Recap. change of variables 3D.

$$x = g(u,v,w)$$
$$y = h(u,v,w)$$
$$z = j(u,v,w)$$

☆☆ $$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_G f(g(u,v,w)\,h(u,v,w)\,j(u,v,w)) \cdot |J(u,v,w)|\,du\,dv\,dw$$ ☆☆

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{w} \end{vmatrix}$$

Eg. spherical coordinates

$$\boxed{x = \rho\cos\theta\sin\varphi}$$
$$y = \rho\sin\theta\sin\varphi$$
$$z = \rho\cos\varphi$$

$$\rho\underbrace{\cos\theta\sin\varphi}_{\text{coefficient}} \qquad \boxed{\rho\cos\theta}\cos\varphi \quad \text{coefficient}$$

$$\boxed{-\rho\sin\varphi\sin\theta} \quad \text{coefficient}$$

$$J(\rho,\theta,\varphi) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\varphi\sin\theta & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ & & -\rho\sin\varphi \end{vmatrix}$$

$$\begin{vmatrix} \cos\psi & 0 & -\rho\sin\psi \end{vmatrix}$$

$$= \cos\psi\left(-\rho^2\sin^2\theta\sin\psi\cos\psi - \rho^2\cos^2\theta\sin\psi\cos\psi\right) +$$
$$- \rho\sin\psi\left(\rho\cos^2\theta\sin^2\psi - (-\rho\sin^2\theta\sin^2\psi)\right)$$
$$= -\cos\psi\,\rho^2\sin\psi\cos\psi(\sin^2\theta+\cos^2\theta) - \rho^2\sin^3\psi(\cos^2\theta+\sin^2\theta)$$
$$= -\rho^2\cos^2\psi\sin\psi - \rho^2\sin^3\psi$$
$$= -\rho^2\sin\psi(\cos^2\psi+\sin^2\psi)$$
$$= \underline{-\rho^2\sin\psi} \Rightarrow dx\,dy\,dz$$
$$\therefore \rho^2\sin\psi\,d\rho\,d\theta\,d\psi$$

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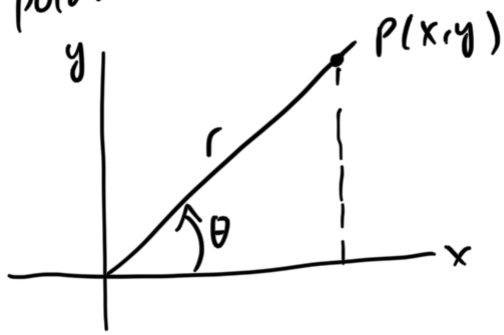
① polar coordinates



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \text{polar} \to \text{cartesian}$$

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}\right\} \text{cartesian} \to \text{polar}$$

Note $\tan\theta$ has period of π

principal value for inverse

$$-\frac{\pi}{2} < \tan^{-1}\left(\frac{y}{x}\right) < \frac{\pi}{2}$$

$$\theta = \begin{cases} \arctan\frac{y}{x} & \text{if } x>0 \\ \arctan\frac{y}{x}+\pi & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0 \quad y>0 \\ -\frac{\pi}{2} & \text{if } x=0 \quad y<0 \end{cases}$$

Ex. Find a) Cartesian coordinate of P where P is $(2, \frac{\pi}{3})$

b) Polar coordinate of Q where Q is $(\underline{\underline{-1,-1}})$

Ans

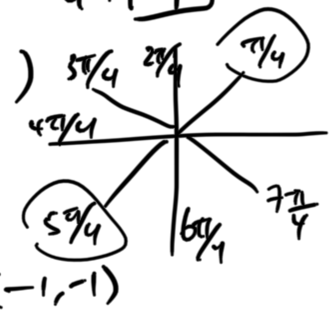
a) $x = 2\cos\frac{\pi}{3} = 1$

$y = 2\sin\frac{\pi}{3} = \sqrt{3}$

b) $r = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta = \frac{-1}{-1} = 1 \quad \theta = \frac{\pi}{4}, \boxed{\frac{5\pi}{4}}$

$\therefore (\sqrt{2}, \frac{5\pi}{4})$



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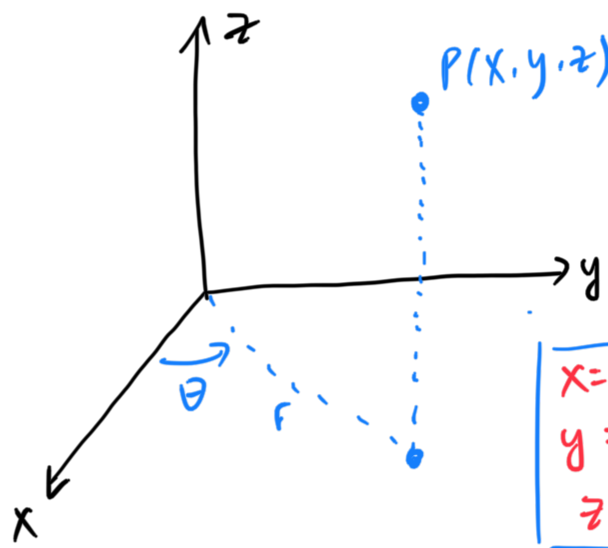
cylindrical coordinates



$P(x,y,z) \to (\underbrace{r,\theta}_{\text{polar coordinate of } (x,y)}, z)$

This is the vertical projection along z-axis of P onto xy-plane

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}}$$

$r^2 = x^2 + y^2$

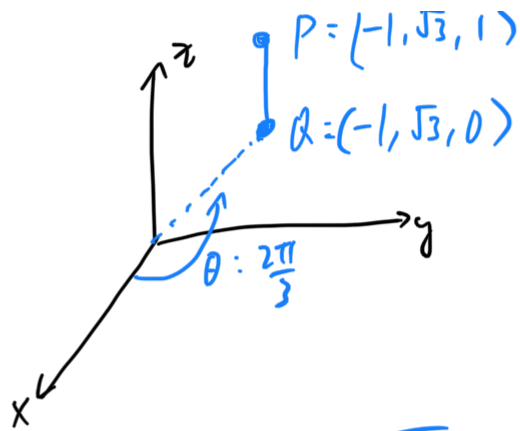
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ or $\tan^{-1}\left(\frac{y}{x}\right) + \pi$

Q: (a) Plot the pt w/ cylindrical coordinates $(2, \frac{2\pi}{3}, 1)$

find its rectangular coordinates

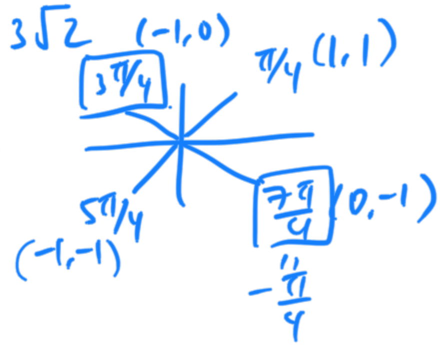
(b) Find cylindrical coordinates of the point with rectangular coordinates $(\underline{3,-3}, -7)$

(a)



b) $r = \sqrt{x^2+y^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-3}{3}\right) = -1 \quad -\frac{\pi}{4}, \frac{3\pi}{4}$

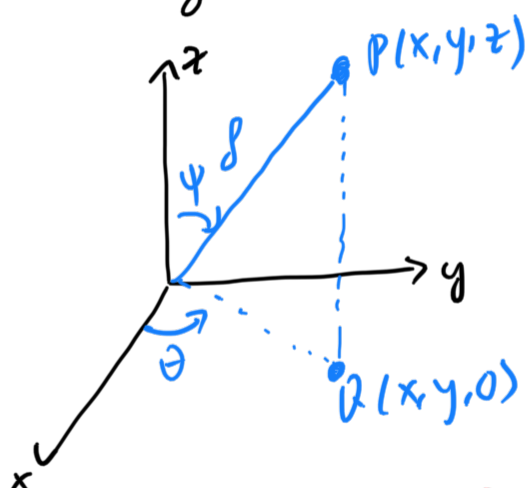


$= (3\sqrt{2}, -\frac{\pi}{4}, -7)$

Spherical coordinates

$P(x,y,z) \to (\rho, \theta, \varphi)$

$\rho > 0, \; 0 \le \theta \le 2\pi, \; 0 \le \varphi \le \pi$



$\rho > 0$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi$

ρ = distance (P, O)

θ = Angle from zx plane counterclockwise to the half plane originating from x-axis and containing P.

φ - angle from positive z-axis to vector $\vec{OP}$

characteristics.

Note P is on z-axis $\varphi = 0$

φ increases from 0 to $\frac{\pi}{2}$ as P moves towards the xy plane

φ keeps increasing as P moves below xy plane.

ψ reaches max value π when ρ is on the negative z-axis.

Convert rectangular $\leftrightarrow$ cylindrical $\leftrightarrow$ spherical

$$x = r\cos\theta = \rho\sin\psi\cos\theta$$
$$y = r\sin\theta = \rho\sin\psi\sin\theta$$
$$z = z \quad = \rho\cos\psi$$

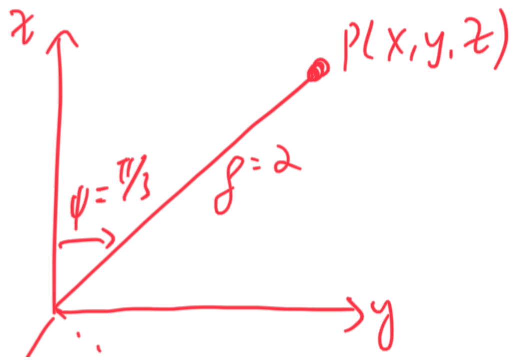
$$\rho = \sqrt{x^2+y^2+z^2} \qquad \tan\theta = \frac{y}{x} \qquad \cos\psi = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x>0 \\ \tan^{-1}\left(\frac{y}{x}\right)+\pi & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0,\ y>0 \\ -\frac{\pi}{2} & \text{if } x=0,\ y<0 \end{cases}$$

$$\psi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

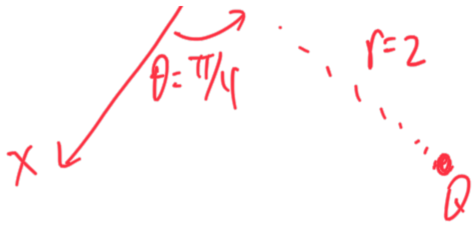
Ex. a) plot the pt w/ spherical coordinates $(\underset{\rho}{2}, \underset{\theta}{\frac{\pi}{4}}, \underset{\psi}{\frac{\pi}{3}})$
find its rectangular coordinates
(b) Find the spherical coordinates for the point w/ rectangular coordinates $(\underset{x}{0}, \underset{y}{2\sqrt{3}}, \underset{z}{-2})$

Ans (a)



$$x = \rho\sin\psi\cos\theta = 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{6}}{2}$$

$$y = \rho\sin\psi\sin\theta = 2\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{6}}{2}$$



$$z = \rho\cos\phi = 2\cos\frac{\pi}{3} = 1$$

(b) $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(2\sqrt{3})^2+4} = \sqrt{16} = 4$

$\theta = \frac{\pi}{2}$ since $(0, 2\sqrt{3}, -2)$ is on positive y-axis

$\cos\psi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \psi = \frac{2\pi}{3}$

$\therefore \left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)$ spherical coordinates