MA2001 **Mid-term Test** **Semester A, 2015/2016**

Name: ______________________________ Student ID: ____________________

Answer ALL questions. (Full marks: 100)

Total Marks: __________

Q1: ______ Q2: ______

Q3: ______

Question 1 (40 marks)

Suppose $z^3 - 2yz + x^2 = 0$ determines $z = z(x, y)$ as a function of x, y locally at $(x, y, z) = (1, 1, 1)$.

(a) Find the linear approximation (tangent plane approximation) of z at $(x, y, z) = (1, 1, 1)$.
(b) Find the quadratic surface approximation of z at $(x, y, z) = (1, 1, 1)$.

(a) $$z^3 - 2yz + x^2 = 0 \Rightarrow \begin{cases} 3z^2 z_x - 2y z_x + 2x = 0 \\ 3z^2 z_y - 2z - 2y z_y = 0 \end{cases} \Rightarrow \begin{cases} z_x = \dfrac{-2x}{3z^2 - 2y} \\ z_y = \dfrac{2z}{3z^2 - 2y} \end{cases}$$

At $(x,y) = (1,1)$

$$\begin{cases} z_x(1,1) = -2 \\ z_y(1,1) = 2 \end{cases} \Rightarrow \underline{\text{linear appr.}}\quad z(x,y) \approx 1 + z_x(1,1)(x-1) + z_y(1,1)(y-1)$$
$$= 1 - 2(x-1) + 2(y-1)$$
$$= 1 - 2x + 2y$$

(b) $$\begin{cases} 3z^2 z_x - 2y z_x + 2x = 0 \\ 3z^2 z_y - 2z - 2y z_y = 0 \end{cases} \Rightarrow \begin{cases} 6z(z_x)^2 + 3z^2 \underline{z_{xx}} - 2y\, \underline{z_{xx}} + 2 = 0 \\ 6z(z_y)\cdot z_x + 3z^2 \underline{z_{xy}} - 2z_x - 2y\, \underline{z_{xy}} = 0 \\ 6z(z_y)^2 + 3z^2 \underline{z_{yy}} - 2z_y - 2z_y - 2y\, \underline{z_{yy}} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} z_{xx} = \dfrac{-2 - 6z\cdot(z_x)^2}{3z^2 - 2y} \\ z_{xy} = \dfrac{2z_x - 6z\cdot z_x \cdot z_y}{3z^2 - 2y} \\ z_{yy} = \dfrac{4z_y - 6z(z_y)^2}{3z^2 - 2y} \end{cases} \Rightarrow \text{At } (x,y,z) = (1,1,1) \begin{cases} z_{xx}(1,1) = -26 \\ z_{xy}(1,1) = 20 \\ z_{yy}(1,1) = -16 \end{cases}$$

$\Rightarrow$ Quadratic appr.

$$z(x,y) \approx 1 + z_x(1,1)(x-1) + z_y(1,1)(y-1) + \frac{1}{2!}\left[z_{xx}(1,1)(x-1)^2 + 2z_{xy}(1,1)(x-1)(y-1) + z_{yy}(1,1)(y-1)^2\right]$$
$$= 1 - 2x + 2y + \frac{1}{2}\left[-26(x-1)^2 + 40(x-1)(y-1) - 16(y-1)^2\right]$$
$$= .$$

Question 2 (30 marks)

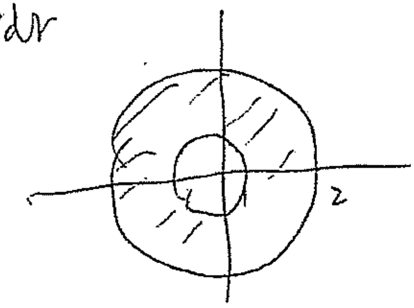
Compute the double integral $\iint_S (x^2+y^2)dxdy$, where S is the ring area between $x^2 + y^2 = 1$ and $x^2+y^2=4$.

$$\iint_S (x^2+y^2)dxdy = \int_1^2\int_0^{2\pi} r^2\cdot r\,d\theta\,dr$$

$$= \int_1^2\int_0^2 r^3 d\theta\,dr = 2\pi\int_1^2 r^3 dr$$

$$= 2\pi\frac{r^4}{4}\Big|_1^2$$

$$= \frac{\pi}{2}(2^4-1) = \frac{15\pi}{2}$$



Question 3 (30 marks)

Let A be a 3×3 matrix given by $A = \begin{pmatrix} -4 & 0 & 2 \\ 4 & 4 & -1 \\ -5 & 0 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of A.

(b) Find the eigenvectors corresponding to each of the eigenvalues.

(c) Diagonalize A by finding an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$;

(d) Calculate P^{-1}.

(e) Find the eigenvalues of $A^2 + 2A + I$, where I is the identity matrix.

(a) $|A-\lambda I| = \begin{vmatrix} -4-\lambda & 0 & 2 \\ 4 & 4-\lambda & -1 \\ -5 & 0 & 3-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} -4-\lambda & 2 \\ -5 & 3-\lambda \end{vmatrix} = (4-\lambda)\left[-(4+\lambda)(3-\lambda)+10\right]$

$= (4-\lambda)\left[-12+\lambda+\lambda^2+10\right] = (4-\lambda)\left[\lambda^2+\lambda-2\right]$

$= (4-\lambda)(\lambda-1)(\lambda+2)$

$\Rightarrow \lambda = 1, -2, 4$

(b) For $\lambda = 1$, $\Rightarrow \vec{v}_1 = t\begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \\ 1 \end{pmatrix}$, $t \neq 0$

For $\lambda = -2$ $\Rightarrow \vec{v}_2 = t\begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$, $t \neq 0$

For $\lambda = 4$ $\Rightarrow \vec{v}_3 = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $t \neq 0$

(c) $P = \begin{pmatrix} \frac{2}{5} & 1 & 0 \\ -\frac{1}{5} & -\frac{1}{2} & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(d) $P^{-1} = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ -\frac{5}{3} & 0 & \frac{5}{3} \\ \frac{5}{3} & 0 & -\frac{2}{3} \end{pmatrix}$

(e) $\lambda_1 = 1^2 + 2\cdot 1 + 1 = 4$

$\lambda_2 = (-2)^2 + 2\cdot(-2) + 1 = 1$

$\lambda_3 = 4^2 + 2\cdot 4 + 1 = 25$.