Vector Algebra

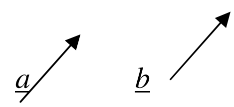
1. Review of Basic Ideas

In engineering and science, physical quantities which are completely specified by their magnitude (size) are known as scalars. Examples are: mass, temperature, volume, resistance, charge, voltage, current, etc.

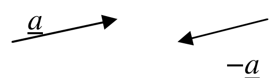
Other quantities possess both magnitude and direction and may be represented geometrically by directed line segments known as vectors. The length of the line is known as the magnitude of the vector and its direction is the direction of the vector. Examples of vector quantities are: velocity, acceleration, force, electric field, magnetic field etc and will be denoted by $\underline{v}$, $\underline{a}$, $\underline{F}$, $\underline{E}$, $\underline{B}$, etc.

1. Two vectors $\underline{a}$ and $\underline{b}$ are equal if they have the same magnitude and direction irrespective of their initial points. We write

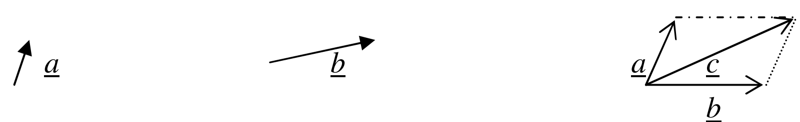
   $\underline{a} = \underline{b}$



2. A vector having the same magnitude as $\underline{a}$ but the opposite direction is denoted by $-\underline{a}$.

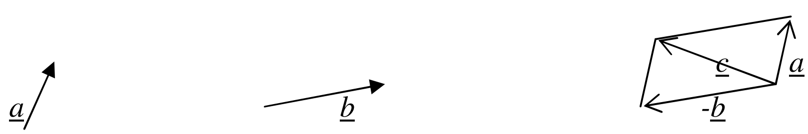


3. Geometrically the sum of two vectors is given by the parallelogram law



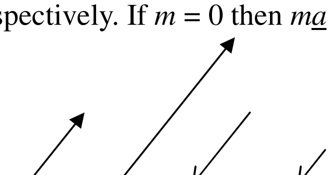
$\underline{c} = \underline{a} + \underline{b}$

4. The difference of two vectors $\underline{a}$ and $\underline{b}$, represented by $\underline{c} = \underline{a} - \underline{b}$ is defined as $\underline{c} = \underline{a} + \left(-\underline{b}\right)$



5. If $\underline{a} = \underline{b}$ then $\underline{a} - \underline{b}$ is the zero vector denoted by $\underline{0}$. This has magnitude 0 but no direction.

6. Multiplication of $\underline{a}$ by a scalar, m, produces a vector $m\underline{a}$ with magnitude m times that of $\underline{a}$ and direction the same as or opposite to that of $\underline{a}$ according to whether m is positive or negative respectively. If $m = 0$ then $m\underline{a} = \underline{0}$.



$\underline{a}$ $\quad 2\underline{a}$ $\quad -\underline{a}$ $\quad -\frac{1}{2}\underline{a}$

7. <u>Unit vectors</u> are vectors with magnitude 1. If $\underline{a}$ is any vector then we usually denote its magnitude by $|\underline{a}|$. A unit vector with the same direction as $\underline{a}$ will be $\dfrac{\underline{a}}{|\underline{a}|}$.

2. Components of a Vector

In a rectangular coordinate system in $3-\text{D}$ Euclidean space R^3, <u>orthogonal</u> (perpendicular) unit vectors in the directions of the positive x, y and z axis are denoted by $i = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $j = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\underline{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ respectively. The vector from the origin O to a point P is known as the <u>position vector</u> of P. If P has Cartesian coordinates $\begin{pmatrix}x\\y\\z\end{pmatrix}$, then the position vector of P, $\underline{r}$, may be written as,

$$\overrightarrow{OP} = \underline{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix} = x\underline{i} + y\underline{j} + z\underline{k}$$

$$\underline{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix} = xi + yj + zk\,.$$

x, y, z are know as the <u>components</u> or <u>coordinates</u> of $\underline{r}$ with respect to the vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$.

If $P:\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$ and $Q:\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}$ are two points, the vector from P to Q, $\overrightarrow{PQ}$ will be

$$\overrightarrow{PQ} = \underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},$$ where the components of $\underline{a}$ are

$a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$ and $a_3 = z_2 - z_1$.

Note that the ordered triple of components of a vector is unique with respect to a given coordinate system.

If $\underline{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\underline{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, in terms of components we have:

Equality :

$\underline{a} = \underline{b} \quad iff \quad a_1 = b_1, a_2 = b_2, a_3 = b_3$

Addition:

$$\underline{a} + \underline{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} = (a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j} + (a_3 + b_3)\underline{k}$$

Scalar Multiplication:

$$m\underline{a} = \begin{pmatrix} ma_1 \\ ma_2 \\ ma_3 \end{pmatrix} = ma_1\underline{i} + ma_2\underline{j} + ma_3\underline{k}$$

Zero Vector :

$$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Magnitude :

$|\underline{a}| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,3}}$ (Pythagoras)

Unit Vector:

$$\frac{\underline{a}}{|\underline{a}|} = \frac{1}{\sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{a_1\underline{i} + a_2\underline{j} + a_3\underline{k}}{\sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}}$$

Notice that the above are also applicable to the *n*-component vectors $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in R^n,\ n \geq 1$**.**

Example

The vector $\underline{a}$ from $P: \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ to $Q: \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ has components

$a_1 = 1 - 3 = -2,\ a_2 = 2 - (-2) = 4,\ a_3 = -4 - 1 = -5$. Hence

$$\underline{a} = \begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix} = -2\underline{i} + 4\underline{j} - 5\underline{k}, \quad |\underline{a}| = \sqrt{(-2)^2 + 4^2 + (-5)^2} = \sqrt{45}$$

And a unit vector in the direction of $\underline{a}$ is $\dfrac{1}{\sqrt{45}}\begin{pmatrix} -2 \\ 4 \\ -5 \end{pmatrix}$

If $\underline{a}$ has the initial point $R:\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then its terminal point is $S:\begin{pmatrix} -1 \\ 6 \\ -2 \end{pmatrix}$.

Example

The vector $\underline{a}$ from $P:\begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}$ to $Q:\begin{pmatrix} 1 \\ 1 \\ -4 \\ 2 \end{pmatrix}$ has components

$a_1 = 1-1 = 0,\ a_2 = 1-2 = -1,\ a_3 = -4-(-2) = -2,\ a_3 = 2-1 = 1$. Hence

$$\underline{a} = \begin{pmatrix} 0 \\ -1 \\ -2 \\ 1 \end{pmatrix}, \quad |\underline{a}| = \sqrt{0^2 + (-1)^2 + (-2)^2 + 1^2} = \sqrt{6}$$

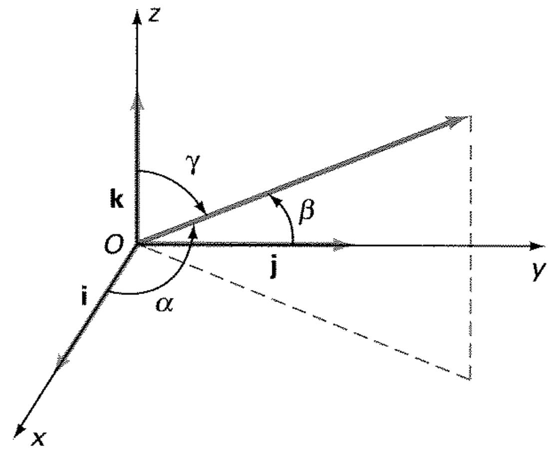
And a unit vector in the direction of $\underline{a}$ is $\dfrac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

If $\underline{a}$ has the terminal point $R:\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$, then its initial point S is:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} - S = \begin{pmatrix} 0 \\ -1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow S = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 0 \end{pmatrix}.$$

Direction Cosines

If $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, the direction of $\underline{r}$ may be specified by the cosines of the angles made by $\underline{r}$ with the 3 coordinate axes.



$$l = \cos\alpha = \frac{x}{|\underline{r}|}$$

$$m = \cos\beta = \frac{y}{|\underline{r}|}$$

$$n = \cos\gamma = \frac{z}{|\underline{r}|}$$

l, m, and n are known as the direction cosines of $\underline{r}$, $l\underline{i} + m\underline{j} + n\underline{k}$ is a unit vector along $\underline{r}$ and

$\underline{r} = |\underline{r}|(l\underline{i} + m\underline{j} + n\underline{k})$

Example

Let $\overrightarrow{OP}$ be $\begin{pmatrix}3\\2\\6\end{pmatrix} = 3\underline{i} + 2\underline{j} + 6\underline{k}$, then $|\underline{r}| = 7,\ l = 3/7,\ m = 2/7,\ n = 6/7$ and

$\alpha = \cos^{-1}(3/7),\ \beta = \cos^{-1}(2/7),\ \gamma = \cos^{-1}(6/7).$

If $\underline{a}, \underline{b}, \underline{c} \in R^n$ are vectors and m, n are scalars (real numbers), then we have

1.	$\underline{a} + \underline{b} = \underline{b} + \underline{a}$	Commutative law of vector addition
2.	$\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$	Associative law of vector addition
3.	$\underline{a} + \underline{0} = \underline{a}$	Existence of $\underline{0}$ as an additive vector identity
4.	$\underline{a} + (-\underline{a}) = \underline{0}$	Existence of additive inverses
5.	$m(\underline{a} + \underline{b}) = m\underline{a} + m\underline{b}$	Scalar distribution over vector addition
6.	$(m + n)\underline{a} = m\underline{a} + n\underline{a}$	Vector distribution over scalar addition
7.	$(mn)\underline{a} = m(n\underline{a})$	Associative law for scalar multiplication
8.	$1\underline{a} = \underline{a}$	Multiplicative scalar identity

3. Vector Products

Let $\underline{a}$ and $\underline{b}$ be two 3-component vectors, their dot product or scalar product, written $\underline{a} \bullet \underline{b}$, is defined as,

$$\underline{a} \bullet \underline{b} = \begin{cases} |\underline{a}||\underline{b}|\cos\theta & \text{if } \underline{a} \neq 0,\ \underline{b} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$, where θ is the angle between $\underline{a}$ and $\underline{b}$

Example

Given non-zero position vectors $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, show that $\underline{a} \bullet \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Proof:

According to Cosine Law, we have $|\underline{b} - \underline{a}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}|\cos\theta$.

$$|\underline{b} - \underline{a}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2, \quad |\underline{a}|^2 = a_1{}^2 + a_2{}^2 + a_3{}^2, \quad |\underline{b}|^2 = b_1{}^2 + b_2{}^2 + b_3{}^2$$

Then we have

$$|\underline{b} - \underline{a}|^2 = |\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}|\cos\theta$$
$$\Rightarrow (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = a_1{}^2 + a_2{}^2 + a_3{}^2 + b_1{}^2 + b_2{}^2 + b_3{}^2 - 2|\underline{a}||\underline{b}|\cos\theta$$
$$\Rightarrow 2a_1 b_1 + 2a_2 b_2 + 2a_3 b_3 = 2|\underline{a}||\underline{b}|\cos\theta$$
$$\Rightarrow \underline{a} \bullet \underline{b} = |\underline{a}||\underline{b}|\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Accordingly, we can define the dot product of two n-component vectors $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in R^n$ as

$$\underline{a} \bullet \underline{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Properties of the dot product $\underline{a} \bullet \underline{b}$, $\underline{a}, \underline{b} \in R^n$

(i) The result is a scalar

(ii) $\underline{a} \bullet \underline{b}$ is zero if $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$ or $\underline{a}$ and $\underline{b}$ are perpendicular (orthogonal)

(iii) $|\underline{a}| = \sqrt{\underline{a} \bullet \underline{a}}$

(iv) $\underline{a} \bullet \underline{b} = \underline{b} \bullet \underline{a}$ (symmetry)

(v) $(m\underline{a} + n\underline{b}) \bullet \underline{c} = m(\underline{a}, \underline{c}) + n(\underline{b}, \underline{c})$ $\forall \underline{a}, \underline{b} \in R^3$ and $m, n \in R$ (Linearity)

(vi) $\underline{a} \bullet \underline{a} \geq 0$ and $\underline{a} \bullet \underline{a} = 0$ iff $\underline{a} = \underline{0}$ (Positive definiteness)

(vii) $|\underline{a} \bullet \underline{b}| \leq |\underline{a}||\underline{b}|$ (Schwartz inequality)

(viii) $|\underline{a}+\underline{b}| \le |\underline{a}| + |\underline{b}|$ (Triangle inequality)

(ix) $|\underline{a}+\underline{b}|^2 + |\underline{a}-\underline{b}|^2 = 2\left(|\underline{a}|^2 + |\underline{b}|^2\right)$

We observe that $\underline{i} \bullet \underline{i} = \underline{j} \bullet \underline{j} = \underline{k} \bullet \underline{k} = 1$ and $\underline{i} \bullet \underline{j} = \underline{j} \bullet \underline{k} = \underline{k} \bullet \underline{i} = 0$

Example

Let $\underline{a} = 5\underline{i} + 4\underline{j} + 2\underline{k}$ and $\underline{b} = 4\underline{i} - 5\underline{j} + 3\underline{k}$, find $\underline{a} \bullet \underline{b}$ and the angle between the vectors.

Solution:

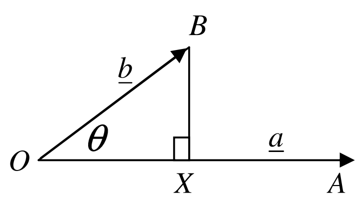
$\underline{a} \bullet \underline{b} = (5\times 4) + (4\times(-5)) + (2\times 3) = 6$. But $|\underline{a}| = \sqrt{45}$, $|\underline{b}| = \sqrt{50}$, hence

$$\cos\theta = \frac{\underline{a} \bullet \underline{b}}{|\underline{a}||\underline{b}|} = \frac{6}{\sqrt{45}\times\sqrt{50}} = \frac{2}{5\sqrt{10}} \Rightarrow \theta = \arccos\left(\frac{2}{5\sqrt{10}}\right)$$

Example

Given two vectors $\underline{a} = \overrightarrow{OA}$, $\underline{b} = \overrightarrow{OB}$, find the projection $\text{pro}_{\underline{a}}\underline{b}$ of $\underline{b}$ in the direction of $\underline{a}$ and the coefficient of $\text{pro}_{\underline{a}}\underline{b}$ (i.e. the coefficient of the projection of $\underline{b}$ in the direction of $\underline{a}$).

Solution:



$$\begin{aligned}\underline{a} \cdot \underline{b} &= |\underline{a}||\underline{b}| \cos\theta \\ &= OA \times OB \cos\theta \\ &= OA \times OX\end{aligned}$$

Then

$$\frac{\underline{a} \bullet \underline{b}}{|\underline{a}|} = OB\cos\theta = OX$$

The projection $\text{pro}_{\underline{a}}\underline{b}$ of $\underline{b}$ in the direction of $\underline{a}$ is:

$$(OB\cos\theta)\frac{\underline{a}}{|\underline{a}|} = \frac{\underline{a} \bullet \underline{b}}{|\underline{a}|}\frac{\underline{a}}{|\underline{a}|} = \frac{\underline{a} \bullet \underline{b}}{|\underline{a}|^2}\underline{a} = \frac{\underline{a} \bullet \underline{b}}{\underline{a} \bullet \underline{a}}\underline{a}.$$

the coefficient of $\text{pro}_{\underline{a}}\underline{b}$ (i.e. the coefficient of the projection of $\underline{b}$ in the direction of $\underline{a}$) is:

$$OX = |\underline{b}|\cos\theta = \frac{1}{|\underline{a}|}\underline{a} \bullet \underline{b} = \underline{b} \bullet \text{ (unit vector in } \underline{a} \text{ direction)}$$

Notice that the <u>coefficient</u> of $\text{pro}_{\underline{a}}\underline{b}$ (i.e. the coefficient of the projection of $\underline{b}$ in the direction of $\underline{a}$) can be negative if the angle θ between $\underline{a}$, $\underline{b}$ is an obtuse angle.

<u>Example</u>

A force $\underline{F} = 2\underline{i} + 3\underline{j} + \underline{k}$ acts on a particle which is displaced though $\underline{d} = \underline{i} - \underline{j} + 2\underline{k}$. Find the coefficient of $\text{pro}_{\underline{d}}\underline{F}$ (i.e. the coefficient of the projection of $\underline{F}$ in the direction of $\underline{d}$) and the work done by the force.

Solution:

Coefficient of $\text{pro}_{\underline{d}}\underline{F}$ is $\underline{F} \bullet \frac{\underline{d}}{|\underline{d}|} = \frac{2-3+2}{\sqrt{6}} = \frac{1}{\sqrt{6}}$

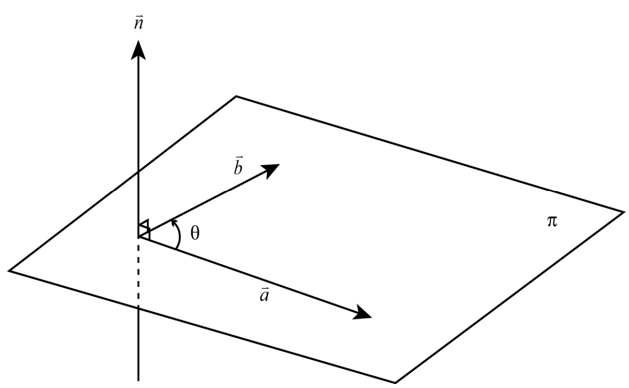
Work done by $\underline{F}$ = (Coefficient of $\underline{F}$ in the direction of $\underline{d}$) multiplied by $|\underline{d}|$

$$= \left(\underline{F} \cdot \frac{\underline{d}}{|\underline{d}|}\right)|\underline{d}| = \underline{F} \cdot \underline{d} = 1$$

Let $\underline{a}$, $\underline{b} \in R^3$ be two three component vectors, the <u>vector product</u> or <u>cross product</u> of $\underline{a}$ and $\underline{b}$, written $\underline{a} \times \underline{b}$, is defined as:

$$\underline{a} \times \underline{b} = \begin{cases} |\underline{a}||\underline{b}|\sin\theta\,\underline{v} & \text{if } \underline{a} \neq 0, \underline{b} \neq 0 \\ \underline{0} & \text{otherwise} \end{cases}$$

, where v a unit vector such that $\underline{a}$, $\underline{b}$, $\underline{v}$ form a right-handed triple, and θ the angle between $\underline{a}$, $\underline{b}$.



Notice that cross product is defined only for 3-component vectors.

Properties of the cross product:

(i) The result is a vector and $\underline{a} \times \underline{b}$ is zero *iff* $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$ or $\underline{a}$ and $\underline{b}$ are parallel.

(ii) $\underline{a} \times \underline{a} = \underline{0}$

(iii) $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

(iv) $|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\theta$= area of parallelogram with sides $\underline{a}$ and $\underline{b}$.

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}(a_2b_3\text{-}a_3b_2) + \underline{j}(a_3b_1\text{-}a_1b_3) + \underline{k}(a_1b_2\text{-}a_2b_1)$$

(Prove using results (vii) and (viii) below)

(v) $\underline{a} \times (\underline{b} + \underline{c}) = (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c})$

(vi) $m\ (\underline{a} \times \underline{b}) = (m\underline{a} \times \underline{b}) = (\underline{a} \times m\underline{b}) = (\underline{a} \times \underline{b})m$

(vii) $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}$, $\quad \underline{i} \times \underline{j} = \underline{k}, \underline{j} \times \underline{k} = \underline{i}, \underline{k} \times \underline{i} = \underline{j}$

Example

If $\underline{a} = 5\underline{i} + 4\underline{j} + 2\underline{k}$ and $\underline{b} = 4\underline{i} - 5\underline{j} + 3\underline{k}$, find $\underline{a} \times \underline{b}$ and a unit vector perpendicular to both $\underline{a}$ and $\underline{b}$.

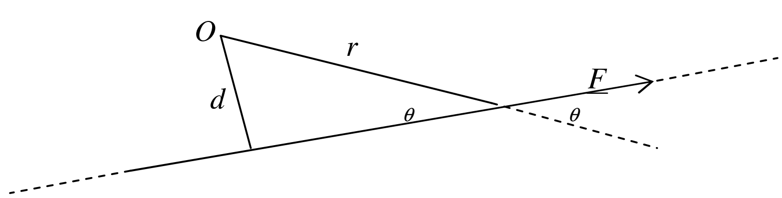
$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 4 & 2 \\ 4 & -5 & 3 \end{vmatrix} = \underline{i}\begin{vmatrix} 4 & 2 \\ -5 & 3 \end{vmatrix} - \underline{j}\begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix} + \underline{k}\begin{vmatrix} 5 & 4 \\ 4 & -5 \end{vmatrix}$$

$$= 22\underline{i} + 7\underline{j} + 41\underline{k}$$

A unit vector is $\pm\dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \pm\dfrac{1}{\sqrt{2214}}(22\underline{i} - 7\underline{j} - 41\underline{k})$

Example – Moment of a force

In mechanics the moment, *m*, of a force $\underline{F}$ about a point 0 is defined as the magnitude of $\underline{F}$ times the perpendicular distance (*d*) from 0 to the line of action, *L*, of $\underline{F}$. Let $\underline{r}$ be the vector from 0 to any point on *L*.



$m = |\underline{F}||\underline{r}|\sin\theta$

$m = |\underline{r} \times \underline{F}|$

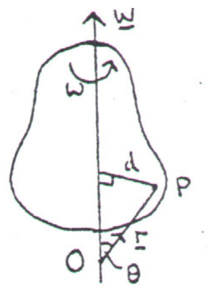
The vector $\underline{m} = \underline{r} \times \underline{F}$ is called the vector moment of $\underline{F}$ about 0. Its direction is along the axis about which $\underline{F}$ has a tendency to produce a rotation.

Example – Magnetic Field

The force $\underline{F}$ experienced by a point charge q moving with velocity $\underline{v}$ in a magnetic field of flux density $\underline{B}$ is given by $\underline{F} = q\,\underline{v} \times \underline{B}$

Example – Rotation

Consider a rigid body rotating with angular speed w about an axis. Let $\underline{w}$ be the vector with magnitude w and direction along the axis such that the rotation of the body appears clockwise looking along this direction.



Let P be any point in the body and d its distance from the axis. Then P has speed wd.

Let P have position vector $\underline{r}$ with respect to some point 0 on the axis. Then

$$d = |\underline{r}|\sin\theta$$

$$wd = |\underline{w}||\underline{r}|\sin\theta = |\underline{w}\times\underline{r}|$$

And the velocity $\underline{v}$ of P is given by $\underline{v} = \underline{w}\times\underline{r}$.

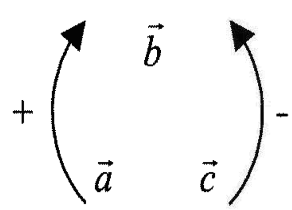
Products of three or more vectors follow naturally:

Consider the triple scalar product $\underline{a}\bullet(\underline{b}\times\underline{c})$. We observe that $\underline{a}\bullet(\underline{b}\times\underline{c})$ is a scalar.

Properties:

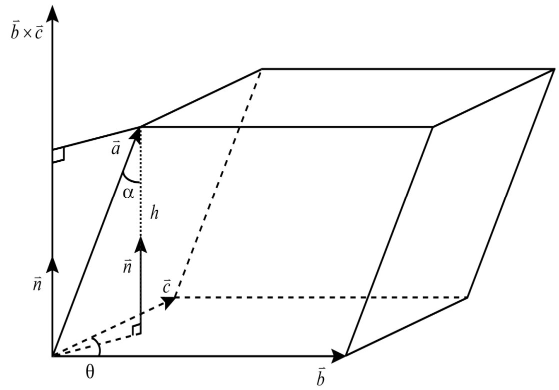
(i) $$\underline{a}\bullet(\underline{b}\times\underline{c}) = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k})\bullet\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(ii) $\underline{a}\bullet(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\bullet\underline{c}$ - property of determinant and the triple scalar product is usually written $(\underline{a}, \underline{b}, \underline{c})$



(iii) $(a,\ b,\ c) = -(b,\ a,\ c) = (a,\ b,\ c) = (b,\ c,\ a) = (c,\ a,\ b)$

(iv) Geometrically the absolute value of ($\underline{a}$, $\underline{b}$, $\underline{c}$) equals the volume of the parallelepiped with $\underline{a}$, $\underline{b}$ and $\underline{c}$ as adjacent edges.



Proof:

Observe that $\vec{b}\times\vec{c}=\left(\left|\vec{b}\right|\left|\vec{c}\right|\sin\theta\right)\vec{n}$, where $\vec{n}$ is a unit vector in the same direction as $\vec{b}\times\vec{c}$ such that $\vec{b},\vec{c},\vec{n}$ form a right-hand triple.

Area of the base of the parallelepiped $=\left|\vec{b}\times\vec{c}\right|=\left|\vec{b}\right|\left|\vec{c}\right|\sin\theta\ (>0)$.

Perpendicular height of the parallelepiped $= h=\left|\vec{a}\right|\cos\alpha=\left|\vec{a}\cdot\vec{n}\right|$.

$\therefore$ Volume of the parallelepiped = Base area of the parallelepiped$\times$its height

$$= \left|\vec{b}\times\vec{c}\right|\left|\vec{a}\cdot\vec{n}\right|=\left(\left|\vec{b}\right|\left|\vec{c}\right|\sin\theta\right)\left|\vec{a}\cdot\vec{n}\right|=\left|\vec{a}\cdot\left(\left|\vec{b}\right|\left|\vec{c}\right|\sin\theta\right)\vec{n}\right|=\left|\vec{a}\cdot\vec{b}\times\vec{c}\right|.$$

(v) Three vectors are coplanar *iff* their triple scalar product is zero.

Consider the <u>triple vector product</u> $\underline{a}\times(\underline{b}\times\underline{c})$. We observe that $\underline{a}\times(\underline{b}\times\underline{c})$ is a vector.

<u>Properties:</u>

(i) Note that $\underline{a}\times(\underline{b}\times\underline{c})\neq(\underline{a}\times\underline{b})\times\underline{c}$ in general e.g. $\underline{i}\times(\underline{j}\times\underline{j})=\underline{0}$ whereas $(\underline{i}\times\underline{j})\times\underline{j}=-\underline{i}$

(ii) $\underline{a}\times(\underline{b}\times\underline{c})=(\underline{a}\bullet\underline{c})\underline{b}-(\underline{a}\bullet\underline{b})\underline{c}$ (prove by expanding both sides in components – straightforward but tedious)

<u>Some Vector Identities:</u>

(a) $(\underline{a}\times\underline{b})\bullet(\underline{c}\times\underline{d})=(\underline{a}\bullet\underline{c})(\underline{b}\bullet\underline{d})-(\underline{a}\bullet\underline{d})(\underline{b}\bullet\underline{c})$

(b) $(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d})=(\underline{a},\underline{b},\underline{d})\underline{c}-(\underline{a},\underline{b},\underline{c})\underline{d}$

(c) $(\underline{a}\times\underline{b})\bullet(\underline{b}\times\underline{c})\times(\underline{c}\times\underline{a})=(\underline{a},\underline{b},\underline{c})^2$

Example

Prove identity (a) above.

$$\begin{aligned}(\underline{a}\times\underline{b})\bullet(\underline{c}\times\underline{d}) &= \underline{a}\bullet\left[\underline{b}\times(\underline{c}\times\underline{d})\right] && \text{triple scalar product of } \underline{a}, \underline{b},\ \underline{c}\times\underline{d}\\ &= \underline{a}\bullet\left[(\underline{b}\bullet\underline{d})\underline{c}-(\underline{b}\bullet\underline{c})\underline{d}\right] && \text{property (ii)}\\ &= (\underline{a}\bullet\underline{c})(\underline{b}\bullet\underline{d})-(\underline{a}\bullet\underline{d})(\underline{b}\bullet\underline{c})\end{aligned}$$

4. Linear Dependence and Independence

If $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k$ are any k n-component vectors, then an expression of the form $\sum_{i=1}^{k} m_i \underline{a}_i$, ($m_1, m_2, \ldots, m_k$ are any k scalars) is called a linear combination of $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k$.

$\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k$ is linearly dependent if at least one of the vectors can be represented as a linear combination of the others. Otherwise the set is linearly independent.

Examples

The vectors $\underline{a} = 3\underline{i} + 5\underline{j} - 2\underline{k}$, $\underline{b} = 4\underline{j} + 2\underline{k}$ and $\underline{c} = \underline{i} + \underline{j} - \underline{k}$ are linearly dependent since $\underline{a} = \frac{1}{2}\underline{b} + 3\underline{c}$

Hence vector $\underline{a}$ lies in the plane of the vectors $\underline{b}$ and $\underline{c}$. However, the vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ are linearly independent.

An equivalent definition is: A set of k n-components vectors is linearly independent *iff* $\sum_{i=1}^{k} m_i \underline{a}_i = \underline{0}$ implies $m_1 = m_2 = \cdots = m_k = 0$, that is, the vector equation $\sum_{i=1}^{k} m_i \underline{a}_i = \underline{0}$ has the trivial solution $m_1 = m_2 = \cdots = m_k = 0$ only.

Proof of equivalence :

Assume $m_p \neq 0$ for some $1 \le p \le k$, then $\sum_{i=1}^{k} m_i \underline{a}_i = \underline{0}$ *iff* $m_p \underline{a}_p = -\sum_{\substack{i=1\\ i\neq p}}^{k} m_i \underline{a}_i$

iff $\underline{a}_p = -\sum_{\substack{i=1\\ i\neq p}}^{k} \frac{m_i}{m_p} \underline{a}_i$ *iff* $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k\}$ is linearly dependent.

If two vectors in 3-D space are linearly dependent they must be collinear. If three vectors in 3-D space are linearly dependent they must either be collinear of coplanar. Hence three vectors form a linearly independent set *iff* their triple scalar product is not zero.

Four or more vectors in 3-D space will always be linearly dependent.

Example

If $\underline{a} = 3\underline{i} + 5\underline{j} - 2\underline{k}$, $\underline{b} = 4\underline{j} + 2\underline{k}$, $\underline{c} = \underline{i} + \underline{j} - \underline{k}$, $\underline{c} = \underline{i} + \underline{j} - \underline{k}$

$$(\underline{a},\underline{b},\underline{c}) = \begin{vmatrix} 3 & 5 & -2 \\ 0 & 4 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

And hence $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly dependent.

Example

Show that the four 4-component vectors, $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ in R^4 are linearly independent.

Proof:

Consider the vector equation $x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 = \underline{0}$, that is,

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2 = x_3 = x_4 = 0$$

Example

Given any four 3-component vectors, $\underline{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix}, \underline{v}_4 = \begin{pmatrix} v_{14} \\ v_{24} \\ v_{34} \end{pmatrix}$, show that they must be dependent.

Proof:

$$x_1 \underline{v}_1 + x_2 \underline{v}_2 + x_3 \underline{v}_3 + x_4 \underline{v}_4 = x_1 \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} + x_2 \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} + x_3 \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} + x_4 \begin{pmatrix} v_{14} \\ v_{24} \\ v_{34} \end{pmatrix} = \underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 v_{11} + x_2 v_{12} + x_3 v_{13} + x_4 v_{14} \\ x_1 v_{21} + x_2 v_{22} + x_3 v_{23} + x_4 v_{24} \\ x_1 v_{31} + x_2 v_{32} + x_3 v_{33} + x_4 v_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} v_{11} x_1 + v_{12} x_2 + v_{13} x_3 + v_{14} x_4 = 0 \\ v_{21} x_1 + v_{22} x_2 + v_{23} x_3 + v_{24} x_4 = 0 \\ v_{31} x_1 + v_{32} x_2 + v_{33} x_3 + v_{34} x_4 = 0 \end{cases}$$

$$\begin{cases} v_{11}x_1 + v_{12}x_2 + v_{13}x_3 + v_{14}x_4 = 0 \\ v_{21}x_1 + v_{22}x_2 + v_{23}x_3 + v_{24}x_4 = 0 \\ v_{31}x_1 + v_{32}x_2 + v_{33}x_3 + v_{34}x_4 = 0 \end{cases}$$ is a homogeneous system in unknowns x_1, x_2, x_3, x_4 and since there are more unknowns than equations, there must exist infinitely many solutions for x_1, x_2, x_3, x_4, thus, there must exist non-trivial solutions for x_1, x_2, x_3, x_4. It therefore follows that

$$\underline{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix}, \underline{v}_3 = \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix}, \underline{v}_4 = \begin{pmatrix} v_{14} \\ v_{24} \\ v_{34} \end{pmatrix}$$ must be linearly dependent.

In R^n, $n \geq 1$ there are *n* linearly independent *n*-component vectors , for instance, $e_1, \cdots, e_n$,

whereas any set of *n* + 1 or more *n*-component vectors is linearly dependent.

Consider $\{\underline{v}_1, \underline{v}_2, ..., \underline{v}_m\}$ in R^n, $n \geq 1$, if $\underline{v}_1, \underline{v}_2, ..., \underline{v}_m$ are linearly independent and each *n*-component vector $\underline{v} \in R^n$ is a linear combination of $\underline{v}_1, \underline{v}_2, ..., \underline{v}_m$ then $\{\underline{v}_1, \underline{v}_2, ..., \underline{v}_m\}$ is called a <u>basis</u> of

R^n, $n \geq 1$, for instance, $\{e_1, \cdots, e_n\}$ is a basis of R^n. Note, however, that a basis is not unique.

If $\{\underline{v}_1, \underline{v}_2, ..., \underline{v}_m\}$ is a basis of R^n, then $m = n$, that is, every basis of R^n contains *n* vectors and we say that R^n has <u>dimension *n*</u> .

$\{\underline{v}_1, \underline{v}_2, ..., \underline{v}_m\}$ in R^n is said to be <u>orthogonal</u> if $\underline{v}_i \bullet \underline{v}_j = 0$ if $i \neq j$. $\{\underline{v}_1, \underline{v}_2, ..., \underline{v}_m\}$ in R^n is

<u>orthonormal</u> if $\{\underline{v}_1, \underline{v}_2, ..., \underline{v}_m\}$ in R^n is <u>orthogonal</u> and $\|\underline{v}_i\|^2 = \underline{v}_i \bullet \underline{v}_i = 1$ for $i = 1, 2, ..., m$.

n-component non-zero vectors which are orthogonal are also linearly independent (can you prove this?) but the converse is not true (give an example).

<u>Example</u>

(i) For R^3, the dimension of R^3 is 3, as expected, and the 3 vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$, which are linearly independent, form an orthonormal basis for R^3. Any vector $\underline{v}$ may be written as a linear combination of $\underline{i}$, $\underline{j}$ and $\underline{k}$.

(ii) The vectors $\underline{i} + \underline{j}$, $2\underline{i} - \underline{j}$ and $\underline{k}$ also form a basis for R^3 since they are linearly independent and any vector in R^3 may be expressed as a linear combination of these vectors, eg. $-4\underline{i} + 5\underline{j} + 6\underline{k} = 2(\underline{i} + \underline{j}) - 3(2\underline{i} - \underline{j}) + 6\underline{k}$. However they are not useful in practice since they are not orthogonal.

(iii) The vectors $\underline{a} = 3\underline{i} + 5\underline{j} - 2\underline{k}$, $\underline{b} = 4\underline{j} + 2\underline{k}$ and $\underline{c} = \underline{i} + \underline{j} - \underline{k}$ of the previous example do not form a basis for R^3 since they are linearly dependent. The vector $\underline{d} = \underline{i} + \underline{j} + \underline{k}$, for example, cannot be expressed as a linear combination of $\underline{a}$, $\underline{b}$ and $\underline{c}$.

(iv) In R^n, the n vectors $\{e_1, e_2, \cdots, e_n\}$, that is, $\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$ are linearly independent and thus form a basis of R^n, called the <u>standard basis</u> and also they are orthonormal therefore form an orthonormal basis of R^n.