Tutorial 3

1. Consider that you have 5 HKD100 notes and you want to divide among four people, say, A, B, C, and D. Determine the number of ways in doing this. Note that each person can receive 0 to 5 notes.

2. A computer program consists of 2 modules. The first and second modules has probabilities 0.2 and 0.4, respectively, to contain an error, and these two error events are independent. An error in the first module alone causes the program to crash with probability 0.5. While for the second module, this probability is 0.8. If there are errors in both modules, the program crashes with probability 0.9.

   Suppose the program crashed. Find the probability that there are errors in both modules.

3. Suppose 0.01% of bits transmitted through a digital transmission are received in error. Bits are transmitted until the first error occurs. Find the probability that there is no error in the first 10000 bit transmission.

4. With the use of the binomial theorem:

$$(\alpha+\beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \beta^k$$

Prove that the sum of all PMFs of a binomial random variable with parameters n and p is 1.

5. Consider the binomial distribution with parameters n and p. When n is fixed, what is the value of p if the probability of 0 success is equal to the probability of 1 success?

6. An airline knows 5% of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

Solution

1.

Since 0 note can be obtained, a possible arrangement for A, B, C, and D, can be 5,0,0,0 (AAAAA). Note also that this is not identical to 0,0,0,5 (DDDDD).

Hence it can be viewed as calculating the number of combinations with replacement. That is, selecting $k = 5$ from $N = 4$ when the order does not matter (e.g., AABBB = ABBBA):

$$C_r(4,5) = \frac{(4+5-1)!}{(4-1)!5!} = 56$$

2.

Let A, B, and C the events of having an error in first module, having an error in second module, and crash, respectively. Our task is to find $P(A \cap B|C)$.

From the given information, we have:

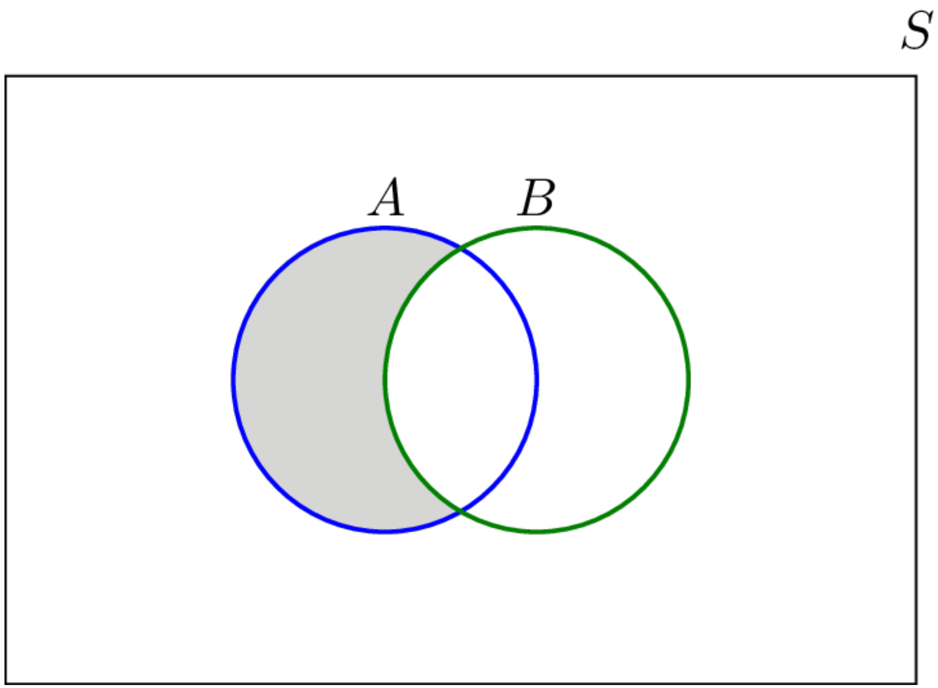
$$P(A) = 0.2$$
$$P(B) = 0.4$$
$$P(A \cap B) = P(A)P(B) = 0.08$$
$$P(C|A \setminus B) = 0.5$$
$$P(C|B \setminus A) = 0.8$$
$$P(C|A \cap B) = 0.9$$

We may notice that Bayes' rule can be applied:

$$P(A \cap B|C) = \frac{P(C|A \cap B) \cdot P(A \cap B)}{P(C)}$$

Based on the law of total probability, $P(C)$ can be computed as:

$$P(C) = P(C|A \setminus B)P(A \setminus B) + P(C|B \setminus A)P(B \setminus A) + P(C|A \cap B)P(A \cap B) + P(C|\overline{A \cup B})P(\overline{A \cup B})$$



Note that $A \setminus B$, $B \setminus A$, $A \cap B$, and $\overline{A \cup B}$ are pairwise disjoint, and their union is the sample space.

Recall $P(A \cap B) = P(A)P(B) = 0.08$

$$P(A \setminus B) = P(A) - P(A \cap B) = 0.2 - 0.08 = 0.12$$
$$P(B \setminus A) = P(B) - P(A \cap B) = 0.4 - 0.08 = 0.32$$

Hence

$$P(\overline{A \cup B}) = 1 - 0.08 - 0.12 - 0.32 = 0.48$$

In fact, we do not need computing $P(\overline{A \cup B})$ because $P(C|\overline{A \cup B})$ must be 0.

Hence we get $P(A \cap B|C)$:

$$\frac{P(C|A \cap B) \cdot P(A \cap B)}{P(C)} = \frac{0.9 \cdot 0.08}{0.5 \cdot 0.12 + 0.8 \cdot 0.32 + 0.9 \cdot 0.08 + 0} = 0.1856$$

3.

Let the probability of error be p. Here, $p = 0.0001$ and the probability that there is an error in the first or second or up to the 10000th bit is:

$$p(1) + \cdots + p(10000)$$

We may make use of the CDF of geometric RV:

$$F(r) = P(X \leq r) = 1 - (1-p)^r$$

Then the required probability is:

$$1 - F(10000) = (1-p)^{10000} = 0.9999^{10000} = 0.3679$$

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>> (0.9999)^(10000)

ans = 0.3679
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4.

Recall the PMF of binomial RV:

$$p(r) = P(X = r) = C(n,r)p^r(1-p)^{n-r}, \quad 0 \leq r \leq n$$

The sum of all PMFs is then:

$$\sum_{r=0}^{n} C(n,r)p^r(1-p)^{n-r}$$

Applying the binomial theorem with $\alpha = p$ and $\beta = 1 - p$ yields:

$$(p + 1 - p)^n = 1$$

5.

Recall the PMF of binomial RV:

$$p(r) = P(X = r) = C(n,r)p^r(1-p)^{n-r}, \quad 0 \le r \le n$$

Now we need $p(0) = p(1)$:

$$C(n,0)p^0(1-p)^{n-0} = C(n,1)p^1(1-p)^{n-1} \Rightarrow 1-p = np \Rightarrow p = \frac{1}{n+1}$$

When n is larger, we need smaller value of probability of success to make $p(0) = p(1)$.

We may also realize that the PMF plot of binomial distribution can have different shapes by considering different values of p. Note the value of r when the PMF reaches its maximum.

```
n=50;
p=0.5;
for k=0:50;
    P(k+1)=nchoosek(n,k)*p^k*(1-p)^(n-k);
end
stem(0:50,P)
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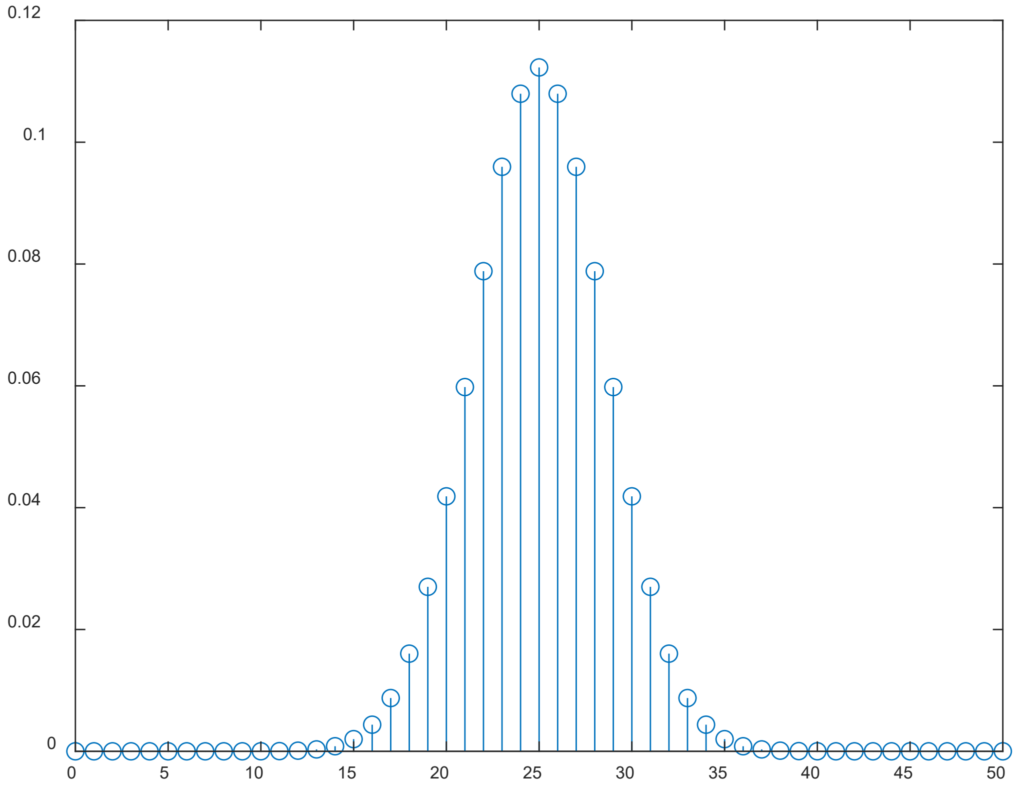

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C(1)=P(1);
for k=1:50;
    C(k+1)=C(k)+P(k+1);
end
stem(0:50,C)
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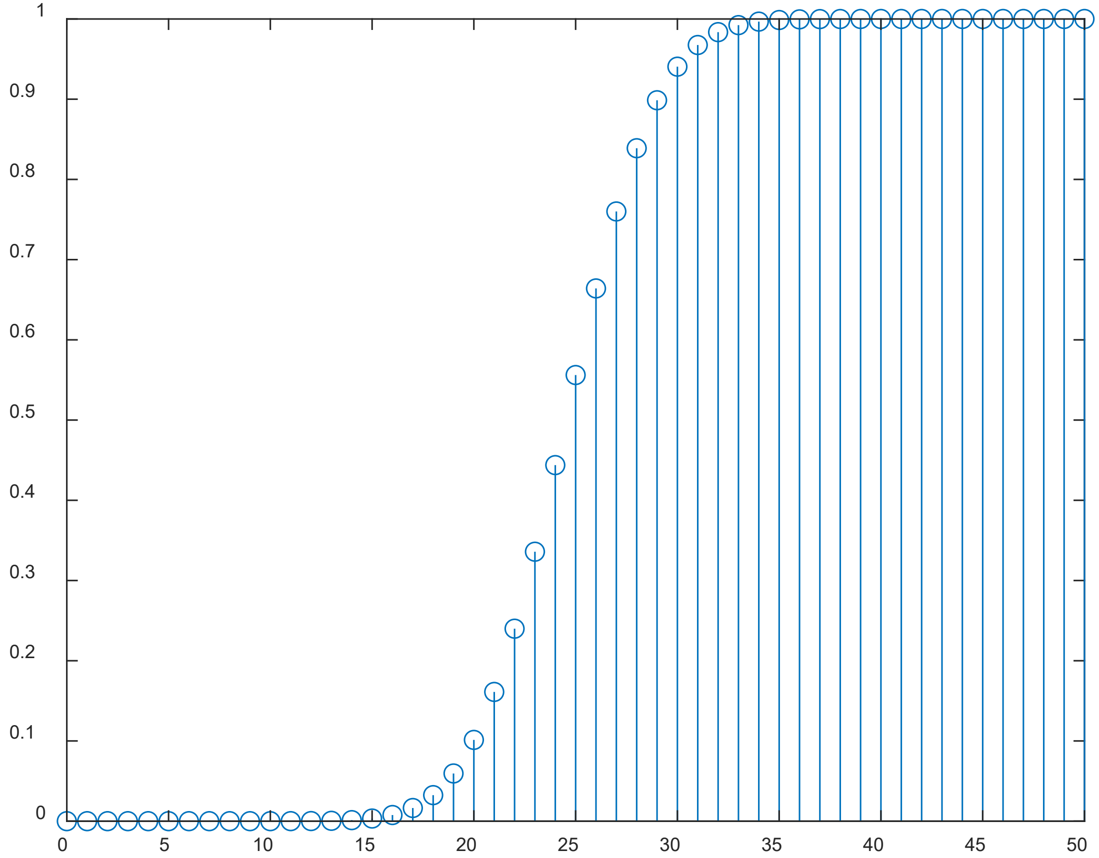
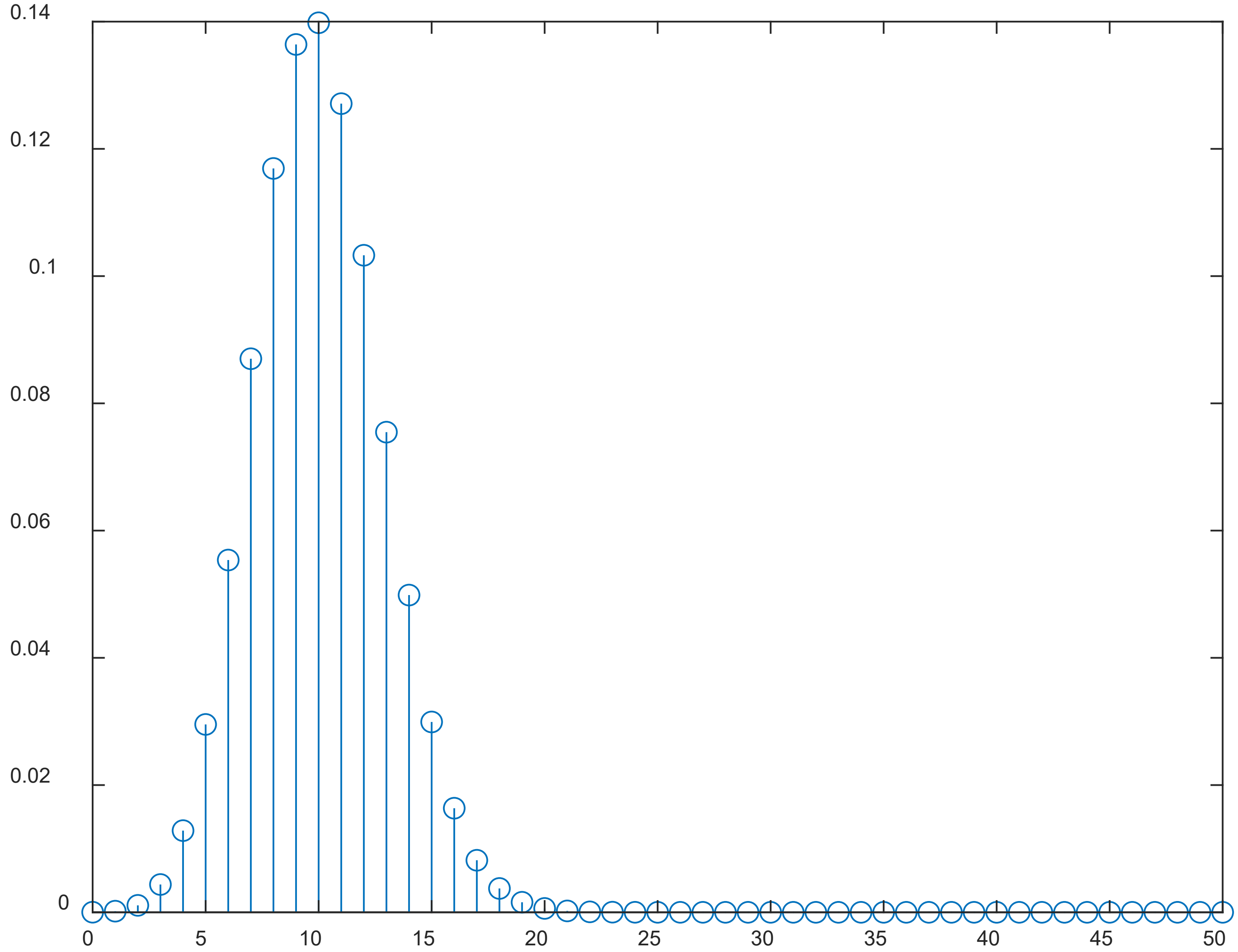
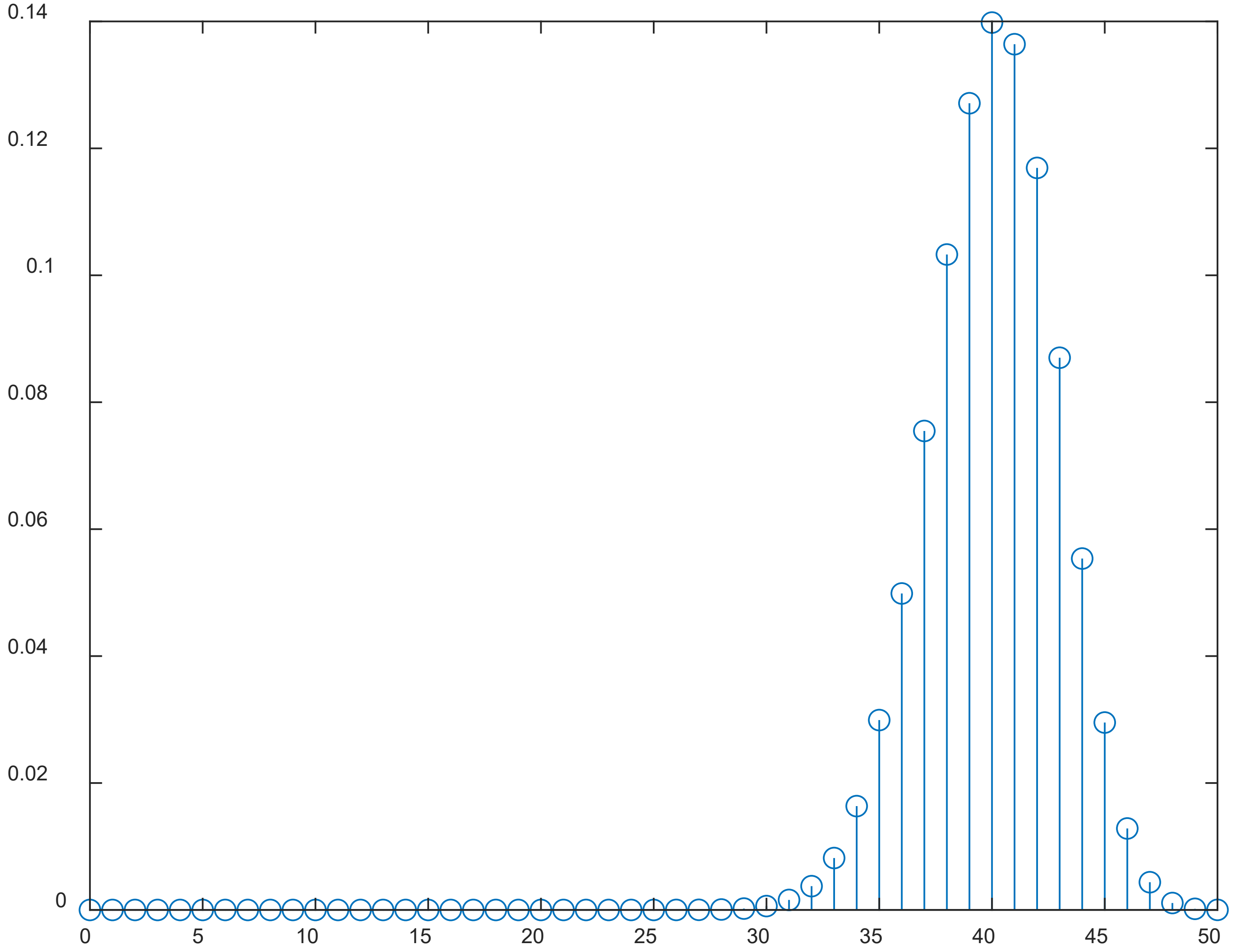


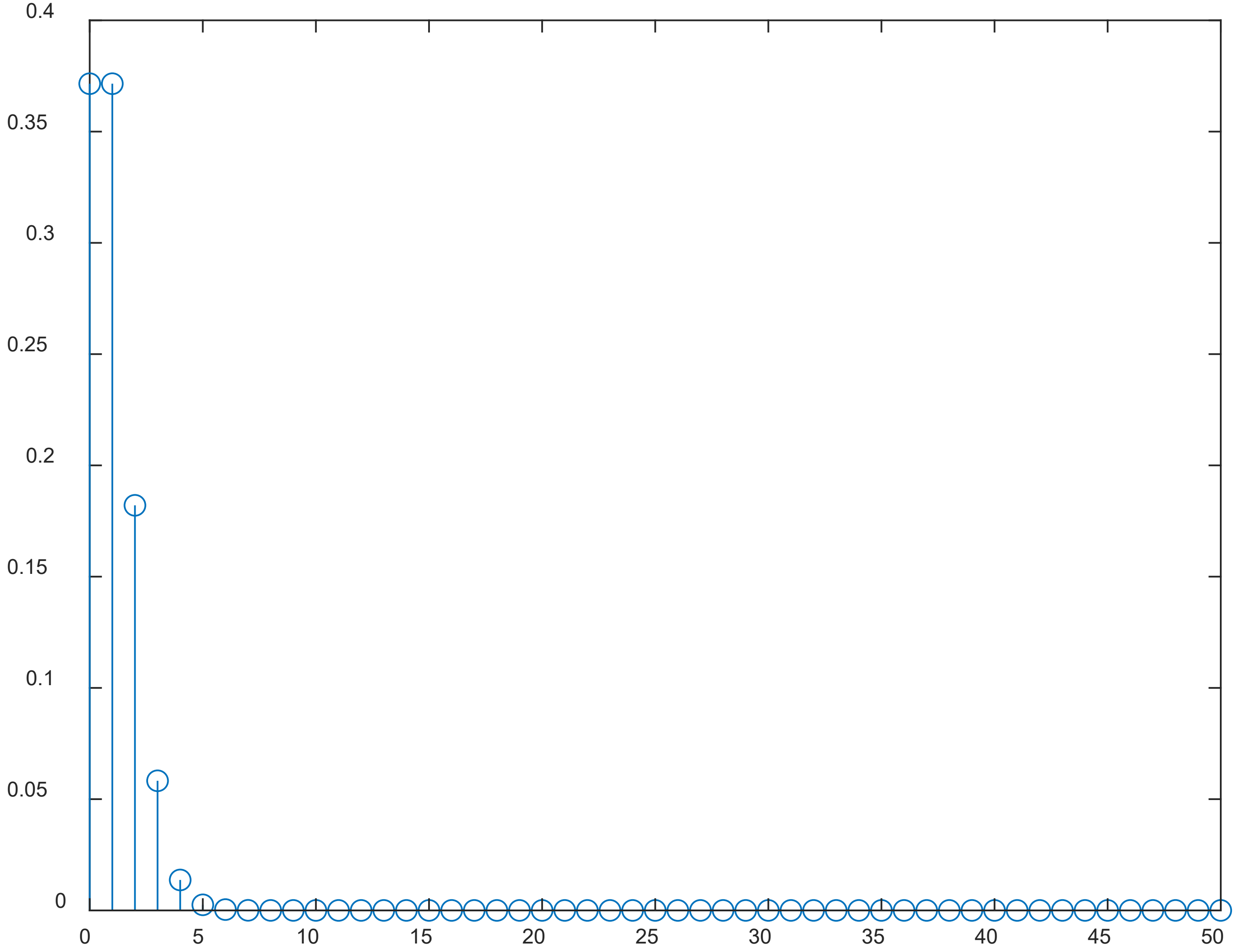
$n = 50$ and $p = 0.2$



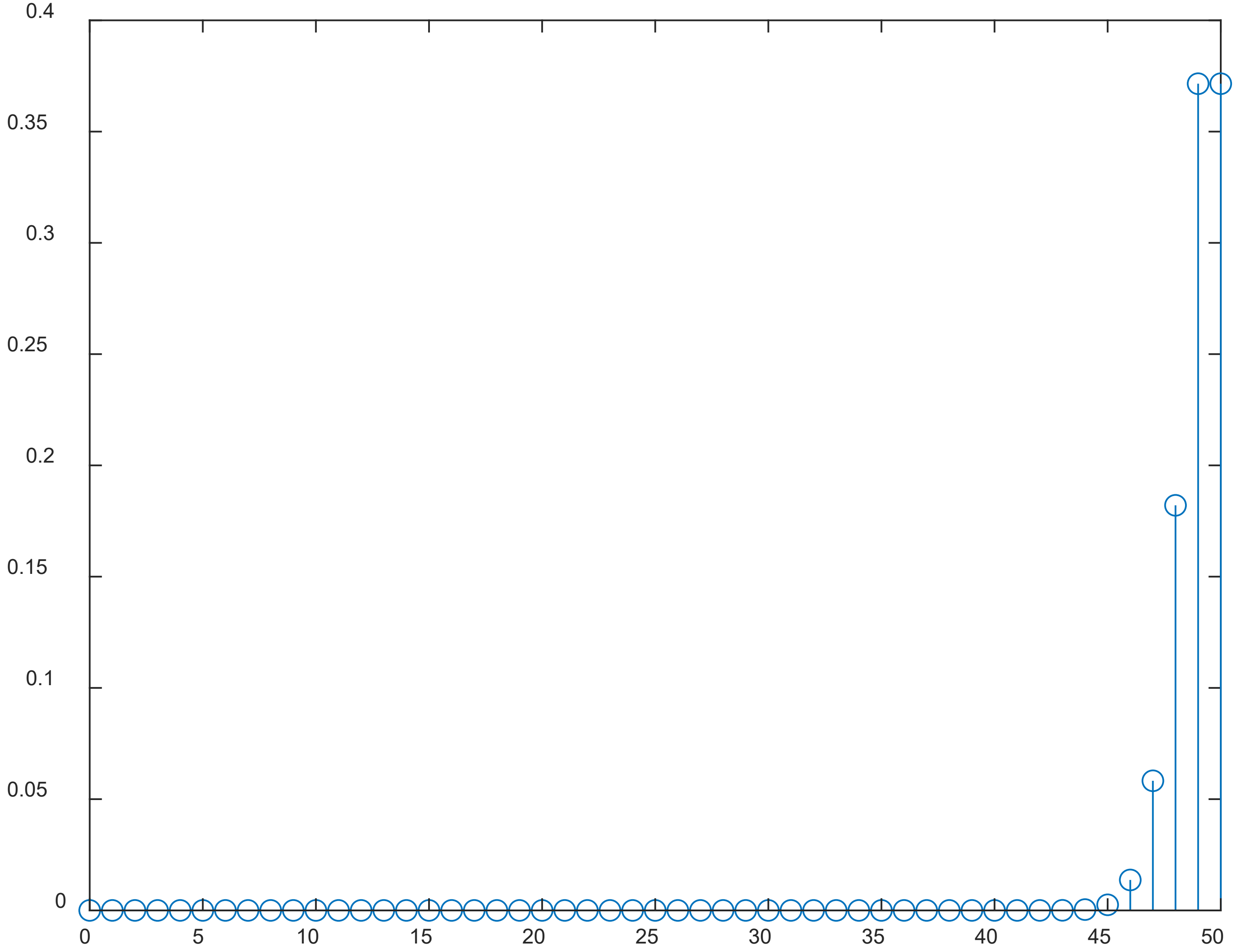
$n = 50$ and $p = 0.8$



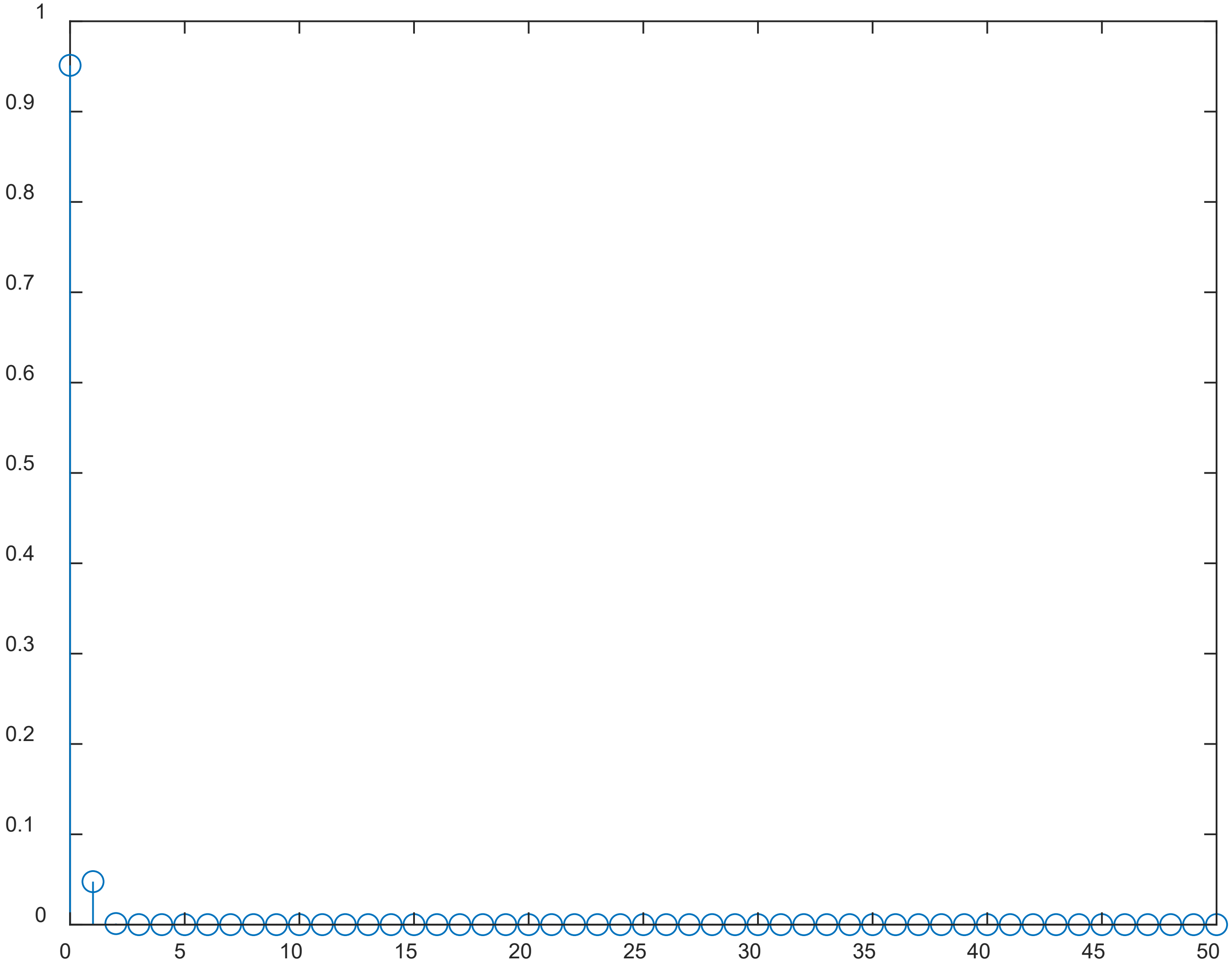
$n = 50$ and $p = 1/51$



$n = 50$ and $p = 50/51$



$n = 50$ and $p = 0.001$



6.

We can apply the binomial distribution for the probability computation. Let $p=0.95$ be the probability of success or show up. It is required that the number of passengers should be 0 to 50. That is, the probability of having 51 or 52 is not allowed, implying the required probability is:

$$1-C(52,0)0.95^{52}-C(52,1)0.95^{51}(0.05)=0.7405$$

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>> 1-(0.95)^(52)-52*(0.95)^(51)*0.05

ans = 0.7405
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