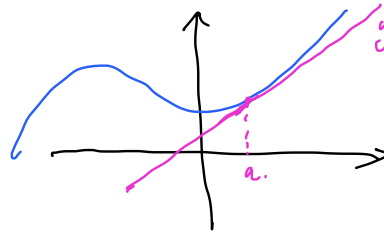


# 1 Taylor Series

## Single variable Case:

Let  $f$  be a function, which has continuous derivatives at  $x = a$  up to any order.  
 $y = f(x)$



- Linear approximation:  
(Linearization)

$$T_1(x) = f(a) + f'(a)(x - a).$$

eg.  $y = e^x$   $x=0$   
 $f(0) = f'(0) = 1 = f''(0)$

$$T_1(x) = 1 + 1(x-0) = x+1$$

$$T_2(x) = 1 + x + \frac{1}{2}(x-0)^2$$

$2! = 1 \cdot 2$   
 $3! = 1 \cdot 2 \cdot 3$   
 $n! = 1 \cdot 2 \cdot 3 \cdots n$

- Quadratic approximation:

$$T_2(x) = f(a) + \underbrace{f'(a)}_{T_1(x)}(x - a) + \frac{f''(a)}{2!}(x - a)^2.$$

- Cubic approximation:

$$T_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3.$$

- Taylor Series of order  $n$ :

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

$$= f(a) + \sum_{k=1}^n \frac{1}{k!} [x-a] \frac{d^k}{dx^k} f(a)$$

- When  $a = 0$ , Maclaurin Series of order  $n$ :

$$M_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

**Example.** Find linear, quadratic and cubic approximation of  $f(x) = e^x$

and  $g(x) = \sin x$  at  $x = 0$ .

$$\left\{ \begin{array}{l} T_1(x) = 1 + x \\ T_2(x) = 1 + x + \frac{1}{2}x^2 \\ T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 \end{array} \right.$$

$$T_n(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$$

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

$$\left\{ \begin{array}{l} g(0) = 0 \\ g'(0) = 1 \\ g''(0) = 0 \\ g^{(3)}(0) = -1 \\ g^{(4)}(0) = 0 \\ \vdots \end{array} \right.$$

$$T_1(x) = 0 + \frac{1}{1!}x$$

$$T_2(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2$$

$$T_3(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3$$

$$T_n(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

## Single variable Case:

If a function  $f(x)$  of one variable is differentiable through order  $n + 1$  at  $x = \underline{a}$ , then for any real number  $x$  such that  $a - l < x < a + l$ , we have

$$\underline{f(x)} = \underline{T_n(x)} + \underline{R_n(x)}$$

where  $T_n(x)$  is the Taylor series of  $f$  at  $x = a$  of order  $n$  and

$$\underline{R_n(x)} = \frac{f^{(n+1)}[\underline{a} + \theta(x - \underline{a})]}{(n+1)!} (x - a)^{n+1}$$

for some  $0 < \theta < 1$ .

$$f(x) - \underline{T_n(x)} = \frac{f^{(n+1)}(a + \theta(x-a))}{(n+1)!} (x-a)^{n+1}$$

⇓

$$|f(x) - T_n(x)| \leq \frac{\max_{z \in D} f^{(n+1)}(z)}{(n+1)!} |x-a|^{n+1}$$

for all  $|x-a| \leq c$ ,  $D = \{z : |z-a| \leq c\}$ .

eg.  $f(x) = e^x$ .  $T_2(x) = 1 + x + \frac{1}{2!}x^2$ .

let  $\underline{|x-0| \leq 0.1}$ .

$$|\underline{f(x)} - \underline{T_2(x)}| \leq \max_{|z| \leq 0.1} \frac{f^{(3)}(z)}{3!} |x|^3$$

$$\leq \frac{\max_{|z| \leq 0.1} e^z}{3!} \cdot 0.1^3$$

$$\leq \frac{e^{0.1}}{6} \cdot 0.001$$

## Multi-Variable Case:

Given a two-variable function  $z = f(x, y)$ , its linearization or linear approximation of  $f$  at  $P_0(a, b)$  is the function

$$L(x, y) = f(P_0) + \frac{\partial f}{\partial x}(P_0)(x - a) + \frac{\partial f}{\partial y}(P_0)(y - b),$$

which is the Tangent plane of the surface of  $f$  at  $P_0(a, b)$  with equation

$$z = f(P_0) + \frac{\partial f}{\partial x}(P_0)(x - a) + \frac{\partial f}{\partial y}(P_0)(y - b).$$

**Example**  $z = f(x, y) = 9 - x^2 - y^2$ . Find the tangent plane of  $f$  at  $P(\underline{1}, \underline{2}, 4)$ . linearization of  $f$ .

$$\begin{aligned} f(P) &= 4 = f(1, 2) \\ f_x(x, y) &= 0 - 2x = 0 \\ f_y(x, y) &= -2y \\ f_x(1, 2) &= -2 \cdot 1 = -2 \\ f_y(1, 2) &= -2 \cdot 2 = -4 \end{aligned}$$

$$L(x, y) = f(P) + f_x(P)(x - 1) + f_y(P)(y - 2)$$

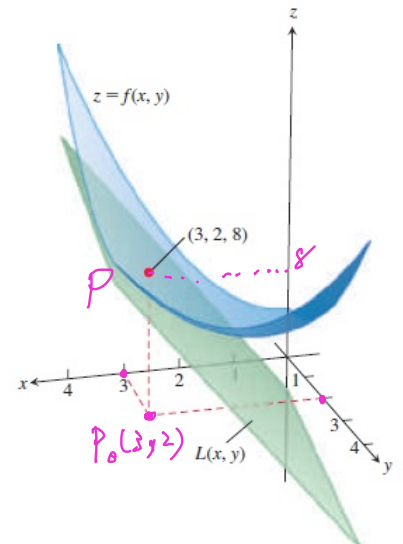
$$= 4 - 2(x - 1) - 4(y - 2)$$

**Example** Find the linear approximation of  $z = f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$  at the point  $(\underline{3}, \underline{2})$ .

$$\begin{cases} f(P) = 8 \\ f_x(P) = 4 \\ f_y(P) = -1 \end{cases}$$

$$L(x, y) = f(P) + f_x(P)(x - 3) + f_y(P)(y - 2)$$

$$= 8 + 4(x - 3) + (-1)(y - 2)$$



Given a two-variable function  $z = f(x, y)$ , its **quadratic approximation** of  $f$  at  $P(a, b)$  is the function

$$\begin{aligned}
 Q(x, y) &= f(a, b) + \frac{\partial f}{\partial x}(P)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) \\
 &\quad + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial x^2}(P)(x - a)^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(P)(x - a)(y - b) + \frac{\partial^2 f}{\partial y^2}(y - b)^2 \right] \\
 &= f(a, b) + \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right] f \\
 &\quad + \frac{1}{2!} \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right]^2 f
 \end{aligned}$$

$\frac{1}{2!} \left[ (x-a)^2 \frac{\partial^2}{\partial x^2} + 2(x-a)(y-b) \frac{\partial^2}{\partial x \partial y} + (y-b)^2 \frac{\partial^2}{\partial y^2} \right]$

**Cubic Approximation** of  $z = f(x, y)$  at  $P_0(a, b)$

$$\begin{aligned}
 C(x, y) &= f(P) + \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right] f(P) \\
 &\quad + \frac{1}{2!} \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right]^2 f(P) \\
 &\quad + \frac{1}{3!} \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right]^3 f(P) \\
 &= Q(x, y) + \frac{1}{3!} \left[ (x-a)^3 f_{xxx}(P) + 3(x-a)^2(y-b) f_{xxy}(P) + 3(x-a)(y-b)^2 f_{xyy}(P) + (y-b)^3 f_{yyy}(P) \right]
 \end{aligned}$$

**Taylor Series** of  $f(x, y)$  up to  $n$  order at  $P_0(a, b)$

$$\begin{aligned}
 T_n(x, y) &= f(P) + \frac{1}{1!} \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right] f(P) \\
 &\quad + \frac{1}{2!} \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right]^2 f(P) \\
 &\quad + \dots + \frac{1}{n!} \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right]^n f(P) \\
 &= f(P) + \sum_{k=1}^n \frac{1}{k!} \left[ (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right]^k f(P)
 \end{aligned}$$

Particularly, when  $P(0, 0)$ , it's called **Maclaurin Series**.

**Example** Find the quadratic approximation of  $f(x, y) = e^x \sin y$  at  $(0, 0)$ .

$$\begin{aligned} f(P) &= 0 \\ f_x &= e^x \sin y & f_x(P) &= 0 \\ f_y &= e^x \cos y & f_y(P) &= 1 \\ f_{xx} &= e^x \sin y & f_{xx}(P) &= 0 \\ f_{xy} &= e^x \cos y & f_{xy}(P) &= 1 \\ f_{yy} &= -e^x \sin y & f_{yy}(P) &= 0 \end{aligned}$$

$$\begin{aligned} Q(x, y) &= f(P) + f_x(P)x + f_y(P)y \\ &\quad + \frac{1}{2!} [x^2 f_{xx}(P) + 2f_{xy}(P)xy + f_{yy}(P)y^2] \\ &= 0 + 0x + y + \frac{1}{2}(0 + 2xy + 0) \\ &= y + xy \end{aligned}$$

**Example**  $z^3 - 2xz + y = 0$  defines an implicit function  $z = f(x, y)$ .  
Find tangent plane approximation of  $z$  at  $(0, 0)$ .

$$\begin{aligned} f(P) &= -1 \leftarrow z^3 - 2 \cdot 0 \cdot z + 0 = 0 \Rightarrow z = -1 \\ f_x(P) &= -\frac{2}{3} \leftarrow \text{Take } \frac{\partial}{\partial x} \text{ for both sides } 3z^2 \cdot \frac{\partial z}{\partial x} - 2(z + x \frac{\partial z}{\partial x}) + 0 = 0 \xrightarrow{x=0, y=1, z=-1} 3 \cdot \frac{\partial z}{\partial x} - 2(-1) = 0 \\ f_y(P) &= \frac{1}{3} \leftarrow \text{Take } \frac{\partial}{\partial y} \text{ for } \dots 3z^2 \cdot \frac{\partial z}{\partial y} - 2x \frac{\partial z}{\partial y} + 1 = 0 \xrightarrow{x=0, y=1, z=-1} 3 \frac{\partial z}{\partial y} + 1 = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{1}{3} \end{aligned}$$

$$L(x, y) = -1 + \frac{2}{3}(x-0) - \frac{1}{3}(y-1)$$

**Example** Find the linear and quadratic approximation of  $f(x, y) = \sin x \sin y$  at the origin  $(0, 0)$ .

$$f(P) = 0$$

$$f_x = \cos x \sin y \quad f_x(P) = 0$$

$$f_y = \sin x \cos y \quad f_y(P) = 0$$

$$f_{xx} = -\sin x \sin y \quad f_{xx}(P) = 0$$

$$f_{xy} = \cos x \cos y \quad f_{xy}(P) = 1$$

$$f_{yy} = -\sin x \sin y \quad f_{yy}(P) = 0$$

$$\begin{cases} L(x, y) = 0 \\ Q(x, y) = 0 + \frac{1}{2!} [0 + 2 \cdot 1 (x-0)(y-0) + 0] \\ = xy \end{cases}$$

$$Q(x, y) = 0 + \frac{1}{2!} [0 + 2 \cdot 1 (x-0)(y-0) + 0] = xy$$

Use the linear  $L$  and quadratic  $Q$  approximation of  $f(x, y) = \sin x \sin y$  at the origin to estimate  $f(0.1, 0.1) = \sin 0.1 \sin 0.1$ .

$$f(0.1, 0.1) \approx L(0.1, 0.1) = 0$$

$$f(0.1, 0.1) \approx Q(0.1, 0.1) = 0.1 \cdot 0.1 = 0.01 \rightarrow (\sin 0.1)^2$$

Estimate the error between  $f$  and  $Q$  in the region  $|x| \leq 0.1$  and  $|y| \leq 0.1$ .

$$|f(x, y) - T_2(x, y)| \leq \frac{M}{3!} (|x-0| + |y-0|)^3$$

$$\begin{aligned} \text{where } M &= \max \{|f_{xxx}|, |f_{xyx}|, |f_{yyx}|, |f_{yyy}|\} \\ &= \max \{|-\cos x \sin y|, |-\sin x \cos y|, \dots\} \\ &\leq 1 \end{aligned}$$

$$\begin{aligned} |f - Q| &\leq \frac{1}{3!} (|x| + |y|)^3 \\ &= \frac{1}{6} (0.1 + 0.1)^3 = \frac{0.008}{6} \quad \text{when } |x| \leq 0.1, |y| \leq 0.1 \end{aligned}$$

## Remark

- When  $(x, y)$  is around  $(a, b)$ ,  $L(x, y)$  provides a good approximation for  $f(x, y)$ . i.e.  $f(x, y) \approx L(x, y)$ .

More precisely,

$$|f(x, y) - L(x, y)| \leq \frac{M}{2!} (|x - a| + |y - b|)^2$$

where  $M = \max\left\{\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}\right\}$ .

- When  $(x, y)$  is around  $(a, b)$ ,  $T_n(x, y)$  provides a good approximation for  $f(x, y)$ . i.e.  $f(x, y) \approx T_n(x, y)$ .

More precisely,

$$|f(x, y) - T_n(x, y)| \leq \frac{M}{(n+1)!} (|x - a| + |y - b|)^{n+1}$$

where  $M = \max_{s+t=n+1} \left\{ \frac{\partial^{n+1} f}{\partial x^s \partial y^t} \right\}$ .

- When  $(x, y)$  is far away from  $(a, b)$ ,  $T_n(x, y)$  might be very different from  $f(x, y)$ .

### Example

Find the tangent plane and quadratic surface approximations given by Taylor's Theorem at the point  $(1, 0)$  for the cone  $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$ . Hence estimate  $f(0.9, 0.1)$ .



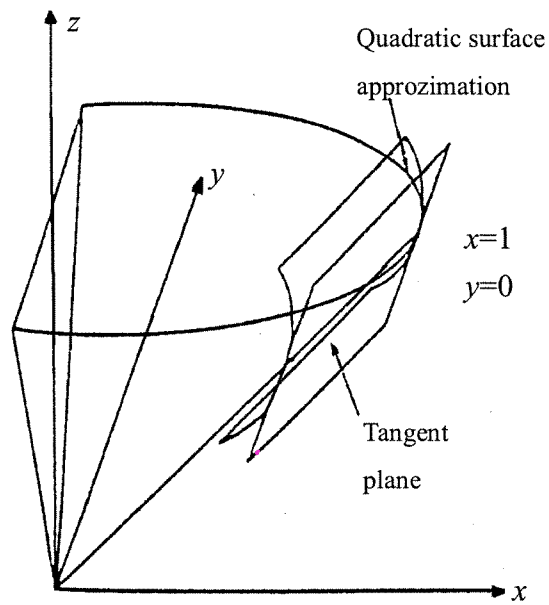
### Example

Let  $f(x)$  be a smooth function, that is,  $f(x)$  has continuous derivatives up to any order.

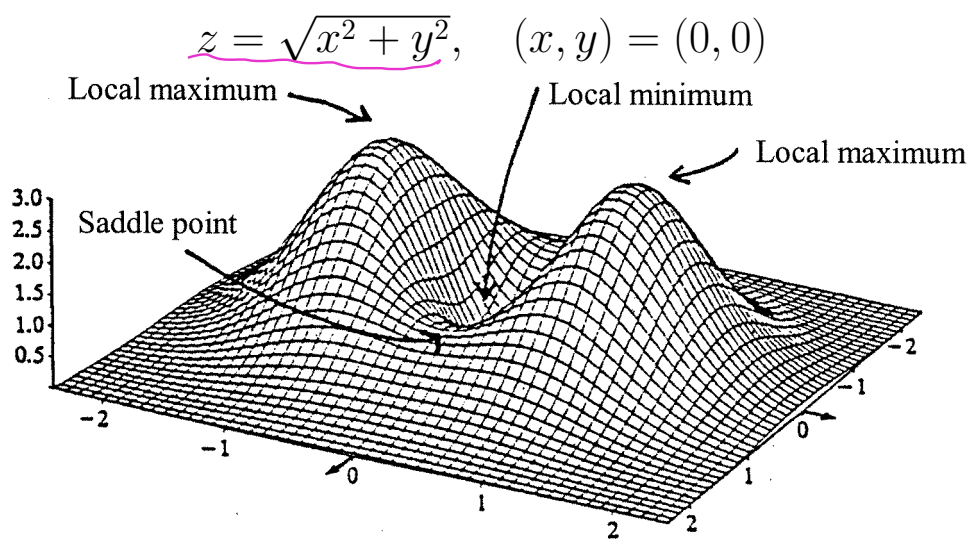
Then the Taylor series of  $f(x)$  about the point  $a$  up to order 2 is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots$$

Hence or otherwise, find the Taylor series of  $g(x, y, z) = \cos(x + y + z) - \cos x \cos y \cos z$  about the point  $(0, \frac{\pi}{2}, \frac{\pi}{2})$  up to order 2 .



Linear and quadratic approximations to the cone



$$z = f(x, y) = (x^2 + 3y^2) e^{1-x^2-y^2}$$