MA1200 Calculus and Basic Linear Algebra I

Lecture Note 2

Sets and Functions

Set Notation

A *set* A is a collection of distinct objects (they can be numbers, letters or anything you like). An object inside the set is called an *element* of the set A.

Some examples of sets

$$\underbrace{A}_{set} = \overbrace{\{1,3,5,7,9\}}^{elements} \quad \text{(set of all odd numbers between 1 and 10)}$$

$B = \{1,2,3,4,5 \ldots\ldots\}$ (set of all positive integers)

$C = \{0, +3, -3, +6, -6, +9, -9, \ldots\}$ (set of multiple of 3)

$D = \{all\ real\ numbers\} = \mathbb{R}$ (set of real numbers)

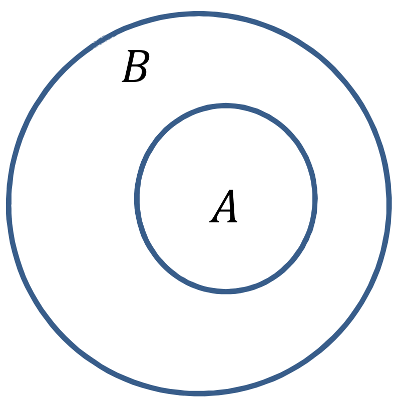
Mathematically, we write

- $p \in B$ if the element p is in the set B ("∈" means "belongs to") and
- $p \notin B$ if the element p is NOT in the set B.

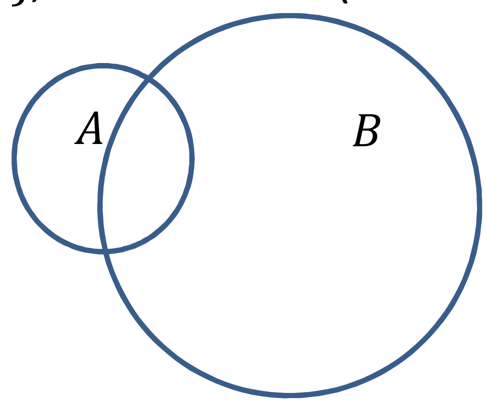
For example : If $E = \{2,3,4,5\}$, then $3 \in E$ and $\sqrt{6} \notin E$.

- (Equality of sets) We say two sets A, B are equal (we write $A = B$) only when two sets contain the same elements. For example:
  - ✓ If $A = \{1,2,3\}$ and $B = \{1,2,3\}$, then $A = B$.
  - ✓ If $A = \{1,3,4\}$ and $B = \{1,2,3\}$, then $A \neq B$.

- (Subset) Given two sets A and B, we say A is *subset* of B (denoted by $A \subseteq B$) if every elements in A is also an element in B. For example:
  - ✓ If $A = \{1,3\}$ and $B = \{0,1,3,4\}$, then $A \subseteq B$.
  - ✓ If $A = \{2,4\}$ and $B = \{0,1,3,4\}$, then $A \nsubseteq B$. (since $2 \notin B$)



$A \subseteq B$



$A \nsubseteq B$

General description of sets

In general, we describe the set by mentioning the common properties that the objects in the set have. In particular

$$E = \{x |\, x \; has \; certain \; properties\}.$$

Example 1

$$A = \left\{ x |\, x \; is \; prime \;\; and \;\; \underbrace{0 < x \leq 10}_{\substack{x \; lies \; between \\ 0 \; and \; 10}} \right\} = \{2,3,5,7\}.$$

$$B = \{x |\;\; x > 0 \; and \;\; x \; is \; multiple \; of \; 3\} = \{3,6,9,12, \dots .\}$$

$$\begin{aligned} C &= \{x |\, x^2 \leq 100 \;\; and \;\; x \;\; is \;\; negative \; integer\} \\ &= \{x\!: -10 \leq x \leq 10 \;\; and \;\; x \; is \; negative \; integer\} \\ &= \{-10,-9,-8,-7,-6,-5,-4,-3,-2,-1,0,1,2,3,4,5,6,7,8,9,10\}. \end{aligned}$$

Common sets in Mathematics (or in this course)

ϕ = empty set (the set containing nothing)

$\mathbb{N} = \{1,2,3,4,5,..\}$ (the set of all positive integers)

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ (the set of all integers)

$\mathbb{Q} = \left\{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$, (the set of rational numbers)

$[a, b] = \{x: a \leq x \leq b\}$, $[a, b) = \{a \leq x < b\}$, $(a, \infty) = \{x: x > a\}$,(intervals)

$\mathbb{R}$ = the set of real numbers

$\mathbb{C}$ = the set of all complex numbers

Note: In mathematics, we usually write

"$x \in \mathbb{R}$" to represent "x is real", "$x \in \mathbb{N}$" to represent "x is positive integer",

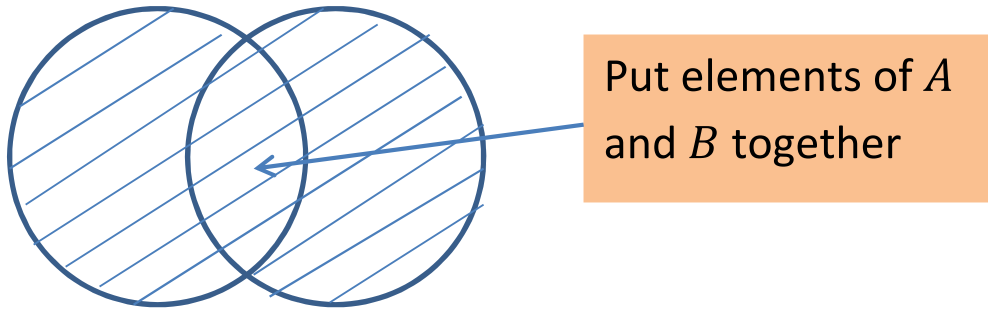
"$x \in [a, b]$" to represent "$a \leq x \leq b$" or "x lies between a and b".

Operation of sets

Given two sets A and B, we define

1. **The union of two sets**, denoted by $A \cup B$, is defined as

$$A \cup B = \{x | x \in A \ \ or \ \ x \in B\}.$$

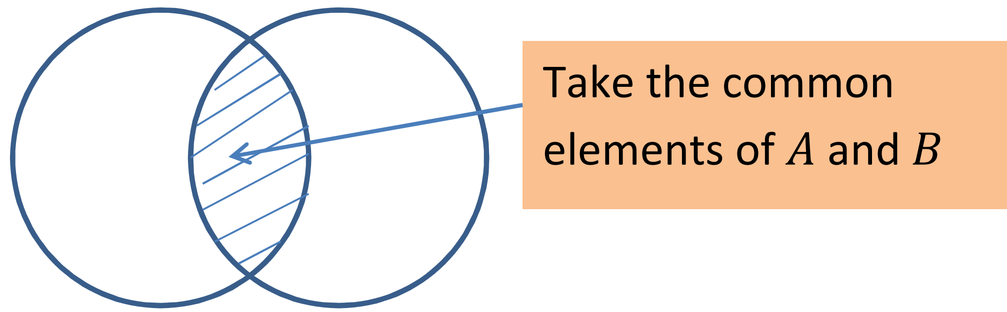


Example

If $A = \{1,3\}$ and $B = \{2,4,5\}$, then $A \cup B = \{1,2,3,4,5\}$.
If $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, then $A \cup B = \{1,2,3,3,4,4,5,6\} = \{1,2,3,4,5,6\}$.
Note: In set notation, repeated elements (say 3,4 in the last example) count only once.

2. **The intersection of two sets**, denoted by $A \cap B$, is defined as

$$A \cap B = \{x | x \in A \;\; and \;\; x \in B\}.$$



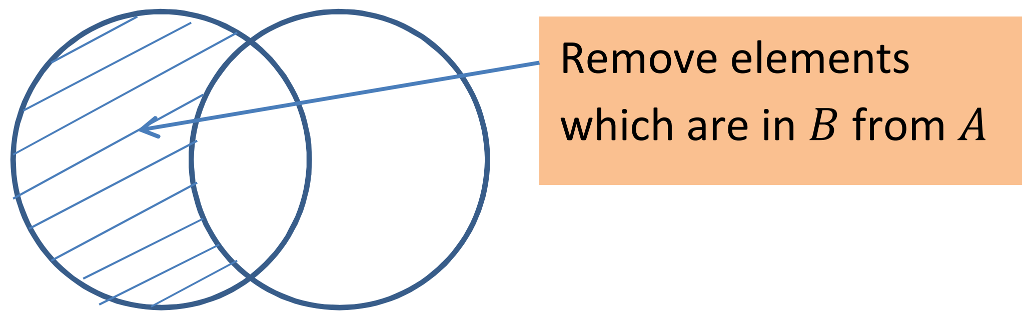
Example

If $A = \{1,3\}$ and $B = \{2,3,4\}$, then $A \cap B = \{3\}$.

If $A = \{2,6,8\}$ and $B = \{\sqrt{3}, \sqrt{7}\}$, then $A \cap B = \phi$ (i.e. there is no common elements between two sets).

3. **The complement of $\boldsymbol{B}$ in $\boldsymbol{A}$**, denoted by $A\backslash B$, is defined as

$$A\backslash B = \{x | x \in A \ \ and \ \ x \notin B\}$$



Example

If $A = \{2,3,4,5,6\}$ and $B = \{1,2,3,4\}$, then $A\backslash B = \{5,6\}$ (since the elements "2,3,4" are in B.

If $A = \{1,3,5\}$ and $B = \{2,4,6\}$, then $A\backslash B = \{1,3,5\}$ (since no elements In A are in B)

If $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5,6\}$, then $A\backslash B = \phi$ (since every element in A is in B also)

Example 2 (More examples)

Compute

(a) $[2,8] \cup (3,10)$
(b) $(3,7) \cap \mathbb{N}$
(c) $\mathbb{N} \backslash \mathbb{Z}$

☺Solution

(a) $[2,8] \cup (3,10) = [2,10)$
(b) $(3,7) \cap \mathbb{N} = \{x:\ 3 < x < 7\ \ and\ \ x\ \ is\ positive\ integer\} = \{4,5,6\}$.
(c) Note that $\mathbb{N} = \{1,2,3,4, \dots..\}$ and $\mathbb{Z} = \{\dots, -2, -1,0,1,2,3,4, \dots.\}$, so every element in $\mathbb{N}$ is also in $\mathbb{Z}$, therefore

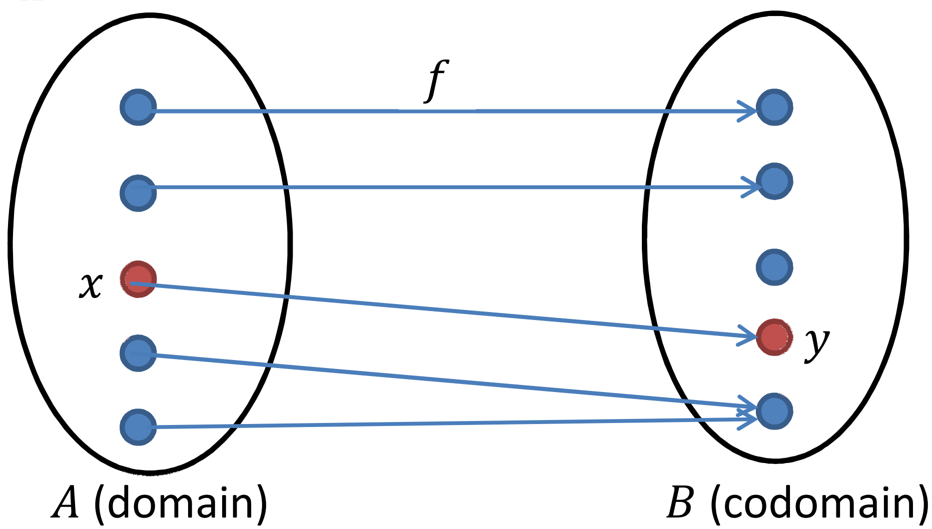
$$\mathbb{N} \backslash \mathbb{Z} = \phi.$$

Functions

A function $f(x)$ from set A to set B, denoted by

$$f\colon \underbrace{A}_{domain} \to \underbrace{B}_{codomain},$$

assigns (maps) each element of A to exactly one element of B. Mathematically, we write $f\underbrace{(x)}_{\in A} = \underbrace{y}_{\in B}$.

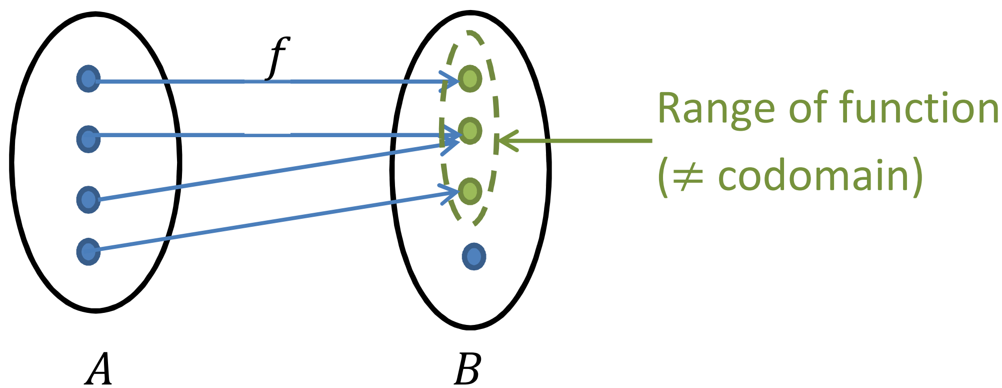


Some terminologies

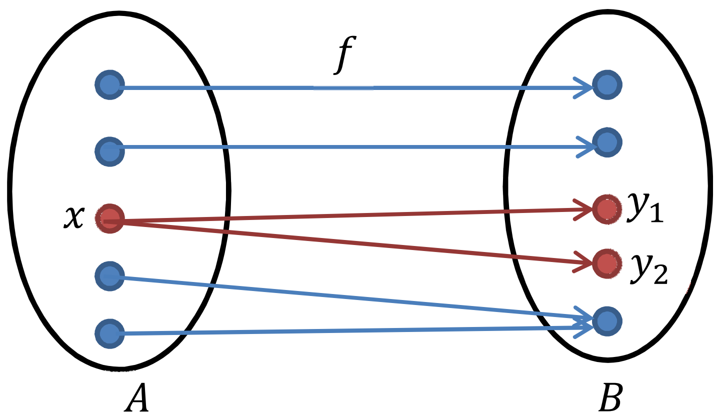
- The *domain* of a function is the collection of numbers that can be "put" (in the sense that the value of $f(x)$ is defined) in the function.

- The *codomain* of a function is the set which all possible outputs of the function $f(x)$ lie in this set.

- The *range* of a function is the collection of all possible outputs of the function (i.e. all possible values of $f(x)$).
  (In general, the range of $f(x)$ does not necessarily cover the whole codomain.)



The following figures shows some examples of non-functions (or not well-defined function)

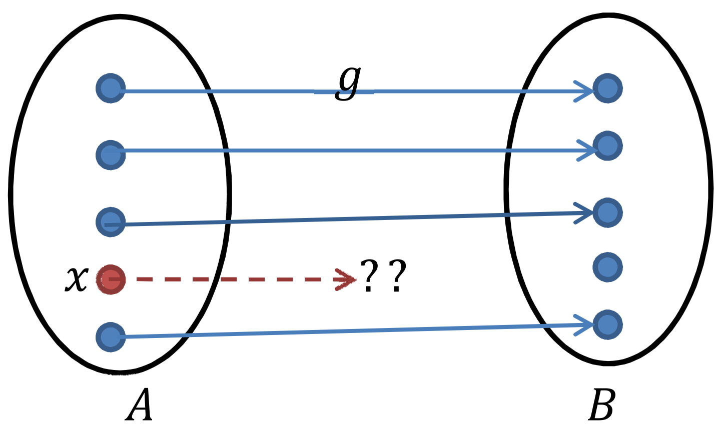


f is not function since $f(x)$ has two possible values y_1 or y_2.

Example:

$f: [0, \infty) \to \mathbb{R}, \quad f(x) = \sqrt{x}$

$f(4) = \sqrt{4}$ can be 2 or -2.



f is not function since $f(x)$ is not defined

Example:

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \log x$

$f(-2) = \log(-2)$ is not defined!

Examples of Functions

$f_1 : \mathbb{R} \to \mathbb{R}$, given by $f_1(x) = 2x$

$f_2 : \{1,2,3, \dots\} \to [1, \infty)$ given by $f_2(x) = 3x$

$f_3 : [0, \infty) \to [0, \infty)$ given by $f_3(x) = \sqrt{x}$ (Here, $\sqrt{x}$ takes zero or positive values)

Examples of non-functions

$g_1 : \mathbb{R} \to \mathbb{R}$ given by $g_2(x) = \frac{1}{(x-1)(x-3)}$

- g_2 is not function since $g_2(1) = \frac{1}{0}$ and $g_2(3) = \frac{1}{0}$ which are not defined.

$g_2 : \mathbb{R} \to (-\infty, 2)$, given by $g_3(x) = 4 - x^2$

- g_3 is not function since $g_3(1) = 4 - (-1)^2 = 3$ which does not lie in the codomain $(-\infty, 2)$ of g_2.

Example 1 (Range of function)

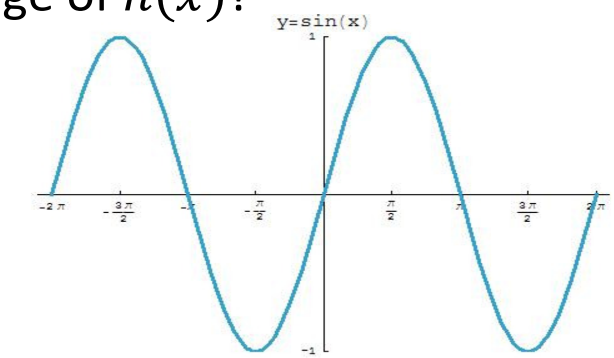
Let $g\colon \underbrace{\mathbb{N}}_{=\{1,2,3,\dots\}} \to \mathbb{R}$ given by $g(x) = 3x$. What is the range of $g(x)$?

☺Solution:

Since $g(1) = 3, g(2) = 6, g(3) = 9, g(4) = 12, \dots$, hence the range of g is $\{3,6,9,12,15,\dots\}$ which is the (positive) multiple of 3.

Example 2 (Range of function)

Let $h\colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = \sin x$. What is the range of $h(x)$?



☺Solution:

By plotting the graph, we see $\sin x$ lies between

-1 and 1. Thus range of $h(x)$ is the interval $[-1,1]$.

Example 3

Find the domain (largest possible domain) for each of the following functions (i.e. find all possible x that can be put in each function)

$(a)\ \ f_1(x) = x^2 - 2x - 3$

$$(b)\ \ f_2(x) = \frac{1}{x^2 - 2x - 3}$$

$$(c)\ \ f_3(x) = \frac{x^2 - 1}{x - 1}$$

$(d)\ \ f_4(x) = \sqrt{4 - x^2}$

☺Solution:

$$\boxed{f_1(x) = x^2 - 2x - 3}$$

One can calculate $x^2 - 2x - 3$ for every real number x, thus the domain of f_1 $= \mathbb{R}$.

$$f_2(x) = \frac{1}{x^2 - 2x - 3}$$

Note that $f_2(x)$ is not defined when $x^2 - 2x - 3 = 0$.

$$x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3 \;\; or \;\; x = -1.$$

Thus the domain of $f_2 = \mathbb{R}\backslash\{1,3\}$.

$$f_3(x) = \frac{x^2 - 1}{x - 1}$$

Note that $f_3(x)$ is not defined when $x - 1 = 0$, i.e. $x = 1$. Thus the domain of $f_3 = \mathbb{R}\backslash\{1\}$).

$$f_4(x) = \sqrt{4 - x^2}$$

Note that $f_4(x)$ is defined only when $4 - x^2 \geq 0$, i.e. $-2 \leq x \leq 2$. Thus the domain of $f_4 = [-2,2]$.

Basic Operation of function

Let $f(x)$ and $g(x)$ be two functions, we define

1. $(f \pm g)(x) = f(x) \pm g(x)$ (addition and subtraction)
2. $(fg)(x) = f(x) \times g(x)$ (multiplication)
3. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (division)
4. $(f \circ g)(x) = f\big(g(x)\big)$ (composition)

   (Note: $(f \circ g)(x) \neq (g \circ f)(x)$ (or $f\big(g(x)\big) \neq g(f(x))$) in general.

Example 4

Let $f(x) = x^2 + 1$ and $g(x) = 1 - x - x^2$. Then

$(f + g)(x) = f(x) + g(x) = x^2 + 1 + (1 - x - x^2) = 2 - x.$

$(fg)(x) = f(x)g(x) = (x^2 + 1)(1 - x - x^2) = \cdots = 1 - x - x^3 - x^4.$

Example 5 (Composition of functions)

We let $f(x) = x^2 + 1$ and $g(x) = \sqrt[3]{x}$. Then

$$(f \circ g)(8) = f(g(8)) = f(\sqrt[3]{8}) = f(2) = 2^2 + 1 = 5.$$

$$(g \circ f)(8) = g(f(8)) = g(8^2 + 1) = g(65) = \sqrt[3]{65} \approx 4.02.$$

[☺Note: $f(g(x)) \neq g(f(x))$ in general]

Example 6 (Composition of functions)

We let $f(x) = 100^x$ and $g(x) = \log_{10} x$. Then

$$(f \circ g)(x) = f(g(x)) = f(\log x) = 100^{\log_{10} x} = 10^{2\log_{10} x} = \left(10^{\log_{10} x}\right)^2 \\ = x^2.$$

$$(g \circ f)(x) = g(f(x)) = g(100^x) = \log_{10} 100^x = \log_{10} 10^{2x} = 2x \log_{10} 10 \\ = 2x.$$

Some commonly used functions used in Mathematics

1. Identity function

An identity function, denoted by $I(x)$ is given by

$$I(x) = x.$$

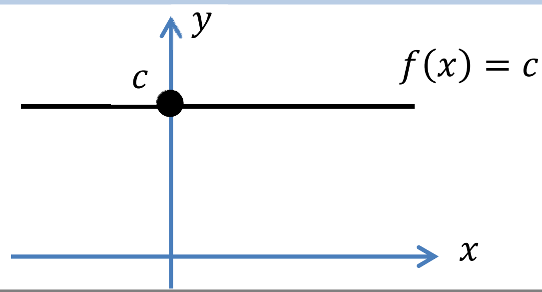
Roughly speaking, an identity function maps x to x itself.

2. Constant function

A constant function $f(x)$ is given by

$$f(x) = c.$$

where c is a fixed real number.

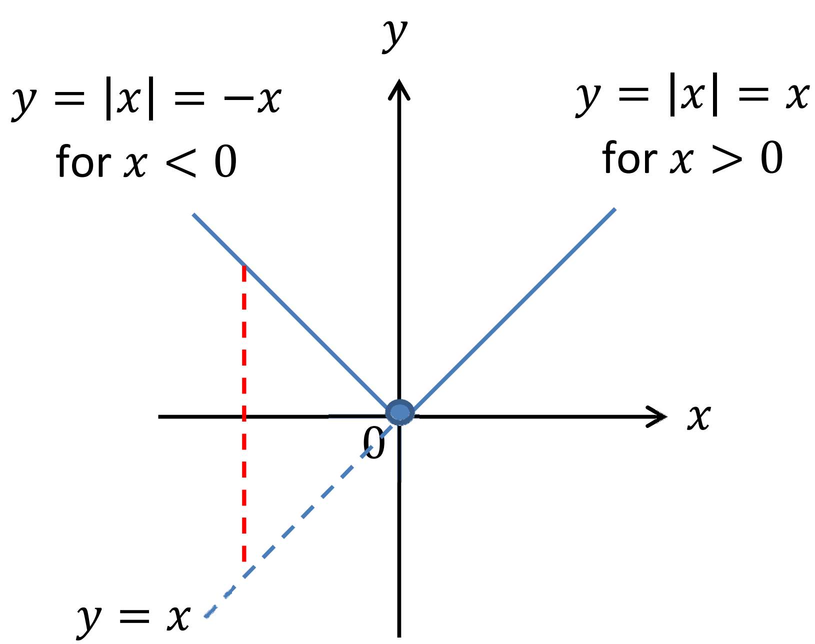


3. Absolute value function

The absolute value function, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x & if \;\; x \geq 0 \\ -x & if \;\; x < 0 \end{cases}$$

- Some Examples: $|5| = 5$, $|0| = 0$, $|-4| = -(-4) = 4$.



Properties of $|x|$

- $|x|^2 = \begin{cases} x^2 & if \;\; x \geq 0 \\ (-x)^2 = x^2 & if \;\; x < 0 \end{cases} = x^2.$
- $|xy| = |x||y|$ and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.
- But $|x + y| \neq |x| + |y|$ and $|x - y| \neq |x| - |y|$ in general !!!!!!

4. Polynomials and rational functions

A *polynomial* $p(x)$ is a function of the following form:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where n is non-negative integer and $a_n, a_{n-1}, \dots, a_0$ are fixed numbers.

A *rational function* $r(x)$ is the quotient or ratio of two polynomials

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are two polynomials and $q(x) \neq 0$.

- $x^4 - 3x + 1$ is polynomial,
- $x^{\sqrt{2}} - 2x$ and $x^{-2} = \frac{1}{x^2}$ are NOT polynomials.
- $\frac{x^2+1}{x^3-x-1}$ is rational function but $\frac{x+\cos x}{1-x^5}$ is NOT rational function.

(We will discuss their properties in Chapter 3)

5. Trigonometric functions

Six basic trigonometric functions

$$\sin x\,, \cos x\,, \tan x\,, \csc x = \frac{1}{\sin x}, \sec x = \frac{1}{\cos x}, \cot x = \frac{1}{\tan x}.$$

*Here, x is measured in radian ($1° = \frac{\pi}{180}$ (rad)).

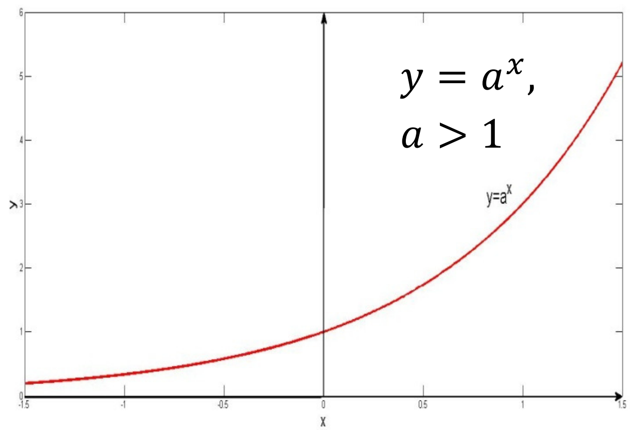
The properties of trigonometric functions will be discussed in Chapter 4.

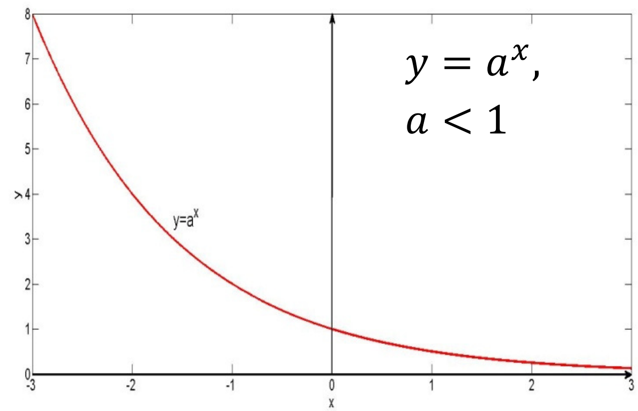
6. Exponential functions

The exponential function $f(x)$ is of the following form:

$$f(x) = a^x$$

where $a > 0$ is constant and $a \neq 1$ (When $a = 1$, $f(x) = 1^x = 1$ which is a constant function).





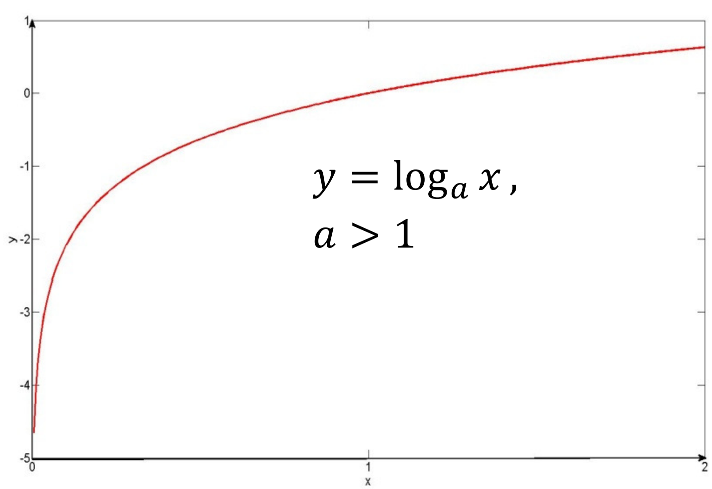
The properties of exponential functions will be discussed in Chapter 5.

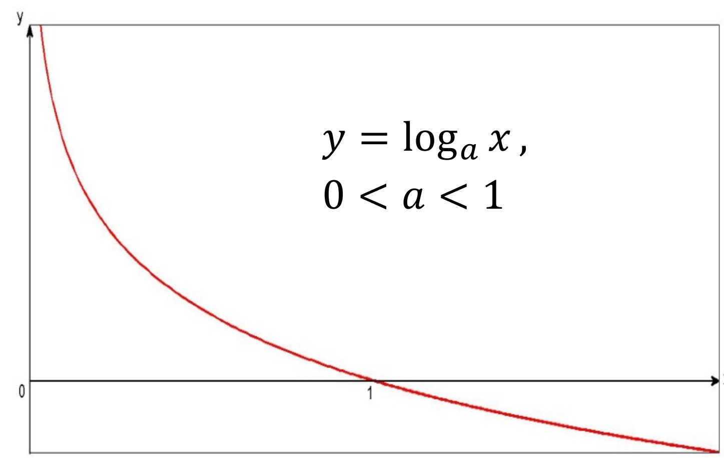
7. Logarithmic functions

The logarithmic function, denoted by $y = \log_a x \;\; (x > 0)$, is the number satisfying

$$a^y = x.$$

where $a > 0$ is constant and $a \neq 1$. (Here, a is called base)



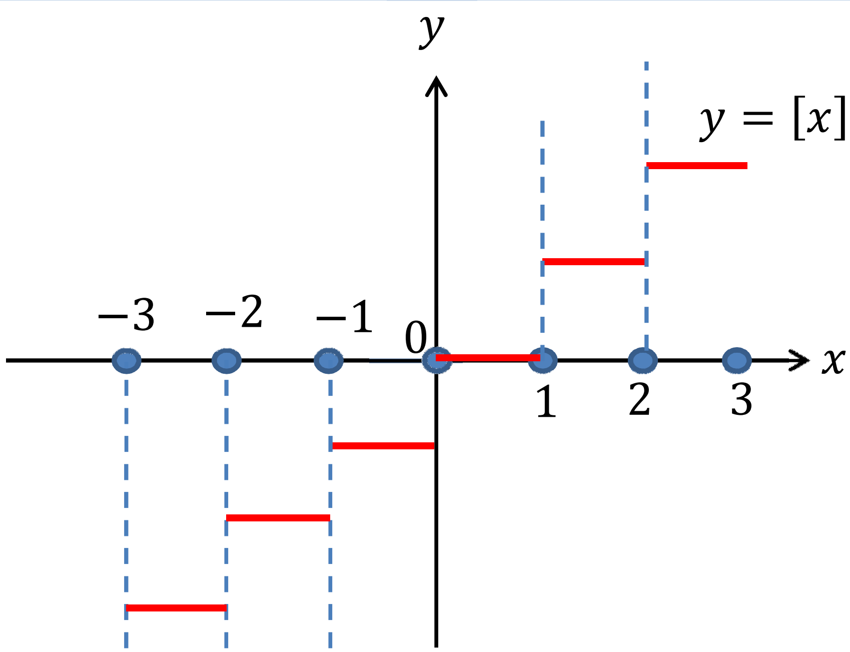


The properties of logarithmic functions will be discussed in Chapter 5.

8. Greatest Integer function (Less important)

Let $[x]$ be the greatest integer less than or equal to x, e.g. $[7.2]=7$, $[7.9]=7$ and $[7]=7$. The greatest integer function $g(x)$ is defined as

$$y=g(x)=[x].$$



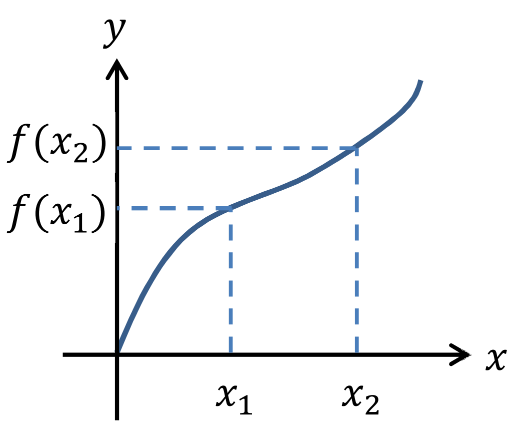
- There is a "jump" (discontinuous) at the integer points.

Special types of functions

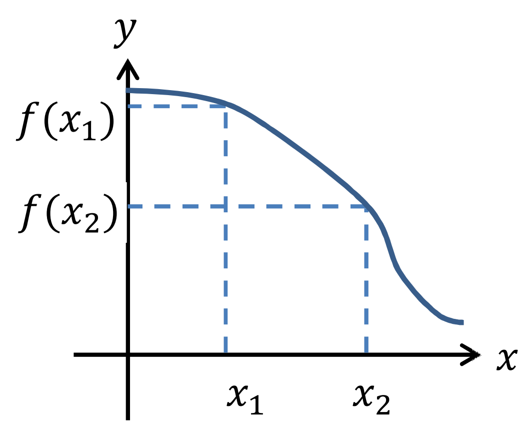
1. Monotone Functions

We say a function is *monotonic increasing* *(resp. monotonic decreasing)* if for any $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$ (resp. $f(x_1) \geq f(x_2)$).

We say a function is *strictly increasing* *(resp. monotonic decreasing)* if for any $x_1 < x_2$, we have $f(x_1) < f(x_2)$. (resp. $f(x_1) > f(x_2)$).



Increasing



Decreasing

Example 5

Determine whether the following functions are monotonic

(a) $f(x) = -2x + 3$, (b) $g(x) = 5^x$, (c) $h(x) = \sin x$

☺Solution:

- $f(x) = -2x + 3$ is *strictly decreasing*

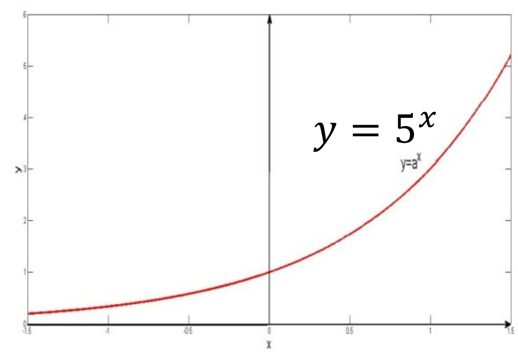
  Since for any $x_1 < x_2$, we have

$$f(x_1) = -2x_1 + 3 > -2x_2 + 3 = f(x_2)$$

- $g(x) = 5^x$ is *strictly increasing*.

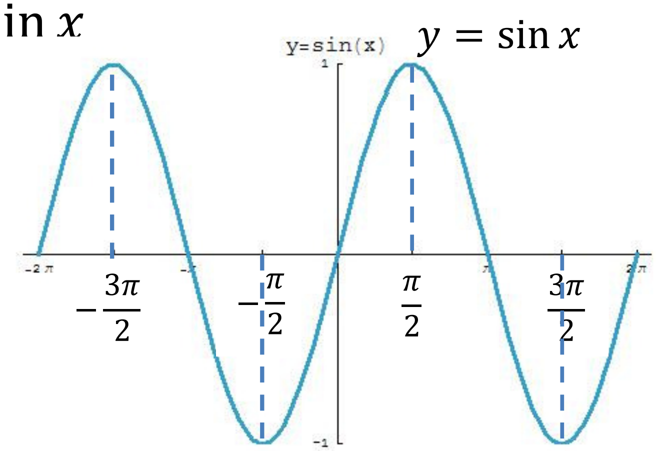
  To see this, for any $x_1 < x_2$, we consider

$$\frac{g(x_2)}{g(x_1)} = \frac{5^{x_2}}{5^{x_1}} = 5^{\overbrace{x_2 - x_1}^{>0}} > 1 \Rightarrow g(x_2) < g(x_1)$$



- $h(x) = \sin x$ is neither increasing nor decreasing.

  One can see this easily from the graph of $y = \sin x$



Remarks

Although $h(x)$ is not monotonic over the domain $\mathbb{R}$, $h(x)$ becomes monotonic if we choose a smaller domain. For example:

- $y = \sin x$ is strictly increasing over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,
- $y = \sin x$ is strictly decreasing over $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

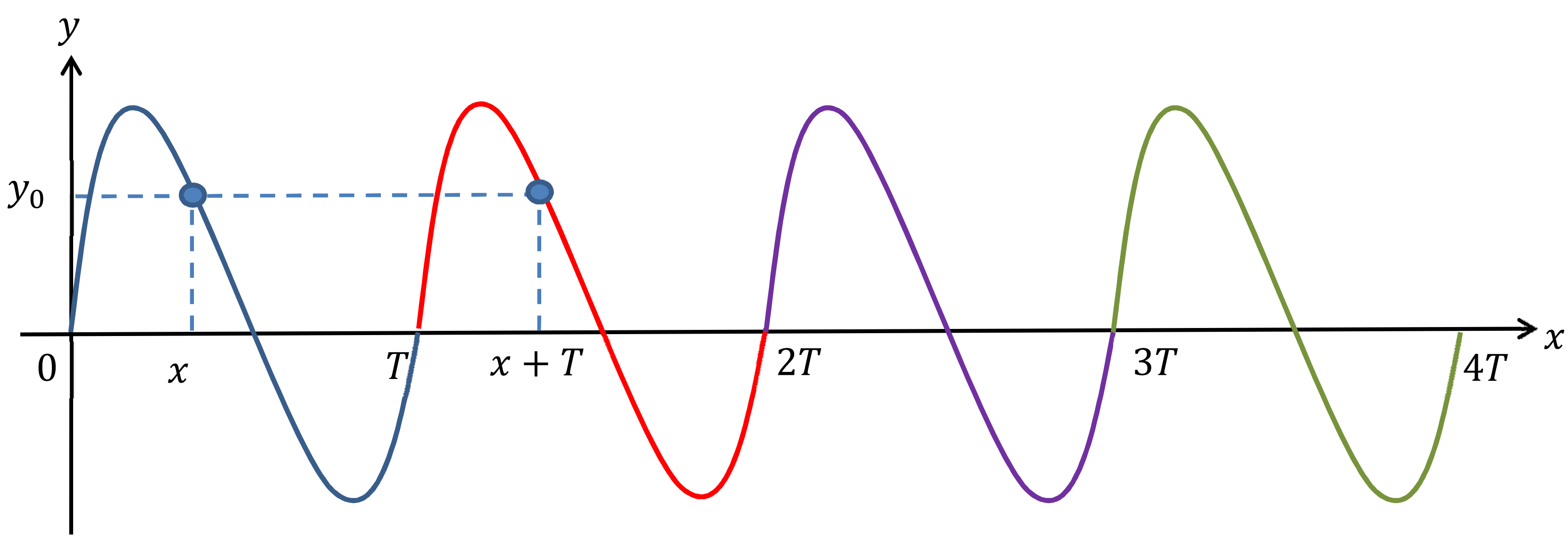
2. Periodic function

We say a function $f(x)$ is *periodic with period* $T > 0$ if

$$f(x+T) = f(x)$$

for all x.

(Here, T should be the <u>smallest</u> number such that $f(x+T) = f(x)$.)

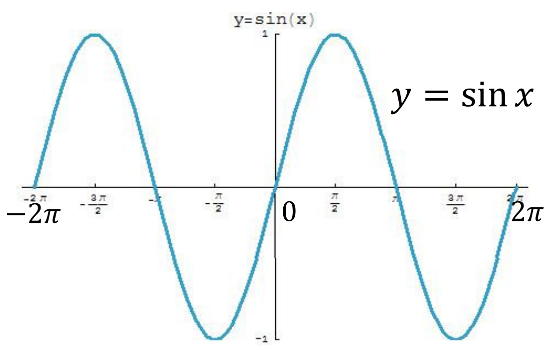


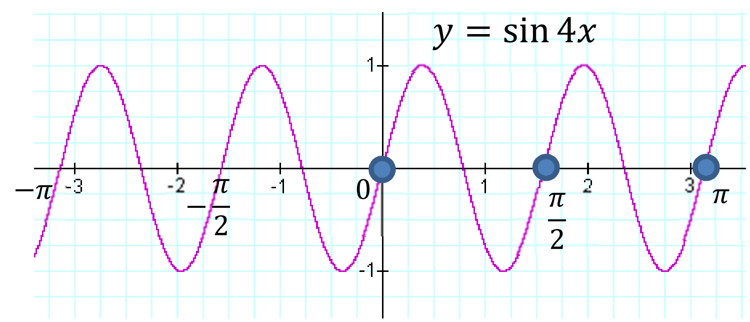
Example 6

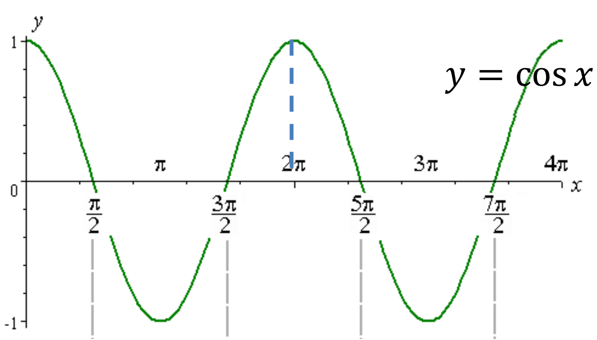
The functions $f(x) = \sin x$, $g(x) = \cos x$ are periodic with period 2π (or $360°$).

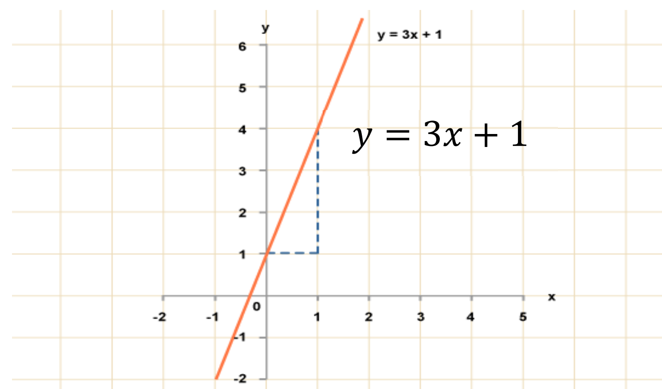
The function $h(x) = \sin 4x$ is periodic with period $\frac{\pi}{2}$ (or $90°$)

The function $j(x) = 3x + 1$ is not periodic.





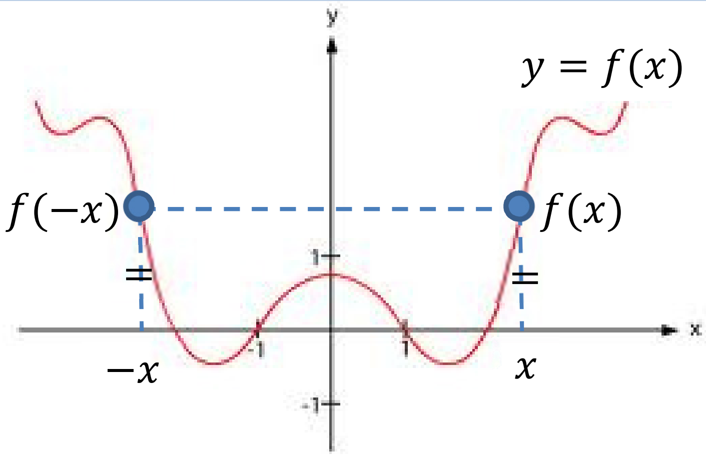




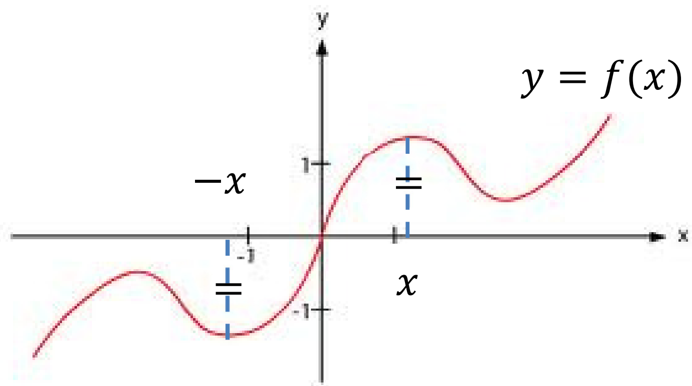
3. Even and odd functions

We say a function $f(x)$ is *even functions* if $f(-x) = f(x)$ for all x.

We say a function $f(x)$ is odd functions if $f(-x) = -f(x)$ for all x.



Even function



Odd function

- The graph of an even function is <u>symmetric about y-axis</u>.
- The graph of odd function is <u>symmetric about the origin</u>.

Example 7

- The function $f(x) = \cos x$ is even function since $f(-x) = \cos(-x) = \cos x = f(x)$.
- The function $f(x) = \sin x$ is odd function since $f(-x) = \sin(-x) = -\sin x = -f(x)$.
- The function $f(x) = \frac{a^x + a^{-x}}{2}$ is even function since $f(-x) = \frac{a^{-x} + a^{-(-x)}}{2} = \frac{a^{-x} + a^x}{2} = f(x)$.
- The function $f(x) = x^2 + x - 1$ is neither even or odd.
  To see this, we observe that $f(2) = 5$ and $f(-2) = 1$. It is neither
  $f(2) \neq f(-2)$ (not even function) nor $f(2) \neq -f(-2)$ (not odd function)

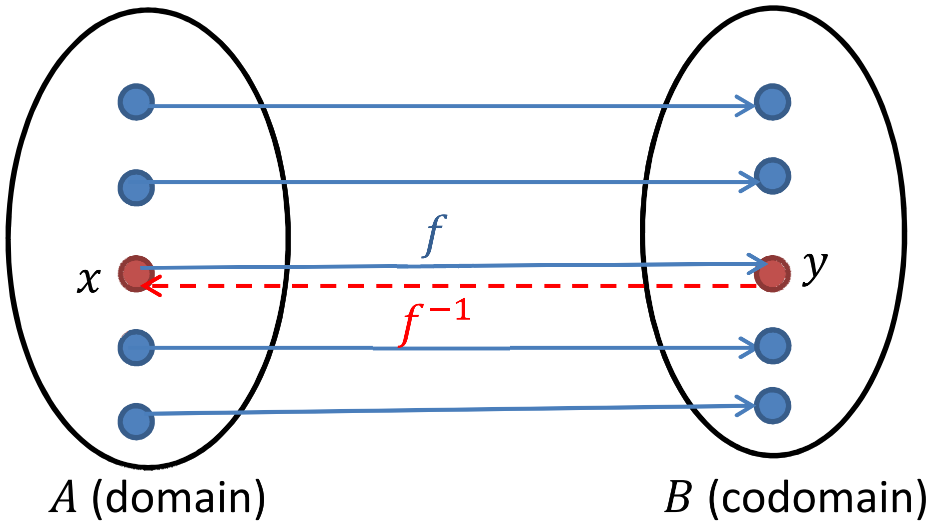
Classwork:

Determine whether each of the following functions are even or odd.

$f_1(x) = x^2,\ f_2(x) = -x,\ f_3(x) = x \sin x,\ f_4(x) = 3.$

Inverse Function

Recall that a function $f: A \to B$ takes an element x in domain A and assigns it to another element $y = f(x)$ in the codomain B.



Question:

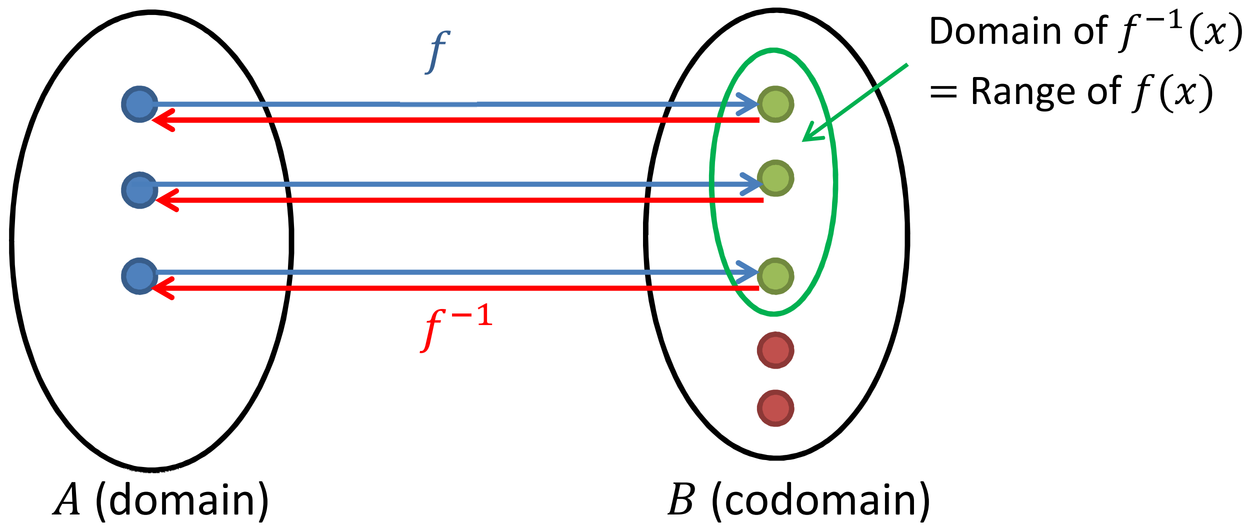
Given the value of y, what is the value of x such that

$$f(x) = y?$$

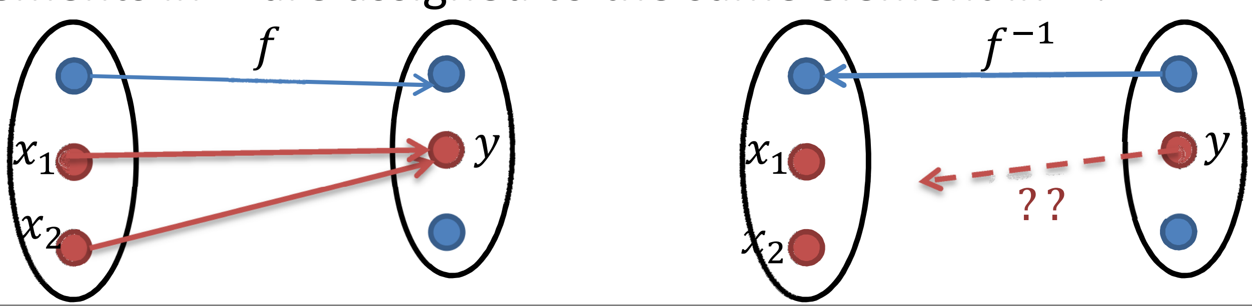
The inverse function of $f(x)$, denoted by f^{-1} tries to takes an element in the range of $f(x)$ back to the element x in the domain A (see the red dash arrow).

Mathematically, the inverse of f^{-1} satisfies $f^{-1}\big(f(x)\big) = x,\ f\big(f^{-1}(y)\big) = y$.

- The domain of the inverse $f^{-1}(x)$ is the range of $f(x)$ (may not necessarily to be the whole codomain)



- The inverse function $f^{-1}(x)$ does not exist if there are more than 2 elements in A are assigned to the same element in B.



To make sure that the inverse of a function exists, we require that there are no two elements x_1 and x_2 such that $f(x_1) = f(x_2)$. The function satisfying this requirement is called one-to-one.

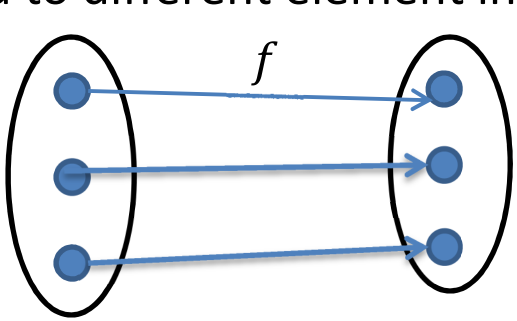
Definition (One-to-one)

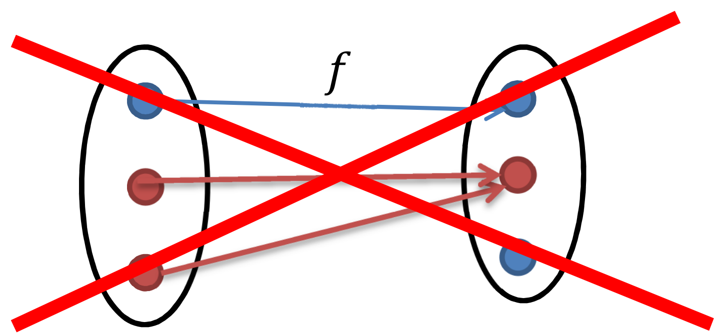
We say a function $f: A \to B$ is *one-to-one* if for any $x_1, x_2 \in A$ and $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

Or equivalently, if $f(x_1) = f(x_2)$, then it must be that $x_1 = x_2$.

- If a function is one-to-one, then the inverse f^{-1} exists.

In other word, one-to-one requires that different elements in A should be assigned to different element in B.





Example 8

We let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) = 2x - 3$. Show that $f(x)$ is one-to-one and find its inverse.

☺Solution

$f(x)$ is one-to-one

Note that for any $x_1 \neq x_2$

$$f(x_1) = 2x_1 - 3 \neq 2x_2 - 3 = f(x_2).$$

So that $f(x)$ is one-to-one and its inverse f^{-1} exists.

Find f^{-1}

Let $y = f(x) = 2x - 3$, then we express x in terms of y:

$$y = 2x - 3 \Rightarrow x = \frac{y+3}{2} \Rightarrow f^{-1}(y) = \frac{y+3}{2}.$$

Example 9

(a) Does the inverse of $g_1 : \mathbb{R} \to [0, \infty)$ given by $g_1(x) = x^2$ exist?

(b) Does the inverse of $g_2 : [0, \infty) \to [0, \infty)$ given by $g_2(x) = x^2$ exist?

☺Solution:

(a) Note that $g_1(-1) = g_1(1) = 1$, so $g_1(x)$ is not one-to-one and the inverse of g_1 does not exist.

(b) For any $x_1, x_2 \in [0, \infty)$ and $x_1 \neq x_2$, since $g_2(x)$ is strictly increasing on $[0, \infty)$, then we must have $g(x_1) \neq g(x_2)$. So $g_2(x)$ is one-to-one.

Therefore, the inverse of $g_2(x)$ exists and $g_2^{-1} : [0, \infty) \to [0, \infty)$ is given by

$$y = x^2 \Rightarrow x = g_2^{-1}(y) = \sqrt{y}.$$

Example 10

(a) Does the inverse of $h_1: \mathbb{R} \to \mathbb{R}$ given by $h_1(x) = \sin x$ exists?

(b) Does the inverse of $h_2: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ given by $h_2(x) = \sin x$ exists?

☺Solution:

(a) Note that $h_1\left(\frac{\pi}{4}\right) = h_2\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$, so $h_1(x) = \sin x$ is not one-to-one and its inverse does not exist.

(b) Since $h_2(x) = \sin x$ is increasing over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then for any $x_1, x_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x_1 \neq x_2$, we must have $h_2(x_1) \neq h_2(x_2)$. So $h_2(x)$ is one-to-one and the inverse of $h_2(x)$ exists.

The inverse $h_2^{-1}: \underbrace{[-1,1]}_{not\ \mathbb{R}!} \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is given by

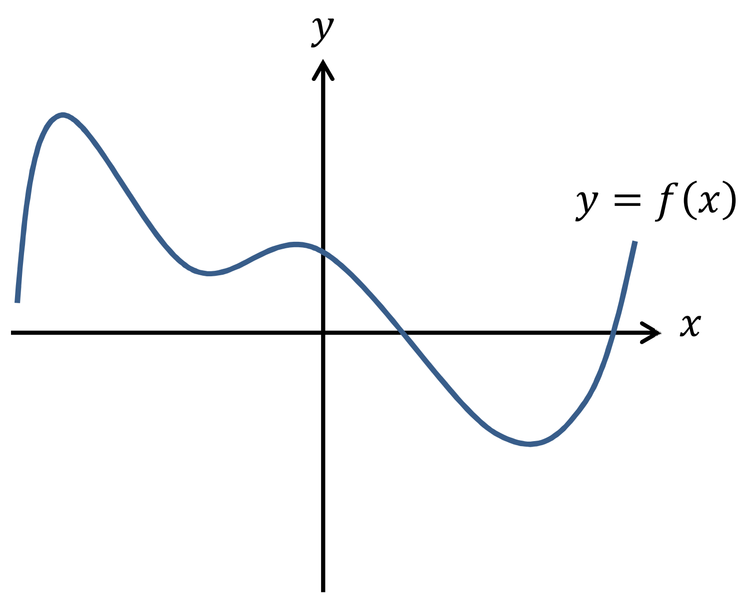
$$h_2^{-1}(x) = \sin^{-1} x.$$

Some other common inverse functions used in Mathematics

$f(x)$	Inverse of $f(x)$
$f_1: \mathbb{R} \to [0, \infty)$, $f_1(x) = 10^x$	$f_1^{-1}(x) = \log_{10} x$
$f_2: \mathbb{R} \to [0, \infty)$, $f_2(x) = e^x$	$f_2^{-1}(x) = \ln x \ (= \log_e x)$
$f_3: [0, \infty) \to [0, \infty)$ $f_3(x) = x^2$	$f_3^{-1}(x) = \sqrt{x}$
$f_4: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1)$, $f_3(x) = \sin x$	$f_4^{-1}(x) = \sin^{-1} x \ (= \arcsin x)$
$f_5: [0, \pi] \to [-1,1)$, $f_4(x) = \cos x$	$f_5^{-1}(x) = \cos^{-1} x \ (= \arccos x)$
$f_6(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, $f_5(x) = \tan x$	$f_6^{-1}(x) = \tan^{-1} x \ (= \arctan x)$

Transformation of Functions

- Geometrically, one can "visualize" a function $f(x)$ by plotting the graph of $y = f(x)$. One can obtain more information (say maximum, minimum, monotonicity (increasing/decreasing) about the function by observing its graph.



- It is easy to plot the graph of some simple functions (or elementary function) such as $y = ax + b$, $y = x^2$, $y = \sin x$, $y = |x|$, $y = e^x$ etc.

- It is not straightforward to plot the graph of more complicated functions like $y = -(2x-3)^2 + 5$, $y = 3|3x-4|$ and $y = e^{1-3x}$ because the inclusion of some extra parameters.

- Of course, one can plot these graphs by plugging in some values x and obtain the coordinates of points $(x, y) = \left(x, f(x)\right)$ on the curve. Then we may obtain the graph by connecting these points. However, it is not efficient and the graph obtained may not be accurate.

- One can observe that the functions are similar to $y = x^2$, $y = |x|$ and $y = e^x$. It may be possible that the graphs of those functions using these graphs of simple functions (geometric transformation technique).

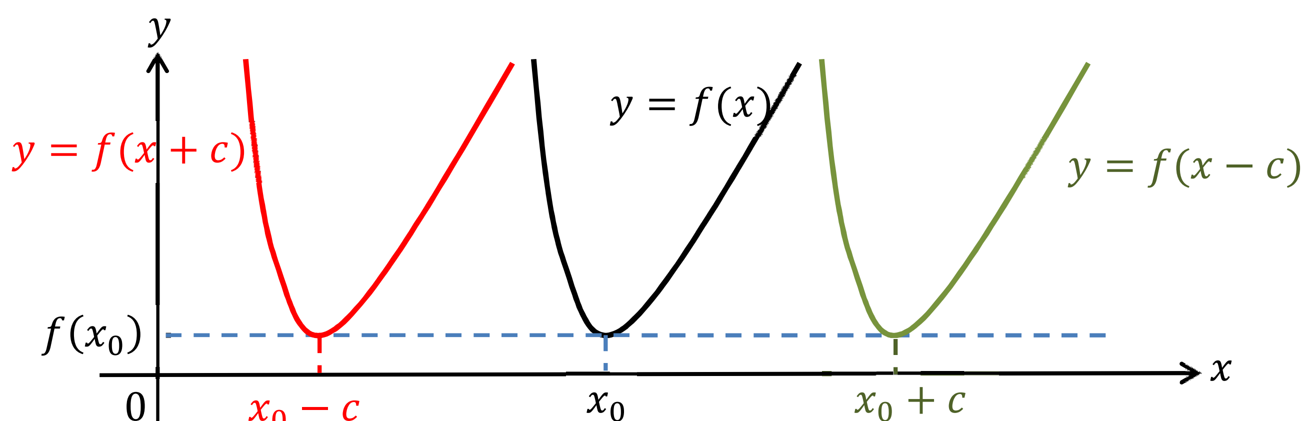
Types of transformation

Roughly speaking, there are two types of transformation: (1) Transformation on x and (2) Transformation on $f(x)$ (or y).

Type 1: Transformation on $\boldsymbol{x}$

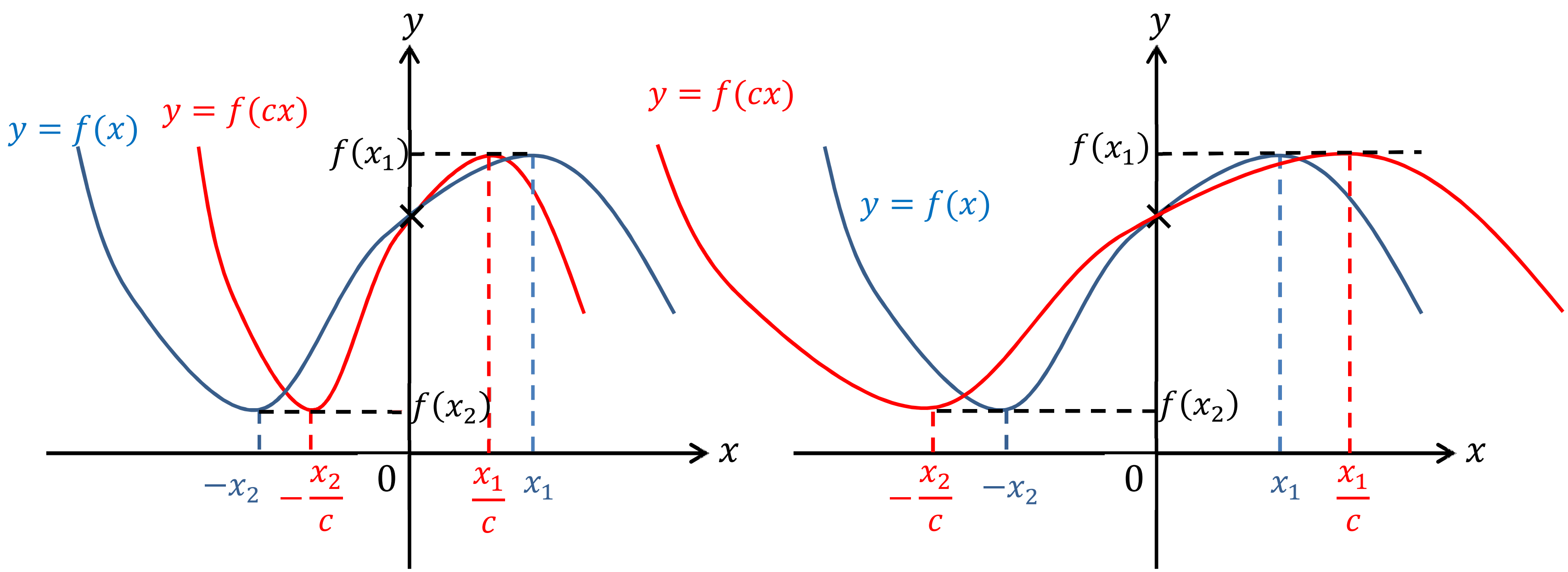
1. Horizontal transformation (The graph of $\boldsymbol{y = f(x + c)},\ \boldsymbol{y = f(x - c)}$)

- The graph of $y = f(x + c)$ can be obtained by shifting the graph of $y = f(x)$ by c units to the *left*.
- The graph of $y = f(x - c)$ can be obtained from the graph of $y = f(x)$ by c units to the *right*.



2. Horizontal stretching/shrinking (The graph of $y = f(cx)$)

- For $c > 0$, the graph of $y = f(cx)$ can be obtained by dividing the x-coordinate of each point on the graph by c.



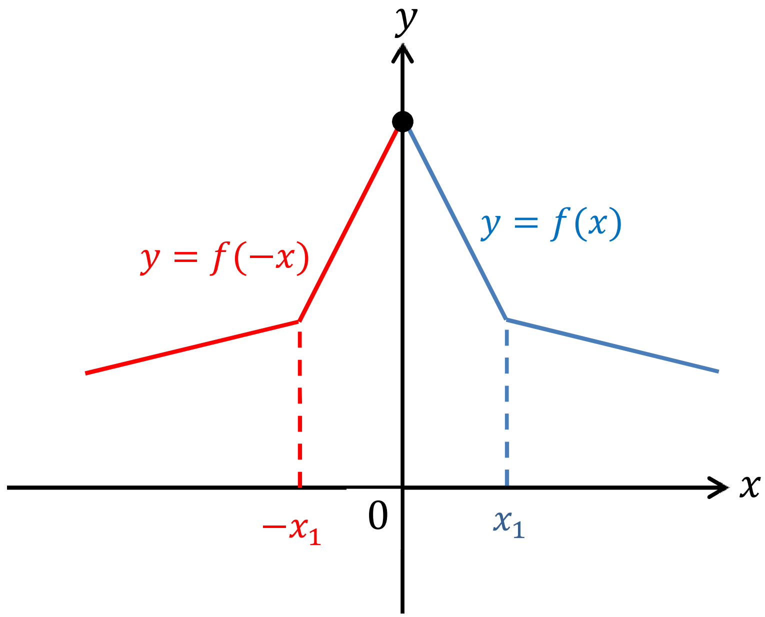
$c > 1$
(Shrinking)

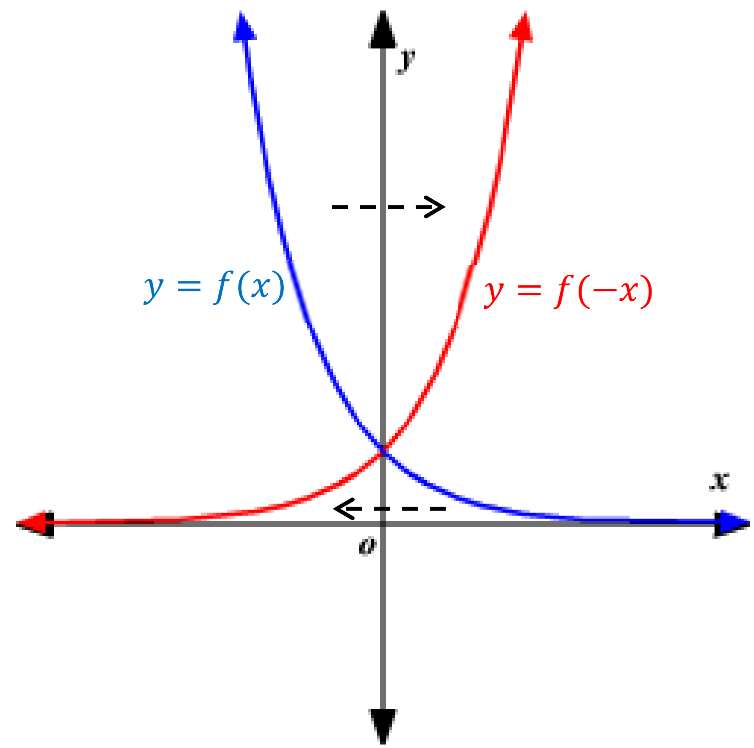
$c < 1$
(Stretching)

- Note that $y = f(x)$ and $y = f(cx)$ intersect at $x = 0$.

3. Reflection about y-axis (The graph of $y = f(-x)$)

- The graph of $y = f(-x)$ can be obtained by reflecting the graph $y = f(x)$ about y-axis.



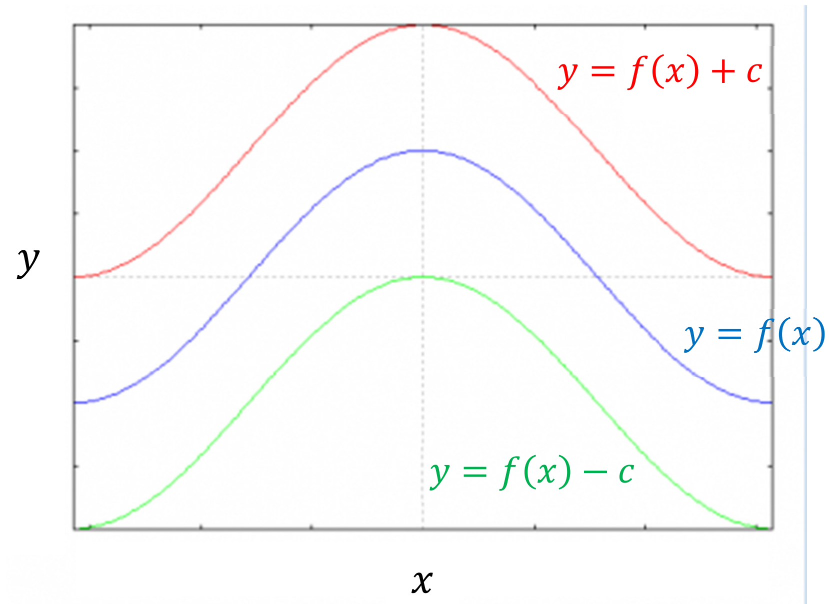


- Note that $y = f(x)$ and $y = f(-x)$ intersect at $x = 0$.

Type 2: Transformation on $f(x)$ (or y)

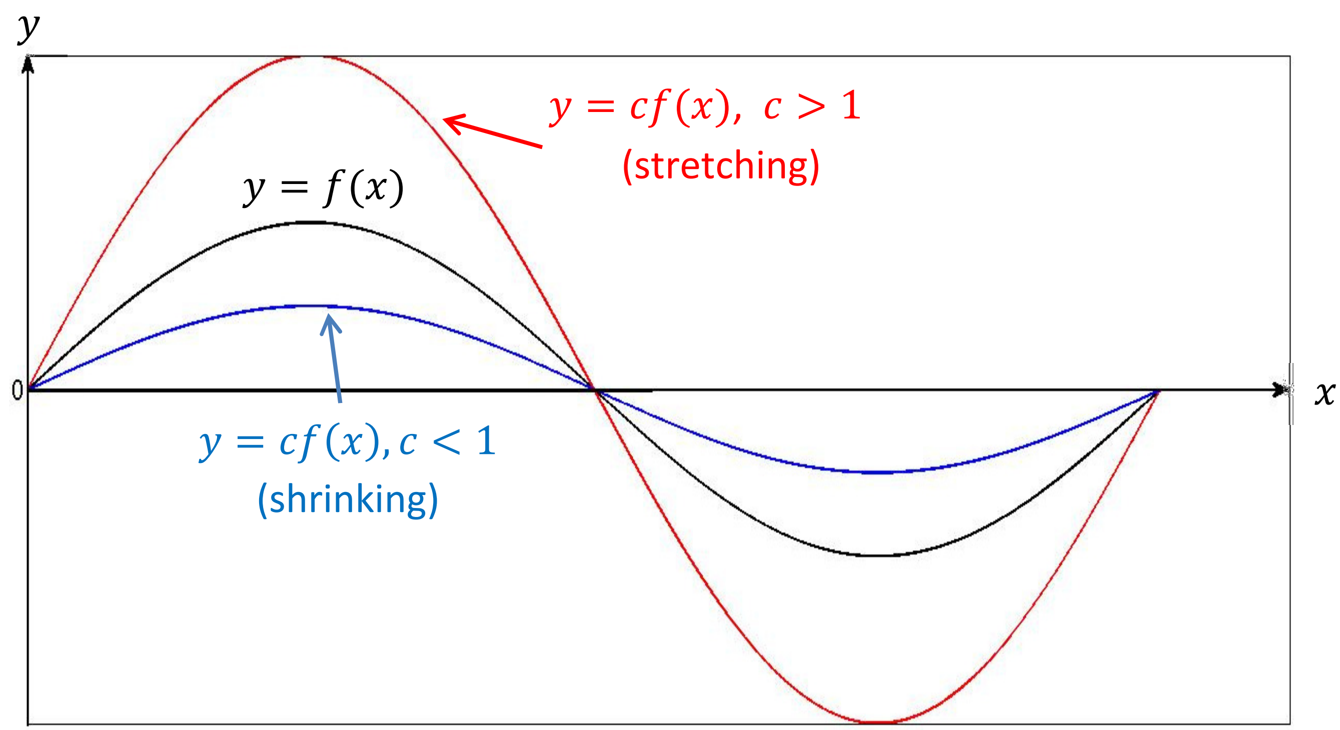
4. Vertical transformation (The graph of $y = f(x) + c$, $y = f(x) - c$)

- The graph of $y = f(x) + c$ can be obtained by shifting the graph of $y = f(x)$ by c units *upward*.
- The graph of $y = f(x) - c$ can be obtained from the graph of $y = f(x)$ by c units *downward*.



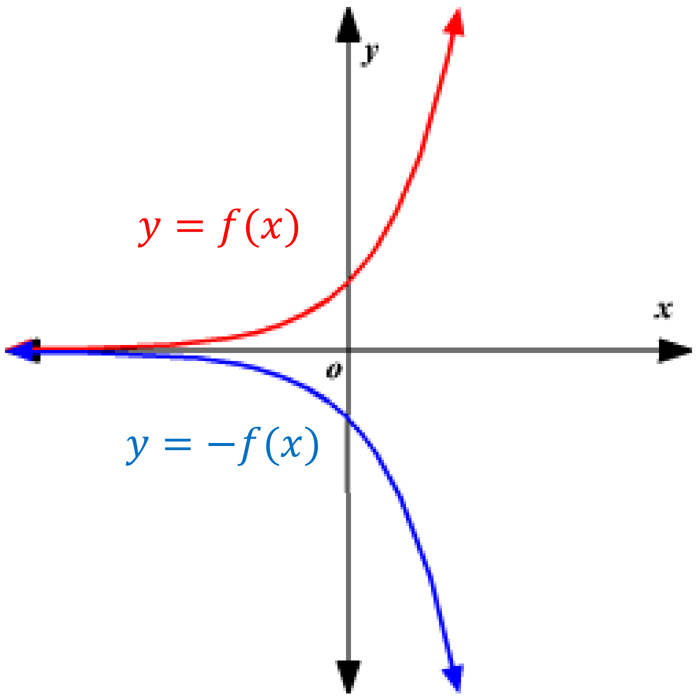
5. Vertical stretching/shrinking (The graph of $y = cf(x)$)

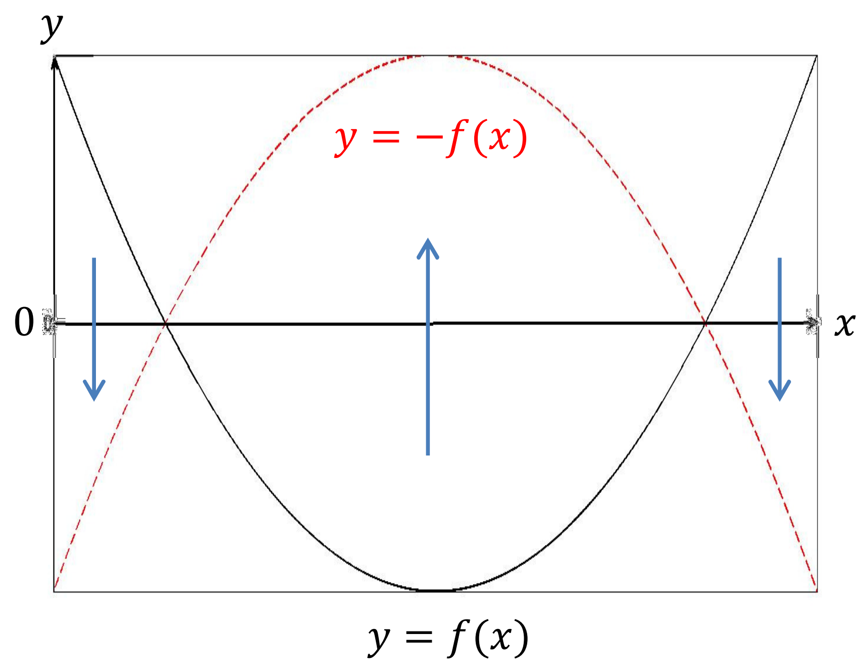
- For $c > 0$, the graph of $y = cf(x)$ can be obtained by multiplying the y-coordinate of each point on the graph by c.



6. Reflection about x-axis (The graph of $y = -f(x)$)

- The graph of $y = -f(x)$ can be obtained by reflecting the graph $y = f(x)$ about x-axis.





Example 11

Using the graph of $y = \sin x$

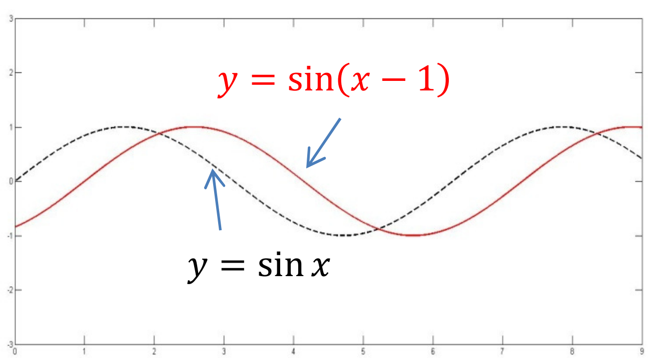
(a) Sketch the graph of $y = 3\sin(x-1)$.

(b) Sketch the graph of $y = |\sin 2x|$

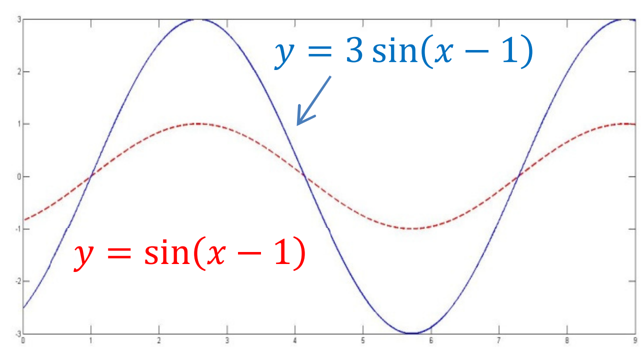
☺Solution

(a) The graph of $y = 3\sin(x-1)$ can be obtained from the graph of $y = \sin x$ using the following procedures

1. Shift $y = \sin x$ by 1 unit to the right and obtain $y = \sin(x-1)$

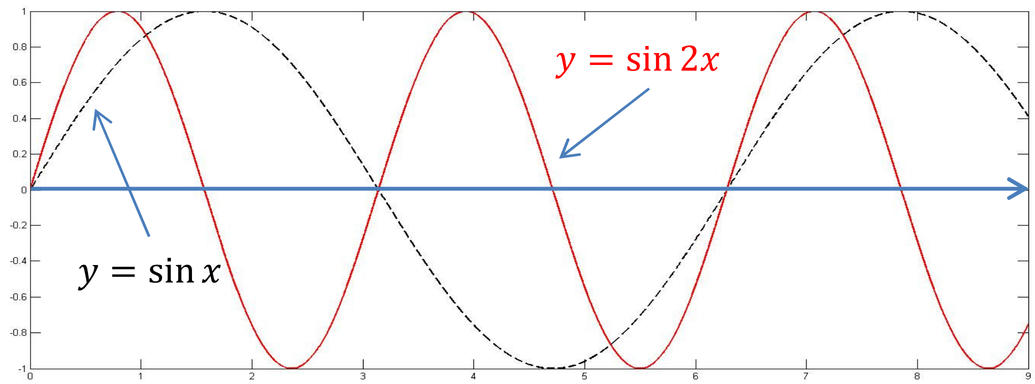


2. Multiply the y-coordinate of $y = \sin(x-1)$ by 3 and obtain $y = 3\sin(x-1)$.

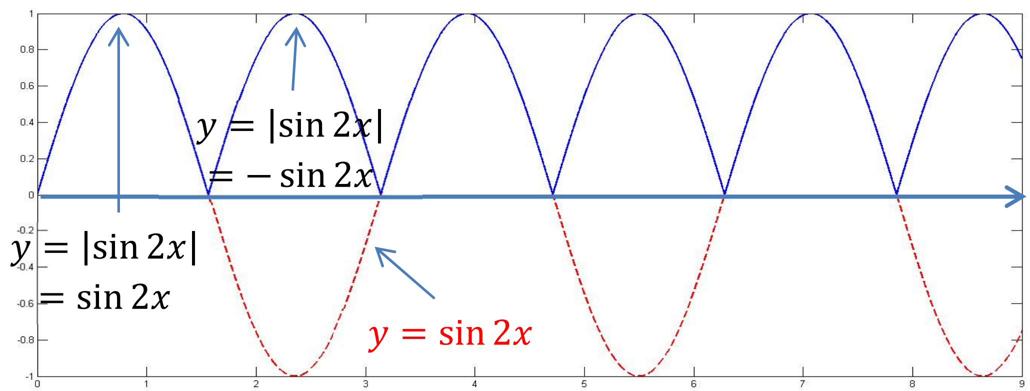


(b) The graph of $y = |\sin 2x|$ can be obtained from the graph of $y = \sin x$ using the following procedure:

1. Obtain $\sin 2x$ by dividing the x-coordinate of the graph of $y = \sin x$ by 2



2. Obtain $y = |\sin 2x|$ by reflecting the <u>negative part</u> of $y = \sin 2x$ about x-axis.



Recall that

$$|y| = \begin{cases} y & if \ \ y \geq 0 \\ -y & if \ \ y < 0 \end{cases}$$

Example 12

Using the graph of $y = x^2$

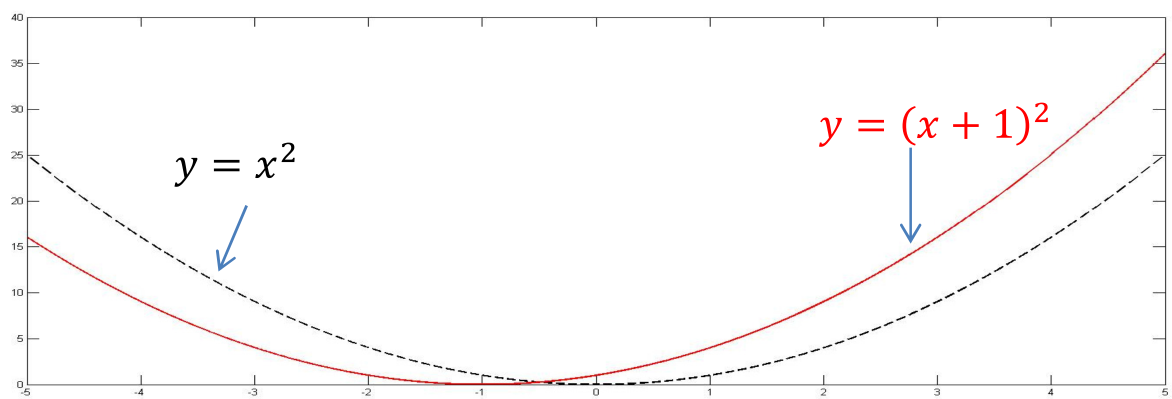
(a) Sketch the graph of $y = 2(x+1)^2 + 5$.

(b) Sketch the graph of $y = -x^2 + 6x - 1$.

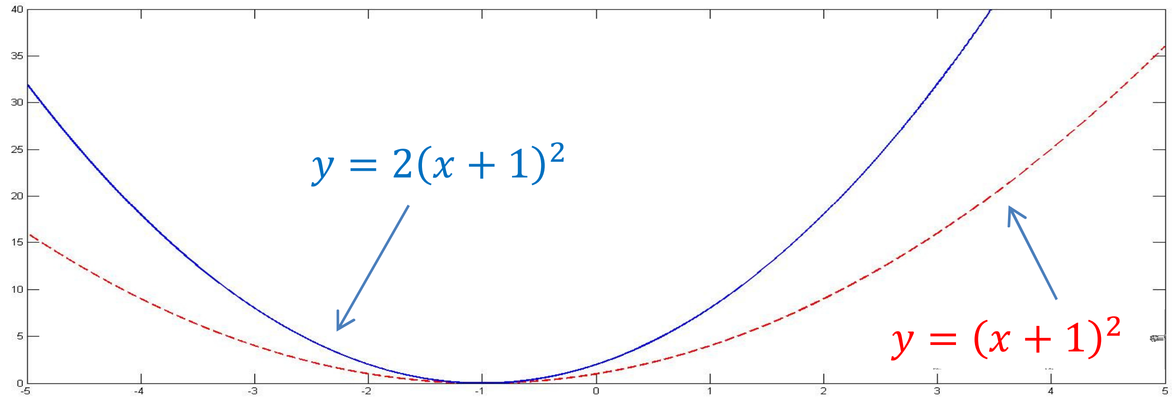
☺Solution

(a) The graph of $y = 2(x+1)^2 + 5$ can be obtained from the graph of $y = x^2$ using the following procedure

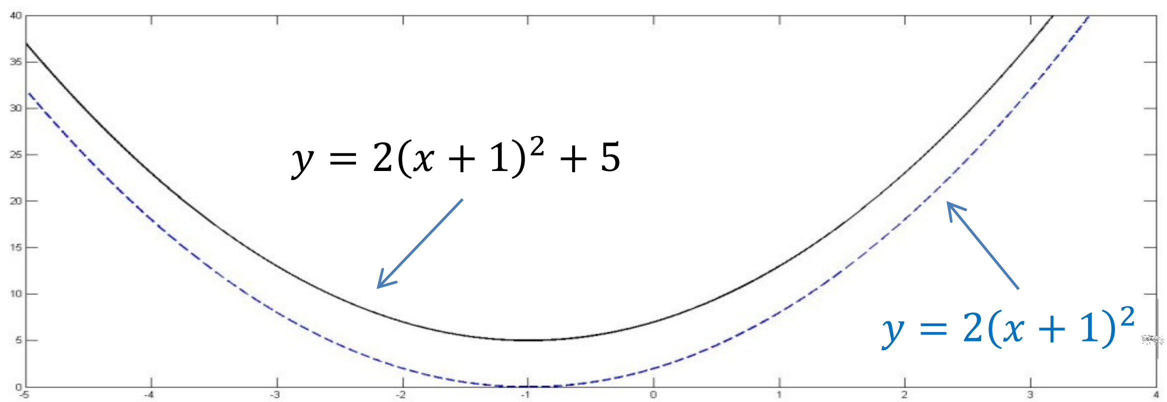
1. Obtain $y = (x+1)^2$ by shifting $y = x^2$ to the left by 1 units



2. Obtain $y = 2(x+1)^2$ by multiplying the y-coordinate of $y = (x+1)^2$ by 2.



3. Obtain $y = 2(x+1)^2 + 5$ by shifting $y = 2(x+1)^2$ upward by 5 units.



(b) We need to use rewrite the equation into the form $a(x-h)^2+b$ using completing square techniques. Note that

$$-x^2+6x-1=-(x^2-6x)-1=-\left(\underbrace{x^2-2(3)x+3^2}_{a^2-2ab+b^2}-3^2\right)-1$$

$$=-(x-3)^2+8.$$

Then one can obtain the graph $y=-x^2+6x-1=-(x-3)^2+8$ from the graph $y=x^2$ by the following procedure (left as exercise):

1. Obtain $y=(x-3)^2$ by shifting the graph $y=x^2$ to the right by 3 units.
2. Obtain $y=-(x-3)^2$ by reflecting the graph $y=(x-3)^2$ about x-axis.
3. Obtain $y=-(x-3)^2+8$ by shifting the graph $y=-(x-3)^2$ upwards by 8 units.