City University of Hong Kong,

EE3210 Signals & Systems

Lecture note
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Semester A, 2021/22

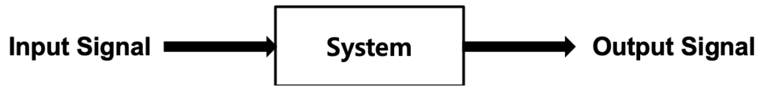
1 Signals and Systems

Major References:

- *Chapter 1, Signals and Systems by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall*
- *Chapter 1, Schaum's Outline of Signals and Systems, 2nd Edition, 2010, McGraw-Hill*

Introduction

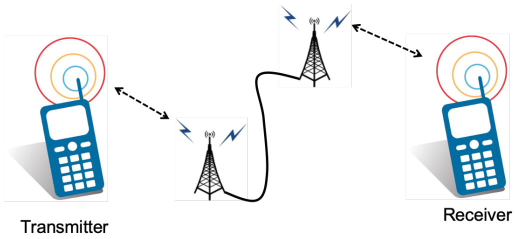
- **Objective**: Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



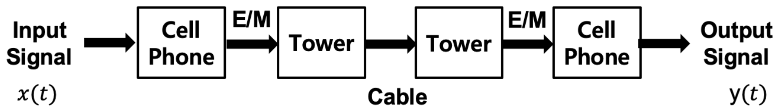
- **System level approach**
  - **Abstraction**
    - Identify the system input and output signals → characterize the signal types
    - Write input-output relation of the system → Operational transformations (e.g., Fourier analysis, Laplace Transformation).
    - Characterize the system types by the input-output relation → system types
  - **Modular design**
    - Break down the system into a number of interconnected subsystems (**module**)
    - Each module performs some specific task.
    - Focus on the **flow of signal**, abstract everything else away
  - **Composite system**
    - Determine the input-output relationship between each modules
    - Combine the components (**module**) to composite the overall system
    - **Component** and **composite systems** have the same form, and are analyzed with same methods.

Example 1.1

Let's consider a typical mobile communication between the transmitter and receiver. Abstract the system input and output signals, then determine the input-output relationship.



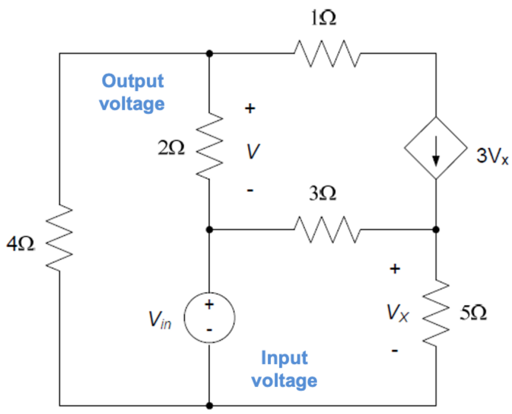
Sol) We first describe each module as a cascade of component systems.



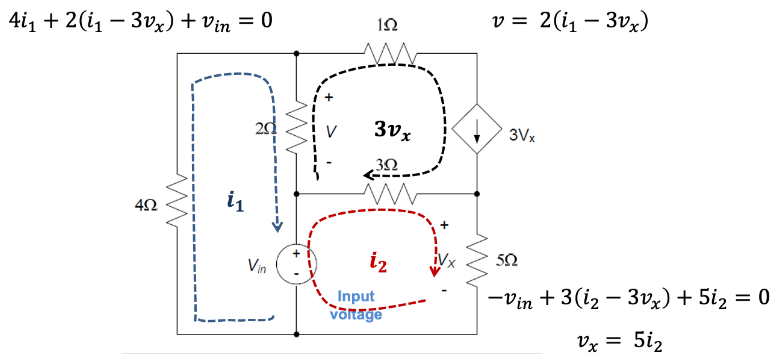
Then we combine the modules into a composite system.

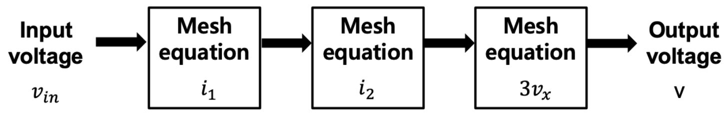
Example 1.2

Determine the input-output relationship of the following electrical circuit.

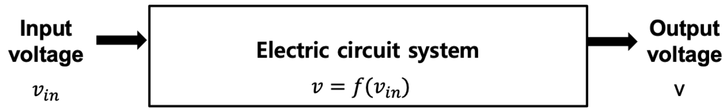


Sol) We first characterize the input and output signals, then perform modular design by breaking the circuit into several modular circuits





Then we combine the modules into a composite system



1.1 Classification of Signal Types

- **[Def]** **Signal** $x(t)$ is a function of an independent variable t representing a physical behavior of the phenomenon.

1. Continuous-Time (CT) and Discrete-Time (DT) Signals

A signal $x(t)$ is a **continuous-time signal** if t is a continuous variable. If t is a discrete variable, i.e., $x(t)$ is defined at discrete times, then $x(t)$ is a **discrete-time signal**.

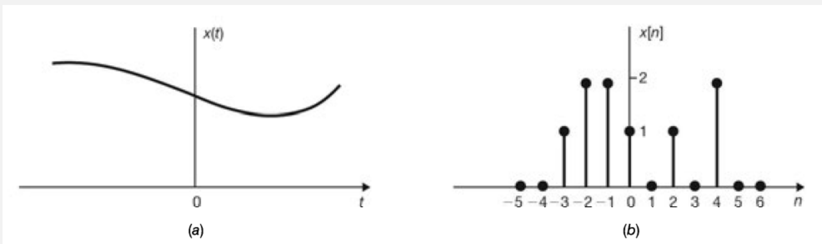


Fig. 1 (a) Continuous-time and (b) discrete-time signals

DT signal $x[n]$ may be obtained by *sampling* a CT signal $x(t)$ such as

$$x_n = x\,[n] = x\,(t_n)$$

Two representations of the DT signals

- Functional form. For example,

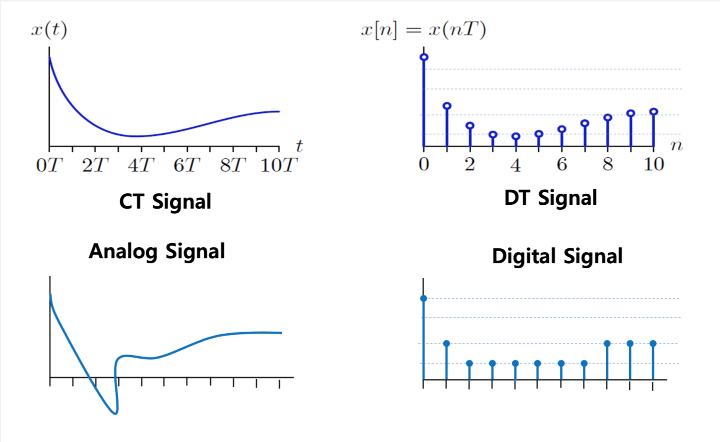
$$x_n = x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- Sequence form. For instance,

$$\{x_n\} = \{\ldots, 0, 0, 1, 2, 2, 1, 0, 1, 0, 2, 0, 0, \ldots\}$$

2. Analog and Digital Signals

A CT signal $x(t)$ is an **analog signal** if $x(t)$ can take on any value in the continuous interval (a, b). A DT signal $x[n]$ is a **digital signal** if $x[n]$ can take on only a finite number of distinct values.



3. Periodic and Aperiodic (or nonperiodic) Signals

A signal $x(t)$ (or $x[n]$) is **periodic** with period T (or N) if there is a positive non-zero value of T (or N) for which the following equality holds

$$\begin{aligned} x(t+T) &= x(t), \quad \text{for CT signal } x(t), \\ x[n+N] &= x[n], \quad \text{for DT signal } x[n] \end{aligned} \tag{1.1}$$

Any signal that is not periodic is called a **nonperiodic** (or **aperiodic**) signal.

- Property 1. For a given periodic signal, $\{T, 2T, 3T, \ldots, mT, \ldots\}$ are all available period at all t and any integer m.
- Property 2. **Fundamental period** T_0 is the smallest positive value of the period T for which (1.1) holds.

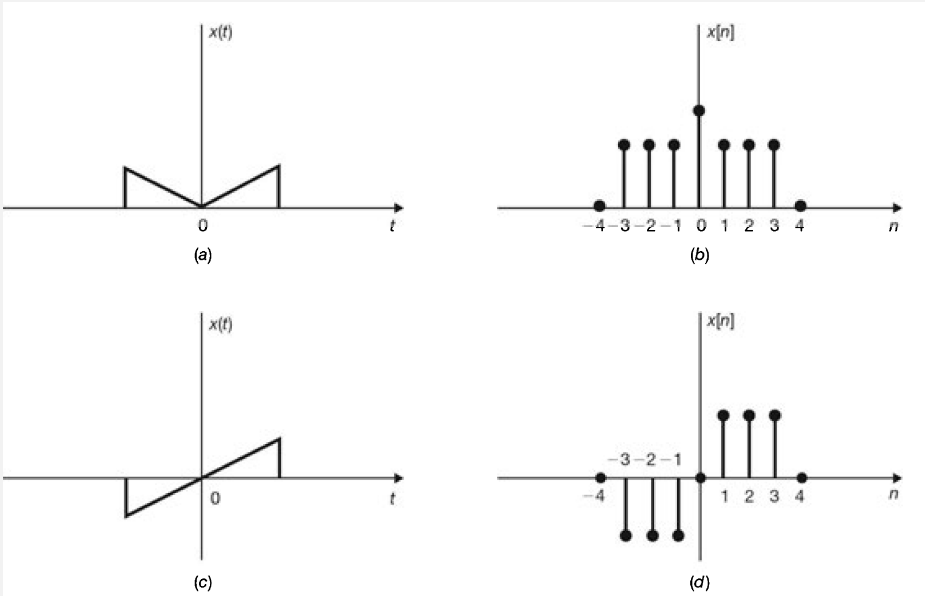
4. Even and Odd Signals

A signal is referred to as an **even signal** if

$$x(-t) = x(t) \quad \text{for CT signal } x(t), \quad x[-n] = x[n] \quad \text{for DT signal } x[n] \tag{1.2}$$

A signal is referred to as an **odd signal** if

$$x(-t) = -x(t) \quad \text{for CT signal } x(t), \quad x[-n] = -x[n] \quad \text{for DT signal } x[n] \tag{1.3}$$



- Property 1. Any signal $x(t)$ or $x[n]$ can be expressed as a sum of two signals

$$\begin{aligned} x(t) &= x_e(t) + x_o(t) \quad \text{for CT signal}, \\ x[n] &= x_e[n] + x_o[n] \quad \text{for DT signal}, \end{aligned} \tag{1.4}$$

  where $x_e(t)$ (or $x_e[n]$) is the even part, $x_o(t)$ (or $x_o[n]$) is the odd part, and these components are related to the original signal $x(t)$ (or $x[n]$) as follows

$$\begin{aligned} x_e(t) &= \frac{1}{2}[x(t) + x(-t)], \\ x_o(t) &= \frac{1}{2}[x(t) - x(-t)]. \end{aligned} \tag{1.5}$$

- Property 2. Product of signals
  - Even signal × Even signal = Even signal, Odd × odd = Even signal
  - Even signal × odd signal = odd signal

5. Energy and Power Signals

Energy E of a signal $x(t)$ (or $x[n]$) is defined as

$$\mathbf{E} = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{for CT signal,} \quad \mathbf{E} = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad \text{for DT signal,} \tag{1.6}$$

whereas the Power P of a signal is defined as follows

$$\mathbf{P} = \begin{cases} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt & \text{for CT signal,} \\ \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 & \text{for DT signal} \end{cases} \tag{1.7}$$

- **Energy signal** has finite energy and zero power

$$0 < \mathbf{E} < \infty, \quad \mathbf{P} = 0.$$

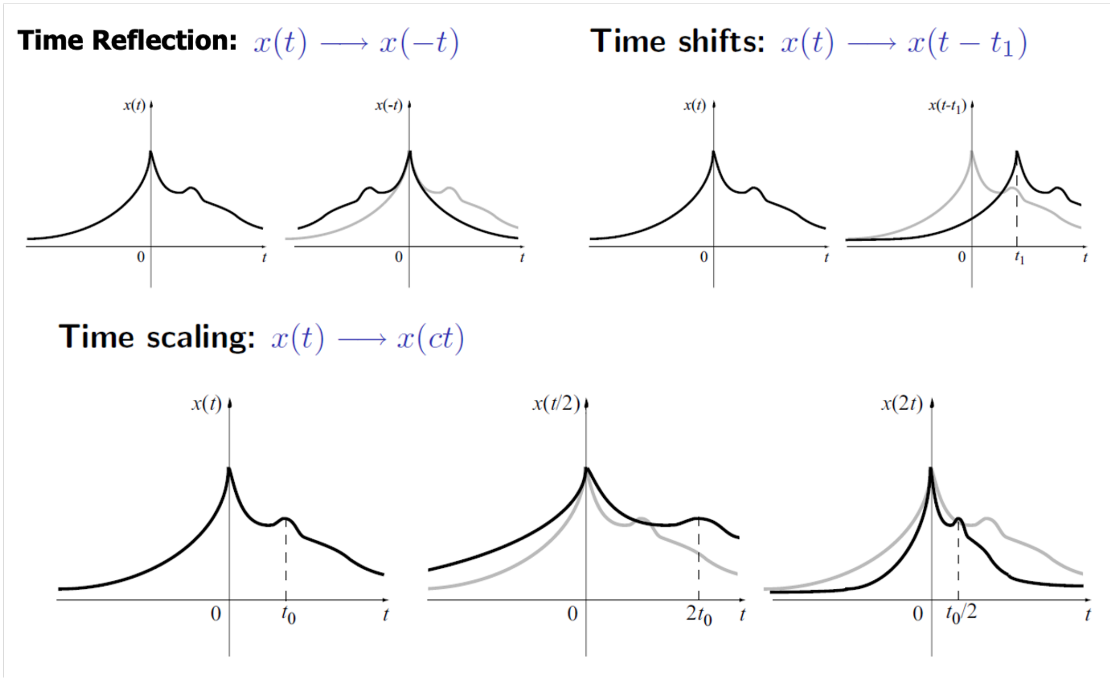
- **Power signal** has finite power and infinite energy

$$0 < \mathbf{P} < \infty, \quad \mathbf{E} = \infty.$$

- Signals that satisfy neither property are neither energy signals nor power signals.

1.2 Basic Signal Operations

- **Time Reversal**: Flip the signal around the vertical axis $x(t) \longrightarrow x(-t)$
- **Time Shifts**: Shift the signal to left or right $x(t) \longrightarrow x(t - t_0)$
  - **Right-shift** if $t_0 > 0$, **Left-shift** if $t_0 < 0$.
- **Time Scaling**: Linearly stretch or compress the signal $x(t) \longrightarrow x(ct)$
  - **Compression** if $|c| > 1$, **Expansion** if $|c| < 1$.
- **Affine Transformation**: $x(t) \longrightarrow x(\alpha t + \beta) = x(\alpha(t + \beta/\alpha))$ for any real α, β
  - Step 1. **Scale by** α. If $\alpha < 0$, reflection across y-axis
  - Step 2. **Shift by** $-\beta/\alpha$.
    - If α and β have different signs, right-shift.
    - If α and β have same signs, left shift.



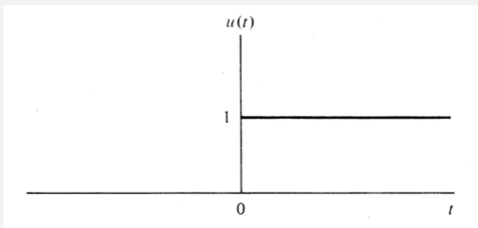
1.3 Example of Important Signals

1. **Unit Step Function** (also referred as *Heaviside unit function*)

- **Definition**

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0, \end{cases} \tag{1.8}$$

- **Properties**
  - Aperiodic signal
  - Power signal $\mathbf{P} = 1/2$
  - Infinite Energy $\mathbf{E} = \infty$



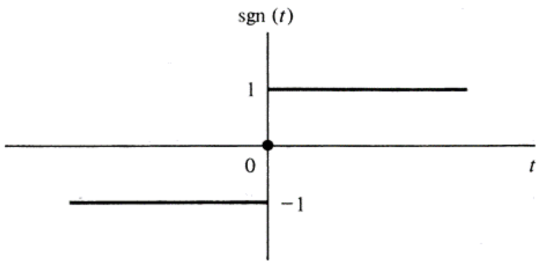
Functions related to the step function $u(t)$

a) **Signum Function**

- Definition

$$sgn(t) = 2u(t) - 1 = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

- Properties
  - Aperiodic & odd signal
  - Power signal $\mathbf{P} = 1$
  - Infinite Energy $\mathbf{E} = \infty$

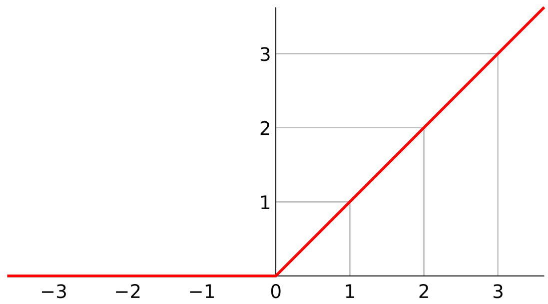


b) **Ramp Function**

- Definition

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}, \qquad \int_{-\infty}^{t} u(\tau)\, d\tau = r(t)$$

- Properties
  - Aperiodic
  - Infinite Power $\mathbf{P} = \infty$
  - Infinite Energy $\mathbf{E} = \infty$

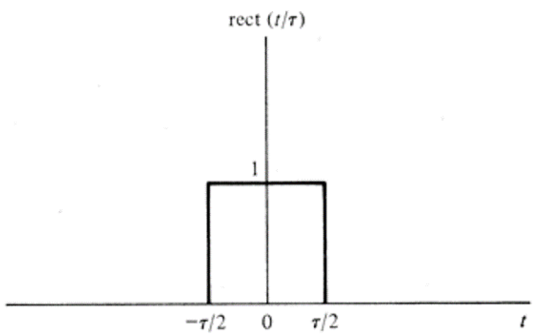


c) **Rectangular Pulse**

- Definition

$$rect(t/\tau) = \begin{cases} 1, & |t| < \frac{\tau}{2} \\ 0, & |t| > \frac{\tau}{2} \end{cases}, \qquad rect(t/\tau) = u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) \quad (1.9)$$

- Properties
  - Aperiodic & Even signal
  - Zero Power $\mathbf{P} = 0$
  - Energy Signal $\mathbf{E} = \tau$



2. **Unit Impulse Function** (also referred as *Direc delta function*)

- **Definition**

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{1.10}$$

- **Properties**
  - **Sampling Property**

$$x(t)\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

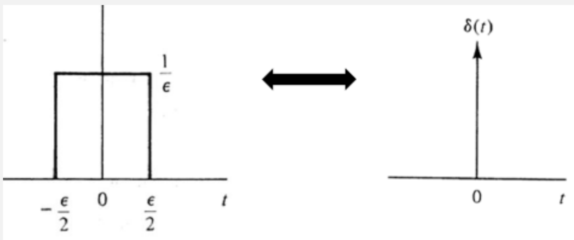
  - **Sifting Property**

$$\int_a^b x(t)\delta(t - t_0)\, dt = \begin{cases} x(t_0), & \text{if } a < t_0 < b \\ 0, & \text{otherwise} \end{cases}$$

  - Impulse function is the *building block of any signal*, *i.e.*, arbitrary signal can be respresented as an infinite sum of impulse function and signal amplitude.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t - \tau)\, d\tau \tag{1.11}$$

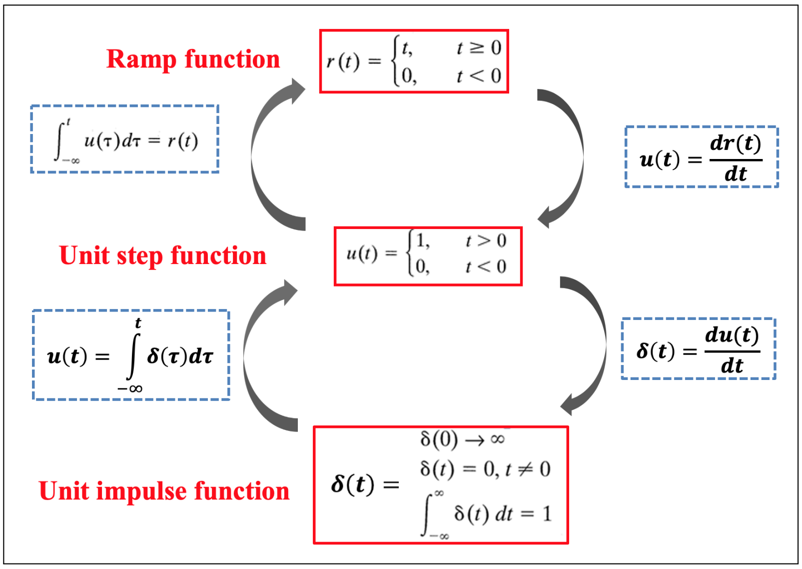
Relationship between Rectangular Pulse and Impulse Function



- $\delta_\epsilon(t) = \frac{1}{\epsilon} rect\left(\frac{t}{\epsilon}\right)$
- $\delta_\epsilon(0) = \frac{1}{\epsilon}$
- $\delta_\epsilon(t) = 0,\ |t| > \frac{\epsilon}{2}$
- $\int_{-\infty}^{\infty} \delta_\epsilon(t)\, dt = 1$
- $\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t)$
- $\delta(0) \to \infty$
- $\delta(t) = 0, t \neq 0$
- $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

Additional Properties of Unit impulse function

- Scaling Property: $\delta\left(at\right) = \frac{1}{|a|}\delta\left(t\right)$
- Even Function: $\delta\left(-t\right) = \delta\left(t\right)$
- Derivative and Integral: $u(t) = \int_{-\infty}^{t} \delta\left(\tau\right) d\tau, \quad \delta\left(t\right) = \dfrac{du\left(t\right)}{dt}$



3. **Complex Exponential Function**

- **Definition**

$$e^{jw_0 t} = \cos\left(w_0 t\right) + j\sin\left(w_0 t\right)$$

- **Properties**
  - Periodic with $T = \frac{2\pi n}{w_0}$ where n is an integer
  - Fundamental period $T_0 = \frac{2\pi}{|w_0|}$
  - Infinite Energy $\mathbf{E} = \infty$
  - Finite power $\mathbf{P} = \lim\limits_{T_0 \to \infty} \dfrac{1}{T_0} \displaystyle\int_0^{T_0} |e^{jw_0 t}|^2 dt = \lim\limits_{T_0 \to \infty} \dfrac{1}{T_0} \displaystyle\int_0^{T_0} 1 \cdot dt = 1$

4. **Sinusoidal Function**

$$A\cos\left(w_0 t + \theta\right) \quad \text{or} \quad A\sin\left(w_0 t + \theta\right),$$

where A is the *amplitude*, θ is the *phase angle*, w_0 is the *radian frequency* with

$$\text{Fundamental period } T_0 = \frac{2\pi}{w_0} \text{ (sec)}, \quad \text{Fundamental frequency } f_0 = \frac{1}{T_0} \text{ hertz (Hz)}$$

1.4 Classification of System Types

- **[Def]** A *system* is a mathematical model of a physical process that relates the *input signal* to the *output signal* in the form $y = Tx$.

1. Invertible and Noninvertible System

A system is said to be **invertible** if distinct inputs lead to distinct outputs. Otherwise, the system is said to be **noninvertible**.

[Examples]

Invertible System

- $y(t) = 2x(t) \leftrightarrow w(t) = \frac{1}{2}y(t)$
- $y[n] = \sum_{k=-\infty}^{n} x[k] \leftrightarrow w[n] = y[n] - y[n-1]$

Noninvertible System

- $y[n] = 0$
- $y(t) = x^2(t)$

2. Memory and Memoryless System

A system is said to be **memoryless** if the output at any time depends only on the input at that same time. Otherwise, the system is said to have **memory**.

[Examples]

Memoryless System

- $y(t) = Rx(t)$
- $y[n] = \left(2x[n] - x^2[n]\right)^2$
-

System with Memory

- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- $y[n] = x[n-1]$
- $y(t) = \frac{1}{c}\int_{-\infty}^{t} x(\tau)d\tau$

3. Causal and Noncausal System

A system is said to be **causal** if its output at the present time depends on only the present and/or past values of the input. If its output at the present time depends on future values of the input, the system is known as **noncausal**.

[Examples]

Causal System

- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- $y(t) = x^2(t)$

Noncausal System

- $y[n] = x[n] + x[n+2]$
- $y[n] = x[-n]$ or $y(t) = x(t+1)$

$*$ **Note)** All memoryless systems are causal, but not vice versa.

4. Linear and Nonlinear System

A system is said to be **linear** if the following superposition property (1.12) holds for a given operator $\mathbf{T}$. If the system does not satisfy (1.12), it is a **nonlinear system**.

$$\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 \mathbf{T}\{x_1\} + \alpha_2 \mathbf{T}\{x_2\} \tag{1.12}$$

[Examples]

Linear System

- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- $y(t) = tx(t)$

Nonlinear System

- $y(t) = x^2(t)$
- $y[n] = 2x[n] + 3$

$*$ **Note)** For a linear system, zero input always yields a zero output.

5. Time-invariant and Time-Varying System

A system is **time-invariant** if a time-shift of the input causes a corresponding shift in the output. In other words, the system response is independent of time.

$$\text{If } y(t) = \mathbf{T}\{x(t)\}, \text{ then } y(t - t_0) = \mathbf{T}\{x(t - t_0)\} \tag{1.13}$$

[Examples]

Time invariant System

- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- $y[n] = x[n - n_0]$

Time varying System

- $y(t) = x(2t)$
- $y[n] = nx[n]$

LTI System

Linear time-invariant (LTI) system: A system that is linear and also time-invariant.

6. Stable and Unstable System

A system is **stable** if every bounded input produces a bounded output for all time.

$$\text{If } |x(t)| < A, \text{ then } |y(t)| < B \text{ where } |A| < \infty, |B| < \infty \tag{1.14}$$

[Examples]

Stable System

- $y(t) = x^2(t)$
- $y[n] = x[n] + x[n+2]$

Unstable System

- $y[n] = \frac{1}{x[n]}$
- $y[n] = nx[n]$

1.5 Examples

[Example 1-1] Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$

b) $x(t) = \sin\left(\frac{2\pi t}{3}\right)$

c) $x(t) = \cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{4}\right)$

d) $x(t) = \cos(t) + \sin\left(\sqrt{2}t\right)$

e) $x(t) = \sin^2(t)$

f) $x(t) = e^{j\left[\frac{\pi}{2}t - 1\right]}$

g) $x(t) = \cos\left(2t + \frac{\pi}{4}\right)$

h) $x(t) = \cos^2(t)$

i) $x(t) = (\cos(2\pi t))\, u(t)$

j) $x(t) = e^{j\pi t}$

Solution) To solve this type of problem, try to find the minimum T that satisfy $x(t+T) = x(t)$. For instance, in (a), if the following equality holds with a nonzero constant T, then it is periodic

$$\cos\left(t + \frac{\pi}{4}\right) = \cos\left(t + T + \frac{\pi}{4}\right) \quad \longrightarrow \quad \cos(t') = \cos(t' + T), \tag{1.15}$$

where we used a *change of variable* $t' = t + \frac{\pi}{4}$ in the second equality. Since the minimum T that satisfy (1.15) is 2π, (a) is a periodic signal with period $T = 2\pi$. Similarly, for (b),

$$\sin\left(\frac{2\pi t}{3}\right) = \sin\left(\frac{2\pi t}{3} + \frac{2\pi T}{3}\right) \quad \longrightarrow \quad \frac{2\pi T}{3} = 2\pi, \tag{1.16}$$

and by denoting $t' = \frac{2\pi t}{3}$, the minimum T that satisfy (1.16) is 3.

For (c) and (d), we can use (1.18); The period T_1 for $\cos\left(\frac{\pi t}{3}\right)$ in (c) is $T_1 = 6$ and T_2 for $\sin\left(\frac{\pi t}{4}\right)$ is $T_2 = 8$. Since $T_1/T_2 = 3/4$, (c) is a periodic signal with period $T = 24$. In (d), the period T_1 for $\cos(t)$ is $T_1 = 2\pi$ and T_2 for $\sin\left(\sqrt{2}t\right)$ is $T_2 = \sqrt{2}\pi$. Since $T_1/T_2 = \sqrt{2}$, (d) is aperiodic signal.

For (e) and (h), convert $x(t)$ as follows, then apply similar approach as (a).

$$\cos^2(t) = \frac{1}{2}(1 + \cos(2t)), \quad \sin^2(t) = \frac{1}{2}(1 - \cos(2t)), \tag{1.17}$$

and the remaining can be solved using similar method. The solutions are summarized below.

a) Periodic with $T = 2\pi$

b) Periodic with $T = 3$

c) Periodic with $T = 24$

d) Aperiodic

e) Periodic with $T = \pi$

f) Periodic with $T = 4$

g) Periodic with $T = \pi$

h) Periodic with $T = \pi$

i) Aperiodic

j) Periodic with $T = 2$

Sum of Periodic Signals

- Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2, respectively. The sum $x(t) = x_1(t) + x_2(t)$ is periodic if and only if the following condition holds
$$\frac{T_1}{T_2} = \frac{k}{m} = \text{rational number} \tag{1.18}$$
where the fundamental period T is the least common multiple of T_1 and T_2.

- Let $x_1[n]$ and $x_2[n]$ be periodic sequence with fundamental periods N_1 and N_2, respectively. The sum $x[n] = x_1[n]+x_2[n]$ is periodic given the following condition
$$mN_1 = kN_2 = N \tag{1.19}$$
where the fundamental period N is the least common multiple of N_1 and N_2.
Refer [Schaum's text, Problem 1.14 & 1.15]

[Example 1-2] Determine whether the following signals are energy signals, power signals, or neither.

a) $x(t) = e^{-at}u(t), a > 0$

b) $x(t) = A\cos(\omega_0 t + \theta)$

Solution) To solve this type of problem, **(Step 1.)** you need to calculate the energy **E** first. If **E** is finite, the signal is a Energy signal. Otherwise, **(Step 2.)** if **E** is infinite, you need to calculate the power **P** as well. If **P** is finite, the signal is a Power signal. Otherwise, if **P** is infinite, then it is neither a energy nor a power signal. For example, in (a),

$$\mathbf{E} = \int_{-\infty}^{\infty} e^{-2at}u(t)dt = \int_{0}^{\infty} e^{-2at}dt = \frac{1}{2a}, \tag{1.20}$$

where we used the definition of the step function in the second equality. Since $\frac{1}{2a}$ is finite, $x(t)$ in (a) is a energy signal. For a periodic signal, the integration interval T in (1.7) is equal to the period. In (b), the period is $T = \frac{2\pi}{\omega_0}$ and the signal power can be calculated as follows

$$\begin{aligned}\mathbf{P} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2\cos^2(\omega_0 t + \theta)\,dt = \lim_{T\to\infty}\frac{A^2}{2\pi}\int_\theta^{2\pi+\theta}\cos^2(l)\,dl \\ &= \lim_{T\to\infty}\frac{A^2}{4\pi}\int_\theta^{2\pi+\theta}[1+\cos(2l)]\,dl = \frac{A^2}{2},\end{aligned} \tag{1.21}$$

where we used $T = \frac{2\pi}{\omega_0}$ and *a change of variable*, $l = \omega_0 t + \theta$ or $\omega_0 dt = dl$, in the second equality, then applied the Cosine rule $\cos^2(t) = \frac{1}{2}(1+\cos(2t))$ in the third equality. Since $\frac{A^2}{2}$ is finite, $x(t)$ in (b) is a power signal. In summary, the solutions are

a) Energy signal

b) Power signal

Definition of Energy and Power Signals

- **Energy signal** has finite energy and zero power, i.e., $0 < \mathbf{E} < \infty,\ \mathbf{P} = 0$
- **Power signal** has finite power and infinite energy, i.e., $0 < \mathbf{P} < \infty,\ \mathbf{E} = \infty$, where

$$\mathbf{E} = \int_{-\infty}^{\infty} |x(t)|^2 dt, \quad \mathbf{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

Properties of Periodic Signals

The following equalities hold for a periodic signal $x(t+T) = x(t)$

$$\int_{\alpha}^{\beta} x(t)dt = \int_{\alpha+T}^{\beta+T} x(t)dt, \quad \int_{0}^{T} x(t)dt = \int_{a}^{a+T} x(t)dt,$$

$$\mathbf{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T_0} \int_{0}^{T_0} |x(t)|^2 dt,$$

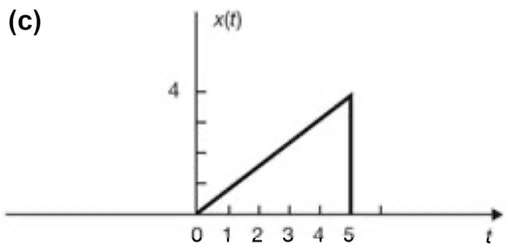
where T_0 is the fundamental period and α, β, a are arbitrary real valued constants.
Refer [Schaum's text, Problem 1.17 & 1.18]

[Example 1-3] Determine the even and odd component of the following signals

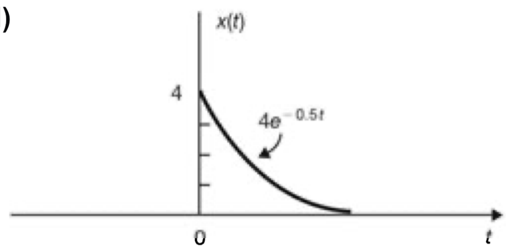
a) $x(t) = u(t)$

b) $x(t) = \sin\left(\omega_0 t + \frac{\pi}{4}\right)$

(c)



(d)



Solution) To solve this type of problem, you need to apply (1.22). In (a), $x(-t) = u(-t) = 1$ for $t < 0$ and $u(-t) = 0$ for $t > 0$. Then, the following results can be derived

$$x_e(t) = \frac{1}{2}[u(t) + u(-t)] = \frac{1}{2},$$

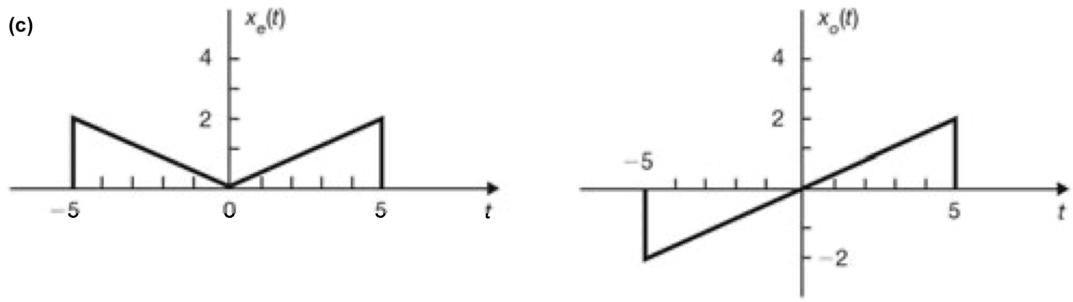
$$x_o(t) = \frac{1}{2}[u(t) - u(-t)] = \frac{1}{2} sgn(t) = \begin{cases} 0.5, & t > 0, \\ -0.5, & t < 0 \end{cases}$$

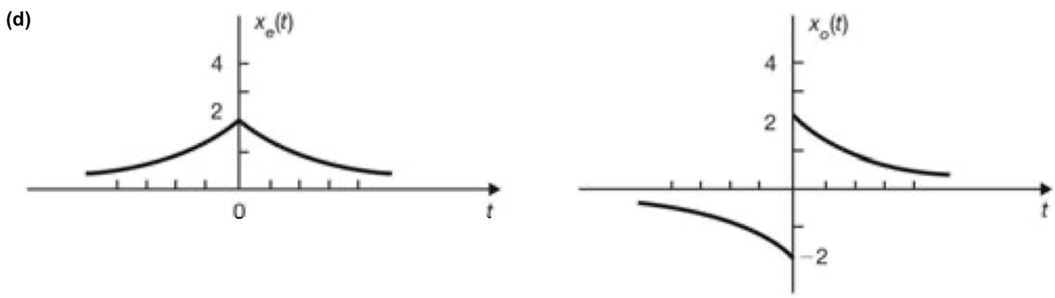
In (b), we first use Sine rule to expand the Sine function, then the following reults can be derived.

$$\sin\left(\omega_0 t+\frac{\pi}{4}\right)=\sin\left(\omega_0 t\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\omega_0 t\right)\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\left(\sin\left(\omega_0 t\right)+\cos\left(\omega_0 t\right)\right).$$

$$x_e\left(t\right)=\frac{1}{\sqrt{2}}\cos\left(\omega_0 t\right),\quad x_o\left(t\right)=\frac{1}{\sqrt{2}}\sin\left(\omega_0 t\right).$$

Similarly, the even and odd component of (c) and (d) can be found as follows





Even and Odd Component

Any signal $x(t)$ can be expressed as a sum of two signals

$$x\left(t\right)=x_e\left(t\right)+x_o\left(t\right),$$

where $x_e\left(t\right)$ and $x_o\left(t\right)$ are related to the original signal $x(t)$ as follows

$$x_e\left(t\right)=\frac{1}{2}\left[x\left(t\right)+x\left(-t\right)\right],\quad x_o\left(t\right)=\frac{1}{2}\left[x\left(t\right)-x\left(-t\right)\right]. \tag{1.22}$$

[Example 1-4] **[Part 1]** Sketch the following signals.

a) $x_1(t)=u(t)+5u\left(t-1\right)-2u\left(t-2\right)$

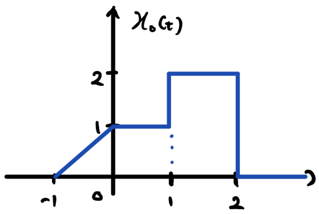
b) $x_2(t)=r(t)-r\left(t-1\right)-u\left(t-2\right)$

c) $x_3(t)=u(t)u\left(a-t\right),\ a>0$

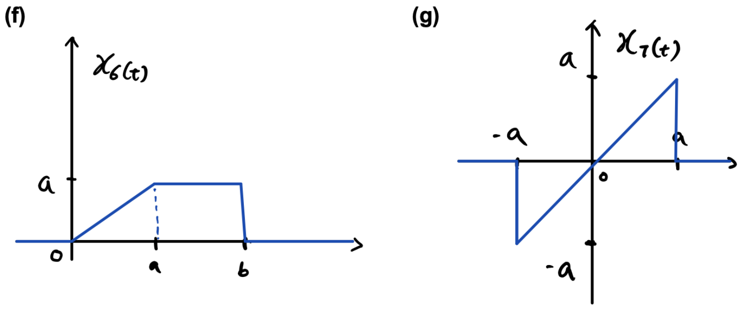
d) $x_4(t)=x_0(t)u(1-t)$

e) $x_5(t) = x_0(t)\,[u(t) - u\,(t-1)]$

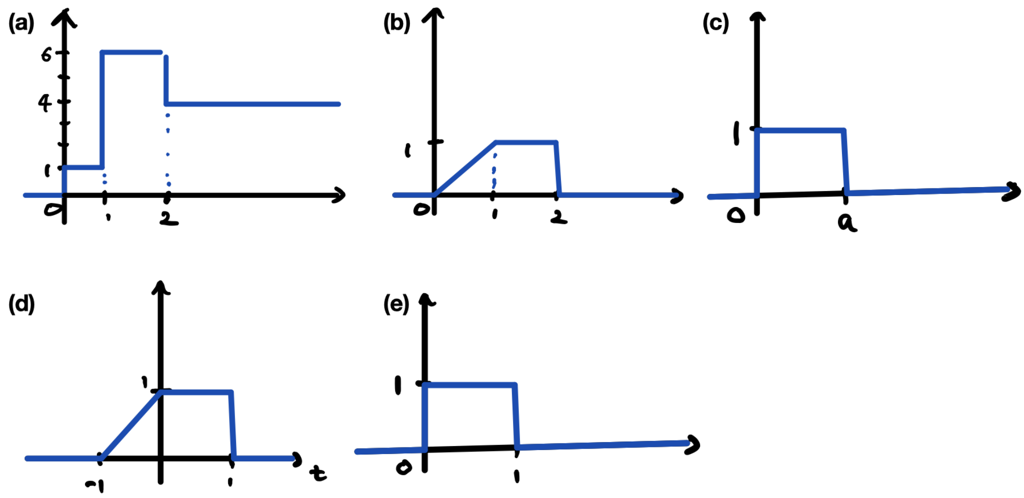
where the signal $x_0(t)$ is plotted below.



[Part 2] For each of the signals plotted below, write an expression in terms of unit step and unit ramp functions.



Solution)[Part 1]



[Part 2]

(f) $x_6(t) = r(t) - r\,(t-a) - au\,(t-b)$, (g) $x_7(t) = (r(t) - r\,(-t))\,(u\,(t+a) - u\,(t-a))$

- **Time Reversal**: Flip the signal around the vertical axis $x(t) \longrightarrow x(-t)$
- **Time Shifts**: Shift the signal to left or right $x(t) \longrightarrow x(t-t_0)$
  - **Right-shift** if $t_0 > 0$, **Left-shift** if $t_0 < 0$.
- **Time Scaling**: Linearly stretch or compress the signal $x(t) \longrightarrow x(ct)$
  - **Compression** if $|c| > 1$, **Expansion** if $|c| < 1$.

[Example 1-5] Evaluate the following integrals.

a) $\int_{-\infty}^{t} \cos(\tau)\, u(\tau) d\tau$

b) $\int_{-\infty}^{t} \cos(\tau)\, \delta(\tau) d\tau$

c) $\int_{-\infty}^{\infty} \cos(t) u(t-1)\, \delta(t) dt$

d) $\int_{0}^{2\pi} t \sin\left(\frac{t}{2}\right) \delta(t-\pi)$

e) $\int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1)\, dt$

f) $\int_{-3}^{2} \left[\exp(1-t) + \sin\left(\frac{2\pi t}{3}\right)\right] \delta\left(t - \frac{3}{2}\right) dt$

Solution)

(a)

$$\int_{-\infty}^{t} \cos(\tau)\, u(\tau) d\tau = \begin{cases} \text{If } t > 0, \int_0^t \cos(\tau)\, d\tau = \sin(t) \\ \text{If } t < 0, 0 \end{cases} = u(t)\sin(t)$$

(b)

$$\int_{-\infty}^{t} \cos(\tau)\, \delta(\tau) d\tau = \begin{cases} \text{If } t > 0, \cos 0 = 1 \\ \text{If } t < 0, 0 \end{cases} = u(t)$$

(c)

$$\int_{-\infty}^{\infty} \cos(t) u(t-1)\, \delta(t) dt = \cos(0) u(-1) = 0$$

(d)

$$\int_{0}^{2\pi} t \sin\left(\frac{t}{2}\right) \delta(t-\pi) = \pi \sin\left(\frac{\pi}{2}\right) = \pi$$

(e)

$$\int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1)\, dt = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$$

(f)

$$\int_{-3}^{2} \left[\exp(1-t) + \sin\left(\frac{2\pi t}{3}\right)\right] \delta\left(t - \frac{3}{2}\right) dt = \exp\left(-\frac{1}{2}\right) + \sin(\pi) = \exp(-0.5)$$

Properties of Unit impulse function

- $\int_a^b x(t)\delta(t-t_0)\,dt = \begin{cases} x(t_0), & \text{if } a < t_0 < b \\ 0, & \text{otherwise} \end{cases}$
- $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$
- $\delta(at) = \frac{1}{|a|}\delta(t), \quad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau, \quad \delta(t) = \frac{du(t)}{dt}$

[Example 1-6] Determine whether the following system is (*i*) memoryless, (*ii*) causal, (*iii*) linear, (*iv*) time-invariant, or (*v*) stable. *Refer [Schaum's text, Problem 1.33, 1.34, 1.36, 1.38]*

a) $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$

b) $y(t) = x(t)\cos(\omega_0 t)$

c) $y[n] = x[n-1]$

d) $y[n] = nx[n]$

Solution) In (a), the output depends on the past input, so it is not memoryless system. The output depends on the present and past values of the input, so it is a Causal system. To test linearlity, substitute $x(t) \leftarrow \alpha_1 x_1(t) + \alpha_2 x_2(t)$ as the input, where $y_1(t)$ and $y_2(t)$ is the corresponding output of $x_1(t)$ and $x_2(t)$, respectively. Then,

$$\begin{aligned} y(t) &= \frac{1}{C}\int_{-\infty}^{t} [\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)]\,d\tau \\ &= \alpha_1 \left[\frac{1}{C}\int_{-\infty}^{t} x_1(\tau)\,d\tau\right] + \alpha_2 \left[\frac{1}{C}\int_{-\infty}^{t} x_2(\tau)\,d\tau\right] = \alpha_1 y_1(t) + \alpha_2 y_2(t), \end{aligned}$$

so the superposition property holds, which indicates a linear system. To test time-invariance, input time shifted signal $x(t-t_0)$. If the corresponding output is $y(t-t_0)$, then it is a time invariant system.

$$\frac{1}{C}\int_{-\infty}^{t} x(\tau - t_0)\,d\tau = \frac{1}{C}\int_{-\infty}^{t-t_0} x(l)\,dl = y(t-t_0),$$

by using *a change of variable* $l = \tau - t_0$ in the first equality. Hence, it is a time-invariant system. For stability, (a) can be easily proved to be a unstable by substituting a unit step function $x(t) = u(t)$ as the input, which achieves unbounded $y(t) = \frac{tu(t)}{C}$. The remaining can be proved using similar method. The solutions are summarized below.

a) memory, causal, linear, time-invariant, unstable

b) memoryless, causal, linear, time-variant, stable

c) memory, causal, linear, time-invariant, stable

d) memoryless, causal, linear, time-variant, unstable.

System Characterization

1. **Memoryless System**; output at any time depends only on the input at that same time
2. **Causal System**; output at the present time depends only on the present and/or past input values
3. **Linear System**; the superposition property holds, i.e., $\mathrm{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 \mathrm{T}\{x_1\} + \alpha_2 \mathrm{T}\{x_2\}$
4. **Time-invariant System**; time-shift of the input causes a same amount of shifting in the output
5. **Stable System**; If $|x(t)| < A$, then $|y(t)| < B$ where $|A| < \infty, |B| < \infty$
6. **LTI System**; A system that is linear and also time-invariant

2 Linear Time-Invariant Systems

Major References:

- *Chapter 2, Signals and Systems by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall*
- *Chapter 2, Schaum's Outline of Signals and Systems, 2nd Edition, 2010, McGraw-Hill*

2.1 Convolution

2.1.1 Convolution Integral of CT Signal

1. **Definition**
   Convolution Integral of two continuous-time signals $x(t)$ and $y(t)$ is defined by

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau. \tag{2.1}$$

   Convolution $x(t) \; * \; y(t)$ represents the degree to which $x\&y$ overlap at t as y sweeps across the domain t.

   Step. 1) $y(\tau)$ is time-reversed, then shifted by t; $y(\tau) \longrightarrow y(-\tau) \longrightarrow y(t-\tau)$

   Step. 2) $x(\tau)$ and $y(t-\tau)$ are multiplied, then integrated over τ

   Step. 3) Convolution will remain zero as long as $x\&y$ do not overlap

   Step. 4) Sweep $y(t-\tau)$ from $t = -\infty$ to $t = \infty$ to produce the entire output

2. **Properties of the Convolution Integral**
   The convolution integral has the following properties. Refer [Schaum's text, Problem 2.1] for the proof.
   a) Commutative
   $$x(t) * y(t) = y(t) * x(t)$$
   b) Associative
   $$\{x(t) * y_1(t)\} * y_2(t) = x(t) * \{y_1(t) * y_2(t)\}$$
   c) Distributive
   $$x(t) * \{y_1(t) + y_2(t)\} = x(t) * y_1(t) + x(t) * y_2(t)$$

3. **Additional Properties**
   Refer [Schaum's text, Problem 2.2, 2.8] for the proof.
   a) $x(t) * \delta(t) = x(t)$
   b) $x(t) * \delta(t-t_0) = x(t-t_0)$
   c) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$
   d) $x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau) d\tau$
   e) If $x(t)$ and $y(t)$ are periodic signals with a common period T, the convolution in (2.1) does not converge. Instead, we define the *periodic convolution* $f(t) = x(t) \circledast y(t)$, where $f(t)$ is periodic with period T.

$$\begin{aligned} f(t) = x(t) \circledast y(t) &= \int_{0}^{T} x(\tau)\, y(t-\tau)\, d\tau \\ &= \int_{a}^{a+T} x(\tau)\, y(t-\tau)\, d\tau \quad \text{for arbitrary } a \end{aligned} \tag{2.2}$$

2.1.2 Convolution Sum of DT Signal

1. **Definition**
   Convolution Sum of two discrete-time sequence $x[n]$ and $y[n]$ is defined by

$$z[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k] \tag{2.3}$$

Step. 1) $y[k]$ is time-reversed, then shifted by n; $y[k] \to y[-k] \to y[n-k]$
Step. 2) $x[k]$ and $y[n-k]$ are multiplied, then summed over all k
Step. 3) Convolution will remain zero as long as $x\&y$ do not overlap
Step. 4) Sweep $y[n-k]$ from $n = -\infty$ to $n = \infty$ to produce the entire output

2. **Properties of the Convolution Sum**
   The convolution sum has the following properties. Refer [Schaum's text, Problem 2.26] for the proof.
   a) Commutative
   $$x[n] * y[n] = y[n] * x[n]$$
   b) Associative
   $$\{x[n] * y_1[n]\} * y_2[n] = x[n] * \{y_1[n] * y_2[n]\}$$
   c) Distributive
   $$x[n] * \{y_1[n] + y_2[n]\} = x[n] * y_1[n] + x[n] * y_2[n]$$

3. **Additional Properties**
   Refer [Schaum's text, Problem 2.27, 2.31] for the proof.
   a) $x[n] * \delta[n] = x[n]$
   b) $x[n] * \delta[n-n_0] = x[n-n_0]$
   c) $x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$
   d) $x[n] * u[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$
   e) If $x[n]$ and $y[n]$ are periodic sequence with a common period N, the convolution in (2.3) does not converge. Instead, we define the *periodic convolution* $f[n] = x[n] \circledast y[n]$, where $f[n]$ is periodic with period N.

$$f[n] = x[n] \circledast y[n] = \sum_{k=0}^{N-1} x[k]\, y[n-k] \tag{2.4}$$

[Example 2-1] Evaluate the following convolutions

1. $u(t+a) * u(t+b)$
2. $rect(t/\tau) * rect(t/\tau)$
3. $rect(t/\tau) * u(t)$
4. $x(t) * y(t)$ where $x(t) = \begin{cases} 1 & \text{for } 0 < t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $y(t) = \begin{cases} 1 & \text{for } 0 < t < 2 \\ 0 & \text{otherwise} \end{cases}$
5. $rect(t/\tau) * \delta_T(t)$ where $\tau < T$ and $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ is the *unit impulse train*

Refer [Schaum's text, Problem 2.6, 2.7, 2.8]

Solution) Example 2-1. 1) The convolution integral is given by

$$
\begin{aligned}
u(t+a) * u(t+b) &= \int_{-\infty}^{\infty} u(\tau + a)\, u(-\tau + b + t)\, d\tau = (t+a+b)\, u(t+a+b) \\
&= \begin{cases} \displaystyle\int_{-a}^{b+t} 1 \cdot d\tau = (t+a+b) & \text{if } t+a+b > 0 \\ 0 & \text{if } t+a+b < 0 \end{cases}
\end{aligned} \tag{2.5}
$$

where the product of two step functions $u(\tau + a)\, u(-\tau + b + t)$ has a non-zero value at $\tau + a > 0$ and $-\tau + b + t > 0$. As shown in Fig. 2.1, if $-a < b + t$, then the product of two step functions overlap each other within the interval $-a < \tau < b + t$. If $b + t < -a$, there is no overlap and the integral in (2.5) becomes zero.

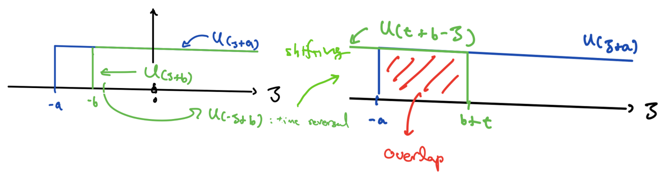


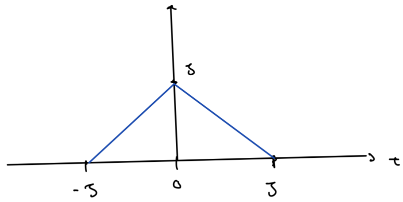
Figure 2.1:

Example 2-1. 2) The convolution can be expanded by expressing the rectangular pulse signal via the step fucntion

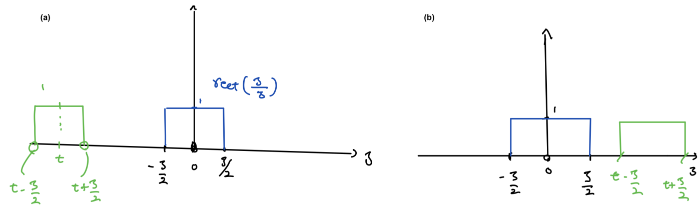
$$
\begin{aligned}
rect(t/\tau) * rect(t/\tau) &= \left\{u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)\right\} * \left\{u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)\right\} \\
&= (t+\tau)\, u(t+\tau) - 2tu(t) + (t-\tau)\, u(t-\tau),
\end{aligned} \tag{2.6}
$$

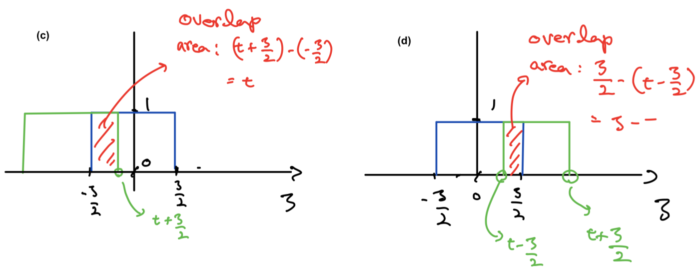
where we used (1.9) in the first equality and the result from [Example 2-1. 1] in the second equality. The last expression has four separate intervals with different values, which is a *triangular pulse signal* with maximum magnitude τ at $t = 0$ and width 2τ (two times larger than the rectangular pulse signal $rect(t/\tau)$).

$$
\begin{cases} 0 & \text{if } t < -\tau \\ t+\tau & \text{if } -\tau < t < 0 \\ t+\tau-2t = \tau - t & \text{if } 0 < t < \tau \\ \tau - t + t - \tau = 0 & \text{if } t > \tau \end{cases}
$$



(b) can also be solved using the direct definition of the convolution where we multiply one original signal to another signal that is time reversed, then time-shifted by t, which is plotted below.





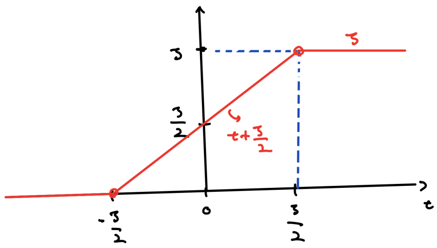
- For sub-figure (a) and (b), there is no overlap between two signals, hence the convolution is zero. These cases correspond to the condition $t+\frac{\tau}{2} < -\frac{\tau}{2}$ and $t-\frac{\tau}{2} > \frac{\tau}{2}$, *i.e.*, $t < -\tau$ for sub-figure (a) and $t > \tau$ for (b).
- For sub-figure (c), if $t+\frac{\tau}{2} > -\frac{\tau}{2}$ and $t-\frac{\tau}{2} < -\frac{\tau}{2}$, the overlap area is $t+\frac{\tau}{2}-\left(-\frac{\tau}{2}\right) = t$, which is the convolution in the interval $-\tau < t < 0$.
- For sub-figure (d), if $t-\frac{\tau}{2} < \frac{\tau}{2}$ and $t+\frac{\tau}{2} > \frac{\tau}{2}$, the overlap area is $\frac{\tau}{2}-\left(t-\frac{\tau}{2}\right) = \tau - t$, which is the convolution in the interval $0 < t < \tau$.

Example 2-1. 3) The convolution can be expanded in terms of the step fucntion as follows

$$rect\,(t/\tau) * u(t) = \left\{u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)\right\} * u(t) = \left(t+\frac{\tau}{2}\right)u\left(t+\frac{\tau}{2}\right) - \left(t-\frac{\tau}{2}\right)u\left(t-\frac{\tau}{2}\right), \tag{2.7}$$

where we applied [Example 2-1. 1] in the second equality. The last expression has three separate intervals with different values as follows

$$\begin{cases} 0 & \text{if } t < -\dfrac{\tau}{2} \\ t+\dfrac{\tau}{2} & \text{if } -\dfrac{\tau}{2} < t < \dfrac{\tau}{2} \\ t+\dfrac{\tau}{2}-\left(t-\dfrac{\tau}{2}\right) = \tau & \text{if } t > \dfrac{\tau}{2} \end{cases}$$

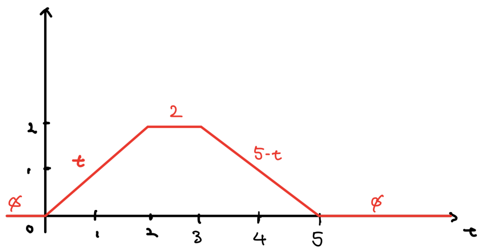


Example 2-1. 4) The convolution can be expanded in terms of the step fucntion as follows

$$\begin{aligned} x(t) * y(t) &= \{u\,(t) - u\,(t-3)\} * \{u\,(t) - u\,(t-2)\} \\ &= tu(t) - (t-3)\,u\,(t-3) - (t-2)\,u\,(t-2) + (t-5)\,u\,(t-5)\,, \end{aligned} \tag{2.8}$$

where we applied [Example 2-1. 1] in the second equality. The last expression has five separate intervals with different values as follows

$$\begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 < t < 2 \\ t-(t-2) = 2 & \text{if } 2 < t < 3 \\ 2-(t-3) = 5-t & \text{if } 3 < t < 5 \\ 5-t-(t-5) = 0 & \text{if } t > 5 \end{cases}$$



Example 2-1. 5)

$$rect\,(t/\tau) * \left[\sum_{n=-\infty}^{\infty} \delta\,(t-nT)\right] = \sum_{n=-\infty}^{\infty} rect\,(t/\tau) * \delta\,(t-nT) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT}{\tau}\right) \tag{2.9}$$

where we used distributive property in the second equality and $x(t) * \delta\,(t-t_0) = x\,(t-t_0)$ in the last equality.

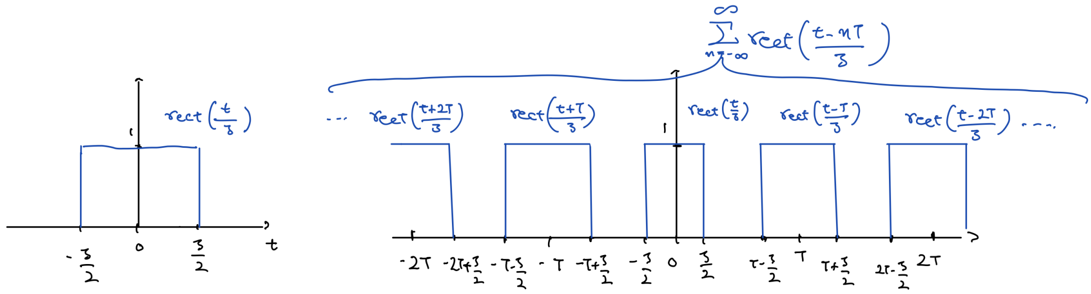


Figure 2.2:

As shown in Fig. 2.2, the time-shifted rectangular pulse signal do not overlap each other as far as $\tau < T$ is satisfied. However, if $\tau > T$, then the time-shifted rectangular pulse signals will overlap each other.

Important Convolution Pairs

1. $u\,(t+a) * u\,(t+b) = (t+a+b)\,u\,(t+a+b)$
2. $rect\,(t/\tau) * rect\,(t/\tau) = \begin{cases} 0 & \text{if } t < -\tau \\ \tau + t & \text{if } -\tau < t < 0 \\ \tau - t & \text{if } 0 < t < \tau \\ 0 & \text{if } t > \tau \end{cases}$

[Example 2-2] Evaluate the following convolutions

1. $x(t) * y(t)$ where $x(t) = u(t)$ and $y(t) = e^{-\alpha t}u(t)$, $\alpha > 0$
2. $x(t) * y(t)$ where $x(t) = e^{-\alpha t}u(t)$ and $y(t) = e^{\alpha t}u\,(-t)$, $\alpha > 0$

Refer [Schaum's text, Problem 2.4, 2.5]

Solution) Example 2-2. 1)

$$x(t) * y(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}u\,(\tau)\,u\,(t-\tau)\,d\tau = \begin{cases} \int_0^t e^{-\alpha\tau}d\tau = \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right), & \text{if } t > 0 \\ 0, & \text{if } t < 0 \end{cases} \tag{2.10}$$

$$= \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)u(t)$$

Example 2-2. 2)

$$x(t) * y(t) = \int_{-\infty}^{\infty} e^{\alpha\tau}u\,(-\tau)\,e^{-\alpha(t-\tau)}u\,(t-\tau)\,d\tau \tag{2.11}$$

where the product of two step functions $u\,(-\tau)\,u\,(t-\tau)$ is determined by the magnitude of t as shown in Fig. 2.3. Then (2.11) can be derived as follows

$$\begin{cases} e^{-\alpha t}\int_{-\infty}^{0} e^{2\alpha\tau}d\tau = \frac{1}{2\alpha}e^{-\alpha t}, & \text{if } t > 0 \\ e^{-\alpha t}\int_{-\infty}^{t} e^{2\alpha\tau}d\tau = \frac{1}{2\alpha}e^{\alpha t}, & \text{if } t < 0 \end{cases} \Rightarrow \quad \frac{1}{2\alpha}e^{-\alpha|t|} \tag{2.12}$$

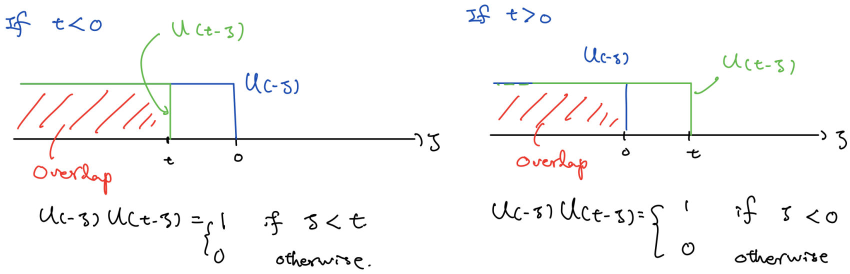


Figure 2.3:

2.2 LTI System Response

Linear Time-Invariant (LTI) System, represented by $\mathbf{T}\{\cdot\}$, satisfy the following two attributes.

- Linearity: $\mathbf{T}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \alpha_1 \mathbf{T}\{x_1(t)\} + \alpha_2 \mathbf{T}\{x_2(t)\}$
- Time-Invariance: $\mathbf{T}\{x(t-t_0)\} = y(t-t_0)$

2.2.1 Response of a CT LTI System

1. **Impulse Response**
   - *Impulse Response* is defined as the output of a system when the input is a impulse signal $\delta(t)$.

$$h(t) = \mathbf{T}\{\delta(t)\} \tag{2.13}$$

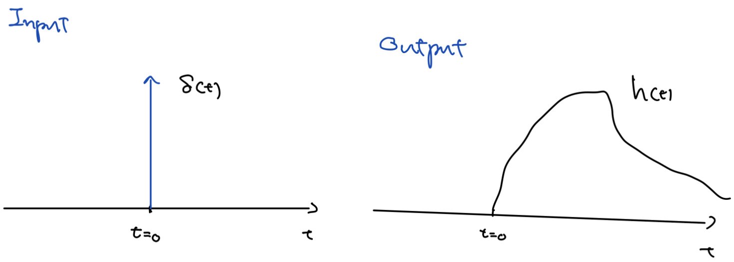


Figure 2.4:

   - *The output of any CT-LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$*

$$y(t) = x(t) * h(t) \tag{2.14}$$

Proof) Arbitrary input $x(t)$ can be expressed in terms of the imulse signal $\delta(t)$ using convolution

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau \tag{2.15}$$

USing the properties of an LTI system, the output to an arbitrary input $x(t)$ can be expressed as

$$\begin{aligned} y(t) = \mathbf{T}\{x(t)\} &= \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\, d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = x(t) * h(t), \end{aligned} \tag{2.16}$$

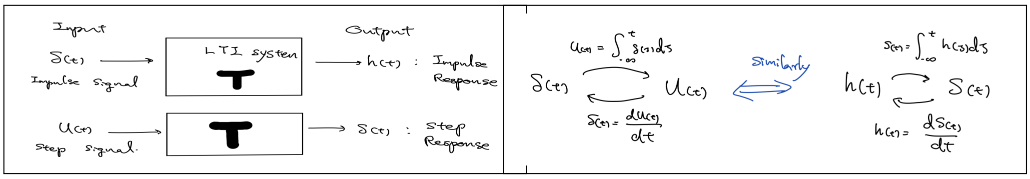
by using (2.15) in the second equality, linearity in the third, and time-invariance in the fourth equality.

2. **Step Response**
   - *Step Response* is defined as the output of a system when the input is a step signal $u(t)$.

$$s(t) = \mathbf{T}\left\{u(t)\right\} \tag{2.17}$$

   - *Step response* can be obtained by integrating the impulse response $h(t)$. Similarly, the impulse response $h(t)$ can be determined by differentiating the step response $h(t)$.

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u\left(t-\tau\right) d\tau = \int_{-\infty}^{t} h(\tau)d\tau \quad \Leftrightarrow \quad h(t) = \frac{ds(t)}{dt} \tag{2.18}$$



3. **Properties of CT LTI System**
   - **Memoryless or Memory System:** The output of a memoryless system depends only on the present input, so that the input-output relationship can be represented in the form $y(t) = Kx(t)$. Since $h(t)$ is the system output for an impulse signal input $\delta(t)$, the impulse response $h(t)$ can be expressed as $h(t) = K\delta(t)$. Hence, the following statement holds

$$\text{If } h(t_0) \neq 0 \text{ for } t_0 \neq 0\text{, then it is a LTI system with memory.} \tag{2.19}$$

   - **Causality:** For a causal system, the output at a present instant do not anticipate input from future instants. In other words, the input that occurs before $t < t_0$ can only determine the output with $t < t_0$.

$$x(t),\ t < t_0 \quad \Leftrightarrow \quad y(t),\ t < t_0 \tag{2.20}$$

   Thus, in a causal system, it is impossible to obtain an output before an input is applied. Since the impulse response is defined as $h(t) = \mathbf{T}\left\{\delta(t)\right\}$, the impulse response $h(t)$ is zero for $t < 0$

$$\delta(t) = 0,\ t < 0 \quad \Leftrightarrow \quad h(t) = 0,\ t < 0 \tag{2.21}$$

   Due to (2.21), the output of a causal LTI system can be expressed as follows

$$\text{Causal LTI System, Arbitrary Input:} \quad y(t) = \int_{0}^{\infty} h(\tau)x\left(t-\tau\right) d\tau = \int_{-\infty}^{t} x(\tau)h\left(t-\tau\right) d\tau \tag{2.22}$$

   Furthermore, if we define a *causal signal* to achieve the following condition

$$x(t) = 0, \qquad t < 0, \tag{2.23}$$

   the output of causal signal input on a causal LTI system can be expressed as follows

$$\text{Causal LTI System, Causal Input:} \quad y(t) = \int_{0}^{t} h(\tau)x\left(t-\tau\right) d\tau = \int_{0}^{t} x(\tau)h\left(t-\tau\right) d\tau \tag{2.24}$$

   - **Stability:** An LTI system is stable if its impulse response $h(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty \tag{2.25}$$

Proof) If $|x(t)| \leq k_1 < \infty$ for any t, then the output can be bounded as follows

$$\begin{aligned} |y(t)| &= |\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau| \leq \int_{-\infty}^{\infty} |h(\tau)\, x(t-\tau)| d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)| d\tau \leq k_1 \int_{-\infty}^{\infty} |h(\tau)| d\tau \end{aligned} \tag{2.26}$$

If $\int_{-\infty}^{\infty} |h(\tau)| d\tau \leq k_2 < \infty$, then $|y(t)| \leq k_1 \times k_2 < \infty$ and the system is stable.

2.2.2 Response of a DT LTI System

1. **Impulse Response**
   - *Impulse Response* is the output of a system when the input is a impulse signal $\delta[n]$

$$h[n] = \mathbf{T}\{\delta[n]\} \tag{2.27}$$

   - *The output of any DT-LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$*

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \tag{2.28}$$

2. **Step Response**
   - *Step Response* is the system output for a step signal input $u[n]$, *i.e.*, $s[n] = \mathbf{T}\{u[n]\}$
   - *Step response* can be obtained by calculating the cumulative sum of the impulse response $h[n]$. Similarly, the impulse response $h[n]$ can be determined by differentiating the step response $h[n]$.

$$\begin{aligned} s[n] &= h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k] u[n-k] = \sum_{k=-\infty}^{n} h[k], \\ h[n] &= s[n] - s[n-1] \end{aligned} \tag{2.29}$$

3. **Properties of DT LTI System**
   - **Memory System**

$$\text{If } h[n_0] \neq 0 \text{ for } n_0 \neq 0, \text{ then it is a LTI system with memory.} \tag{2.30}$$

   - **Causality:** The causality condition for a DT-LTI system is given by

$$h[n] = 0, \;\; n < 0 \tag{2.31}$$

   Due to (2.31) and the definition of causal signal, the output of a causal LTI system can be expressed as

$$\begin{aligned} &\text{Causal LTI System, Arbitrary Input:} \quad y[n] = \sum_{k=0}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^{n} x[k] h[n-k], \\ &\text{Causal LTI System, Causal Input:} \quad y[n] = \sum_{k=0}^{n} h[k] x[n-k] = \sum_{k=0}^{n} x[k] h[n-k] \end{aligned} \tag{2.32}$$

   - **Stability:** An LTI system is stable if its impulse response $h(t)$ is absolutely summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \tag{2.33}$$

[Example 2-3] Consider a CT-LTI system with step response $s(t) = e^{-t}u(t)$. Determine the output of the system for input signal $x(t) = u(t-1) - u(t-3)$.
Refer [Schaum's text, Problem 2.10]

Solution) Based on the definition of a step response $s(t) = \mathbf{T}\{u(t)\}$, the output of the LTI system is given by

$$y(t) = s(t-1) - s(t-3) = e^{-(t-1)}u(t-1) - e^{-(t-3)}u(t-3), \tag{2.34}$$

where we used linearity and time-invariance of the LTI system in the second equality.

[Example 2-4] Consider a CT-LTI system described by

$$y(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau)\, d\tau. \tag{2.35}$$

Find the impulse response $h(t)$ of the system and answer whether this system is causal or not.
Refer [Schaum's text, Problem 2.11]

Solution) Since the impulse response is the output for the impulse signal input, the following equality holds

$$h(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \delta(\tau)\, d\tau = \begin{cases} \frac{1}{T} & \text{if } t - \frac{T}{2} < 0 \text{ and } t + \frac{T}{2} > 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.36}$$

where we applied the sifting property of $\delta(t)$ in the second equality. The impulse response $h(t)$ is plotted in Fig. 2.5. Since $h(t) \neq 0$ for $-\frac{T}{2} < t < 0$, this system is not causal.

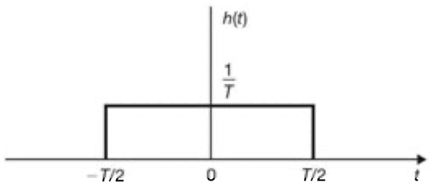


Figure 2.5:

[Example 2-5] Consider CT-LTI systems composed of two component blocks where the impulse response of each block is given by $h_1(t) = e^{-2t}u(t)$ and $h_2(t) = 2e^{-t}u(t)$. For (a) cascade and (b) parallel connection case, find the impulse response $h(t)$ of the overall system and answer whether the overall system is stable or not.
Refer [Schaum's text, Problem 2.14, 2.53]

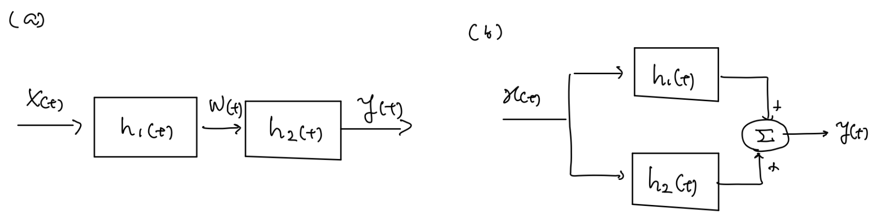


Figure 2.6:

Solution) For (a) cascaded connection case, the impulse response can be derived as follows

$$
\begin{aligned}
h(t) = h_1(t) * h_2(t) &= \int_{-\infty}^{\infty} 2e^{-\tau} u(\tau)\, e^{-2(t-\tau)} u(t-\tau)\, d\tau \\
&= 2e^{-2t} \int_{-\infty}^{\infty} e^{\tau} u(\tau)\, u(t-\tau)\, d\tau = \begin{cases} 2e^{-2t} \int_0^t e^{\tau} d\tau = 2\left(e^{-t} - e^{-2t}\right) & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases} \\
&= 2\left(e^{-t} - e^{-2t}\right) u(t)
\end{aligned}
\tag{2.37}
$$

To check stability, we need to verify whether $h(t)$ is absolutely integrable.

$$
\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_0^{\infty} 2\left(e^{-t} - e^{-2t}\right) d\tau = 2\left[\int_0^{\infty} e^{-\tau} d\tau - \int_0^{\infty} e^{-2\tau} d\tau\right] = 1 < \infty \tag{2.38}
$$

For (b) parallel connection case, the overall impulse response is given by

$$
h(t) = h_1(t) + h_2(t) = \left(e^{-2t} + 2e^{-t}\right) u(t), \tag{2.39}
$$

and the stability test can be performed as below

$$
\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_0^{\infty} \left(e^{-2\tau} + 2e^{-\tau}\right) d\tau = \frac{1}{2} + 2 < \infty, \tag{2.40}
$$

which indicates that the parallel connection system in (b) is stable.

[Example 2-6] For the following impulse responses, determine whether the given LTI system is causal and stable.

a) $h(t) = e^{-3t} \sin(t)\, u(t)$

b) $h(t) = \delta(t) + e^{-3t} u(t)$

c) $h[n] = \delta[n+1]$

d) $h[n] = \left(-\frac{1}{2}\right)^n u[n-1]$

Solution) For causality, we need to check whether the impulse response has a non-zero value on $t < 0$ (or $n < 0$). Then, it is clear that, except (c), the other systems are all causal system. For stability test, we need to check whether the given impulse response is absolutely integrable (or absolutely summable).

$$
\begin{aligned}
&\text{(a)} \quad \int_0^{\infty} e^{-3t} |\sin(t)|\, dt \le \int_0^{\infty} e^{-3t} dt = \frac{1}{3} < \infty \\
&\text{(b)} \quad \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = 1 + \int_0^{\infty} e^{-3t} dt = \frac{4}{3} < \infty \\
&\text{(c)} \quad \sum_{n=-\infty}^{\infty} |h[n]| = 1 < \infty, \qquad \text{(d)} \quad \sum_{n=1}^{\infty} \left|\left(-\frac{1}{2}\right)^n\right| = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 < \infty
\end{aligned}
\tag{2.41}
$$

Hence, the solutions are summarized below.

a) Causal, Stable

b) Causal, Stable

c) Noncausal, Stable

d) Causal, Stable

[Example 2-7] Compute the output $y[n]$ of a DT-LTI system for the given impulse response and the input signals

a) $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$

b) $x[n] = u[n]$ and $h[n] = \alpha^n u[n]$, where $0 < \alpha < 1$

Refer [Schaum's text, Problem 2.28, 2.29]

Solution) In (a), the convolution sum between the input signal and the impulse response is given by

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k \beta^{n-k} u[k]u[n-k] \\ &= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k u[n] = \begin{cases} \beta^n \dfrac{1-(\alpha/\beta)^{n+1}}{1-\alpha/\beta} u[n] = \dfrac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha} u[n] & \text{if } \alpha \neq \beta \\ \beta^n (n+1) u[n] & \text{if } \alpha = \beta \end{cases} \end{aligned} \tag{2.42}$$

For (b), we can apply [Example 2-7. a] by substituting $\alpha \leftarrow 1$ and $\beta \leftarrow \alpha$ into (2.42). Then, the output is given by

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} u[n] \tag{2.43}$$

[Example 2-8] Compute the DT convolution of the following sequences

- $x[0] = 0.5, x[1] = 2, x[n] = 0$ otherwise.
- $h[0] = h[1] = h[2] = 1$, $h[n] = 0$ otherwise.

Compute $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

Solution) Using the convolution sum $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1] \tag{2.44}$$

Substituting the values of $h[n]$ into $y[n]$, we have

- $y[0] = 0.5h[0] + 2h[0-1] = 0.5 \times 1 = 0.5$
- $y[1] = 0.5h[1] + 2h[0] = 0.5 \times 1 + 2 \times 1 = 2.5$
- $y[2] = 0.5h[2] + 2h[1] = 0.5 \times 1 + 2 \times 1 = 2.5$
- $y[3] = 0.5h[3] + 2h[2] = 0.5 \times 0 + 2 \times 1 = 2$
- $y[n] = 0$ for $n < 0$ and $n \geq 4$.