

HW 2-3 Solution

Q1. a)

$$((j2\pi f) + 2) Y(f) = (1 + j2\pi f) X(f)$$

freq response: $\Rightarrow H(f) = \frac{Y(f)}{X(f)} = 1 - \frac{1}{1 + j2\pi f}$

impulse response: $h(t) = \delta(t) - e^{-2t} u(t)$

Step response

$$\begin{aligned} Y(f) &= H(f) \mathcal{F}(u(t)) = H(f) \left(\frac{1}{2} \delta(f) + \frac{1}{j2\pi f} \right) \\ &= \frac{1}{2} \left(\frac{1}{2} \delta(f) + \frac{1}{j2\pi f} \right) + \frac{1}{2} \left(\frac{1}{2 + j2\pi f} \right) \end{aligned}$$

$$y(t) = \frac{1}{2} u(t) + \frac{1}{2} e^{-2t} u(t)$$

b) $((j2\pi f)^2 + 6(j2\pi f) + 8) Y(f) = 2X(f)$

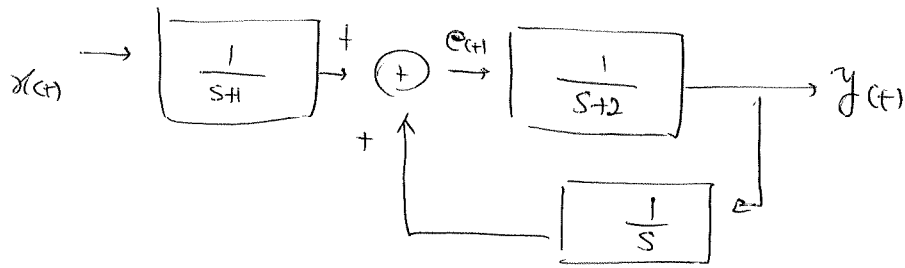
$$\Rightarrow H(f) = \frac{Y(f)}{X(f)} = \frac{1}{2 + j2\pi f} - \frac{1}{4 + j2\pi f}$$

$$h(t) = e^{-2t} u(t) - e^{-4t} u(t)$$

Step response

$$\begin{aligned} Y(f) &= H(f) \mathcal{F}(u(t)) = \frac{1}{4} \left(\frac{1}{2} \delta(f) + \frac{1}{j2\pi f} \right) - \frac{1}{2} \cdot \frac{1}{2 + j2\pi f} + \frac{1}{4} \frac{1}{4 + j2\pi f} \\ y(t) &= \frac{1}{4} u(t) - \frac{1}{2} e^{-2t} u(t) + \frac{1}{4} e^{-4t} u(t) \end{aligned}$$

Q2.



$$\begin{cases} E(s) = \frac{1}{s+1} X(s) + \frac{1}{(s+2)s} E(s) & \text{--- (1)} \\ Y(s) = \frac{1}{s+2} E(s) & \text{--- (2)} \end{cases}$$

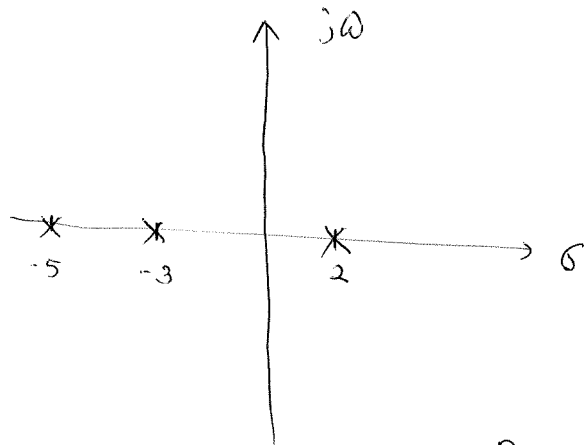
$$\Rightarrow \text{From (1), } \left(1 - \frac{1}{s(s+2)}\right) E(s) = \frac{1}{s+1} X(s) \quad \text{--- (3)}$$

Substitute (3) to (2)

$$\begin{aligned} Y(s) &= \frac{1}{(s+2)} \cdot \frac{1}{(s+1)} \cdot \frac{1}{1 - \frac{1}{s(s+2)}} X(s) \\ &= \frac{s}{s^3 + 3s^2 + s - 1} X(s) \end{aligned}$$

$$\Rightarrow \therefore H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^3 + 3s^2 + s - 1}$$

Q3. Three poles at $s = 2$, $s = -3$, $s = -5$



a) Possible ROC =
$$\begin{cases} \text{Re}(s) > 2 \\ -3 < \text{Re}(s) < 2 \\ -5 < \text{Re}(s) < -3 \\ \text{Re}(s) < -5 \end{cases}$$

- b)
- | | | |
|--------------------------|---|---------------------|
| $\text{Re}(s) > 2$ | : | Unstable Causal |
| $-3 < \text{Re}(s) < 2$ | : | Stable, Noncausal |
| $-5 < \text{Re}(s) < -3$ | : | Unstable, Noncausal |
| $\text{Re}(s) < -5$ | : | Unstable, Noncausal |

Q 4

a)

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

$$= - \sum_{n=-\infty}^{\infty} a^n u[-n-1] e^{-j2\pi f n}$$

$$= - \sum_{n=-\infty}^{-1} (a e^{-j2\pi f})^n = - \sum_{l=1}^{\infty} \left(\frac{1}{a} e^{j2\pi f} \right)^l \quad \left(\begin{array}{l} \text{by using} \\ \text{change of variable} \\ n = -l \end{array} \right)$$

$$= - \frac{\frac{1}{a} e^{j2\pi f}}{1 - \frac{1}{a} e^{j2\pi f}}$$

$$\text{when } \frac{1}{|a|} < 1 \Leftrightarrow |a| > 1$$

$$= \frac{1}{1 - a e^{-j2\pi f}} \quad \text{and } |a| > 1$$

b)

$$X(f) = \sum_{n=0}^{N-1} e^{-j2\pi f n} = \frac{1 - e^{-j2\pi f N}}{1 - e^{-j2\pi f}} = e^{-j\pi f (N-1)} \cdot \frac{\sin(\pi f N)}{\sin(\pi f)}$$

Q 5.

a)

$$\sum_{n=-\infty}^{\infty} x[n-n_0] e^{-j2\pi f n} = \left\{ \sum_{l=-\infty}^{\infty} x[l] e^{-j2\pi f l} \right\} e^{-j2\pi f n_0}$$

use change of variable $n - n_0 = l$

$$\Rightarrow X(f) \cdot e^{-j2\pi f n_0}$$

b)

$$\sum_{n=-\infty}^{\infty} e^{j2\pi f_0 n} x[n] e^{-j2\pi f n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi (f-f_0) n}$$

$$= X(f-f_0)$$

c)

$$\frac{d}{df} \left(X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n} \right)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} (-j2\pi n) x[n] e^{-j2\pi f n} = \frac{dX(f)}{df}$$

$$\sum_{n=-\infty}^{\infty} (n x[n]) e^{-j2\pi f n} = \frac{j}{2\pi} \times \frac{dX}{df}$$

Therefore

$$n x[n] \longleftrightarrow \frac{j}{2\pi} \frac{dX}{df}$$

Q 6.

$$a) \mathcal{F}(\text{sinc}^2(4t)) = \frac{1}{4} \text{tri}\left(\frac{f}{4}\right).$$

$$b) \mathcal{F}(\text{sgn}(t)) = \frac{1}{j\pi f}$$

$$c) \mathcal{F}(e^{-2t} u(t)) = \frac{1}{2 + j2\pi f}$$

$$d) \mathcal{F}\left(\text{rect}\left(\frac{t}{3}\right)\right) = 3 \text{sinc}(3f)$$

Q 7.

$$a) \quad X(s) = \frac{1}{s+5}, \quad \operatorname{Re}(s) > -5$$

$$b) \quad X(s) = \frac{1}{s^2}, \quad \operatorname{Re}(s) > 0.$$

$$c) \quad X_1(s) = \frac{1}{s+3}, \quad \operatorname{Re}(s) > -3$$

$$\mathcal{L}(x_1(t-4)) = e^{-4s} X_1(s) = \frac{e^{-4s}}{s+3}, \quad \operatorname{Re}(s) > -3$$

$$d) \quad X(s) = \frac{s^2 + 2s + 5}{(s+3)(s+5)^2}, \quad \operatorname{Re}(s) > -3$$

$$= \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$$

$$x(t) = \left[2e^{-3t} - (1+10t)e^{-5t} \right] u(t)$$

Q 8.

$$X(z) = \sum_{n=-\infty}^{\infty} \sin(\omega_0 n) (r z^{-1})^n$$

$$= \frac{1}{2j} \sum_{n=-\infty}^{\infty} [e^{j\omega_0 n} - e^{-j\omega_0 n}] (r z^{-1})^n$$

$$= \frac{1}{2j} \left[\sum_{n=-\infty}^{\infty} (e^{j\omega_0} r z^{-1})^n - \sum_{n=-\infty}^{\infty} (e^{-j\omega_0} r z^{-1})^n \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega_0} r z^{-1}} - \frac{1}{1 - e^{-j\omega_0} r z^{-1}} \right] \quad \text{and } |r z^{-1}| < 1$$

$$= \frac{2j \sin(\omega_0) (r z^{-1})}{2j (1 - 2 \cos \omega_0 r z^{-1} + (r z^{-1})^2)}$$

$$\Rightarrow \text{ROC: } |z| > |r|$$

$$\therefore X(z) = \frac{\sin(\omega_0) \cdot r z^{-1}}{1 - 2 \cos \omega_0 \cdot r z^{-1} + (r z^{-1})^2} \quad \text{and ROC } |z| > |r|$$

Q 9.

$$a) X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = 5z^2 + 3z^1 + 4 + z^{-1} - 2z^{-3}$$

$$ROC = \{0 < |z| < \infty\}$$

$$b) x[n] = \frac{1}{3} \cdot \frac{1}{4^n} u[n] - \frac{2}{2^n} u[n] + \frac{8}{3} u[n]$$

$$X(z) = \frac{1}{3} \underbrace{\frac{z}{z - \frac{1}{4}}}_{ROC: |z| > \frac{1}{4}} - 2 \cdot \underbrace{\frac{z}{z - 2}}_{ROC: |z| > 2} + \frac{8}{3} \underbrace{\frac{z}{z - 1}}_{ROC: |z| > 1}$$

$$ROC: |z| > \frac{1}{4}$$

$$ROC: |z| > 2$$

$$ROC: |z| > 1$$

both forms are ok.

$$= \frac{z(z^2 - \frac{9}{2}z + \frac{3}{2})}{(z - \frac{1}{4})(z - 2)(z - 1)}$$

$$\text{with } ROC = \{|z| > 2\}$$

$$c) X(z) = \log\left(\frac{1}{1-z}\right) \quad |z| < 1$$

$$= -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n \xrightarrow{\text{change of variable } n=-l} -\sum_{l=-\infty}^{-1} \frac{1}{l} z^{-l}$$

Since $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$, the inverse z -transform is

$$x[n] = -\frac{1}{n} u[-n-1]$$

$$d) X(z) = \frac{3}{z-4}, \quad |z| > 4$$

$$= 3z^{-1} \cdot \frac{z}{z-4} \longrightarrow z^{-1}(X(z)) = x[n] = 3 \cdot 4^{n-1} u[n-1]$$

Q 10.

$$Y(z) = 5 \left(\frac{Y(z)}{z} + y[-1] \right) + 6 \left(\frac{Y(z)}{z^2} + \frac{y[-1]}{z} + y[-2] \right)$$
$$= 3 \left(\frac{X(z)}{z} + x[-1] \right) + 5 \left(\frac{X(z)}{z^2} + \frac{x[-1]}{z} + x[-2] \right)$$

→ by using the initial condition

$$Y(z) = 5 \left(\frac{Y(z)}{z} + \frac{11}{6} \right) + 6 \left(\frac{Y(z)}{z^2} + \frac{11}{6z} + \frac{37}{36} \right) = \frac{3X(z)}{z} + \frac{5X(z)}{z^2} \quad \text{--- (1)}$$

where the input is

$$X(z) = \frac{z}{z-0.5} \quad \text{--- (2)}$$

→ Substitute (2) into (1) and by using partial fraction

$$Y(z) = \frac{26}{15} \cdot \frac{z}{z-0.5} - \frac{7}{3} \frac{z}{z-2} + \frac{18}{5} \frac{z}{z-3}$$

z^{-1} ↓

$$\rightarrow y[n] = \frac{26}{15} (0.5)^n u[n] - \frac{7}{3} 2^n u[n] + \frac{18}{5} 3^n u[n]$$