Take Home Assignment MA2001 #2

Due: Oct 27th, 6PM

For each of the following questions, write down your solution with details of steps. Marks will not be given if only final answers are provided.

1. Suppose $3z^3 - 2yz + x^2 = 2$ determines the function $z = z(x, y)$ as a function of x, y locally at $(x, y, z) = (1, 1, 1)$.

   (a) Find the linear approximation of z at $(x, y, z) = (1, 1, 1)$.

   (b) Find the quadratic surface approximation of z at $(x, y, z) = (1, 1, 1)$.

2. It is given that $f(x, y) = e^{2x} \sin(2y)$.

   (a) Use Taylor's formula to find a linear approximation of $f(x, y)$ at the origin.

   (b) Estimate the error in the linear approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

3. Find the stationary points of the function $f(x, y) = xye^{-2(x^2+y^2)}$ and determine their nature.

4. Let $f(x, y) = x^2 - xy + y^2 - y$. Find the directions $\vec{u}$ and the values of $D_{\vec{u}}f(1, -1)$ for which

   (a) $D_{\vec{u}}f(1, -1)$ is the largest;

   (b) $D_{\vec{u}}f(1, -1)$ is the smallest;

   (c) $D_{\vec{u}}f(1, -1) = 0$;

   (d) $D_{\vec{u}}f(1, -1) = 4$;

   (e) $D_{\vec{u}}f(1, -1) = -3$.

5. **Discovery Question**. (Read Lecture Note Chapter 2 Page 46). Consider a point $P(x_0, y_0)$ and a parabola $y = ax^2 + bx + c$. The value

$$[y_0 - (ax_0^2 + bx_0 + c)]^2$$

is called the square of the vertical displacement of the data point $P(x_0, y_0)$ from the parabola $y = ax^2 + bx + c$. Determine the parabola $y = ax^2 + bx + c$ such that the sum S of the squares of the vertical displacements of the data points

$$\begin{aligned}
P_1(x_1, y_1) &= P_1(1, 1.2),\\
P_2(x_2, y_2) &= P_2(-1, 1.4),\\
P_3(x_3, y_3) &= P_3(2, 4.2),\\
P_4(x_4, y_4) &= P_4(-2, 4.4)
\end{aligned}$$

from the parabola is the smallest. In other words, determine the values of a, b, c such that S is the smallest where

$$S(a, b, c) = \sum_{i=1}^{4} (y_i - ax_i^2 - bx_i - c)^2.$$

Hint: Solve the question by the following steps:

(a) Find the only stationary point (a_0, b_0, c_0) of S.

(b) Use the Taylor series for triple variables, you can expand S at (a_0, b_0, c_0) as

$$\begin{aligned}
S(a, b, c) &= S(a_0, b_0, c_0)\\
&\quad + [S_a(a_0, b_0, c_0)(a - a_0) + S_b(a_0, b_0, c_0)(b - b_0) + S_c(a_0, b_0, c_0)(c - c_0)]\\
&\quad + \frac{1}{2}\left[S_{aa}(a_0, b_0, c_0)(a - a_0)^2 + S_{bb}(a_0, b_0, c_0)(b - b_0)^2 + S_{cc}(a_0, b_0, c_0)(c - c_0)^2\right.\\
&\quad + 2S_{ab}(a_0, b_0, c_0)(a - a_0)(b - b_0)\\
&\quad + 2S_{ac}(a_0, b_0, c_0)(a - a_0)(c - c_0)\\
&\quad \left. + 2S_{bc}(a_0, b_0, c_0)(b - b_0)(c - c_0)\right]
\end{aligned}$$

Rewrite the above $S(a, b, c) - S(a_0, b_0, c_0)$ in terms of the **quadratic form**:

$$S(a, b, c) - S(a_0, b_0, c_0) = \frac{1}{2} \begin{bmatrix} a - a_0 & b - b_0 & c - c_0 \end{bmatrix} \begin{bmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ab} & S_{bb} & S_{bc} \\ S_{ac} & S_{bc} & S_{cc} \end{bmatrix} \begin{bmatrix} a - a_0 \\ b - b_0 \\ c - c_0 \end{bmatrix}.$$

(c) Show that the matrix

$$A = \begin{bmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ab} & S_{bb} & S_{bc} \\ S_{ac} & S_{bc} & S_{cc} \end{bmatrix}$$

is **positive definite**. Then

$$S(a, b, c) - S(a_0, b_0, c_0) \geq 0$$

for all a, b, c. Therefor proving that (a_0, b_0, c_0) is a global minimum. There are many ways to show that a (symmetric) matrix is positive definite. Please refer to page 346 of the reference book [Advanced Engineering Mathematics (10th ed.) by Erwin Kreyszig, Wiley 2011].