

Frequency response

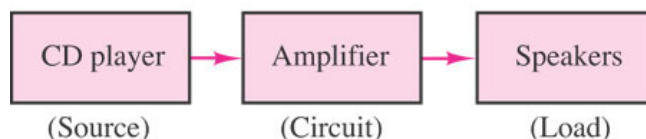
- 1) Changing the frequency affects the currents and voltages in a circuit
- 2) This is due to changes in the impedances of the various components in a circuit
- 3) This affects the working frequency range of a particular device or circuit
- 4) Hence it is important to find out the frequency response of a circuit
- 5) The frequency response of a circuit is a measure of the variation of a load-related voltage or current in relation to the input frequency
- 6) We typically express this in terms of variation in output voltage over the source:

$$H_V(j\omega) = \frac{V_L(j\omega)}{V_S(j\omega)}$$

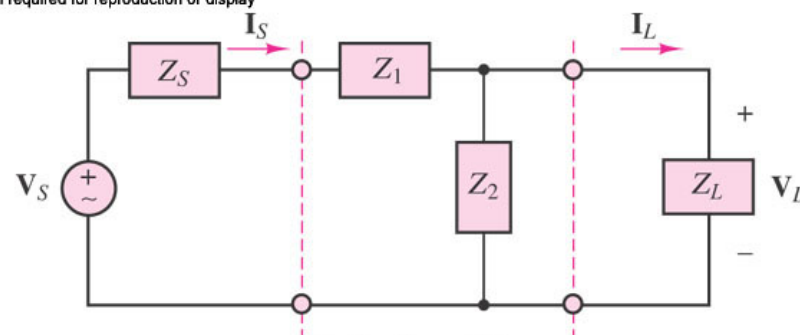
How does V_L change relative to V_S for different frequencies?
How does V_L change with respect to phase and magnitude?

Copyright©The McGraw-Hill Companies, Inc. Permission required for reproduction or display

**Voltage divider rule used almost always
to derive expression of output/input**

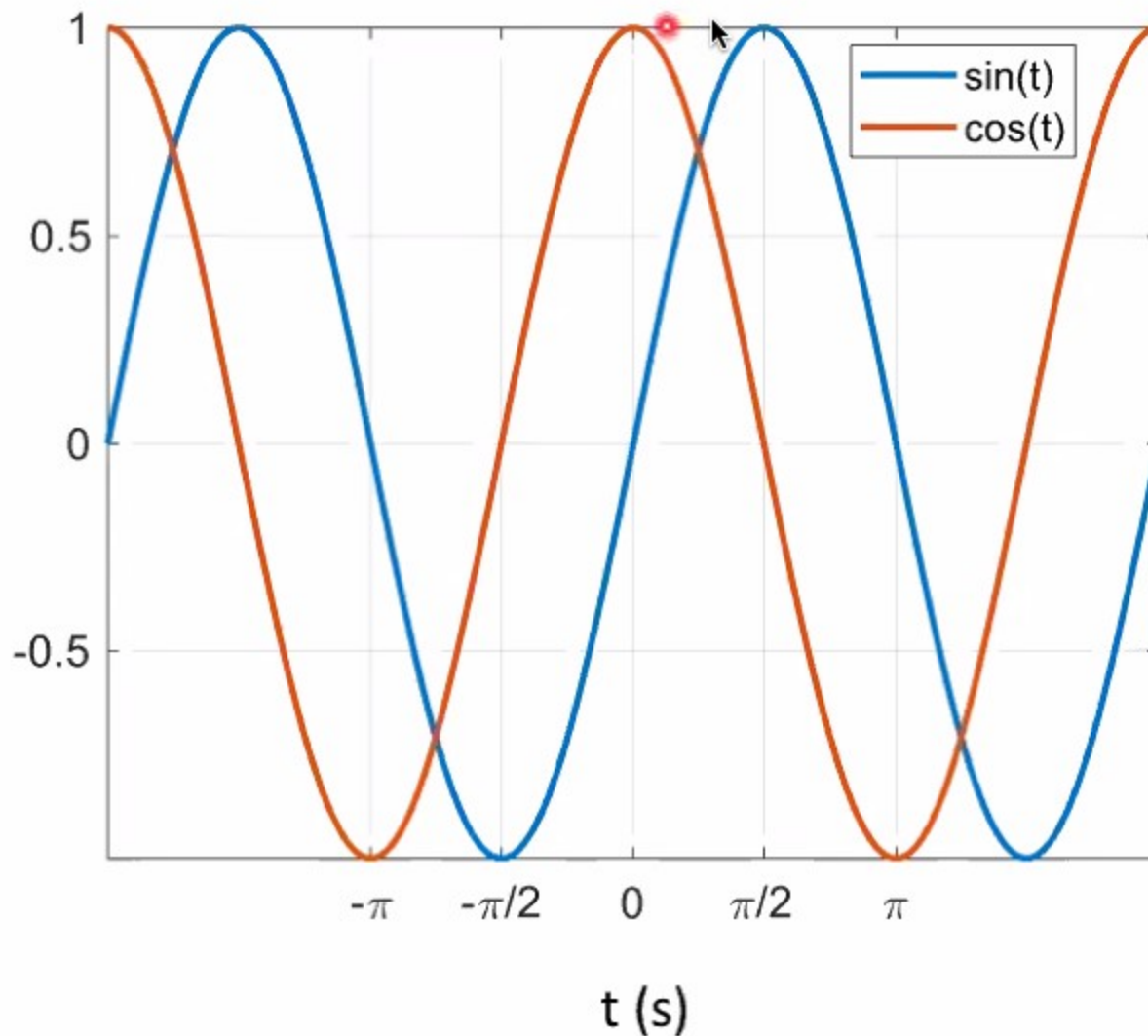


A physical system



A circuit model

Review: Phase

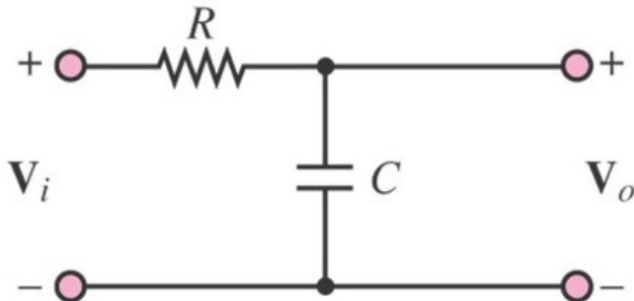


- $\sin\left(\frac{\pi}{2}\right) = \cos(0) = 1$
- $\sin(t) = \cos\left(t - \frac{\pi}{2}\right)$
- $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$

Low pass filter

Let us consider the response of the output V_o in relation to the input V_i . We keep the amplitude of V_i constant but vary its frequency ω .

By voltage divider rule:
$$\frac{V_o}{V_i}(j\omega) = \frac{1/j\omega C}{R + 1/j\omega C}$$
$$= \frac{1}{1 + j\omega CR}$$



Note that CR is a constant based on the circuit values unlike ω , which is a variable.

Re-write CR as a constant with the same unit as angular frequency:

$$\omega_c = \frac{1}{RC} \quad \text{This frequency is called the cutoff radian frequency (ω_c) and is a **CONSTANT**}$$

Sub back into above equation:
$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$

Analyze the response of low pass filter

$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$

Both the phase and magnitude of V_o/V_i will change when ω is allowed to vary.

Magnitude

When $\omega \rightarrow 0$:

$$V_o/V_i \rightarrow 1/1 \Rightarrow |V_o/V_i| \rightarrow 1$$

When $\omega \rightarrow \text{Infinity}$:

$$V_o/V_i \rightarrow \underline{\hspace{2cm}} \Rightarrow |V_o/V_i| \rightarrow 0$$

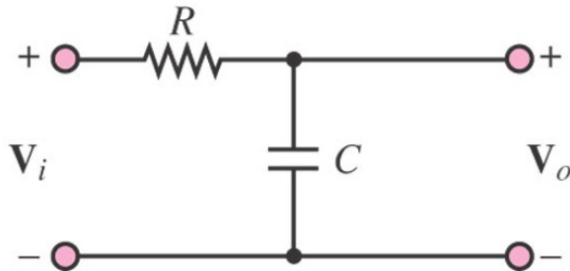
Phase

When $\omega \rightarrow 0$:

$$V_o/V_i \rightarrow 1\angle 0^\circ / 1\angle 0^\circ \Rightarrow \angle(V_o/V_i) \rightarrow 0^\circ$$

When $\omega \rightarrow \text{Infinity}$:

$$V_o/V_i \rightarrow \underline{\hspace{2cm}} \Rightarrow \angle(V_o/V_i) \rightarrow -90^\circ$$

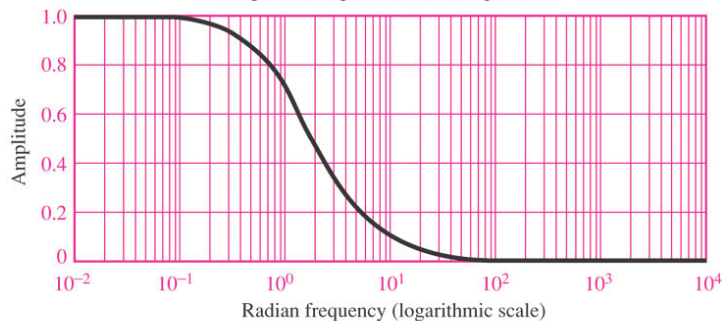


Sketch the response of low pass filter

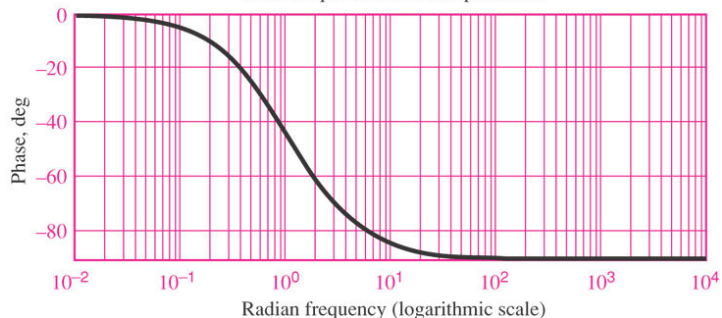
Allows lower frequency signals to pass and filters off higher frequency signals

Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display

Magnitude response of RC low-pass filter



Phase response of RC low-pass filter



Observations:

When ω approaches zero, magnitude of V_o/V_i approaches 1 and its phase is close to zero

When ω becomes large, magnitude of V_o/V_i approaches zero and its phase is close to $-\pi/2$

Allows lower frequency signal to pass and filters off higher frequency signals

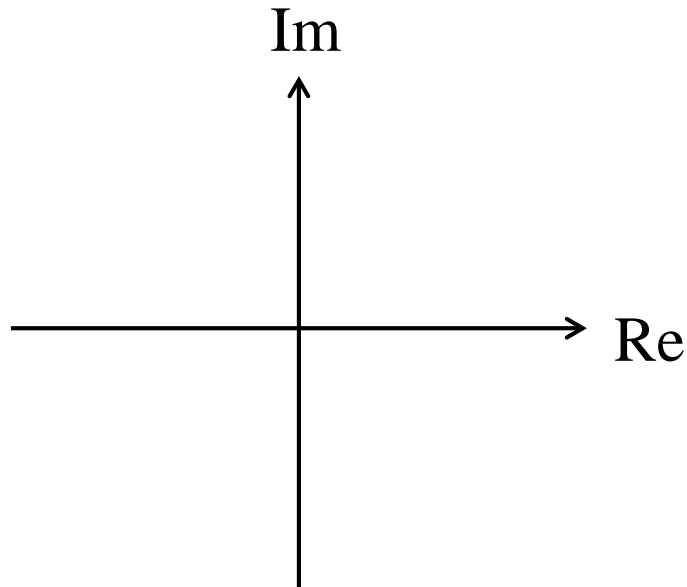
What about in between these two extremes, around ω_c ?

The above graphs are presentations of semi-log plots

Semi-log plots: y-axis follows a linear scale, x-axis follows a logarithmic scale

Logarithmic scale (base 10): Between each interval on axis, we increase/decrease by a factor of 10 (it is the power/index that changes)

At the cut off frequency



Plot of denominator for V_o/V_i

$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$

At $\omega = \omega_c$:

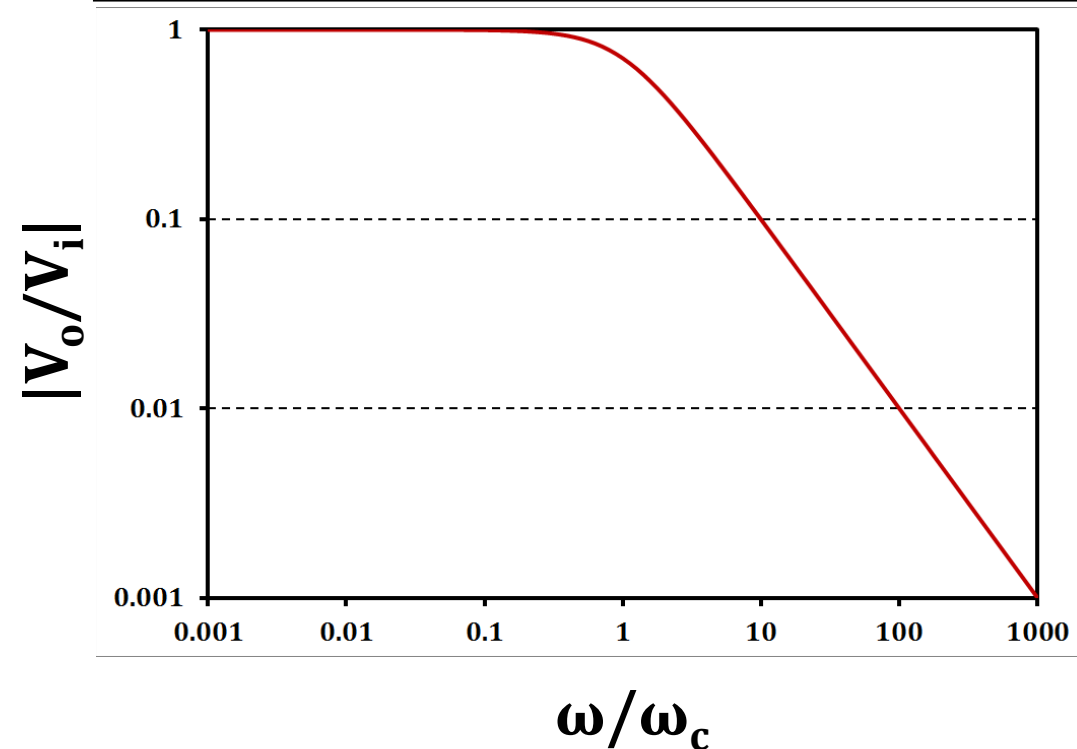
Denominator: $1 + j$

Denominator as a phasor: $\sqrt{2} \angle 45^\circ$

$$\frac{V_o}{V_i} = \frac{1}{\sqrt{2}} \angle -45^\circ$$

At $\omega = \omega_c$, V_o/V_i drops to $1/\sqrt{2}$ of the maximum and has a phase of -45°

Log-Log Plot for Magnitude (Low Pass)



Two parts of the curve:

When $\omega \ll \omega_c$:

$|V_o/V_i|$ stays flat close to 1

When $\omega \gg \omega_c$:

$|V_o/V_i|$ decreases with ω ;
x10 time reduction for every x10
increase in ω

Change of $|V_o/V_i|$ with ω seen as a
linear slope on the log-log plot

Change between the 2 parts occurs
at ω_c

Log-log plot: **Both** the y-axis and x-axis are on logarithmic scales (base 10).
This means moving by 1 interval on either axis, the value increases or decreases
by a factor of 10.

Note that on a log scale, one never arrives at zero/infinity.

Bode plot for Low Pass

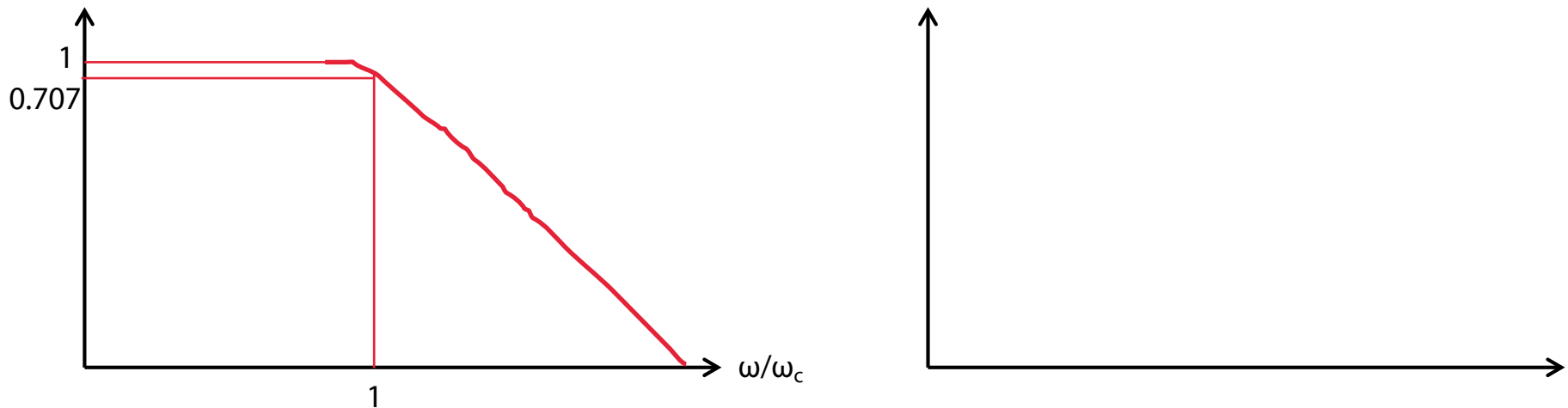
Bode plot typically comes as a **pair of graphs**:

(1) Log-log plot of magnitude ratio of V_o/V_i vs. frequency

(Log-log plot: Both x and y axes are on logarithmic scales)

(2) Semi-log plot of the phase of V_o/V_i vs. frequency

(Semi-log plot: Linear scale for y-axis (Phase) and log scale for x-axis)



Magnitude

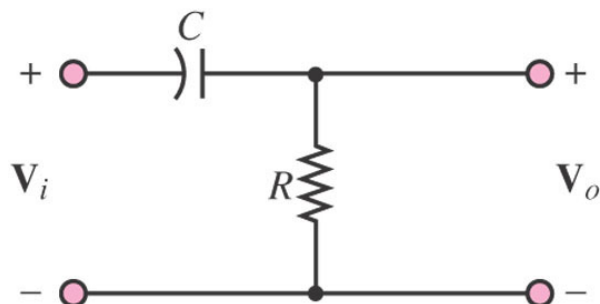
$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j\omega/\omega_c}$$

Phase

High pass filter

Let us consider the response of the output V_o in relation to the input V_i .
We keep the amplitude of V_i constant but vary its frequency ω .

By voltage divider rule:
$$\frac{V_o}{V_i}(j\omega) = \frac{R}{R + 1/j\omega C}$$
$$= \frac{1}{1 + 1/j\omega CR}$$



Once again we re-write RC to define the cut off radian frequency, ω_c whereby $\omega_c = 1/RC$:

Sub back into above equation:
$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + (\omega_c / j\omega)}$$

Analyze response of high pass filter

$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}$$

Both the phase and magnitude of V_o/V_i will change when ω is allowed to vary.

Magnitude

When $\omega \rightarrow 0$:

$$V_o/V_i \rightarrow 0/1 \Rightarrow |V_o/V_i| \rightarrow 0$$

When $\omega \rightarrow \text{Infinity}$:

$$V_o/V_i \rightarrow \underline{\hspace{2cm}} \Rightarrow |V_o/V_i| \rightarrow 1$$

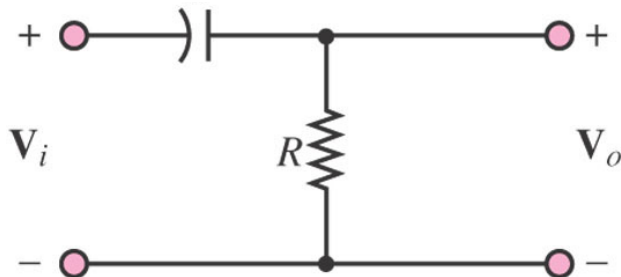
Phase

When $\omega \rightarrow 0$:

$$V_o/V_i \rightarrow (j0)/(1) \rightarrow 0\angle 90^\circ / 1\angle 0^\circ \Rightarrow \angle(V_o/V_i) \rightarrow 90^\circ$$

When $\omega \rightarrow \text{Infinity}$:

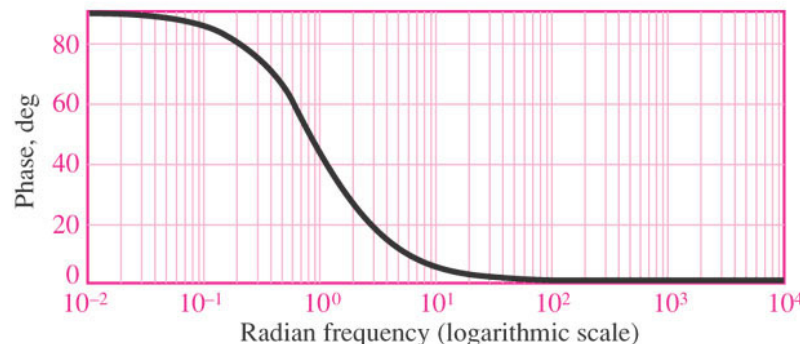
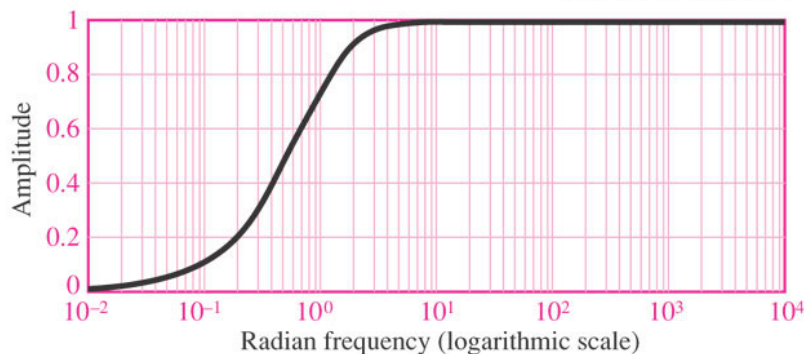
$$V_o/V_i \rightarrow \underline{\hspace{2cm}} \Rightarrow \angle(V_o/V_i) \rightarrow 0^\circ$$



Sketch response of high pass filter

Allows higher frequency signals to pass and filters off lower frequency signals

Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display



Semi-log plots: y-axis on linear scale, x-axis on logarithmic scale

Observations:

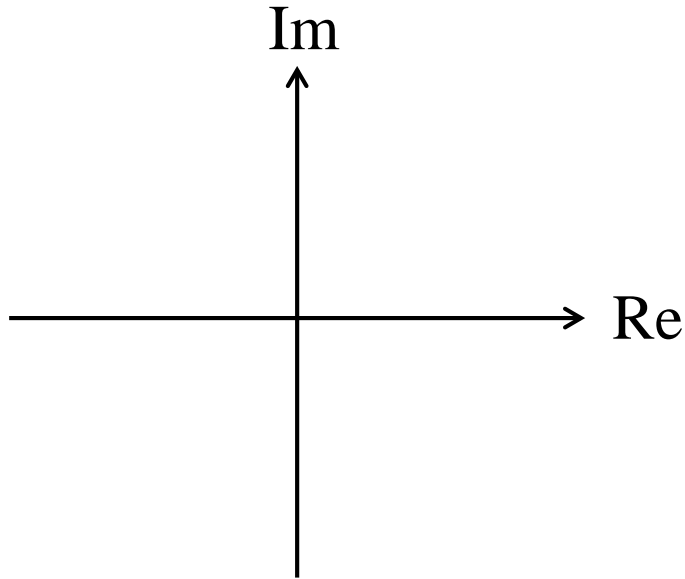
When ω approaches zero, V_o/V_i approaches zero and phase is close to $\pi/2$

When ω becomes large, V_o/V_i approaches 1 and phase is close to 0

Allows higher frequency signals to pass and filters off lower frequency signals

What about in between these two extremes, around ω_c ?

At the cut off frequency



Plot of denominator for V_o/V_i

Note that numerator adds a phase shift of 90° from j

$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}$$

At $\omega = \omega_c$:

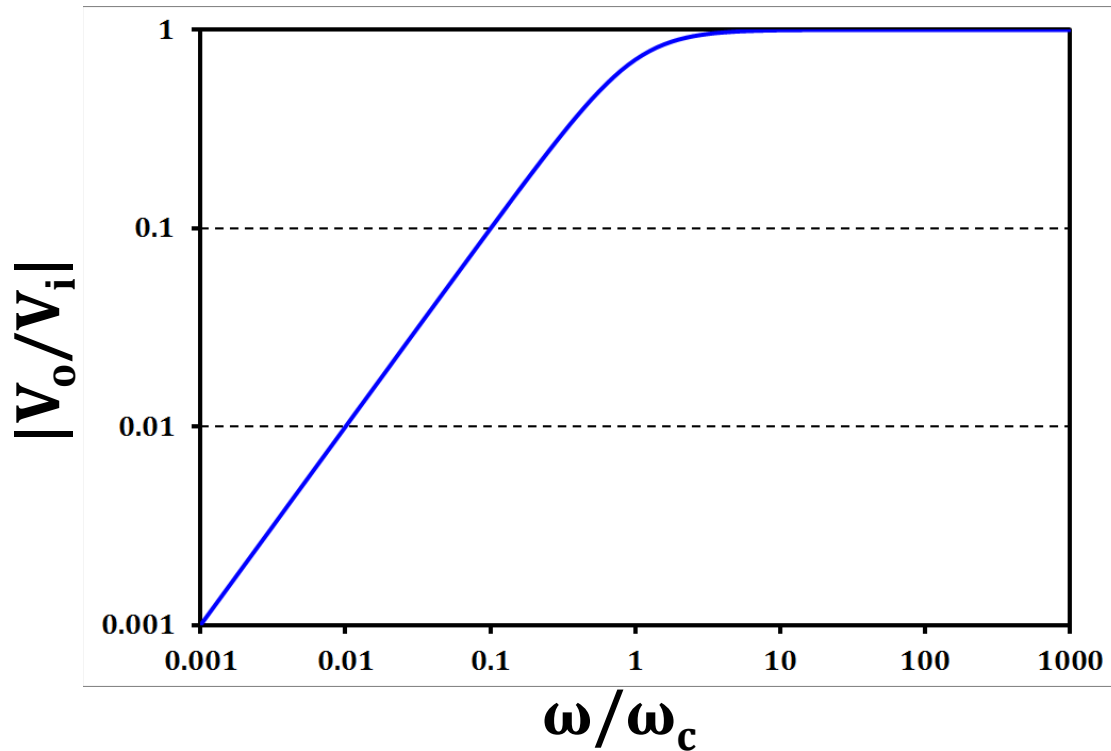
Denominator: $1 + j$

Denominator as a phasor: $\sqrt{2} \angle 45^\circ$

$$\frac{V_o}{V_i} =$$

At $\omega = \omega_c$, V_o/V_i is once again $1/\sqrt{2}$ of the maximum and has a phase of 45°

Log-Log Plot for $|V_o/V_i|$ (High Pass)



When $\omega \gg \omega_c$:
 $|V_o/V_i|$ stays flat close to 1

When $\omega \ll \omega_c$:
 $|V_o/V_i|$ decreases with ω ;
x10 time reduction for every x10
reduction in ω

Seen as a linear slope on the log-
log plot

Bode plot for High Pass

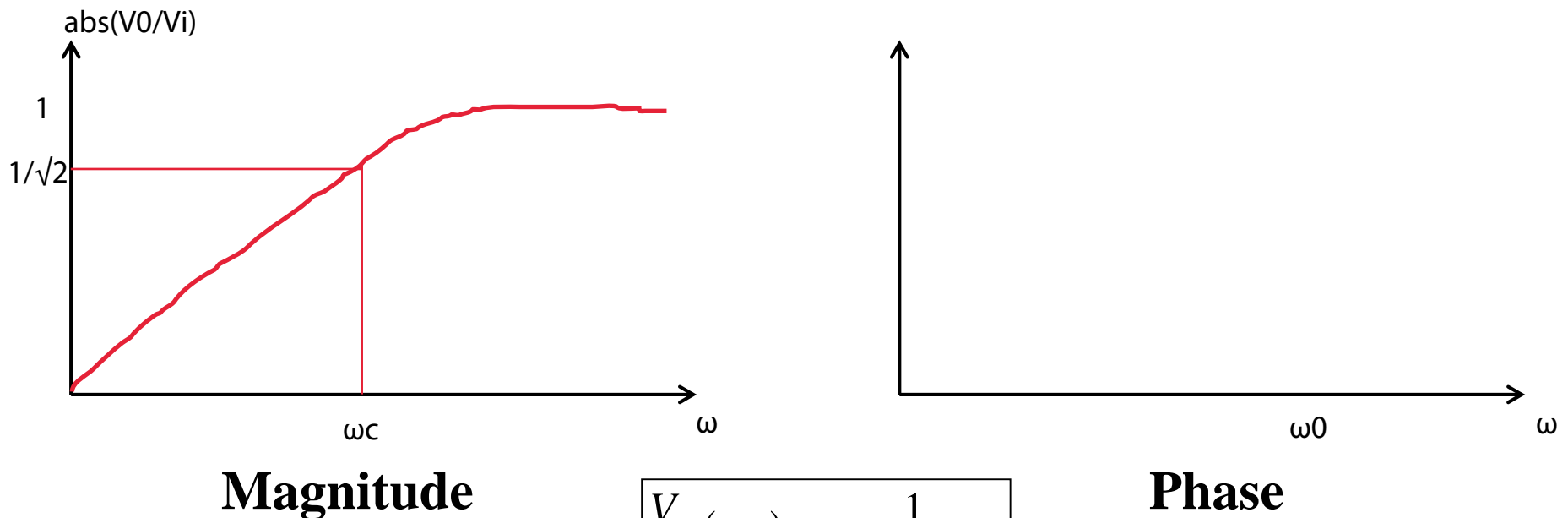
Bode plot typically comes as a **pair of graphs**:

(1) Log-log plot of magnitude ratio of V_o/V_i vs. frequency

(Log-log plot: Both x and y axes are on logarithmic scales)

(2) Semi-log plot of the phase of V_o/V_i vs. frequency

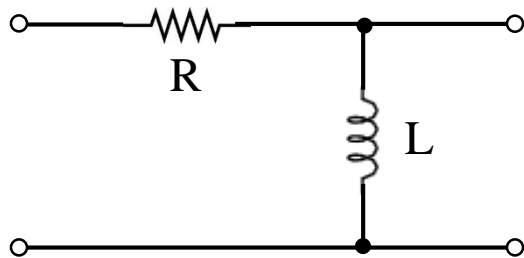
(Semi-log plot: Linear scale for y-axis (Phase) and log scale for x-axis)



$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + \omega_c / j\omega}$$

Other examples on filters 1

Determine the frequency response characteristics (low pass or high pass) for the following filter circuits. Hence draw the bode plot of the filter.



Bode Plots



Magnitude

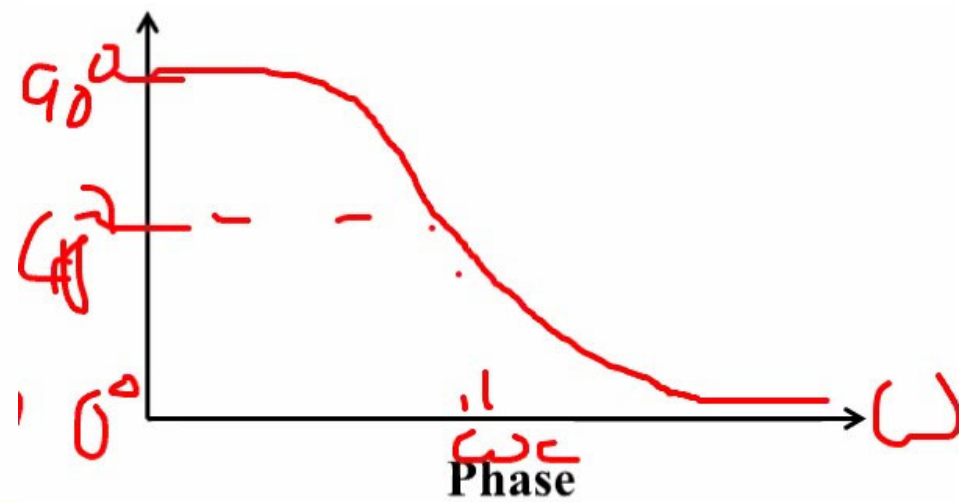
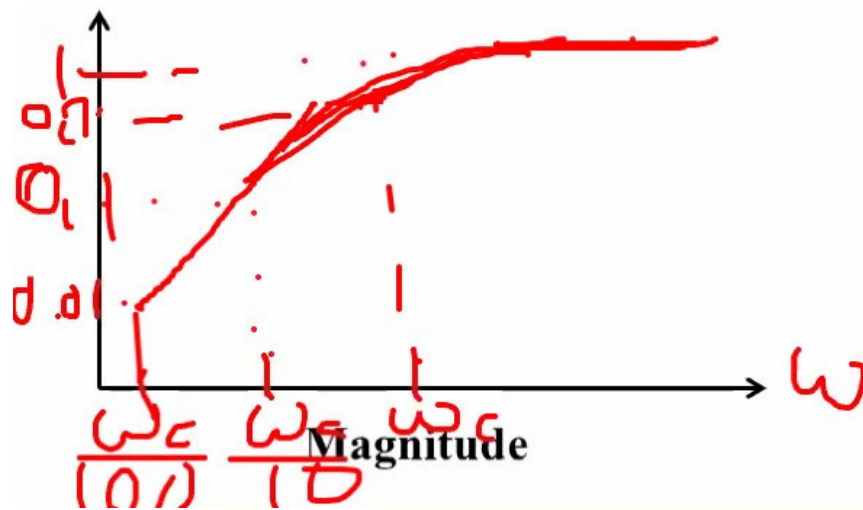


Phase

$$\frac{V_o}{V_i} = \frac{j\omega L}{R + j\omega L}$$

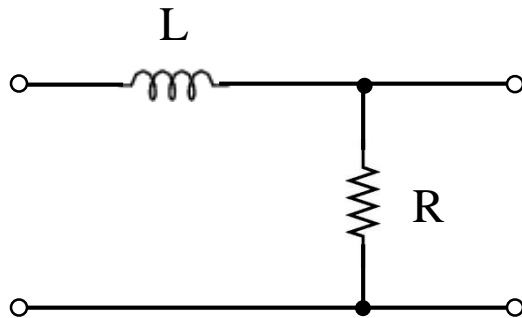
$$\omega_c = \frac{R}{L}$$

$$\frac{V_o}{V_i} = \frac{j\omega(\frac{L}{R})}{1 - j\omega(\frac{L}{R})}$$



Other examples on filters 2

Determine the frequency response characteristics (low pass or high pass) for the following filter circuits. Hence draw the bode plot of the filter.



Bode Plots



Magnitude

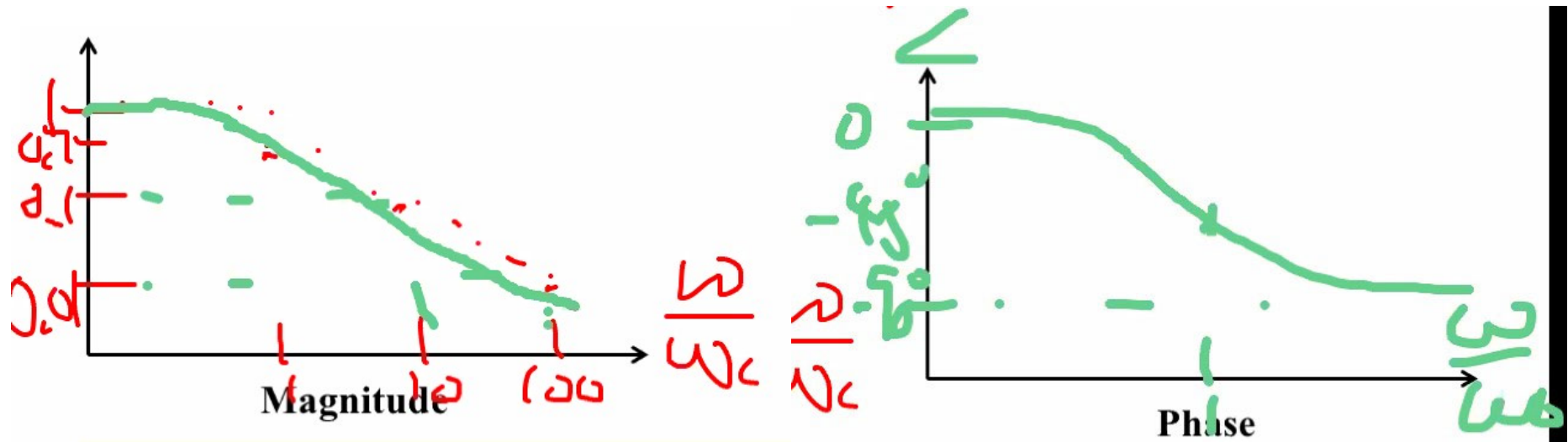


Phase

$$\frac{V_o}{V_i} = \frac{R}{R + j\omega L}$$

$$\omega_c = \frac{R}{L}$$

$$\frac{V_o}{V_i} = \frac{1}{1 + j\omega(\frac{L}{R})}$$

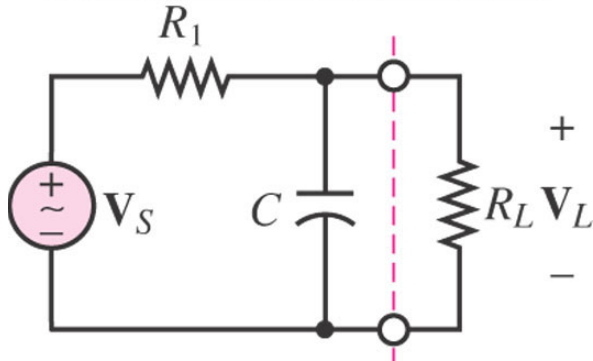


Frequency response example 1

Compute the frequency response of V_L/V_s for the following circuit:

$$R_1 = 1\text{k}\Omega; C = 10\mu\text{F}; R_L = 10\text{k}\Omega$$

Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display



First find the combined impedance of the capacitor C and resistor R_L in parallel:

$$Z_{RC} = \frac{R_L}{(1 + j\omega R_L C)} = \frac{10}{(1 + j0.1\omega)} \text{ k}\Omega$$

Now apply voltage divider rule:

$$\begin{aligned} \frac{V_L}{V_s}(j\omega) &= \frac{Z_{RC}}{Z_{RC} + R_1} = \frac{10/(1 + j0.1\omega)}{[10/(1 + j0.1\omega)] + 1} \\ &= \frac{10}{11 + j0.1\omega} = \frac{100}{110 + j\omega} \end{aligned}$$

Frequency response example 1

$$\frac{V_L}{V_S}(j\omega) = \frac{100}{110 + j\omega} = \left(\frac{100}{110}\right) \left(\frac{1}{1 + j\omega/110}\right)$$

When $\omega \rightarrow 0$:

$$V_o/V_i \rightarrow 100/110$$

$$|V_o/V_i| \rightarrow 100/110 = 10/11$$

$$\angle(V_o/V_i) \rightarrow 0^\circ$$

When $\omega \rightarrow \text{Infinity}$:

$$V_o/V_i \rightarrow$$

$$|V_o/V_i| \rightarrow 0$$

$$\angle(V_o/V_i) \rightarrow -90^\circ$$



Magnitude



Phase

Frequency response example 1

$$\frac{V_L}{V_S}(j\omega) = \frac{100}{110 + j\omega} = \left(\frac{100}{110}\right) \left(\frac{1}{1 + j\omega/110}\right)$$

$$\omega_c = 110 \text{ rad/s}$$

When $\omega \rightarrow 0$:

$$V_o/V_i \rightarrow 100/110$$

$$|V_o/V_i| \rightarrow 100/110 = 10/11$$

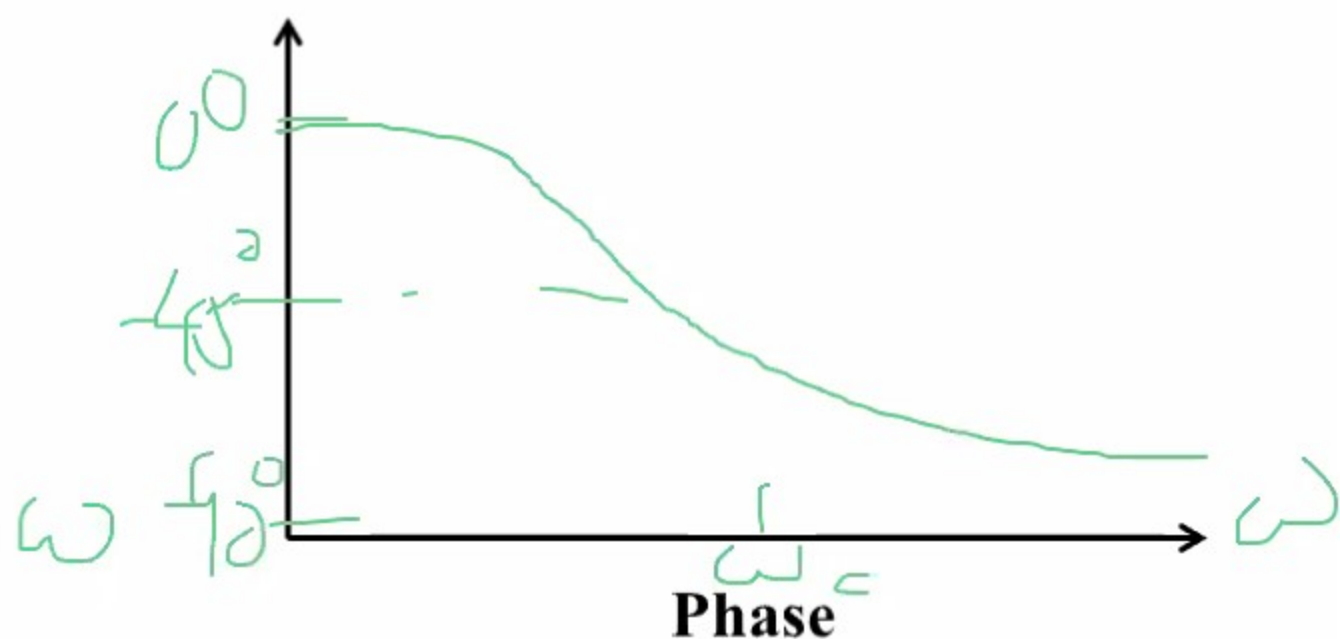
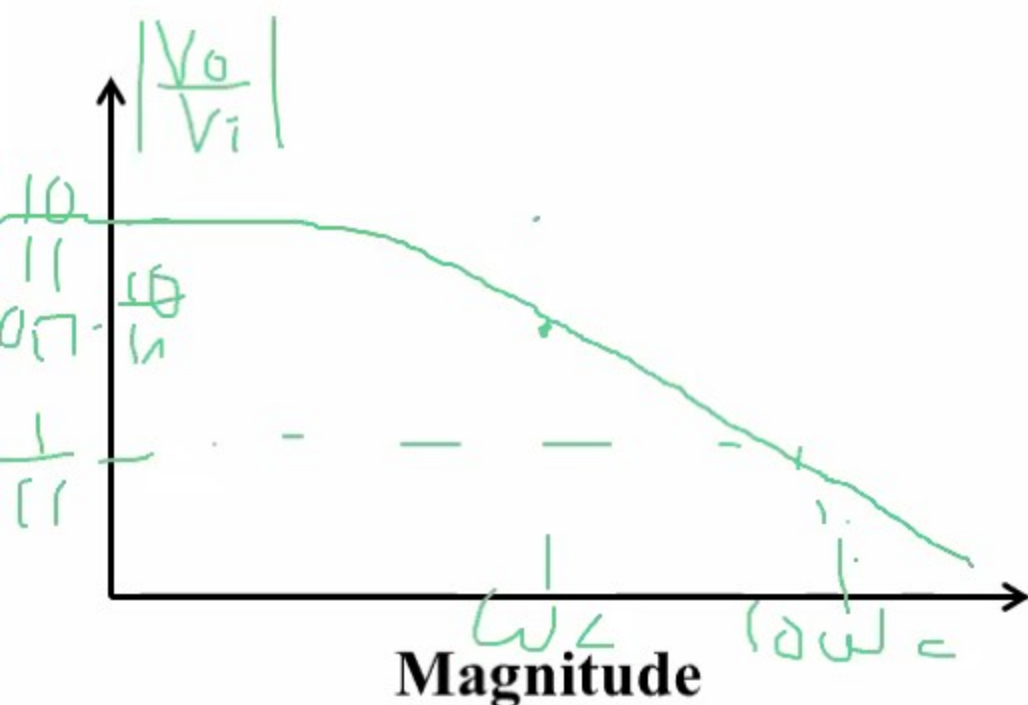
$$\angle(V_o/V_i) \rightarrow 0^\circ$$

When $\omega \rightarrow \text{Infinity}$:

$$V_o/V_i \rightarrow$$

$$|V_o/V_i| \rightarrow 0$$

$$\angle(V_o/V_i) \rightarrow -90^\circ$$

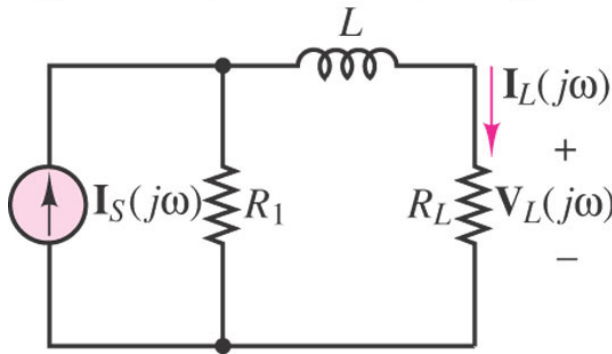


Frequency response example 2

Compute the frequency response of V_L/I_S for the following circuit:

$$R_1 = 1\text{k}\Omega; L = 2\text{mH}; R_L = 4\text{k}\Omega$$

Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display



First find the combined impedance of the inductor and resistor in parallel:

$$Z_{RL} = R_L + j\omega L = 4000 + j(2 \times 10^{-3})\omega \Omega$$

Now apply current divider rule:

$$\begin{aligned} \frac{V_L}{I_S}(j\omega) &= \left(\frac{I_L}{I_S} \right) R_L = \left(\frac{R_1}{R_1 + Z_{RL}} \right) R_L \\ &= \frac{(1000)(4000)}{1000 + 4000 + j(2 \times 10^{-3})\omega} = \frac{4 \times 10^6}{5000 + j(2 \times 10^{-3})\omega} \\ &= \frac{800}{1 + j(4 \times 10^{-7})\omega} \end{aligned}$$

Frequency response example 2

$$\frac{V_L}{I_s}(j\omega) = \frac{800}{1 + j(4 \times 10^{-7})\omega}$$

When $\omega \rightarrow 0$:

$$V_L/I_s \rightarrow 800$$

$$|V_L/I_s| \rightarrow 800$$

$$\angle(V_L/I_s) \rightarrow 0^\circ$$

When $\omega \rightarrow \text{Infinity}$:

$$V_L/I_s \rightarrow$$

$$|V_L/I_s| \rightarrow 0$$

$$\angle(V_L/I_s) \rightarrow -90^\circ$$



Magnitude



Phase

Frequency response example 2

$$\frac{V_L}{I_S}(j\omega) = \frac{800}{1 + j(4 \times 10^{-7})\omega}$$

$$\omega = \frac{1}{4 \times 10^{-7}} = 0,25 \cdot 10^7 \text{ rad/s}$$

When $\omega \rightarrow 0$:

$$V_L/I_S \rightarrow 800$$

$$|V_L/I_S| \rightarrow 800$$

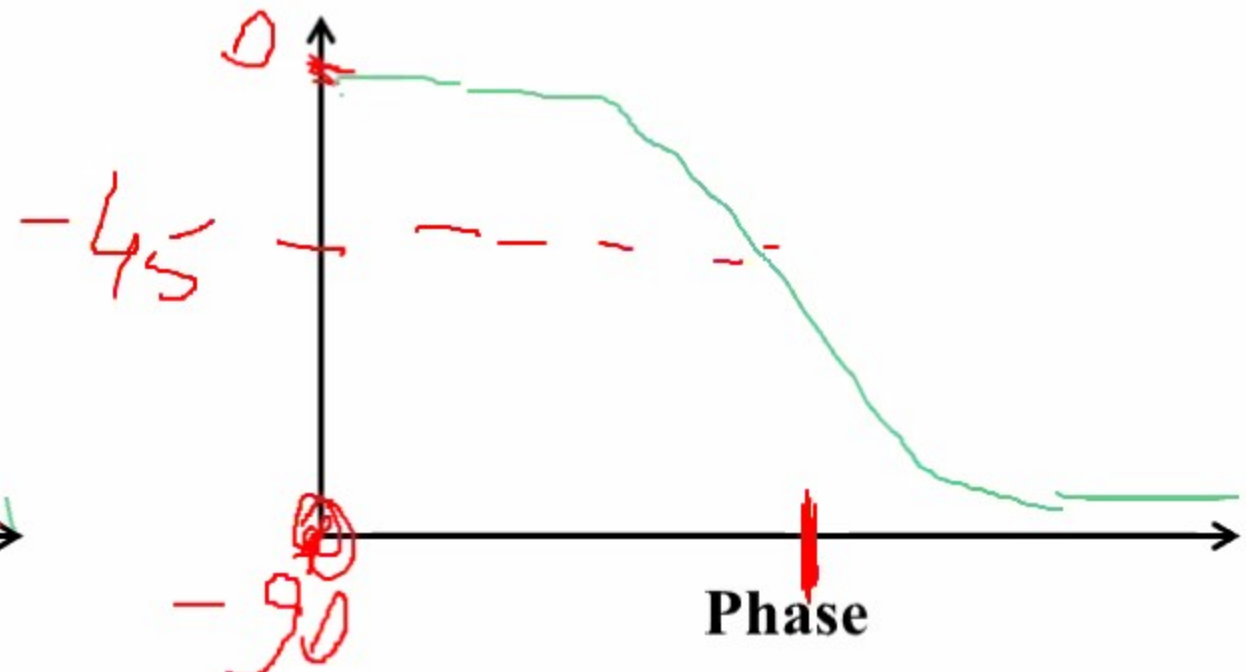
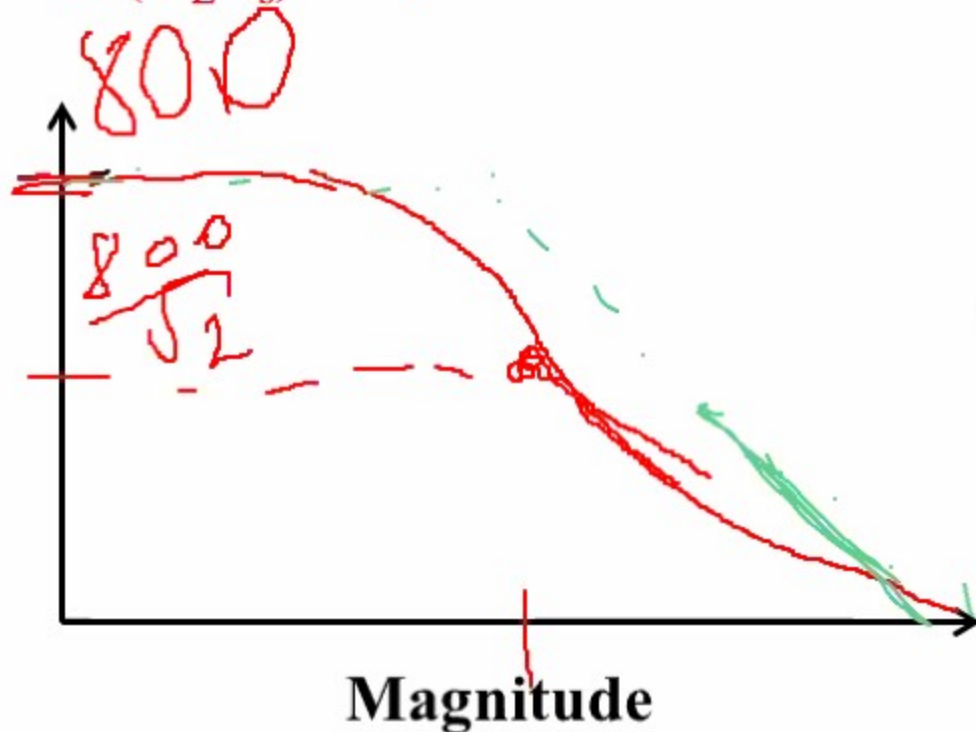
$$\angle(V_L/I_S) \rightarrow 0^\circ$$

When $\omega \rightarrow \text{Infinity}$:

$$V_L/I_S \rightarrow$$

$$|V_L/I_S| \rightarrow 0$$

$$\angle(V_L/I_S) \rightarrow -90^\circ$$



General form for low pass filter

$$\frac{V_o}{V_i}(j\omega) = A \left[\frac{1}{1 + j(\omega/\omega_c)} \right]$$

A is a constant and real number

A defines the magnitude of V_o/V_i in the **pass band**



Magnitude



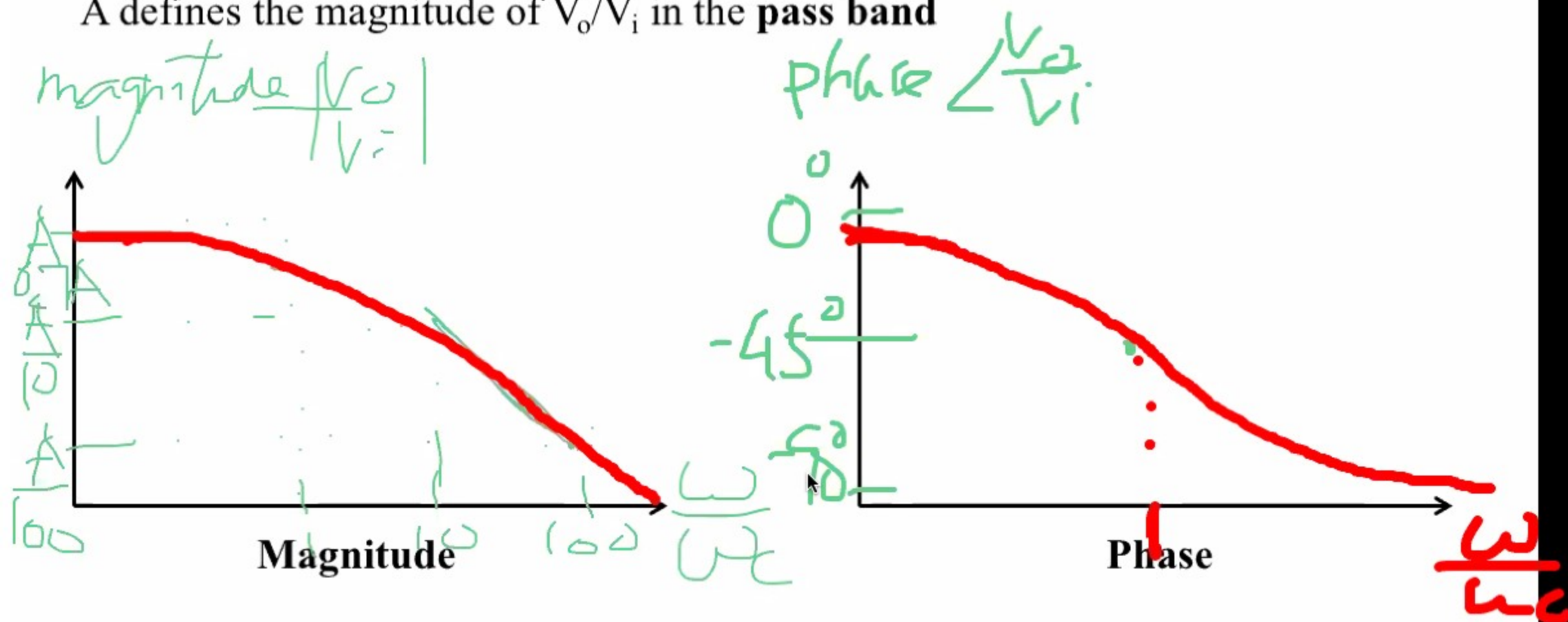
Phase

General form for low pass filter

$$\frac{V_o}{V_i}(j\omega) = A \left[\frac{1}{1 + j(\omega/\omega_c)} \right]$$

A is a constant and real number

A defines the magnitude of V_o/V_i in the **pass band**



General form for high pass filter

$$\frac{V_o}{V_i}(j\omega) = A \left[\frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)} \right]$$

A is a constant and real number

A defines the magnitude of V_o/V_i in the **pass band**



Magnitude



Phase

General form for high pass filter

$$\frac{V_o}{V_i}(j\omega) = A \left[\frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)} \right]$$

A is a constant and real number

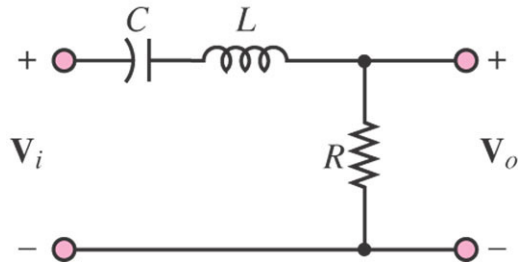
A defines the magnitude of V_o/V_i in the **pass band**



RLC Series Resonator (Bandpass filter)

Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display

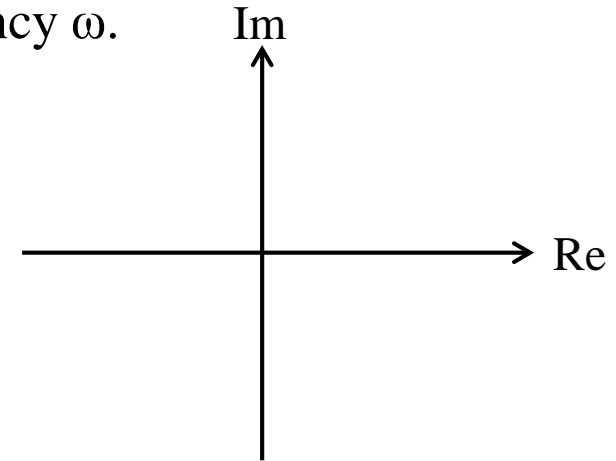
RLC bandpass filter. The circuit preserves frequencies within a band.



Let us consider the response of the output V_o in relation to the input V_i . We keep the amplitude of V_i constant but vary its frequency ω .

By voltage divider rule:

$$\begin{aligned}\frac{V_o}{V_i}(j\omega) &= \frac{R}{R + j\omega L + 1/j\omega C} \\ &= \frac{1}{1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)}\end{aligned}$$



$|V_o/V_i|$ is **max** when **denominator is minimized**

Plot of denominator for V_o/V_i

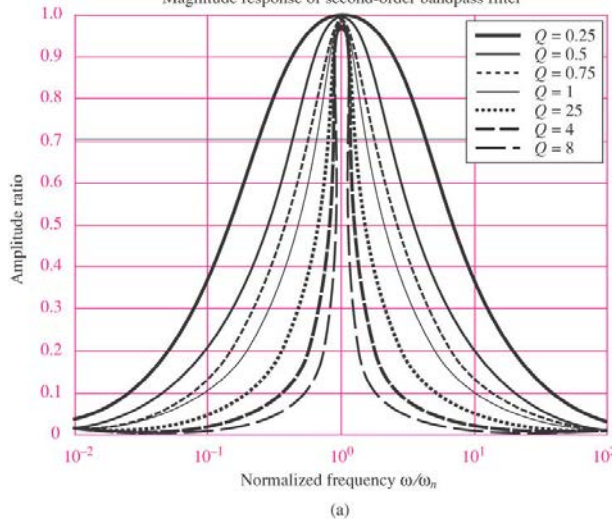
We can see that the output will be at its maximum when the imaginary part of the denominator is zero:

$$(\omega L/R) - [1/(\omega RC)] = 0 \rightarrow \omega^2 = 1/(LC)$$

When this happens, the impedances of the capacitor and inductor are equal and opposite. This is known as resonance. Max value of V_o/V_i for all frequencies is 1 in this case.

Quality factor

Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display
Magnitude response of second-order bandpass filter



- (1) There is no “flat” part in the frequency response curve
- (2) Response peaks at one frequency: $\omega = 1/\sqrt{LC}$, this is known at the resonance frequency, ω_0
- (3) For frequencies move further away from ω_0 (whether higher or lower), $|V_o/V_i|$ gets increasingly smaller

$$\frac{V_o}{V_i}(j\omega) = \frac{j\omega CR}{1 - \omega^2 LC + j\omega CR}$$

Make the subst. using: $\omega_0 = \frac{1}{\sqrt{LC}}$

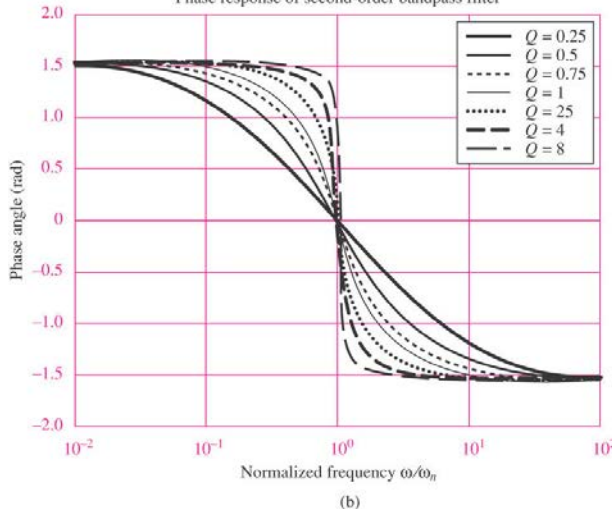
$$\frac{V_o}{V_i}(j\omega) = \frac{j\left(\frac{\omega}{\omega_0}\right) \frac{R}{\sqrt{L/C}}}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\left(\frac{\omega}{\omega_0}\right) \frac{R}{\sqrt{L/C}}}$$

If we define: $Q = \frac{\sqrt{L/C}}{R}$

$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_0)(1/Q)}{1 - (\omega/\omega_0)^2 + j(\omega/\omega_0)(1/Q)}$$

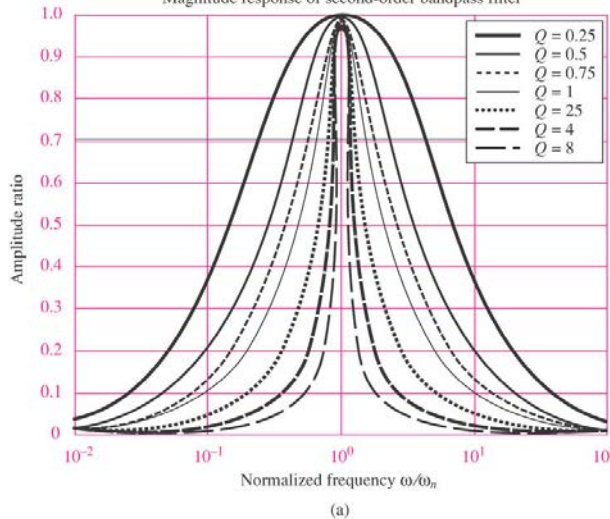
Q is known as the **Quality Factor** and it describes the sharpness or width of the peak relative to ω_0

Phase response of second-order bandpass filter



Frequency response

Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display.
Magnitude response of second-order bandpass filter



As Q increases, the resonance peak becomes sharper
In this sense, the resonator only responds at the resonance frequency, ω_0

Frequencies outside ω_0 are filtered out

The resonator works like frequency selector

$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_0)(1/Q)}{1 - (\omega/\omega_0)^2 + j(\omega/\omega_0)(1/Q)} \quad \frac{V_o}{V_i}(j\omega) = \frac{R}{R + j\omega L + 1/j\omega C}$$

Magnitude

When $\omega \rightarrow 0$ or $\rightarrow \text{Infinity}$: $|V_o/V_i| \rightarrow 0$

When $\omega = \omega_0$: $|V_o/V_i| = 1$

Phase

When $\omega \rightarrow 0$: $\angle(V_o/V_i) = 90^\circ$ since $V_o/V_i \rightarrow j\omega CR$

When $\omega \rightarrow \text{Infinity}$: $\angle(V_o/V_i) = -90^\circ$ since $V_o/V_i \rightarrow R/j\omega L$

When $\omega = \omega_0$: $\angle(V_o/V_i) = 0$ since $V_o/V_i \rightarrow 1$

