vector field

Def vector field.

A vector field on 2 or 3 D space is a function $\vec{F}$ that assigns to each point (x,y) or (x,y,z) a 2 or 3D vector given by $\vec{F}(x,y)$ or $\vec{F}(x,y,z)$.

Direction and magnitude.

Application: Magnetic field, electrical field, flow of fluid.

standard notation

scalar functions

2D $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$

3D $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

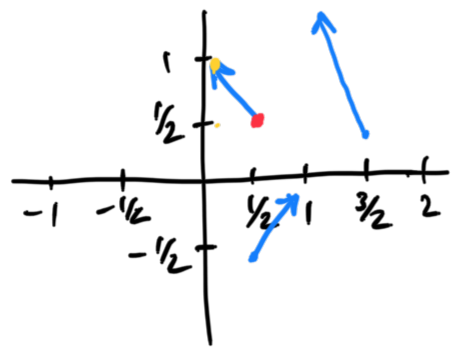
Ex. Sketch the vector field.

(a) $\boxed{\vec{F}(x,y) = -y\vec{i} + x\vec{j}}$

$\vec{F}\left(\frac{1}{2},\frac{1}{2}\right) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$

$\vec{F}\left(\frac{1}{2},-\frac{1}{2}\right) = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$

$\vec{F}\left(\frac{3}{2},\frac{1}{4}\right) = -\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$

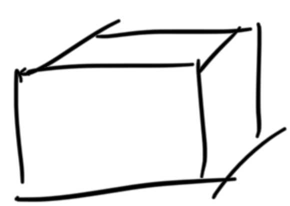


Ex.(b) $\vec{F}(x,y,z) = 2x\vec{i} - 2y\vec{j} - 2x\vec{k}$

(1)

$\vec{F}(1,-3,2) = 2\vec{i} + 6\vec{j} - 2\vec{k}$

$\vec{F}(0,5,3) = -10\vec{j}$



Gradient ∇ (nabla) "del" grad

Maxwell's equation (Relates electric field $\vec{E}$ and the magnetic field $\vec{B}$)

$$\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right)\vec{i} - \left(\frac{\partial E_3}{\partial x} - \frac{\partial E_1}{\partial z}\right)\vec{j} + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)\vec{k} = \frac{1}{c}\left(\frac{\partial B_1}{\partial t}\vec{i} + \frac{\partial B_2}{\partial t}\vec{j} + \frac{\partial B_3}{\partial t}\vec{k}\right)$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$$

like differential operator $\frac{\partial}{\partial x}$

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

① Gradient of a scalar-valued function $f(x,y,z)$ is the vector field

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Note input: scalar valued function

output: vector valued function

② Divergence of a vector field $\vec{F}(x,y,z)$ is the scalar-valued function

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

input vector valued. output - scalar-valued function

③ The curl of the vector field $\vec{F}(x,y,z)$ is the vector field

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

input vector valued

vector valued.

④ Laplacian of a scalar-valued function $f(x,y,z)$

... scalar-valued

is scalar-valued

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian of a vector field $\vec{F}(x, y, z)$ is vector field.

$$\Delta \vec{F} = \nabla^2 \vec{F} = \nabla \cdot \nabla \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2}$$

* Application : divergence and curl – Maxwell's equation *

$$\nabla \cdot \vec{E} = 4\pi \rho$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = \frac{4\pi}{c}\vec{J}$$

$\vec{E}$ – electric field

$\vec{B}$ – magnetic field

J – current density

c = speed of light

ρ = charge of density

Vector Identities

① $\frac{d}{dx}(af(x) + bg(x)) = a\frac{df}{dx}(x) + b\frac{dg}{dx}(x)$ linearity

② $\frac{d}{dx}(f(x)g(x)) = g(x)\frac{df}{dx}(x) + f(x)\frac{dg}{dx}(x)$ product rule.

Gradient Identities

* ① $\nabla(f+g) = \nabla f + \nabla g$

* ② $\nabla(cf) = c\nabla f$ c – any constant

* ③ $\nabla(fg) = (\nabla f)g + f(\nabla g)$

④ $\nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2}$ where $g(x) \neq 0$

⑤ $\nabla(\vec{F}\cdot\vec{G}) = \vec{F}\times(\nabla\times\vec{G}) - (\nabla\times\vec{F})\times\vec{G} + (G\cdot\nabla)F + (F\cdot\nabla)G$

Divergence Identities

① $\nabla\cdot(\vec{F}+\vec{G}) = \nabla\cdot\vec{F} + \nabla\cdot\vec{G}$

② $\nabla\cdot(c\vec{F}) = c\nabla\cdot\vec{F}$ c = constant

③ $\nabla\cdot(f\vec{F}) = (\nabla f)\cdot\vec{F} + f\nabla\cdot\vec{F}$

④ $\nabla\cdot(\vec{F}\times\vec{G}) = (\nabla\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\nabla\times\vec{G})$

Curl Identities

① $\nabla\times(\vec{F}+\vec{G}) = \nabla\times\vec{F} + \nabla\times\vec{G}$

② $\nabla\times c\vec{F} = c\nabla\times\vec{F}$ for any c=constant

③ $\nabla\times(f\vec{F}) = (\nabla f)\times\vec{F} + f\nabla\times\vec{F}$

④ $\nabla\times(\vec{F}\times\vec{G}) = \vec{F}(\nabla\cdot\vec{G}) - (\nabla\cdot\vec{F})\vec{G} + (\vec{G}\cdot\nabla)\vec{F} - (\vec{F}\cdot\nabla)\vec{G}$

Laplacian Identities

① $\nabla^2(f+g) = \nabla^2 f + \nabla^2 g$

② $\nabla^2(cf) = c\nabla^2 f$ for any c = constance

③ $\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$

Degree Two Identities. (Advance)

① $\nabla\cdot(\nabla\times\vec{F}) = 0$ divergence of curl

② $\nabla\times(\nabla f) = 0$ curl of gradient

③ $\ldots = \nabla f\cdot(\nabla g\times\nabla h)$

(3) $\nabla \cdot (f\{\nabla g \times \nabla h\}) = \nabla f \cdot (\nabla g \ldots)$

(4) $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

(5) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$ curl of curl

Memory aid.

1) If LHS is a vector (scalar), then the RHS must be a vector (scalar)

2) the only valid products of 2 vectors are the dot and cross products

3) the product of a scalar with either a scalar or a vector cannot be either a dot or cross product

4) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (cross product is antisymmetric)

Ex. $\underbrace{\nabla \cdot (f\vec{F})}_{\text{scalar}} = \underbrace{(\nabla f) \cdot \vec{F} + f\nabla \cdot \vec{F}}_{\text{scalar}}$ (Div #3)

LHS (1) scalar

(2) (Derivative of product f and $\vec{F}$)

RHS scalar

(product rule the sum of 2 terms one with $\vec{F}$ multiplying a derivative of f, the other one with f multiplying a derivative of $\vec{F}$.)

(3) The derivative acting on f must be ∇f because $\nabla \cdot f$ and $\nabla \times f$ are not well-defined. To end up with scalar, rather than a vector, we must take the dot product ∇f and $\vec{F}$ so $\nabla f \cdot \vec{F}$

(4) The derivative acting on $\vec{F}$ must be either $\nabla \cdot \vec{F}$ or $\nabla \times \vec{F}$. We also need to multiply by the scalar

$\nabla \times F$. We also ...

f and end up with a scalar. So the derivative must be scalar. ie. $\nabla \cdot \vec{F}$ and that $f(\nabla \cdot \vec{F})$

$$\therefore \quad \nabla \cdot (f\vec{F}) = (\nabla f) \cdot \vec{F} + f \nabla \cdot \vec{F}$$

Please recall.

1) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
2) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c}$

Interpretation of the Gradient. $\nabla f(r_0)$

Story time: suppose we are moving through space and that your position at time t is $r(t) = (x(t), y(t), z(t))$

As you move along, you measure the temperature.

If the temperature at position (x, y, z) is $f(x, y, z)$ then the temperature that you measure at time t is

$f(x(t), y(t), z(t))$.

Rate of change of temperature that you feel

$$\frac{d}{dt} f(x(t), y(t), z(t))$$

$$= \frac{\partial f}{\partial x}(x(t), y(t), z(t)) \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t), z(t)) \frac{dy}{dt}(t) + \frac{\partial f}{\partial z}(x(t), y(t), z(t)) \frac{dz}{dt}(t) \quad \text{(chain rule)}$$

$$= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= |\nabla f(\vec{r}(t))|\, |\vec{r}\,'(t)| \cos\theta$$

Remark

where θ is the angle between the gradient vector $\nabla f(\vec{r}(t))$ ...

and the velocity vector $\vec{r}'(t)$

This is the rate of change per unit time.

Rate of change per unit distance travelled by moving with speed one so that $|\vec{r}'(t)| = 1$.

$$\frac{d}{dt} f(\vec{r}(t)) = |\nabla f(\vec{r}(t))| \cos\theta$$

$\therefore$ If at a given moment $t = t_0$, you are at $\vec{r}(t_0) = \vec{r}_0$

then $$\frac{d}{dt} f(\vec{r}(t))\Big|_{t=t_0} = |\nabla f(\vec{r}_0)| \cos\theta$$

Remark: Recall θ = angle between our direction of motion and the gradient vector $\nabla f(\vec{r}_0)$.

So to maximize the rate of change of temperature that we feel as we pass through $\vec{r}_0$, we should choose our direction of motion to be the direction of the gradient vector $\nabla f(\vec{r}_0)$

$\nabla f(\vec{r}_0)$ has direction of maximum rate of change of f at $\vec{r}_0$

has magnitude of maximum rate of change (per unit distance of f at $\vec{r}_0$)