Solutions to HW7

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. The solution to problem 5.8.3 is mine.

Problem 4.1.1 •

Random variables X and Y have the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & x \geq 0; \\ & y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $P[X \leq 2, Y \leq 3]$?

(b) What is the marginal CDF, $F_X(x)$?

(c) What is the marginal CDF, $F_Y(y)$?

Problem 4.1.1 Solution

(a) *Using Definition 4.1* The probability $P[X \leq 2, Y \leq 3]$ can be found be evaluating the joint CDF $F_{X,Y}(x,y)$ at $x = 2$ and $y = 3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2,3) = (1-e^{-2})(1-e^{-3}) \tag{1}$$

(b) *By Theorem 4.1* To find the marginal CDF of X, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 1-e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(c) Likewise *by Theorem 4.1* for the marginal CDF of Y, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 1-e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 4.1.2 •

Express the following extreme values of $F_{X,Y}(x,y)$ in terms of the marginal cumulative distribution functions $F_X(x)$ and $F_Y(y)$.

(a) $F_{X,Y}(x,-\infty)$

(b) $F_{X,Y}(x,\infty)$

(c) $F_{X,Y}(-\infty,\infty)$

(d) $F_{X,Y}(-\infty, y)$

(e) $F_{X,Y}(\infty, y)$

Problem 4.1.2 Solution

(a) Because the probability that any random variable is less than $-\infty$ is zero, we have *(also by Theorem 4.1d)*

$$F_{X,Y}(x, -\infty) = P[X \leq x, Y \leq -\infty] \leq P[Y \leq -\infty] = 0 \tag{1}$$

(b) The probability that any random variable is less than infinity is always one. *See also Theorem 4.1.b.*

$$F_{X,Y}(x, \infty) = P[X \leq x, Y \leq \infty] = P[X \leq x] = F_X(x) \tag{2}$$

(c) Although $P[Y \leq \infty] = 1$, $P[X \leq -\infty] = 0$. Therefore the following is true. *(Theorem 4.1b and the definition of the CDF)*

$$F_{X,Y}(-\infty, \infty) = P[X \leq -\infty, Y \leq \infty] \leq P[X \leq -\infty] = 0 \tag{3}$$

(d) Part (d) follows the same logic as that of part (a). *Theorem 4.1d*

$$F_{X,Y}(-\infty, y) = P[X \leq -\infty, Y \leq y] \leq P[X \leq -\infty] = 0 \tag{4}$$

(e) Analogous to Part (b), we find that *(See also Theorem 4.1c.)*

$$F_{X,Y}(\infty, y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = F_Y(y) \tag{5}$$

Problem 4.2.1 •
Random variables X and Y have the joint PMF

$$P_{X,Y}(x, y) = \begin{cases} cxy & x = 1, 2, 4; \quad y = 1, 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[Y < X]$?

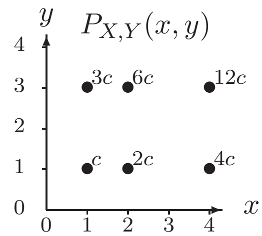
(c) What is $P[Y > X]$?

(d) What is $P[Y = X]$?

(e) What is $P[Y = 3]$?

Problem 4.2.1 Solution

In this problem, it is helpful to label points with nonzero probability on the X, Y plane:



(a) We must choose c so the PMF sums to one:

$$\sum_{x=1,2,4} \sum_{y=1,3} P_{X,Y}(x, y) = c \sum_{x=1,2,4} x \sum_{y=1,3} y \tag{1}$$

$$= c\,[1(1+3) + 2(1+3) + 4(1+3)] = 28c \tag{2}$$

Thus $c = 1/28$.

(b) The event $\{Y < X\}$ has probability

$$P[Y < X] = \sum_{x=1,2,4} \sum_{y<x} P_{X,Y}(x, y) = \frac{1(0) + 2(1) + 4(1+3)}{28} = \frac{18}{28} \tag{3}$$

(c) The event $\{Y > X\}$ has probability

$$P[Y > X] = \sum_{x=1,2,4} \sum_{y>x} P_{X,Y}(x, y) = \frac{1(3) + 2(3) + 4(0)}{28} = \frac{9}{28} \tag{4}$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y = X] = \sum_{x=1,2,4} \sum_{y=x} P_{X,Y}(x, y) = \frac{1(1) + 2(0)}{28} = \frac{1}{28} \tag{5}$$

The indirect way is to use the previous results and the observation that

$$P[Y = X] = 1 - P[Y < X] - P[Y > X] = (1 - 18/28 - 9/28) = 1/28 \tag{6}$$

(e)

$$P[Y = 3] = \sum_{x=1,2,4} P_{X,Y}(x, 3) = \frac{(1)(3) + (2)(3) + (4)(3)}{28} = \frac{21}{28} = \frac{3}{4} \tag{7}$$

Problem 4.4.1 •

Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

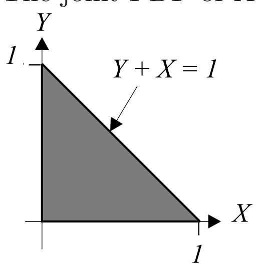
(a) What is the value of the constant c?

(b) What is $P[X \leq Y]$?

(c) What is $P[X + Y \leq 1/2]$?

Problem 4.4.1 Solution

(a) The joint PDF of X and Y is



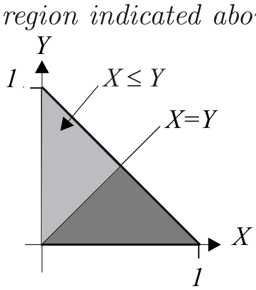
$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x,y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

To find the constant c we integrate over the region shown. *Note that we are integrating a constant over a region of area 1/2, so we should expect the result of the integration to be $c/2$.* This gives

$$\int_0^1 \int_0^{1-x} c\,dy\,dx = cx - \frac{cx}{2}\Big|_0^1 = \frac{c}{2} = 1 \tag{2}$$

Therefore $c = 2$.

(b) To find the $P[X \leq Y]$ we look to integrate over the area indicated by the graph *This time we're determining the probability by integrating the constant PDF over 1/2 the region indicated above, so we should expect the probability to be 1/2.*

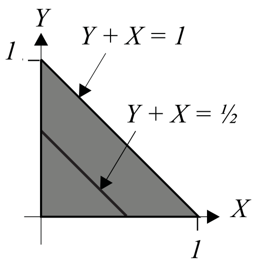


$$P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} dy\,dx \tag{3}$$

$$= \int_0^{1/2} (2 - 4x)\,dx \tag{4}$$

$$= 1/2 \tag{5}$$

(c) The probability $P[X + Y \leq 1/2]$ can be seen in the figure. *Here we integrate the constant PDF over 1/4 of the original region so we should expect the probability to be 1/4.* Here we can set up the following integrals



$$P[X+Y \leq 1/2] = \int_0^{1/2} \int_0^{1/2-x} 2\,dy\,dx \tag{6}$$
$$= \int_0^{1/2} (1-2x)\,dx \tag{7}$$
$$= 1/2 - 1/4 = 1/4 \tag{8}$$

Problem 5.1.1 •

Every laptop returned to a repair center is classified according its needed repairs: (1) LCD screen, (2) motherboard, (3) keyboard, or (4) other. A random broken laptop needs a type i repair with probability $p_i = 2^{4-i}/15$. Let N_i equal the number of type i broken laptops returned on a day in which four laptops are returned.

(a) Find the joint PMF
$$P_{N_1,N_2,N_3,N_4}(n_1,n_2,n_3,n_4)$$

(b) What is the probability that two laptops require LCD repairs?

(c) What is the probability that more laptops require motherboard repairs than keyboard repairs?

Problem 5.1.1 Solution

The repair of each laptop can be viewed as an independent trial with four possible outcomes corresponding to the four types of needed repairs. *My first question here was whether a laptop could need two repairs. I resolved this by summing the probabilities assigned to each of the individual pairs and determining that the sum was one, so there were no laptops requiring two repairs. The problem would be much more complicated if this were not the case.*

(a) Since the four types of repairs are mutually exclusive choices and since 4 laptops are returned for repair, the joint distribution of $N_1, \ldots, N_4$ is the multinomial PMF

$$P_{N_1,\ldots,N_4}(n_1,\ldots,n_4) = \binom{4}{n_1,n_2,n_3,n_4} p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4} \tag{1}$$
$$= \begin{cases} \frac{4!}{n_1!n_2!n_3!n_4!}\left(\frac{8}{15}\right)^{n_1}\left(\frac{4}{15}\right)^{n_2}\left(\frac{2}{15}\right)^{n_3}\left(\frac{1}{15}\right)^{n_4} & n_1+\cdots+n_4=4; n_i \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) Let L_2 denote the event that exactly two laptops need LCD repairs. Thus $P[L_2] = P_{N_1}(2)$. Since each laptop requires an LCD repair with probability $p_1 = 8/15$, the number of LCD repairs, N_1, is a binomial $(4, 8/15)$ random variable with PMF

$$P_{N_1}(n_1) = \binom{4}{n_1}(8/15)^{n_1}(7/15)^{4-n_1} = \binom{4}{n_1}(8/7)^{n_1}(7/15)^4 \tag{3}$$

The probability that two laptops need LCD repairs is

$$P_{N_1}(2) = \binom{4}{2}(8/15)^2(7/15)^2 = 0.3717 \tag{4}$$

(c) A repair is type (2) with probability $p_2 = 4/15$. A repair is type (3) with probability $p_3 = 2/15$; otherwise a repair is type "other" *[Note that this is not the same "other" as the fourth type of repairs in the problem statement. This "other" includes both types 1 and types 4.]* with probability $p_o = 9/15$. Define X as the number of "other" repairs needed. The joint PMF of X, N_2, N_3 is the multinomial PMF

$$P_{N_2,N_3,X}(n_2,n_3,x) = \binom{4}{n_2,n_3,x}\left(\frac{4}{15}\right)^{n_2}\left(\frac{2}{15}\right)^{n_3}\left(\frac{9}{15}\right)^{x} \tag{5}$$

However, Since $X + 4 - N_2 - N_3$, we observe that

$$P_{N_2,N_3}(n_2,n_3) = P_{N_2,N_3,X}(n_2,n_3,4-n_2-n_3) \tag{6}$$
$$= \binom{4}{n_2,n_3,4-n_2-n_3}\left(\frac{4}{15}\right)^{n_2}\left(\frac{2}{15}\right)^{n_3}\left(\frac{9}{15}\right)^{4-n_2-n_3} \tag{7}$$
$$= \left(\frac{9}{15}\right)^4\binom{4}{n_2,n_3,4-n_2-n_3}\left(\frac{4}{9}\right)^{n_2}\left(\frac{2}{9}\right)^{n_3} \tag{8}$$

Similarly, since each repair is a motherboard repair with probability $p_2 = 4/15$, the number of motherboard repairs has binomial PMF

$$P_{N_2}(n_2) = \binom{4}{n_2}\left(\frac{4}{15}\right)^{n_2}\left(\frac{11}{15}\right)^{4-n_2} = \binom{4}{n_2}\left(\frac{4}{11}\right)^{n_2}\left(\frac{11}{15}\right)^{4} \tag{9}$$

Finally, the probability that more laptops require motherboard repairs than keyboard repairs is

$$P[N_2 > N_3] = P_{N_2,N_3}(1,0) + P_{N_2,N_3}(2,0) + P_{N_2,N_3}(2,1) + P_{N_2}(3) + P_{N_2}(4) \tag{10}$$

where we use the fact that if $N_2 = 3$ or $N_2 = 4$, then we must have $N_2 > N_3$. Inserting the various probabilities, we obtain

$$P[N_2 > N_3] = P_{N_2,N_3}(1,0) + P_{N_2,N_3}(2,0) + P_{N_2,N_3}(2,1) + P_{N_2}(3) + P_{N_2}(4) \tag{11}$$

Plugging in the various probabilities yields $P[N_2 > N_3] = 8{,}656/16{,}875 \approx 0.5129$.

Problem 4.3.1 •

Given the random variables X and Y in Problem 4.2.1, find

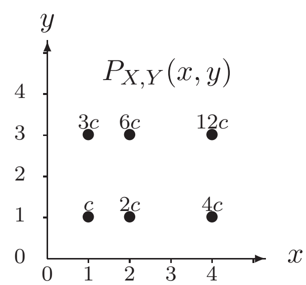
(a) The marginal PMFs $P_X(x)$ and $P_Y(y)$,

(b) The expected values $E[X]$ and $E[Y]$,

(c) The standard deviations σ_X and σ_Y.

Problem 4.3.1 Solution

On the X, Y plane, the joint PMF $P_{X,Y}(x, y)$ is



By choosing $c = 1/28$, the PMF sums to one.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=1,3} P_{X,Y}(x, y) = \begin{cases} 4/28 & x = 1 \\ 8/28 & x = 2 \\ 16/28 & x = 4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$P_Y(y) = \sum_{x=1,2,4} P_{X,Y}(x, y) = \begin{cases} 7/28 & y = 1 \\ 21/28 & y = 3 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) The expected values of X and Y are

$$E[X] = \sum_{x=1,2,4} x P_X(x) = (4/28) + 2(8/28) + 4(16/28) = 3 \tag{3}$$

$$E[Y] = \sum_{y=1,3} y P_Y(y) = 7/28 + 3(21/28) = 5/2 \tag{4}$$

(c) The second moments are

$$E\left[X^2\right] = \sum_{x=1,2,4} x P_X(x) = 1^2(4/28) + 2^2(8/28) + 4^2(16/28) = 73/7 \tag{5}$$

$$E\left[Y^2\right] = \sum_{y=1,3} y P_Y(y) = 1^2(7/28) + 3^2(21/28) = 7 \tag{6}$$

The variances are

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 10/7 \qquad \text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 3/4 \tag{7}$$

The standard deviations are $\sigma_X = \sqrt{10/7}$ and $\sigma_Y = \sqrt{3/4}$.

Problem 4.3.3 •
For $n = 0, 1, \ldots$ and $0 \leq k \leq 100$, the joint PMF of random variables N and K is

$$P_{N,K}(n,k) = \frac{100^n e^{-100}}{n!} \binom{100}{k} p^k (1-p)^{100-k}.$$

Otherwise, $P_{N,K}(n,k) = 0$. Find the marginal PMFs $P_N(n)$ and $P_K(k)$.

Problem 4.3.3 Solution
We recognize that the given joint PMF is written as the product of two marginal PMFs $P_N(n)$ and $P_K(k)$ where

$$P_N(n) = \sum_{k=0}^{100} P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{n!} & n = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \begin{cases} \binom{100}{k} p^k (1-p)^{100-k} & k = 0, 1, \ldots, 100 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Problem 4.5.1 •
Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the region of nonzero probability.

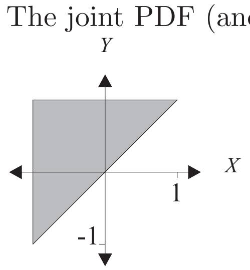
(b) What is $P[X > 0]$?

(c) What is $f_X(x)$?

(d) What is $E[X]$?

Problem 4.5.1 Solution

(a) The joint PDF (and the corresponding region of nonzero probability) are



$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(b)

$$P[X > 0] = \int_0^1 \int_x^1 \frac{1}{2}\, dy\, dx = \int_0^1 \frac{1-x}{2}\, dx = 1/4 \tag{2}$$

This result can be deduced by geometry. The shaded triangle of the X, Y plane corresponding to the event $X > 0$ is 1/4 of the total shaded area.

(c) For $x > 1$ or $x < -1$, $f_X(x) = 0$. For $-1 \leq x \leq 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_x^1 \frac{1}{2}\, dy = (1-x)/2. \tag{3}$$

The complete expression for the marginal PDF is

$$f_X(x) = \begin{cases} (1-x)/2 & -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

(d) From the marginal PDF $f_X(x)$, the expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{1}{2}\int_{-1}^{1} x(1-x)\, dx \tag{5}$$

$$= \left.\frac{x^2}{4} - \frac{x^3}{6}\right|_{-1}^{1} = -\frac{1}{3}. \tag{6}$$

Problem 5.3.2 ■

A wireless data terminal has three messages waiting for transmission. After sending a message, it expects an acknowledgement from the receiver. When it receives the acknowledgement, it transmits the next message. If the acknowledgement does not arrive, it sends the message again. The probability of successful transmission of a message is p independent of other transmissions. Let $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}'$ be the 3-dimensional random vector in which K_i is the total number of transmissions when message i is received successfully. (K_3 is the total number of transmissions used to send all three messages.) Show that

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3(1-p)^{k_3-3} & k_1 < k_2 < k_3; \\ & k_i \in \{1, 2 \ldots\}, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 5.3.2 Solution

Since J_1, J_2 and J_3 are independent, we can write

$$P_{\mathbf{K}}(\mathbf{k}) = P_{J_1}(k_1)\, P_{J_2}(k_2 - k_1)\, P_{J_3}(k_3 - k_2) \tag{1}$$

Since $P_{J_i}(j) > 0$ only for integers $j > 0$, we have that $P_{\mathbf{K}}(\mathbf{k}) > 0$ only for $0 < k_1 < k_2 < k_3$; otherwise $P_{\mathbf{K}}(\mathbf{k}) = 0$. Finally, for $0 < k_1 < k_2 < k_3$,

$$P_{\mathbf{K}}(\mathbf{k}) = (1-p)^{k_1-1}p(1-p)^{k_2-k_1-1}p(1-p)^{k_3-k_2-1}p \tag{2}$$

$$= (1-p)^{k_3-3}p^3 \tag{3}$$

I solved this differently. I first defined new random variables N_1, N_2, and N_3 to be the number of transmissions needed for successful receipt of message i, $i \in \{1,2,3\}$. Then for each N_i,

$$P_{N_i}(n_i) = (1-p)^{n_i}\, p \tag{4}$$

where we have no $\binom{x}{y}$ because we know which of the transmissions is the successful one. It is the last. Now the successes or failures of the transmissions being independent, we have for n_1, n_2, and n_3 positive integers,

$$P_{N_1,N_2,N_3}(n_1,n_2,n_3) = (1-p)^{n_1-1}\, p\, (1-p)^{n_2-1}\, p\, (1-p)^{n_3-1}\, p = (1-p)^{n_1+n_2+n_3-3}\, p^3 \tag{5}$$

so with $K_1 = N_1$, $K_2 = N_1 + N_2$, and $K_3 = N_1 + N_2 + N_3$, we have

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} (1-p)^{k_3-3}\, p^3 & k_1 < k_2 < k_3, \qquad \text{integers} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Problem 4.10.2 •

X and Y are independent, identically distributed random variables with PMF

$$P_X(k) = P_Y(k) = \begin{cases} 3/4 & k = 0, \\ 1/4 & k = 20, \\ 0 & \text{otherwise.} \end{cases}$$

Find the following quantities:

$$\begin{array}{lll} E[X], & \text{Var}[X], & \\ E[X+Y], & \text{Var}[X+Y], & E\left[XY2^{XY}\right]. \end{array}$$

Problem 4.10.2 Solution

Using the following probability model

$$P_X(k) = P_Y(k) = \begin{cases} 3/4 & k = 0 \\ 1/4 & k = 20 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can calculate the requested moments.

$$E[X] = 3/4 \cdot 0 + 1/4 \cdot 20 = 5 \tag{2}$$

$$\text{Var}[X] = 3/4 \cdot (0-5)^2 + 1/4 \cdot (20-5)^2 = 75 \tag{3}$$

$$E[X+Y] = E[X] + E[X] = 2E[X] = 10 \tag{4}$$

Since X and Y are independent, Theorem 4.27 yields

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] = 2\,\text{Var}[X] = 150 \tag{5}$$

Since X and Y are independent, $P_{X,Y}(x,y) = P_X(x)P_Y(y)$ and

$$E\left[XY2^{XY}\right] = \sum_{x=0,20}\sum_{y=0,20} XY2^{XY} P_{X,Y}(x,y) = (20)(20)2^{20(20)} P_X(20)\, P_Y(20) \tag{6}$$

$$= 6.46 \times 10^{121} \tag{7}$$

Problem 4.10.7 •

X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & y \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

(a) What is $P[X > Y]$?

(b) What is $E[XY]$?

(c) What is $\text{Cov}[X, Y]$?

Problem 4.10.7 Solution

X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) To calculate $P[X > Y]$, we use the joint PDF $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.

$$P[X > Y] = \iint_{x>y} f_X(x) f_Y(y) \, dx \, dy \tag{2}$$

$$= \int_0^\infty \frac{1}{2}e^{-y/2} \int_y^\infty \frac{1}{3}e^{-x/3} \, dx \, dy \tag{3}$$

$$= \int_0^\infty \frac{1}{2}e^{-y/2}e^{-y/3} \, dy \tag{4}$$

$$= \int_0^\infty \frac{1}{2}e^{-(1/2+1/3)y} \, dy = \frac{1/2}{1/2 + 2/3} = \frac{3}{7} \tag{5}$$

(b) Since X and Y are exponential random variables with parameters $\lambda_X = 1/3$ and $\lambda_Y = 1/2$, Appendix A tells us that $E[X] = 1/\lambda_X = 3$ and $E[Y] = 1/\lambda_Y = 2$. Since X and Y are independent, the correlation is $E[XY] = E[X]E[Y] = 6$.

(c) Since X and Y are independent, $\text{Cov}[X, Y] = 0$.

Problem 5.4.2 •

In Problem 5.1.1, are N_1, N_2, N_3, N_4 independent?

Problem 5.4.2 Solution

The random variables N_1, N_2, N_3 and N_4 are *dependent*. To see this we observe that $P_{N_i}(4) = p_i^4$. However,

$$P_{N_1,N_2,N_3,N_4}(4, 4, 4, 4) = 0 \neq p_1^4 p_2^4 p_3^4 p_4^4 = P_{N_1}(4) P_{N_2}(4) P_{N_3}(4) P_{N_4}(4). \tag{1}$$

Note that this can also be determined by calculating the marginals and checking to see whether the product of the marginals equals the original function (more work, but just as effective).

Problem 5.4.5 •

The PDF of the 3-dimensional random vector $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} e^{-x_3} & 0 \leq x_1 \leq x_2 \leq x_3, \\ 0 & \text{otherwise.} \end{cases}$$

Are the components of $\mathbf{X}$ independent random variables?

Problem 5.4.5 Solution

This problem can be solved without any real math. Some thought should convince you that for any $x_i > 0$, $f_{X_i}(x_i) > 0$. Thus, $f_{X_1}(10) > 0$, $f_{X_2}(9) > 0$, and $f_{X_3}(8) > 0$. Thus $f_{X_1}(10)f_{X_2}(9)f_{X_3}(8) > 0$. However, from the definition of the joint PDF

$$f_{X_1,X_2,X_3}(10, 9, 8) = 0 \neq f_{X_1}(10)\, f_{X_2}(9)\, f_{X_3}(8)\,. \tag{1}$$

It follows that X_1, X_2 and X_3 are dependent. Readers who find this quick answer dissatisfying are invited to confirm this conclusions by solving Problem 5.4.6 for the exact expressions for the marginal PDFs $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, and $f_{X_3}(x_3)$. *Solution to 5.4.6. is included below.*

Problem 5.4.6 ■

The random vector $\mathbf{X}$ has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} e^{-x_3} & 0 \leq x_1 \leq x_2 \leq x_3, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PDFs $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, and $f_{X_3}(x_3)$.

Problem 5.4.6 Solution

This problem was not assigned but contains the straightforward solution to problem 5.4.5, hence is included here in the solutions.

We find the marginal PDFs using Theorem 5.5. First we note that for $x < 0$, $f_{X_i}(x) = 0$. For $x_1 \geq 0$,

$$f_{X_1}(x_1) = \int_{x_1}^{\infty} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_2 = \int_{x_1}^{\infty} e^{-x_2}\, dx_2 = e^{-x_1} \tag{1}$$

Similarly, for $x_2 \geq 0$, X_2 has marginal PDF

$$f_{X_2}(x_2) = \int_0^{x_2} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_1 = \int_0^{x_2} e^{-x_2}\, dx_1 = x_2 e^{-x_2} \tag{2}$$

Lastly,

$$f_{X_3}(x_3) = \int_0^{x_3} \left(\int_{x_1}^{x_3} e^{-x_3}\, dx_2 \right) dx_1 = \int_0^{x_3} (x_3 - x_1) e^{-x_3}\, dx_1 \tag{3}$$

$$= -\frac{1}{2}(x_3 - x_1)^2 e^{-x_3} \Big|_{x_1=0}^{x_1=x_3} = \frac{1}{2} x_3^2 e^{-x_3} \tag{4}$$

The complete expressions for the three marginal PDFs are

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$f_{X_2}(x_2) = \begin{cases} x_2 e^{-x_2} & x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$f_{X_3}(x_3) = \begin{cases} (1/2)x_3^2 e^{-x_3} & x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In fact, each X_i is an Erlang $(n, \lambda) = (i, 1)$ random variable.

Problem 4.6.1 •

Given random variables X and Y in Problem 4.2.1 and the function $W = X - Y$, find

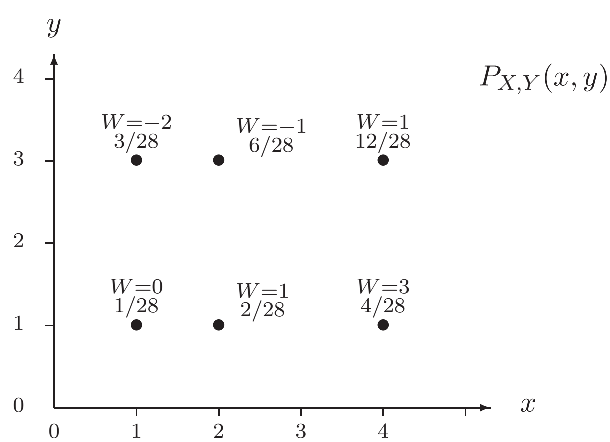
(a) The probability mass function $P_W(w)$,

(b) The expected value $E[W]$,

(c) $P[W > 0]$.

Problem 4.6.1 Solution

In this problem, it is helpful to label possible points X, Y along with the corresponding values of $W = X - Y$. From the statement of Problem 4.6.1,



(a) To find the PMF of W, we simply add the probabilities associated with each possible value of W:

$$P_W(-2) = P_{X,Y}(1,3) = 3/28 \qquad P_W(-1) = P_{X,Y}(2,3) = 6/28 \tag{1}$$

$$P_W(0) = P_{X,Y}(1,1) = 1/28 \qquad P_W(1) = P_{X,Y}(2,1) + P_{X,Y}(4,3) \tag{2}$$

$$P_W(3) = P_{X,Y}(4,1) = 4/28 \qquad = 14/28 \tag{3}$$

For all other values of w, $P_W(w) = 0$.

(b) The expected value of W is

$$E[W] = \sum_w w P_W(w) \tag{4}$$
$$= -2(3/28) + -1(6/28) + 0(1/28) + 1(14/28) + 3(4/28) = 1/2 \tag{5}$$

(c) $P[W > 0] = P_W(1) + P_W(3) = 18/28$.

Problem 5.5.2 ■

In the message transmission problem, Problem 5.3.2, the PMF for the number of transmissions when message i is received successfully is

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3(1-p)^{k_3-3} & k_1 < k_2 < k_3; \\ & k_i \in \{1, 2 \ldots\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $J_3 = K_3 - K_2$, the number of transmissions of message 3; $J_2 = K_2 - K_1$, the number of transmissions of message 2; and $J_1 = K_1$, the number of transmissions of message 1. Derive a formula for $P_{\mathbf{J}}(\mathbf{j})$, the PMF of the number of transmissions of individual messages.

Problem 5.5.2 Solution

The random variable J_n is the number of times that message n is transmitted. Since each transmission is a success with probability p, independent of any other transmission, the number of transmissions of message n is independent of the number of transmissions of message m. That is, for $m \neq n$, J_m and J_n are independent random variables. Moreover, because each message is transmitted over and over until it is transmitted succesfully, each J_m is a geometric (p) random variable with PMF

$$P_{J_m}(j) = \begin{cases} (1-p)^{j-1}p & j = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus the PMF of $\mathbf{J} = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}'$ is

$$P_{\mathbf{J}}(\mathbf{j}) = P_{J_1}(j_1) P_{J_2}(j_2) P_{J_3}(j_3) = \begin{cases} p^3(1-p)^{j_1+j_2+j_3-3} & j_i = 1, 2, \ldots; \\ & i = 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 4.7.1 •

For the random variables X and Y in Problem 4.2.1, find

(a) The expected value of $W = Y/X$,

(b) The correlation, $E[XY]$,

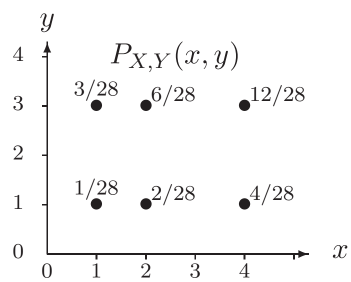
(c) The covariance, $\text{Cov}[X, Y]$,

(d) The correlation coefficient, $\rho_{X,Y}$,

(e) The variance of $X + Y$, $\text{Var}[X + Y]$.

(Refer to the results of Problem 4.3.1 to answer some of these questions.)

Problem 4.7.1 Solution



In Problem 4.2.1, we found the joint PMF $P_{X,Y}(x, y)$ as shown. Also the expected values and variances were

$$E[X] = 3 \qquad \text{Var}[X] = 10/7 \tag{1}$$

$$E[Y] = 5/2 \qquad \text{Var}[Y] = 3/4 \tag{2}$$

We use these results now to solve this problem.

(a) Random variable $W = Y/X$ has expected value

$$E[Y/X] = \sum_{x=1,2,4} \sum_{y=1,3} \frac{y}{x} P_{X,Y}(x, y) \tag{3}$$

$$= \frac{1}{1}\frac{1}{28} + \frac{3}{1}\frac{3}{28} + \frac{1}{2}\frac{2}{28} + \frac{3}{2}\frac{6}{28} + \frac{1}{4}\frac{4}{28} + \frac{3}{4}\frac{12}{28} = 15/14 \tag{4}$$

(b) The correlation of X and Y is

$$r_{X,Y} = \sum_{x=1,2,4} \sum_{y=1,3} xy P_{X,Y}(x, y) \tag{5}$$

$$= \frac{1 \cdot 1 \cdot 1}{28} + \frac{1 \cdot 3 \cdot 3}{28} + \frac{2 \cdot 1 \cdot 2}{28} + \frac{2 \cdot 3 \cdot 6}{28} + \frac{4 \cdot 1 \cdot 4}{28} + \frac{4 \cdot 3 \cdot 12}{28} \tag{6}$$

$$= 210/28 = 105/14 \tag{7}$$

Recognizing that $P_{X,Y}(x, y) = xy/28$ yields the faster calculation

$$r_{X,Y} = E[XY] = \sum_{x=1,2,4} \sum_{y=1,3} \frac{(xy)^2}{28} \tag{8}$$

$$= \frac{1}{28} \sum_{x=1,2,4} x^2 \sum_{y=1,3} y^2 \tag{9}$$

$$= \frac{1}{28}(1 + 2^2 + 4^2)(1^2 + 3^2) = 210/28 = 105/14 \tag{10}$$

(c) The covariance of X and Y is

$$\text{Cov}[X, Y] = E[XY] - E[X] E[Y] = \frac{15}{2} - 3\frac{5}{2} = 0 \tag{11}$$

(d) Since X and Y have zero covariance, the correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = 0. \tag{12}$$

(e) Since X and Y are uncorrelated, the variance of $X+Y$ is

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] = \frac{61}{28}. \tag{13}$$

Problem 5.6.1 •

Random variables X_1 and X_2 have zero expected value and variances $\text{Var}[X_1] = 4$ and $\text{Var}[X_2] = 9$. Their covariance is $\text{Cov}[X_1, X_2] = 3$.

(a) Find the covariance matrix of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$.

(b) X_1 and X_2 are transformed to new variables Y_1 and Y_2 according to

$$\begin{aligned} Y_1 &= X_1 - 2X_2 \\ Y_2 &= 3X_1 + 4X_2 \end{aligned}$$

Find the covariance matrix of $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}'$.

Problem 5.6.1 Solution

(a) The coavariance matrix of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$ is

$$\mathbf{C_X} = \begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1, X_2] \\ \text{Cov}[X_1, X_2] & \text{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}. \tag{1}$$

(b) From the problem statement,

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \mathbf{X} = \mathbf{AX}. \tag{2}$$

By Theorem 5.13, $\mathbf{Y}$ has covariance matrix

$$\mathbf{C_Y} = \mathbf{AC_XA}' = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}. \tag{3}$$

Problem 4.11.1 •
Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = ce^{-(x^2/8)-(y^2/18)}.$$

What is the constant c? Are X and Y independent?

Problem 4.11.1 Solution

$$f_{X,Y}(x,y) = ce^{-(x^2/8)-(y^2/18)} \tag{1}$$

The omission of any limits for the PDF indicates that it is defined over all x and y. We know that $f_{X,Y}(x,y)$ is in the form of the bivariate Gaussian distribution so we look to Definition 4.17 and attempt to find values for σ_Y, σ_X, $E[X]$, $E[Y]$ and ρ. First, we know that the constant is

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \tag{2}$$

Because the exponent of $f_{X,Y}(x,y)$ doesn't contain any cross terms we know that ρ must be zero, and we are left to solve the following for $E[X]$, $E[Y]$, σ_X, and σ_Y:

$$\left(\frac{x-E[X]}{\sigma_X}\right)^2 = \frac{x^2}{8} \qquad \left(\frac{y-E[Y]}{\sigma_Y}\right)^2 = \frac{y^2}{18} \tag{3}$$

From which we can conclude that

$$E[X] = E[Y] = 0 \tag{4}$$
$$\sigma_X = \sqrt{8} \tag{5}$$
$$\sigma_Y = \sqrt{18} \tag{6}$$

Putting all the pieces together, we find that $c = \frac{1}{24\pi}$. Since $\rho = 0$, we also find that X and Y are independent.

Problem 5.7.1 •
$\mathbf{X}$ is the 3-dimensional Gaussian random vector with expected value $\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix}4 & 8 & 6\end{bmatrix}'$ and covariance

$$\mathbf{C}_{\mathbf{X}} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}.$$

Calculate

(a) the correlation matrix, $\mathbf{R}_{\mathbf{X}}$,

(b) the PDF of the first two components of $\mathbf{X}$, $f_{X_1,X_2}(x_1,x_2)$,

(c) the probability that $X_1 > 8$.

Problem 5.7.1 Solution

(a) From Theorem 5.12, the correlation matrix of $\mathbf{X}$ is

$$\mathbf{R}_X = \mathbf{C}_X + \boldsymbol{\mu}_X \boldsymbol{\mu}_X' \tag{1}$$

$$= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \begin{bmatrix} 4 & 8 & 6 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 16 & 32 & 24 \\ 32 & 64 & 48 \\ 24 & 48 & 36 \end{bmatrix} = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix} \tag{3}$$

(b) Let $\mathbf{Y} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$. Since $\mathbf{Y}$ is a subset of the components of $\mathbf{X}$, it is a Gaussian random vector with expected velue vector

$$\boldsymbol{\mu}_Y = \begin{bmatrix} E[X_1] & E[X_2] \end{bmatrix}' = \begin{bmatrix} 4 & 8 \end{bmatrix}'. \tag{4}$$

and covariance matrix

$$\mathbf{C}_Y = \begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1, X_2] \\ \mathbf{C}_{X_1} X_2 & \text{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \tag{5}$$

We note that $\det(\mathbf{C}_Y) = 12$ and that

$$\mathbf{C}_Y^{-1} = \frac{1}{12} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}. \tag{6}$$

This implies that

$$(\mathbf{y} - \boldsymbol{\mu}_Y)' \mathbf{C}_Y^{-1} (\mathbf{y} - \boldsymbol{\mu}_Y) = \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} y_1 - 4 \\ y_2 - 8 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} y_1/3 + y_2/6 - 8/3 \\ y_1/6 + y_2/3 - 10/3 \end{bmatrix} \tag{8}$$

$$= \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16 y_1}{3} - \frac{20 y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3} \tag{9}$$

The PDF of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi\sqrt{12}} e^{-(\mathbf{y}-\boldsymbol{\mu}_Y)' \mathbf{C}_Y^{-1} (\mathbf{y}-\boldsymbol{\mu}_Y)/2} \tag{10}$$

$$= \frac{1}{\sqrt{48\pi^2}} e^{-(y_1^2 + y_1 y_2 - 16 y_1 - 20 y_2 + y_2^2 + 112)/6} \tag{11}$$

Since $\mathbf{Y} = \begin{bmatrix} X_1, X_2 \end{bmatrix}'$, the PDF of X_1 and X_2 is simply

$$f_{X_1,X_2}(x_1, x_2) = f_{Y_1,Y_2}(x_1, x_2) = \frac{1}{\sqrt{48\pi^2}} e^{-(x_1^2 + x_1 x_2 - 16 x_1 - 20 x_2 + x_2^2 + 112)/6} \tag{12}$$

(c) We can observe directly from $\boldsymbol{\mu}_X$ and $\mathbf{C}_X$ that X_1 is a Gaussian $(4, 2)$ random variable. Thus,

$$P[X_1 > 8] = P\left[\frac{X_1 - 4}{2} > \frac{8 - 4}{2}\right] = Q(2) = 0.0228 \tag{13}$$

Problem 5.8.1 •

Consider the vector $\mathbf{X}$ in Problem 5.7.1 and define the average to be $Y = (X_1+X_2+X_3)/3$. What is the probability that $Y > 4$?

Problem 5.8.1 Solution

We can use Theorem 5.16 since the scalar Y is also a 1-dimensional vector. To do so, we write

$$Y = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \mathbf{X} = \mathbf{A}\mathbf{X}. \tag{1}$$

By Theorem 5.16, Y is a Gaussian vector with expected value

$$E[Y] = \mathbf{A}\boldsymbol{\mu}_X = (E[X_1] + E[X_2] + E[X_3])/3 = (4+8+6)/3 = 6 \tag{2}$$

and covariance matrix

$$\mathbf{C}_Y = \text{Var}[Y] = \mathbf{A}\mathbf{C}_X\mathbf{A}' \tag{3}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{2}{3} \tag{4}$$

Thus Y is a Gaussian $(6, \sqrt{2/3})$ random variable, implying

$$P[Y > 4] = P\left[\frac{Y-6}{\sqrt{2/3}} > \frac{4-6}{\sqrt{2/3}}\right] = 1 - \Phi(-\sqrt{6}) = \Phi(\sqrt{6}) = 0.9928 \tag{5}$$

Problem 5.8.3 ■

For the vector of daily temperatures $\begin{bmatrix} T_1 & \cdots & T_{31} \end{bmatrix}'$ and average temperature Y modeled in Quiz 5.8, we wish to estimate the probability of the event

$$A = \left\{ Y \leq 82, \min_i T_i \geq 72 \right\}$$

To form an estimate of A, generate 10,000 independent samples of the vector $\mathbf{T}$ and calculate the relative frequency of A in those trials.

Problem 5.8.3 Solution

We are given that the random temperature vector has covariance

$$\text{Cov}[T_i, T_j] = C_T[i-j] = \frac{36}{1+|i-j|} \tag{1}$$

where i and j indicate the ith and jth days of the month and corresponding elements of the temperature vector. Thus

$$\mathbf{C}_\mathbf{T} = \begin{bmatrix} C_T[0] & C_T[1] & \cdots & C_T[30] \\ C_T[1] & C_T[0] & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_T[1] \\ C_T[30] & \cdots & C_T[1] & C_T[0] \end{bmatrix}. \tag{2}$$

Applying the method discussed on p. 236 of the textbook, we use the matlab script below to generate the 10,000 samples and estimate the relative frequency of occurance of the event A.

```
%
% Solves problem 5.8.3 of Yates and Goodman.  3/09/06 sk
%
% 5.8.3  Generate 10,000 random samples of the vector T of length 31
% having E[T_i]=80 and Cov[T_i,T_j]=36/(1+|i-j|).  Let Y = sum_i(T_i)/31.
% Let A be the event that Y \leq 82 AND min_i(T_i) \geq 72
%
   muT = 80*ones(31,1);
% First generate covariance matrix C.
   for iindex=1:31,
       for jindex = 1:31,
   C(iindex,jindex)=36/(1+abs(iindex-jindex));
       end;
   end;
% Then generate samples.  To avoid having a 10,000x31 variable in our
% workspace, after generating each sample, let's keep only Y and min(T).
   [u,d,v]=svd(C); % as shown on page 236.
   for index = 1:10000,
       T = v*(d^(0.5))*randn(31,1) + muT;
       Y(index) = sum(T)/31;
       minT(index) = min(T);
   end;
% Finally count the number of times A occurs out of the 10,000 samples.
   for index = 1:10000,
       if and((Y(index)<= 82),(minT(index) >= 72)),
          A(index) = 1;
       else
          A(index) = 0;
       end;
   end;
% Now estimate P[A].

disp(['On the basis of this experiment, we estimate P[A] to be  ',...
       num2str((sum(A)/10000))]);
```

The output obtained is

```
>> p5_8_3
On the basis of this experiment, we estimate P[A] to be  0.0719
```