

1 Set Notation

A set is a collection of distinct objects called *elements* or *members* of that set. For example, $A = \{1, 2, 3, 4, 5\}$ is a set and a list of all its elements is given. In general, we use the notation $\{x/x \text{ possesses certain properties}\}$ to denote a set of objects that share some common properties. Also, if e is an element of a set A , we write $e \in A$ (read as *e belongs to A*).

Illustration

Let V be the set of all vowels of the English alphabets, then

$$V = \{a, e, i, o, u\}$$

u is an element of the set V . However, p is NOT an element of the set V .

We can use $u \in V$ (read as *u belongs to V*) to show that u is an element of V , and use $p \notin V$ to show that p is NOT an element of V .

Some notations of the sets commonly used in Mathematics:

Z the set of all integers (the set that contains all integers), i.e. $\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$

R the set of all real numbers (the set that contains all real numbers)

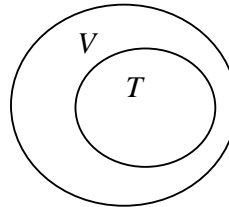
ϕ (called a *null set* or *empty set*) a set that contains no element

Two sets are equal if they contain the same elements.

e.g. If $A = \{3, 5, 7\}$ and $B = \{3, 5, 7\}$, then we can write $A = B$.

The relationships among sets can be conveniently illustrated by *Venn diagrams*.

e.g. Let $V = \{a, e, i, o, u\}$ and $T = \{a, u\}$. The figure on the right shows the Venn diagram:



Subset

Given two sets A and B , we say that A is a *subset* of B (denoted by $A \subset B$) if all elements of A belong to B . In the above case, T is a subset of V and therefore we can write $T \subset V$. For example, we can write $\mathbf{Z} \subset \mathbf{R}$ to indicate that the set of all integers is a subset of the set of all real numbers.

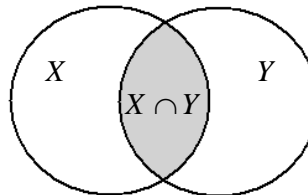
Operations of Sets

It is often necessary to combine two or more sets to form new sets. This is done by *set operations*.

(a) Intersection

The *intersection* of two sets X and Y is a set whose elements belong to both X and Y . It is denoted by $X \cap Y$.

e.g. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $Y = \{5, 6, 7, 8, 9, 10, 11, 12\}$. We see that the elements 5, 6, 7, 8 belong to both X and Y . Therefore, $X \cap Y = \{5, 6, 7, 8\}$. The figure on the right shows the Venn diagram:



The shaded part represents $X \cap Y$.

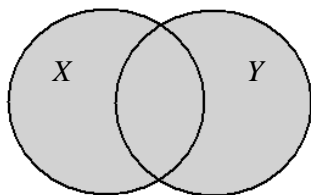
e.g. For the sets $A = \{1, 2, 3, 4\}$ and $B = \{9, 10, 11, 12\}$, no object belongs to both A and B . Therefore, $A \cap B = \phi$ (empty set).

(b) Union

The *union* of two sets X and Y is a set whose elements belong to either X or Y or both of them. It is denoted by $X \cup Y$.

e.g. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $Y = \{5, 6, 7, 8, 9, 10, 11, 12\}$.

Then $X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.



The shaded part represents $X \cup Y$.

Question: Let $A = \{2, 4, 6, 8\}$ and $B = \{-3, 6, 8, 12, 4\}$. Write the set described by each of the following. List all the elements in the set.

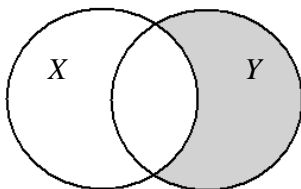
- (i) $A \cup B$ (ii) $A \cap B$ (iii) $B \cap \mathbf{Z}$ (iv) $B \cap \mathbf{R}$

(c) Complements

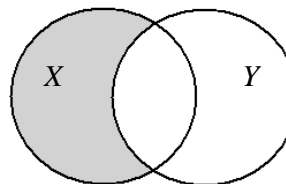
The *complement* of X with respect to Y is a set whose elements belong to Y but not belong to X . It is denoted by $Y \setminus X$.

e.g. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $Y = \{5, 6, 7, 8, 9, 10, 11, 12\}$.

Then $Y \setminus X = \{9, 10, 11, 12\}$ and $X \setminus Y = \{1, 2, 3, 4\}$.



The shaded part represents $Y \setminus X$.



The shaded part represents $X \setminus Y$.

2 Intervals

We can also use the notation $\{x \mid x \text{ processes certain properties}\}$ to denote a set of objects that share some common properties. Sets with infinitely many elements are often denoted by this method.

e.g. $\{x \mid x \text{ is the outcome of throwing a die}\}$ is the set $\{1, 2, 3, 4, 5, 6\}$.

$\{x \mid x \text{ is a prime number}\}$ is the set that contains all prime numbers.

$\{x \mid x > 0 \text{ and } x \text{ is divisible by } 3\}$ is the set $\{3, 6, 9, 12, 15, 18, \dots\}$.

$\{x \mid x = 3m \text{ and } m \in \mathbf{Z}\}$ is the set that contains all multiples of 3.

$\mathbf{Z} = \{x \mid x \text{ is an integer}\}$

$\{x \mid x \text{ is a real number and } 3 < x < 7\}$ is the set of real numbers which are smaller than 7 and greater than 3.

As mentioned, we use the symbol \mathbf{R} to denote the set which contains exactly all the real numbers. Also, the following symbols are frequently used to describe the corresponding subsets of real numbers (a, b are two distinct real numbers):

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$$

$$[a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}$$

$$[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$$

$$[a, \infty) = \{x \in \mathbf{R} \mid x \geq a\}$$

$$(-\infty, a) = \{x \in \mathbf{R} \mid x < a\}$$

(The other subsets like $(a, b], (a, \infty), (-\infty, a], (-\infty, \infty)$ are defined similarly). These sets are usually called *intervals*. In our discussion, most of the sets we consider are intervals.

Example 1

Use set notations to represent each of the following sets.

(a) The set of integers which are smaller than -6 and greater than -13.

(b) The set of integers which are greater than 2 and smaller than 30.

Solutions

(a) $\{-12, -11, -10, -9, -8, -7\}$ or $\{x \mid x \in \mathbf{Z} \text{ and } -13 < x < -6\}$ or $\{x \in \mathbf{Z} \mid -13 < x < -6\}$

(b) $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$
or $\{x \mid x \in \mathbf{Z} \text{ and } 2 < x < 30\}$ or $\{x \in \mathbf{Z} \mid 2 < x < 30\}$

Example 2

Use bounded intervals to represent each of the following sets.

(a) The set of real numbers which are greater than -3 and are smaller than or equal to 6.

(b) The set of real numbers which are smaller than -6.

(c) $[2, 8] \cap (3, 10)$

(d) $[2, 8] \cup (3, 10)$

Solutions

(a) $[-3, 6]$ (it represents the set $\{x \mid x \in \mathbf{R} \text{ and } -3 < x \leq 6\}$, i.e. $\{x \in \mathbf{R} \mid -3 < x \leq 6\}$.)

(b) $(-\infty, -6)$ (it represents the set $\{x \mid x \in \mathbf{R} \text{ and } x < -6\}$, i.e. $\{x \in \mathbf{R} \mid x < -6\}$.)

(c) $(3, 8]$

(d) $[2, 10)$

Question: Use bounded intervals to represent each of the following sets.

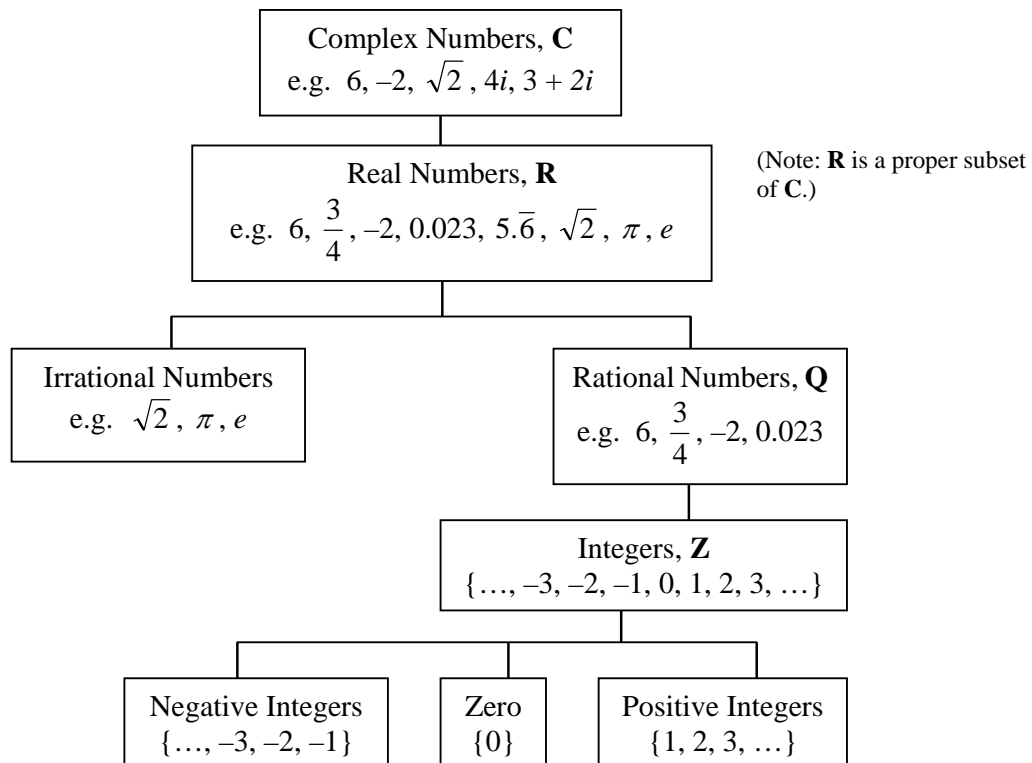
- (i) $[-2, 3] \cup (3, \infty)$ (ii) $(-\infty, 6] \cap (3, \infty)$ (iii) $(-\infty, 6] \cup (3, \infty)$

The Number System

The following presents the description of some types of real numbers:

- *Natural Numbers* are the numbers 1, 2, 3,
- All natural numbers, together with 0, -1, -2, forms the set of *integers*. $\{1, 2, 3, 4, \dots\}$ are the set of positive integers (also called the set of natural numbers) and $\{\dots, -3, -2, -1\}$ are the set of negative integers. 0 is neither positive nor negative.
- *Rational numbers* are numbers that can be represented in the form $\frac{p}{q}$, where q is non-zero and p and q are both integers. In particular, all integers are rational (pick $q = 1$). Other examples of rational numbers are: $\frac{3}{2}, -\frac{5}{7}, -\frac{343}{11}$.
- *Irrational numbers* are real numbers with non-repeating decimals. Examples are:
 $e = 2.71828\dots$ (this special number is called the natural number)
 $\pi = 3.14159\dots$ (pi, the ratio of a circle's perimeter to its diameter)
 $\sqrt{2} = 1.4142\dots$

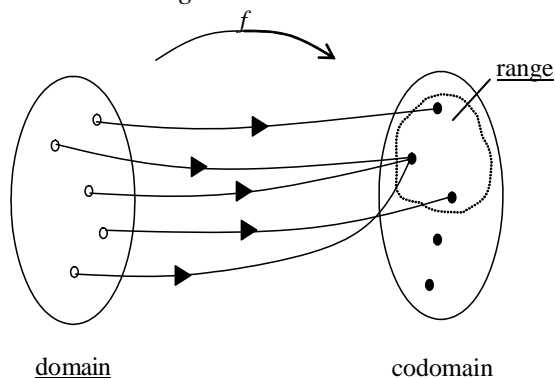
The following tree diagram represents the relations between different types of numbers:



3 Functions of a single real variable (p.157 – p.158, p.173 – p.176, p.193 – p.197)

A. Definition of a Function

A function f is a rule of correspondence that associates with each object x in one set A (called the *domain* of f) a unique (exactly one) value $f(x)$ from a second set B (called the *codomain* of f). The set of values so obtained is called the *range* of the function.

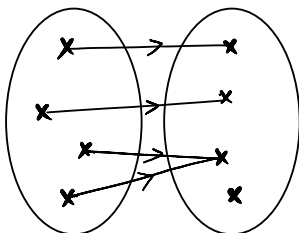


It is customary to write f as:

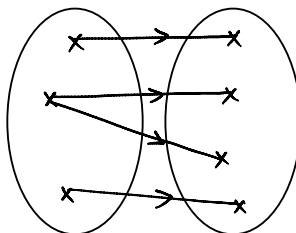
$$f : A \rightarrow B$$

In this course, we will mainly study those functions whose domains and codomains are subsets of \mathbf{R} , the set of real numbers. Moreover, when the rule for a function is given by an equation of the form $y = f(x)$ (for example: $y = x^2 + 1$), x is often called the *independent variable* and y the *dependent variable*.

The following diagram shows the interpretation:



A well-defined function



Not a function

The following equations define y as a function of x (where x is any real numbers):

$$y = x^2 - 5x + 1, \quad y = -x^3 + x^2 - 3x + 2, \quad x + y = 4$$

$$y = \sin x, \quad y = \cos x, \quad y = \tan x \quad (\text{These are examples of } \textit{Trigonometric Functions}.)$$

$y = e^x, \quad y = 10^x, \quad y = \ln x$ (for $x > 0$), $y = \log x$ (for $x > 0$) (More properties of these functions will be discussed later.)

The following equations do not define y as a function of x (where x is any real numbers). (Why?)

$$x + y^2 = 4, \quad x^2 + y^2 = 9$$

Question: If x is any real numbers, does $y = \sqrt{x}$ define y as a function of x ?

How about if x is any non-negative real numbers?

Sometimes, we write $f(x) = x^2 - 5x + 1$ instead of $y = x^2 - 5x + 1$ to better indicate the relation that the value of y depends on the inputting value of x .

Example 3

It is given the function $f(x) = ax^2 + 3x + c$, where a and c are constants. If $f(0) = 5$ and $f(3) = 32$, find the values of a and c .

Solution

$$f(0) = 5 \quad a(0)^2 + 3(0) + c = 5 \quad \therefore c = 5$$

$$f(3) = 32 \quad a(3)^2 + 3(3) + 5 = 32 \quad \therefore a = 2$$

Example 4

Determine the largest possible domain and the largest possible range for each of the following functions.

$$(a) \ f(x) = x^2 \quad (b) \ f(x) = 25 - x \quad (c) \ f(x) = \sqrt{x+4} \quad (d) \ f(x) = 3 + \frac{1}{x-5}$$

Solutions

(a) For the function $f(x) = x^2$, it is well-defined for any real numbers x . Therefore, the largest possible domain is the set of all real numbers, i.e. \mathbf{R} .

For any x , $x^2 \geq 0$. Therefore, the largest possible range is the set of all non-negative real numbers, i.e. $[0, \infty)$.

(b) For the function $f(x) = 25 - x$, it is well-defined for any real numbers x . Therefore, the largest possible domain is the set of all real numbers, i.e. \mathbf{R} .

For any x , $25 - x$ is a real number which can be any number in \mathbf{R} . Therefore, the largest possible range is \mathbf{R} .

(c) For the function $f(x) = \sqrt{x+4}$, it is well-defined as long as $x+4 \geq 0$, i.e. $x \geq -4$.

\therefore The largest possible domain is $[-4, \infty)$.

For any x , $\sqrt{x+4}$ is a non-negative real number.

\therefore The largest possible range is $[0, \infty)$.

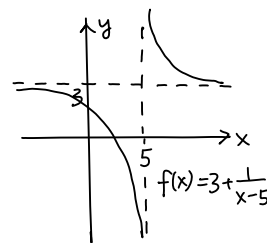
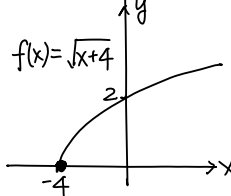
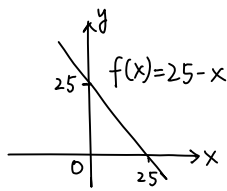
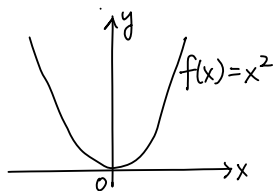
(d) For the function $f(x) = 3 + \frac{1}{x-5}$, it is well-defined as long as $x-5 \neq 0$, i.e. $x \neq 5$.

\therefore The largest possible domain is $\mathbf{R} \setminus \{5\}$.

For any x , $3 + \frac{1}{x-5}$ is a real number EXCEPT 3.

\therefore The largest possible range is $\mathbf{R} \setminus \{3\}$.

Remark: The following show the graphs of the above functions.



Question:

Determine the largest possible domain and largest possible range for the function $f(x) = 5 + \sin x$.

B. Operations on functions

Consider the functions f and g with formulas $f(x) = \frac{x^2 - 1}{2}$ and $g(x) = x^9 - x + 1$.

We can make a new function of $f + g$, where

$$(f + g)(x) = f(x) + g(x) = \frac{x^2 - 1}{2} + x^9 - x + 1.$$

Clearly, x must be a number which belongs to both the domains of f and g . Similarly, we can define the functions $f - g$, fg and f / g as follows:

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x) \times g(x)$$

$$(f / g)(x) = \frac{f(x)}{g(x)} \text{ (defined only for those } x \text{ with } g(x) \neq 0)$$

C. Composition of functions

Let $f : A \rightarrow B$, $g : B \rightarrow C$ be two functions. We define the *composite* of g with f by

$$(g \circ f)(x) = g(f(x)).$$

Note that the domain of this function $g \circ f$ is A (and its codomain is C). In general $g \circ f$ and $f \circ g$ (if both are defined) are two different functions.

Example 5

Let $f : \mathbf{R} \rightarrow \mathbf{R}$, $f(x) = x^2$ and $g : \mathbf{R} \rightarrow \mathbf{R}$, $g(x) = x + 1$. Find (a) $f \circ g$, (b) $g \circ f$.

Solution

$$(a) \quad f \circ g : \mathbf{R} \rightarrow \mathbf{R} \text{ is} \quad (f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2.$$

$$(b) \quad g \circ f : \mathbf{R} \rightarrow \mathbf{R} \text{ is} \quad (g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1. \quad \text{Note that } g \circ f \neq f \circ g.$$

4 Elementary Functions

In this section we will introduce different types of functions that are frequently used.

A. Some typical examples of functions (p.169, p.182 – p.187, p.206 – p.209)

A function of the form $f(x) = k$ where k is a fixed real number, is called a *constant function*.

The function $f(x) = x$ is called the *identity function*.

A function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ (where the a_i 's are real numbers and n is a non-negative integer) is called a *polynomial function*. If $a_n \neq 0$, n is the *degree* of the polynomial function. In particular,

$f(x) = ax + b$ ($a \neq 0$) is called a linear function

$f(x) = ax^2 + bx + c$ ($a \neq 0$) is called a quadratic function.

Quotients of polynomial functions are called *rational functions*. That is, f is a rational function if it is of the form

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0},$$

where a_i 's and b_j 's are real numbers and both n and m are non-negative integers.

The trigonometric functions, inverse trigonometric functions, exponential functions and logarithmic functions and their properties will be discussed later. The following trigonometric identities will be discussed later.

Odd-Even Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Addition Identities

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Double-Angle Identities

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

Half-Angle Identities

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

Sum Identities

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

Product Identities

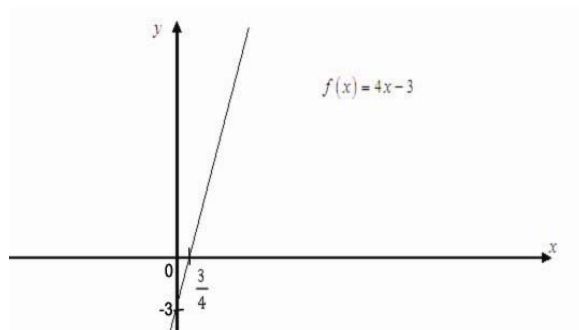
$$\sin x \sin y = -\frac{1}{2} [\cos(x+y) - \cos(x-y)]$$

$$\cos x \cos y = \frac{1}{2} [\cos(x+y) + \cos(x-y)]$$

$$\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

B. Even and Odd Functions (p.178 – p.179, p.188 – p.189)

When both the domain and codomain of a function consist of real numbers, we can picture the function by drawing its graph on a coordinate plane. For example, the graph of the function $f(x) = 4x - 3$ is shown below:



We can often predict the symmetries of the graph of a function by inspecting the formula for the function.

If $f(-x) = f(x)$, then the graph is symmetric with respect to y-axis. Such a function is called an *even function*.

If $f(-x) = -f(x)$, the graph is symmetric with respect to the origin. We call such a function an *odd function*.

Example 6

Determine whether each of the functions are odd or even or neither of them.

(a) $h(x) = x^2$ (b) $f(x) = x^3$ (c) $g(x) = \frac{x^4 + x^2 + 1}{x^2}$ (d) $m(x) = x^5 + x^4 + 5$

Solutions

(a) $h(-x) = (-x)^2 = x^2 = h(x) \quad \therefore$ It is an even function.

(b) $f(-x) = (-x)^3 = -x^3 = -f(x) \quad \therefore$ It is an odd function.

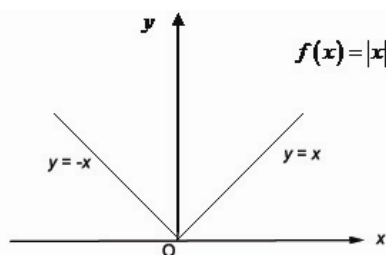
(c) $g(-x) = \frac{(-x)^4 + (-x)^2 + 1}{(-x)^2} = \frac{x^4 + x^2 + 1}{x^2} = g(x) \quad \therefore$ It is an even function.

(d) $m(-x) = (-x)^5 + (-x)^4 + 5 = -x^5 + x^4 + 5$ which is neither $m(x)$ nor $-m(x)$
 \therefore It is neither an odd nor even function.

The *absolute value function*, defined by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is an example of even function.



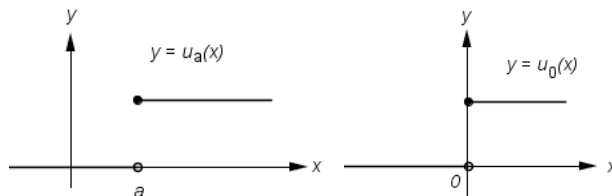
The absolute value function is also an example of **piecewise-defined function**, i.e. a function that is described by using different formulas on different parts of its domain.

The *unit step function* at $x = a$ (where $a \geq 0$), is the function defined as

$$u_a(x) = \begin{cases} 0 & \text{if } x < a, \\ 1 & \text{if } x \geq a. \end{cases}$$

In particular, when $a = 0$, we write $u_0(x)$ as $u(x)$.

The unit step function is also an example of piecewise-defined function.



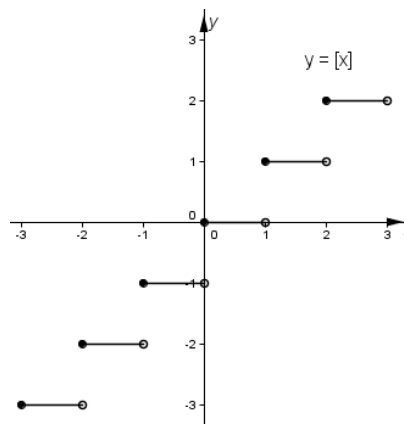
The *greatest integer function* is the function

$$f(x) = [x] = \text{the greatest integer smaller than or equal to } x.$$

For example, $f(3.1) = [3.1] = 3$, $f(2) = [2] = 2$,

$$f(-3.1) = [-3.1] = -4.$$

The greatest integer function is sometimes denoted as $f(x) = \lfloor x \rfloor$.



Question: What is the graph of the *least integer function* defined as

$$f(x) = \lceil x \rceil = \text{the least integer greater than or equal to } x?$$

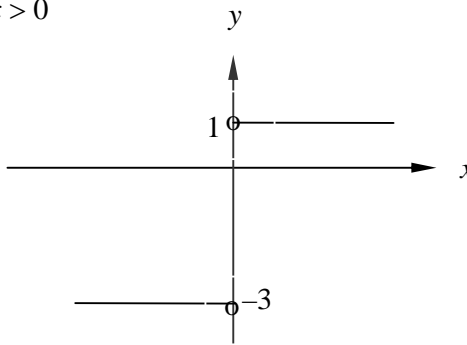
Example 7

Sketch the graph of $y = \frac{2x}{|x|} - 1$ for $x \neq 0$.

Solution

$$\text{Note that } y = \begin{cases} \frac{2x}{|x|} - 1, & x < 0 \\ -x & x < 0 \\ \frac{2x}{|x|} - 1, & x > 0 \\ x & x > 0 \end{cases}, \text{ i.e. } y = \begin{cases} -3, & x < 0 \\ 1, & x > 0 \end{cases}$$

The graph of $y = \frac{2x}{|x|} - 1$ is as follows:



□

C. Periodic functions and Increasing/Decreasing Functions

A function $f(x)$ is *periodic* with *period* T if $f(x+T) = f(x)$ for any x which is contained in the domain of f . If we look at the graph of such f , we will find that the graph of f ‘repeat’ itself periodically.

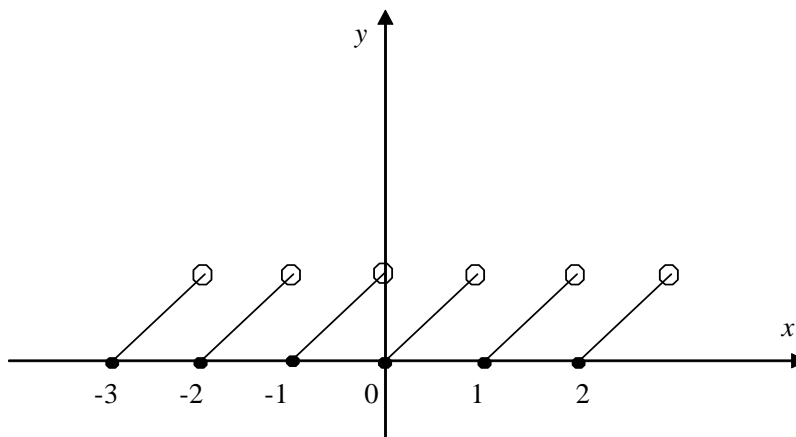
Illustration

The function $\sin x$ is periodic with period 2π since $\sin(x+2\pi) = \sin x$.

The function $f(x) = x - [x]$ is periodic with period 1. It is because

$$f(x+1) = (x+1) - [x+1] = x+1 - [x] - 1 = x - [x] = f(x)$$

for any $x \in \mathbf{R}$. The graph of $f(x)$ is shown below:



□

A function f is said to be *monotonic increasing* (resp. *monotonic decreasing*) if the following condition is satisfied:

$$\begin{aligned} f(x_1) &\geq f(x_2) \text{ whenever } x_1 > x_2 \\ \text{[resp. } f(x_1) &\leq f(x_2) \text{ whenever } x_1 > x_2 \text{]} \end{aligned}$$

Furthermore, if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$, we call this f a *strictly increasing function*. Of course, we can define *strictly decreasing functions* similarly.

Illustration

$f(x) = x^3$ is strictly increasing over \mathbf{R} . For any two given real numbers x_1, x_2 with $x_1 > x_2$, we have

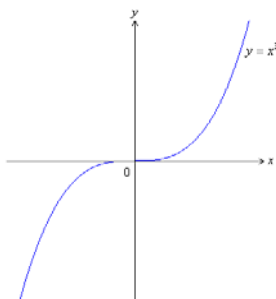
$$f(x_1) - f(x_2) = x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = \frac{1}{2}(x_1 - x_2)[(x_1 + x_2)^2 + x_1^2 + x_2^2] > 0$$

(since $x_1 > x_2$). That is, $f(x_1) > f(x_2)$ whenever $x_1 > x_2$.

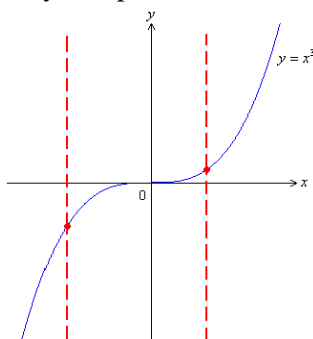
Before ending this section, we want to point out there are functions like $e^x, \sin^{-1} x, \log x, \cosh x \cdots$ etc.

5 Inverse Functions

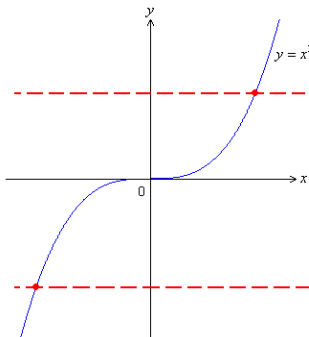
Consider the function $f(x) = x^3$.



$f(x)$ has only one value for each value of x in its domain (that is \mathbf{R}). In geometric terms, any vertical line meets the graph of f at only one point as shown below.



For this function $f(x) = x^3$, any horizontal line also meets the graph at only one point, as shown in the following graph.



This means that different values of x always give different values to $f(x)$. Such a function is said to be one-to-one.

Definition:

A function f is *one-to-one* if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 belong to the domain of f and $x_1 \neq x_2$. Equivalently, if

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

Illustration

$f(x) = x^3$ is one-to-one.

The equation $y = x^3$ has a unique solution x for every given value of y in the range of f .

$$x = y^{\frac{1}{3}}$$

This equation defines x as a function of y . We call this new function the inverse of f and denote it f^{-1} . Thus,

$$f^{-1}(y) = y^{\frac{1}{3}}.$$

Whenever a function f is one-to-one, for any number y in its range there will always exist a single number x in its domain such that $y = f(x)$. Since x is determined uniquely by y , it is a function of y . We write

$$x = f^{-1}(y)$$

and call f^{-1} the *inverse* of f .

We usually like to write functions with the independent variable x rather than y , so we reverse the roles of x and y .

Theorem:

A function f has an inverse if and only if it is one-to-one.

In practice, the inverse function f^{-1} is calculated from f by the following procedure:

- 1) check whether the function $y = f(x)$ is one-to-one,
- 2) solve x in terms of y (if possible),
- 3) rewrite the independent variable as x and the dependent variable as y .

Illustration

Given $f : \mathbf{R} \rightarrow \mathbf{R}$ where $f(x) = 2x - 3$.

The function $f(x)$ is one-to-one.

$$y = 2x - 3$$

$$\Rightarrow x = \frac{y + 3}{2}$$

$$\therefore \text{The inverse function is } f^{-1}(x) = \frac{x + 3}{2}.$$

Illustration

Let $f : \mathbf{R} \rightarrow [0, \infty)$ and $f(x) = x^2$. Then f has no inverse function since it is not one-to-one as it is observed that $f(x)$ takes the same value twice for $x \neq 0$. For example, $f(-1) = 1 = f(1)$.

Illustration

Let $g : [0, \infty) \rightarrow [0, \infty)$ where $g(x) = x^2$. Then g is one-to-one, so it has an inverse $g^{-1}(x) = \sqrt{x}$.

Note: Do not confuse the -1 in f^{-1} with an exponent. The inverse f^{-1} is NOT the reciprocal $\frac{1}{f}$, which can be written as $(f(x))^{-1}$.

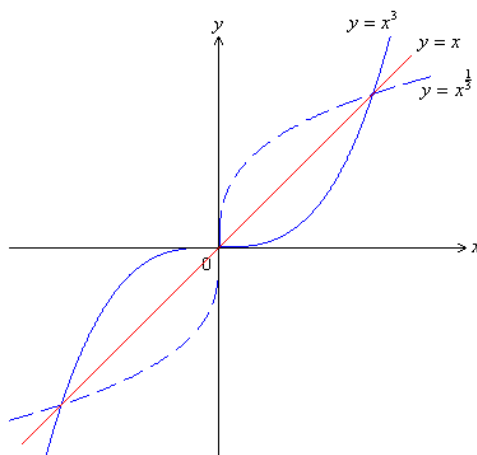
$$\therefore \boxed{f^{-1}(x) \neq (f(x))^{-1}}.$$

Properties of inverse functions

1. $y = f^{-1}(x) \Leftrightarrow x = f(y)$.
2. The domain of f^{-1} is the range of f .
3. The range of f^{-1} is the domain of f .
4. $f^{-1}(f(x)) = x$ for all x in the domain of f .
5. $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} .
6. $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f (i.e. the inverse of f^{-1} is f .)
7. The graph of f^{-1} is the reflection of the graph f in the line $y = x$.

E.g. $f(x) = x^3$

$$f^{-1}(x) = x^{\frac{1}{3}}$$



Some common examples of

1. Let $f : \mathbf{R} \rightarrow [0, \infty)$ and $f(x) = 10^x$. Then the inverse of f is $f^{-1}(x) = \log_{10} x$.
2. Let $g : \mathbf{R} \rightarrow [0, \infty)$ and $g(x) = e^x$. Then the inverse of g is $g^{-1}(x) = \ln x$.
3. Let $h : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$ and $h(x) = \sin x$. Then the inverse of h is $h^{-1}(x) = \sin^{-1} x$.
4. Let $f : [0, \pi] \rightarrow [-1, 1]$ and $f(x) = \cos x$. Then the inverse of f is $f^{-1}(x) = \cos^{-1} x$.
5. Let $g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbf{R}$ and $g(x) = \tan x$. Then the inverse of g is $g^{-1}(x) = \tan^{-1} x$.