Tutorial 8

1. Consider two random variables X and Y with joint probability mass function (PMF) given in the following table:

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.01	0	0
$X = 1$	0.09	0.09	0
$X = 2$	0	0	0.81

(a) Compute the correlation of X and Y, i.e., $r_{X,Y} = \mathbb{E}\{XY\}$.
(b) Compute $\mathrm{cov}(X, Y)$.
(c) Compute correlation coefficient $\rho_{X,Y}$.

2. Express $\mathrm{cov}(X+Y, X+Y)$ in terms of $\mathrm{var}(X)$, $\mathrm{var}(Y)$ and $\mathrm{cov}(X,Y)$.

3. Prove the following property of correlation coefficient of random variables X and Y with variances σ_X^2 and σ_Y^2:

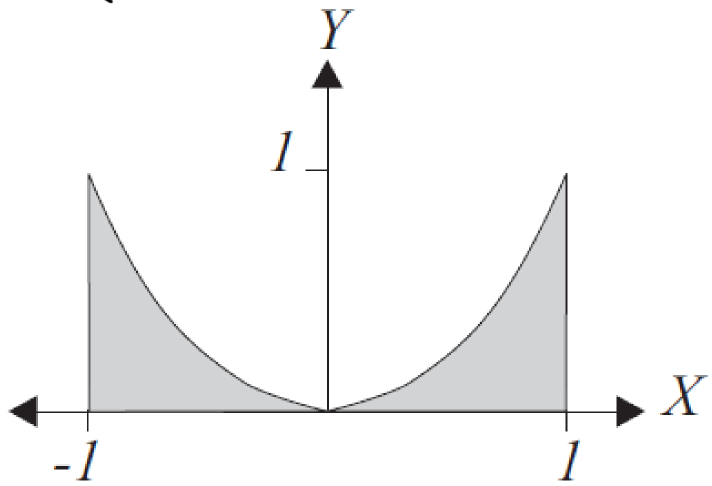
$$-1 \leq \rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} \leq 1$$

Hint: Let $W = X - aY$ where $a \in \mathbb{R}$ and then consider $\mathrm{var}(W)$ with suitable values of a.

4. A random variable X is transformed to another random variable $Y = aX + b$ where a and b are constants. Suppose $a < 0$. Determine $\rho_{X,Y}$.

5. Random variables X and Y have the following joint probability density function (PDF):

$$P_{XY}(x,y) = \begin{cases} 5x^2/2, & -1 \le x \le 1; 0 \le y \le x^2 \\ 0, & \text{otherwise} \end{cases}$$



(a) Compute $\mathbb{E}\{X\}$ and $\mathrm{var}(X)$.
(b) Compute $\mathbb{E}\{Y\}$ and $\mathrm{var}(Y)$.
(c) Compute $\mathrm{cov}(X,Y)$.
(d) Compute $\mathbb{E}\{X+Y\}$ and $\mathrm{var}(X+Y)$.

Solution

1.(a)

$$r_{X,Y} = \mathbb{E}\{XY\} = \sum_{x=0}^{2}\sum_{y=0}^{2} xyP_{XY}(x,y)$$
$$= (1)(1)(0.09) + (2)(2)(0.81) = 3.33$$

1.(b)

The marginal PMFs are:

$$p(x) = \begin{cases} 0.01, & x = 0 \\ 0.18, & x = 1 \\ 0.81, & x = 2 \\ 0, & \text{otherwise} \end{cases} \qquad p(y) = \begin{cases} 0.1, & y = 0 \\ 0.09, & y = 1 \\ 0.81, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

Then we compute the expected values:

$$\mathbb{E}\{X\} = (1)(0.18) + (2)(0.81) = 1.8$$
$$\mathbb{E}\{Y\} = (1)(0.09) + (2)(0.81) = 1.71$$

Using (3.21), we have:

$$\text{cov}(X,Y)=\mathbb{E}\{XY\}-\mathbb{E}\{X\}\mathbb{E}\{Y\}=0.252$$

1.(c)

$$\begin{aligned}\mathbb{E}\{X^2\} &= (1)^2(0.18)+(2)^2(0.81)=3.42\\ \mathbb{E}\{Y^2\} &= (1)^2(0.09)+(2)^2(0.81)=3.33\end{aligned}$$

Applying (2.23) yields:

$$\text{var}(X)=\mathbb{E}\{X^2\}-(\mathbb{E}\{X\})^2=0.18$$
$$\text{var}(Y)=\mathbb{E}\{Y^2\}-(\mathbb{E}\{Y\})^2=0.4059$$

Using (3.25), we have

$$\rho_{X,Y}=\frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}}=\frac{0.252}{\sqrt{0.18}\sqrt{0.4059}}=0.9323$$

2.

According to the commutative and distributive properties of covariance, we get:

$$
\begin{aligned}
\operatorname{cov}(X+Y, X+Y) &= \operatorname{cov}(X, X) + \operatorname{cov}(X, Y) + \operatorname{cov}(Y, X) + \operatorname{cov}(Y, Y) \\
&= \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X, Y)
\end{aligned}
$$

which aligns with (3.23) for $n = 2$.

3.
Let $W = X - aY$ where $a \in \mathbb{R}$. Applying the result in Question 2, the variance of W is:

$$\text{var}(W) = \text{var}(X) + a^2\text{var}(Y) - 2a\text{cov}(X, Y)$$

As variance must be nonnegative, we have:

$$\begin{aligned} \text{var}(W) \geq 0 &\Rightarrow \text{var}(X) + a^2\text{var}(Y) \geq 2a\text{cov}(X, Y) \\ &\Rightarrow \sigma_X^2 + a^2\sigma_Y^2 \geq 2a\text{cov}(X, Y) \end{aligned}$$

Note that the inequality holds for all $a \in \mathbb{R}$. Set $a = \sigma_X/\sigma_Y > 0$:

$$2\sigma_X^2 \geq 2\sigma_X/\sigma_Y \cdot \text{cov}(X, Y) \Rightarrow \sigma_X\sigma_Y \geq \text{cov}(X, Y) \Rightarrow \rho_{X,Y} \leq 1$$

We then set $a = -\sigma_X/\sigma_Y < 0$:

$$-2\sigma_X^2 \leq 2\sigma_X/\sigma_Y \cdot \text{cov}(X, Y) \Rightarrow -\sigma_X\sigma_Y \leq \text{cov}(X, Y) \Rightarrow \rho_{X,Y} \geq -1$$

Combining the results yields $1 \geq \rho_{X,Y} \geq -1$.

4.

From the results of Question 4 in Tutorial 6, we have:

$$\mathbb{E}\{Y\} = \mu_y = \mathbb{E}\{aX + b\} = \mathbb{E}\{aX\} + \mathbb{E}\{b\} = a\mathbb{E}\{X\} + b = a\mu_x + b$$

$$\text{var}(Y) = \sigma_y^2 = \mathbb{E}\{(Y - \mu_y)^2\} = a^2\text{var}(X) = a^2\sigma_x^2$$

According to (3.21), we obtain:

$$\begin{aligned}\text{cov}(X,Y) &= \mathbb{E}\{(X-\mu_x)(aX + b - a\mu_x - b)\} \\ &= \mathbb{E}\{(X-\mu_x)(aX - a\mu_x)\} = a\mathbb{E}\{(X-\mu_x)^2\} = a\sigma_x^2\end{aligned}$$

From (3.25), the correlation coefficient is:

$$\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} = \frac{a\sigma_x^2}{\sqrt{\sigma_x^2}\sqrt{a^2\sigma_x^2}} = \frac{a}{|a|}$$

When $a < 0$, we get $\rho_{X,Y} = -1$.

5.(a)

$$\mathbb{E}\{X\} = \int_{-1}^{1}\int_{0}^{x^2} x\frac{5x^2}{2}dydx = \int_{-1}^{1}\frac{5x^5}{2}dx = -\frac{5x^6}{12}\bigg|_{-1}^{1} = 0$$

Since $\mathbb{E}\{X\} = 0$, $\mathrm{var}(X) = \mathbb{E}\{X^2\}$:

$$\mathrm{var}(X) = \mathbb{E}\{X^2\} = \int_{-1}^{1}\int_{0}^{x^2} x^2\frac{5x^2}{2}dydx = \int_{-1}^{1}\frac{5x^6}{2}dx = \frac{5x^7}{14}\bigg|_{-1}^{1} = \frac{5}{7}$$

5.(b)

$$\mathbb{E}\{Y\} = \int_{-1}^{1}\int_{0}^{x^2} y\frac{5x^2}{2}dydx = \frac{5}{14}$$

$$\mathbb{E}\{Y^2\} = \int_{-1}^{1}\int_{0}^{x^2} y^2\frac{5x^2}{2}dydx = \frac{5}{27}$$

Hence

$$\mathrm{var}(Y) = \mathbb{E}\{Y^2\} - (\mathbb{E}\{Y\})^2 = 0.0576$$

5.(c)
Since $\mathbb{E}\{X\} = 0$, we have:

$$\mathrm{cov}(X,Y) = \mathbb{E}\{XY\} - \mathbb{E}\{X\}\mathbb{E}\{Y\} = \mathbb{E}\{XY\}$$

Hence

$$\mathrm{cov}(X,Y) = \mathbb{E}\{XY\} = \int_{-1}^{1}\int_{0}^{x^2} xy\frac{5x^2}{2}dydx = \int_{-1}^{1}\frac{5x^7}{4}dx = 0$$

5.(d)

$$\mathbb{E}\{X+Y\} = \mathbb{E}\{X\} + \mathbb{E}\{Y\} = \frac{5}{14}$$

$$\mathrm{var}(X+Y) \;=\; \mathrm{var}(X) + \mathrm{var}(Y) + 2\mathrm{cov}(X,Y) = 0.7719$$