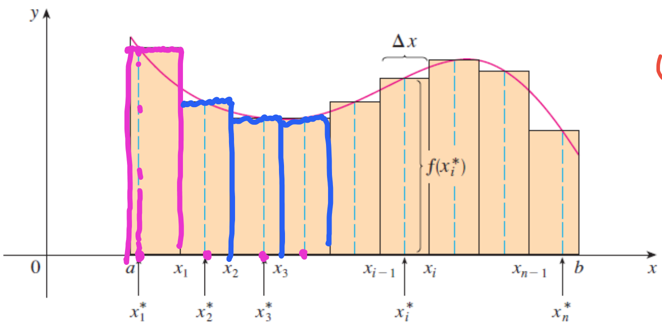
Chapter 3. Multiple Integral

1 Single-Variable Case:

- Definition and Interpretation:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} [\,f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x] = \lim_{n\to\infty} \sum_{j=1}^{n} \underbrace{f(x_j^*)}_{\text{height}}\ \underbrace{\Delta x_j}_{\text{base}}$$

= Area of the region between the curve f and x-axis.



- Computation:

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is an antiderivative of f, i.e. $F'(x) = f(x)$.

eg. $F(x) = \frac{x^3}{3}$ is an antiderive of x^2

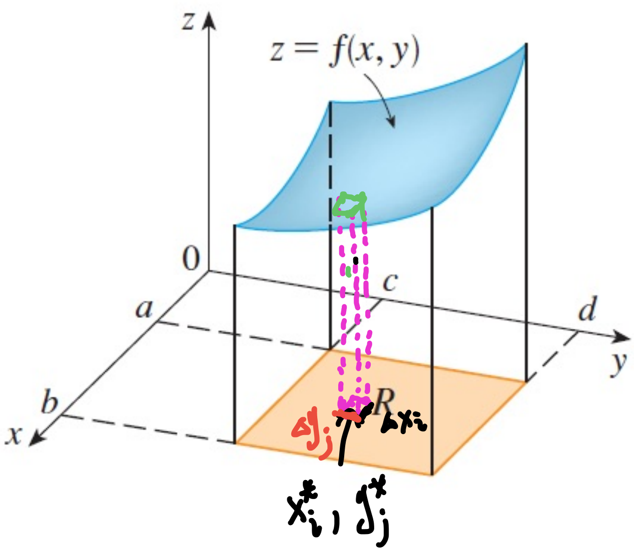
$$\int_1^2 x^2 dx = \left(\frac{x^3}{3}\right)\Big|_{x=2} - \left(\frac{x^3}{3}\right)\Big|_{x=1}$$

$$= \frac{2^3}{3} - \frac{1}{3} = \frac{7}{3}$$

2 Two-Variable Case (Double Integral):

$z = f(x, y).$

2.1 Definition and Interpretation:

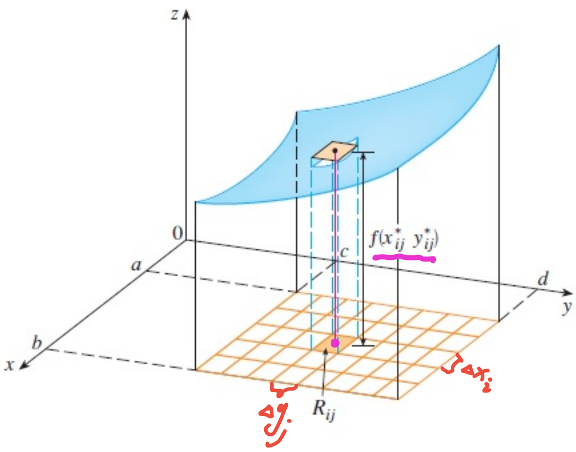


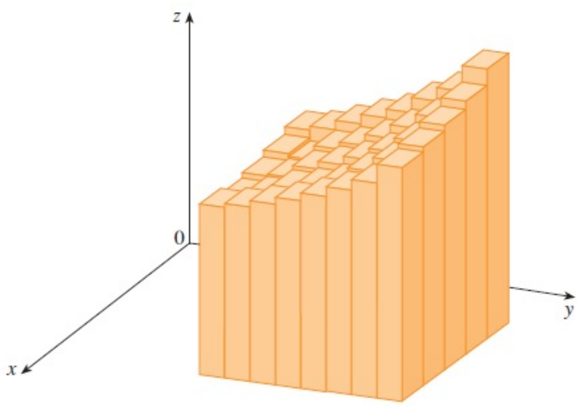
$$V_{ij} = \underbrace{f(x_i^*, y_j^*)}_{\text{height.}} \cdot \underbrace{\Delta x_i \ \Delta y_j}_{\text{base}}$$

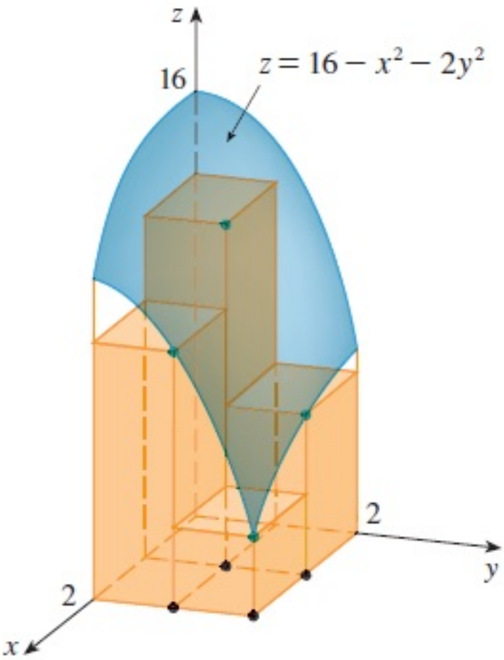
$$\iint_R f(x,y)\,dx\,dy = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \underbrace{f(x_i^*, y_j^*)\,\Delta x_i \cdot \Delta y_j}$$

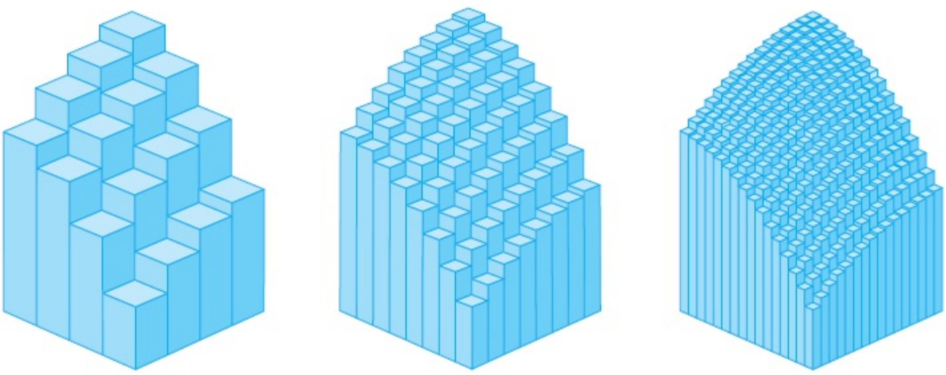
$=$ Volume of the solid below the surface of f and above xy-plane.

double integral of f on region R.









(a) $m = n = 4$, $V \approx 41.5$

(b) $m = n = 8$, $V \approx 44.875$

(c) $m = n = 16$, $V \approx 46.46875$

How to evaluate $\iint_R f(x,y)\,dxdy$

2.2 Computation of Double Integrals:

Case 1: R is an rectangle: $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

$$\iint_R f(x,y)\,dxdy = \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)(\Delta x_i)(\Delta y_j).$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\lim_{m\to\infty}\sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta y_j\right]\Delta x_i$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\int_c^d f(x_i^*, y)\,dy\right]\Delta x_i$$

$$\overset{\Delta}{=} \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx \leftarrow \text{iterated integral in } dy\,dx$$

Similarly,

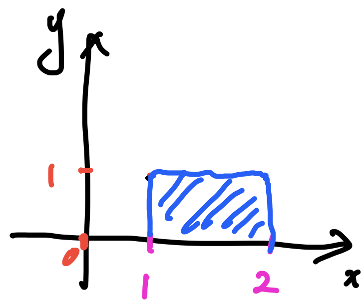
$$\iint_R f(x,y)\,dxdy = \lim_{m\to\infty}\sum_{j=1}^{m}\left[\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_j^*)\,\Delta x_i\right]\Delta y_j = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

iterated integral in $dx\,dy$

Example Use iterated integral in two different orders to evaluate

$$\iint_R (2xy + y^2)\,dxdy$$

with $R = [1,2]\times[0,1]$.



$$\iint_R (2xy + y^2)\,dxdy = \int_0^1\int_1^2 (2xy + y^2)\,dx\,dy.$$

treat y as constant $= \int_0^1 \left[x^2y + xy^2\right]\Big|_{x=1}^{x=2} dy.$

$$= \int_0^1 \left[2^2y + 2y^2 - y - y^2\right] dy$$

$$= \int_0^1 (3y\ {}^3 + y^2)\,dy = \left(\frac{3y^2}{2} + \frac{y^3}{3}\right)\Big|_{y=0}^{y=1}$$

$$= \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$$

$$\iint_R (2xy + y^2)\, dx\,dy = \int_1^2 \left[\int_0^1 \underline{(2xy + y^2)}\, dy \right] dx$$

treat x as constant

$$= \int_1^2 \left[\left(xy^2 + \frac{y^3}{3} \right) \Bigg|_{y=0}^{y=1} \right] dx$$

$$= \int_1^2 \left(x + \frac{1}{3} - 0 \right) dx$$

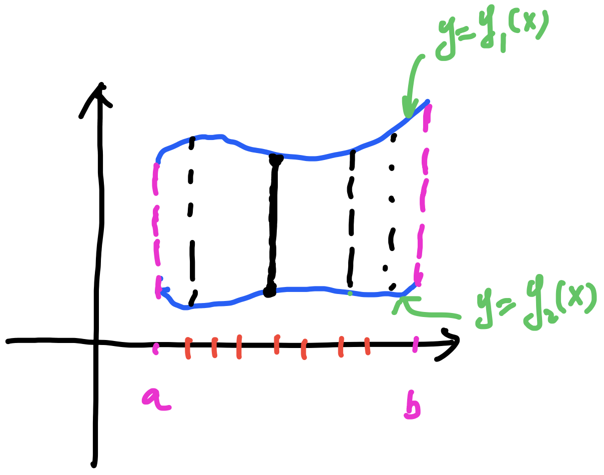
$$= \left(\frac{x^2}{2} + \frac{1}{3}x \right) \Bigg|_{x=1}^{x=2}$$

$$= \left(\frac{2^2}{2} + \frac{2}{3} \right) - \left(\frac{1}{2} + \frac{1}{3} \right)$$

$$= \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$$

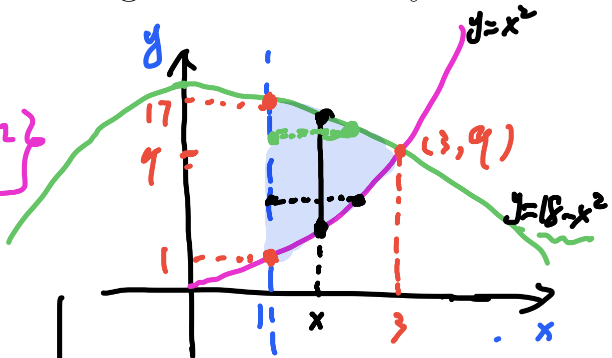
Case 2. R is <u>vertically</u> simple (vertical segments are easily bounded in x)

$R = \{(x,y) : a \le x \le b,\ y_2(x) \le y \le y_1(x)\}$



$$\iint_R f(x,y)\,dxdy = \int_a^b \left[\int_{y=y_2(x)}^{y=y_1(x)} f(x,y)\,dy\right] dx$$

Example Evaluate $\iint_R 2x^2y\,dxdy$, with R as a region bounded by $x = 1, x = 3, y = x^2$ and $y = -x^2 + 18$.



$R = \{(x,y) : 1 \le x \le 3,\ x^2 \le y \le 18 - x^2\}$

$$\iint_R 2x^2y\,dxdy = \int_1^3 \left[\int_{x^2}^{18-x^2} 2x^2y\,dy\right] dx$$

$$= \int_1^3 \left[x^2y^2 \Big|_{y=x^2}^{y=18-x^2}\right] dx$$

$$= \int_1^3 \left[x^2(18-x^2)^2 - x^2\cdot x^2\right] dx$$

$$= \dots = \frac{5328}{2}$$

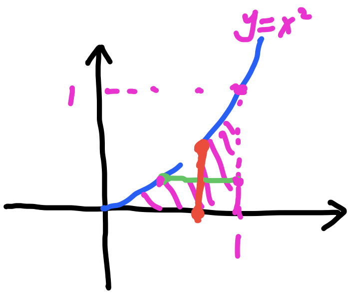
$$\iint_R 2x^2y\,dxdy = \int_1^{17}\left[\int_{\bigcirc}^{\bigcirc} 2x^2y\,dx\right] dy$$

$$= \int_1^9 \left[\int_1^{\sqrt{y}} 2x^2y\,dx\right] dy + \int_9^{17}\left[\int_1^{\sqrt{18-y}} 2x^2y\,dx\right] dy$$

$$= \dots =$$

Exercise Evaluate $\iint_R xy^2\,dxdy$, with R as a region bounded by $x = 1, y = 0$ and $y = x^2$.

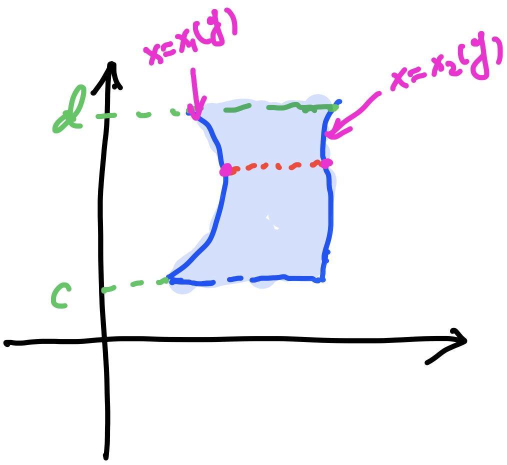
$$\iint_R xy^2\,dxdy = \int_0^1 \int_{y=0}^{y=x^2} xy^2\,dy\,dx$$



$$= \int_{0}^{1} \int_{x=\sqrt{y}}^{x=1} x y^2 \, d\underline{x} \, dy$$

Case 3. R is horizontally simple (horizontal segments are easily bounded in y)

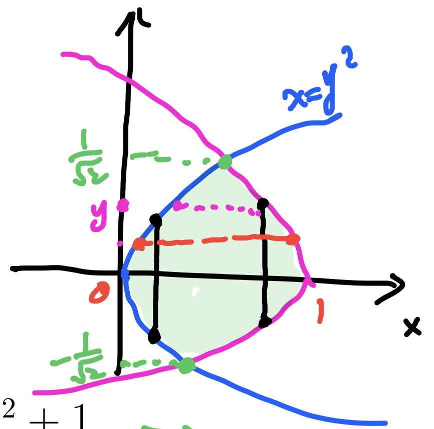
$R = \{(x,y) : c \le y \le d,\ x_1(y) \le x \le x_2(y)\}$.



$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\right] dy$$

Example Evaluate

$$\iint_R xy^2\,dx\,dy$$



, with R as a region bounded by $x = y^2$ and $x = -y^2 + 1$.

$\Rightarrow y^2 = -y^2 + 1$

$\Rightarrow y = \pm\sqrt{\frac{1}{2}}$

$R = \{(x,y) : -\frac{1}{\sqrt{2}} \le y \le \frac{1}{\sqrt{2}},\ y^2 \le x \le 1-y^2\}$

$$\iint_R xy^2\,dx\,dy = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[\int_{y^2}^{1-y^2} xy^2\,dx\right] dy$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[\frac{x^2}{2}y^2 \Big|_{x=y^2}^{x=1-y^2}\right] dy$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left[\frac{(1-y^2)^2}{2}y^2 - \frac{(y^2)^2}{2}y^2\right] dy = \cdots$$

$R = \{ 0 \le x \le 1 \,,\; x \le y \le \sqrt{x} \}.$

Example Show that

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)\; dy\; dx = \int_0^1 \int_{y^2}^{y} f(x,y)\; dx\; dy.$$

$R = \{ 0 \le y \le 1 \,,\; y^2 \le x \le y \}$