CITY UNIVERSITY OF HONG KONG

Course code and title	:	MA1200 Calculus and Basic Linear Algebra I
Session	:	Semester A, 2012/2013
Time allowed	:	Two hours

This paper has **SIX** pages (including this cover page).

A brief table of derivatives is attached on pages 5 and 6.

Instructions to candidates:

1. This paper has **TEN** questions.
2. Attempt **ALL** questions in Section A and B.
3. Each question in Section A carries 9 marks.
4. Each question in Section B carries 15 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Section A
Answer **ALL** questions in this section.

Question 1

Find, in radians, the general solution of the equation
$$\sqrt{2}\sin x = \tan x,$$
and give all the values of x which lie between 0 and 2π.

(9 marks)

Question 2

Let $f:\mathbf{R}\to\mathbf{R}$ and $f(x)=\dfrac{1}{1-x^2}$ for $x\in\mathbf{R}\setminus\{-1,1\}$.

(a) Show that $f(x)$ is not one-to-one.

(2 marks)

(b) Show that the function $f(x)$ has no value between 0 and 1.

(4 marks)

(c) Find the range of $f(x)$.

(3 marks)

Question 3

Let $g(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0. \end{cases}$ Determine whether $g(x)$ is differentiable at $x=0$, if so, find the value of the first derivative there.

(Hint: Since $\sin\theta$ is bounded by -1 and 1 for all θ, it follows that $-|x|\le x\sin\dfrac{1}{x}\le|x|$ for all real values of x.)

(9 marks)

Question 4

(a) Evaluate $\displaystyle\lim_{x\to\infty}\frac{3x^2-x+2}{x^3+5}$.

(4 marks)

(b) Let $F(x)=\begin{cases} \dfrac{h(x)+\cos x}{x-\pi}, & \text{if } x\neq\pi \\ h'(\pi), & \text{if } x=\pi, \end{cases}$ where $h(x)$ is differentiable everywhere and $h(\pi)=1$. Determine whether $F(x)$ is continuous at $x=\pi$. Give your reason.

(5 marks)

Question 5

Differentiate with respect to x

(a) $\sqrt{1+x^2}+\log_e(1+x^2)$,

(3 marks)

(b) $\tan^{-1}(\sinh x)$,

(3 marks)

(c) $2^{\sqrt{x}}$.

(3 marks)

Question 6

Express $\dfrac{2x+11}{(x-2)(x^2+1)}$ in partial fractions.

(9 marks)

Question 7

(a) Show from first principles that $\dfrac{\mathrm{d}}{\mathrm{d}x}(e^{ax})=ae^{ax}$, where a is a constant.

(Hint: You may use without proof the exponential theorem

$e^x=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots+\dfrac{x^r}{r!}+\ldots,\quad -\infty<x<\infty$.)

(4 marks)

(b) The parametric equations of a curve are $x=e^{3t}$, $y=t^2$, where t is the parameter and $t\geq 0$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of t.

(5 marks)

Question 8

Let $y=x^{\frac{2}{3}}e^x$ for $x\in\mathbf{R}$.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for $x\neq 0$.

(4 marks)

(b) Show that there is a turning point on the curve $y=x^{\frac{2}{3}}e^x$ when $x=-\dfrac{2}{3}$, and determine whether it is a local minimum point or a local maximum point.

(5 marks)

Section B

Answer **ALL** questions in this section.

Question 9

The equation of a conic is

$$41x^2 - 24xy + 34y^2 - 50 = 0 . \text{——} (1)$$

Let xy coordinate system be rotated anti-clockwise through an angle θ (without translation of axes), resulting in a new $x'y'$ coordinates system with $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

(a) Show that equation (1) may be transformed into an equation of the form $ax'^2 + by'^2 = 1$, where a, b are constants.

(6 marks)

(b) Draw a rough sketch showing how the graph of the conic with respect to the axes of coordinates.

(3 marks)

(c) Find the equation of the tangent to the conic (1) at the point $(\frac{3\sqrt{2}}{5}, \frac{4\sqrt{2}}{5})$.

(6 marks)

Question 10

For any non-negative integer n, the Legendre polynomial $P_n(x)$ is defined by

$$P_n(x) = \frac{1}{2^n(n!)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x^2-1)^n], \text{ for } -1 \le x \le 1 .$$

Show that

(a) $y = P_n(x)$ satisfies the equation $(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + n(n+1)y = 0$,

(Hint: Let $u = \frac{1}{2^n(n!)}(x^2-1)^n$. Show that $(x^2-1)\frac{\mathrm{d}u}{\mathrm{d}x} = 2nxu$.)

(8 marks)

(b) $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

(Hint: Consider $2^n(n!)P_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x+1)^n(x-1)^n]$, and using Leibnitz' rule.)

(7 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	Derivative of y with respect to x
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d f(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	Derivative of y with respect to x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \text{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u\frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u\frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \text{sech}^2 u\frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\text{cosech}^2 u\frac{du}{dx}$
$y = \text{sech}\, u$	$\frac{dy}{dx} = -\text{sech}\, u \tanh u\frac{du}{dx}$
$y = \text{cosech}\, u$	$\frac{dy}{dx} = -\text{cosech}\, u \coth u\frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \text{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y = \text{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

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