CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester A, 2013/2014

Time allowed : Two hours

This paper has **SIX** pages (including this cover page).

A brief table of derivatives is attached on pages 5 and 6.

Instructions to candidates:

1. This paper has **TEN** questions.
2. Attempt **ALL** questions in Section A and B.
3. Each question in Section A carries 9 marks.
4. Each question in Section B carries 15 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Section A

Answer **ALL** questions in this section. Each question carries 9 marks.

Question 1

(a) Evaluate $\lim_{x \to 0} \dfrac{1-e^{-x}}{\log_e(1-x)}$.

(4 marks)

(b) Let $f(x) = x^{\frac{2}{3}}$ for $x \in \mathbf{R}$. Determine whether $f(x)$ is differentiable at $x = 0$. Give your reason.

(5 marks)

Question 2

(a) If $y = \sin 2x$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ and then conjecture the formula for $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$.

(6 marks)

(b) Find the value of $\dfrac{\mathrm{d}^{15}}{\mathrm{d}x^{15}}(\sin 2x)$ when $x = \dfrac{\pi}{6}$.

(3 marks)

Question 3

Differentiate with respect to x:

(a) $\dfrac{3}{x^2+1} + \sqrt{x^2+1}$,

(3 marks)

(b) $x^2 \sec(x^2)$,

(3 marks)

(c) $\left(\dfrac{x^2-2}{x}\right)^{3x}$.

(3 marks)

Question 4

(a) If $x^3 + y^3 = k$, where k is a constant, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y and show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{2kx}{y^5}.$$

(4 marks)

(b) If $x = 3\cos t - \cos^3 t$, $y = 3\sin t - \sin^3 t$, where t is a variable, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in its simplest form in terms of t and show that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\cot^2 t \operatorname{cosec}^5 t$.

(5 marks)

Question 5

Express $\dfrac{12x+6}{(x-1)^2(x^2+5)}$ in partial fractions.

(9 marks)

Question 6

(a) Find the equation of the straight line through $P(-1, 4)$ perpendicular to the line L, $x+2y+3=0$.

(5 marks)

(b) Hence, find the coordinates of the foot of the perpendicular from point P to the line L.

(4 marks)

Question 7

(a) Express $\sin x-\sqrt{3}\cos x$ in the form $r\sin(x-\phi)$, where $r>0$ and $0<\phi<\dfrac{\pi}{2}$.

(Hint: $\sin(A-B)=\sin A\cos B-\cos A\sin B$)

(2 marks)

(b) Find, in radians, the general solution of the equation $\sin x-\sqrt{3}\cos x=-1$.

(3 marks)

(c) Find the greatest and least values of the function $g(x)=\dfrac{1}{\sin x-\sqrt{3}\cos x+3}$ for $x\in\mathbf{R}$.

(4 marks)

Question 8

Find the dimensions of the cylinder of maximum volume that can be inscribed in a right circular cone of height 15 cm and radius of base 7 cm as shown in Figure 1.

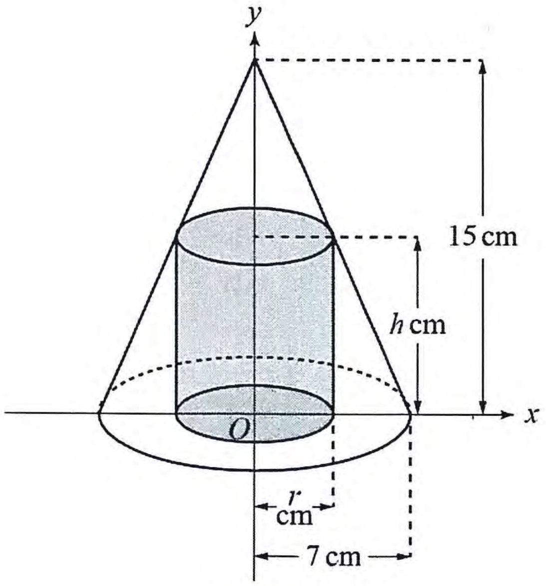

Figure 1.

(9 marks)

Section B

Answer **ALL** questions in this section. Each question carries 15 marks.

Question 9

(a) Show that the equation $x^2 - 9y^2 + 2x + 36y - 44 = 0$ represents a hyperbola whose centre is at the point $C(-1, 2)$.
(Hint: You may use the method of completing the square.)

(3 marks)

(b) Find the coordinates of the foci of the hyperbola and the coordinates of the points where its asymptotes cut the x-axis.

(7 marks)

(c) Find the equation of the tangent to the hyperbola at the point $Q(5, 2+\sqrt{3})$.

(5 marks)

Question 10

Let $y = \cosh(\sin^{-1} x)$, where $\sin^{-1} x$ denotes the principal value of the inverse sine function, and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

(a) Show that $(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$.

(5 marks)

(b) Hence, deduce that, $y^{(n+2)}(0) = (n^2+1)y^{(n)}(0)$ for $n = 1, 2, 3, \ldots$,

where $y^{(n)}(0)$ denotes the value of $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ when $x = 0$.

(5 marks)

(c) Hence, or otherwise, find the Maclaurin expansion for $\cosh(\sin^{-1} x)$ as far as the term in x^6.

(5 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	Derivative of y with respect to x
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d f(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\operatorname{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \operatorname{cosec} u$	$\frac{dy}{dx} = -\operatorname{cosec} u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	Derivative of y with respect to x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u \frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u \frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \operatorname{sech}^2 u \frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\operatorname{cosech}^2 u \frac{du}{dx}$
$y = \operatorname{sech} u$	$\frac{dy}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx}$
$y = \operatorname{cosech} u$	$\frac{dy}{dx} = -\operatorname{cosech} u \coth u \frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

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