Convolution Sum

- The output of a discrete-time LTI system with impulse response $h[n]$ to input $x[n]$ is given as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

Let $m = n - k$. We have $k = n - m$, and

$$y[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] = h[n] * x[n]$$

Example 1

- Given:
  - $x[0] = 0.5$, $x[1] = 2$, $x[n] = 0$ otherwise.
  - $h[0] = h[1] = h[2] = 1$, $h[n] = 0$ otherwise.
- Find $y[n]$.
- Solution: Using the convolution sum: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
  - $y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$
  - Substituting the values of $h[n]$ into $y[n]$, we have
    - $y[0] = 0.5h[0] + 2h[-1] = 0.5\times 1 + 2\times 0 = 0.5$,
    - $y[1] = 0.5h[1] + 2h[0] = 0.5\times 1 + 2\times 1 = 2.5$
    - $y[2] = 0.5h[2] + 2h[1] = 0.5\times 1 + 2\times 1 = 2.5$,
    - $y[3] = 0.5h[3] + 2h[2] = 0.5\times 0 + 2\times 1 = 2$
    - $y[n] = 0$ for $n < 0$ and $n \geq 4$.

The Motivating Example (revisit)

- The impulse response:
- $h[n] = \left(\frac{1}{2}\right)^n u[n]$

$= \left(\frac{1}{2}\right)^n$ for $n = 0,1,2,\dots$

The input signal:

$$x[n] = \begin{cases} 2 & n = 0 \\ 1 & n = 1 \\ 3 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

For $n = 0$, $y[0] = x[0]h[0] + x[1]h[0-1] + x[2]h[0-2]$
$= 2\times 1 = 2$

For $n = 1$, $y[1] = x[0]h[1] + x[1]h[1-1] + x[2]h[1-2]$
$= 2\times\frac{1}{2} + 1\times 1 = 2$

Cont

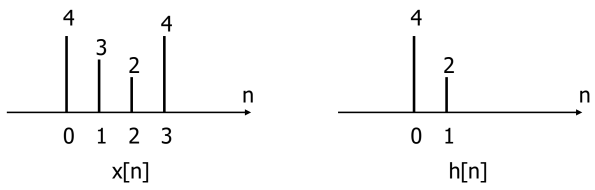
For $n = 2$, $y[2] = x[0]h[2-0] + x[1]h[2-1] + x[2]h[2-2]$
$= 2\times\left(\frac{1}{2}\right)^2 + 1\times\frac{1}{2} + 3\times 1 = 4$

For $n = 3$, $y[3] = x[0]h[3-0] + x[1]h[3-1] + x[2]h[3-2]$
$= 2\times\left(\frac{1}{2}\right)^3 + 1\times\left(\frac{1}{2}\right)^2 + 3\times\frac{1}{2} = 2$

For $n = 4$, $y[4] = x[0]h[4-0] + x[1]h[4-1] + x[2]h[4-2]$
$= 2\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2$
$= \frac{1}{8} + \frac{1}{8} + \frac{3}{4} = 1$

For $n \geq 2$, $y[n] = 2\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + 3\left(\frac{1}{2}\right)^{n-2} = 4\left(\frac{1}{2}\right)^{n-2}$

Example 2



Find $y[n] = x[n] * h[n]$.

We will use different methods to find the answer.

Example 2: Analytical Method

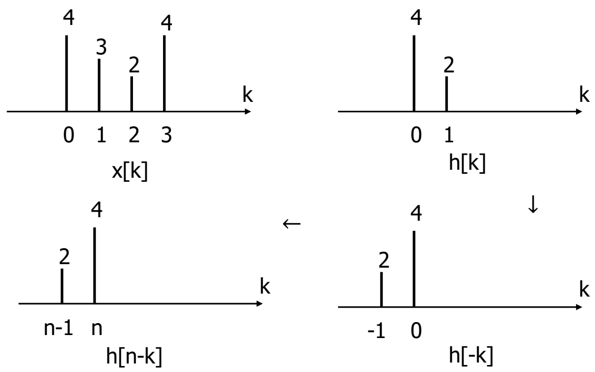
$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= h[0]x[n] + h[1]x[n-1] \\
&= 4x[n] + 2x[n-1]
\end{aligned}
$$

$y[n] = 0$ for $n < 0$ because $x[n]$ and $x[n-1]$ are zero for $n < 0$

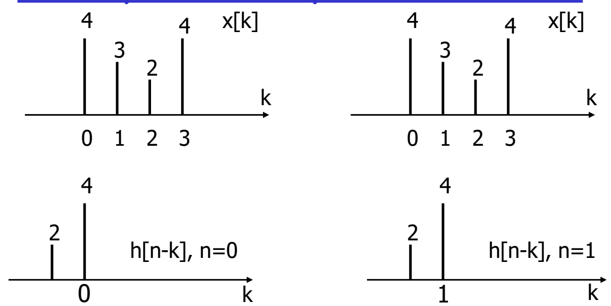
$$
\begin{aligned}
y[0] &= 4x[0] + 2x[-1] \\
&= 4 \times 4 = 16 \\
y[1] &= 4x[1] + 2x[0] \\
&= 4 \times 3 + 2 \times 4 = 20 \\
y[2] &= 4x[2] + 2x[1] \\
&= 4 \times 2 + 2 \times 3 = 14 \\
y[3] &= 4x[3] + 2x[2] \\
&= 4 \times 4 + 2 \times 2 = 20 \\
y[4] &= 4x[4] + 2x[3] \\
&= 4 \times 0 + 2 \times 4 = 8 \\
y[5] &= 4x[5] + 2x[4] \\
&= 4 \times 0 + 2 \times 0 = 0
\end{aligned}
$$

$y[n] = 0,$ for $n < 0$ or $n \geq 5$.

Example 2: Graphical Method



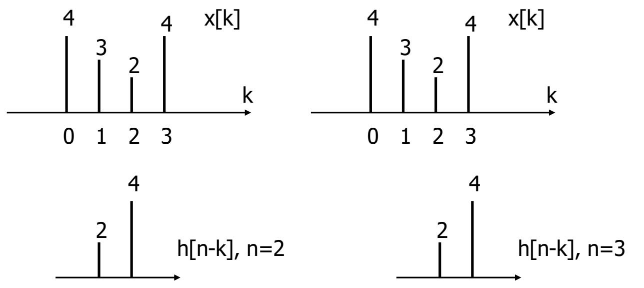
Example 2: Graphical Method



$y[0] = 4 \times 4 = 16$

$y[1] = 4 \times 2 + 3 \times 4 = 20$

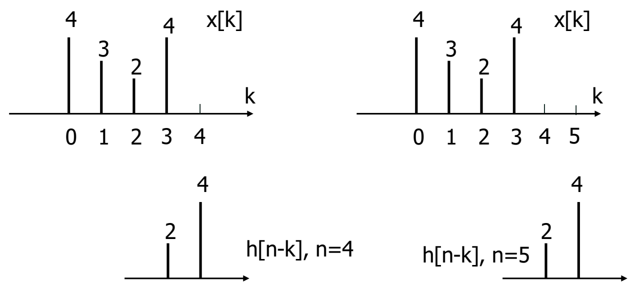
Example 2 (cont.)



y[2] = 3×2+2×4 = 14   y[3] = 2×2+4×4 = 20

57

Example 2 (cont.)

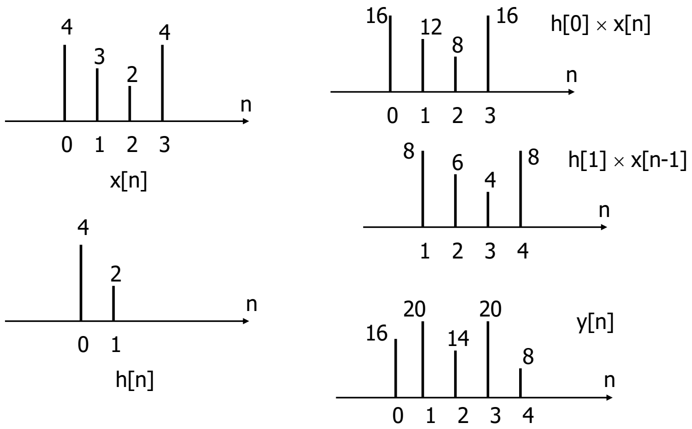


y[4] = 4×2 = 8   y[n] = 0 when n ≥ 5

For n<0, h[n-k] and x[k] do not have overlap giving the product zero.
Thus y[n]=0

58

Example 2: Superposition Method



59

Matrix Method

- The linear convolution of $\{h[n], n = 0,1,\dots,N-1\}$ and $\{x[n], n = 0,1,\dots,M-1\}$ with respective lengths N and M has length of $(M+N-1)$.
- Set $M = 4$ and $N = 2$. We have

$x[n] = \{x[0], x[1], x[2], x[3]\}$, $h[n] = \{h[0], h[1]\}$

The number of output samples is 4+2−1=5

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} x[0] & 0 \\ x[1] & x[0] \\ x[2] & x[1] \\ x[3] & x[2] \\ 0 & x[3] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ 0 & h[1] & h[0] & 0 \\ 0 & 0 & h[1] & h[0] \\ 0 & 0 & 0 & h[1] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

60

Example 2 Matrix Method

- $\{\underset{\uparrow}{4},2\}*\{\underset{\uparrow}{4},3,2,4\}$
- Number of output samples is 2+4−1=5
- $$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \end{bmatrix} = \begin{bmatrix} 16 \\ 20 \\ 14 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 3 & 4 \\ 2 & 3 \\ 4 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

General case

- Consider

$\{h[n], n = q, q+1, \dots, N+q-1\}$ and
$\{x[n], n = p, p+1, \dots, M+p-1\}$
with respective lengths N and M,
the output has length of $(M+N-1)$.

- The output $y[n] = x[n] * h[n]$ is given as

$$y[n] = \{y[p+q], y[p+q+1], \dots, y[M+N+p+q-2]\}$$

Example

- $\{\underset{\uparrow}{4},2\}*\{4,\underset{\uparrow}{3},2,4\}$
- $q = 0, p = -1, N = 2, M = 4$
  output length= 2+4−1=5
- The first output sample is at $p + q = -1$,
  while the last output sample is at
  $$M + N + p + q - 2 = 4 + 2 - 1 + 0 - 2 = 3$$
- $$\begin{bmatrix} y[-1] \\ y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} 16 \\ 20 \\ 14 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 3 & 4 \\ 2 & 3 \\ 4 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$