CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester A, 2014/2015

Time allowed : Two hours

This paper has **SEVEN** pages (including this cover page).

A brief table of derivatives is attached on pages 6 and 7.

Instructions to candidates:

1. This paper has **TEN** questions.
2. Attempt **ALL** questions in Section A and B.
3. Each question in Section A carries 9 marks.
4. Each question in Section B carries 15 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Section A
Answer **ALL** questions in this section. Each question carries 9 marks.

Question 1

Differentiate the following functions with respect to x:

(a) $(7-5x)^4+\dfrac{3}{1+4x^2}$;

(3 marks)

(b) $\sinh^2 x-\cosh^2 x$;

(3 marks)

(c) $\dfrac{(2x+3)^6}{(x^2+1)e^{4x}}$.

(3 marks)

(Your results may be left in an unsimplified form.)

Question 2

(a) Given that $y=e^{\sqrt{3}x}\cos x$, express $\dfrac{dy}{dx}$ in the form $r\,e^{\sqrt{3}x}\cos(x+\phi)$, where r>0 and $0<\phi<\dfrac{\pi}{2}$, and state the numerical value of r and ϕ. Express $\dfrac{d^2y}{dx^2}$ in similar form.
(Hint: $\cos(A+B)=\cos A\cos B-\sin A\sin B$.)

(4 marks)

(b) A curve has parametric equations

$$x=2t-\log_e(2t) \ ,$$
$$y=t^2-\log_e(t^2) \ ,$$

where t is the parameter and $t>0$.

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t .

(5 marks)

Question 3

(a) Prove from first principles that $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$ and deduce by implicit differentiation that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$.

(Hint: $\cos A - \cos B = -2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$.)

(5 marks)

(b) Let $T_n(x) = \cos\left(n\cos^{-1} x\right)$, for $x \in [-1,1]$, where n is a non-negative integer.

Show that $\left(1-x^2\right)\frac{\mathrm{d}^2 T_n(x)}{\mathrm{d}x^2} - x\frac{\mathrm{d}T_n(x)}{\mathrm{d}x} + n^2 T_n(x) = 0$.

(4 marks)

Question 4

The displacement, at time t seconds, of a particle moving along the y-axis is given by $y = p\cos(qt)$ metres, where p and q are positive constants.

Find the velocity and acceleration of the particle at time t seconds, and show that $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + q^2 y = 0$.

Find the maximum velocity and the maximum acceleration, stating, in each case, the first time t seconds for which it occurs.

(9 marks)

Question 5

(a) Let

$$f(x) = \begin{cases} (x-1)^2 & , \ \textit{if} \ x < 3 \\ c & , \ \textit{if} \ x = 3 \\ 4x-8 & , \ \textit{if} \ x > 3 \end{cases} .$$

(i) Evaluate the limit $\lim_{x \to 3} f(x)$.

(ii) Find the value of c for which $f(x)$ is continuous at $x = 3$. Give your reason.

(5 marks)

(b) Let $g(x) = x^{\frac{2}{3}}$ for $x \in \mathrm{R}$. Determine whether $g(x)$ is differentiable at $x = 0$. Give your reason.

(4 marks)

Question 6

Express $\dfrac{9x+64}{(5x-2)(x^2-x+7)}$ in partial fractions.

(9 marks)

Question 7

Two functions are defined on the domain $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ by

$$F(x)=\sin x \quad \text{and} \quad G(x)=\cos x \ .$$

Explain why one of these functions has an inverse while the other does not.

When the domain is restricted to $\left[0,\dfrac{\pi}{2}\right]$, calculate $(F\circ G^{-1})(\frac{1}{2})$.

(9 marks)

Question 8

Use the derivative test, find the coordinates of the local extremal points and the points of inflexion of the curve $y=2x^6-3x^4$.

(9 marks)

Section B
Answer **ALL** questions in this section. Each question carries 15 marks.

Question 9

(a) Starting from the formula $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$,
prove that $\tan 4\theta=\dfrac{4t(1-t^2)}{1-6t^2+t^4}$, where $t=\tan\theta$.
Deduce that, when $\theta=\tan^{-1}(\frac{1}{5})$, $\tan(4\theta-\frac{\pi}{4})=\frac{1}{239}$.

(5 marks)

(b) If $y=\tan^{-1}x$, prove that $(1+x^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$. By repeated differentiation of this result and use the Maclaurin theorem, or otherwise, prove that the first three non-zero terms in the series expansion of $\tan^{-1}x$ are $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}+\cdots\cdots$.

(7 marks)

(c) Using the results in parts (a) and (b), find an approximation to the value of π , giving 5 decimal places in your answer.

(3 marks)

Question 10

(a) Show that the equation $9x^2+16y^2-36x-32y-92=0$ represents an ellipse whose centre is at the point $C(2,1)$.
(Hint: You may use the method of completing the square.)

(2 marks)

(b) Find the value of the eccentricity, e .
(Hint: The eccentricity of the ellipse $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, $a>b>0$ is given by $b^2=a^2(1-e^2)$.)

(2 marks)

(c) Find the coordinates of any intersections of the curve with the axes and the coordinates of the vertices of the ellipse.

(4 marks)

(d) Sketch the graph of the ellipse.

(2 marks)

(e) Find the equation of the tangent to the ellipse at the point $P(4,1+\dfrac{3\sqrt{3}}{2})$.

(5 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	**Derivative of y with respect to x**
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d\,f(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	**Derivative of** y **with respect to** x
$y = \cot^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$
$y = \sec^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \sinh u$	$\dfrac{dy}{dx} = \cosh u\dfrac{du}{dx}$
$y = \cosh u$	$\dfrac{dy}{dx} = \sinh u\dfrac{du}{dx}$
$y = \tanh u$	$\dfrac{dy}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$
$y = \coth u$	$\dfrac{dy}{dx} = -\operatorname{cosech}^2 u\dfrac{du}{dx}$
$y = \operatorname{sech} u$	$\dfrac{dy}{dx} = -\operatorname{sech} u \tanh u\dfrac{du}{dx}$
$y = \operatorname{cosech} u$	$\dfrac{dy}{dx} = -\operatorname{cosech} u \coth u\dfrac{du}{dx}$
$y = \sinh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$
$y = \cosh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \tanh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$
$y = \coth^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{\lvert u\rvert\sqrt{u^2+1}}\dfrac{du}{dx}$

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