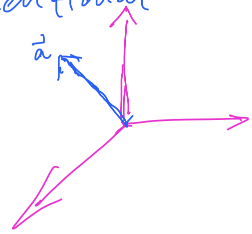
Chapter 0 Review of Linear Algebra

1. Vectors

* $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$

eg. $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \longrightarrow$ three dimentional vector.

$|\vec{a}| = \sqrt{1^2+0^2+2^2} = \sqrt{5}$



* magnitude (length) $|\vec{a}| = \sqrt{a_1^2+a_2^2+\cdots+a_n^2}$

* Unit vector $|\vec{a}|=1$, standard unit vector

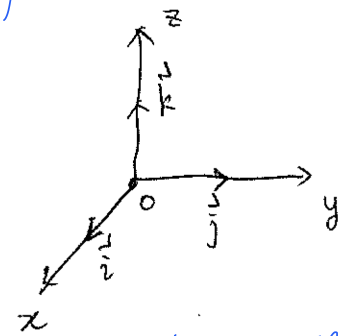
$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

Zero vector $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

* $\vec{a} \neq 0$, unit vector $\dfrac{\vec{a}}{|\vec{a}|}$

$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ normalize $\dfrac{\vec{v}}{|\vec{v}|} = \begin{pmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{pmatrix}$

* $\mathbb{R}^3$: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$



$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

2. Vector Operations

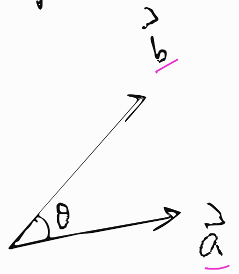
linear operation:

* $\vec{a} \pm \vec{b}$
* $\underset{\substack{\uparrow \\ \text{scalar}}}{m}\vec{a}$

$3 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 \\ 3\cdot 0 \\ 3\cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$

* $m(\vec{a} \pm \vec{b}) = m\vec{a} \pm m\vec{b}$.
* $(m \pm n)\vec{a} = m\vec{a} \pm n\vec{a}$.

3. Scalar product, Cross product and Triple scalar product



Scalar product,

$$* \quad \vec{a}\cdot\vec{b} = \sum_{i=1}^{n} a_i b_i = a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

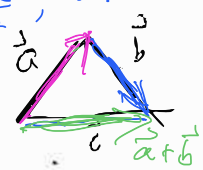
$$= |\vec{a}||\vec{b}|\cos\theta$$

eg. $\vec{a} = \begin{pmatrix}1\\0\\2\end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix}-1\\3\\5\end{pmatrix}$

$\vec{a}\cdot\vec{b} = 1\cdot(-1) + 0\cdot 3 + 2\cdot 5 = 9$

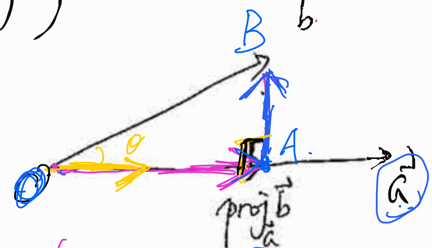
If $\vec{a}\cdot\vec{b} = 0$, then $\theta = \frac{\pi}{2}$, $\vec{a}\perp\vec{b}$.

$$* \quad |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$$



$$* \quad |\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}| \quad \text{(Schwartz inequality)}$$

$$* \quad |\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}| \quad \text{(Triangle inequality)}$$



$*$ $proj_{\vec{a}}\vec{b}$: projection vector of $\vec{b}$ on $\vec{a}$

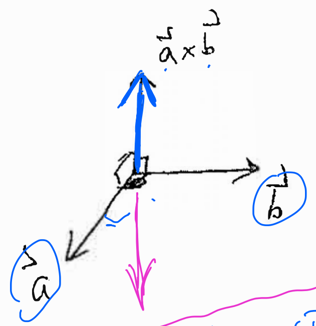
$$proj_{\vec{a}}\vec{b} = \underbrace{\left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)}_{\text{Scalar}} \underset{\uparrow\ \text{vector}}{\vec{a}} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \left(\frac{\vec{a}}{|\vec{a}|}\cdot\vec{b}\right)\frac{\vec{a}}{|\vec{a}|}$$

Example: Compute $proj_{\vec{a}}\vec{b}$ given $\vec{a} = \begin{pmatrix}4\\0\\3\\0\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}3\\3\\1\\2\end{pmatrix}$

Solution:

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} = \frac{15}{25}\begin{pmatrix}4\\0\\3\\0\end{pmatrix} = \frac{3}{5}\begin{pmatrix}4\\0\\3\\0\end{pmatrix} = \begin{pmatrix}\frac{12}{5}\\0\\\frac{9}{5}\\0\end{pmatrix}$$

$*$ **Cross product**: $\vec{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$, $\vec{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} \in \mathbb{R}^3$



$$\vec{a}\times\vec{b} = \underbrace{\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}_{3\times3}} \neq \vec{b}\times\vec{a}$$

$$\boxed{\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})}$$

$\vec{a}\times\vec{b}$ { magnitude $|\vec{a}||\vec{b}|\sin\theta$ ✓

direction: $\vec{a}\times\vec{b} \perp \vec{a}$, $\vec{b}$ right-hand rule. ✓

$\vec{a} = \begin{pmatrix}1\\0\\2\end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix}-1\\1\\3\end{pmatrix}$ $\quad$ - -

$$\underline{\vec{a}\times\vec{b}} = \det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 2\\ -1 & 1 & 3\end{pmatrix} = -2\cdot\vec{i} + (-1)\begin{vmatrix}1 & 2\\ -1 & 3\end{vmatrix}\vec{j} + \begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}\vec{k} = -2\vec{i} - 5\vec{j} + \vec{k}$$

vector

notation

$\vec{a} \in R^n$, n-dimensional vector. $= \begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}$.

eg, $\vec{a} \in R^3$ $\quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$.

magnitude (length) : $|\vec{a}| = \sqrt{\sum_{j=1}^{n} a_j^2}$.

unit vector : $|\vec{a}| = 1$ then $\vec{a}$ is called unit

normalize a vector : If $\vec{a} \neq \vec{0}$, then $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector.

standard ~~u~~ unit vector : $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ... $\vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

zero vector : $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

operations (I)

linear operation

$\vec{a} \pm \vec{b}$.

$m\vec{a}$

$m(\vec{a} \pm \vec{b}) = m\vec{a} + m\vec{b}$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$.

Jan 16 (Thur),

operations (II)

Scalar product (Inner product)

$\vec{a} \cdot \vec{b} = \sum_{j=1}^{n} a_j b_j \qquad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

cross product.

$\vec{a} \times \vec{b}$ is still a vector whose magnitude is $|\vec{a}||\vec{b}| \sin\theta$.

direction is ~~orthog~~ perpenticular with both $\vec{a}$, $\vec{b}$.

properties of a set of vectors.

linear dependent / linear independent.

orthogonality.

$$\underbrace{\begin{pmatrix}1\\2\\4\end{pmatrix}}_{\vec{u}} = \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{\vec{a}} + 2\underbrace{\begin{pmatrix}0\\1\\2\end{pmatrix}}_{\vec{b}}$$

$$\overset{v_3}{\vec{u}} = \overset{v_1}{\vec{a}} + 2\overset{v_2}{\vec{b}}$$

$\{\vec{u}, \vec{a}, \vec{b}\}$ is linear dependent.

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is linear dependent $\Longleftrightarrow$ $a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{0}$ and ~~at~~ at least one of coefficients is nonzero

if $\vec{v}_j = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_{j-1}\vec{v}_{j-1} + a_{j+1}\vec{v}_{j+1} + \dots + a_m\vec{v}_m$

$$\vec{a} + 2\vec{b} - \vec{u} = \vec{0}$$

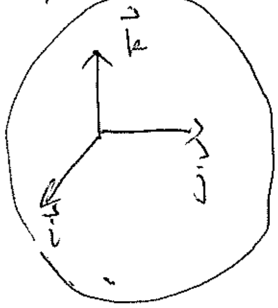
linear dependent.

$\vec{v}_j = a_1\vec{v}_1 + \dots + a_{j-1}\vec{v}_{j-1} + \dots + a_m\vec{v}_m$ $\Longleftrightarrow$ $a_1\vec{v}_1 + \dots + a_m\vec{v}_m = \vec{0}$

nontrivial linear combination.

4 Linear dependence and Linear Independence

* $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_k$ are $\boxed{\text{linearly dependent}}$ if there exists $1 \le i \le k$ such that $\vec{a}_i = \sum_{\substack{j=1 \\ j \ne i}}^{k} m_j \vec{a}_j$. If no vector can be represented by other vectors, then $\vec{a}_1, \cdots, \vec{a}_k$ are said to be $\boxed{\text{linearly independent}}$.



* $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_k$ are linearly independent iff.

$$\boxed{\sum_{j=1}^{k} m_j \vec{a}_j = 0 \Rightarrow m_j = 0, \ j = 1, \cdots, k.}$$

* Ex. Are $\vec{a} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ linearly indep?

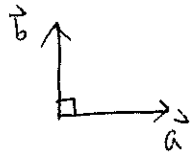
$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} 3 & 5 & -2 \\ 0 & 4 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c} \text{ are linearly dependent!}$$

Q: Are 4 vectors in $\mathbb{R}^3$ linear independent or not?

Thm: $(n+1)$ vectors in $\mathbb{R}^n$ are always linearly dependent!

5. Orthogonality

* $\vec{a}$ and $\vec{b}$ are said to be orthogonal if $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow (\vec{a} \perp \vec{b})$



* $\{a_1, \cdots, a_k\}$ is said to be orthogonal if

$$a_i \perp a_j \text{ for } 1 \le i, j \le k \text{ and } i \ne j.$$

* $\{a_1, \cdots, a_k\}$ is said to be orthonormal if it is orthogonal and

$$|\vec{a}_j| = 1, \quad j = 1, \cdots, k.$$

If $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is orthogonal, then it must be linear independent.
(the converse is not necessary).

eg. $\{\vec{i}, \vec{j}, \vec{k}\}$ is orthonormal

$$x_1\vec{i} + x_2\vec{j} + x_3\vec{k} = \vec{0} \Rightarrow x_1\begin{pmatrix}1\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\Leftrightarrow \begin{pmatrix}x_1+0+0\\0+x_2+0\\0+0+x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow x_1 = x_2 = x_3 = 0.$$

eg. $\vec{v}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ is linear independent. but not orthogonal.

$\vec{v}_1 \cdot \vec{v}_2 = 1 \neq 0$ so $\vec{v}_1, \vec{v}_2$ not orthogonal.

$x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{0} \Rightarrow \begin{pmatrix}x_1\\2x_1\\0\end{pmatrix} + \begin{pmatrix}x_2\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \checkmark \Rightarrow \underline{x_1 = x_2 = 0}$, so $\vec{v}_1, \vec{v}_2$ linear indep.

$0\vec{v}_1 + 0\vec{v}_2 = \vec{0}$

nontrivial linear combination of

$\underline{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}}.$

$$\underline{2}\begin{pmatrix}1\\2\\3\end{pmatrix} \underline{-}\begin{pmatrix}4\\5\\6\end{pmatrix} \underline{+}\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

2 $\quad$ -1 $\quad$ +1

6. Matrix (Square matrix) $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\times 1}$

* $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n} = (\vec{a}_1, \cdots, \vec{a}_n) = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$$

column vector, row vector, entry

* Diagonal matrix eg: $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}_{3\times 3}$

$$D = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

identity matrix

Zero matrix $O = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$

7. Matrix Operations

* $A \pm B$, cA

* Transpose A^T.

eg. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{3\times 3}$, $A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

$A^T = A$ symmetric matrix eg. $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 5 \\ 0 & 5 & 3 \end{pmatrix}$

* $\boxed{\text{product}}$.

$$C = AB = (c_{ij})$$

$$\underbrace{C}_{m\times l} = \underbrace{A}_{m\times n}\,\underbrace{B}_{n\times l}$$

$AB \neq BA$ in general.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kl}$$

eg $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}_{2\times 3} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}_{3\times 1} = \begin{pmatrix} 1\cdot 1 + 2\cdot 3 + 3\cdot(-1) \\ -1\cdot 1 + 0\cdot 3 + 4\cdot(-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}_{2\times 1}$

* **Determinant** (square matrix) A: $n\times n$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}_{2\times 2} = \boxed{a_{11}a_{22} - a_{12}a_{21}}$$

$$\det(A) = |A|$$

eg. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$

$|A| = 1\cdot 4 - 2\cdot 3 = -2$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}_{3\times 3} = (-1)^{1+1} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} + (-1)^{1+2} a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33}\end{vmatrix}$$

$B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 0 & 2 & 1\end{pmatrix}$

$\det(B) = |B| = 1\cdot\begin{vmatrix} 4 & 0 \\ 2 & 1\end{vmatrix} - 2\begin{vmatrix} 3 & 0 \\ 0 & 1\end{vmatrix} + (-1)\cdot\begin{vmatrix} 3 & 4 \\ 0 & 2\end{vmatrix} = \cdots$

$A = (a_{ij})_{1\le i,j\le n}$

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

Cofactor

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Cofactor of a_{ij} — Minor of a_{ij}

$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & \cancel{a_{1j}} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \cancel{a_{i1}} & \cdots & \cancel{a_{ij}} & \cdots & \cancel{a_{in}} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & \cancel{a_{nj}} & \cdots & a_{nn}\end{vmatrix}.$$

* $\det(A) = \det(A^T)$, $\boxed{\det(AB) = \det A\cdot\det B}$

* **Rank**: rank A is the maximal number of linearly independent row vectors

$\mathrm{rank}(A) = \mathrm{rank}(A^T)$

eg. $A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & -1 \end{pmatrix}_{2\times 3}$ $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}_{3\times 3}$

$$(AB)_{2\times 3} = \begin{pmatrix} 1\cdot 1+1\cdot 2+3\cdot(-1) & 1\cdot 0+1\cdot 1+3\cdot 1 & (1,1,3)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} \\ (2\ 3\ -1)\begin{pmatrix}1\\2\\-1\end{pmatrix} & (2\ 3\ -1)\begin{pmatrix}0\\1\\1\end{pmatrix} & (2\ 3\ -1)\begin{pmatrix}1\\0\\0\end{pmatrix} \end{pmatrix}$$

$\underset{3\times 3}{B}\ \underset{2\times 3}{A}$ is not defined

$$= \begin{pmatrix} 0 & 4 & 1 \\ 9 & 2 & 2 \end{pmatrix}_{2\times 3}$$

Note : a vector $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ can be regarded as a matrix with <u>only one column</u>

eg. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix}_{3\times 3}$ $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}_{3\times \underline{1}}$ ✓

$$A\vec{x} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 0 + 3\cdot 2 \\ 4\cdot 1 + 1\cdot 0 + 0\cdot 2 \\ 2\cdot 1 + (-1)\cdot 0 + 2\cdot 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}.$$

Note: usually, $AB \neq BA$.

- Determinant

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. $\det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$.

$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}_{3\times 3}$. $\det(A) = 1\cdot\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} - 0\begin{vmatrix} -1 & 1 \\ 0 & 4 \end{vmatrix} + 2\begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix}$.

$= \ldots$