MA2001 Vector Differential Calculus

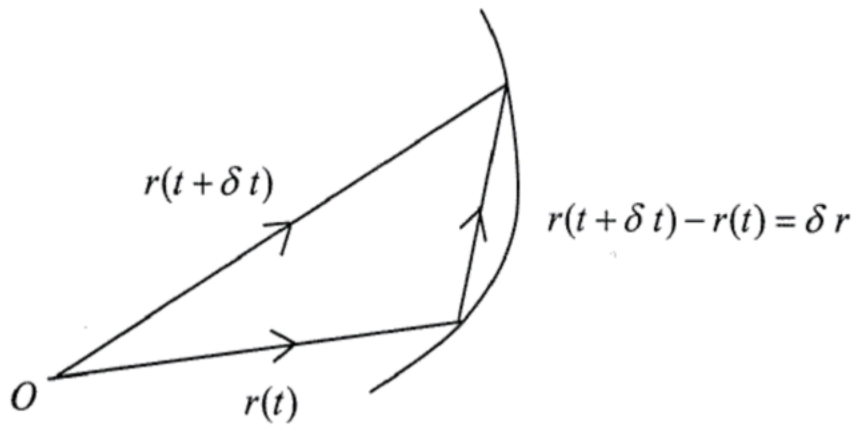
1. Differentiation of a Vector Function

If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is a vector function of a scalar variable t then the derivative of $\vec{r}(t)$ with respect to t is:

$$\frac{d\vec{r}(t)}{dt} = \lim_{\delta t \to 0}\left[\frac{\vec{r}(t+\delta t)-\vec{r}(t)}{\delta t}\right]$$
$$= \lim_{\delta t \to 0}\frac{x(t+\delta t)-x(t)}{\delta t}\vec{i} + \lim_{\delta t \to 0}\frac{y(t+\delta t)-y(t)}{\delta t}\vec{j} + \lim_{\delta t \to 0}\frac{z(t+\delta t)-z(t)}{\delta t}\vec{k}$$
$$= \frac{dx}{dt}(t)\vec{i} + \frac{dy}{dt}(t)\vec{j} + \frac{dz}{dt}(t)\vec{k}$$

Example

If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is the position vector of a moving point, where t represents time,



$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0}\frac{\delta\vec{r}}{\delta t}$ represents the velocity of the moving point and is seen to be a vector tangential to the path of the point at $\vec{r}(t)$. Similarly, $\frac{d^2\vec{r}}{dt^2}$ represents the acceleration of the moving point.

□

Example

Let C be a space curve with a parametric representation as $C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, a \le t \le b$, the number t is called a parameter.

C is a space curve of class $C^{(1)}$ (a smooth space curve) if $x(t), y(t), z(t)$, $a \le t \le b$ are continuous and $x'(t), y'(t), z'(t), a \le t \le b$ exist and are continuous, by $x'(a), y'(a), z'(a), x'(b), y'(b), z'(b)$ we mean, respectively, right-hand and left-hand derivatives, in addition, $\frac{d\vec{r}}{dt}(t) \neq \overrightarrow{0}, a \leq t \leq b$.

The length l of the curve C is $l = \int_a^b \left|\frac{d\vec{r}}{dt}(t)\right| dt$.

Every smooth cure C has a parametric representation of particular geometric interest. It is called the standard representation and is defined as follows.

Let $\vec{r}(t)$ represent C on $a \leq t \leq b$ and let $s(t) = \int_a^t \left|\frac{d\vec{r}}{du}(u)\right| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du, a \leq t \leq b$,

which is the length of the part of C represented on $a \leq l \leq t$. Clearly, $s(a) = 0, s(b) = l$.

Observe that $s'(t) = |\vec{r}'(t)| > 0$, so the equation $s = s(t)$ can be solved for t, that is, the function has an inverse, say $t = t(s)$ of class $C^{(1)}$ for $0 \leq s \leq l$.

Consider $\vec{r}(s) = \vec{r}[t(s)] = x[t(s)]\vec{i} + y[t(s)]\vec{j} + z[t(s)]\vec{k}$ which is the standard representation of C.

Then $\vec{r}\,'(s) = \vec{r}\,'(t)t\,'(s) \underset{\substack{t'(s)=\frac{1}{s'(t)} \\ s'(t)\neq\vec{0}}}{=} \overrightarrow{r}\,'(t)\frac{1}{s'(t)} = \frac{1}{|\vec{r}\,'(t)|}\vec{r}\,'(t)$

We find that $|\vec{r}\,'(s)| = \left|\frac{1}{|\vec{r}\,'(t)|}\vec{r}\,'(t)\right| = \frac{|\vec{r}\,'(t)|}{|\vec{r}\,'(t)|} = 1$, $\vec{r}\,'(s)$is a unit tangent vector at $\vec{r}(s)$.

$\square$

<u>Example</u>

A particle moves such that its position vector at time t is $\vec{r}(t) = e^{-t}\vec{i} + 2\cos 3t\vec{j} + 3\sin 3t\vec{k}$

Determine its velocity $\vec{v}$ and acceleration $\vec{a}$ at time $t = 0$.

Solution:

$\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\vec{i} - 6\sin 3t\vec{j} + 9\cos 3t\vec{k}, \vec{v}(0) = -\vec{i} + 9\vec{k}$, magnitude $\sqrt{82}$, direction $\frac{1}{\sqrt{82}}(-\vec{i} + 9\vec{k})$.

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\vec{i} - 18\cos 3t\vec{j} - 27\sin 3t\vec{k}, a(0) = \vec{i} - 18\vec{j}$, magnitude $\sqrt{325}$, direction $\frac{1}{\sqrt{325}}(\vec{i} - 18\vec{j})$.

$\square$

The usual rules of differentiation apply

$$\frac{d[\vec{a}(t)+\vec{b}(t)]}{dt} = \frac{d\vec{a}(t)}{dt} + \frac{d\vec{b}(t)}{dt}$$

$$\frac{d[c\vec{a}(t)]}{dt} = c\frac{d\vec{a}(t)}{dt}, \quad c- \text{ constant}$$

and also

$$\frac{d[\vec{a}(t)\cdot\vec{b}(t)]}{dt} = \frac{d\vec{a}(t)}{dt} \cdot \vec{b}(t) + \vec{a}(t) \cdot \frac{d\vec{b}(t)}{dt}$$

$$\frac{d[\vec{a}(t)\times\vec{b}(t)]}{dt} = \frac{d\vec{a}(t)}{dt} \times \vec{b}(t) + \vec{a}(t) \times \frac{d\vec{b}(t)}{dt}$$

If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ where a_1, a_2, a_3 are functions of several variables $t_1, t_2, \cdots, t_n$ then partial derivatives $\frac{\partial \vec{a}}{\partial t_i} = \frac{\partial a_1}{\partial t_i}\vec{i} + \frac{\partial a_2}{\partial t_i}\vec{j} + \frac{\partial a_3}{\partial t_i}\vec{k}, 1 \le i \le n$ may be defined in the obvious way. Also, vector integration may be defined.

<u>Example</u>

If a particle moves with velocity $\vec{v}(t) = 2\sin t\vec{i} + (\cos t - 1)\vec{j} + 6t\vec{k}$ and $\vec{r} = \overrightarrow{0}$ at $t = 0$, determine its position vector $\vec{r}$ at time t.

Solution:

$\vec{r} = \int \frac{d\vec{r}}{dt}dt = \int \vec{v}dt = \int [2\sin t\vec{i} + (\cos t - 1)\vec{j} + 6t\vec{k}]dt$

$= \vec{i}\int 2\sin tdt + \vec{j}\int(\cos t - 1)dt + \vec{k}\int 6tdt$

$= (-2\cos t)\vec{i} + (\sin t - t)\vec{j} + 3t^2\vec{k} + C$

At $t = 0, 0 = -2\vec{i} + C$, hence $C = 2\vec{i}$ and $\vec{r} = (2 - 2\cos t)\vec{i} + (\sin t - t)\vec{j} + 3t^2\vec{k}$.

$\square$

<u>Example</u>

Let $\vec{r}$ in polar form be the position vector of a moving particle in R^2. Find the velocity and the acceleration of $\vec{r}$ in polar form.

Solution:

Represent $\vec{r}$ in polar form, then $\vec{r} = r\vec{R}$, where $\vec{R} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ a unit vector and $r = |\vec{r}|$.

The velocity $\vec{r}$ of the moving particle in polar form is:

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\vec{R})}{dt} = \frac{dr}{dt}\vec{R} + r\frac{d\vec{R}}{dt} = \frac{dr}{dt}\vec{R} + r\frac{d\vec{R}}{d\theta}\frac{d\theta}{dt}$

Let $\frac{d\vec{R}}{d\theta} = \begin{pmatrix} -\sin\theta \\ \\ \cos\theta \end{pmatrix} = \vec{P}$. Notice that $\vec{R}$ is perpendicular to $\vec{P}$.

Then $\vec{v} = \frac{dr}{dt}\vec{R} + r\frac{d\theta}{dt}\vec{P}$.And $\vec{v}$ is the linear combination of two perpendicular unit vectors.

The acceleration $\vec{a}$ of the moving particle in polar form is:

$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(\frac{dr}{dt}\vec{R}+r\frac{d\theta}{dt}\vec{P}\right)}{dt} = \frac{d^2r}{dt^2}\vec{R} + \frac{dr}{dt}\frac{d\vec{R}}{d\theta}\frac{d\theta}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{P} + r\frac{d^2\theta}{dt^2}\vec{P} + r\frac{d\theta}{dt}\frac{d\vec{P}}{d\theta}\frac{d\theta}{dt}$

$= \frac{d^2r}{dt^2}\vec{R} + 2\frac{dr}{dt}\frac{d\theta}{dt}\vec{P} + r\frac{d^2\theta}{dt^2}\vec{P} + r\frac{d\theta}{dt}\frac{d\vec{P}}{d\theta}\frac{d\theta}{dt} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\vec{R} + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\vec{P}$

, where $\frac{d\vec{P}}{d\theta} = \begin{pmatrix} -\cos\theta \\ \\ -\sin\theta \end{pmatrix} = -\vec{R}$

$\square$

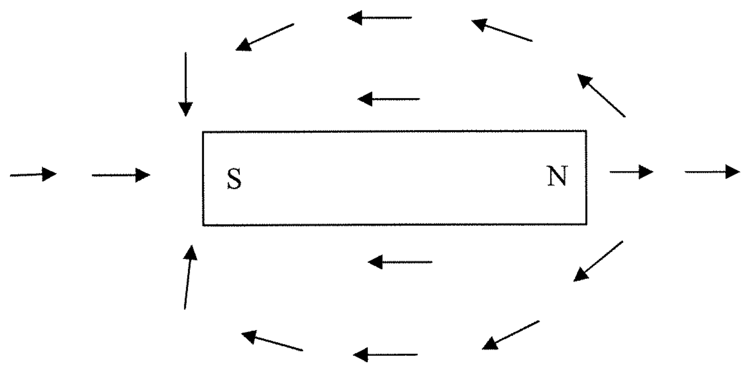
2. Vector Fields

If at each point (x, y, z) there is an associated vector $\vec{v}(x, y, z) = v_1(x, y, z)\vec{i} + v_2(x, y, z)\vec{j} + v_3(x, y, z)\vec{k}$

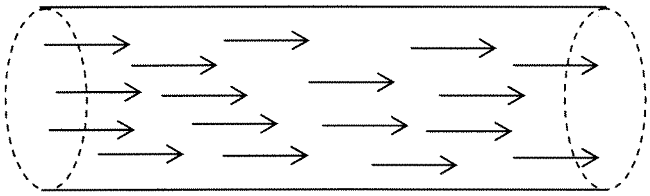
then $\vec{v}(x, y, z)$ is a vector function and we say that we have a vector field.

Example

(i) A magnetic field $\vec{B}$ in a region of space, $\vec{B} = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$



(ii) The velocity field of water flowing in a pipe, $\vec{v}(x, y, z)$.



(iii) The earths gravitational field.

$\square$

3. <u>The Vector Operator</u> ∇

Recall that for any scalar field $\varphi(x, y, z)$ we may define its <u>gradient</u> $\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} + \frac{\partial \varphi}{\partial z}\vec{k}$.

This is also an example of a vector field which gives the magnitude and direction of the greatest rate of change of φ at a point.

If we define the vector operator $\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ (analogous to $D \equiv \frac{d}{dx}$), known as del(or nabla), then $\operatorname{grad} \varphi = \nabla \varphi$.

We may also apply the operator ∇ to a vector function

$\vec{v}(x, y, z) = v_1(x, y, z)\vec{i} + v_2(x, y, z)\vec{j} + v_3(x, y, z)\vec{k}$.

4. <u>Divergence of a Vector Field</u>

If $\vec{v}(x, y, z) = v_1(x, y, z)\vec{i} + v_2(x, y, z)\vec{j} + v_3(x, y, z)\vec{k}$ is a vector field, then its <u>divergence</u> is defined as $\operatorname{div} \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$.

In terms of the operator ∇ we may write

$$\nabla \cdot \vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[v_1(x, y, z)\vec{i} + v_2(x, y, z)\vec{j} + v_3(x, y, z)\vec{k}\right]$$

$$= \frac{\partial v_1(x,y,z)}{\partial x} + \frac{\partial v_2(x,y,z)}{\partial y} + \frac{\partial v_3(x,y,z)}{\partial z} = div\vec{v}$$

Note $\nabla \cdot \vec{v}$ is a <u>scalar</u> function.

Example

Let $\vec{v} = x^2 z\vec{i} - 2y^3 z\vec{j} + xyz^2\vec{k}$, find $\nabla \cdot \vec{v}$

Solution:

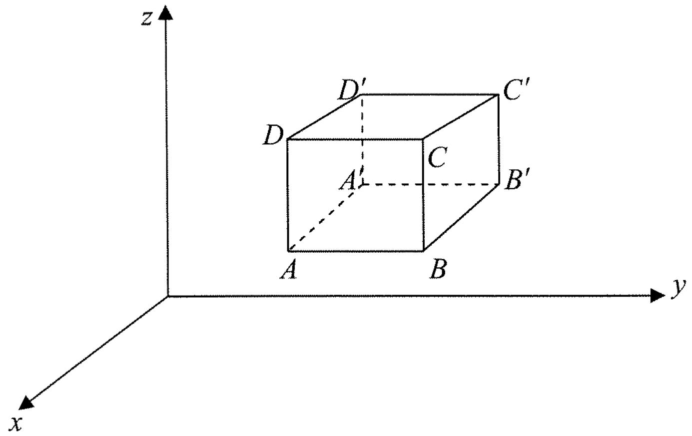
$$\nabla \cdot \vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2 z\vec{i} - 2y^3 z\vec{j} + xyz^2\vec{k}\right) = 2xz - 6y^2 z + 2xyz$$

□

Physical Interpretation

The idea of divergence is probably most easily illustrated by considering the motion of an incompressible fluid (for example water) where we take the vector filed to be the velocity $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ of the fluid.

Consider a small imaginary rectangular box of dimensions $\delta x, \delta y$ and δz centered at (x, y, z) in the fluid.



We consider the flow of fluid through this box. The volume of fluid crossing a surface in unit time is an example of flux. Taking the midpoint of a face as a representative point, the flux through the face $ABCD$ out of the box is $v_1\left(x + \frac{\delta x}{2}, y, z\right)\delta y\delta z$ (v_2 and v_3 are parallel to $ABCD$ and make no contribution) Similarly, the flux through face $A'B'C'D'$ into the box is $v_1\left(x - \frac{\delta x}{2}, y, z\right)\delta y\delta z$.

The net flux out of the box is $\left[v_1\left(x + \frac{\delta x}{2}, y, z\right) - v_1\left(x - \frac{\delta x}{2}, y, z\right)\right]\delta y\delta z$.

And

$$\left[v_1\left(x+\tfrac{\delta x}{2},y,z\right)-v_1\left(x-\tfrac{\delta x}{2},y,z\right)\right]\delta y\delta z$$

$$=\left[\left(x_1(x,y,z)+\frac{\frac{\delta x}{2}}{1!}\frac{\partial v_1(x,y,z)}{\partial x}+\frac{\left(\frac{\delta x}{2}\right)^2}{2!}\frac{\partial^2 v_1(x,y,z)}{\partial x^2}+\frac{\left(\frac{\delta x}{2}\right)^3}{3!}\frac{\partial^3 v_1(x,y,z)}{\partial x^3}+\cdots\right)\right.$$

$$\left.-\left(v_1(x,y,z)+\frac{\left(-\frac{\delta x}{2}\right)}{1!}\frac{\partial v_1(x,y,z)}{\partial x}+\frac{\left(-\frac{\delta x}{2}\right)^2}{2!}\frac{\partial^2 v_1(x,y,z)}{\partial x^2}+\frac{\left(-\frac{\delta x}{2}\right)^3}{3!}\frac{\partial^3 v_1(x,y,z)}{\partial x^3}+\cdots\right)\right]\delta y\delta z$$

$$=\left[\frac{2}{1!}\left(\frac{\delta x}{2}\right)\frac{\partial v_1(x,y,z)}{\partial x}+\frac{2}{3!}\left(\frac{\delta x}{2}\right)^3\frac{\partial^3 v_1(x,y,z)}{\partial x^3}+\cdots\right]\delta y\delta z$$

$$=\left[\frac{\partial v_1(x,y,z)}{\partial x}+\frac{(\delta x)^2}{3!2^2}\frac{\partial^3 v_1(x,y,z)}{\partial x^3}+\cdots\right]\delta x\delta y\delta z$$

Similarly by considering the two other pairs of faces, we have that the total flux out of the box is

$$\left[\frac{\partial(v_1(x,y,z))}{\partial x}+\frac{\partial(v_2(x,y,z))}{\partial y}+\frac{\partial(v_3(x,y,z))}{\partial z}\right]\delta x\delta y\delta z$$

$$+\left[\frac{(\delta x)^2}{3!2^2}\frac{\partial^3 v_1(x,y,z)}{\partial x^3}+\frac{(\delta y)^2}{3!2^2}\frac{\partial^3 v_2(x,y,z)}{\partial y^3}+\frac{(\delta z)^2}{3!2^2}\frac{\partial^3 v_3(x,y,z)}{\partial z^3}+\cdots\right]\delta x\delta y\delta z$$

If we consider the flux <u>per unit volume</u>, we have

$$\left[\frac{\partial v_1(x,y,z)}{\partial x}+\frac{\partial v_2(x,y,z)}{\partial y}+\frac{\partial v_3(x,y,z)}{\partial z}\right]+\left[\frac{2}{3!}\frac{(\delta x)^2}{2^3}\frac{\partial^3 v_1(x,y,z)}{\partial x^3}+\frac{2}{3!}\frac{(\delta y)^2}{2^3}\frac{\partial^3 v_2(x,y,z)}{\partial y^3}+\frac{2}{3!}\frac{(\delta z)^2}{2^3}\frac{\partial^3 v_3(x,y,z)}{\partial z^3}+\cdots\right]$$

Take the limit as the volume of the box shrinks to zero we obtain

$$\operatorname{div}\vec{v}=\lim_{\delta x\to 0,\delta y\to 0,\delta z\to 0}\left[\frac{\partial v_1(x,y,z)}{\partial x}+\frac{\partial v_2(x,y,z)}{\partial y}+\frac{\partial v_3(x,y,z)}{\partial z}\right]$$

$$+\lim_{\delta x\to 0,\delta y\to 0,\delta z\to 0}\left[\frac{2}{3!}\frac{(\delta x)^2}{2^3}\frac{\partial^3 v_1(x,y,z)}{\partial x^3}+\frac{2}{3!}\frac{(\delta y)^2}{2^3}\frac{\partial^3 v_2(x,y,z)}{\partial y^3}+\frac{2}{3!}\frac{(\delta z)^2}{2^3}\frac{\partial^3 v_3(x,y,z)}{\partial z^3}+\cdots\right]$$

$$=\frac{\partial v_1(x,y,z)}{\partial x}+\frac{\partial v_2(x,y,z)}{\partial y}+\frac{\partial v_3(x,y,z)}{\partial z}$$

As a measure of the net outward flux per unit volume at a point (x,y,z), positive divergence indicates a source of fluid at the point whilst negative divergence indicates a sink.

If $\vec{E}$ is an electric field vector, $\operatorname{div}\vec{E}$ measures the charge density at a point.

If for a vector field $\vec{v}, \operatorname{div}\vec{v} = 0$ then $\vec{v}$ is said to be solenoidal, that is, no net outflow or inflow of fluid/ no electric charge.

Example

Find the electric field at position $\vec{r}$ due to a point charge at the origin.

Solution:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} = \frac{Q}{4\pi\varepsilon_0}\frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$
$$\Rightarrow \operatorname{div}\vec{E} = \nabla\cdot\vec{E} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\right]$$
$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r^3} + x\frac{-3}{2}\frac{2x}{r^5} + \frac{1}{r^3} + y\frac{-3}{2}\frac{2y}{r^5} + \frac{1}{r^3} + z\frac{-3}{2}\frac{2z}{r^5}\right) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{3}{r^3} - \frac{3\left(x^2+y^2+z^2\right)}{r^5}\right] = 0$$

$, \operatorname{div}\vec{E} = 0$ except at the origin.

Hence $\vec{E}$ is solenoidal $(\vec{r} \neq \overrightarrow{0})$ which is consistent with the above interpretation.

□

5. Curl of a Vector Field

We define

$$\text{curlv} = \nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$
$$= \vec{i}\frac{\partial}{\partial x}\times\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right) + \vec{j}\frac{\partial}{\partial y}\times\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right) + \vec{k}\frac{\partial}{\partial z}\times\left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right)$$
$$= \frac{\partial v_1}{\partial x}\vec{i}\times\vec{i} + \frac{\partial v_2}{\partial x}\vec{i}\times\vec{j} + \frac{\partial v_3}{\partial x}\vec{i}\times\vec{k} + \frac{\partial v_1}{\partial y}\vec{j}\times\vec{i} + \frac{\partial v_2}{\partial y}\vec{j}\times\vec{j} + \frac{\partial v_3}{\partial y}\vec{j}\times\vec{k} + \frac{\partial v_1}{\partial z}\vec{k}\times\vec{i} + \frac{\partial v_2}{\partial z}\vec{k}\times\vec{j} + \frac{\partial v_3}{\partial z}\vec{k}\times\vec{k}$$
$$= \vec{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \vec{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Note $\nabla\times\vec{v}$ is a vector function.

Example

Let $\vec{v} = x^2 z\vec{i} - 2y^3 z^2\vec{j} + xyz^2\vec{k}$, find curl$\tilde{\text{v}}$.

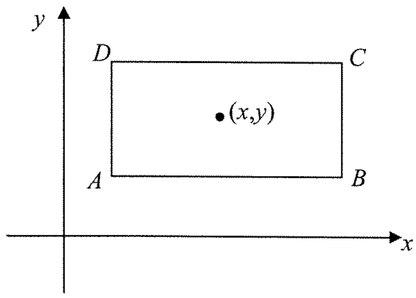
Solution:

$$\text{curl}\tilde{\text{v}} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & -2y^3 z^2 & xyz^2 \end{vmatrix} = \left(xz^2 + 4y^3 z\right)\vec{i} + \left(x^2 - yz^2\right)\vec{j}.$$

□

Physical Interpretation

Again we shall illustrate the idea of curl using the motion of an incompressible fluid with velocity $\vec{v} = v_1\vec{i} + v_2\vec{j}$. Consider just the xy -plane and a small imaginary rectangle of sides δx and δy centered at(x, y).



Flow along AB = length of $AB\times$ velocity component along $AB = v_1\left(x, y - \frac{\delta y}{2}\right)\delta x$ and flow along $CD = -v_1\left(x, y + \frac{\delta y}{2}\right)\delta x$.

Sum $= \left[v_1\left(x, y - \frac{\delta y}{2}\right) - v_1\left(x, y + \frac{\delta y}{2}\right)\right]\delta x$

$= \Bigg[v_1(x,y) + \left(-\frac{\delta y}{2}\right)\frac{\partial v_1(x,y)}{\partial y} + \frac{\left(-\frac{\delta y}{2}\right)^2}{2!}\frac{\partial^2 v_1(x,y)}{\partial y^2} + \cdots$

$- \Bigg(v_1(x,y) + \frac{\delta y}{2}\frac{\partial v_1(x,y)}{\partial y} + \frac{\left(\frac{\delta y}{2}\right)^2}{2!}\frac{\partial^2 v_1(x,y)}{\partial y^2} + \cdots\Bigg]\delta x = -\frac{\partial v_1(x,y)}{\partial y}\delta x\delta y - \frac{(\delta y)^3}{3!2^2}\frac{\partial^3 v_1(x,y)}{\partial y^3}\delta x\delta y - \cdots$

Similarly the sum of flows along $BC + DA$ is:

$\frac{\partial v_2(x,y)}{\partial x}\delta x\delta y + \frac{(\delta x)^3}{3!2^2}\frac{\partial^3 v_2(x,y)}{\partial y^3}\delta x\delta y + \cdots$

The total circulation (net flow around a closed path) around the rectangle is:

$\left[\frac{\partial v_2(x,y)}{\partial x} - \frac{\partial v_1(x,y)}{\partial y}\right]\delta x\delta y + \left[\frac{(\delta x)^3}{3!2^2}\frac{\partial^3 v_2(x,y)}{\partial y^3} - \frac{(\delta y)^3}{3!2^2}\frac{\partial^3 v_1(x,y)}{\partial y^3}\right]\delta x\delta y +$

If we consider circulation per unit area, we have

$\left[\frac{\partial v_2(x,y)}{\partial x} - \frac{\partial v_1(x,y)}{\partial y}\right] + \left[\frac{(\delta x)^3}{3!2^2}\frac{\partial^3 v_2(x,y)}{\partial y^3} - \frac{(\delta y)^3}{3!2^2}\frac{\partial^3 v_1(x,y)}{\partial y^3}\right] + \cdots$

Taking the limit as the area of the rectangle shrinks to zero, we have

$\lim_{\delta x\to 0, \delta y\to 0}\left[\frac{\partial v_2(x,y)}{\partial x} - \frac{\partial v_1(x,y)}{\partial y}\right] + \lim_{\delta x\to 0, \delta y\to 0}\left[\frac{(\delta x)^3}{3!2^2}\frac{\partial^3 v_2(x,y)}{\partial y^3} - \frac{(\delta y)^3}{3!2^2}\frac{\partial^3 v_1(x,y)}{\partial y^3}\right] + \cdots$

$= \frac{\partial v_2(x,y)}{\partial x} - \frac{\partial v_1(x,y)}{\partial y}$

So curl $\vec{v} = \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\vec{k}$

If this quantity is positive/negative/zero a small paddle wheel placed in the fluid at the point P would rotate anticlockwise/clockwise/remain stationary.

By analogy with a rigid body rotating about an axis with angular velocity ω for which the angular velocity vector is $\omega\vec{n}$, where $\vec{n}$ is a unit vector along the axis of rotation, we may associate with the rotational effect of the fluid a vector.

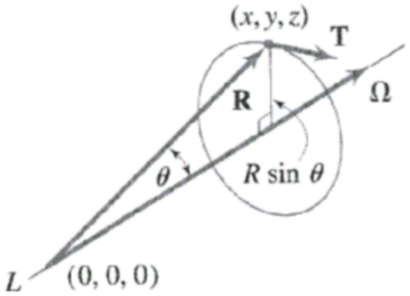
$$\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\vec{k}$$

This is the third component of curl $\vec{v}$. Similarly, by considering flow in 3 -dimensions we may derive the full expression for curl $\vec{v}$ which gives a measure of the rotational effect of the fluid flow at a point.

If $\vec{H}$ is a magnetic field vector, $curl\vec{H}$ measures the current density at a point.

Example: (An Interpretation of Curl)

Angular Velocity Ω as the Curl of the Tangential Velocity $\vec{T}$. Suppose a body rotates with uniform angular speed ω about a line L. See the figure below.



The angular velocity vector Ω has magnitude ω and is directed along L as a right-handed screw would progress if given the same sense of rotation as the object.Let L go through the origin and let $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$ for any point (x, y, z) on the rotating object. Let $\vec{T}(x, y, z)$ be the tangential velocity. Then $|\vec{T}| = \omega|\vec{R}|\sin\theta = |\Omega \times \vec{R}|$, where θ is the angle between R and L. Since T and $\Omega \times \vec{R}$ are in the same direction, then $\vec{T} = \Omega \times \vec{R}$.

Now, write $\Omega = a\vec{i} + b\vec{j} + c\vec{k}$.

Then $\vec{T} = \Omega \times \vec{R} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k}.$

But

$$\operatorname{curl}\vec{T} = \nabla \times \vec{T} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix}$$

$$= \left[\frac{\partial(ay-bx)}{\partial y} - \frac{\partial(cx-az)}{\partial z}\right]\vec{i} - \left[\frac{\partial(ay-bx)}{\partial x} - \frac{\partial(bz-cy)}{\partial z}\right]\vec{j} + \left[\frac{\partial(cx-az)}{\partial x} - \frac{\partial(bz-cy)}{\partial y}\right]\vec{k}$$

$$= 2a\vec{i} + 2b\vec{j} + 2c\vec{k} = 2\Omega$$

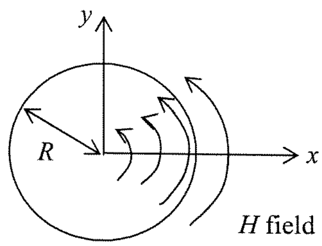
Therefore, $\Omega = \frac{1}{2}\operatorname{curl}\vec{T} = \frac{1}{2}\nabla \times \vec{T}$

The angular velocity of a uniformly rotating body is a scalar multiple of the curl of the tangential velocity. Because of this interpretation, curl was once written *rot* (for rotation), particularly in British books on mechanics. This is also the motivation for the term irrotational for a vector field whose curl is zero.

$\square$

Example

Consider a current-carrying conductor with circular cross-section of radius R. Then the magnetic field $\vec{H}$ is $\vec{H} = \begin{cases} \frac{-y\vec{i}+x\vec{j}}{2\pi r^2} & \text{outside conductor} \\ \frac{-y\vec{i}+x\vec{j}}{2\pi R^2} & \text{inside conductor} \end{cases}$, where $r = (x^2+y^2)^{\frac{1}{2}}$



Outside the conductor,

$$\operatorname{curl}\vec{H} = \nabla \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{2\pi(x^2+y^2)} & \frac{x}{2\pi(x^2+y^2)} & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + \left(\frac{1}{2\pi r^2} - \frac{2x^2}{2\pi r^4} + \frac{1}{2\pi r^2} - \frac{2y^2}{2\pi r^4}\right)\vec{k} = \vec{0}$$

Inside the conductor,

$$\operatorname{curl}\vec{H} = \nabla\times\vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{2\pi R^2} & \frac{x}{2\pi R^2} & 0 \end{vmatrix} = \left(\tfrac{1}{2\pi R^2} + \tfrac{1}{2\pi R^2}\right)\vec{k}$$

$= \frac{1}{\pi R^2}\vec{k} = J-$ the current density in the conductor.

□

If for a vector field $\vec{v}$, curl $\vec{v} = \overrightarrow{0}$, then $\vec{v}$ is said to be irrotational.

Examples

(i) The magnetic field above, outside the conductor, is irrotational.

(ii) The electric field E due to a point charge is irrotational (exercise).

□

6. Vector Identities

Some useful results involving the vector operator ∇ are:

f and g are scalar fields, $\vec{v}$ and $\vec{w}$ are vector fields.

(i) $\nabla(f+g) = \nabla f + \nabla g$

(ii) $\nabla(fg) = f\nabla g + g\nabla f$

(iii) $\nabla\cdot(\vec{v}+\vec{w}) = \nabla\cdot\vec{v} + \nabla\cdot\vec{w}$

(iv) $\nabla\cdot(\overrightarrow{fv}) = \nabla f\cdot\vec{v} + f\nabla\cdot\vec{v}$

(v) $\nabla\times(\vec{v}+\vec{w}) = \nabla\times\vec{v} + \nabla\times\vec{w}$

(vi) $\nabla\times(\overrightarrow{fv}) = \nabla f\times\vec{v} + f\nabla\times\vec{v}$

(vii) $\nabla\cdot(\vec{v}\times\vec{w}) = \vec{w}\cdot\nabla\times\vec{v} - \vec{v}\cdot\nabla\times\vec{w}$

(viii) $\nabla\times(\vec{v}\times\vec{w}) = (\vec{w}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{w} + (\nabla\cdot\vec{w})\vec{v} - (\nabla\cdot\vec{v})\vec{w}$

(ix) $\nabla\times\nabla f = \overrightarrow{0}$

(x) $\nabla\cdot(\nabla\times\vec{v}) = 0$

(xi) $\nabla\cdot\nabla f = \nabla\cdot\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Let $v(x,y,z)$ be a scalar field. Define the Laplacian $\nabla^2 v$ of v by $\nabla^2 v \equiv \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$.

In del notation $\nabla^2 v = \nabla \cdot \nabla v . \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ can be applied to a vector field $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

and $\nabla^2 \vec{v} \equiv \frac{\partial^2 \vec{v}}{\partial x^2} + \frac{\partial^2 \vec{v}}{\partial y^2} + \frac{\partial^2 \vec{v}}{\partial z^2} = \frac{\partial^2 v_1}{\partial x^2}\vec{i} + \frac{\partial^2 v_2}{\partial x^2}\vec{j} + \frac{\partial^2 v_3}{\partial x^2}\vec{k} + \frac{\partial^2 v_1}{\partial y^2}\vec{i} + \frac{\partial^2 v_2}{\partial y^2}\vec{j} + \frac{\partial^2 v_3}{\partial y^2}\vec{k} + \frac{\partial^2 v_1}{\partial z^2}\vec{i} + \frac{\partial^2 v_2}{\partial z^2}\vec{j} + \frac{\partial^2 v_3}{\partial z^2}\vec{k}$.

Example

Prove (vi) $\nabla \times (\overrightarrow{fv}) = \nabla f \times \vec{v} + f \nabla \times \vec{v}$

Proof:

$$\nabla \times (f\vec{v}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & fv_2 & fv_3 \end{vmatrix} = \vec{i}\left[\frac{\partial}{\partial y}(fv_3) - \frac{\partial}{\partial z}(fv_2)\right] + \vec{j}\left[\frac{\partial}{\partial z}(f_1) - \frac{\partial}{\partial x}(fv_3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(f_2) - \frac{\partial}{\partial y}(f_1)\right]$$

$$= \vec{i}\left(\frac{\partial f}{\partial y}v_3 + f\frac{\partial v_3}{\partial y} - \frac{\partial f}{\partial z}v_2 - f\frac{\partial v_2}{\partial z}\right) + \vec{j}\left(f\frac{\partial v_1}{\partial z} + \frac{\partial f}{\partial z}v_1 - f\frac{\partial v_3}{\partial x} - \frac{\partial f}{\partial x}v_3\right)$$

$$+\vec{k}\left(f\frac{\partial v_2}{\partial x} + v_2\frac{\partial f}{\partial x} - f\frac{\partial v_1}{\partial y} - v_1\frac{\partial f}{\partial y}\right) = \vec{i}\left(\frac{\partial f}{\partial y}v_3 - \frac{\partial f}{\partial z}v_2\right) + \vec{f}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)$$

$$+\vec{j}\left(\frac{\partial f}{\partial z}v_1 - \frac{\partial f}{\partial x}v_3\right) + f\vec{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \vec{k}\left(v_2\frac{\partial f}{\partial x} - v_1\frac{\partial f}{\partial y}\right) + f\vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = \nabla f \times \vec{v} + f\nabla \times \vec{v}$$

$\square$

Example

Prove (ix) curl(grad f) $= \nabla \times (\nabla f) = \overrightarrow{0}$.

Proof:

$$\nabla \times (\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right] - \vec{j}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\right] + \vec{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]$$

$$= \vec{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - \vec{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \vec{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = \overrightarrow{0}$$

$\square$

The converse of this result is also true, that is, every irrotational vector field is the gradient of some scalar field, and has important applications in electromagnetism and fluid dynamics.

Similarly the converse of result (x), which says that every solenoidal vector field is the curl of some vector field, arises in applications.

<u>Example</u>

Show that $\nabla \cdot \vec{F} = 0$ (where $\vec{F} = f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k}$ is some differentiable vector field) implies and is implied by $\vec{F} = \nabla \times \vec{H}$ (where $\vec{H} = h_1(x,y,z)\vec{i} + h_2(x,y,z)\vec{j} + h_3(x,y,z)\vec{k}$ is another differentiable vector field).

Proof:

$$\vec{F} = \nabla \times \vec{H} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ h_1 & h_2 & h_3 \end{pmatrix} = \left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\vec{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right)\vec{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\vec{k}$$

Then

$$\nabla \cdot \vec{F} = \nabla \cdot \nabla \times \vec{H} = \frac{\partial}{\partial x}\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)$$

$$= \frac{\partial^2 h_3}{\partial x \partial y} - \frac{\partial^2 h_3}{\partial y \partial x} + \frac{\partial^2 h_2}{\partial z \partial x} - \frac{\partial^2 h_2}{\partial x \partial z} + \frac{\partial^2 h_1}{\partial y \partial z} - \frac{\partial^2 h_1}{\partial z \partial y}$$

If $\frac{\partial^2 h_3}{\partial x \partial y}, \frac{\partial^2 h_3}{\partial y \partial x}, \frac{\partial^2 h_2}{\partial z \partial x}, \frac{\partial^2 h_2}{\partial x \partial z}, \frac{\partial^2 h_1}{\partial y \partial z}, \frac{\partial^2 h_1}{\partial z \partial y}$ are continuous, then $\frac{\partial^2 h_3}{\partial x \partial y} = \frac{\partial^2 h_3}{\partial y \partial x}, \frac{\partial^2 h_2}{\partial z \partial x} = \frac{\partial^2 h_2}{\partial x \partial z}, \frac{\partial^2 h_1}{\partial y \partial z} = \frac{\partial^2 h_1}{\partial z \partial y}$.

It follows that $\nabla \cdot \vec{F} = \frac{\partial^2 h_3}{\partial x \partial y} - \frac{\partial^2 h_3}{\partial y \partial x} + \frac{\partial^2 h_2}{\partial z \partial x} - \frac{\partial^2 h_2}{\partial x \partial z} + \frac{\partial^2 h_1}{\partial y \partial z} - \frac{\partial^2 h_1}{\partial z \partial y} = 0$

Suppose $\nabla \cdot \vec{F} = 0$, that is $\frac{\partial}{\partial x} f_1(x,y,z) + \frac{\partial}{\partial y} f_2(x,y,z) + \frac{\partial}{\partial z} f_3(x,y,z) = 0$

Consider $\vec{F} = \nabla \times \vec{H} = \left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\vec{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right)\vec{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\vec{k}$

$$\text{Solve} \begin{cases} f_1 = \frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \\ f_2 = \frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \\ f_3 = \frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \end{cases}.$$

Let $h_1 \equiv 0$, then $\begin{cases} f_1 = \frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \\ f_2 = -\frac{\partial h_3}{\partial x} \\ f_3 = \frac{\partial h_2}{\partial x} \end{cases}$

$f_3 = \frac{\partial h_2}{\partial x} \Rightarrow h_2(x,y,z) = \int_{x_0}^{x} f_3(t,y,z)dt . f_2 = -\frac{\partial h_3}{\partial x} \Rightarrow h_3(x,y,z) = -\int_{x_0}^{x} f_2(t,y,z)dt + k(y,z)$

Then

$f_1(x,y,z) = \frac{\partial\left[-\int_{x_0}^{x} f_2(t,y,z)dt + k(y,z)\right]}{\partial y} - \frac{\partial\left(\int_{x_0}^{x} f_3(t,y,z)dt\right)}{\partial z}$

$= -\int_{x_0}^{x} \frac{\partial f_2(t,y,z)}{\partial y}dt + \frac{\partial k(y,z)}{\partial y} - \int_{x_0}^{x} \frac{\partial f_3(t,y,z)}{\partial z}dt = -\int_{x_0}^{x} \left[\frac{\partial f_2(t,y,z)}{\partial y} + \frac{\partial f_3(t,y,z)}{\partial z}\right] dt + \frac{\partial k(y,z)}{\partial y}$

$\underset{\frac{\partial f_1(t,y,z)}{\partial x} + \frac{\partial f_2(t,y,z)}{\partial y} + \frac{\partial f_3(t,y,z)}{\partial z} = 0}{=} -\int_{x_0}^{x} -\frac{\partial f_1(t,y,z)}{\partial t}dt + \frac{\partial k(y,z)}{\partial y} = \int_{x_0}^{x} \frac{\partial f_1(t,y,z)}{\partial t}dt + \frac{\partial k(y,z)}{\partial y}$

$= f_1(x,y,z) - f_1(x_0,y,z) + \frac{\partial k(y,z)}{\partial y} \Rightarrow \frac{\partial k(y,z)}{\partial y} = f_1(x_0,y,z) \Rightarrow k(y,z) = \int_{y_0}^{y} f_1(x_0,t,z)\, dt$

So $h_3(x,y,z) = -\int_{x_0}^{x} f_2(t,y,z)dt + \int_{y_0}^{y} f_1(x_0,t,z)\, dt$

Finally, $\vec{H} = \vec{j} \int_{x_0}^{x} f_3(t,y,z)dt + \vec{k}\left[-\int_{x_0}^{x} f_2(t,y,z)dt + \int_{y_0}^{y} f_1(x_0,t,z)\, dt\right]$

$\square$

Example

Let $\vec{F} = (y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}$. Find a vector field $\vec{H} = h_1(x,y,z)\vec{i} + h_2(x,y,z)\vec{j} + h_3(x,y,z)\vec{k}$ with $h_1 \equiv 0$ such that $\vec{F} = \nabla \times \vec{H}$.

Solution:

$$\nabla \times \vec{H} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ h_1 & h_2 & h_3 \end{pmatrix} = \left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\vec{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right)\vec{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\vec{k}$$

$\vec{F} = \nabla \times \vec{H}$ and $h_1 \equiv 0 \Rightarrow (y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k} = \left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\vec{i} - \frac{\partial h_3}{\partial x}\vec{j} + \frac{\partial h_2}{\partial x}\vec{k}$

That is, $\begin{cases} y - z = \frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \\ z - x = -\frac{\partial h_3}{\partial x} \\ x - y = \frac{\partial h_2}{\partial x} \end{cases}$

$x - y = \frac{\partial h_2}{\partial x} \Rightarrow h_2 = \frac{x^2}{2} - yx, z - x = -\frac{\partial h_3}{\partial x} \Rightarrow h_3 = \frac{x^2}{2} - zx + C(y,z).$

$y - z = \frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} = \frac{\partial C}{\partial y} \Rightarrow C(y,z) = \frac{y^2}{2} - zy$. Thus, $h_3 = \frac{x^2}{2} + \frac{y^2}{2} - zx - zy$.

So, $\vec{H} = \left(\frac{x^2}{2} - yx\right)\vec{j} + \left(\frac{x^2}{2} + \frac{y^2}{2} - zx - zy\right)\vec{k}$. (Notice that $\vec{H}$ is <u>not unique</u>.)