Solution

1.

Let X be the random variable representing the number of box(es) to be opened. Hence the admissible values of X are 1, 2, …, 11.

The probability that the ring is found in the first attempt is then:

$$P(X=1)=\frac{1}{11}$$

The ring is found in the second attempt means that it is not found in the first attempt. Hence the probability is:

$$P(X=2)=\frac{10}{11}\cdot\frac{1}{10}=\frac{1}{11}$$

We can easily deduce that

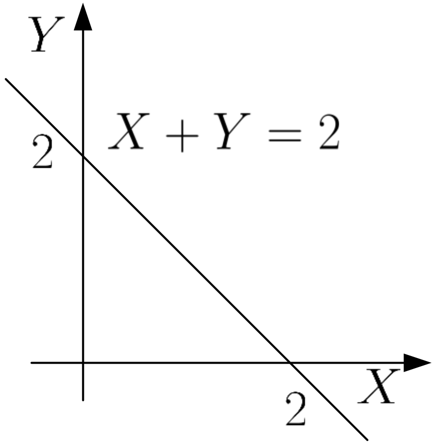
$$P(X=i)=\begin{cases}\frac{1}{11}, & i=1,\cdots,11\\ 0, & \text{otherwise}\end{cases}$$

Hence the expected number of boxes to be opened, i.e., $\mathbb{E}\{X\}$, is:

$$\mathbb{E}\{X\}=\sum_{i=1}^{11} X\cdot P(X=i)=\sum_{i=1}^{11} X\cdot\frac{1}{11}=\frac{11(11+1)}{2}\cdot\frac{1}{11}=6$$

2.(a)

The joint PDF can be illustrated as:



As the total probability should be 1, we have:

$$= \int_0^2 \int_0^{2-x} c dy dx = c2x - c\frac{x^2}{2}\Big|_0^2 = 1 \Rightarrow c = 0.5$$

Alternatively, the area of the non-zero region is 2x2/2=2, while its PDF is constant with value c. We can also get

$$2c = 1 \Rightarrow c = 0.5$$

2.(b)
The marginal PDF of X is:

$$P_X(x) = \int_0^{2-x} 0.5dy = \begin{cases} 1-0.5x, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, the marginal PDF of Y is computed as:

$$P_Y(x) = \int_0^{2-y} 0.5dx = \begin{cases} 1-0.5y, & 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

2.(c)

$$\mathbb{E}\{X\} = \int_{-\infty}^{\infty} x P_X(x) dx = \int_0^2 x(1-0.5x)dx = \frac{1}{2}x^2 - \frac{1}{6}x^3 \Big|_0^2 = \frac{2}{3}$$

$$\mathbb{E}\{X^2\} = \int_{-\infty}^{\infty} x^2 P_X(x) dx = \int_0^2 x^2(1-0.5x)dx = \frac{1}{3}x^3 - \frac{1}{8}x^4 \Big|_0^2 = \frac{2}{3}$$

$$\mathrm{var}(X) = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$$

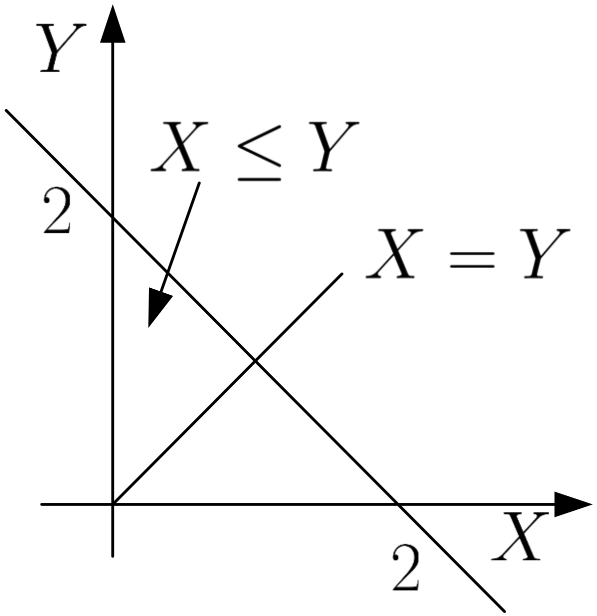
2.(d)

$$\begin{aligned}\mathbb{E}\{XY\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyP_{XY}(x,y)dxdy = \frac{1}{2}\int_0^2\int_0^{2-x} xydydx \\ &= \frac{1}{2}\int_0^2 x\cdot\frac{1}{2}y^2\Big|_0^{2-x} dx = \frac{1}{2}\int_0^2 x\cdot\frac{1}{2}(2-x)^2 dx \\ &= \frac{1}{4}\int_0^2 [x^3-4x^2+4x]dx \\ &= \frac{1}{4}\left[\frac{1}{4}x^4-\frac{4}{3}x^3+2x^2\Big|_0^2\right] = \frac{1}{3}\end{aligned}$$

2.(e)

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)} = \begin{cases}\dfrac{1}{2-y}, & 0\le x\le 2-y \\ \text{otherwise}\end{cases}$$

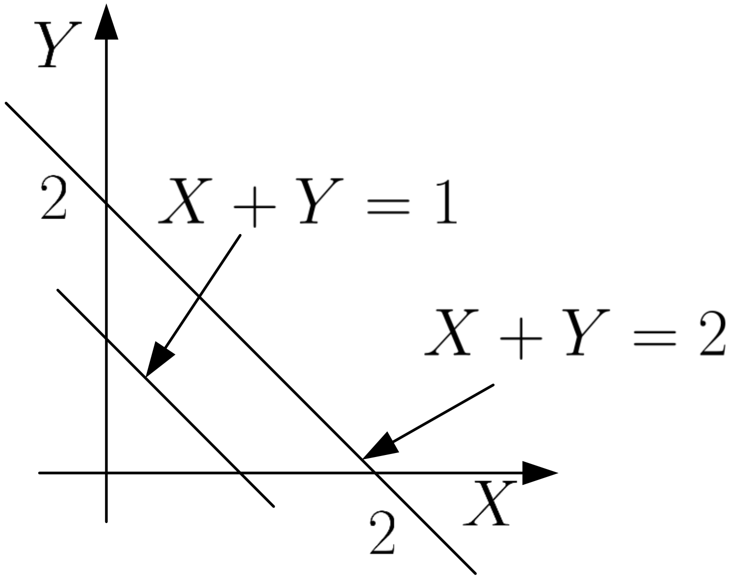
2.(f)



$$P(X \leq Y) = \frac{1}{2}\int_0^1 \int_x^{2-x} dydx = \frac{1}{2}\int_0^1 [2-2x]dx = \frac{1}{2}\left[2x - x^2\right]\Big|_0^1 = \frac{1}{2}$$

Alternatively, it can be seen that the area bounded by the triangle of $X \leq Y$ corresponds to half of the non-zero region. Hence $P(X \leq Y) = 0.5$.

2.(g)

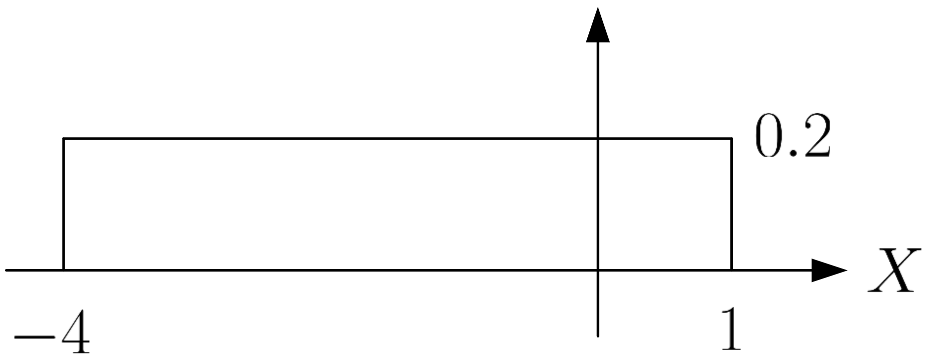


$$P(X \leq Y) = \frac{1}{2}\int_0^1 \int_0^{1-x} dydx = \frac{1}{2}\int_0^1 [1-x]dx = \frac{1}{2}\left[x - x^2/2\Big|_0^1\right] = \frac{1}{4}$$

Alternatively, it can be seen that the area bounded by the triangle of $X+Y \leq 1$ corresponds to 1/4 of the non-zero region. Hence $P(X+Y \leq 1) = 0.25$.

3.

The PDF of X is illustrated as:



It is seen that $P(X \geq 0) = 0.2$ and $P(X < 0) = 0.8$. Hence:

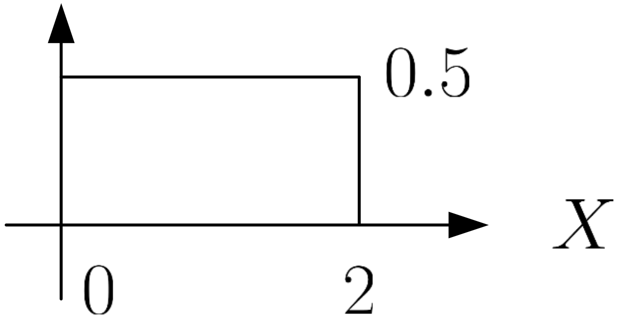
$$P_Y(y) = \begin{cases} 0.2, & y = 1 \\ 0.8, & y = 0 \end{cases}$$

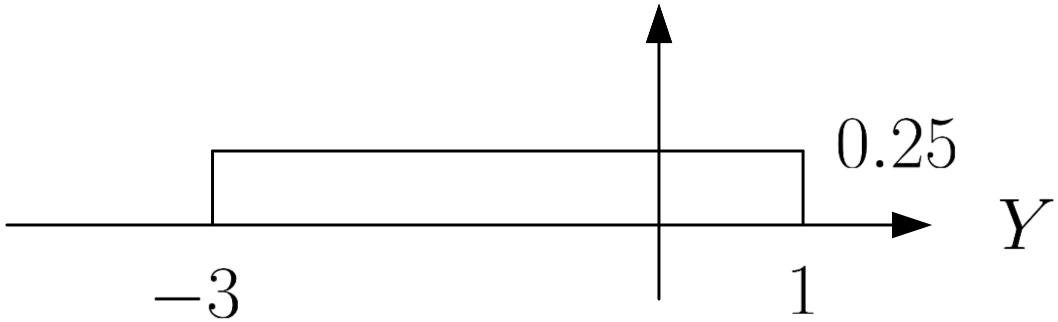
Alternatively, we can compute them as follows:

$$P_Y(0) = P(X < 0) = \int_{-\infty}^{0} P_X(x)dx = \int_{-4}^{0} 0.2dx = 0.8$$

$$P_Y(1) = P(X \geq 0) = \int_{0}^{\infty} P_X(x)dx = \int_{0}^{1} 0.2dx = 0.2$$

4.(a)





$$\mathbb{E}\{X\} = \int_{-\infty}^{\infty} x P_X(x) dx = \int_0^2 0.5 x dx = 0.25 x^2 \Big|_0^2 = 1$$

$$\mathbb{E}\{Y\} = \int_{-\infty}^{\infty} y P_Y(y) dx = \int_{-3}^1 0.25 y dy = 0.125 y^2 \Big|_{-3}^1 = -1$$

Hence:

$$\mathbb{E}\{Z\} = \mathbb{E}\{X + 2Y\} = \mathbb{E}\{X\} + 2\mathbb{E}\{Y\} = 1 + 2(-1) = -1$$

4.(b)

$$\mathbb{E}\{Z^2\} = \mathbb{E}\{(X+2Y)^2\} = \mathbb{E}\{X^2\} + 4\mathbb{E}\{X\}\mathbb{E}\{Y\} + 4\mathbb{E}\{Y^2\}$$

Note that $\mathbb{E}\{XY\} = \mathbb{E}\{X\}\mathbb{E}\{Y\}$ because of independence between X and Y.

$$\mathbb{E}\{X^2\} = \int_{-\infty}^{\infty} x^2 P_X(x)dx = 0.5\int_0^2 x^2 dx = \frac{1}{6}x^3\Big|_0^2 = \frac{4}{3}$$

$$\mathbb{E}\{Y^2\} = \int_{-\infty}^{\infty} y^2 P_Y(y)dy = 0.25\int_{-3}^1 y^2 dy = \frac{1}{12}y^3\Big|_{-3}^1 = \frac{7}{3}$$

Hence

$$\begin{aligned}\mathbb{E}\{Z^2\} &= \mathbb{E}\{X^2\} + 4\mathbb{E}\{X\}\mathbb{E}\{Y\} + 4\mathbb{E}\{Y^2\} \\ &= \frac{4}{3} + 4\cdot 1\cdot -1 + 4\frac{7}{3} = \frac{20}{3}\end{aligned}$$

4.(c)

$$\begin{aligned}\mathbb{E}\{Z^3\} &= \mathbb{E}\{(X+2Y)^3\} \\ &= \mathbb{E}\{X^3\} + 6\mathbb{E}\{X^2\}\mathbb{E}\{Y\} + 12\mathbb{E}\{X\}\mathbb{E}\{Y^2\} + 8\mathbb{E}\{Y^3\}\end{aligned}$$

$$\mathbb{E}\{X^3\} = \int_{-\infty}^{\infty} x^3 P_X(x)dx = 0.5\int_0^2 x^3 dx = \frac{1}{8}x^4\bigg|_0^2 = 2$$

$$\mathbb{E}\{Y^3\} = \int_{-\infty}^{\infty} y^3 P_Y(y)dy = 0.25\int_{-3}^{1} y^3 dy = \frac{1}{16}y^4\bigg|_{-3}^{1} = -5$$

$$\begin{aligned}\mathbb{E}\{Z^3\} &= \mathbb{E}\{X^3\} + 6\mathbb{E}\{X^2\}\mathbb{E}\{Y\} + 12\mathbb{E}\{X\}\mathbb{E}\{Y^2\} + 8\mathbb{E}\{Y^3\} \\ &= 2 + 6\left(\frac{4}{3}\right)(-1) + 12(1)\left(\frac{7}{3}\right) + 8(-5) = -18\end{aligned}$$

5.

Denote head and tail as H and T, respectively. We list out the possible outcomes first.

Outcome	Probability	X	Y
HH	$(1-p)^2$	0	1
HT	$p(1-p)$	1	0
TH	$p(1-p)$	1	1
TT	p^2	2	0

The joint PMF can be tabulated in the following table:

$P_{XY}(x,y)$	$Y=0$	$Y=1$
$X=0$	0	$(1-p)^2$
$X=1$	$p(1-p)$	$p(1-p)$
$X=2$	p^2	0

6.(a)
$X \sim \mathcal{N}(0, 0.5)$ means that the mean and variance are $\mu = 0$ and $\sigma^2 = 0.5$. Hence we have:

$$p_X(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{\pi}} e^{-u^2} du$$

6.(b)
Y is non-negative. For $Y \ge 0$:

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= P(X \le \sqrt{y}) - P(X \le -\sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

While $P_Y(y) = 0$ for $Y < 0$. Combining the results, we have:

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}), & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

6.(c)
Differentiating the CDF yields the PDF, i.e.,

$$P_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(\sqrt{y}) - \frac{d}{dy}F_X(-\sqrt{y})$$

Applying

$$\frac{d}{dy}\left(\int_{-\infty}^{g(y)} f(u)du\right) = f(g(y)) \cdot \frac{dg(y)}{dy}, \quad g(y) = \pm\sqrt{y} \Rightarrow \frac{d}{dy}g(y) = \pm\frac{1}{2\sqrt{y}}$$

We obtain for $Y \geq 0$:

$$\frac{d}{dy}F_X(\sqrt{y}) = \frac{1}{2\sqrt{y}}P_X(\sqrt{y}), \quad f(\cdot) = P_X(\cdot)$$

$$\frac{d}{dy}F_X(-\sqrt{y}) = -\frac{1}{2\sqrt{y}}P_X(-\sqrt{y}) = -\frac{1}{2\sqrt{y}}P_X(\sqrt{y})$$

Hence

$$\begin{aligned} P_Y(y) &= \frac{1}{2\sqrt{y}}P_X(\sqrt{y}) - \left(-\frac{1}{2\sqrt{y}}P_X(\sqrt{y})\right) = \frac{1}{\sqrt{y}}P_X(\sqrt{y}) \\ &= \frac{1}{\sqrt{\pi y}}e^{-y} \end{aligned}$$

Combining the results, we have:

$$P_Y(y) = \begin{cases} \dfrac{1}{\sqrt{\pi y}}e^{-y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$