MA2001 Solutions Assignment for Chapter 3 Multiple Integrals

1. Change the order of the integration in $\int_0^1\left[\int_0^{2y} f(x,y)dx\right]dy+\int_1^3\left[\int_0^{3-y} f(x,y)dx\right]dy$.

Solution:
Let R be region of the integration. Observe that R is bounded by $x=0,\ x=2y,\ x=3-y$.

$\begin{cases} x=2y \\ x=3-y \end{cases} \Rightarrow 2y=3-y \Leftrightarrow y=1$. And $\begin{cases} y=1 \\ x=3-y \end{cases} \Rightarrow x=2$.

It follows that $x=2y,\ x=3-y$ intercept at the point $(2,1)$.

$$\int_0^1\left[\int_0^{2y} f(x,y)dx\right]dy+\int_1^3\left[\int_0^{3-y} f(x,y)dx\right]dy=\iint_R f(x,y)dxdy=\int_0^2\left[\int_{\frac{x}{2}}^{3-x} f(x,y)dy\right]dx.$$

2. Evaluate the following double integrals:

(a) $\iint_S xy\ dxdy$, where S is the region bounded by the lines $x = 0$, $x = 1$, $y = x^2$ and $y = 4$.

(b) $\iint_S x^2\ dxdy$, where S is the region bounded by $y=2x$ and $x^2+y=8$.

Sol:

(a)

$$\iint_S xy\ dxdy=\int_0^1\left[\int_{x^2}^4 xydy\right]dx=\int_0^1 \frac{xy^2}{2}\Bigg|_{x^2}^{4}dx=\int_0^1\frac{16x-x^5}{2}dx=\left[4x^2-\frac{x^6}{12}\right]\Bigg|_0^1=4-\frac{1}{12}=\frac{47}{12}$$

(b)

$$\begin{cases} y=2x \\ x^2+y=8 \end{cases} \Leftrightarrow \begin{cases} y=2x \\ x^2+2x-8=0 \end{cases} \Leftrightarrow \begin{cases} y=2x \\ (x+4)(x-2)=0 \end{cases} \Leftrightarrow \begin{cases} y=2x \\ x=-4 \text{ or } x=2 \end{cases} \Leftrightarrow \begin{cases} x=-4 \\ y=-8 \end{cases} \text{or} \begin{cases} x=2 \\ y=4 \end{cases}$$

$$\iint_S x^2\ dxdy=\int_{-4}^2\left[\int_{2x}^{8-x^2} x^2dy\right]dx=\int_{-4}^2 x^2y\Bigg|_{2x}^{8-x^2}dx=\int_{-4}^2 x^2\left(8-x^2-2x\right)dx=\left[\frac{8x^3}{3}-\frac{x^5}{5}-\frac{x^4}{2}\right]\Bigg|_{-4}^{2}$$

$$=\frac{64}{3}-\frac{32}{5}-8-\left(\frac{-512}{3}-\frac{-1024}{5}-128\right)=192-211.2+120=100.8$$

*3. Evaluate $\iint_S xy\ \ dxdy$ where S is the region enclosed by the 4 parabolas $y^2=x$, $y^2=2x$, $x^2=y$, $x^2=2y$ using the change of variable $u=\frac{x^2}{y}, v=\frac{y^2}{x}$.

Sol:

Under the transformations $u=\dfrac{x^2}{y}, v=\dfrac{y^2}{x}$, the boundaries $y^2=x$ goes to $v=1$, $y^2=2x$ to $v=2$, $x^2=y$ to $u=1$ and $x^2=2y$ to $u=2$. And the region S in x-y plane which are enclosed by the 4 parabolas $y^2=x$, $y^2=2x$, $x^2=y$, $x^2=2y$ goes to the region $\overline{S}$ in u-v plane which are enclosed by the lines $u=1, u=2, v=1, v=2$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\det\begin{pmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x}\\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y}\end{pmatrix}}=\frac{1}{\det\begin{pmatrix}\dfrac{2x}{y} & -\dfrac{y^2}{x^2}\\ -\dfrac{x^2}{y^2} & \dfrac{2y}{x}\end{pmatrix}}=\frac{1}{3}$$

Therefore,

$$\iint_S xy\ dxdy=\iint_{\overline{S}} uv|J|dudv=\int_1^2\left[\int_1^2\frac{1}{3}uvdv\right]du=\int_1^2\frac{1}{2}udu=\frac{3}{4}$$

4. Evaluate $E_z=\dfrac{\sigma_0 z}{4\pi\,\varepsilon_0}\iint_S\dfrac{1}{(x^2+y^2+z^2)^{3/2}}\ dxdy$ where S is the disc $x^2+y^2\le a^2$, which represents the z-component of the electric field at the point (0, 0, z) due to a uniformly charged circular disc lying in $x^2+y^2\le a^2$, $z=0$.

Sol:

$$x^2+y^2=r^2, dxdy=rdrd\theta$$

$$E_z=\frac{\sigma_0 z}{4\pi\,\varepsilon_0}\iint_S\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\ dxdy=\frac{\sigma_0 z}{4\pi\,\varepsilon_0}\int_0^a\left[\int_0^{2\pi}\frac{rd\theta}{(r^2+z^2)^{\frac{3}{2}}}\right]dr$$

$$=\frac{2\pi\sigma_0 z}{4\pi\,\varepsilon_0}\int_0^a\frac{rdr}{(r^2+z^2)^{\frac{3}{2}}}=\frac{\sigma_0 z}{2\,\varepsilon_0}\left.\frac{-1}{(r^2+z^2)^{\frac{1}{2}}}\right|_0^a=-\frac{\sigma_0 z}{2\,\varepsilon_0}\left(\frac{1}{(a^2+z^2)^{\frac{1}{2}}}-\frac{1}{|z|}\right)$$

*5. Let R be the region bounded by $x+y=1$, $x=0$, $y=0$. Show that

$\iint_R\cos\left(\dfrac{x-y}{x+y}\right)dx\,dy=\dfrac{\sin 1}{2}$, using the substitution $x-y=u$, $x+y=v$.

Sol:

$$x+y=1\to v=1, x=0\to u+v=0, y=0\to u-v=0$$

$$\begin{matrix}x-y=u\\ x+y=v\end{matrix}\to\begin{matrix}x=\dfrac{u+v}{2}\\ y=\dfrac{v-u}{2}\end{matrix},\quad \text{Jacobian}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=1/2$$

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dx\,dy = \iint_{R'} \frac{1}{2}\cos\left(\frac{u}{v}\right) du\,dv = \int_0^1\int_{-v}^{v} \frac{1}{2}\cos\left(\frac{u}{v}\right) dudv = \frac{\sin 1}{2}$$

*6. Evaluate $\iint_S e^{xy}\ dxdy$ where S is the region enclosed by $xy=1$, $xy=2$, $y=x$, $y=4x$ using the change of variable $xy=u$, $\frac{y}{x}=v$.

Sol:

Under the transformations $xy=u$, $\frac{y}{x}=v$,

$xy=1$ goes to $u=1$, $xy=2$ goes to $u=2$, $y=x$ goes to $v=1$, $y=4x$ goes to $v=4$.

The region S in x-y plane which is enclosed by $xy=1$, $xy=2$, $y=x$, $y=4x$ consists of two parts, say S_1 & S_2 , that is, $S=S_1\cup S_2$. S_1 is in the first quadrant of xy-plane and S_2 in the third quadrant. Under the transformations $xy=u$, $\frac{y}{x}=v$, both S_1 & S_2 go to the region $\overline{S}$ in u-v plane which is enclosed by the lines $u=1, u=2, v=1, v=4$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\det\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}}=\frac{1}{\det\begin{pmatrix} y & -\frac{y}{x^2}\\ x & \frac{1}{x}\end{pmatrix}}=\frac{1}{\frac{y}{x}+\frac{y}{x}}=\frac{1}{2v}$$

$$\iint_S e^{xy}\ dS = 2\iint_{\overline{S}} e^u\left|\frac{1}{2v}\right| dudv = 2\int_1^2\left[\int_1^4 e^u\,\frac{1}{2v}\,dv\right]du=\int_1^2\left(e^u \ln v\right)\Big|_1^4 du=\int_1^2 (\ln 4)e^u\,du=\left(e^2-e\right)\ln 4$$

*7. Use the change of variables $x+y=u$, $x-y=v$ to evaluate $\iint_{|x|+|y|\le 1} e^{(x-y)}dxdy$.

Sol:

$|x|+|y|\le 1$ is equivalent to the region bounded by $x+y=1, -x-y=1, -x+y=1, x-y=1$.

Under the transformations: $x+y=u$, $x-y=v$,

$x+y=1$ goes to $u=1$, $-x-y=1$ goes to $u=-1$, $-x+y=1$ goes to $v=-1$ and $x-y=1$ goes to $v=1$.

The Jacobian $J=\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\det\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}}=-\frac{1}{2}$.

So

$$\iint_{|x|+|y|\le 1} e^{(x-y)}dxdy = \iint_{\substack{-1\le x\le 1\\ -1\le y\le 1}} e^v |J| dudv = \int_{-1}^{1}\left[\int_{-1}^{1}\frac{e^v}{2}\,dv\right]du=\int_{-1}^{1}\frac{e^v}{2}\Big|_{-1}^{1}du=\frac{1}{2}\int_{-1}^{1}\left(e-e^{-1}\right)du=e-e^{-1}.$$

8. An iterated integral like $\int_0^1\left[\int_0^{\frac{1-x}{2}}\left(\int_0^{1-x-2y} f(x,y,z)dz\right)dy\right]dx$ is called an iterated integral with order $dzdydx$.

Change the order of the iterated integral $\int_0^1\left[\int_0^{\frac{1-x}{2}}\left(\int_0^{1-x-2y} f(x,y,z)dz\right)dy\right]dx$ to an equivalent iterated integral with order $dxdzdy$.

Solution:

Suppose we evaluate the triple integral of the function $f(x,y,z)$ in a solid V.

Let the projection of V onto x-y plane is σ_{xy}.

σ_{xy} is the set $\left\{(x,y): 0\le x\le 1,\ 0\le y\le \frac{1-x}{2}\right\}$, which is the triangle enclosed by $x=0,\ y=0,\ y=\frac{1-x}{2}$.

V is the set $\left\{(x,y,z): 0\le x\le 1,\ 0\le y\le \frac{1-x}{2}, 0\le z\le 1-x-2y\right\}$, which is the solid enclosed by $x=0,\ y=0,\ z=0,\ z=1-x-2y$.

Now, suppose the projection of V onto y-z plane is σ_{yz}. Let $x=0$, $z=1-x-2y \underset{x=0}{\Rightarrow} z=1-2y$. Therefore, σ_{yz} is the set $\left\{(y,z): 0\le y\le \frac{1}{2},\ 0\le z\le 1-2y\right\}$.

V can also be described as the set $\left\{(x,y,z): 0\le y\le \frac{1}{2},\ 0\le z\le 1-2y,\ 0\le x\le 1-2y-z\right\}$.

$$\int_0^1\left[\int_0^{\frac{1-x}{2}}\left(\int_0^{1-x-2y} f(x,y,z)dz\right)dy\right]dx=\int_0^{\frac{1}{2}}\left[\int_0^{1-2y}\left(\int_0^{1-2y-z} f(x,y,z)dx\right)dz\right]dy$$

9. Let V be the region in the first octant, where $x,y,z\ge 0$ bounded by $x^2+y^2=1$, $x=0,\ y=0,\ z=0,\ z=1$. Using cylindrical polar coordinate, compute $\iiint_V xydxdydz$.

Solution:

The projection of V on xy-plane is $\begin{cases} x^2+y^2\le 1 \\ x,y\ge 0\end{cases}$. Then

$$\iiint_V xydxdydz=\int_0^1\left[\int_0^{\frac{\pi}{2}}\left(\int_0^1 r^3\cos\theta\sin\theta dz\right)d\theta\right]dr=\int_0^1\left(\int_0^{\frac{\pi}{2}} r^3\cos\theta\sin\theta d\theta\right)dr=\int_0^1\left(\frac{r^3}{2}\sin^2\theta\right)\Bigg|_0^{\frac{\pi}{2}}d\theta$$

$$=\int_0^1\frac{r^3}{2}dr=\frac{r^4}{8}\Bigg|_0^1=\frac{1}{8}$$

10. (Optional) In a sample model of the charge distribution around the positively charged (Q) nucleus of the hydrogen

atom the charge density at the point (x, y, z) in the electron cloud is $f(x,y,z) = \frac{-Q}{\pi\ a^3} e^{-\frac{2\sqrt{x^2+y^2+z^2}}{a}}$

where a is the Bohr radius. Determine the total charge in the electron cloud.

Solution:

Under the transformations $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$, R^3 is transformed into V': $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \\ 0 \le \rho < \infty \end{cases}$.

$$q = \iiint_{R^3} \left(\frac{-Q}{\pi\ a^3} e^{-\frac{2\sqrt{x^2+y^2+z^2}}{a}} \right) dxdydz = \iiint_{V'} \frac{-Q}{\pi\ a^3} e^{-\frac{2\rho}{a}} \rho^2 \sin\phi d\theta d\phi d\rho$$

$$= \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} \left(\int_0^\infty \frac{-Q}{\pi\ a^3} e^{-\frac{2\rho}{a}} \rho^2 \sin\phi d\rho \right) d\theta d\phi = \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} \left[\int_0^\infty \frac{Q}{2\pi\ a^2} \sin\phi\rho^2 d\left(e^{-\frac{2\rho}{a}} \right) \right] d\theta d\phi$$

$$= \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} \frac{Q}{2\pi\ a^2} \sin\phi \left[\rho^2 e^{-\frac{2\rho}{a}} \Big|_0^\infty - \int_0^\infty e^{-\frac{2\rho}{a}} d\left(\rho^2\right) \right] d\theta d\phi$$

$$= \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} \frac{Q}{2\pi\ a^2} \sin\phi \left(-\int_0^\infty 2e^{-\frac{2\rho}{a}} \rho d\rho \right) d\theta d\phi = \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} \frac{Q}{2\pi\ a} \sin\phi \left[\int_0^\infty \rho d\left(e^{-\frac{2\rho}{a}} \right) \right] d\theta d\phi$$

$$= \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} \frac{Q}{2\pi\ a} \sin\phi \left(\rho e^{-\frac{2\rho}{a}} \Big|_0^\infty - \int_0^\infty e^{-\frac{2\rho}{a}} d\rho \right) d\theta d\phi = \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} \frac{Q}{2\pi\ a} \sin\phi \left(-\int_0^\infty e^{-\frac{2\rho}{a}} d\rho \right) d\theta d\phi$$

$$= \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} \frac{Q}{4\pi} \sin\phi \left(e^{-\frac{2\rho}{a}} \Big|_0^\infty \right) d\theta d\phi = \iint_{0\le\theta\le2\pi,\ 0\le\phi\le\pi} -\frac{Q}{4\pi} \sin\phi d\theta d\phi$$

$$= \int_0^\pi \left(\int_0^{2\pi} \frac{-Q}{4\pi} \sin\phi d\theta \right) d\phi = \int_0^\pi \frac{-Q}{2} \sin\phi d\phi = \frac{-Q}{2} \cos\phi \Big|_0^\pi = -Q$$

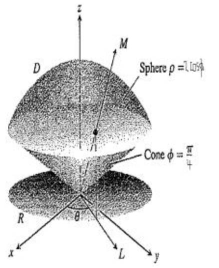
Note: To evaluate $\rho^2 e^{-\frac{2r}{a}} \Big|_0^\infty$, differentiate with respect to ρ twice the top and the bottom of $\frac{\rho^2}{e^{\frac{2\rho}{a}}}$

$$\rho^2 e^{-\frac{2r}{a}} \Big|_0^\infty = \lim_{\rho\to\infty} \frac{\rho^2}{e^{\frac{2\rho}{a}}} = \lim_{b\to\infty} \frac{2\rho}{\frac{2}{a} e^{\frac{2\rho}{a}}} = \lim_{b\to\infty} \frac{2}{\frac{4}{a^2} e^{\frac{2\rho}{a}}} = 0$$

11. (Optional) Let V be the region enclosed by both the surfaces: $\begin{cases} x^2+y^2+(z-1)^2=1 \\ z\ge 1 \end{cases}$ and $x^2+y^2=z^2$.

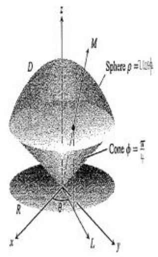
Using spherical polar coordinate, compute $\iiint\limits_V z\,dxdydz$.

Solution:



We integrate over the solid enclosed by the upper hemisphere $\begin{cases} x^2+y^2+(z-1)^2=1 \\ z\ge 1 \end{cases}$ with center at $(0,0,1)$ and radius 1 and by the cone $x^2+y^2=z^2$.

See the figure below.



$$\iiint\limits_V z\,dxdydz=\int_0^{\frac{\pi}{4}}\left(\int_0^{2\cos\phi}\left[\int_0^{2\pi}(\rho\cos\phi)\left(\rho^2\sin\phi\right)d\theta\right]d\rho\right)d\phi=\int_0^{\frac{\pi}{4}}\left(\int_0^{2\cos\phi}2\pi\rho^3\sin\phi\cos\phi\,d\rho\right)d\phi$$

$$=\int_0^{\frac{\pi}{4}}\left(\frac{\pi\rho^4}{2}\sin\phi\cos\phi\right)\Bigg|_0^{2\cos\phi}d\phi=\int_0^{\frac{\pi}{4}}8\pi\sin\phi\cos^5\phi\,d\phi=-\int_0^{\frac{\pi}{4}}8\pi\cos^5\phi\,d\left(\cos\phi\right)$$

$$=-\frac{4\pi\cos^6\phi}{3}\Bigg|_0^{\frac{\pi}{4}}=-\frac{4\pi}{3}\left[\left(\frac{1}{\sqrt{2}}\right)^6-1\right]=-\frac{4\pi\left(\frac{1}{8}-1\right)}{3}=\frac{7\pi}{6}$$

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