Tutorial 6

1. A discrete random variable X has the probability mass function (PMF) $p(k)=1/5$, $k=0,1,2,3,4$. Define another random variable as $Y=\sin(\pi/2\cdot X)$. Compute $\mathbb{E}\{Y\}$.

2. Let X be a continuous random variable with probability density function (PDF):

$$p(x)=\begin{cases}\dfrac{3}{x^4}, & x\geq 1\\ 0, & x<1\end{cases}$$

   Determine $\mathbb{E}\{X\}$ and $\mathrm{var}(X)$.

3. Let Y be a random variable transformed from $X \sim \mathcal{U}(0,1)$ via $Y = e^X$. Find the cumulative density function (CDF) and PDF of Y.

4. Suppose a random variable X has mean μ_x and variance σ_x^2. Let $Y = aX + b$ where a and b are constants. Determine the mean and variance of Y in terms of μ_x and σ_x^2.

   Then write down the MATLAB command to generate a Gaussian random variable $Y \sim \mathcal{N}(1,2)$ with the use of `randn`.

5. Let X be a continuous random variable with probability density function:

$$p(x) = \begin{cases} x^2 \left(2x + \dfrac{3}{2}\right), & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

If $Y = \dfrac{2}{X} + 3$, determine $\mathrm{var}(Y)$.

Solution

1.

Applying (2.24), we have:

$$\mathbb{E}\{g(X)\} = \sum_{k=0}^{4} g(x)p(k) = \sum_{k=0}^{4} \sin(\pi k/2) \cdot \frac{1}{5} = \frac{1}{5}(0+1+0-1+0) = 0$$

Alternatively, we can find the PMF of y using $Y = \sin(\pi/2 \cdot X)$:

$$x = 0 \Rightarrow y = \sin(\pi/2 \cdot 0) = 0$$
$$x = 1 \Rightarrow y = \sin(\pi/2 \cdot 1) = 1$$
$$x = 2 \Rightarrow y = \sin(\pi/2 \cdot 2) = 0$$
$$x = 3 \Rightarrow y = \sin(\pi/2 \cdot 3) = -1$$
$$x = 4 \Rightarrow y = \sin(\pi/2 \cdot 4) = 0$$

Hence $p_Y(-1) = 1/5$, $p_Y(0) = 3/5$, and $p_Y(1) = 1/5$

$$\mathbb{E}\{Y\} = \sum_{k=-1}^{1} y p_Y(k) = 0$$

2.

According to (2.21):

$$\mathbb{E}\{X\} = \int_{-\infty}^{\infty} x p(x) dx = \int_{1}^{\infty} x \cdot \frac{3}{x^4} dx = \int_{1}^{\infty} \frac{3}{x^3} dx = -\frac{3}{2} x^{-2} \Big|_{1}^{\infty} = \frac{3}{2}$$

$$\mathbb{E}\{X^2\} = \int_{-\infty}^{\infty} x^2 p(x) dx = \int_{1}^{\infty} x^2 \cdot \frac{3}{x^4} dx = \int_{1}^{\infty} \frac{3}{x^2} dx = -3x^{-1}\Big|_{1}^{\infty} = 3$$

Applying (2.23), we have:

$$\mathrm{var}(X) = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

3.
Following Example 2.26:

$$F_X(x) = \begin{cases} 0, & 0 \leq x \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

That is, $F_X(x) = P(X \leq x) = x$ for $0 < x < 1$.

Let $Y = e^X$. As $X \in (0,1)$, we have $Y \in (1,e)$. Hence we know $F_Y(1) = P(Y \leq 1) = 0$ and $F_Y(e) = P(Y \leq e) = 1$, and we only investigate the range in $(1,e)$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y) \\ &= F_X(\ln y), \quad 0 < \ln y < 1 \\ &= \ln y \end{aligned}$$

Combining the results, we have:

$$F_Y(y) = \begin{cases} 0, & 1 \leq y \\ \ln y, & 1 < y < e \\ 1, & y \geq e \end{cases}$$

Applying (2.10), we get:

$$p_Y(y) = \frac{d}{dy} \ln y = \frac{1}{y}, \quad 1 < y < e$$

Hence:

$$p_Y(y) = \begin{cases} \frac{1}{y}, & 1 < y < e \\ 0, & \text{otherwise} \end{cases}$$

Writing $Y = g(X)$, the PDF can also be obtained as:

$$p_Y(y) = p_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

Now $Y = g(X) = e^X \Rightarrow g^{-1}(y) = x = \ln y$. Again, we know that Y only has values between $(1, e)$. We then compute:

$$p_X(g^{-1}(y)) = p_X(\ln y) = 1, \quad 0 < \ln y < 1$$

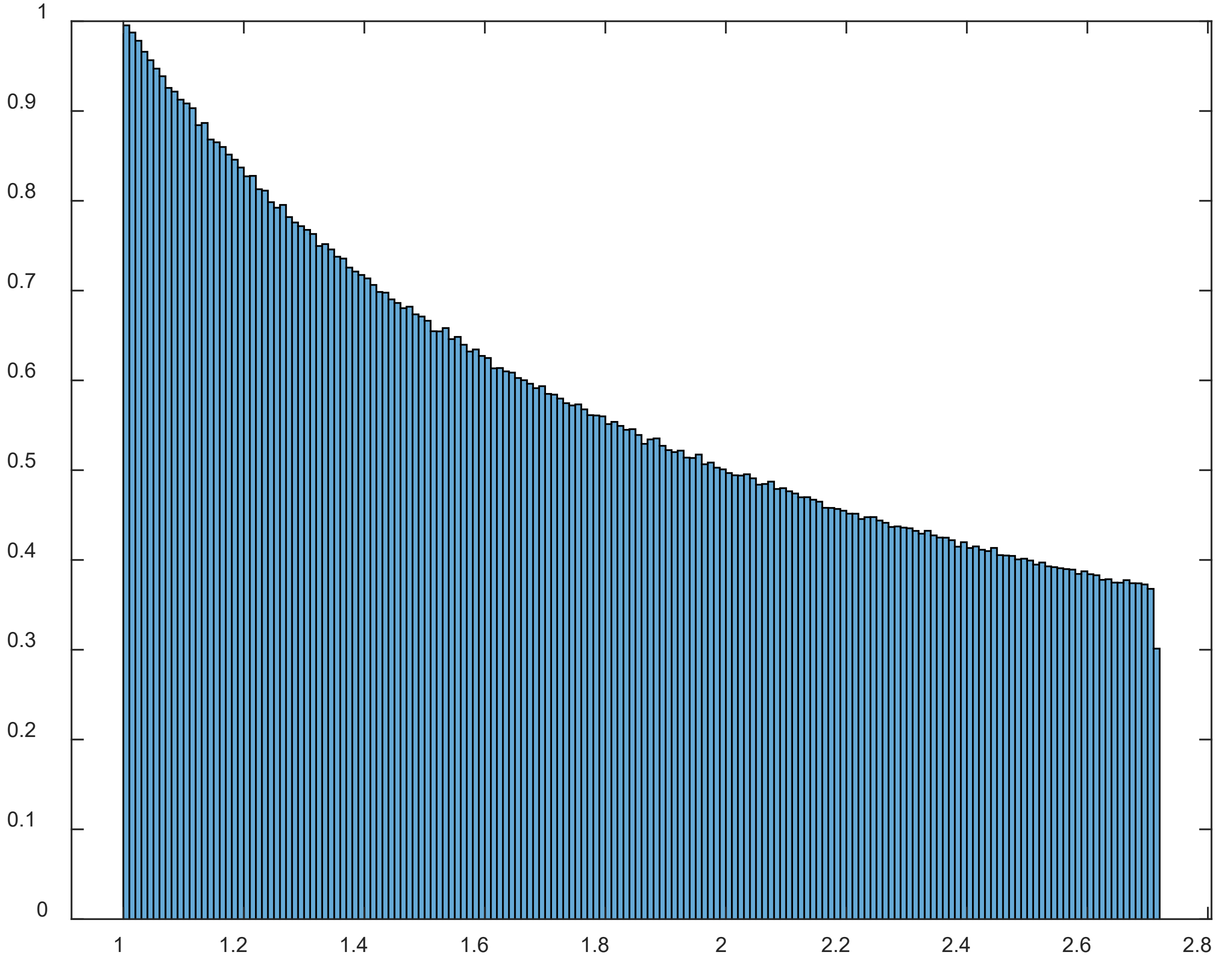
$$\frac{dg^{-1}(y)}{dy} = \frac{d \ln y}{dy} = \frac{1}{y}$$

Combining the results, we get:

$$p_Y(y) = 1 \cdot \frac{1}{y} = \frac{1}{y}, \quad 1 < y < e$$

Integrating $p_Y(y)$ with respect to y yields the same CDF $F_Y(y)$.

```
x=rand([1,10000000]);
y=exp(x);
histogram(y,'Normalization','pdf')
```



4.

$$\mathbb{E}\{Y\} = \mu_y = \mathbb{E}\{aX + b\} = \mathbb{E}\{aX\} + \mathbb{E}\{b\} = a\mathbb{E}\{X\} + b = a\mu_x + b$$

The same result can be obtained by following Example 2.28 or applying directly (2.27).

$$\begin{aligned}
\mathrm{var}(Y) = \sigma_y^2 &= \mathbb{E}\{(Y - \mu_y)^2\} \\
&= \mathbb{E}\{(aX + b - (a\mu_x + b))^2\} \\
&= \mathbb{E}\{(aX - a\mu_x))^2\} \\
&= a^2\mathbb{E}\{(X - \mu_x))^2\} \\
&= a^2\mathrm{var}(X) \\
&= a^2\sigma_x^2
\end{aligned}$$

From the above results, if $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$, then

$$Y \sim \mathcal{N}(a\mu_x + b, a^2\sigma_x^2)$$

The MATLAB command `randn` generates $X \sim \mathcal{N}(0,1)$. To produce $Y \sim \mathcal{N}(1,2)$, a and b are computed as:

$$a\mu_x + b = 1 \Rightarrow a \cdot 0 + b = 1 \Rightarrow b = 1$$

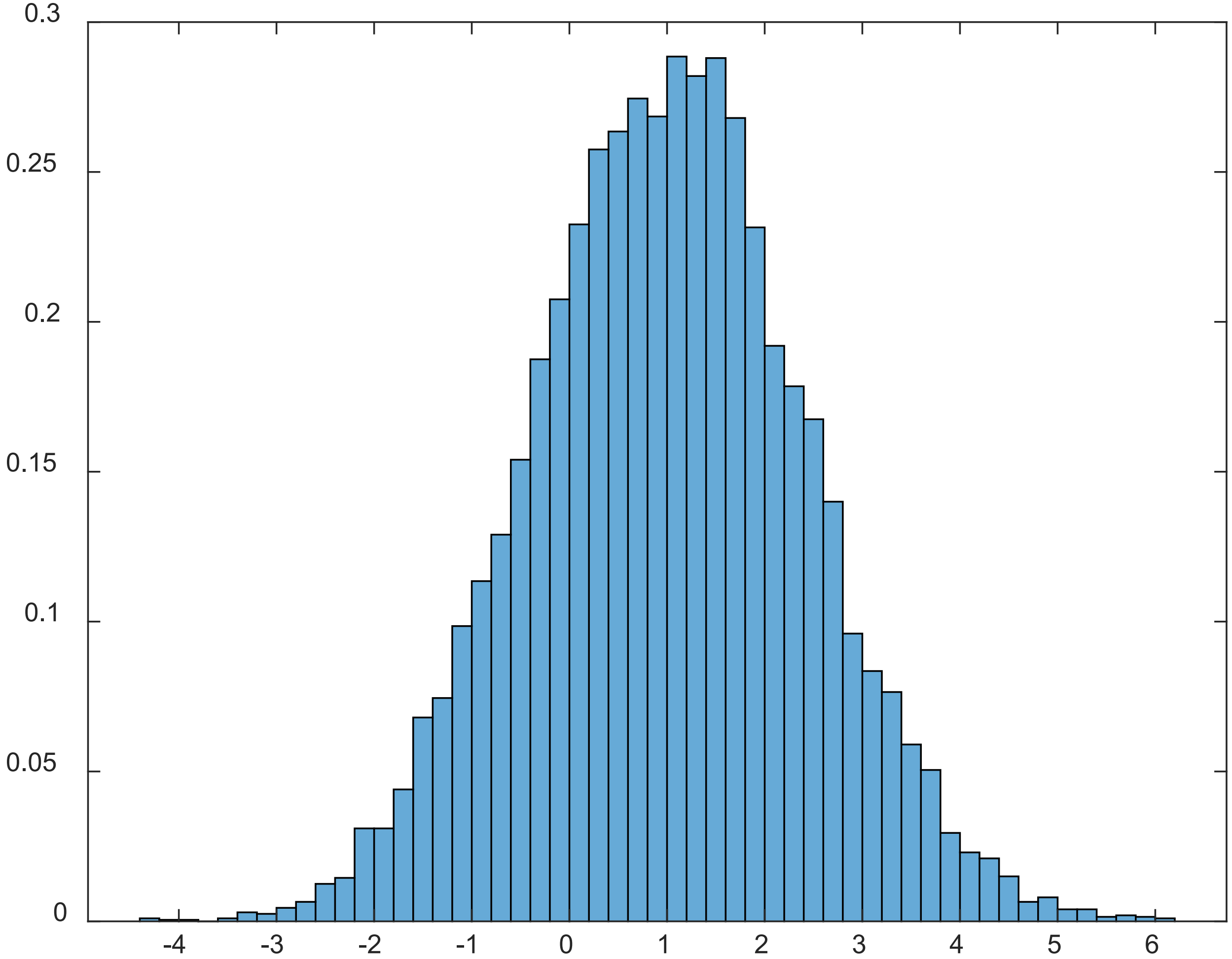
$$a^2\sigma_x^2 = 2a^2 \cdot 1 = 2a = \sqrt{2}$$

Hence the MATLAB command is `sqrt(2)*randn+1`

Following Example 2.25:

```
Y= sqrt(2)*randn(1,10000)+1;
mean(Y)
= 1.0023
mean((Y-mean(Y)).*(Y-mean(Y)))
= 1.9659
```

```
histogram(Y,'Normalization','pdf')
```



We see the mean is shifted to 1 and there is a wider spread.

5.

According to the results in Question 4, we have:

$$\mathrm{var}(Y) = 2^2 \mathrm{var}\left(\frac{1}{X}\right) = 4\mathrm{var}\left(\frac{1}{X}\right)$$

Then we apply (2.23):

$$\mathrm{var}\left(\frac{1}{X}\right) = \mathbb{E}\left\{\frac{1}{X^2}\right\} - \left(\mathbb{E}\left\{\frac{1}{X}\right\}\right)^2$$

Considering $g(X) = 1/X$ and $g(X) = 1/X^2$, and applying (2.25):

$$\mathbb{E}\left\{\frac{1}{X}\right\} = \int_{-\infty}^{\infty} \frac{1}{x} p(x) dx = \int_0^1 x\left(2x + \frac{3}{2}\right) dx = \int_0^1 \left(2x^2 + \frac{3}{2}x\right) dx = \frac{17}{12}$$

$$\mathbb{E}\left\{\frac{1}{X^2}\right\} = \int_{-\infty}^{\infty} \frac{1}{x^2} p(x) dx = \int_0^1 \left(2x + \frac{3}{2}\right) dx = \int_0^1 \left(2x + \frac{3}{2}\right) dx = \frac{5}{2}$$

Combining the results yields:

$$\mathrm{var}(Y) = \frac{71}{36}$$