Tutorial 7

1. Consider two random variables X and Y with joint probability mass function (PMF) given in the following table:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

(a) Find $P(X=0, Y \leq 1)$.
(b) Find the marginal PMFs of X and Y.
(c) Are X and Y independent? Briefly explain your answer.

2. Consider two random variables X and Y with the following joint PMF:

$$P_{XY}(x,y) = \begin{cases} cxy, & x = 1,2,4,\ y = 1,3 \\ 0, & \text{otherwise} \end{cases}$$

where c is an unknown constant.

(a) Find the value of c.
(b) Find $P(Y = 3)$.
(c) Find $P(Y < X)$.
(d) Find $P(Y > X)$.
(e) Find $P(X = Y)$.
(f) Find the marginal PMFs of X and Y.
(g) Are X and Y independent? Briefly explain your answer.
(h) Find $\mathbb{E}\{X\}$ and $\mathbb{E}\{Y\}$.
(i) Find $\mathrm{var}(X)$ and $\mathrm{var}(Y)$.

3. Consider two independent Bernoulli variables X and Y with success probabilities p_1 and p_2, respectively. Find the joint PMF and cumulative density function (CDF) for X and Y.

4. A bag consists of 100 balls where 40 are blue and 60 are red. 10 balls are randomly chosen at the same time. Let random variables X and Y be the numbers of blue and red balls. Find the joint PMF of X and Y. Which combination of blue and red balls has the highest probability?

Solution

1.(a)

$$
\begin{aligned}
P(X=0, Y \leq 1) &= P(X=0, Y=0) + P(X=0, Y=1) \\
&= \frac{1}{6} + \frac{1}{4} = \frac{5}{12}
\end{aligned}
$$

1.(b)

$$
\begin{aligned}
P(X=0) &= P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=1) \\
&= \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}
\end{aligned}
$$

$$
\begin{aligned}
P(X=1) &= P(X=1, Y=0) + P(X=1, Y=1) + P(X=1, Y=1) \\
&= \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}
\end{aligned}
$$

Combining the results yields:

$$p(x) = \begin{cases} \dfrac{13}{24}, & x = 0 \\ \dfrac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly,

$$p(y) = \begin{cases} \dfrac{7}{24}, & y = 0 \\ \\ \dfrac{5}{12}, & y = 1 \\ \\ \dfrac{7}{24}, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

1.(c)

If X and Y are independent, this means:

$$p(x,y) = p(x)p(y)$$

for all x and y. Consider when $x = 0$ and $y = 0$, the joint PMF is:

$$p(0,0) = \frac{1}{6}$$

which cannot be equal to the product of

$$P_X(0) = \frac{13}{24} \quad \text{and} \quad P_Y(0) = \frac{7}{24}$$

Hence X and Y are not independent.

2.(a)

As the sum of all PMFs is equal to 1, we have:

$$\sum_{x=1,2,4}\sum_{y=1,3} cxy = c\sum_{x=1,2,4} x \sum_{y=1,3} y = c[1(1+3)+2(1+3)+4(1+3)] = 1$$

$$\Rightarrow 28c = 1 \Rightarrow c = \frac{1}{28}$$

2.(b)

$$P(Y=3) = \frac{1}{28}\sum_{y=3} y \sum_{x=1,2,4} x = \frac{1}{28}3(1+2+4) = \frac{3}{4}$$

2.(c)

$$P(Y<X) = \frac{1}{28}\sum_{x=1,2,4} x \sum_{y<x} y = \frac{[1(0)+2(1)+4(1+3)]}{28} = \frac{9}{14}$$

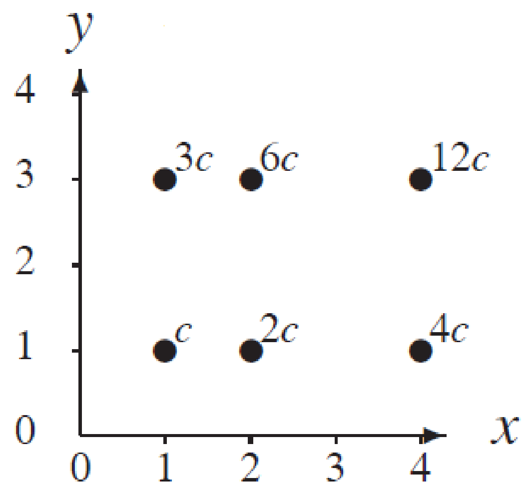
2.(d)

$$P(Y > X) = \frac{1}{28} \sum_{x=1,2,4} x \sum_{y>x} y = \frac{[1(3) + 2(3) + 4(0)]}{28} = \frac{9}{28}$$

2.(e)

$$P(Y = X) = 1 - P(Y < X) - P(Y > X) = 1 - \frac{9}{14} - \frac{9}{28} = \frac{1}{28}$$

Alternatively, the problem can be solved by first showing the nonzero PMFs in a 2-D plot:



2.(f)
From the plot, we can easily obtain:

$$p(x) = \begin{cases} \frac{1}{7}, & x = 1 \\ \frac{2}{7}, & x = 2 \\ \frac{4}{7}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we also obtain:

$$p(y) = \begin{cases} \frac{1}{4}, & y = 1 \\ \frac{3}{4}, & y = 3 \\ 0, & \text{otherwise} \end{cases}$$

2.(g)

As $p(x,y)=p(x)p(y)$ holds for all pairs of x and y, X and Y are independent.

It can also be observed from

$$P_{XY}(x,y)=\frac{xy}{28}=\frac{x}{7}\cdot\frac{y}{4}=P_X(x)\cdot P_Y(y), \quad x=1,2,4,\ y=1,3$$

2.(h)

Using (2.19), the expected values are computed as:

$$\mathbb{E}\{X\}=\sum_{x=1,2,4} xp(x)=(1)\frac{1}{7}+(2)\frac{2}{7}+(4)\frac{4}{7}=3$$

$$\mathbb{E}\{Y\}=\sum_{y=1,3} yp(y)=(1)\frac{1}{4}+(3)\frac{3}{4}=\frac{5}{2}$$

2.(i)

$$\mathbb{E}\{X^2\} = \sum_{x=1,2,4} x^2 p(x) = (1)^2\frac{1}{7} + (2)^2\frac{2}{7} + (4)^2\frac{4}{7} = \frac{73}{7}$$

$$\mathbb{E}\{Y^2\} = \sum_{y=1,3} y^2 p(y) = (1)\frac{1}{4} + (3)^2\frac{3}{4} = 7$$

Applying (2.23) yields:

$$\mathrm{var}(X) = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 = \frac{10}{7}$$

$$\mathrm{var}(Y) = \mathbb{E}\{Y^2\} - (\mathbb{E}\{Y\})^2 = \frac{3}{4}$$

3.

We know that both X and Y only have values 0 or 1. Hence the PMF is:

$$P(X=0,Y=0)=(1-p_1)(1-p_2)$$
$$P(X=1,Y=0)=p_1(1-p_2)$$
$$P(X=0,Y=1)=(1-p_1)p_2$$
$$P(X=1,Y=1)=p_1p_2$$

Note that when $p=p_1=p_2$, it will reduce to the PMF of binomial distribution with $n=2$.

For the CDF, it is clear that

$$F(x,y)=0, \quad x<0$$
$$F(x,y)=0, \quad y<0$$
$$F(x,y)=1, \quad x\geq 1,\ y\geq 1$$

For $0 \leq x < 1$ and $y \leq 1$, we have:

$$P(X = 0, Y \leq 1) = (1 - p_1)(1 - p_2) + (1 - p_1)p_2 = 1 - p_1$$

For $x \leq 1$ and $0 \leq y < 1$, we have:

$$P(X \leq 1, Y = 0) = (1 - p_1)(1 - p_2) + p_1(1 - p_2) = 1 - p_2$$

For $0 \leq x < 1$ and $0 \leq y < 1$, we have:

$$P(X = 0, Y = 0) = (1 - p_1)(1 - p_2)$$

Combining them yields:

$$F(x,y)=\begin{cases}0, & x<0\\ 0, & y<0\\ 1-p_1, & 0\le x<1,\ y\le 1\\ 1-p_2, & x\le 1,\ 0\le y<1\\ (1-p_1)(1-p_2), & 0\le x<1,\ 0\le y<1\\ 1, & x\ge 1,\ y\ge 1\end{cases}$$

4.

Here, there is a constraint of $X+Y=10$. Hence the combination of (X,Y) includes (0,10), (1,9), …, (10,0). For example, the probability of having 0 blue ball and 10 red balls is:

$$\frac{\binom{60}{10}}{\binom{100}{10}}$$

Similarly, the probability of having 1 blue ball and 9 red balls is:

$$\frac{\binom{40}{1}\cdot\binom{60}{9}}{\binom{100}{10}}$$

We can write the joint PMF as:

$$p(i,j) = \begin{cases} \dfrac{\dbinom{40}{i} \cdot \dbinom{60}{j}}{\dbinom{100}{10}}, & i+j=10, 10 \geq i \geq 0, 10 \geq j \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Intuitively, the probability of getting a blue ball is 4/10 while that of red ball is 6/10. Hence we expect the highest chance is:

$$p(i,j) = \frac{\dbinom{40}{4} \cdot \dbinom{60}{6}}{\dbinom{100}{10}} = 0.2643$$

We can compute all PMF values to check it. If we vary the value of i in the range of $10 \geq i \geq 0$, then the value of j is fixed because of the relationship of $i + j = 10$.

```
>> for i=0:10
(nchoosek(40,i)*nchoosek(60,10-i))/nchoosek(100,10)
end
ans = 0.0044
ans = 0.0342
ans = 0.1153
ans = 0.2204
ans = 0.2643
ans = 0.2076
ans = 0.1081
ans = 0.0369
ans = 0.0079
ans = 9.4778e-04
ans = 4.8969e-05
```