2 Linear Time-Invariant Systems

Major References:

- *Chapter 2, Signals and Systems by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall*
- *Chapter 2, Schaum's Outline of Signals and Systems, 2nd Edition, 2010, McGraw-Hill*

2.1 Convolution

2.1.1 Convolution Integral of CT Signal

1. **Definition**
   Convolution Integral of two continuous-time signals $x(t)$ and $y(t)$ is defined by

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau. \tag{2.1}$$

   Convolution $x(t) * y(t)$ represents the degree to which $x\&y$ overlap at t as y sweeps across the domain t.

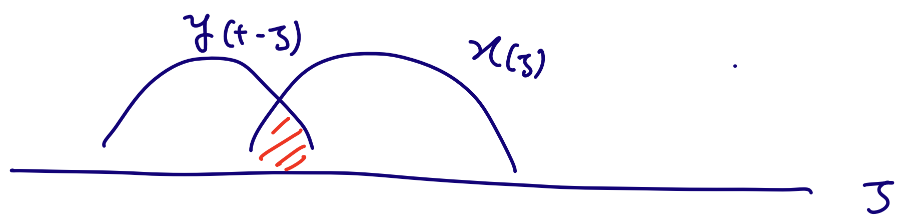
   Step. 1) $y(\tau)$ is time-reversed, then shifted by t; $y(\tau) \to y(-\tau) \to y(t-\tau)$
   Step. 2) $x(\tau)$ and $y(t-\tau)$ are multiplied, then integrated over τ
   Step. 3) Convolution will remain zero as long as $x\&y$ do not overlap
   Step. 4) Sweep $y(t-\tau)$ from $t=-\infty$ to $t=\infty$ to produce the entire output

2. **Properties of the Convolution Integral**
   The convolution integral has the following properties. Refer [Schaum's text, Problem 2.1] for the proof.
   a) Commutative
$$x(t) * y(t) = y(t) * x(t)$$
   b) Associative
$$\{x(t) * y_1(t)\} * y_2(t) = x(t) * \{y_1(t) * y_2(t)\}$$
   c) Distributive
$$x(t) * \{y_1(t) + y_2(t)\} = x(t) * y_1(t) + x(t) * y_2(t)$$

3. **Additional Properties**
   Refer [Schaum's text, Problem 2.2, 2.8] for the proof.
   a) $x(t) * \delta(t) = x(t)$
   b) $x(t) * \delta(t-t_0) = x(t-t_0)$
   c) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$
   d) $x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau) d\tau$
   e) If $x(t)$ and $y(t)$ are periodic signals with a common period T, the convolution in (2.1) does not converge. Instead, we define the *periodic convolution* $f(t) = x(t) \circledast y(t)$, where $f(t)$ is periodic with period T.

$$\begin{aligned} f(t) = x(t) \circledast y(t) &= \int_{0}^{T} x(\tau)\, y(t-\tau)\, d\tau \\ &= \int_{a}^{a+T} x(\tau)\, y(t-\tau)\, d\tau \quad \text{for arbitrary } a \end{aligned} \tag{2.2}$$

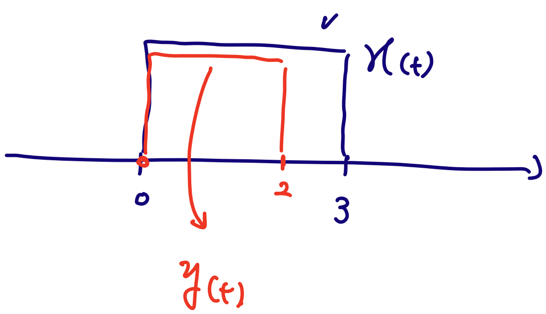
$$\boxed{\begin{aligned} z(t) &= x(t) * y(t) \\ &= \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \end{aligned}}$$

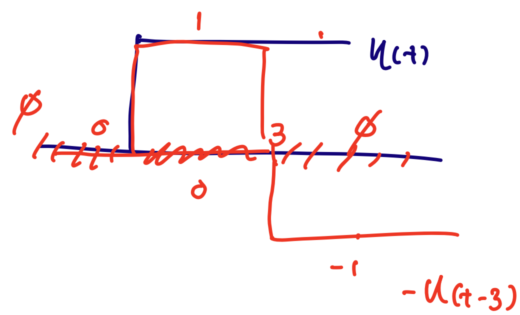


Ex 2-1)

4). $x(t) * y(t)$

$$\boxed{\begin{aligned} &u(t+a) * u(t+b) \\ &= (t+a+b)\, u(t+a+b) \end{aligned}}$$





$$\Rightarrow x(t) * y(t) = \big(u(t) - u(t-3)\big) * \big(u(t) - u(t-2)\big)$$

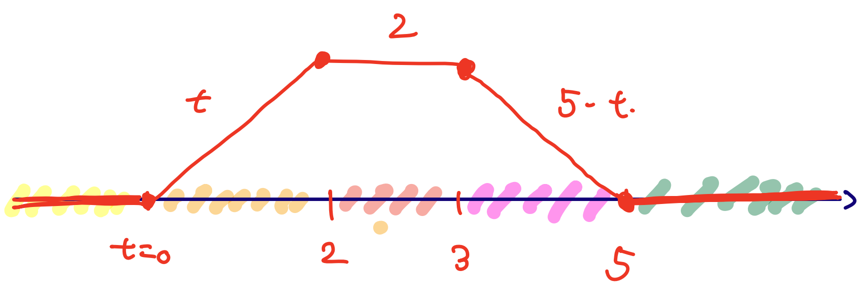
$$= u(t) * u(t) \xrightarrow{a=b=0} t\, u(t)$$

$$- u(t) * u(t-2) \xrightarrow{a=0,\, b=-2} -(t-2)\, u(t-2)$$

$$- u(t-3) * u(t) \xrightarrow{a=-3,\, b=0} -(t-3)\, u(t-3)$$

$$+ u(t-3) * u(t-2) \xrightarrow[b=-2]{a=-3,} (t-5)\, u(t-5)$$

$$\Rightarrow x(t) * y(t) = \underline{t\,u(t) - (t-2)\,u(t-2) - (t-3)\,u(t-3) + (t-5)\,u(t-5)}$$



$t<0$, ϕ.

$t>5$, $\cancel{t}\cdot 1 - (\cancel{t}-\cancel{2})\cdot 1 - (\cancel{t}-\cancel{3})\cdot 1 + (\cancel{t}-\cancel{5})\cdot 1$.

$= \phi$

$0<t<2 \Rightarrow t\cdot 1 - \phi = t.$

$2<t<3 \Rightarrow t - (t-2) = 2$

$3<t<5 \Rightarrow \underline{t - (t-2)} - (t-3).$

$= 2 - (t-3) = 5 - t.$

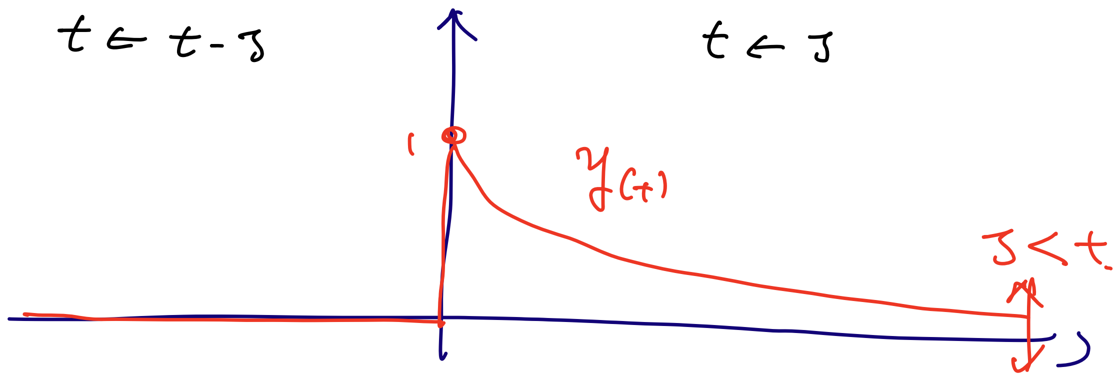
Ex 2-2)

$$= \boxed{\int_{-\infty}^{\infty} y(\tau)\, x(t-\tau)\, d\tau} \checkmark$$

1. $$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$x(t) = u(t)$, $\quad y(t) = e^{-\alpha t} u(t)$, $\alpha > 0$

$t \leftarrow t-\tau$ $\qquad\qquad$ $t \leftarrow \tau$



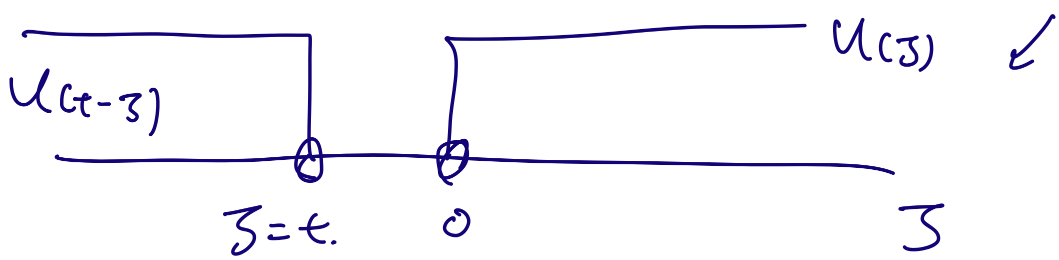
$g(\tau)$ $\quad t-\tau>0$

$$x(t) * y(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}\, \boxed{u(\tau)\, u(t-\tau)}\, d\tau$$

$\tau>0$, $\quad t-\tau>0$

$\Leftrightarrow \quad \tau < t$

$$g(\tau) = \begin{cases} 1, & 0<\tau<t \quad \text{if } t>0 \\ 0, & \text{if } t \le 0. \end{cases}$$

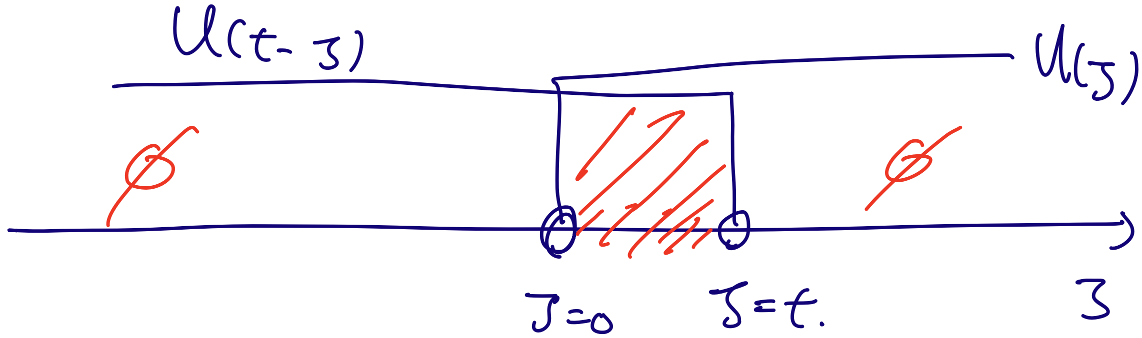


$$\Rightarrow \int_0^t e^{-\alpha\tau} \cdot 1 \cdot d\tau \qquad \text{if } t>0.$$

$$= \frac{1}{\alpha} e^{-\alpha\tau}\Big|_t^0 = \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \cdot \phi \cdot d\tau = \phi \qquad \text{if } t<0.$$

$$\frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) u(t) = x(t) * y(t)$$



$$\underbrace{U(\tau)}_{=1}\cdot\underbrace{U(t-\tau)}_{=1} = 1$$

$t \leftarrow t-\tau$

$$\boxed{x(t) = e^{-\alpha t}\,U(t)}$$

$e^{-\alpha\cdot 0}\,U(0) = 1$

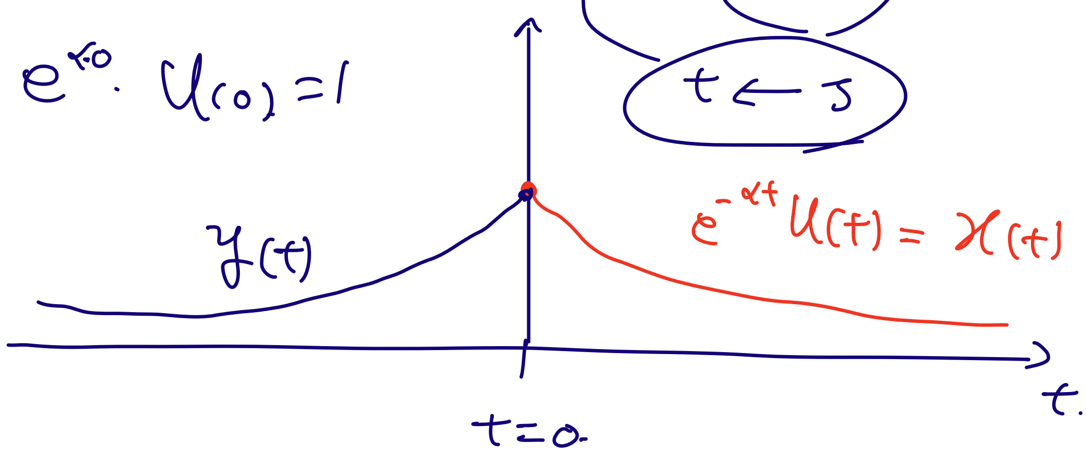
$\alpha > 0$

$$y(t) = \boxed{e^{\alpha t}\,U(-t)}$$

$-t > 0$

$t \leftarrow \tau$

$e^{\alpha\cdot 0}\cdot U(0) = 1$



$x(t) * y(t)$

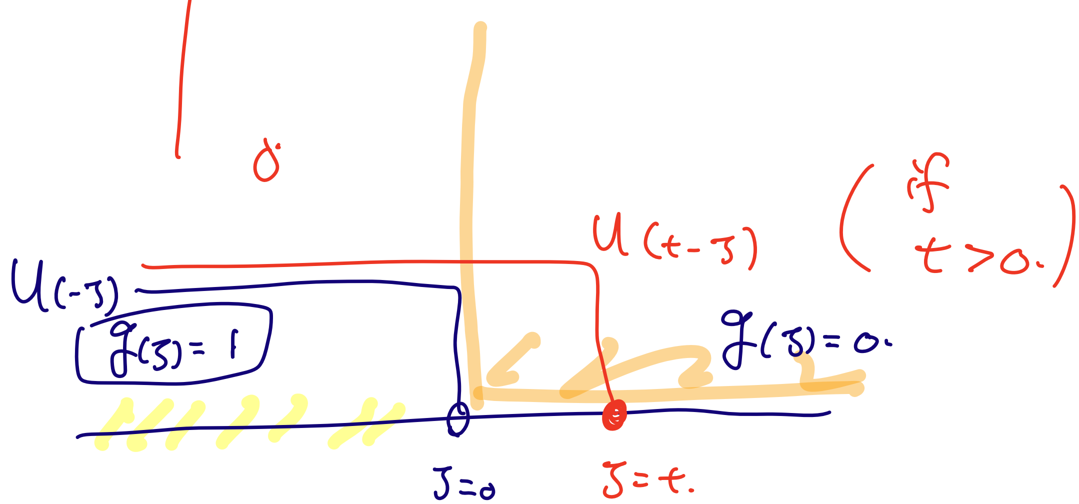
$$= \int_{-\infty}^{\infty} \left(e^{\alpha\zeta} U(-\zeta)\right) \left(e^{-\alpha(t-\zeta)} U(t-\zeta)\right) d\zeta$$

$$\exp(\alpha\zeta - \alpha t + \alpha\zeta) = e^{2\alpha\zeta} \cdot \boxed{e^{-\alpha t}}$$

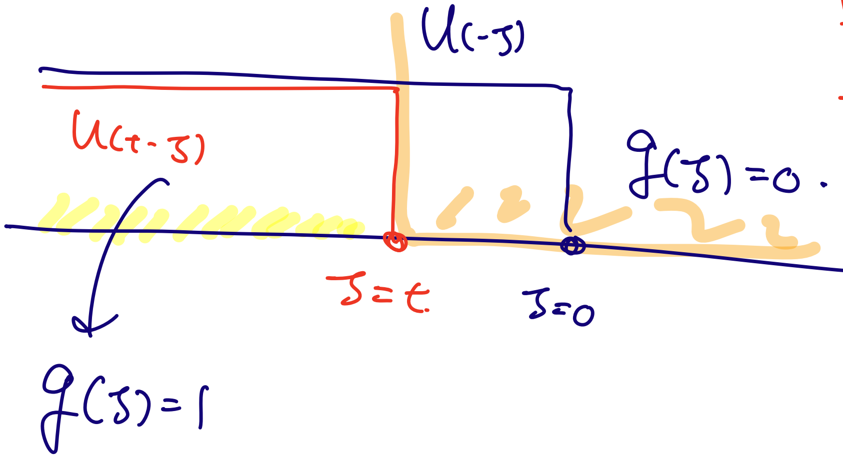
$\boxed{\zeta < 0}$ $\quad -\zeta > 0$ $\qquad \boxed{\zeta < t}$ $\quad t - \zeta > 0$

$$= e^{-\alpha t} \int_{-\infty}^{\infty} e^{2\alpha\zeta} \cdot \boxed{\underbrace{U(-\zeta)\, U(t-\zeta)}_{= g(\zeta)}}\, d\zeta \quad (1)$$

$$g(\zeta) = \begin{cases} 1 \\ \\ 0 \end{cases}$$



if $t<0$



if $t>0$.

$$① = e^{-\alpha t}\int_{-\infty}^{0} e^{2\alpha \tau}\, d\tau$$

$$= \boxed{e^{-\alpha t}\cdot \boxed{\frac{1}{2\alpha}}}$$

if $t<0$.

$$① = e^{-\alpha t}\int_{-\infty}^{t} e^{2\alpha \tau}\, d\tau$$

$$= e^{-\alpha t}\cdot \frac{1}{2\alpha} e^{2\alpha t}$$

$$= \boxed{\boxed{\frac{1}{2\alpha}} e^{-\alpha t}}$$

$$\Rightarrow x(t) * y(t) = \frac{1}{2\alpha} e^{-\alpha|t|}.$$