Take Home Assignment MA2001 #3

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Q5-Q7 are optional related to Chapter 5 that won't be counted into assessment.

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For each of the following questions, write down your solution with details of steps. Marks will not given if only final answers are provided.

1. Evaluate $\iint_S e^{xy}dxdy$, where S is the region enclosed by $xy = 1, xy = 2, y = x, y = 4x$ using the change of variable $xy = u$, $\frac{y}{x} = v$.

   Solution. Under the transformations $xy = u, y/x = v$, we have $xy = 1$ goes to $u = 1$, $xy = 2$ goes to $u = 2$, $y = x$ goes to $v = 1$, and $y = 4x$ goest to $v = 4$.

   The region S in xy-plane which is enclosed by $xy = 1$, $xy = 2$, $y = x$, $y = 4x$ consists of two parts, say $S = S_1 \cup S_2$. S_1 is in the first quadrant of xy-plane and S_2 is in the third quadrant. Under the transformations, both S_1 and S_2 go to the region S_{uv} in the uv-plane which is enclosed by the lines $u = 1, u = 2, v = 1, v = 4$. The Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} y & -y/x^2 \\ x & 1/x \end{vmatrix}} = \frac{1}{y/x + x/y} = \frac{1}{2v}.$$

   Hence

$$\iint_S e^{xy}dxdy = 2\iint_{S_{uv}} e^u \left|\frac{1}{2v}\right| dudv = 2\int_1^2 \left[\int_1^4 e^u \frac{1}{2v}dv\right] du$$
$$= \int_1^2 e^u \ln v\Big|_1^4 du = \ln 4 \int_1^2 e^u du = (e^2 - e)\ln 4.$$

2. Compute the following multiple integrals using suitable method.

   (a) $\displaystyle\iint_R x^3dxdy$, where R is the region bounded by x-axis, y-axis, $x = 2$, $y = 1 + x$, and $y = 3 - x$.

   (b) $\displaystyle\iiint_V \frac{1}{\sqrt{4 - x^2 - y^2}}dxdydz$, where V is the region which is bounded above by a sphere $x^2 + y^2 + z^2 = 4$ and is bounded below by a plane $z = 1$.

Solution.

(a)

$$\iint_R x^3 dxdy = \int_0^2 \int_0^1 x^3 dydx + \int_1^2 \int_{y-1}^{3-y} x^3 dxdy$$
$$= \frac{1}{4}\left(x^4\Big|_0^2 + \int_1^2 [(3-y)^4 - (y-1)^4]dy \right)$$
$$= 4 + \frac{1}{4}\left[\frac{-(3-y)^5}{5} - \frac{(y-1)^5}{5}\right]\Big|_1^2 = 4 + \frac{3}{2} = \frac{11}{2}.$$

(b)

$$\iiint_V \frac{1}{\sqrt{4-x^2-y^2}} dxdydz = \iint_{\sigma_{xy}} \int_1^{\sqrt{4-x^2-y^2}} \frac{1}{\sqrt{4-x^2-y^2}} dzdxdy$$
$$= \iint_{\sigma_{xy}} \frac{1}{\sqrt{4-x^2-y^2}} (\sqrt{4-x^2-y^2} - 1)dxdy$$
$$= \iint_{\sigma_{xy}} (1 - \frac{1}{\sqrt{4-x^2-y^2}})dxdy$$
$$= \int_0^{2\pi} \int_0^{\sqrt{3}} (1 - \frac{1}{\sqrt{4-r^2}})rdrd\theta$$
$$= \int_0^{2\pi} d\theta \times \int_0^{\sqrt{3}} (r - \frac{r}{\sqrt{4-r^2}})dr$$
$$= 2\pi \left(\frac{r^2}{2}\Big|_0^{\sqrt{3}} + \sqrt{4-r^2}\Big|_0^{\sqrt{3}} \right)$$
$$= 2\pi(3/2 + 1 - 2)$$
$$= \pi.$$

3. Find $grad f = \nabla f$ for $f(x, y, z) = x^2 + y^2 + z^2$. Hence calculate

(a) the directional derivative of f at $(1, 1, 1)$ in the direction of the unit vector $\frac{1}{3}(2, 2, 1)$;

(b) the maximum rate of change of the function at $(1, 1, 1)$ and its direction.

Solution.

(a) $D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = (f_x, f_y, f_z)\Big|_{(1,1,1)} \cdot \frac{1}{3}(2, 2, 2) = \frac{1}{3}(2x, 2y, 2z)\Big|_{(1,1,1)} \cdot (2, 2, 1) = \frac{1}{3}(2, 2, 2) \cdot (2, 2, 1) = \frac{1}{3}(4 + 4 + 2) = 10/3.$

(b) The maximal rate of change at $(1, 1, 1)$ is $|\nabla f(1, 1, 1)| = |(2, 2, 2)| = \sqrt{12} = 2\sqrt{3}$ and the direction is $\frac{\nabla f(1,1,1)}{|\nabla f(1,1,1)|} = \frac{(2,2,2)}{\sqrt{12}} = \frac{1}{\sqrt{3}}(1, 1, 1)$.

4. Let $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-2z)\vec{j} + (4x+cy+2z)\vec{k}$ be a vector field on $\mathbb{R}^3$, where a, b, and c are real constants.

(a) Find the values of a, b, and c such that $\vec{F}$ is irrotational.

(b) With the values of a, b, and c obtained in (a), determine a potential function φ on $\mathbb{R}^3$ for which $\nabla\varphi = \vec{F}$.

Solution.

(a)

$$\begin{aligned}\text{curl}\vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-2z & 4x+cy+2z \end{vmatrix} \\ &= (c+2)\vec{i} - (4-a)\vec{j} + (b-2)\vec{k}\end{aligned}$$

implies $a = 4, b = 2, c = -2$.

(b) Solving φ from $\varphi_x = F_1 = (x+2y+4z)$, $\varphi_y = F_2 = 2x-3y-2z$, and $\varphi_z = 4x-2y+2z$. We get $\varphi(x,y,z) = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - 2yz + 4xz + constant.$

5. (optional) Compute the following line integrals using suitable method.

(a) $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = 3x\vec{i} + 4xy\vec{j}$ and C is the boundary curve of the region in the first quadrant bounded by x-axis, $y = x$, and a circle $x^2 + y^2 = 1$.

(b) $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = [2xz^2\cos(1+x^2+3y^3)]\vec{i} + [9y^2z^2\cos(1+x^2+3y^3)]\vec{j} + [2z\sin(1+x^2+3y^3)]\vec{k}$ and C is the path moving from a point $(0,1,2)$ and $(3,4,7)$ along a straight line.

Solution.

(a). By Green's theorem, we have

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy \\ &= \iint_S 4y\,dxdy \\ &= \int_0^{\pi/4}\int_0^1 4r\sin\theta\, r\,dr\,d\theta \\ &= 4\int_0^1 r^2 dr \times \int_0^{\pi/4} \sin\theta\, d\theta \\ &= 4\left(r^3/3\Big|_0^1\right) \times \left(\cos\theta\Big|_0^{\pi/4}\right) \\ &= \frac{4-2\sqrt{2}}{3}.\end{aligned}$$

(b) Note that $\vec{F}$ is irrotational and satisfying $\nabla\varphi = \vec{F}$ for $\varphi(x, y, z) = z^2 \sin(1 + x^2 + 3y^3)$. Hence, $\int_C \vec{F} \cdot d\vec{r} = \varphi(3, 4, 7) - \varphi(0, 1, 2) = 49 \sin 202 - 4 \sin 4$.

6. (optional) Compute the following surface integrals using suitable method.

(a) $\iint_S (x^2 + y^2) dS$, where S is the part of the surface $z = 9 - y$ lying inside the cylinder $x^2 + y^2 = 1$.

(b) $\iint_S \vec{F} \cdot \vec{n} dS$, where $\vec{F} = y\vec{i} + x\vec{j}$ and S is the part of the cone $z = \sqrt{x^2 + y^2}$ lying inside the cylinder $x^2 + y^2 = 9$. (Here, $\vec{n}$ is upward pointing normal).

(c) $\iint_S \vec{F} \cdot \vec{n} dS$, where $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ and S is the boundary surface of the region bounded by the cone $z = \sqrt{\frac{x^2+y^2}{3}}$ and the upper-half sphere $x^2 + y^2 + z^2 = 8$. (Here, $\vec{n}$ is outward-pointing normal).

Solution.

(a) $dS = \sqrt{1 + Z_x^2 + Z_y^2} dxdy = \sqrt{2} dxdy$. Hence

$$\iint_S (x^2 + y^2) dS = \iint_{\sigma_{xy}} (x^2 + y^2)\sqrt{2} dxdy = \int_0^1 \int_0^{2\pi} r^2 \sqrt{2} r dr d\theta = \frac{\pi}{\sqrt{2}}.$$

(b) By the divergence theorem,

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_S \operatorname{div}\vec{F} dxdydz = 0$$

since $\operatorname{div}\vec{F} = 0$.

(c) By the divergence theorem,

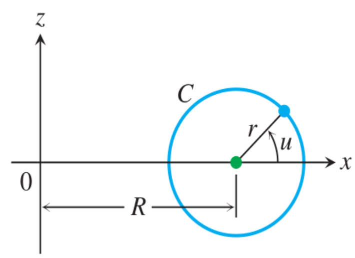
$$\begin{aligned}
\iint_S \vec{F} \cdot \vec{n} dS &= \iiint_S \operatorname{div}\vec{F} dxdydz \\
&= 3 \iiint_V dxdydz \\
&= 3 \int_0^{2\sqrt{2}} \int_0^{2\pi} \int_0^{\pi/3} \rho^2 \sin\varphi d\rho d\varphi d\theta \\
&= 3 \int_0^{2\pi} d\theta \int_0^{\pi/3} \sin\varphi d\varphi \int_0^{2\sqrt{2}} \rho^2 d\rho \\
&= 3 \times 2\pi \times (-\cos\varphi)\Big|_0^{\pi/3} \times \frac{\rho^3}{3}\Big|_0^{2\sqrt{2}} \\
&= 16\sqrt{2}\pi.
\end{aligned}$$

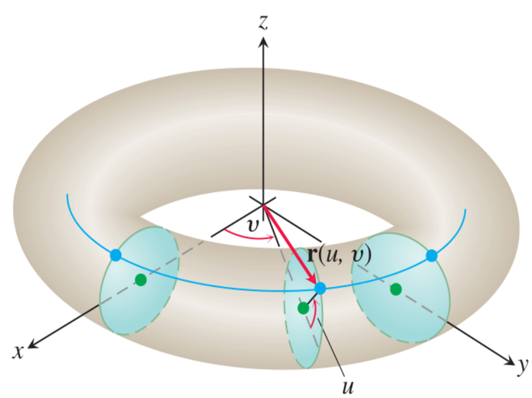
7. (*Discovery Question) Read Lecture Note Chapter 5 Section 3 on Surface Given Parametrically by Three Equations or Chapter 16.5 of the book [Thomas's Calculus.(13th ed.) Wesley, 2014]. Do the following exercise.

   A *torus of revolution* (doughnut) is obtained by rotating a circle C in the xz-plane about the z-axis in the space (See Figure). If C has radius $r > 0$ and center $(R, 0, 0)$ $(R > r)$, show that a parameterization of the torus is

$$\vec{r}(u, v) = ((R + r\cos u)\cos v)\vec{i} + ((R + r\cos u)\sin v)\vec{j} + (r\sin u)\vec{k},$$

   where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$ are the angles in the figure, and show that the surface area of the torus is $A = 4\pi^2 Rr$.





   Solution. Let (x, y, z) be a point on the torus. Then, from the figure, it is easy to see that

$$z = r\sin u, \quad 0 \le u \le 2\pi.$$

   On the other hand, the projection of the position vector $x\vec{i} + y\vec{j} + z\vec{k}$ on to the xy-plane gives rise to a vector of length $\rho = (R + r\cos u)$. Since v is the angle between the x-axis and the projection vector $x\vec{i} + y\vec{j}$, by the polar coordinate relation $x = \rho\cos v$ and $y = \rho\sin v$, we obtain

$$x = (R + r\cos u)\cos v, \qquad y = (R + r\cos u)\sin v, \qquad 0 \le v \le 2\pi.$$

Consequently, the parameterization of the torus using the parameter u, v is given by

$$\vec{r}(u,v) = ((R + r\cos u)\cos v)\vec{i} + ((R + r\cos u)\sin v)\vec{j} + (r\sin u)\vec{k},$$

where $0 \leq u, v \leq 2\pi$. That is

$$\begin{cases} x = x(u,v) = (R + r\cos u)\cos v, \\ y = y(u,v) = (R + r\cos u)\sin v, \\ z = z(u,v) = r\sin u, \end{cases} \qquad 0 \leq u, v \leq 2\pi.$$

The area of the parameterization surface is given by

$$A = \iint_S |\vec{r}_u \times \vec{r}_v| dudv = \iint_S \sqrt{EG - F^2} dudv$$

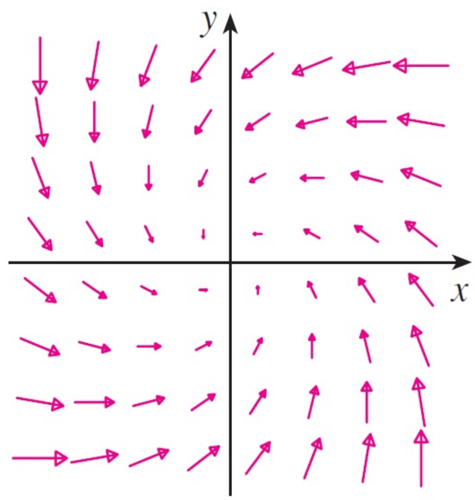
where

$$\begin{aligned} E &= x_u^2 + y_u^2 + z_u^2 = r^2, \\ G &= x_v^2 + y_v^2 + z_v^2 = (R + r\cos u)^2, \\ F &= x_u x_v + y_u y_v + z_u z_v = 0. \end{aligned}$$

Hence,

$$A = \int_0^{2\pi} \int_0^{2\pi} \sqrt{r^2(R + r\cos u)^2} dudv = \int_0^{2\pi} \int_0^{2\pi} r(R + r\cos u) dudv = 4\pi^2 Rr.$$

8. (*Discovery Question) Determine if the vector field shown in the figure is conservative or solenoidal? (For a reference, see Section 16.3 of *CALCULUS-Early Transcendentals* 6th edition by James Stewart)



Solution. No. Since an obvious rotation exists in the field, it is not irrotational and thus not conservative. To see it is not solenoidal, one can check a small box area around point like (1,1) through which the outflow is smaller than the inflow, $\mathrm{div}(\vec{F}) < 0$.