EE3210 Signals & Systems

Due on 11:00 AM, May 15, 2021

Homework #2, 3

1. Total mark is 200 points (= 20 points per problem $\times$ 10 problems)
2. Submission due by 11:00 AM, May 15, 2021. We will not accept late submission.
3. Online submission through Canvas
   - Scan or taking a photo of your answer sheet, then upload to Canvas

Problem 1

Let's consider LTI systems described by the following differential equations. Derive the frequency responses $H(f)$, the corresponding impulse response $h(t)$, and the step responses using Fourier transform.

a)

$$\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

b)

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

HW 2-3 Solution

Q1. a)

$$((j2\pi f)+2)\,Y(f) = (1+j2\pi f)\,X(f)$$

freq response: $\Rightarrow H(f) = \dfrac{Y(f)}{X(f)} = 1 - \dfrac{1}{1+j2\pi f}$

impulse response: $h(t) = \delta(t) - e^{-2t}u(t)$

Step response

$$Y(f) = H(f)\,\mathcal{F}(u(t)) = H(f)\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) + \frac{1}{2}\left(\frac{1}{2+j2\pi f}\right)$$

$$y(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{-2t}u(t)$$

b) $$((j2\pi f)^2 + 6(j2\pi f)+8)\,Y(f) = 2X(f).$$

$$\Rightarrow H(f) = \frac{Y(f)}{X(f)} = \frac{1}{2+j2\pi f} - \frac{1}{4+j2\pi f}$$

$$h(t) = e^{-2t}u(t) - e^{-4t}u(t)$$

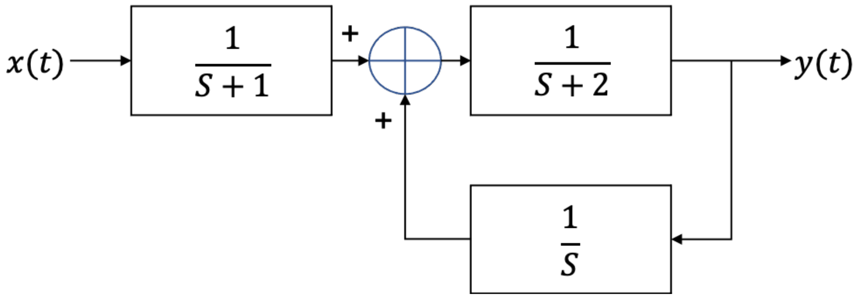
Step response

$$Y(f) = H(f)\,\mathcal{F}(u(t)) = \frac{1}{4}\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) - \frac{1}{2}\cdot\frac{1}{2+j2\pi f} + \frac{1}{4}\frac{1}{4+j2\pi f}$$

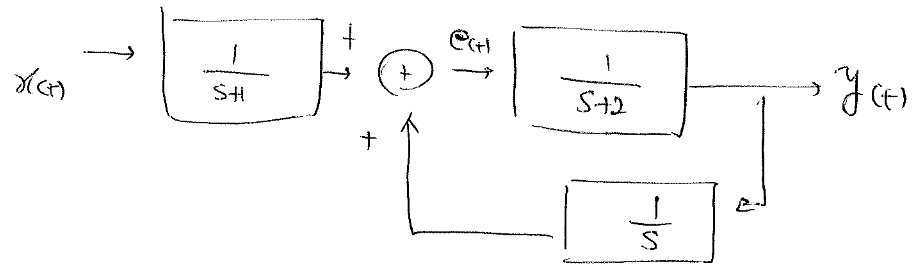
$$y(t) = \frac{1}{4}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

Problem 2

Determine the transfer function (or system function) $H(s)$ of the following system model.



Q2.



$$\begin{cases} E(S) = \dfrac{1}{S+1} X(S) + \dfrac{1}{(S+2)S} E(S) & -① \\ Y(S) = \dfrac{1}{S+2} E(S) & -② \end{cases}$$

$\Rightarrow$ From ①, $\left(1 - \dfrac{1}{S(S+2)}\right) E(S) = \dfrac{1}{S+1} X(S) \quad -③$

Substitute ③ to ②

$$Y(S) = \frac{1}{(S+2)} \cdot \frac{1}{(S+1)} \cdot \frac{1}{1 - \frac{1}{S(S+2)}} X(S)$$

$$= \frac{S}{S^3+3S^2+S-1} X(S)$$

$\Rightarrow \therefore H(S) = \dfrac{Y(S)}{X(S)} = \dfrac{S}{S^3+3S^2+S-1}$

Problem 3

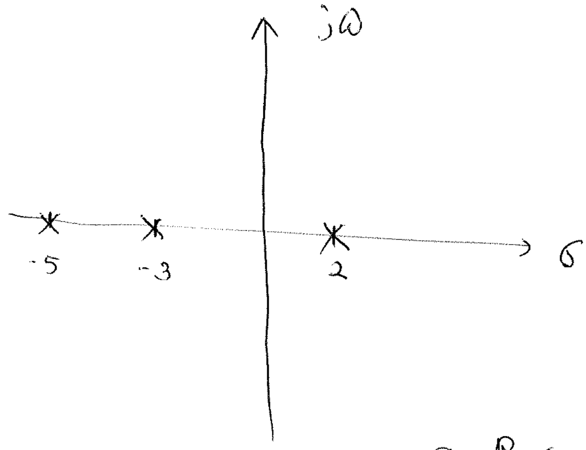
Let us consider LTI system with a following transfer function $H(s)$

$$H(s) = \frac{4}{(S+5)(S+3)(S-2)}.$$

a) Indicate all possible ROC that can be associated with the given $H(s)$.

b) For each ROC, specify whether the associated system is stable and/or causal.

Q3. Three poles at $S=2$, $S=-3$, $S=-5$



a) Possible ROC = $\begin{cases} Re(s) > 2 \\ -3 < Re(s) < 2 \\ -5 < Re(s) < -3 \\ Re(s) < -5 \end{cases}$

b)

$Re(s) > 2$: Unstable Causal

$-3 < Re(s) < 2$: Stable, Noncausal

$-5 < Re(s) < -3$: Unstable, Noncausal

$Re(s) < -5$: Unstable, Noncausal

Problem 4

Find the discrete time Fourier transform of the following signals

a)

$$x[n] = -a^n u\left[-n-1\right], \quad a \text{ is real}$$

b)

$$x[n] = u\left[n\right] - u\left[n-N\right]$$

Q 4

a)

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

$$= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] e^{-j2\pi f n}$$

$$= -\sum_{n=-\infty}^{-1} \left(a e^{-j2\pi f}\right)^n = -\sum_{l=1}^{\infty} \left(\frac{1}{a} e^{j2\pi f}\right)^l \quad \left(\begin{array}{c}\text{by using} \\ \text{change of variable} \\ n = -l\end{array}\right)$$

$$= -\frac{\frac{1}{a} e^{j2\pi f}}{1 - \frac{1}{a} e^{j2\pi f}} \quad \text{when} \quad \frac{1}{|a|} < 1 \Leftrightarrow |a| > 1$$

$$= \frac{1}{1 - a e^{-j2\pi f}} \quad \text{and} \quad |a| > 1$$

b)

$$X(f) = \sum_{n=0}^{N-1} e^{-j2\pi f n} = \frac{1 - e^{-j2\pi f N}}{1 - e^{-j2\pi f}} = e^{-j\pi f (N-1)} \cdot \frac{\sin(\pi f N)}{\sin(\pi f)}$$

Problem 5

Prove the following four properties of discrete time Fourier transform based on the definition of DTFT

$$X(f) = \sum_{n=-\infty}^{n=\infty} x[n]e^{-j2\pi fn}$$

a) Time shifting property

$$x[n-n_0] \leftrightarrow e^{-j2\pi fn_0}X(f)$$

b) Frequency shifting property

$$e^{j2\pi f_0 n}x[n] \leftrightarrow X(f-f_0)$$

c) Differentiation

$$nx[n] \leftrightarrow \frac{j}{2\pi} \cdot \frac{dX(f)}{df}$$

Q 5.

a)

$$\sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j2\pi f n} = \underbrace{\left\{ \sum_{l=-\infty}^{\infty} x[l]\, e^{-j2\pi f l} \right\}}^{X(f)} \cdot e^{-j2\pi f n_0}$$

use change of variable $n - n_0 = l$

$$\Rightarrow \quad X(f) \cdot e^{-j2\pi f n_0}$$

b)

$$\sum_{n=-\infty}^{\infty} e^{j2\pi f_0 n}\, x[n]\, e^{-j2\pi f n} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi (f-f_0) n}$$

$$= X(f-f_0)$$

c)

$$\frac{d}{df}\left(X(f) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi f n} \right)$$

$$\Rightarrow \quad \sum_{n=-\infty}^{\infty} (-j2\pi n)\, x[n]\, e^{-j2\pi f n} = \frac{dX(f)}{df}$$

$$\sum_{n=-\infty}^{\infty} (n\, x[n])\, e^{-j2\pi f n} = \frac{j}{2\pi} \times \frac{dX}{df}.$$

Therefore

$$n\, x[n] \longleftrightarrow \frac{j}{2\pi} \frac{dX}{df}$$

Problem 6

Derive the Fourier transform of the following signals.

a)
$$x(t) = \mathrm{sinc}^2(4t)$$

b)
$$x(t) = \mathrm{sgn}(t)$$

c)
$$x(t) = e^{-2t}u(t)$$

d)
$$x(t) = \mathrm{rect}\left(\frac{t}{3}\right)$$

Q 6.

a) $\mathcal{F}(\operatorname{sinc}^2(4t)) = \frac{1}{4}\operatorname{tri}\left(\frac{f}{4}\right).$

b) $\mathcal{F}(\operatorname{sgn}(t)) = \frac{1}{j\pi f}$

c) $\mathcal{F}(e^{-2t}u(t)) = \frac{1}{2+j2\pi f}$

d) $\mathcal{F}\left(\operatorname{rect}\left(\frac{t}{3}\right)\right) = 3\operatorname{sinc}(3f)$

Problem 7

a) Derive (bilateral) Laplace transform of $x(t) = e^{-5t}u(t)$ and define the ROS.

b) Derive (bilateral) Laplace transform of $x(t) = tu(t)$ and define the ROS.

c) Derive (bilateral) Laplace transform of $x(t) = x_1(t-4)$ where $x_1(t) = e^{-3t}u(t)$

d) Find the inverse Laplace transform of the following $X(s)$

$$X(s) = \frac{S^2 + 2S + 5}{(S+3)(S+5)^2}, \quad \mathrm{Re}(S) > -3$$

Q 7.

a) $X(s) = \frac{1}{s+5}$, $Re(s) > -5$

b) $X(s) = \frac{1}{s^2}$, $Re(s) > 0$.

c) $X_1(s) = \frac{1}{s+3}$, $Re(s) > -3$

$\mathcal{L}(x_1(t-4)) = e^{-4s} X_1(s) = \frac{e^{-4s}}{s+3}$, $Re(s) > -3$

d) $X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}$, $Re(s) > -3$

$= \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$

$x(t) = \left[2e^{-3t} - (1+10t)e^{-5t}\right] u(t)$

Problem 8

Derive the Z-transform of $x[n] = r^n \sin(w_0 n)\, u[n]$.

Q 8.

$$X(z) = \sum_{n=0}^{\infty} \sin(\omega_0 n) \, (r z^{-1})^n$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left[e^{j\omega_0 n} - e^{-j\omega_0 n} \right] (r z^{-1})^n$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} (e^{j\omega_0} r z^{-1})^n - \sum_{n=0}^{\infty} (e^{-j\omega_0} r z^{-1})^n \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega_0} r z^{-1}} - \frac{1}{1 - e^{-j\omega_0} r z^{-1}} \right] \quad \& \quad |r z^{-1}| < 1$$

$\Rightarrow$ ROC: $|z| > |r|$

$$= \frac{2j \sin(\omega_0)(r z^{-1})}{2j \left(1 - 2\cos\omega_0 \, r z^{-1} + (r z^{-1})^2 \right)}$$

$$\therefore \quad X(z) = \frac{\sin(\omega_0) \cdot r z^{-1}}{1 - 2\cos\omega_0 \cdot r z^{-1} + (r z^{-1})^2}$$ and ROC $|z| > |r|$

Problem 9

a) Find the Z-transform of the following sequence

$$x[n] = \left\{ 5, 3, \underset{\uparrow}{4}, 1, 0, 2 \right\}$$

b) Find the Z-transform of the following signal

$$x[n] = \frac{1}{3} \cdot \frac{1}{4^n}\ u[n] - \frac{2}{2^n}\ u[n] + \frac{8}{3}\ u[n]$$

c) Find the inverse Z-transform of the following $X(z)$

$$X(z) = \log\left(\frac{1}{1-Z}\right), \quad |Z| < 1$$

d) Find the inverse Z-transform of the following $X(z)$

$$X(z) = \frac{3}{Z-4}, \quad |Z| > 4$$

Q 9.

a) $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = 5z^{2} + 3z^{1} + 4 + z^{-1} - 2z^{-3}$

$ROC = \{0 < |z| < \infty\}$

b) $x[n] = \frac{1}{3} \cdot \frac{1}{4^n} u[n] - \frac{2}{2^n} u[n] + \frac{8}{3} u[n]$

$X(z) = \frac{1}{3} \underbrace{\frac{z}{z-\frac{1}{4}}}_{} - 2 \cdot \underbrace{\frac{z}{z-2}}_{} + \frac{8}{3} \underbrace{\frac{z}{z-1}}_{}$ ← both forms are ok.

ROC: $|z| > \frac{1}{4}$ ROC: $|z| > 2$ ROC: $|z| > 1$

$= \frac{z(z^2 - \frac{9}{2}z + \frac{3}{2})}{(z-\frac{1}{4})(z-2)(z-1)}$ with $ROC = \{|z| > 2\}$

c) $X(z) = \log\left(\frac{1}{1-z}\right)$ $|z| < 1$

$= -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^{n} \xrightarrow{\text{change of variable } n=-l} = -\sum_{l=-\infty}^{-1} \frac{1}{l} z^{-l}$

Since $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$, the inverse z-transform is

$x[n] = -\frac{1}{n} u[-n-1]$

d) $X(z) = \frac{3}{z-4}$, $|z| > 4$

$= 3z^{-1} \cdot \frac{z}{z-4} \longrightarrow \mathcal{Z}^{-1}(X(z)) = x[n] = 3 \cdot 4^{n-1} u[n-1]$

Problem 10

Solve the following difference equation using Unilateral Z-transform

$$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2],$$

where the initial conditions are given by

$$y[-1] = \frac{11}{6}, \quad y[-2] = \frac{37}{36},$$

and the input is $x[n] = 2^{-n}u[n]$.

Q 10.

$$Y(z) - 5\left(\frac{Y(z)}{z} + y[-1]\right) + 6\left(\frac{Y(z)}{z^2} + \frac{y[-1]}{z} + y[-2]\right)$$

$$= 3\left(\frac{X(z)}{z} + x[-1]\right) + 5\left(\frac{X(z)}{z^2} + \frac{x[-1]}{z} + x[-2]\right)$$

$\rightarrow$ by using the initial condition

$$Y(z) - 5\left(\frac{Y(z)}{z} + \frac{11}{6}\right) + 6\left(\frac{Y(z)}{z^2} + \frac{11}{6z} + \frac{37}{36}\right) = \frac{3X(z)}{z} + \frac{5X(z)}{z^2} \quad \text{— ①}$$

where the input is

$$X(z) = \frac{z}{z-0.5} \quad \text{— ②}$$

$\rightarrow$ Substitute ② into ① and by using partial fraction

$$Y(z) = \frac{26}{15}\cdot\frac{z}{z-0.5} - \frac{7}{3}\frac{z}{z-2} + \frac{18}{5}\frac{z}{z-3}$$

z^{-1}

$$\rightarrow y[n] = \frac{26}{15}(0.5)^n u[n] - \frac{7}{3}2^n u[n] + \frac{18}{5}3^n u[n]$$