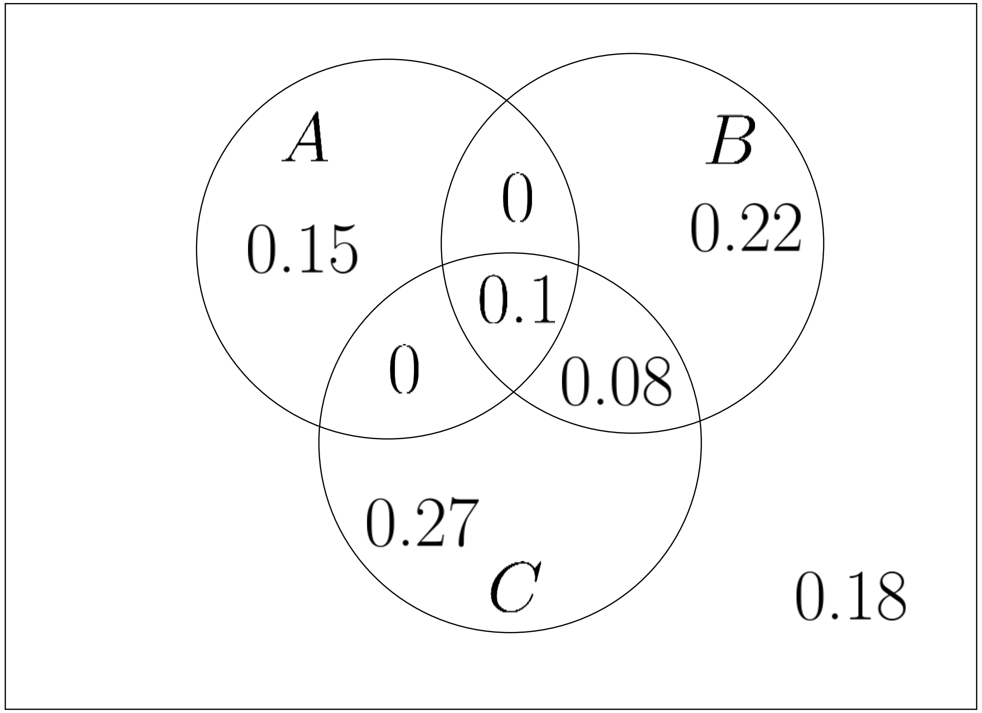
Solution

1.(a)

We construct the Venn diagram to find the probabilities.



As A and B are independent, and so are B and C, we have:

$$P(A \cap B) = P(A)P(B) = 0.25 \cdot 0.4 = 0.1$$
$$P(B \cap C) = P(B)P(C) = 0.4 \cdot 0.45 = 0.18$$

Using $P(A \cap B \cap C) = 0.1$ and $P(A \cap B) = 0.1$, we get:

$$P(A \cap B \cap \overline{C}) = 0$$

Also,

$$A \cap \overline{B} \cap C = \emptyset \Rightarrow P(A \cap \overline{B} \cap C) = 0$$

Then we get:

$$P(A \cap \overline{B} \cap \overline{C}) = 0.25 - 0.1 = 0.15$$

Using $P(B \cap C) = 0.18$, we deduce that

$$P(\overline{A} \cap B \cap C) = 0.18 - 0.1 = 0.08$$

We also get:

$$P(\overline{A} \cap B \cap \overline{C}) = 0.4 - 0.18 = 0.22$$
$$P(\overline{A} \cap \overline{B} \cap C) = 0.45 - 0.18 = 0.27$$

Finally, we have:

$$P(\overline{A \cup B \cup C}) = 1 - 0.45 - 0.15 - 0.22 = 0.18$$

1.(b)

From the Venn diagram, we get:

$$P((A \cup B) \cap C) = 0.18$$

$$P(\overline{A} \cap (\overline{B} \cup C)) = 0.27 + 0.08 + 0.18 = 0.53$$

1.(c)

No. A and C are not independent. It is because

$$P(A \cap C) = 0.1 \neq P(A)P(C) = 0.25 \cdot 0.45 = 0.1125$$

2.(a)

There are 11 characters where both “B” and “I” repeat once, i.e., 2 “B” and 2 “I”. Hence the total number of arrangements is:

$$\frac{11!}{2!2!} = 9979200$$

2.(b)(i)

For each character of the password, there are 26+10=36 choices. Hence the total number of arrangements is:

$$36^6 = 2176782336$$

2.(b)(ii)

Now repeated characters are not allowed, hence the total number of arrangements is:

$$36 \times 35 \times 34 \times 33 \times 32 \times 31 = 1402410240$$

2.(b)(iii)

If the first two characters must be digits, then the total number of combinations with repeated characters allowed is:

$$10^2 \times 36^4 = 167961600$$

If the first two characters must be digits, then the total number of combinations with no repeated characters allowed is:

$$10 \times 9 \times 34 \times 33 \times 32 \times 31 = 100172160$$

3.(a)
Yes. Y is a discrete random variable.

3.(b)
The expected value of Y is computed as:

$$\mathbb{E}\{Y\} = \alpha \sin(90^\circ) + \beta \sin(180^\circ) + 0.4 \sin(270^\circ) = \alpha - 0.4$$

As PMF must be nonnegative and the sum of PMFs should be 1, then we have:

$$0 \leq \alpha \leq 1 - 0.4 \Rightarrow 0 \leq \alpha \leq 0.6$$

Hence

$$-0.4 \leq \mathbb{E}\{Y\} \leq 0.2$$

3.(c)

When $\mathbb{E}\{Y\} = 0.1$, we have:

$$\mathbb{E}\{Y\} = 0.1 = \alpha - 0.4 \Rightarrow \alpha = 0.5$$

Then

$$\beta = 1 - \alpha - 0.4 = 0.1$$

4.

$$\mathbb{E}\{Y\} = \mu_y = \mathbb{E}\{aX + b\} = \mathbb{E}\{aX\} + \mathbb{E}\{b\} = a\mathbb{E}\{X\} + b = a\mu_x + b$$

Given $\mu_x = 2$ and $\mu_y = 10$, hence we have:

$$10 = 2a + b$$

We also have:

$$\mathrm{var}(Y) = \mathbb{E}\{Y^2\} - (\mathbb{E}\{Y\})^2 = \mathbb{E}\{Y^2\} - \mu_y^2$$

$$\mathrm{var}(Y) = a^2\mathrm{var}(X)$$

Given $\mu_y = 10$, $\mathbb{E}\{Y^2\} = 200$, and $\mathrm{var}(X) = 2$, hence we have:

$$a^2\mathrm{var}(X) = \mathbb{E}\{Y^2\} - \mu_y^2 \Rightarrow 2a^2 = 200 - 100 = 100 \Rightarrow a = 5\sqrt{2}$$

After a is obtained, b is computed as:

$$10 = 2a + b \Rightarrow 10 = 10\sqrt{2} + b \Rightarrow b = 10(1 - \sqrt{2})$$

5.(a)(i)

We can apply the binomial distribution for the probability computation. Since there are 4 choices in each question, the probability of success is $p = 0.25$, and probability of failure is 0.75. Hence the mean mark is:

$$10 \cdot 0.25 + 0 \cdot 0.75 = 2.5$$

5.(a)(ii)

To pass the test, the student should answer at least 4 MC questions correctly, i.e., the marks should be between 40 and 100. Hence the probability of having 0, 10, 20, and 30 is not allowed, implying the required probability is:

$$1 - [C(10,0)0.75^{10} + C(10,1)(0.25)(0.75)^9 + C(10,2)(0.25)^2(0.75)^8 + C(10,3)(0.25)^3(0.75)^7] = 0.2241$$

5.(b)(i)

Now there is a penalty for incorrect attempt. The mean mark is now:

$$10 \cdot 0.25 + (-3) \cdot 0.75 = 0.25$$

5.(b)(i)

Since mark reduction is applied for wrong answer, it is expected that more correct answers are needed.

Let r be the minimum number of correct answer, then we can construct:

$$10r + (10 - r)(-3) \geq 40 \Rightarrow r \geq 5.38$$

Hence the student should answer at least 6 MC questions correctly to pass the test.

The required probability is:

$\mathrm{C}(10{,}0)0.25^{10} + C(10,1)(0.75)(0.25)^9 + C(10,2)(0.75)^2(0.25)^8 + C(10,3)(0.75)^3(0.25)^7 + C(10,4)(0.75)^4(0.25)^6 = 0.0197$

5.(c)
The possible values of X are -30, -17, -4, 9, 22, 35, 48, 61, 74, 87, 100.

6.(a)
Denote A as the event that resistor is within gold band tolerance. From the given information, we have:

$$P(A|B_1) = 0.9$$
$$P(A|B_2) = 0.8$$
$$P(A|B_3) = 0.7$$
$$P(A|B_4) = 0.6$$

In one minute, B_1, B_2, B_3, and B_4, produce 300, 400, 500, and 600 resistors, respectively, with a total of 1800. Hence we have:

$$P(B_1) = \frac{300}{1800},\ P(B_2) = \frac{400}{1800},\ P(B_3) = \frac{500}{1800},\ P(B_4) = \frac{600}{1800}$$

Applying the law of total probability, we get:

$$P(A)=\sum_{n=4}^{N}P(B_n)\cdot P(A|B_n)=\frac{1}{18}(0.9\cdot 3+0.8\cdot 4+0.7\cdot 5+0.6\cdot 6)=\frac{13}{18}$$

6.(b)

Now we need to compute $P(B_4|A)$. Using Bayes' rule, we have:

$$P(B_4|A)=\frac{P(B_4)\cdot P(A|B_4)}{P(A)}=\frac{600/1800\cdot 0.6}{13/18}=\frac{18}{65}=0.2769$$