

1. If  $\vec{r}(t) = r \cos(\omega t) \vec{i} + r \sin(\omega t) \vec{j}$  is the position vector of a point at time  $t$ ,  $\vec{v}(t)$  is the velocity vector of  $\vec{r}(t)$  and  $\vec{a}(t)$  is the acceleration vector of  $\vec{r}(t)$ , show that
  - (a)  $\vec{r} \cdot \vec{v} = 0$ ,
  - (b)  $\vec{r} \times \vec{v} = \text{constant vector}$ ,
  - (c)  $\vec{a} = \omega^2 \vec{r}$ .
  
2. (a) Compute the divergence and curl of the vector functions:
  - (i)  $\vec{v} = e^x \cos y \vec{i} + xy^2 \vec{j} + yz^3 \vec{k}$
  - (ii)  $\vec{v} = yz \vec{i} + 3zx \vec{j} + z \vec{k}$
  - (b) (i) Find  $\text{div}(\text{grad } f)$ , for  $f(x, y, z) = 1 - x^2 - 4y^2 + 2z^2$
  - (ii) Find  $\nabla \times \nabla(\nabla \cdot \vec{v})$ , for  $\vec{v}(x, y, z) = e^x \vec{i} + e^y \vec{j} + e^z \vec{k}$
  - (c) Verify the formula  $\text{div}(f\vec{v}) = f \text{div} \vec{v} + \vec{v} \cdot \text{grad } f$  for  $f = e^{xyz}$  and  $\vec{v} = x \vec{i} + y \vec{j} + z \vec{k}$ .
  - (d) Prove that for any vector fields  $\vec{v}$  and  $\vec{w}$  on  $\mathbf{R}^3$ ,
    - (i)  $\text{curl}(\vec{v} + \vec{w}) = \text{curl} \vec{v} + \text{curl} \vec{w}$
    - (ii)  $\text{div}(\text{curl} \vec{v}) = 0$
  
3. It is given that  $\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$  and  $\vec{p} = a \vec{i} + b \vec{j} + c \vec{k}$  is a constant vector and  $\vec{u} = (\vec{p} \cdot \vec{r}) \vec{r}$ .
  - (a) Evaluate  $\vec{u} = (\vec{p} \cdot \vec{r}) \vec{r}$ .
  - (b) Show that
    - (i)  $\nabla \cdot \vec{u} = 4 \vec{p} \cdot \vec{r}$ ,
    - (ii)  $\nabla \times \vec{u} = \vec{p} \times \vec{r}$ ,
    - (iii)  $\nabla \times (\vec{p} \times \vec{r}) = 2 \vec{p}$ .
  
4. Let  $\vec{F}(x, y, z) = (x + 2y + az) \vec{i} + (bx - 3y - z) \vec{j} + (4x + cy + 2z) \vec{k}$  be a vector field on  $\mathbf{R}^3$ , where  $a, b$  and  $c$  are real constants.
  - (a) Find the values of  $a, b$  and  $c$  such that  $\vec{F}$  is irrotational.
  - (b) With the values of  $a, b$  and  $c$  obtained in (a), determine a potential function  $\phi$  on  $\mathbf{R}^3$  for which  $\nabla \phi = \vec{F}$ .
  
5. Let  $\vec{G}(x, y, z) = 3yz \vec{i} + x^2 \vec{j} + x \cos y \vec{k}$  be a vector field on  $\mathbf{R}^3$ .
  - (a) Show that  $\vec{G}$  is solenoidal.
  - (b) Find a vector field  $\vec{F}(x, y, z) = f_1(x, y, z) \vec{i} + f_2(x, y, z) \vec{j}$  on  $\mathbf{R}^3$  such that  $\nabla \times \vec{F} = \vec{G}$ .
  
6. (a) A vector field  $\vec{F}$  is said to be **solenoidal** if  $\nabla \cdot \vec{F} = 0$ . Let  $\vec{F} = (y + z) \vec{i} + (x + z) \vec{j} + (x + y) \vec{k}$ . Show that  $\vec{F}$  is solenoidal.
  - (b) As a consequence of  $\vec{F}$  being solenoidal, there exists a vector field  $\vec{H}$  such that  $\vec{F} = \nabla \times \vec{H}$ . Find a vector field  $\vec{H} = h_1(x, y, z) \vec{i} + h_2(x, y, z) \vec{j} + h_3(x, y, z) \vec{k}$  with  $h_2(x, y, z) \equiv 0$  such that  $\vec{F} = \nabla \times \vec{H}$ .
  - (c) Observe that if  $\phi$  is a scalar field and  $\vec{H}, \vec{F}$  are vector fields such that  $\vec{F} = \nabla \times \vec{H}$ , then we have  $\nabla \times (\vec{H} + \nabla \phi) = \nabla \times \vec{H} + \nabla \times \nabla \phi = \nabla \times \vec{H} = \vec{F} \dots \dots (I)$ . Using (b) and observation (I), find a vector field  $\vec{G} = g_1(x, y, z) \vec{i} + g_2(x, y, z) \vec{j} + g_3(x, y, z) \vec{k}$  such that  $\vec{F} = \nabla \times \vec{G}$  and  $g_2(x, y, z) = 2y$ .