Solution

1.(a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ae^{-(3x+4y)}\mathrm{d}x\mathrm{d}y = \frac{A}{12} = 1,$$
$$A = 12.$$

1.(b)

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} p(u,v)\mathrm{d}u\mathrm{d}v = \begin{cases} \int_{-\infty}^{y}\int_{-\infty}^{x} 12e^{-(3u+4v)}\mathrm{d}u\mathrm{d}v, \\ 0, \end{cases}$$
$$= \begin{cases} (1-e^{-3x})(1-e^{-4y}), & x > 0,\ y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

1.(c)

$$P(0 \le X < 1,\ 0 \le Y < 2) = \int_0^2\int_0^1 12e^{-(3x+4y)}\mathrm{d}x\mathrm{d}y = (1-e^{-3})(1-e^{-8}) \approx 0.9499.$$

2.

Let $Z = X - Y$. As $\mathbb{E}\{X\} = \mathbb{E}\{Y\} = 0$, we have $\mathbb{E}\{Z\} = 0$.

Because $\mathbb{E}\{Z\} = 0$, and X and Y are independent of each other, we have:

$$\mathbb{E}\{Z^2\} = \mathrm{var}(Z) = \mathbb{E}\{(X-Y)^2\} = \mathbb{E}\{X^2\} + \mathbb{E}\{Y^2\} = 1$$

Then we have

$$\mathrm{var}(|X-Y|) = \mathrm{var}(|Z|) = \mathbb{E}\{|Z|^2\} - [\mathbb{E}\{|Z|\}]^2 = \mathbb{E}\{Z^2\} - [\mathbb{E}\{|Z|\}]^2,$$

Using

$$\mathbb{E}\{|Z|\} = \int_{-\infty}^{+\infty} |z|\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\mathrm{d}z = \frac{2}{\sqrt{2\pi}}\int_0^{+\infty} ze^{-z^2/2}\mathrm{d}z = \sqrt{\frac{2}{\pi}},$$

Therefore,

$$\mathrm{var}(|X-Y|) = 1 - \frac{2}{\pi}$$

Note that the answer can be validated using MATLAB:

```
>> X=sqrt(0.5).*randn(1,1000000);
Y=sqrt(0.5).*randn(1,1000000);
Z=abs(X-Y);
var(Z)
ans = 0.3637
```

3.

By observing that $P_{NK}(n,k)$ can be factorized as:

$$P_{NK}(n,k) = \frac{100^n e^{-100}}{n!} \times \binom{100}{k} p^k (1-p)^{100-k}$$

where one corresponds to the Poisson distribution and another corresponds to the binomial distribution, we easily get:

$$P_N(n) = \begin{cases} \dfrac{100^n e^{-100}}{n!}, & n = 0, 1, \cdots, \\ 0, & \text{otherwise} \end{cases}$$

$$P_K(k) = \begin{cases} \dbinom{100}{k} p^k (1-p)^{100-k}, & k = 0, 1, \cdots, 100 \\ 0, & \text{otherwise} \end{cases}$$

4.

$X \sim \mathcal{U}(0,3)$ and $Y \sim \mathcal{U}(0,3)$ imply

$$p(x) = \begin{cases} \dfrac{1}{3}, & 0 \le x \le 3, \\ 0, & x < 0,\ x > 3. \end{cases} \qquad p(y) = \begin{cases} \dfrac{1}{3}, & 0 \le y \le 3, \\ 0, & y < 0,\ y > 3. \end{cases}$$

Because X and Y are independent of each other, the joint PDF is:

$$p(x,y) = \begin{cases} \dfrac{1}{9}, & 0 \le x \le 3,\ 0 \le y \le 3, \\ 0, & x < 0,\ x > 3,\ y < 0,\ y > 3. \end{cases}$$

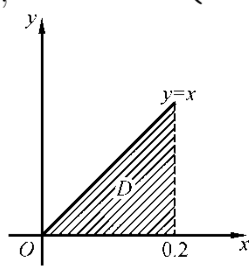
Hence we can easily deduce:

$$P(\max\{X,Y\} \le 1) = \frac{1}{9}$$

5.

Since random variables X, Y are independent of each other, the joint PDF is:

$$p(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} \dfrac{1}{0.2} \cdot 5e^{-5y}, \\ 0, \end{cases} = \begin{cases} 25e^{-5y}, & 0 < x < 0.2 \text{ and } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$



According to the above figure, we have:

$$P(Y \le X) = \int_0^{0.2} \int_0^x 25e^{-5y} dy dx = \int_0^{0.2} (-5e^{5x} + 5) dx = e^{-1} = 0.3679.$$

6.(a)
The area of D is $\int_1^{e^2} \frac{1}{x} dx = \ln e^2 - \ln 1 = 2$. Hence the joint PDF is

$$p(x,y) = \begin{cases} \dfrac{1}{2}, & 1 \le x \le e^2 \text{ and } 0 < y \le \dfrac{1}{x}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence:

$$p(x) = \begin{cases} \displaystyle\int_0^{1/x} \frac{1}{2} dy = \frac{1}{2x}, & 1 \le x \le e^2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p(y) = \begin{cases} \displaystyle\int_1^{e^2} \frac{1}{2} dx = \frac{e^2}{2} - \frac{1}{2} & 0 < y \le \dfrac{1}{e^2}, \\ \displaystyle\int_1^{1/y} \frac{1}{2} dx = \frac{1}{2y} - \frac{1}{2}, & \dfrac{1}{e^2} \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

6.(b)
It is clear that $p(x,y) \ne p(x)p(y)$. Hence X and Y are not independent.

7.

$$\begin{aligned} \mathbb{E}\{\hat{A}\} &= \mathbb{E}\left\{\frac{1}{N-1}\sum_{n=1}^{N} r_n\right\} = \frac{1}{N-1}\sum_{n=1}^{N}\mathbb{E}\{r_n\} = \frac{1}{N-1}\sum_{n=1}^{N}\mathbb{E}\{A+w_n\} \\ &= \frac{1}{N-1}\sum_{n=1}^{N}(A+\mathbb{E}\{w_n\}) = \frac{1}{N-1}\sum_{n=1}^{N}(A+0) = \frac{N}{N-1}A \end{aligned}$$

$$\begin{aligned} \mathrm{var}(\hat{A}) &= \mathbb{E}\left\{\left(\frac{1}{N-1}\sum_{n=1}^{N} r_n - \mathbb{E}\{A\}\right)^2\right\} \\ &= \mathbb{E}\left\{\left(\frac{1}{N-1}\sum_{n=1}^{N} r_n - \frac{N}{N-1}A\right)^2\right\} \\ &= \frac{1}{(N-1)^2}\mathbb{E}\left\{\left(\sum_{n=1}^{N} w_n + nA - nA\right)^2\right\} = \frac{1}{(N-1)^2}\mathbb{E}\left\{\left(\sum_{n=1}^{N} w_n\right)^2\right\} \\ &= \frac{1}{(N-1)^2}\sum_{n=1}^{N}\sum_{m=1}^{N}\mathbb{E}\left\{w_n w_m\right\} = \frac{1}{(N-1)^2}\sum_{n=1}^{N}\mathbb{E}\left\{w_n^2\right\} = \frac{N\sigma_w^2}{(N-1)^2} \end{aligned}$$

We then apply (3.29) to obtain:

$$\mathrm{MSE}(\hat{A}) = \mathrm{var}(\hat{A}) + (A - \mathbb{E}\{A\})^2 = \frac{N\sigma_w^2}{(N-1)^2} + \frac{A^2}{(N-1)^2}$$

8.
Given the event $A = \{\min(X,Y) > 5\}$, we first compute $P(A)$:

$$P(A) = P(X > 5, Y > 5) = \sum_{x=6}^{10}\sum_{y=6}^{10} 0.01 = 0.25$$

Hence $P_{XY|A}(x,y)$ is:

$$P_{XY|A}(x,y) = \begin{cases} 0.04, & x = 6, 7, \cdots 10,\ y = 6, 7, \cdots, 10 \\ 0, & \text{otherwise} \end{cases}$$

9.
Given the event $A = \{X + Y \leq 1\}$, we first compute $P(A)$:

$$P(A) = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)} dy dx = 1 - 3e^{-2} + 2e^{-3}$$

Hence $P_{XY|A}(x,y)$ is:

$$P_{XY|A}(x,y) = \begin{cases} \dfrac{6e^{-(2x+3y)}}{1 - 3e^{-2} + 2e^{-3}}, & x + y \leq 1,\ x \geq 0,\ y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

10.
We compute the marginal PDF $P_Y(y)$ first. For $0 \leq y \leq 1$, we have:

$$P_Y(y) = \int_0^1 (x+y) dx = \frac{2y+1}{2}$$

Hence:

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)} = \begin{cases} \dfrac{2(x+y)}{2y+1}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Due to the symmetry between X and Y, we have:

$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)} = \begin{cases} \dfrac{2(x+y)}{2x+1}, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$