

2. Dynamic Circuits: First-Order Transient

I. Components

- Capacitors (C) [Section 6.2, 6.3]
- Inductors (L) [Section 6.4, 6.5]
- C & L vs. R \rightarrow Storage vs. Dissipation [Section 6.1]

Alexander & Sadiku,
“Fundamentals of Electric Circuits”
5th Edition Chapters 6, 9

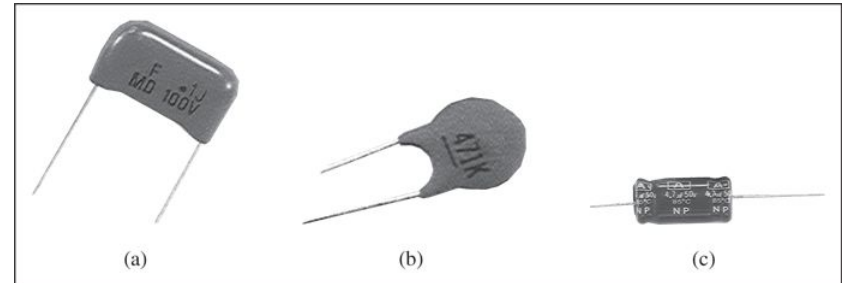
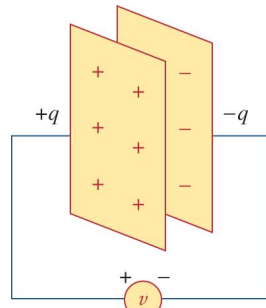
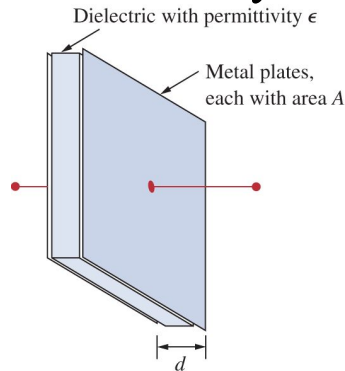
II. Simple RC and RL Circuits

III. Transient Solutions

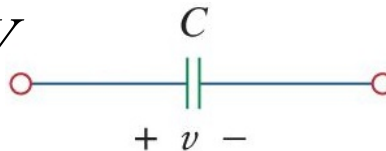
Capacitor and Capacitance

Capacitor

- Simplest way to form a capacitor is to sandwich an insulator (technical term is dielectric) between a pair of parallel conducting plates
- Hence the symbol for a capacitor is two parallel lines separated by a gap



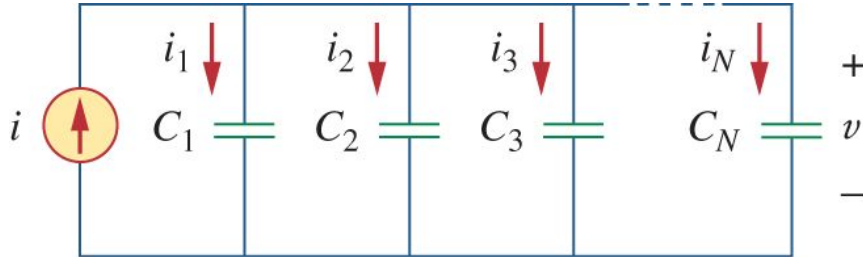
$$Q = CV$$



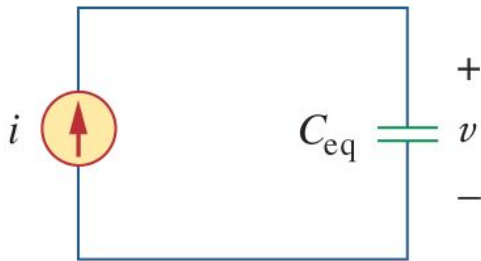
Capacitance

- Commonly symbolized by the letter C with unit of Farads (F)
- Relates the amount of charge stored for a given voltage applied
- No energy dissipated unlike resistors
- Energy is stored (in keeping the plates apart)

Capacitors in parallel



(a)

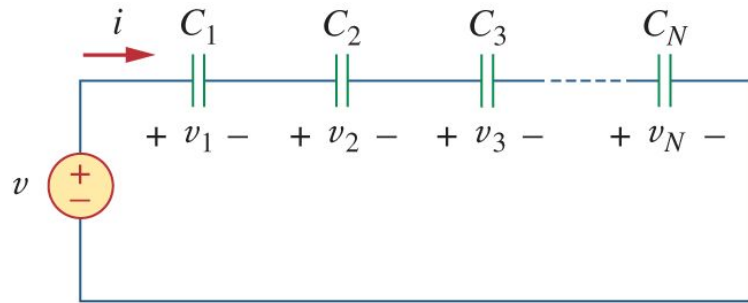


(b)

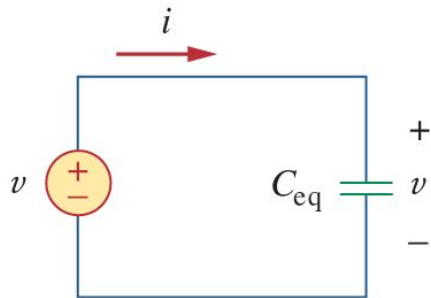
$$C_{eq} = C_1 + C_2 + C_3 + \dots + C_N$$

What is the value of Q_{eq} ?

Capacitors in series



(a)

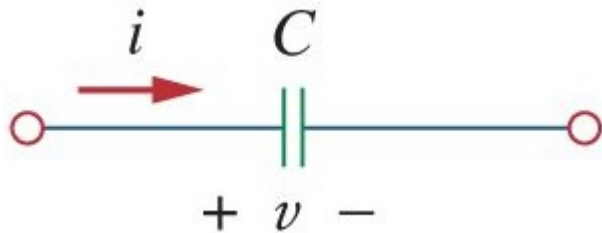


(b)

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_N}$$

I-V relation in a capacitor

If the voltage across the capacitor is time-varying, then the charge stored on the capacitor must also be time-varying:



$$Q(t) = CV(t) \quad \text{Differentiating with respect to time (t):}$$

$$i(t) = C \frac{dV(t)}{dt}$$

Current depends on the
rate of change of voltage

Case study to consider:

Given that $C = 1 \text{ nF}$

If $V = 1 \text{ V} \rightarrow Q = \underline{1 \text{ nC}}$

If we reverse V , such that now $V = -1 \text{ V} \rightarrow Q = \underline{\hspace{2cm}}$

If the above change was made gradually over 1 ms , what would be the resulting current?

$$dV/dt = (-1) - 1 / 1\text{e-}3 \rightarrow i = -2\text{e-}6\text{A}$$

Response in Capacitor

$$\begin{array}{c} + \\ V \\ - \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \circ \\ \circ \end{array} \quad Q(t) = CV(t) \xrightarrow[\text{with respect to time}]{\text{Differentiate}} \boxed{i(t) = C \frac{dV(t)}{dt}}$$

If voltage is constant with time: $dV/dt = 0 \rightarrow I = 0$

No voltage change \rightarrow No current

If voltage changes with time: $dV/dt \neq 0 \rightarrow I \neq 0$

Voltage change → Charge changes → Current

Summary: Capacitor response to DC

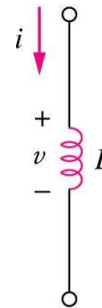
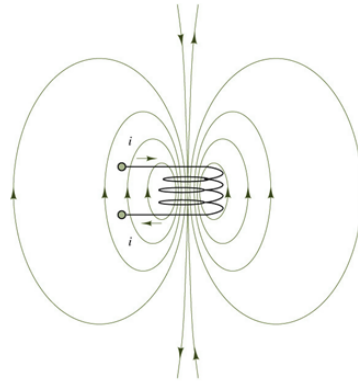
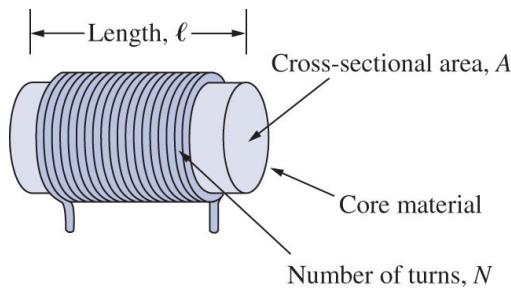
If applied voltage is DC

- Insulating dielectric blocks the current from flowing through
- Plates will charge up
- At DC, capacitor blocks current from flowing through

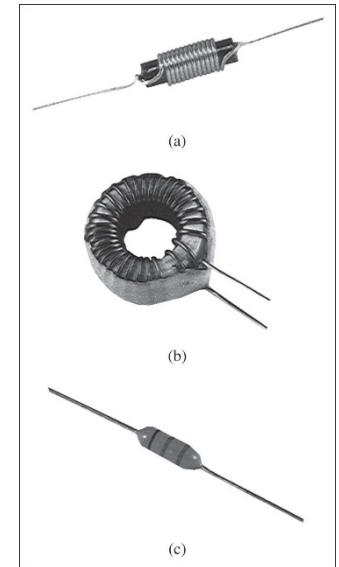
Inductors and Inductance

Inductor

- Simplest way to form an inductor is to winding a coil around a core that concentrates magnetic field lines (flux)
- Hence symbol of an inductor is coil between two terminals



$$\phi = Li$$

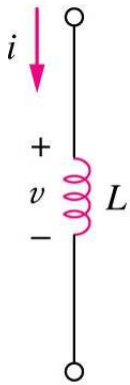


Inductance

- Inductance is commonly symbolized by letter L with unit of Henrys (H)
- Passing a current through an inductor produces a magnetic flux (ϕ) that is related to the inductance (L)
- No energy dissipated unlike resistors (note that wires are assumed to have no resistance by definition)

I-V relation in an inductor

If the current through an inductor is time-varying, then the generated voltage must also be time-varying:



$$\text{Flux: } \phi = Li$$

$$\text{Faraday's Law: } v(t) = \frac{d\phi(t)}{dt}$$

$$\text{Differentiating with respect to time (t): } v(t) = L \frac{di(t)}{dt}$$

Voltage depends on the **rate of change** of current

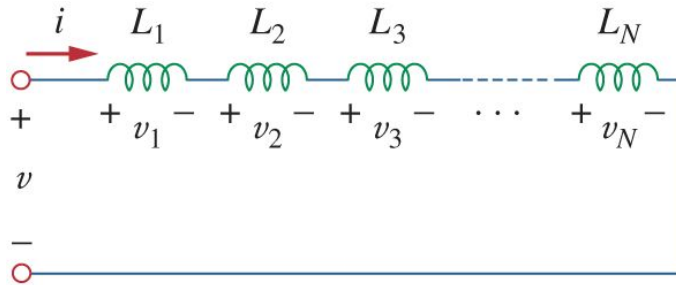
If current is constant with time $\rightarrow di/dt = 0 \rightarrow V = 0$ (No voltage)

No current change \rightarrow No voltage difference

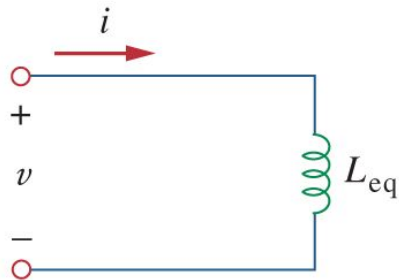
If current changes with time $\rightarrow di/dt \neq 0 \rightarrow V \neq 0$ (There is voltage)

Current change \rightarrow Voltage difference

Inductors in series



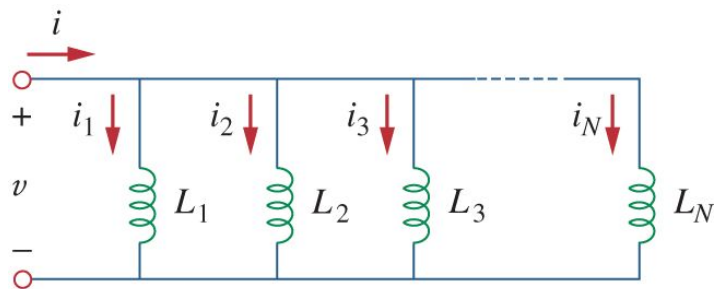
(a)



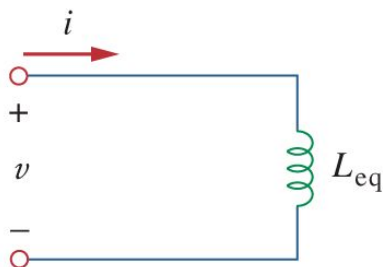
(b)

$$L_{eq} = L_1 + L_2 + L_3 + \dots + L_N$$

Inductors in parallel



(a)



(b)

$$\frac{1}{L_{eq}} \int v(t) dt = \frac{1}{L_1} \int v(t) dt + \frac{1}{L_2} \int v(t) dt + \frac{1}{L_3} \int v(t) dt + \cdots + \frac{1}{L_N} \int v(t) dt$$

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_N}$$

Important characteristics of the basic elements

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i	$v = iR$	$v = \frac{1}{C} \int_{t_0}^t i(\tau) d\tau + v(t_0)$	$v = L \frac{di}{dt}$
i-v	$i = v/R$	$i = C \frac{dv}{dt}$	$i = \frac{1}{L} \int_{t_0}^t v(\tau) d\tau + i(t_0)$
Two in series	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
At DC	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly	Not applicable	v	i

When a circuit has C and/or L, the circuit becomes dynamic.

- Voltage and/or current is a function of time.
- Voltage and/or current is described by differential equation.
- The circuit has
 - › **transient response** (circuit response immediately after certain initial condition) AND
 - › **steady state response** (as $t \rightarrow \infty$)

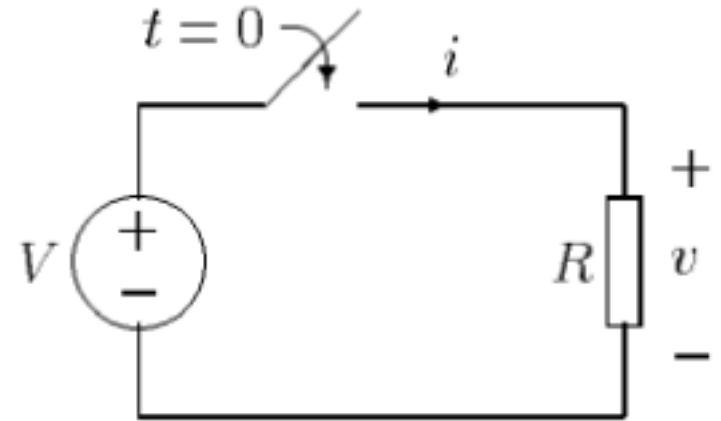


Note that pure resistive circuits have no transient!

When we turn on the switch at $t = 0$, potential difference across the resistor R becomes V immediately.

For all $t \geq 0$,

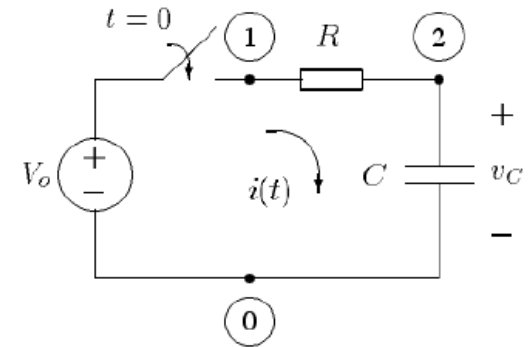
- $v = V = iR$
- $i = V/R$



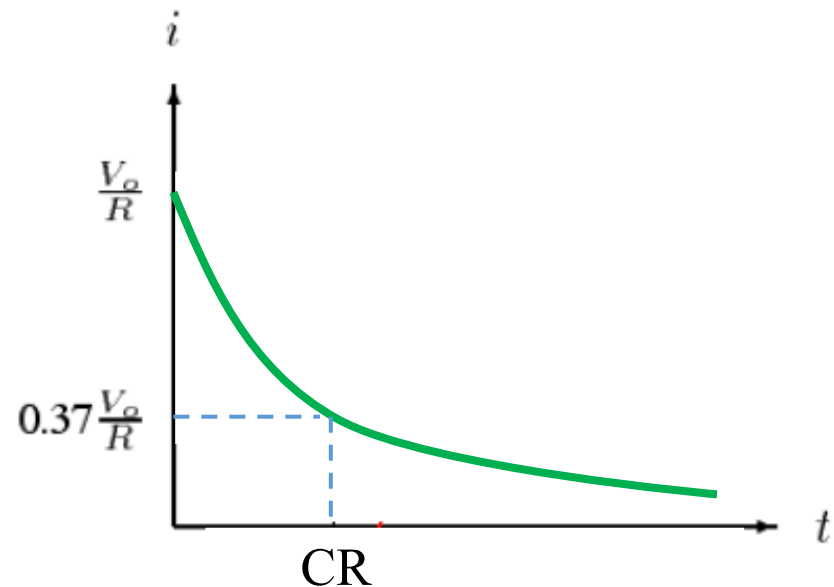
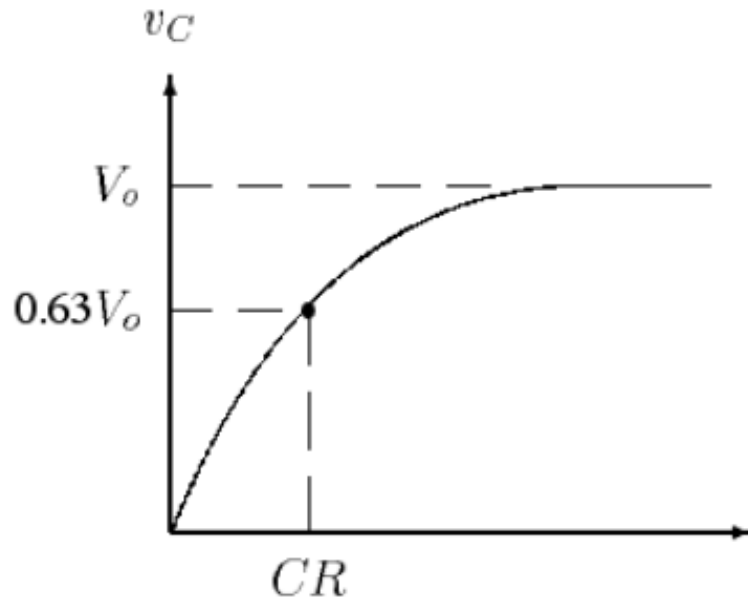
Simple first-order RC circuit

- At $t = 0$, $v_c(0) = 0$
- After $t = 0$, the circuit is closed:
 - › $i(t) = \frac{v_R}{R} = C \frac{dv_c}{dt}$, and
 - › $v_R(t) = V_0 - v_c(t)$
- Hence, $\frac{V_0 - v_c(t)}{CR} = \frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} + \frac{v_c}{CR} = \frac{V_0}{CR}$
- The general solution is $v_c(t) = Ae^{-\frac{t}{CR}} + V_0$ for $t \geq 0$ (A is a constant)
- As the initial condition is $v_c(0^+) = 0$, $A + V_0 = 0 \Rightarrow A = -V_0$
- Hence,

$$v_c(t) = V_0 \left(1 - e^{-\frac{t}{CR}} \right)$$



Transient response of a simple RC circuit



$$v_c(t) = V_0(1 - e^{-\frac{t}{CR}}) \quad \Rightarrow \quad i(t) = C \frac{dv_c}{dt} = \frac{V_0}{R} e^{-\frac{t}{CR}}$$

When $t = CR$,

$$v_c(CR) = V_0(1 - e^{-1}) = 0.63V_0 \quad \text{and} \quad i(CR) = \frac{V_0}{R} e^{-1} = 0.37V_0$$

Simple first-order RL circuit

- When $t < 0$, short-circuit at switch, $i_L = 0$

- For $t \geq 0$,

- $v_L(t) = L \left(\frac{di_L}{dt} \right)$, and

- $v_R(t) = (I_0 - i_L(t))R$

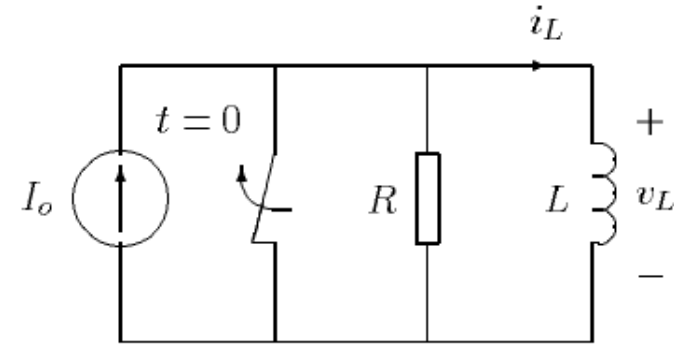
- Hence, $(I_0 - i_L(t))R = L \left(\frac{di_L}{dt} \right) \Rightarrow \frac{di_L}{dt} + \frac{R}{L} i_L(t) = \frac{RI_0}{L}$

- The general solution is $i_L(t) = Ae^{-\frac{Rt}{L}} + I_0$ where A is a constant

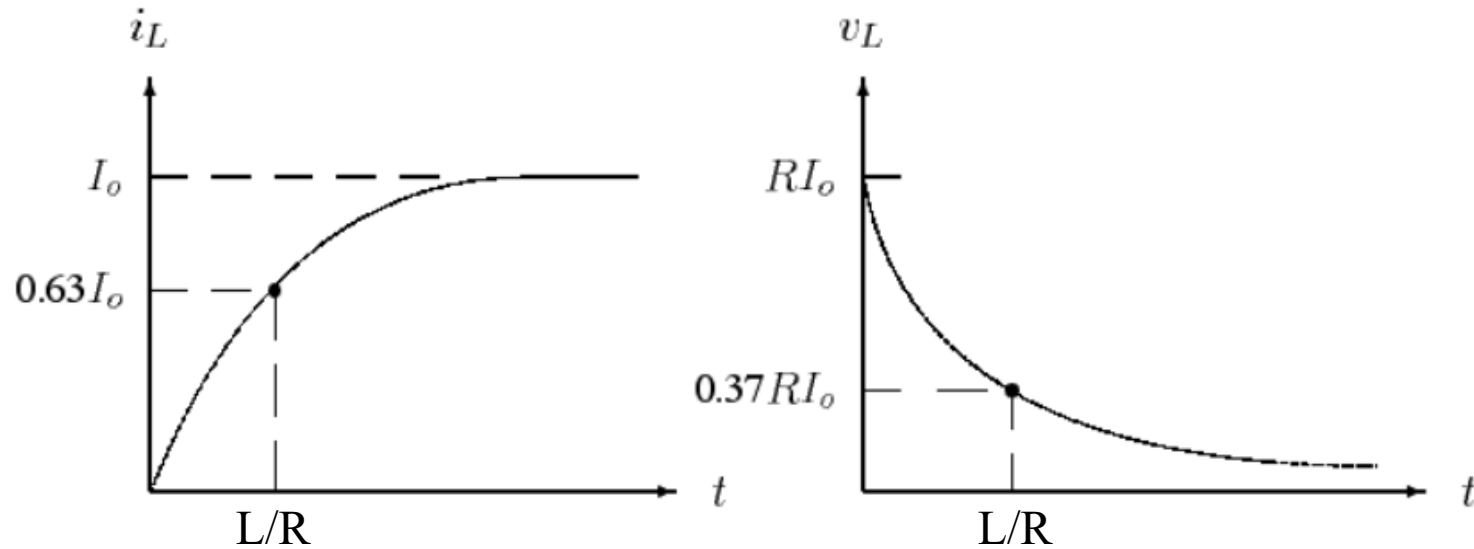
- $i_L(0^+) = 0 = A + I_0 \Rightarrow A = -I_0$

- Hence, for $t \geq 0$, we obtain

$$i_L(t) = I_0 \left(1 - e^{-\frac{Rt}{L}} \right)$$

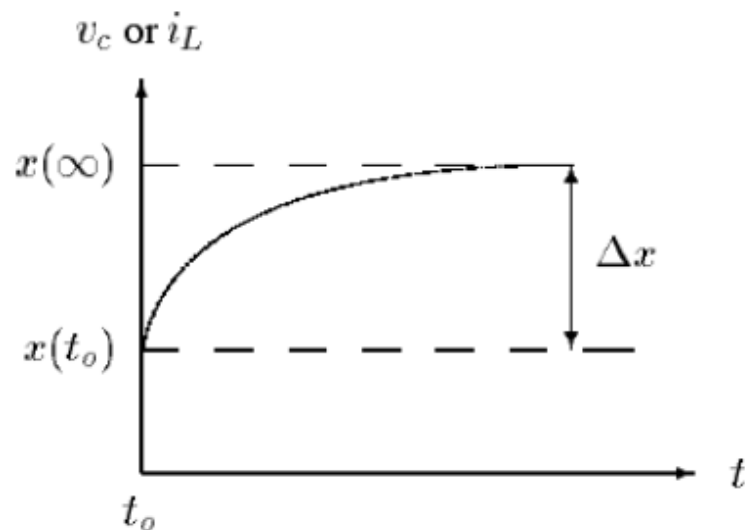


Transient response of a simple RL circuit

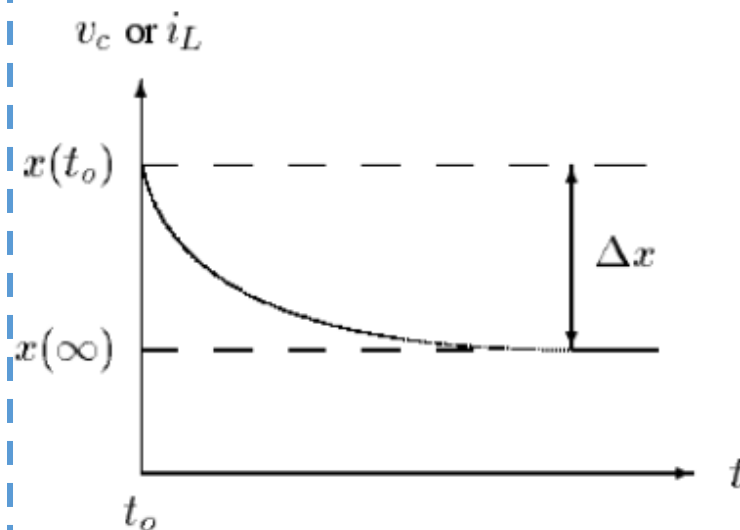


$$i_L(t) = I_o \left(1 - e^{-\frac{Rt}{L}}\right) \quad \Rightarrow \quad v_L(t) = L \frac{di_L(t)}{dt} = RI_o e^{-\frac{Rt}{L}}$$

General first-order solution



Case A



Case B

$$x(t) = x(t_0) + \Delta x(1 - e^{(-t-t_0)/\tau})$$

$$x(t) = x(\infty) + \Delta x e^{(-t-t_0)/\tau}$$

1. Find the time constant τ
2. Find initial value $x(t_0)$ and final value $x(\infty)$
3. Determine whether Case A or Case B expression shall be used

Time constant τ

- Simple 1st order RC circuit

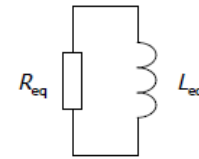
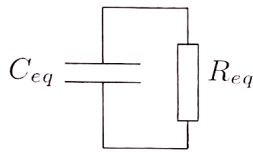
$$\tau = CR$$

- Simple 1st order RL circuit

$$\tau = \frac{L}{R}$$

To find τ from an equivalent simple RC or RL circuit

1. Short-circuit all voltage sources and open-circuit all current sources
2. Place switches in their final positions
3. Reduce resistances to one equivalent resistance R_{eq} , if possible



Reduce capacitances to one equivalent capacitance C_{eq} (if possible) \rightarrow

Time constant of any RC circuit

$$\tau = C_{eq}R_{eq}$$

Reduce inductances to one equivalent inductance L_{eq} (if possible) \rightarrow

Time constant of any RL circuit

$$\tau = \frac{L_{eq}}{R_{eq}}$$

Example 1: Initial and final values given

- At $t = 0$, the switch is thrown to the right
- Initial and final values of capacitor voltage are:

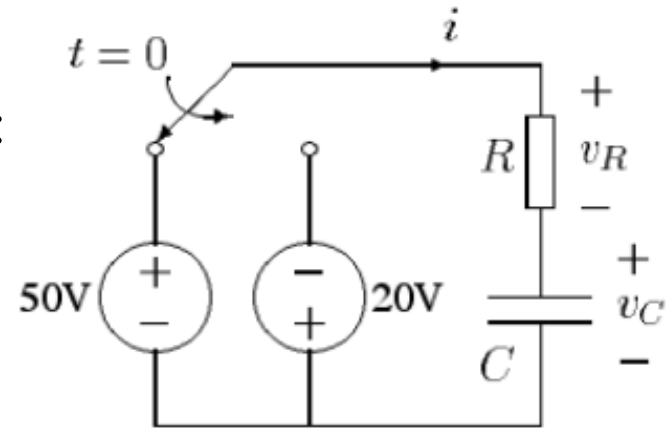
$$v_c(0^+) = 50V \text{ and } v_c(\infty) = -20V$$

- Decrease in $v_c \rightarrow$ Case B ($\tau = CR$)

$$v_c(t) = -20 + 70e^{-\frac{t}{CR}}$$

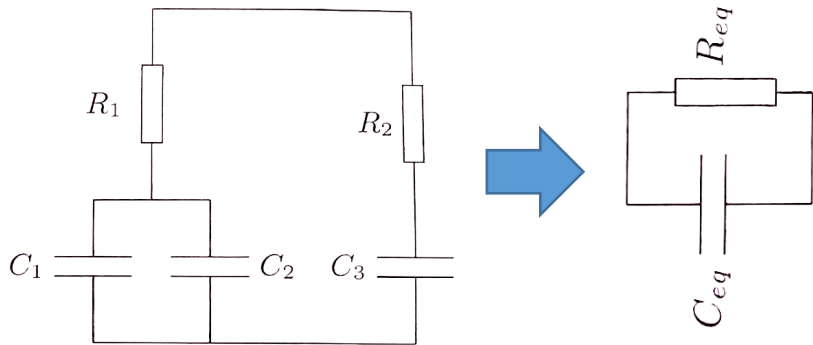
- $v_R(t) = -20 - v_c(t) = -70e^{-\frac{t}{CR}}$

- $i(t) = \frac{v_R(t)}{R} = -\frac{70}{R}e^{-\frac{t}{CR}}$



Example 2 (non-trivial boundary conditions)

- At $t = 0$, the switch is thrown to the right

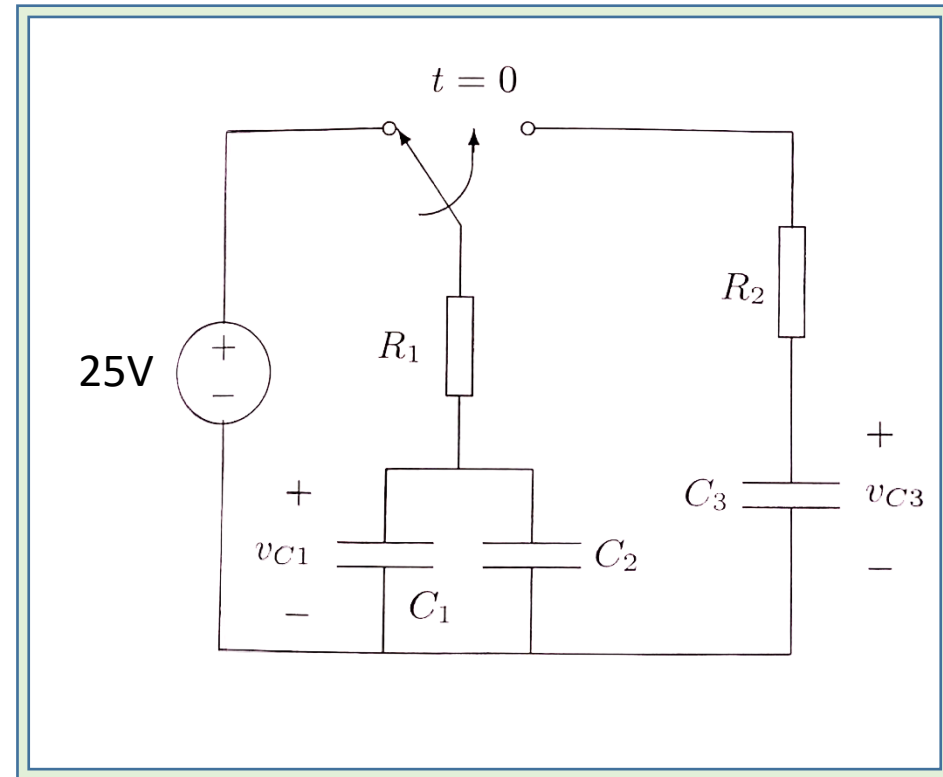


→ Find equivalent resistance:

$$R_{eq} = R_1 + R_2$$

→ Find equivalent capacitance

$$\frac{1}{C_{eq}} = \frac{1}{C_1 + C_2} + \frac{1}{C_3} \Rightarrow C_{eq} = \frac{C_3(C_1 + C_2)}{(C_1 + C_2 + C_3)}$$



Example 2 (non-trivial boundary conditions)

- Time constant $\tau = \frac{C_3(C_1+C_2)(R_1+R_2)}{C_1+C_2+C_3}$

- Initial values are

$$v_{c_1}(0^+) = 25, \quad v_{c_2}(0^+) = 25, \quad v_{c_3}(0^+) = 0$$

- To find the final values after the switch throw to right

$$(C_1 + C_2) \frac{dv_{c_1}}{dt} + C_3 \frac{dv_{c_3}}{dt} = 0$$

$$\Rightarrow (C_1 + C_2)v_{c_1}(t) + C_3v_{c_3}(t) = \text{constant} \quad \text{for all } t \geq 0$$

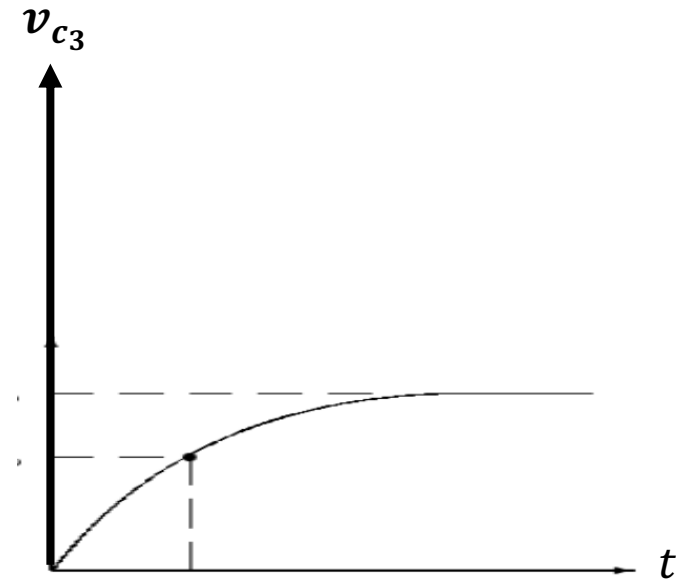
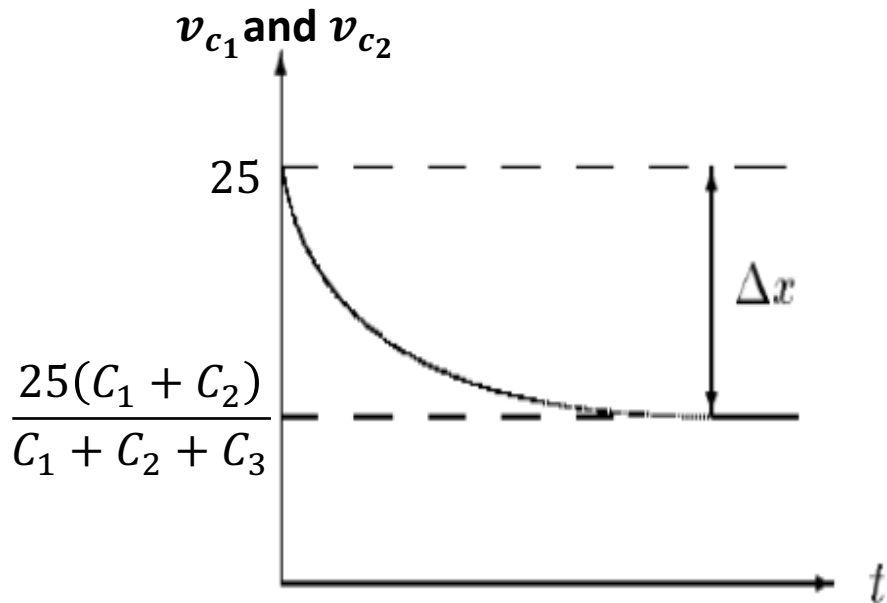
- $v_{c_1}(0^+) = 25$ and $v_{c_3}(0^+) = 0 \Rightarrow 25(C_1 + C_2) = \text{constant}$

$$\Rightarrow (C_1 + C_2)v_{c_1}(t) + C_3v_{c_3}(t) = 25(C_1 + C_2)$$

- The final values must satisfy $v_{c_1}(\infty) = v_{c_3}(\infty) \Rightarrow$

$$v_{c_1}(\infty) = v_{c_2}(\infty) = v_{c_3}(\infty) = \frac{25(C_1 + C_2)}{C_1 + C_2 + C_3}$$

Example 2 (non-trivial boundary conditions)



■ v_{c1} and v_{c2} : Case B

$$v_{c1}(t) = v_{c2}(t)$$

$$= \frac{25(C_1 + C_2)}{C_1 + C_2 + C_3} + \frac{25C_3}{C_1 + C_2 + C_3} e^{-\frac{t}{C_{eq}R_{eq}}}$$

■ v_{c3} : Case A

$$v_{c3}(t) = \frac{25(C_1 + C_2)}{C_1 + C_2 + C_3} (1 - e^{-\frac{t}{C_{eq}R_{eq}}})$$