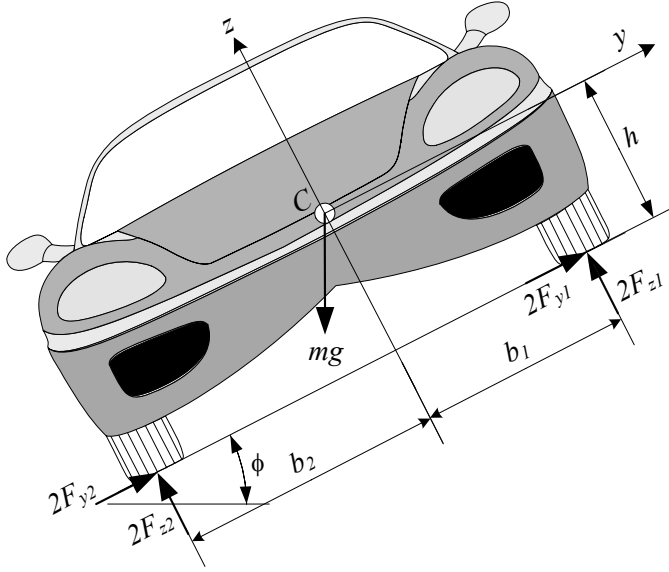


FIGURE 2.13. Normal force under the uphill and downhill tires of a vehicle, parked on banked road.

we can write

$$2F_{y1} + 2F_{y2} - mg \sin \phi = 0 \quad (2.204)$$

$$2F_{z1} + 2F_{z2} - mg \cos \phi = 0 \quad (2.205)$$

$$2F_{z1} b_1 - 2F_{z2} b_2 + 2(F_{y1} + F_{y2}) h = 0. \quad (2.206)$$

We assumed the force under the lower tires, front and rear, are equal, and also the forces under the upper tires, front and rear are equal. To calculate the reaction forces under each tire, we may assume the overall lateral force $F_{y1} + F_{y2}$ as an unknown. The solution of these equations provide the lateral and reaction forces under the upper and lower tires.

$$F_{z1} = \frac{1}{2} mg \frac{b_2}{w} \cos \phi - \frac{1}{2} mg \frac{h}{w} \sin \phi \quad (2.207)$$

$$F_{z2} = \frac{1}{2} mg \frac{b_1}{w} \cos \phi + \frac{1}{2} mg \frac{h}{w} \sin \phi \quad (2.208)$$

$$F_{y1} + F_{y2} = \frac{1}{2} mg \sin \phi \quad (2.209)$$

At the ultimate angle $\phi = \phi_M$, all wheels will begin to slide simultaneously and therefore,

$$F_{y1} = \mu_{y1} F_{z1} \quad (2.210)$$

$$F_{y2} = \mu_{y2} F_{z2}. \quad (2.211)$$

The equilibrium equations show that

$$2\mu_{y_1}F_{z_1} + 2\mu_{y_2}F_{z_2} - mg \sin \phi = 0 \quad (2.212)$$

$$2F_{z_1} + 2F_{z_2} - mg \cos \phi = 0 \quad (2.213)$$

$$2F_{z_1}b_1 - 2F_{z_2}b_2 + 2(\mu_{y_1}F_{z_1} + \mu_{y_2}F_{z_2})h = 0. \quad (2.214)$$

Assuming

$$\mu_{y_1} = \mu_{y_2} = \mu_y \quad (2.215)$$

will provide

$$F_{z_1} = \frac{1}{2}mg \frac{b_2}{w} \cos \phi_M - \frac{1}{2}mg \frac{h}{w} \sin \phi_M \quad (2.216)$$

$$F_{z_2} = \frac{1}{2}mg \frac{b_1}{w} \cos \phi_M + \frac{1}{2}mg \frac{h}{w} \sin \phi_M \quad (2.217)$$

$$\tan \phi_M = \mu_y. \quad (2.218)$$

These calculations are correct as long as

$$\tan \phi_M \leq \frac{b_2}{h} \quad (2.219)$$

$$\mu_y \leq \frac{b_2}{h}. \quad (2.220)$$

If the lateral friction μ_y is higher than b_2/h then the car will roll downhill. To increase the capability of a car moving on a banked road, the car should be as wide as possible with a mass center as low as possible. ■

Example 65 *Tire forces of a parked car in a banked road.*

A car having

$$\begin{aligned} m &= 980 \text{ kg} \\ h &= 0.6 \text{ m} \\ w &= 1.52 \text{ m} \\ b_1 &= b_2 \end{aligned} \quad (2.221)$$

is parked on a banked road with $\phi = 4^\circ$. The forces under the lower and upper tires of the car are:

$$\begin{aligned} F_{z_1} &= 2265.2 \text{ N} \\ F_{z_2} &= 2529.9 \text{ N} \\ F_{y_1} + F_{y_2} &= 335.3 \text{ N} \end{aligned} \quad (2.222)$$

The ratio of the uphill force F_{z_1} to downhill force F_{z_2} depends on only the mass center position.

$$\frac{F_{z_1}}{F_{z_2}} = \frac{b_2 \cos \phi - h \sin \phi}{b_1 \cos \phi + h \sin \phi} \quad (2.223)$$

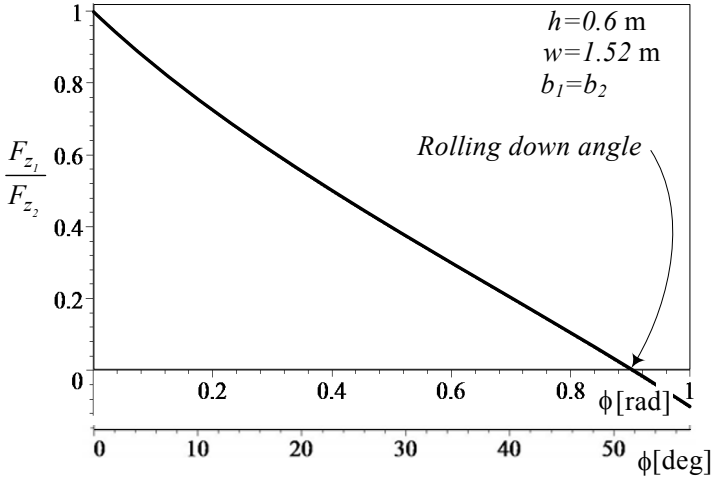


FIGURE 2.14. Illustration of the force ratio F_{z1}/F_{z2} as a function of road bank angle ϕ .

Assuming a symmetric car with $b_1 = b_2 = w/2$ simplifies the equation to

$$\frac{F_{z1}}{F_{z2}} = \frac{w \cos \phi - 2h \sin \phi}{w \cos \phi + 2h \sin \phi}. \quad (2.224)$$

Figure 2.14 illustrates the behavior of force ratio F_{z1}/F_{z2} as a function of ϕ for $h = 0.6 \text{ m}$ and $w = 1.52 \text{ m}$. The rolling down angle $\phi_M = \tan^{-1}(b_2/h) = 51.71 \text{ deg}$ indicates the bank angle at which the force under the uphill wheels become zero and the car rolls down. The negative part of the curve indicates the required force to keep the car on the road, which is not applicable in real situations.

2.6 ★ Optimal Drive and Brake Force Distribution

A certain acceleration a can be achieved by adjusting and controlling the longitudinal forces F_{x1} and F_{x2} . The optimal longitudinal forces under the front and rear tires to achieve the maximum acceleration are

$$\begin{aligned} \frac{F_{x1}}{mg} &= -\frac{1}{2} \frac{h}{l} \left(\frac{a}{g} \right)^2 + \frac{1}{2} \frac{a_2}{l} \frac{a}{g} \\ &= -\frac{1}{2} \mu_x^2 \frac{h}{l} + \frac{1}{2} \mu_x \frac{a_2}{l} \end{aligned} \quad (2.225)$$

$$\begin{aligned}
\frac{F_{x_2}}{mg} &= \frac{1}{2} \frac{h}{l} \left(\frac{a}{g} \right)^2 + \frac{1}{2} \frac{a_1}{l} \frac{a}{g} \\
&= \frac{1}{2} \mu_x^2 \frac{h}{l} + \frac{1}{2} \mu_x \frac{a_1}{l}.
\end{aligned} \tag{2.226}$$

Proof. The longitudinal equation of motion for a car on a horizontal road is

$$2F_{x_1} + 2F_{x_2} = ma \tag{2.227}$$

and the maximum traction forces under each tire is a function of normal force and the friction coefficient.

$$F_{x_1} \leq \pm \mu_x F_{z_1} \tag{2.228}$$

$$F_{x_2} \leq \pm \mu_x F_{z_2} \tag{2.229}$$

However, the normal forces are a function of the car's acceleration and geometry.

$$F_{z_1} = \frac{1}{2} mg \frac{a_2}{l} - \frac{1}{2} mg \frac{h}{l} \frac{a}{g} \tag{2.230}$$

$$F_{z_2} = \frac{1}{2} mg \frac{a_1}{l} + \frac{1}{2} mg \frac{h}{l} \frac{a}{g} \tag{2.231}$$

We may generalize the equations by making them dimensionless. Under the best conditions, we should adjust the traction forces to their maximum

$$\frac{F_{x_1}}{mg} = \frac{1}{2} \mu_x \left(\frac{a_2}{l} - \frac{h}{l} \frac{a}{g} \right) \tag{2.232}$$

$$\frac{F_{x_2}}{mg} = \frac{1}{2} \mu_x \left(\frac{a_1}{l} + \frac{h}{l} \frac{a}{g} \right) \tag{2.233}$$

and therefore, the longitudinal equation of motion (2.227) becomes

$$\frac{a}{g} = \mu_x. \tag{2.234}$$

Substituting this result back into Equations (2.232) and (2.233) shows that

$$\frac{F_{x_1}}{mg} = -\frac{1}{2} \frac{h}{l} \left(\frac{a}{g} \right)^2 + \frac{1}{2} \frac{a_2}{l} \frac{a}{g} \tag{2.235}$$

$$\frac{F_{x_2}}{mg} = \frac{1}{2} \frac{h}{l} \left(\frac{a}{g} \right)^2 + \frac{1}{2} \frac{a_1}{l} \frac{a}{g}. \tag{2.236}$$

Depending on the geometry of the car (h, a_1, a_2), and the acceleration $a > 0$, these two equations determine how much the front and rear driving forces must be. The same equations are applied for deceleration $a < 0$, to

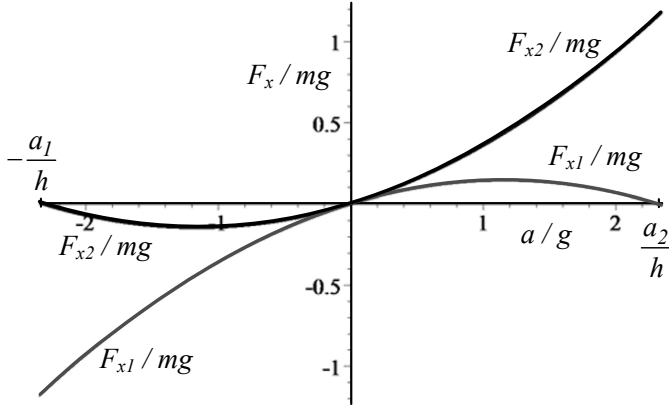


FIGURE 2.15. Optimal driving and braking forces for a sample car.

determine the value of optimal front and rear braking forces. Figure 2.15 represents a graphical illustration of the optimal driving and braking forces for a sample car using the following data:

$$\begin{aligned}
 \mu_x &= 1 \\
 \frac{h}{l} &= \frac{0.56}{2.6} = 0.21538 \\
 \frac{a_1}{l} &= \frac{a_2}{l} = \frac{1}{2}.
 \end{aligned} \tag{2.237}$$

When accelerating $a > 0$, the optimal driving force on the rear tire grows rapidly while the optimal driving force on the front tire drops after a maximum. The value $(a/g) = (a_2/h)$ is the maximum possible acceleration at which the front tires lose their contact with the ground. The acceleration at which front (or rear) tires lose their ground contact is called *tilting acceleration*.

The opposite phenomenon happens when decelerating. For $a < 0$, the optimal front brake force increases rapidly and the rear brake force goes to zero after a minimum. The deceleration $(a/g) = -(a_1/h)$ is the maximum possible deceleration at which the rear tires lose their ground contact.

The graphical representation of the optimal driving and braking forces can be shown better by plotting $F_{x1}/(mg)$ versus $F_{x2}/(mg)$ using (a/g) as a parameter.

$$F_{x1} = \frac{a_2 - \frac{a}{g}h}{a_1 + \frac{a}{g}h} F_{x2} \tag{2.238}$$

$$\frac{F_{x1}}{F_{x2}} = \frac{a_2 - \mu_x h}{a_1 + \mu_x h} \tag{2.239}$$

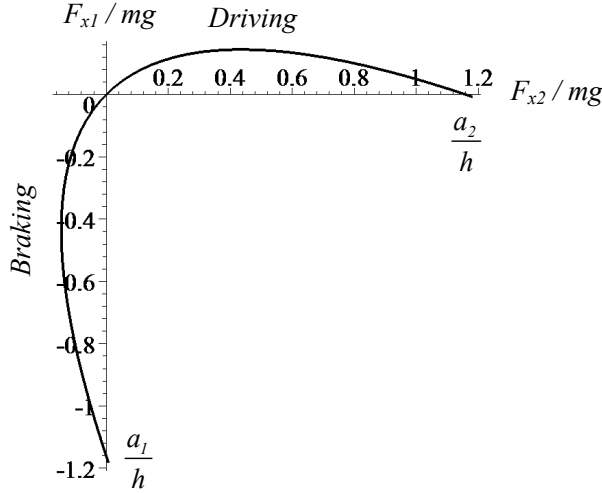


FIGURE 2.16. Optimal traction and braking force distribution between the front and rear wheels.

Such a plot is shown in Figure 2.16. This is a design curve describing the relationship between forces under the front and rear wheels to achieve the maximum acceleration or deceleration.

Adjusting the optimal force distribution is not an automatic procedure and needs a force distributor control system to measure and adjust the forces. ■

Example 66 ★ *Slope at zero.*

The initial optimal traction force distribution is the slope of the optimal curve $(F_{x1}/(mg), F_{x2}/(mg))$ at zero.

$$\begin{aligned}
 \frac{d \frac{F_{x1}}{mg}}{d \frac{F_{x2}}{mg}} &= \lim_{a \rightarrow 0} \frac{-\frac{1}{2} \frac{h}{l} \left(\frac{a}{g}\right)^2 + \frac{1}{2} \frac{a_2}{l} \frac{a}{g}}{\frac{1}{2} \frac{h}{l} \left(\frac{a}{g}\right)^2 + \frac{1}{2} \frac{a_1}{l} \frac{a}{g}} \\
 &= \frac{a_2}{a_1}
 \end{aligned} \tag{2.240}$$

Therefore, the initial traction force distribution depends on only the position of mass center C .

Example 67 ★ *Brake balance and ABS.*

When braking, a car is stable if the rear wheels do not lock. Thus, the rear brake forces must be less than the maximum possible braking force at all time. This means the brake force distribution should always be in the shaded area of Figure 2.17, and below the optimal curve. This restricts the

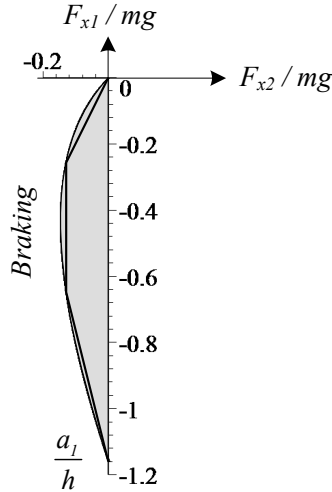


FIGURE 2.17. Optimal braking force distribution between the front and rear wheels, along with a three-line under estimation.

achievable deceleration, especially at low friction values, but increases the stability of the car.

Whenever it is easier for a force distributor to follow a line, the optimal brake curve is underestimated using two or three lines, and a control system adjusts the force ratio F_{x1}/F_{x2} . A sample of three-line approximation is shown in Figure 2.17.

Distribution of the brake force between the front and rear wheels is called **brake balance**. Brake balance varies with deceleration. The higher the stop, the more load will transfer to the front wheels and the more braking effort they can support. Meanwhile the rear wheels are unloaded and they must have less braking force.

Example 68 ★ Best race car.

Racecars always work at the maximum achievable acceleration to finish their race in minimum time. They are usually designed with rear-wheel-drive and all-wheel-brake. However, if an all-wheel-drive race car is reasonable to build, then a force distributor, to follow the curve shown in Figure 2.18, is what it needs to race better.

Example 69 ★ Effect of C location on braking.

Load is transferred from the rear wheels to the front when the brakes are applied. The higher the C , the more load transfer. So, to improve braking, the mass center C should be as low as possible and as back as possible. This is not feasible for every vehicle, especially for forward-wheel drive street cars. However, this fact should be taken into account when a car is

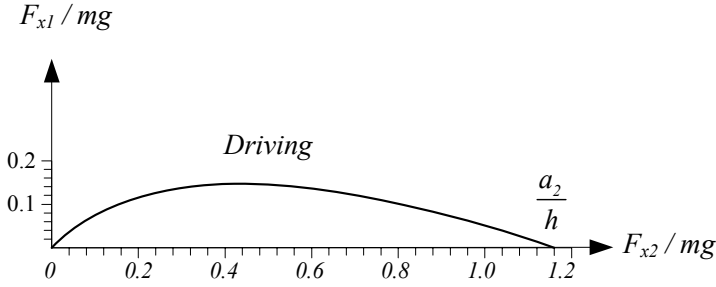


FIGURE 2.18. Optimal traction force distribution between the front and rear wheels.

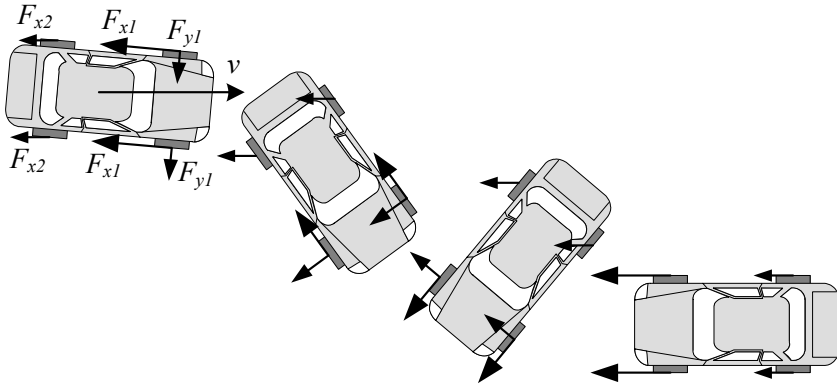


FIGURE 2.19. 180 deg sliding rotation of a rear-wheel-locked car.

being designed for better braking performance.

Example 70 ★ Front and rear wheel locking.

The optimal brake force distribution is according to Equation (2.239) for an ideal F_{x1}/F_{x2} ratio. However, if the brake force distribution is not ideal, then either the front or the rear wheels will lock up first. Locking the rear wheels makes the vehicle unstable, and it loses directional stability. When the rear wheels lock, they slide on the road and they lose their capacity to support lateral force. The resultant shear force at the tireprint of the rear wheels reduces to a dynamic friction force in the opposite direction of the sliding.

A slight lateral motion of the rear wheels, by any disturbance, develops a yaw motion because of unbalanced lateral forces on the front and rear wheels. The yaw moment turns the vehicle about the z -axis until the rear end leads the front end and the vehicle turns 180 deg. Figure 2.19 illustrates a 180 deg sliding rotation of a rear-wheel-locked car.

The lock-up of the front tires does not cause a directional instability, although the car would not be steerable and the driver would lose control.

2.7 ★ Vehicles With More Than Two Axles

If a vehicle has more than two axles, such as the three-axle car shown in Figure 2.20, then the vehicle will be statically indeterminate and the normal forces under the tires cannot be determined by static equilibrium equations. We need to consider the suspensions' deflection to determine their applied forces.

The n normal forces F_{z_i} under the tires can be calculated using the following n algebraic equations.

$$2 \sum_{i=1}^n F_{z_i} - mg \cos \phi = 0 \quad (2.241)$$

$$2 \sum_{i=1}^n F_{z_i} x_i + h(a + mg \sin \phi) = 0 \quad (2.242)$$

$$\frac{F_{z_i}}{k_i} - \frac{x_i - x_1}{x_n - x_1} \left(\frac{F_{z_n}}{k_n} - \frac{F_{z_1}}{k_1} \right) - \frac{F_{z_1}}{k_1} = 0 \quad \text{for } i = 2, 3, \dots, n-1 \quad (2.243)$$

where F_{x_i} and F_{z_i} are the longitudinal and normal forces under the tires attached to the axle number i , and x_i is the distance of mass center C from the axle number i . The distance x_i is positive for axles in front of C , and is negative for the axles in back of C . The parameter k_i is the vertical stiffness of the suspension at axle i .

Proof. For a multiple-axle vehicle, the following equations

$$\sum F_x = ma \quad (2.244)$$

$$\sum F_z = 0 \quad (2.245)$$

$$\sum M_y = 0 \quad (2.246)$$

provide the same sort of equations as (2.126)-(2.128). However, if the total number of axles are n , then the individual forces can be substituted by a summation.

$$2 \sum_{i=1}^n F_{x_i} - mg \sin \phi = ma \quad (2.247)$$

$$2 \sum_{i=1}^n F_{z_i} - mg \cos \phi = 0 \quad (2.248)$$

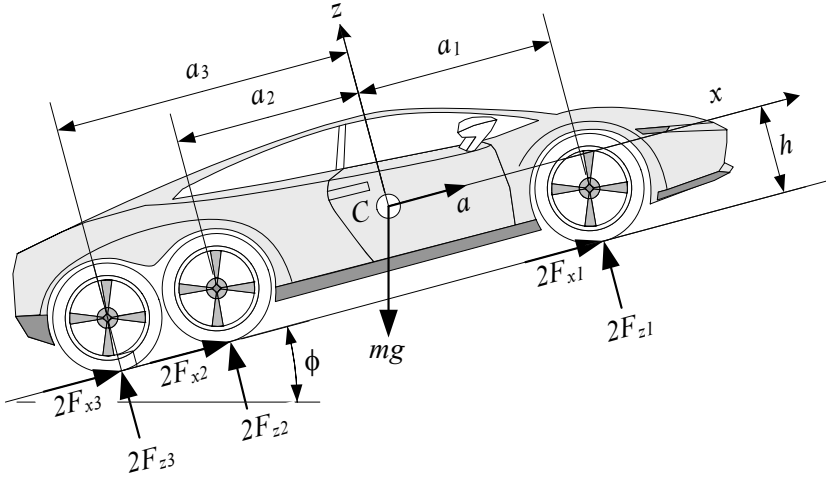


FIGURE 2.20. A three-axle car moving on an inclined road.

$$2 \sum_{i=1}^n F_{z_i} x_i + 2h \sum_{i=1}^n F_{x_i} = 0 \quad (2.249)$$

The overall forward force $F_x = 2 \sum_{i=1}^n F_{x_i}$ can be eliminated between Equations (2.247) and (2.249) to make Equation (2.242). Then, there remain two equations (2.241) and (2.242) for n unknowns F_{z_i} , $i = 1, 2, \dots, n$. Hence, we need $n - 2$ extra equations to be able to find the wheel loads. The extra equations come from the compatibility among the suspensions' deflection.

We ignore the tires' compliance, and use z to indicate the static vertical displacement of the car at C . Then, if z_i is the suspension deflection at the center of axle i , and k_i is the vertical stiffness of the suspension at axle i , the deflections are

$$z_i = \frac{F_{z_i}}{k_i}. \quad (2.250)$$

For a flat road, and a rigid vehicle, we must have

$$\frac{z_i - z_1}{x_i - x_1} = \frac{z_n - z_1}{x_n - x_1} \quad \text{for } i = 2, 3, \dots, n-1 \quad (2.251)$$

which, after substituting with (2.250), reduces to Equation (2.243). The $n - 2$ equations (2.251) along with the two equations (2.241) and (2.242) are enough to calculate the normal load under each tire. The resultant set of equations is linear and may be arranged in a matrix form

$$[A][X] = [B] \quad (2.252)$$

where

$$[X] = \begin{bmatrix} F_{z_1} & F_{z_2} & F_{z_3} & \cdots & F_{z_n} \end{bmatrix}^T \quad (2.253)$$

$$[A] = \begin{bmatrix} 2 & 2 & \cdots & \cdots & \cdots & \cdots & 2 \\ 2x_1 & 2x_2 & \cdots & \cdots & \cdots & \cdots & 2x_n \\ \frac{x_n - x_2}{k_1 l} & \frac{1}{k_2} & \cdots & \cdots & \cdots & \cdots & \frac{x_2 - x_1}{k_n l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{x_n - x_i}{k_1 l} & \cdots & \cdots & \frac{1}{k_i} & \cdots & \cdots & \frac{x_i - x_1}{k_n l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{x_n - x_{n-1}}{k_1 l} & \cdots & \cdots & \cdots & \cdots & \frac{1}{k_{n-1}} & \frac{x_{n-1} - x_1}{k_n l} \end{bmatrix} \quad (2.254)$$

$$l = x_1 - x_n \quad (2.255)$$

$$[B] = \begin{bmatrix} mg \cos \phi & -h(a + mg \sin \phi) & 0 & \cdots & 0 \end{bmatrix}^T. \quad (2.256)$$

■

Example 71 ★ *Wheel reactions for a three-axle car.*

Figure 2.20 illustrates a three-axle car moving on an inclined road. We start counting the axles of a multiple-axle vehicle from the front axle as axle-1, and move sequentially to the back as shown in the figure.

The set of equations for the three-axle car, as seen in Figure 2.20, is

$$2F_{x_1} + 2F_{x_2} + 2F_{x_3} - mg \sin \phi = ma \quad (2.257)$$

$$2F_{z_1} + 2F_{z_2} + 2F_{z_3} - mg \cos \phi = 0 \quad (2.258)$$

$$2F_{z_1}x_1 + 2F_{z_2}x_2 + 2F_{z_3}x_3 + 2h(F_{x_1} + F_{x_2} + F_{x_3}) = 0 \quad (2.259)$$

$$\frac{1}{x_2 - x_1} \left(\frac{F_{z_2}}{k_2} - \frac{F_{z_1}}{k_1} \right) - \frac{1}{x_3 - x_1} \left(\frac{F_{z_3}}{k_3} - \frac{F_{z_1}}{k_1} \right) = 0 \quad (2.260)$$

which can be simplified to

$$2F_{z_1} + 2F_{z_2} + 2F_{z_3} - mg \cos \phi = 0 \quad (2.261)$$

$$2F_{z_1}x_1 + 2F_{z_2}x_2 + 2F_{z_3}x_3 + hm(a + g \sin \phi) = 0 \quad (2.262)$$

$$\begin{aligned} (x_2 k_2 k_3 - x_3 k_2 k_3) F_{z_1} + (x_1 k_1 k_2 - x_2 k_1 k_2) F_{z_3} \\ - (x_1 k_1 k_3 - x_3 k_1 k_3) F_{z_2} = 0. \end{aligned} \quad (2.263)$$

The set of equations for wheel loads is linear and may be rearranged in a matrix form

$$[A][X] = [B] \quad (2.264)$$

where

$$[A] = \begin{bmatrix} 2 & 2 & 2 \\ 2x_1 & 2x_2 & 2x_3 \\ k_2k_3(x_2 - x_3) & k_1k_3(x_3 - x_1) & k_1k_2(x_1 - x_2) \end{bmatrix} \quad (2.265)$$

$$[X] = \begin{bmatrix} F_{z_1} \\ F_{z_2} \\ F_{z_3} \end{bmatrix} \quad (2.266)$$

$$[B] = \begin{bmatrix} mg \cos \phi \\ -hm(a + g \sin \phi) \\ 0 \end{bmatrix}. \quad (2.267)$$

The unknown vector may be found using matrix inversion

$$[X] = [A]^{-1} [B]. \quad (2.268)$$

The solution of the equations are

$$\frac{1}{k_1m} F_{z_1} = \frac{Z_1}{Z_0} \quad (2.269)$$

$$\frac{1}{k_2m} F_{z_2} = \frac{Z_2}{Z_0} \quad (2.270)$$

$$\frac{1}{k_2m} F_{z_3} = \frac{Z_3}{Z_0} \quad (2.271)$$

where,

$$Z_0 = -4k_1k_2(x_1 - x_2)^2 - 4k_2k_3(x_2 - x_3)^2 - 4k_1k_3(x_3 - x_1)^2 \quad (2.272)$$

$$\begin{aligned} Z_1 = & g(x_2k_2 - x_1k_3 - x_1k_2 + x_3k_3)h \sin \phi \\ & + a(x_2k_2 - x_1k_3 - x_1k_2 + x_3k_3)h \\ & + g(k_2x_2^2 - x_1k_2x_2 + k_3x_3^2 - x_1k_3x_3) \cos \phi \end{aligned} \quad (2.273)$$

$$\begin{aligned} Z_2 = & g(x_1k_1 - x_2k_1 - x_2k_3 + x_3k_3)h \sin \phi \\ & + a(x_1k_1 - x_2k_1 - x_2k_3 + x_3k_3)h \\ & + g(k_1x_1^2 - x_2k_1x_1 + k_3x_3^2 - x_2k_3x_3) \cos \phi \end{aligned} \quad (2.274)$$

$$\begin{aligned} Z_3 = & g(x_1k_1 + x_2k_2 - x_3k_1 - x_3k_2)h \sin \phi \\ & + a(x_1k_1 + x_2k_2 - x_3k_1 - x_3k_2)h \\ & + g(k_1x_1^2 - x_3k_1x_1 + k_2x_2^2 - x_3k_2x_2) \cos \phi \end{aligned} \quad (2.275)$$

$$x_1 = a_1 \quad (2.276)$$

$$x_2 = -a_2 \quad (2.277)$$

$$x_3 = -a_3. \quad (2.278)$$

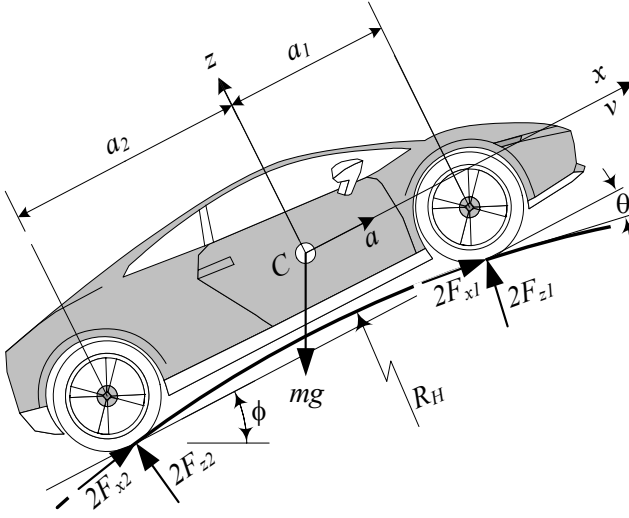


FIGURE 2.21. A cresting vehicle at a point where the hill has a radius of curvature R_h .

2.8 ★ Vehicles on a Crest and Dip

When a road has an outward or inward curvature, we call the road is a crest or a dip. The curvature can decrease or increase the normal forces under the wheels.

2.8.1 ★ Vehicles on a Crest

Moving on the convex curve of a hill is called *cresting*. The normal force under the wheels of a cresting vehicle is less than the force on a flat inclined road with the same slope, because of the developed centrifugal force mv^2/R_H in the $-z$ -direction.

Figure 2.21 illustrates a cresting vehicle at the point on the hill with a radius of curvature R_H . The traction and normal forces under its tires are approximately equal to

$$F_{x_1} + F_{x_2} \approx \frac{1}{2}m(a + g \sin \phi) \quad (2.279)$$

$$F_{z_1} \approx \frac{1}{2}mg \left[\left(\frac{a_2}{l} \cos \phi + \frac{h}{l} \sin \phi \right) \right] - \frac{1}{2}ma \frac{h}{l} - \frac{1}{2}m \frac{v^2}{R_H} \frac{a_2}{l} \quad (2.280)$$

$$F_{z_2} \approx \frac{1}{2}mg \left[\left(\frac{a_1}{l} \cos \phi - \frac{h}{l} \sin \phi \right) \right] + \frac{1}{2}ma \frac{h}{l} - \frac{1}{2}m \frac{v^2}{R_H} \frac{a_1}{l} \quad (2.281)$$

$$l = a_1 + a_2. \quad (2.282)$$

Proof. For the cresting car shown in Figure 2.21, the normal and tangential directions are equivalent to the $-z$ and x directions respectively. Hence, the governing equation of motion for the car is

$$\sum F_x = ma \quad (2.283)$$

$$-\sum F_z = m \frac{v^2}{R_H} \quad (2.284)$$

$$\sum M_y = 0. \quad (2.285)$$

Expanding these equations produces the following equations:

$$2F_{x_1} \cos \theta + 2F_{x_2} \cos \theta - mg \sin \phi = ma \quad (2.286)$$

$$-2F_{z_1} \cos \theta - 2F_{z_2} \cos \theta + mg \cos \phi = m \frac{v^2}{R_H} \quad (2.287)$$

$$2F_{z_1} a_1 \cos \theta - 2F_{z_2} a_2 \cos \theta + 2(F_{x_1} + F_{x_2}) h \cos \theta + 2F_{z_1} a_1 \sin \theta - 2F_{z_2} a_2 \sin \theta - 2(F_{x_1} + F_{x_2}) h \sin \theta = 0. \quad (2.288)$$

We may eliminate $(F_{x_1} + F_{x_2})$ between the first and third equations, and solve for the total traction force $F_{x_1} + F_{x_2}$ and wheel normal forces F_{z_1} , F_{z_2} .

$$F_{x_1} + F_{x_2} = \frac{ma + mg \sin \phi}{2 \cos \theta} \quad (2.289)$$

$$F_{z_1} = \frac{1}{2}mg \left[\left(\frac{a_2}{l \cos \theta} \cos \phi + \frac{h(1 - \sin 2\theta)}{l \cos \theta \cos 2\theta} \sin \phi \right) \right] - \frac{1}{2}ma \frac{h(1 - \sin 2\theta)}{l \cos \theta \cos 2\theta} - \frac{1}{2}m \frac{v^2}{R_H} \frac{a_2}{l \cos \theta} \quad (2.290)$$

$$F_{z_2} = \frac{1}{2}mg \left[\left(\frac{a_1}{l \cos \theta} \cos \phi - \frac{h(1 - \sin 2\theta)}{l \cos \theta \cos 2\theta} \sin \phi \right) \right] + \frac{1}{2}ma \frac{h(1 - \sin 2\theta)}{l \cos \theta \cos 2\theta} - \frac{1}{2}m \frac{v^2}{R_H} \frac{a_1}{l \cos \theta \cos \theta} \quad (2.291)$$

If the car's wheel base is much smaller than the radius of curvature, $l \ll R_H$, then the slope angle θ is too small, and we may use the following trigonometric approximations.

$$\cos \theta \approx \cos 2\theta \approx 1 \quad (2.292)$$

$$\sin \theta \approx \sin 2\theta \approx 0 \quad (2.293)$$

Substituting these approximations in Equations (2.289)-(2.291) produces the following approximate results:

$$F_{x_1} + F_{x_2} \approx \frac{1}{2}m(a + g \sin \phi) \quad (2.294)$$

$$F_{z_1} \approx \frac{1}{2}mg \left[\left(\frac{a_2}{l} \cos \phi + \frac{h}{l} \sin \phi \right) \right] - \frac{1}{2}ma \frac{h}{l} - \frac{1}{2}m \frac{v^2}{R_H} \frac{a_2}{l} \quad (2.295)$$

$$F_{z_2} \approx \frac{1}{2}mg \left[\left(\frac{a_1}{l} \cos \phi - \frac{h}{l} \sin \phi \right) \right] + \frac{1}{2}ma \frac{h}{l} - \frac{1}{2}m \frac{v^2}{R_H} \frac{a_1}{l} \quad (2.296)$$

■

Example 72 ★ *Wheel loads of a cresting car.*

Consider a car with the following specifications:

$$\begin{aligned} l &= 2272 \text{ mm} \\ w &= 1457 \text{ mm} \\ m &= 1500 \text{ kg} \\ h &= 230 \text{ mm} \\ a_1 &= a_2 \\ v &= 15 \text{ m/s} \\ a &= 1 \text{ m/s}^2 \end{aligned} \quad (2.297)$$

which is cresting a hill at a point where the road has

$$\begin{aligned} R_H &= 40 \text{ m} \\ \phi &= 30 \text{ deg} \\ \theta &= 2.5 \text{ deg}. \end{aligned} \quad (2.298)$$

The force information on the car is:

$$\begin{aligned} F_{x_1} + F_{x_2} &= 4432.97 \text{ N} \\ F_{z_1} &= 666.33 \text{ N} \\ F_{z_2} &= 1488.75 \text{ N} \\ mg &= 14715 \text{ N} \\ F_{z_1} + F_{z_2} &= 2155.08 \text{ N} \\ m \frac{v^2}{R_H} &= 8437.5 \text{ N} \end{aligned} \quad (2.299)$$

If we simplifying the results by assuming small θ , the approximate values of the forces are

$$\begin{aligned}
 F_{x_1} + F_{x_2} &= 4428.75 \text{ N} \\
 F_{z_1} &\approx 628.18 \text{ N} \\
 F_{z_2} &\approx 1524.85 \text{ N} \\
 mg &= 14715 \text{ N} \\
 F_{z_1} + F_{z_2} &\approx 2153.03 \text{ N} \\
 m \frac{v^2}{R_H} &= 8437.5 \text{ N.}
 \end{aligned} \tag{2.300}$$

Example 73 ★ *Losing the road contact in a crest.*

When a car goes too fast, it can lose its road contact. Such a car is called a **flying car**. The condition to have a flying car is $F_{z_1} = 0$ and $F_{z_2} = 0$.

Assuming a symmetric car $a_1 = a_2 = l/2$ with no acceleration, and using the approximate Equations (2.280) and (2.281)

$$\frac{1}{2}mg \left[\left(\frac{a_2}{l} \cos \phi + \frac{h}{l} \sin \phi \right) \right] - \frac{1}{2}m \frac{v^2}{R_H} \frac{a_2}{l} = 0 \tag{2.301}$$

$$\frac{1}{2}mg \left[\left(\frac{a_1}{l} \cos \phi - \frac{h}{l} \sin \phi \right) \right] - \frac{1}{2}m \frac{v^2}{R_H} \frac{a_1}{l} = 0 \tag{2.302}$$

we can find the critical minimum speed v_c to start flying. There are two critical speeds v_{c_1} and v_{c_2} for losing the contact of the front and rear wheels respectively.

$$v_{c_1} = \sqrt{2gR_H \left(\frac{h}{l} \sin \phi + \frac{1}{2} \cos \phi \right)} \tag{2.303}$$

$$v_{c_2} = \sqrt{-2gR_H \left(\frac{h}{l} \sin \phi - \frac{1}{2} \cos \phi \right)} \tag{2.304}$$

For any car, the critical speeds v_{c_1} and v_{c_2} are functions of the hill's radius of curvature R_H and the angular position on the hill, indicated by ϕ . The angle ϕ cannot be out of the tilting angles given by Equation (2.141).

$$-\frac{a_1}{h} \leq \tan \phi \leq \frac{a_2}{h} \tag{2.305}$$

Figure 2.22 illustrates a cresting car over a circular hill, and Figure 2.23 depicts the critical speeds v_{c_1} and v_{c_2} at a different angle ϕ for $-1.371 \text{ rad} \leq$

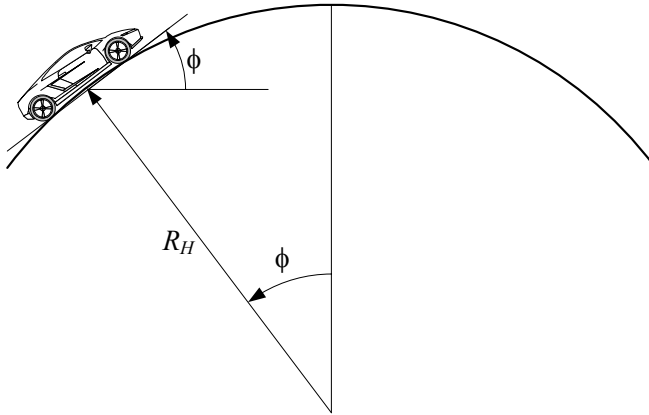


FIGURE 2.22. A cresting car over a circular hill.

$\phi \leq 1.371$ rad. The specifications of the car and the hill are:

$$\begin{aligned}
 l &= 2272 \text{ mm} \\
 h &= 230 \text{ mm} \\
 a_1 &= a_2 \\
 a &= 0 \text{ m/s}^2 \\
 R_H &= 100 \text{ m}.
 \end{aligned}$$

At the maximum uphill slope $\phi = 1.371 \text{ rad} \approx 78.5^\circ$, the front wheels can leave the ground at zero speed while the rear wheels are on the ground. When the car moves over the hill and reaches the maximum downhill slope $\phi = -1.371 \text{ rad} \approx -78.5^\circ$ the rear wheels can leave the ground at zero speed while the front wheels are on the ground. As long as the car is moving uphill, the front wheels can leave the ground at a lower speed while going downhill the rear wheels leave the ground at a lower speed. Hence, at each slope angle ϕ the lower curve determines the critical speed v_c .

To have a general image of the critical speed, we may plot the lower values of v_c as a function of ϕ using R_H or h/l as a parameter. Figure 2.24 shows the effect of hill radius of curvature R_H on the critical speed v_c for a car with $h/l = 0.10123 \text{ mm/mm}$ and Figure 2.25 shows the effect of a car's high factor h/l on the critical speed v_c for a circular hill with $R_H = 100 \text{ m}$.

2.8.2 ★ Vehicles on a Dip

Moving on the concave curve of a hill is called *dipping*. The normal force under the wheels of a dipping vehicle is more than the force on a flat inclined road with the same slope, because of the developed centrifugal

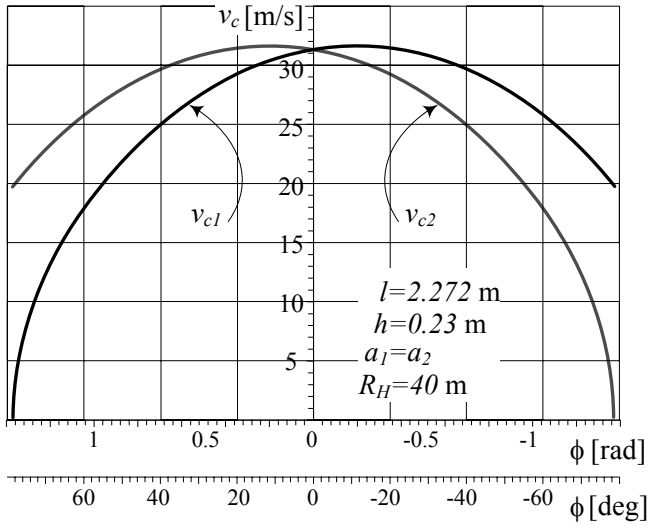


FIGURE 2.23. Critical speeds v_{c1} and v_{c2} at different angle ϕ for a specific car and hill.

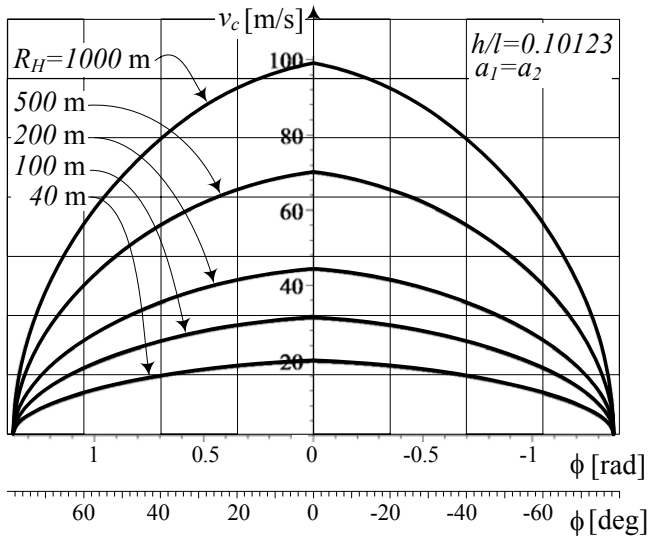


FIGURE 2.24. Effect of hill radius of curvature R_h on the critical speed v_c for a car.

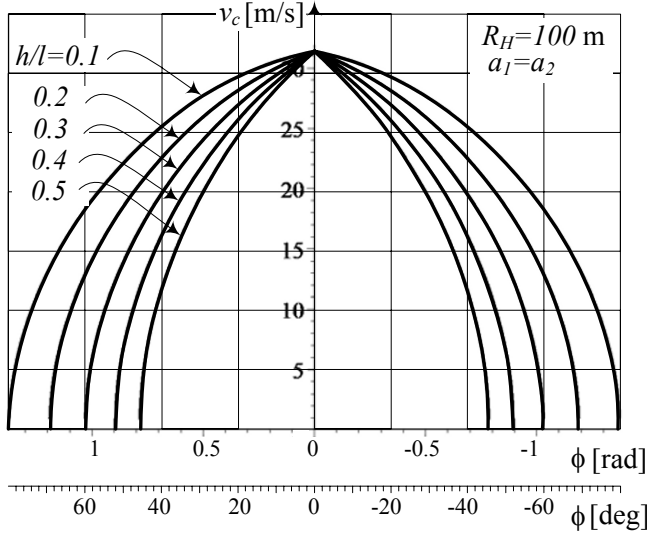


FIGURE 2.25. Effect of a car's height factor h/l on the critical speed v_c for a circular hill.

force mv^2/R_H in the z -direction.

Figure 2.26 illustrates a dipping vehicle at a point where the hill has a radius of curvature R_H . The traction and normal forces under the tires of the vehicle are approximately equal to

$$F_{x_1} + F_{x_2} \approx \frac{1}{2}m(a + g \sin \phi) \quad (2.306)$$

$$F_{z_1} \approx \frac{1}{2}mg \left[\left(\frac{a_2}{l} \cos \phi + \frac{h}{l} \sin \phi \right) - \frac{1}{2}ma \frac{h}{l} + \frac{1}{2}m \frac{v^2}{R_H} \frac{a_2}{l} \right] \quad (2.307)$$

$$F_{z_2} \approx \frac{1}{2}mg \left[\left(\frac{a_1}{l} \cos \phi - \frac{h}{l} \sin \phi \right) + \frac{1}{2}ma \frac{h}{l} + \frac{1}{2}m \frac{v^2}{R_H} \frac{a_1}{l} \right] \quad (2.308)$$

$$l = a_1 + a_2. \quad (2.309)$$

Proof. To develop the equations for the traction and normal forces under the tires of a dipping car, we follow the same procedure as a cresting car. The normal and tangential directions of a dipping car, shown in Figure 2.21, are equivalent to the z and x directions respectively. Hence, the governing

$$F_{z_1} = \frac{1}{2}mg \left[\left(\frac{a_2}{l \cos \theta} \cos \phi + \frac{h(1 - \sin 2\theta)}{l \cos \theta \cos 2\theta} \sin \phi \right) \right] - \frac{1}{2}ma \frac{h(1 - \sin 2\theta)}{l \cos \theta \cos 2\theta} + \frac{1}{2}m \frac{v^2}{R_H} \frac{a_2}{l \cos \theta} \quad (2.317)$$

$$F_{z_2} = \frac{1}{2}mg \left[\left(\frac{a_1}{l \cos \theta} \cos \phi - \frac{h(1 - \sin 2\theta)}{l \cos \theta \cos 2\theta} \sin \phi \right) \right] + \frac{1}{2}ma \frac{h(1 - \sin 2\theta)}{l \cos \theta \cos 2\theta} + \frac{1}{2}m \frac{v^2}{R_H} \frac{a_1}{l \cos \theta \cos \theta} \quad (2.318)$$

Assuming $\theta \ll 1$, these forces can be approximated to

$$F_{x_1} + F_{x_2} \approx \frac{1}{2}m(a + g \sin \phi) \quad (2.319)$$

$$F_{z_1} \approx \frac{1}{2}mg \left[\left(\frac{a_2}{l} \cos \phi + \frac{h}{l} \sin \phi \right) \right] - \frac{1}{2}ma \frac{h}{l} + \frac{1}{2}m \frac{v^2}{R_H} \frac{a_2}{l} \quad (2.320)$$

$$F_{z_2} \approx \frac{1}{2}mg \left[\left(\frac{a_1}{l} \cos \phi - \frac{h}{l} \sin \phi \right) \right] + \frac{1}{2}ma \frac{h}{l} + \frac{1}{2}m \frac{v^2}{R_H} \frac{a_1}{l}. \quad (2.321)$$

■

Example 74 ★ *Wheel loads of a dipping car.*

Consider a car with the following specifications:

$$\begin{aligned} l &= 2272 \text{ mm} \\ w &= 1457 \text{ mm} \\ m &= 1500 \text{ kg} \\ h &= 230 \text{ mm} \\ a_1 &= a_2 \\ v &= 15 \text{ m/s} \\ a &= 1 \text{ m/s}^2 \end{aligned} \quad (2.322)$$

that is dipping on a hill at a point where the road has

$$\begin{aligned} R_H &= 40 \text{ m} \\ \phi &= 30^\circ \\ \theta &= 2.5^\circ. \end{aligned} \quad (2.323)$$

The force information of the car is:

$$\begin{aligned}
 F_{x_1} + F_{x_2} &= 4432.97 \text{ N} \\
 F_{z_1} &= 4889.1 \text{ N} \\
 F_{z_2} &= 5711.52 \text{ N} \\
 mg &= 14715 \text{ N} \\
 F_{z_1} + F_{z_2} &= 10600.62 \text{ N} \\
 m \frac{v^2}{R_H} &= 8437.5 \text{ N}
 \end{aligned} \tag{2.324}$$

If we ignore the effect of θ by assuming $\theta \ll 1$, then the approximate value of the forces are

$$\begin{aligned}
 F_{x_1} + F_{x_2} &= 4428.75 \text{ N} \\
 F_{z_1} &\approx 4846.93 \text{ N} \\
 F_{z_2} &\approx 1524.85 \text{ N} \\
 mg &= 5743.6 \text{ N} \\
 F_{z_1} + F_{z_2} &\approx 10590.53 \text{ N} \\
 m \frac{v^2}{R_H} &= 8437.5 \text{ N.}
 \end{aligned} \tag{2.325}$$

2.9 Summary

For straight motion of a symmetric rigid vehicle, we may assume the forces on the left wheel are equal to the forces on the right wheel, and simplify the tire force calculation.

When a car is accelerating on an inclined road with angle ϕ , the normal forces under the front and rear wheels, F_{z_1} , F_{z_2} , are:

$$F_{z_1} = \frac{1}{2}mg \left(\frac{a_2}{l} \cos \phi - \frac{h}{l} \sin \phi \right) - \frac{1}{2}ma \frac{h}{l} \tag{2.326}$$

$$F_{z_2} = \frac{1}{2}mg \left(\frac{a_1}{l} \cos \phi + \frac{h}{l} \sin \phi \right) + \frac{1}{2}ma \frac{h}{l} \tag{2.327}$$

$$l = a_1 + a_2 \tag{2.328}$$

where, $\frac{1}{2}mg \left(\frac{a_1}{l} \cos \phi \pm \frac{h}{l} \sin \phi \right)$ is the static part and $\pm \frac{1}{2}mg \frac{h}{l} \frac{a}{g}$ is the dynamic part, because it depends on the acceleration a .

2.10 Key Symbols

| | |
|--------------------------------|---|
| $a \equiv \ddot{x}$ | acceleration |
| a_{fwd} | front wheel drive acceleration |
| a_{rwd} | rear wheel drive acceleration |
| a_1 | distance of first axle from mass center |
| a_2 | distance of second axle from mass center |
| a_i | distance of axle number i from mass center |
| a_M | maximum acceleration |
| a, b | arguments for $\text{atan2}(a, b)$ |
| A, B, C | constant parameters |
| b_1 | distance of left wheels from mass center |
| b_1 | distance of hinge point from rear axle |
| b_2 | distance of right wheels from mass center |
| b_2 | distance of hinge point from trailer mass center |
| b_3 | distance of trailer axle from trailer mass center |
| C | mass center of vehicle |
| C_t | mass center of trailer |
| F | force |
| F_x | traction or brake force under a wheel |
| F_{x1} | traction or brake force under front wheels |
| F_{x2} | traction or brake force under rear wheels |
| F_{x_t} | horizontal force at hinge |
| F_z | normal force under a wheel |
| F_{z1} | normal force under front wheels |
| F_{z2} | normal force under rear wheels |
| F_{z3} | normal force under trailer wheels |
| F_{z_t} | normal force at hinge |
| g, \mathbf{g} | gravitational acceleration |
| h | height of C |
| H | height |
| I | mass moment of inertia |
| k_i | vertical stiffness of suspension at axle number i |
| l | wheel base |
| m | car mass |
| m_t | trailer mass |
| M | moment |
| R | tire radius |
| R_f | front tire radius |
| R_r | rear tire radius |
| R_H | radius of curvature |
| t | time |
| $v \equiv \dot{x}, \mathbf{v}$ | velocity |
| v_c | critical velocity |

| | |
|-----------|-------------------------------|
| w | track |
| z_i | deflection of axil number i |
| x, y, z | vehicle coordinate axes |
| X, Y, Z | global coordinate axes |
| θ | road slope |
| ϕ | road angle with horizon |
| ϕ_M | maximum slope angle |
| μ | friction coefficient |

Subscriptions

| | |
|-------|-------------------|
| dyn | dynamic |
| f | front |
| fwd | front-wheel-drive |
| M | maximum |
| r | rear |
| rwd | rear-wheel-drive |
| st | statics |

Exercises

1. Axle load.

Consider a car with the following specifications that is parked on a level road. Find the load on the front and rear axles.

$$m = 1765 \text{ kg}$$

$$l = 2.84 \text{ m}$$

$$a_1 = 1.22 \text{ m}$$

$$a_2 = 1.62 \text{ m}$$

2. Axle load.

Consider a car with the following specification, and find the axles load.

$$m = 1245 \text{ kg}$$

$$a_1 = 1100 \text{ mm}$$

$$a_2 = 1323 \text{ mm}$$

3. Mass center distance ratio.

Peugeot 907 ConceptTM approximately has the following specifications.

$$m = 1400 \text{ kg}$$

$$l = 97.5 \text{ in}$$

Assume $a_1/a_2 \approx 1.131$ and determine the axles load.

4. Axle load ratio.

Jeep Commander XKTM approximately has the following specifications.

$$mg = 5091 \text{ lb}$$

$$l = 109.5 \text{ in}$$

Assume $F_{z_1}/F_{z_2} \approx 1.22$ and determine the axles load.

5. Axle load and mass center distance ratio.

The wheelbase of the 1981 DeLorean SportscarTM is

$$l = 94.89 \text{ in.}$$

Find the axles load if we assume

$$a_1/a_2 \approx 0.831$$

$$mg = 3000 \text{ lb.}$$

6. Mass center height.

McLaren SLR 722 SportscarTM has the following specifications.

$$\begin{array}{ll} \text{front tire} & 255/35ZR19 \\ \text{rear tire} & 295/30ZR19 \end{array}$$

$$\begin{array}{ll} m & = 1649 \text{ kg} \\ l & = 2700 \text{ mm} \end{array}$$

When the front axle is lifted $H = 540$ mm, assume that

$$\begin{array}{ll} a_1 & = a_2 \\ F_{z_2} & = 0.68mg. \end{array}$$

What is the height h of the mass center?

7. A parked car on an uphill road.

Specifications of Lamborghini GallardoTM are

$$\begin{array}{ll} m & = 1430 \text{ kg} \\ l & = 2560 \text{ mm}. \end{array}$$

Assume

$$\begin{array}{ll} a_1 & = a_2 \\ h & = 520 \text{ mm} \end{array}$$

and determine the forces F_{z_1} , F_{z_2} , and F_{x_2} if the car is parked on an uphill with $\phi = 30$ deg and the hand brake is connected to the rear wheels.

What would be the maximum road grade ϕ_M , that the car can be parked, if $\mu_{x_2} = 1$.

8. Parked on an uphill road.

Rolls-Royce PhantomTM has the following specifications

$$\begin{array}{ll} m & = 2495 \text{ kg} \\ l & = 3570 \text{ mm} \\ F_{z_2} & = 0.499mg. \end{array}$$

Assume the car is parked on an uphill road and

$$\begin{array}{ll} a_1 & = a_2 \\ h & = 670 \text{ mm} \\ \phi & = 30 \text{ deg}. \end{array}$$

Determine the forces under the wheels if the car is

- (a) front wheel braking
- (b) rear wheel braking
- (c) four wheel braking.

9. A parked car on an downhill road.

Solve Exercise 7 if the car is parked on a downhill road.

10. Maximum acceleration.

Honda CR-VTM is a midsize SUV car with the following specifications.

$$\begin{aligned} m &= 1550 \text{ kg} \\ l &= 2620 \text{ mm} \end{aligned}$$

Assume

$$\begin{aligned} a_1 &= a_2 \\ h &= 720 \text{ mm} \\ \mu_x &= 0.8 \end{aligned}$$

and determine the maximum acceleration of the car if

- (a) the car is rear-wheel drive
- (b) the car is front-wheel drive
- (c) the car is four-wheel drive.

11. Minimum time for 0 – 100 km/h.

RoadRazerTM is a light weight rear-wheel drive sportscar with

$$\begin{aligned} m &= 300 \text{ kg} \\ l &= 2286 \text{ mm} \\ h &= 260 \text{ mm.} \end{aligned}$$

Assume $a_1 = a_2$. If the car can reach the speed 0 – 100 km/h in $t = 3.2$ s, what would be the minimum friction coefficient?

12. Axle load of an all-wheel drive car.

Acura CourageTM is an all-wheel drive car with

$$\begin{aligned} m &= 2058.9 \text{ kg} \\ l &= 2750.8 \text{ mm.} \end{aligned}$$

Assume $a_1 = a_2$ and $h = 760$ mm. Determine the axles load if the car is accelerating at $a = 1.7$ m/s².

13. A car with a trailer.

Volkswagen TouaregTM is an all-wheel drive car with

$$\begin{aligned} m &= 2268 \text{ kg} \\ l &= 2855 \text{ mm.} \end{aligned}$$

Assume $a_1 = a_2$ and the car is pulling a trailer with

$$\begin{aligned} m_t &= 600 \text{ kg} \\ b_1 &= 855 \text{ mm} \\ b_2 &= 1350 \text{ mm} \\ b_3 &= 150 \text{ mm} \\ h_1 &= h_2. \end{aligned}$$

If the car is accelerating on a level road with acceleration $a = 2 \text{ m/s}^2$, what would be the forces at the hinge.

14. A parked car on a banked road.

Cadillac EscaladeTM is a SUV car with

$$\begin{aligned} m &= 2569.6 \text{ kg} \\ l &= 2946.4 \text{ mm} \\ w_f &= 1732.3 \text{ mm} \\ w_r &= 1701.8 \text{ mm.} \end{aligned}$$

Assume $b_1 = b_2$, $h = 940 \text{ mm}$, and use an average track to determine the wheels load when the car is parked on a banked road with $\phi = 12 \text{ deg}$.

15. ★ A parked car on a banked road with $w_f \neq w_r$.

Determine the wheels load of a parked car on a banked road, if the front and rear tracks of the car are different.

16. Optimal traction force.

Mitsubishi OutlanderTM is an all-wheel drive SUV car with the following specifications.

$$\begin{aligned} m &= 1599.8 \text{ kg} \\ l &= 2669.6 \text{ mm} \\ w &= 1539.3 \text{ mm.} \end{aligned}$$

Assume

$$\begin{aligned} a_1 &= a_2 \\ h &= 760 \text{ mm} \\ \mu_x &= 0.75 \end{aligned}$$

and find the optimal traction force ratio F_{x_1}/F_{x_2} to reach the maximum acceleration.

17. ★ A three-axle car.

Citroën Cruise CrosserTM is a three-axle off-road pick-up car. Assume

$$\begin{aligned}m &= 1800 \text{ kg} \\a_1 &= 1100 \text{ mm} \\a_2 &= 1240 \text{ mm} \\a_3 &= 1500 \text{ mm} \\k_1 &= 12800 \text{ N/m} \\k_2 &= 14000 \text{ N/m} \\k_3 &= 14000 \text{ N/m}\end{aligned}$$

and find the axles load on a level road when the car is moving with no acceleration.

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