MA2001
Review Exercises for Chapter 2 Vector Integral Calculus

Line Integrals

1. Evaluate the following line integrals.

(a) $\int_C xy^2 ds$, where C is the semi-circle $x = -\sqrt{25 - y^2}$ from $(0,5)$ to $(0,-5)$.

Solution:

Parametrize C as $x = 5\cos t$ and $y = 5\sin t$ for $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

Then, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-5\sin t)^2 + (5\cos t)^2} = 5$

i.e. $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 5dt$

$$\int_C xy^2 ds = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (5\cos t)(5\sin t)^2 5dt = 625\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos t \sin^2 t dt$$

$$= 625\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin^2 t \, d\sin t = 625\left[\frac{\sin^3 t}{3}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{1250}{3}$$

(b) $\int_C xyds$, where C is the portion of the parabola $z = y^2$ in the plane from $x = 2$ from $(2,1,1)$ to $(2,3,9)$.

Solution:

Parametrize C as $x = 2$, $y = t$ and $z = t^2$ for $1 \le t \le 3$.

Then, $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(0)^2 + (1)^2 + (2t)^2} = \sqrt{1+4t^2}$

i.e. $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{1+4t^2}\, dt$

$$\int_C xyds = \int_1^3 2t\sqrt{1+4t^2}\, dt = \frac{1}{4}\int_1^3 \sqrt{1+4t^2}\, d\left(1+4t^2\right)$$

$$= \frac{1}{4}\int_1^3 \sqrt{1+4t^2}\, d\left(1+4t^2\right) = \frac{1}{4}\cdot\frac{2}{3}\left[\left(1+4t^2\right)^{\frac{3}{2}}\right]_1^3 = \frac{1}{6}\left(37\sqrt{37} - 5\sqrt{5}\right)$$

2. A wire is in the shape of a straight line segment from $(1,0,1)$ to $(2,-2,2)$. Find its mass if the density ρ at a point (x,y,z) is $\rho(x,y,z)=x^2+z^2$.

Solution:

The mass is given by $\int_C \rho(x,y,z)ds$, where C is the straight line segment from $(1,0,1)$ to $(2,-2,2)$.

Parametrize C as $x=z=1+t$, $y=-2t$ for $0\le t\le 1$.

Then, $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}=\sqrt{1^2+(-2)^2+1^2}=\sqrt{6}$

i.e. $ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}dt=\sqrt{6}dt$

$$\int_C \rho(x,y,z)ds = \int_0^1\left((1+t)^2+(1+t)^2\right)\sqrt{6}dt = 2\sqrt{6}\int_0^1(1+t)^2dt$$

$$=\frac{2\sqrt{6}}{3}\left[(1+t)^3\right]_0^1=\frac{14\sqrt{6}}{3}$$

3. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=(-y\sin z)\vec{i}+(x\sin z)\vec{j}+(xy\cos z)\vec{k}$, C is the circle cut from the cylinder $x^2+y^2=9$ by the plane $z=-2$. C is oriented clockwise as viewed above from the positive direction of the z-axis.

Solution:

Method 1

Note that the cylinder $x^2+y^2=9$ by the plane $z=-2$ intersect at the circle C: $\begin{cases}x^2+y^2=9\\ z=-2\end{cases}$.

Parametrize C as $x=3\cos t$, $y=-3\sin t$, $z=-2$ for $0\le t\le 2\pi$.

Then, $\frac{d\vec{r}(t)}{dt}=\frac{dx}{dt}\vec{i}+\frac{dy}{dt}\vec{j}+\frac{dz}{dt}\vec{k}=(-3\sin t)\vec{i}+(-3\cos t)\vec{j}$

i.e. $d\vec{r}(t)=[(-3\sin t)\vec{i}+(-3\cos t)\vec{j}]dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C[(-y\sin z)\vec{i}+(x\sin z)\vec{j}+(xy\cos z)\vec{k}]\cdot d\vec{r}$$

$$= \int_0^{2\pi}[(3\sin t\sin(-2))(-3\sin t)+(3\cos t\sin(-2))(-3\cos t)]dt$$

$$= \int_0^{2\pi}[9\sin^2 t\sin 2+9\cos^2 t\sin 2]dt$$

$$= 9\sin 2\int_0^{2\pi}dt=18\pi\sin 2$$

Method 2 (To refer to the Remark in P.4 of Notes Chapter 2)

Note that the cylinder $x^2+y^2=9$ by the plane $z=-2$ intersect at the circle C: $\begin{cases}x^2+y^2=9\\ z=-2\end{cases}$.

Parametrize C as $x=3\cos(-t+2\pi)=3\cos t$, $y=3\sin(-t+2\pi)=-3\sin t$, $z=-2$ for $0\le t\le 2\pi$.

Then, $\frac{d\vec{r}(t)}{dt}=\frac{dx}{dt}\vec{i}+\frac{dy}{dt}\vec{j}+\frac{dz}{dt}\vec{k}=(-3\sin t)\vec{i}+(-3\cos t)\vec{j}$

i.e. $d\vec{r}(t)=[(-3\sin t)\vec{i}+(-3\cos t)\vec{j}]dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(-y\sin z)\vec{i} + (x\sin z)\vec{j} + (xy\cos z)\vec{k}]\cdot d\vec{r}$$

$$= \int_0^{2\pi} [(3\sin t\sin(-2))(-3\sin t) + (3\cos t\sin(-2))(-3\cos t)]dt$$

$$= \int_0^{2\pi} [9\sin^2 t\sin 2 + 9\cos^2 t\sin 2]dt$$

$$= 9\sin 2\int_0^{2\pi} dt = 18\pi\sin 2$$

4. Find the work done by the force field $\vec{F}(x,y) = (x^2+xy)\vec{i} + (y-x^2y)\vec{j}$ in moving a particle along the curve $xy=1$ from $(1,1)$ to $\left(2,\frac{1}{2}\right)$ in the first quadrant.

Solution:

<u>*Method 1*</u>

The work done is $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve $xy=1$ from $(1,1)$ to $\left(2,\frac{1}{2}\right)$ in the first quadrant.

Parametrize C as $x=t$, $y=\frac{1}{t}$ for $1\le t\le 2$.

Then, $\frac{d\vec{r}(t)}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} = \vec{i} - \frac{1}{t^2}\vec{j}$

i.e. $d\vec{r}(t) = [\vec{i} - \frac{1}{t^2}\vec{j}]dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(x^2+xy)\vec{i} + (y-x^2y)\vec{j}]\cdot d\vec{r}$$

$$= \int_1^2 [(t^2+1) + \left(\frac{1}{t}-t\right)\left(-\frac{1}{t^2}\right)]dt = \int_1^2 [t^2+1-\frac{1}{t^3}+\frac{1}{t}]dt$$

$$= [\frac{t^3}{3}+t+\frac{1}{2t^2}+\ln t]_1^2 = [\frac{8}{3}+2+\frac{1}{8}+\ln 2]-[\frac{1}{3}+1+\frac{1}{2}]$$

$$= \frac{71}{24}+\ln 2$$

<u>*Method 2*</u>

The work done is $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve $xy=1$ from $(1,1)$ to $\left(2,\frac{1}{2}\right)$ in the first quadrant.

Then, $d\vec{r} = dx\vec{i} + dy\vec{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C [(x^2+xy)\vec{i} + (y-x^2y)\vec{j}]\cdot (dx\vec{i} + dy\vec{j})$$

$$= \int_1^2 \left(x^2 + x\left(\frac{1}{x}\right)\right)dx + \int_1^{\frac{1}{2}} \left(y - \left(\frac{1}{y}\right)^2 y\right)dy = \int_1^2 (x^2+1)dx + \int_1^{\frac{1}{2}} \left(y-\frac{1}{y}\right)dy$$

$$= [\frac{x^3}{3}+x]_1^2 + [\frac{y^2}{2}-\ln y]_1^{\frac{1}{2}} = \frac{71}{24}+\ln 2$$

5. Let $\vec{F}(x,y,z)=(2xy+az)\vec{i}+bx^2\vec{j}+(x+2z)\vec{k}$ be a vector field on $\mathbf{R}^3$, where a and b are real constants.

(a) Find the values of a and b such that $\vec{F}$ is conservative.

(b) With the values of a and b obtained in (a), evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the shorter path from $(5,0,0)$ to $(0,0,5)$ formed by the intersection of the surfaces $x+y+z=5$ and $x^2+y^2+z^2=25$.

Solution:

(a) Since $\vec{F}$ is conservative, we have $\nabla\times\vec{F}=\vec{0}$.

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xy+az & bx^2 & x+2z\end{vmatrix}$$

$$=\left(\frac{\partial(x+2z)}{\partial y}-\frac{\partial(bx^2)}{\partial z}\right)\vec{i}-\left(\frac{\partial(x+2z)}{\partial x}-\frac{\partial(2xy+az)}{\partial z}\right)\vec{j}+\left(\frac{\partial(bx^2)}{\partial x}-\frac{\partial(2xy+az)}{\partial y}\right)\vec{k}$$

$$=(1-a)\vec{j}+(2bx-2x)\vec{k}=\vec{0}$$

$\therefore\quad a=1$ and $b=1$.

(b) Note that $\vec{F}(x,y,z)=(2xy+z)\vec{i}+x^2\vec{j}+(x+2z)\vec{k}$ is conservative.

Let φ be a scalar field on $\mathbf{R}^3$ such that $\nabla\varphi=\vec{F}$, i.e.,

$$\frac{\partial\varphi}{\partial x}=2xy+z\ldots(1),\quad \frac{\partial\varphi}{\partial y}=x^2\ldots(2),\quad \frac{\partial\varphi}{\partial z}=x+2z\ldots(3).$$

From (1), $\varphi(x,y,z)=\int(2xy+z)dx=x^2y+xz+f(y,z)$, where f is a function to be determined.

Then, $\dfrac{\partial\varphi}{\partial y}=x^2+\dfrac{\partial f}{\partial y}$.

Equating this with the equality $\dfrac{\partial\varphi}{\partial y}=x^2$ gives $\dfrac{\partial f}{\partial y}=0$.

It follows that $f(y,z)=g(z)$ for a function g.

i.e. $\varphi(x,y,z)=x^2y+xz+g(z)$

$$\frac{\partial\varphi}{\partial z}=x+g'(z).$$

Comparing with (3), we have $g'(z)=2z$, so that

$g(z)=z^2+c$ (where c is an arbitrary constant.)

A potential function φ is

$\varphi(x,y,z)=x^2y+xz+z^2+c$.

$$\int_C\vec{F}\cdot d\vec{r}=\int_C\nabla\varphi\cdot d\vec{r}=\varphi(0,0,5)-\varphi(5,0,0)$$

$$=(25+c)-c=25$$