EE3210 Signals & Systems
Mid-term Exam 1 (20 points in total) Name: ______________________________

This is a open-book exam. Submission due date is **12 pm, noon, March. 7th, 2020**. Late submission will not be accepted. If you need more space, please feel free to attach additional papers. Once you're finished, scan and upload it to Canvas course website.

Honor Pledge

Please review the following honor code, then sign your name and write down the date.

1. I pledge that the answers in this exam are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,
   (a) I will not plagiarize (copy without citation) from any source;
   (b) I will not communicate or attempt to communicate with any other person during the exam;
   (c) neither will I give or attempt to give assistance to another student taking the exam; and
   (d) I will use only approved devices (e.g., calculators) and/or approved device models.
2. I understand that any act of academic dishonesty can lead to disciplinary action.

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Signature

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Date

1. (4 points) Determine whether the following signals are periodic or aperiodic. If a signal is periodic, determine its fundamental period.

(1 pt) (a) $x(t) = \cos\left(2t + \frac{\pi}{4}\right) + \sin\left(\frac{2}{3}t - 1\right)$

The fundamental period for $\cos(2t+\frac{\pi}{4})$ is $T_1 = \pi$, and $T_2 = 3\pi$ for $\sin(\frac{2t}{3}-1)$. Since $\frac{T_1}{T_2} = \frac{1}{3}$ = rational number, $x(t)$ is periodic with fundamental period $T_0 = 3\pi$

(1 pt) (b) $x(t) =$ Even part of $\cos(4\pi t)\,u(t)$ where $u(t)$ is the unit step function

even part of $\cos(4\pi t)u(t)$ is $x(t) = \frac{1}{2}\left[\cos 4\pi t\, u(t) + \cos(-4\pi t)u(-t)\right] = \frac{1}{2}\cos(4\pi t)$ for all t $-\infty < t < \infty$

$\Rightarrow x(t)$ is periodic with fundamental period $T_0 = \frac{1}{2}$

Determine whether the following signals are energy signals, power signals, or neither.

(1 pt) (c) $x(t) = te^{-at}\,u(t-1)$ → Energy Signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_1^{\infty} \underset{u}{t^2}\, \underset{v'}{e^{-2at}}\, dt = \frac{t^2 e^{-2at}}{2a}\Big|_{\infty}^{1} + \frac{1}{a}\int_1^{\infty} t e^{-2at} dt = \frac{e^{-2a}}{2a} + \frac{1}{a}\left[\frac{te^{-2at}}{2a}\Big|_{\infty}^{1}\right.$$

$$\left. + \frac{1}{2a}\int_1^{\infty} e^{-2at} dt\right] = \frac{e^{-2a}}{2a} + \frac{e^{-2a}}{2a^2} + \frac{1}{4a^3} = \frac{1}{4a^3}\left[1 + 2e^{-2a}(1+a^2)\right] < \infty$$

(1 pt) (d) $x(t) = 2e^{-t}\cos(t)\,u(t)$ → Energy Signal

$$E = 4\int_0^{\infty} e^{-2t}\cos^2 t\, dt = 2\int_0^{\infty} e^{-2t}(1 + \cos 2t)\, dt = 1 + 2\underbrace{\int_0^{\infty} e^{-2t}\cos 2t\, dt}_{\text{denote} = I}$$

$$I = \int_0^{\infty} \underset{v'}{e^{-2t}}\,\underset{u}{\cos 2t}\, dt = \frac{\cos 2t \cdot e^{-2t}}{2}\Big|_{\infty}^{0} - \int_0^{\infty} \underset{u}{\sin 2t} \cdot \underset{v'}{e^{-2t}}\, dt$$

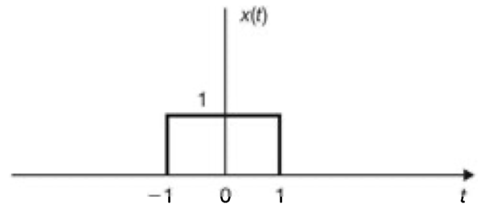
$\frac{1}{2}$

$$= \frac{1}{2} - \left[\cancel{\frac{\sin 2t \cdot e^{-2t}}{2}}\Big|_{\infty}^{0} + \int_0^{\infty} \cos 2t \cdot e^{-2t} dt\right] = \frac{1}{2} - I \Rightarrow I = \frac{1}{4}$$

$\Rightarrow E = 1 + \frac{1}{2} = \frac{3}{2} < \infty$

(Answer Page for Question 1)

2. (4 points) Consider a rectangular pulse signal $x(t) = u(t+1) - u(t-1)$ as plotted below



Compute the convolution of the following pair of signals; $g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau)\, h(t-\tau)\, d\tau$

(1pt) (a) $g(t) = (t+a)\, u(t+a), \quad h(t) = (t+b)\, u(t+b)$

$$(t+a)u(t+a) * (t+b)u(t+b) = \frac{1}{6}(t+b+a)^3 u(t+b+a)$$

$$\Rightarrow \int_{-a}^{t+b} (\tau+a)(t-\tau+b)\, d\tau = \int_{0}^{t+b+a} \ell\,(t+b+a-\ell)\, d\ell$$

(change of variable $\tau + a = \ell$, $d\tau = d\ell$)

(1pt) (b) $g(t) = (t+a)^n\, u(t+a), \quad h(t) = u(t+b)$

$$(t+a)^n u(t+a) * u(t+b) = \frac{1}{n+1}(t+b+a)^{n+1} u(t+b+a)$$

$$\Rightarrow \int_{-a}^{t+b} (\tau+a)^n\, d\tau = \int_{0}^{t+b+a} \ell^n\, d\ell$$

(change of variable $\tau + a = \ell$, $d\tau = d\ell$)

(2pt) (c) Evaluate $g(t) * h(t) = \{x(t) * x(t)\} * \{x(t) * x(t)\}$ where $g(t) = h(t) = x(t) * x(t)$

$$x(t) * x(t) = (t+2)u(t+2) - 2t\,u(t) + (t-2)u(t-2)$$

By using (a), we get

$$g(t) * g(t) = \frac{1}{6}(t+4)^3 u(t+4) - \frac{2}{3}(t+2)^3 u(t+2)$$
$$+ t^3 u(t) - \frac{2}{3}(t-2)^3 u(t-2) + \frac{1}{6}(t-4)^3 u(t-4)$$

$\Rightarrow$ Continue

(Answer Page for Question 2)

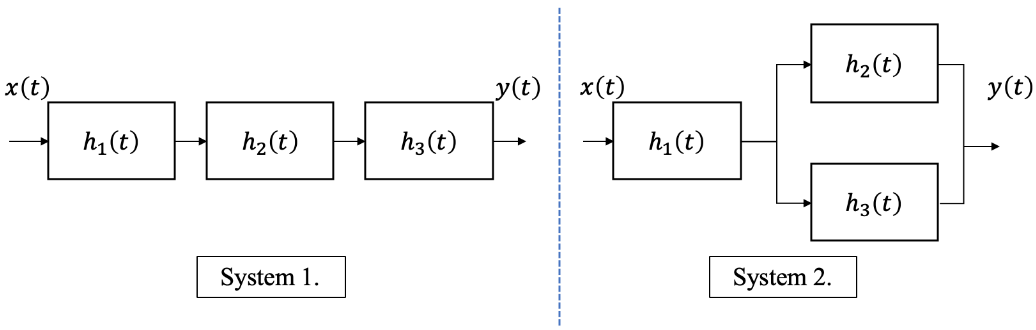
(c)

$$x(t) * x(t) * x(t) * x(t)$$

$$= \begin{cases} \emptyset & t < -4 \\ \frac{1}{6}(t+4)^3 & -4 \le t < -2 \\ \frac{1}{6}\left[-3t^3 + 32 - 12t^2\right] & -2 \le t < 0 \\ \frac{1}{6}\left[3t^3 + 32 - 12t^2\right] & 0 \le t < 2 \\ \frac{1}{6}(4-t)^3 & 2 \le t < 4 \\ \emptyset & t \ge 4 \end{cases}$$

3. (4 points) Consider LTI systems that are composed of three component blocks where the impulse responses of each blocks are give by $h_1(t)$, $h_2(t)$, and $h_3(t)$, respectively.

$$h_1(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right),$$
$$h_2(t) = \delta(t) + 0.5\delta(t-1) + 0.25\delta(t-2),$$
$$h_3(t) = \delta(t-2) + \delta(t-4)$$

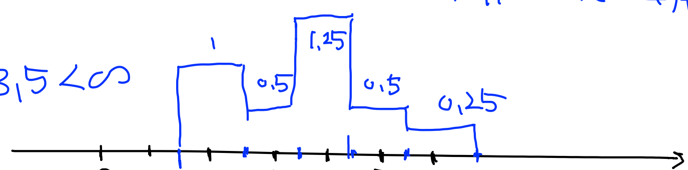


(1 pt) (a) Find the impulse response $h(t)$ of System 1 in the above figure.

(1 pt) (b) Determine whether System 1 is causal or not. Also, answer whether it is a stable system.

$$h(t) = \{h_1(t) * h_2(t)\} * h_3(t) = \{h_1(t) + 0{,}5\,h_1(t-1) + 0{,}25\,h_1(t-2)\} * h_3(t)$$
$$= h_1(t-2) + 0{,}5\,h_1(t-3) + 0{,}25\,h_1(t-4) + h_1(t-4) + 0{,}5\,h_1(t-5) + 0{,}25\,h_1(t-6)$$
$$= h_1(t-2) + 0{,}5\,h_1(t-3) + 1{,}25\,h_1(t-4) + 0{,}5\,h_1(t-5) + 0{,}25\,h_1(t-6)$$

$\int_{-\infty}^{\infty} |h(t)|\,dt = 3{,}5 < \infty$



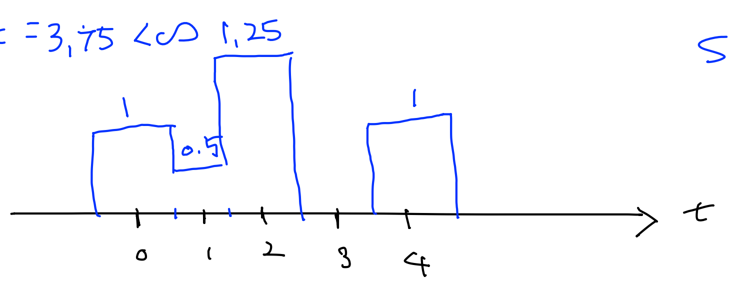
Since $h(t) = 0$ for $t < 0$, System 1 is causal. Also, System 1 is stable

(1 pt) (c) Find the impulse response $h(t)$ of System 2 in the above figure.

(1 pt) (d) Determine whether System 2 is causal or not. Also, answer whether it is a stable system.

$$h(t) = h_1(t) * [h_2(t) + h_3(t)]$$
$$= h_1(t) * [\delta(t) + 0{,}5\,\delta(t-1) + 1{,}25\,\delta(t-2) + \delta(t-4)]$$
$$= h_1(t) + 0{,}5\,h_1(t-1) + 1{,}25\,h_1(t-2) + h_1(t-4)$$

$\int_{-\infty}^{\infty} |h(t)|\,dt = 3{,}75 < \infty$



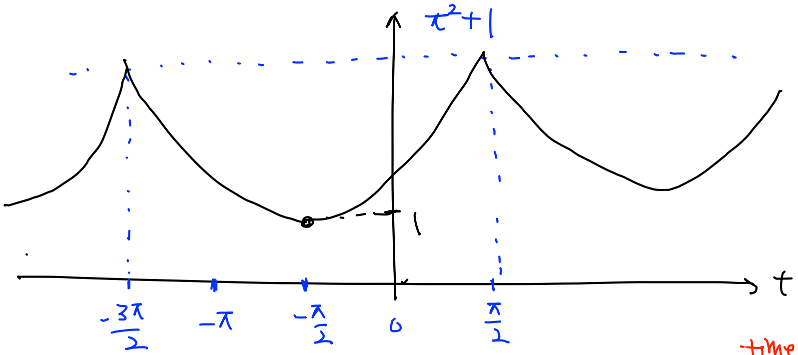
Since $h(t) \neq 0$ for $-\frac{1}{2} < t < 0$, System 2 is non-causal, but it is stable system.

(Answer Page for Question 3)

4. (4 points) Consider a periodic signal $x(t)$ defined by

$$\begin{cases} x(t) = t^2 + \pi t + \frac{4+\pi^2}{4} = \left(t + \frac{\pi}{2}\right)^2 + 1, & -\frac{3\pi}{2} \le t \le \frac{\pi}{2}, \\ x(t+2\pi) = x(t) \end{cases}$$

(1 pts) (a) Sketch $x(t)$ and find its fundamental angular frequency $w_0 = \frac{2\pi}{T_0}$



$T_0 = 2\pi$,

$\omega_0 = \frac{2\pi}{T_0} = 1$

$x(t)$ is obtained from t^2 via two operation

$t^2 \xrightarrow{\text{time-shift}} (t+\frac{\pi}{2})^2 \xrightarrow{\text{add by constant}} (t+\frac{\pi}{2})^2+1$

(2pts) (b) Find the complex exponential Fourier series of $x(t)$.

b) The FS coefficients of t^2 are

$$C_0 = \frac{\pi^2}{3}, \quad C_k = \frac{2(-1)^k}{k^2}$$

After time shift by $-\frac{\pi}{2}$, the FS coefficients of $(t+\frac{\pi}{2})^2$ are

$$C_0 = \frac{\pi^2}{3}, \quad C_k = e^{jk\frac{\pi}{2}} \cdot \frac{2(-1)^k}{k^2} = \frac{2(-j)^k}{k^2}$$

Addition by constant 1, the FS coefficients of $x(t)$ ar

$$C_0 = \frac{\pi^2}{3}+1, \quad C_k = \frac{2(-j)^k}{k^2}$$

(1 pts) (c) Find the trigonometric Fourier series of $x(t)$.

$$C_0 = \frac{\pi^2}{3}+1$$

$$\Rightarrow C_k = \frac{2(-j)^k}{k^2} = \begin{cases} \text{even } k=2m, & \frac{2(-1)^m}{k^2} \\ \text{odd } k=2m+1, & -\frac{2(-1)^m}{k^2}\cdot j \end{cases}$$

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c) $$a_0 = 2C_0 = 2\left(\frac{\pi^2}{3}+1\right), \quad a_k = \begin{cases} \text{even } k=2m, & \frac{(-1)^m}{m^2} \\ \text{odd } k=2m+1, & 0 \end{cases}$$

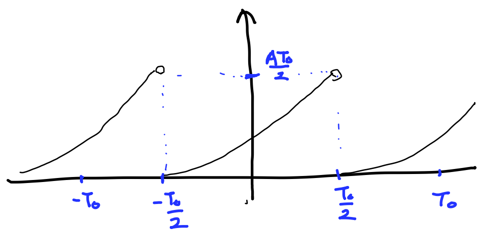
$$b_k = \begin{cases} \text{even } k=2m, & 0 \\ \text{odd } k=2m+1, & \frac{4(-1)^m}{(2m+1)^2} \end{cases}$$

(Answer Page for Question 4)

5. (4 points) Consider a periodic signal $x(t)$ defined by

$$x(t) = \frac{A}{2T_0}\left(t + \frac{T_0}{2}\right)^2, \quad -\frac{T_0}{2} \le t \le \frac{T_0}{2}, \quad \text{and} \quad x(t+T_0) = x(t)$$

(a) Sketch $x(t)$ and find the complex exponential Fourier series of $x(t)$.



$$C_k = \frac{A}{2T_0^2}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left(t+\frac{T_0}{2}\right)^2 e^{-jk\omega_0 t}\,dt$$

$t + \frac{T_0}{2} = x$

$$= \frac{Ae^{jk\pi}}{2T_0^2}\int_0^{T_0} x^2 e^{-jk\omega_0 x}\,dx$$

where $\int_0^{T_0} x^2 e^{-jk\omega_0 x}\,dx = -\frac{T_0^2}{jk\omega_0} - \frac{2T_0}{(jk\omega_0)^2}$

Hence, $C_0 = \frac{AT_0}{6}$, $C_k = \frac{AT_0(-1)^k}{(2\pi k)^2} + j\frac{AT_0(-1)^k}{4\pi k}$

(b) Find the trigonometric Fourier series of $x(t)$.

$$a_0 = 2C_0 = \frac{AT_0}{3}$$

$$a_k = 2\,\mathrm{Re}[C_k] = \frac{AT_0(-1)^k}{2(\pi k)^2}$$

$$b_k = -2\,\mathrm{Im}[C_k] = \frac{AT_0(-1)^{k+1}}{2\pi k}$$

(Answer Page for Question 5)