Definition

The k **th order leading principal minor** of the $n \times n$ symmetric matrix $A = (a_{ij})$ is the determinant of the matrix obtained by deleting the last $n-k$ rows and columns of A (where $k = 1, \ldots, n$) :

$$D_k = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ a_{k1} & a_{k2} & \ldots & a_{kk} \end{vmatrix}.$$

Example: Let

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{pmatrix}$$

The first-order leading principal minor D_1 is the determinant of the matrix obtained from A by deleting the last two rows and columns; that is, $D_1 = 3$. The second-order leading principal minor D_2 is the determinant of the matrix obtained from A by deleting the last row and column; that is

$$D = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}.$$

so that $D_2 = -4$. Finally, the third-order leading principal minor D_3 is the determinant of A, namely -19 . The following result characterizes positive and negative definite quadratic forms (and their associated matrices).

Proposition

Let A be an $n \times n$ symmetric matrix and let D_k for $k = 1, \ldots, n$ be its leading principal minors. Then

- A is positive definite if and only if $D_k > 0$ for $k = 1, \ldots, n$.
- A is negative definite if and only if $(-1)^k D_k > 0$ for $k = 1, \ldots, n$. (That is, if and only if the leading principal minors alternate in sign, starting with negative for D_1.)

In the special case that $n = 2$ these conditions reduce to the previous ones because for

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$
we have $D_1 = a$ and $D_2 = ac - b^2$.

Example

Let $A = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$.

The leading principal minors of A are $D_1 = -3 < 0, D_2 = (-3)(-3) - (2)(2) = 5 > 0$, and $|A| = -25 < 0$.

Thus A is negative definite.

Example

We saw above that the leading principal minors of the matrix

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{pmatrix}.$$

are $D_1 = 3, D_2 = -4$, and $D_3 = -19$. Thus A is neither positive definite nor negative definite. (Note that we can tell this by looking only at the first two leading principal minors-there is no need to calculate D_3.

)

Definition

The k **th order principal minors** of an $n \times n$ symmetric matrix A are the determinants of the $k \times k$ matrices obtained by deleting $n - k$ rows and the corresponding $n - k$ columns of A (where $k = 1, \ldots, n$).

Note that the k th order *leading* principal minor of a matrix is one of its k th order principal minors.

Example

Let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

The first-order principal minors of A are a and c, and the second-order principal minor is the determinant of A, namely $ac - b^2$.

Example

Let $A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & 3 & 2 \end{pmatrix}$

This matrix has 3 first-order principal minors, obtained by deleting

- the last two rows and last two columns
- the first and third rows and the first and third columns
- the first two rows and first two columns

which gives us simply the elements on the main diagonal of the matrix: $3, -1$, and 2 . The matrix also has 3 second-order principal minors, obtained by deleting

- the last row and last column
- the second row and second column
- the first row and first column

which gives us $-4, 2$, and -11. Finally, the matrix has one third-order principal minor, namely its determinant, -19.

The following result gives criteria for semidefiniteness.

Proposition

Let A be an $n \times n$ symmetric matrix. Then

- A is <u>positive semidefinite</u> if and only if all its principal minors are nonnegative.

- A is negative semidefinite if and only if its k th order principal minors are nonpositive for k odd and nonnegative for k even.

Example

Let

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

The two first-order principal minors and 0 and -1, and the second-order principal minor is 0. Thus the matrix is negative semidefinite. (It is not negative definite, because the first leading principal minor is zero.)

Procedure for checking the definiteness of a matrix

Procedure for checking the definiteness of a matrix

- Find the leading principal minors and check if the conditions for positive or negative definiteness are satisfied. If they are, you are done. (If a matrix is positive definite, it is certainly positive semidefinite, and if it is negative definite, it is certainly negative semidefinite.)

- If the conditions are not satisfied, check if they are strictly violated. If they are, then the matrix is indefinite.

- If the conditions are not strictly violated, find all its principal minors and check if the conditions for positive or negative semidefiniteness are satisfied.

Example

Suppose that the leading principal minors of the 3×3 matrix A are $D_1 = 1, D_2 = 0$, and $D_3 = -1$. Neither the conditions for A to be positive definite nor those for A to be negative definite are satisfied. In fact, both conditions are strictly violated (D_1 is positive while D_3 is negative), so the matrix is indefinite.

Example

Suppose that the leading principal minors of the 3×3 matrix A are $D_1 = 1, D_2 = 0$, and $D_3 = 0$. Neither the conditions for A to be positive definite nor those for A to be negative definite are satisfied. But the condition for positive definiteness is not strictly violated. To check semidefiniteness, we need to examine all the principal minors.