Q1: Denote $V_1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) $$AV_1 = \begin{pmatrix} 6 & -2 & -1 \\ -2 & a & -1 \\ -1 & -1 & b \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ a-3 \\ b-2 \end{pmatrix}$$

Since $3 = 3 \times 1$, we have

$$\begin{cases} a-3=3 \\ b-2=3 \end{cases} \Leftrightarrow \begin{cases} a=6 & 5' \\ b=5. & 5' \end{cases}$$

(b). $$A = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -2 & -1 & 5 \end{pmatrix}$$

Find the eigenvalues $\lambda_1 = 3$, $\quad V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad 4'$

$\lambda_2 = 6 \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ $\quad 4'$

$\lambda_3 = 8 \quad V_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $\quad 4'$

$V_1 \cdot V_2 = 0$, $V_2 \cdot V_3 = 0$, $V_1 \cdot V_3 = 0$.

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix} \quad P^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$3'$

(c) As all eigenvalues are positive, A is positive definite. $5'$

(d). $C = A^2 - 3A + I$

$= PDP^T \cdot PDP^T - 3PDP^T + I$ ($P^T P = I$)

$= PD^2P^T - 3PDP^T + PIP^T$

$= P(D^2 - 3D + I)P^T$

Therefore. $\lambda_1 = 3^2 - 3\times3+1 = 1$, $\lambda_2 = 6^2-3\times6+1 = 19$,

$\lambda_3 = 8^2 - 3\times8+1 = 41$. 3'

$V_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $V_2 = \begin{pmatrix}1\\1\\-2\end{pmatrix}$. $V_3 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$ 2'

Q2:

(a). $f(1,2) = 1-8 = -7$ 2'

$\frac{\partial f}{\partial x}(1,2) = 2x\big|_{(1,2)} = 2$ 3'

$\frac{\partial f}{\partial y}(1,2) = -3y^2\big|_{(1,2)} = -12$ 3'

$L(x,y) = -7 + 2(x-1) - 12(y-2) = 2x - 12y + 15$ 2'

(b)

$\begin{cases}\frac{\partial f}{\partial x} = 2x = 0\\ \frac{\partial f}{\partial y} = -3y^2 = 0\end{cases} \iff \begin{cases}x=0\\y=0.\end{cases}$

point $(0,0)$ is the stationary point. 5'

And $f_{xx}(0,0) = 2$, $f_{yy}(0,0) = -6y\big|_{(0,0)} = 0$, $f_{xy}(0,0) = 0$

$\Delta = f_{xx}f_{yy} - f_{xy}^2 = 0-0 = 0$. 5'

Therefore, point $(0,0)$ is inconclusive. 5' #

Q3:

1(a) Since $z = f(x,y)$, take derivative w.r.t x and y respectively on both sides:

$$\begin{cases} 2x + 6z \cdot f_x = 0 \\ 12y^2 + 6z \cdot f_y = 0 \end{cases} \Rightarrow \begin{cases} f_x = -\frac{x}{3z} \\ f_y = -\frac{12y^2}{6z} = -2\frac{y^2}{z}. \end{cases}$$ 5'

At the given point $(2,1)$, we have $z = f(2,1) = -6$ locally, then

$$\begin{cases} f_x(2,1) = \frac{1}{9} \\ f_y(2,1) = \frac{1}{3} \end{cases}, \quad \text{i.e.} \quad \nabla f(2,1) = \left(\frac{1}{9}, \frac{1}{3}\right)$$ 2'

Then $D_{\vec{v}} f(2,1) = \left(\frac{1}{9}, \frac{1}{3}\right) \cdot \frac{(-1,1)}{\|(-1,1)\|} = \frac{\sqrt{2}}{9}$ 3'

1(b). Choose $\vec{u}$ as a unit vector and denote the angle between ∇f and $\vec{u}$ as θ.

$$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u} = \|\nabla f(2,1)\| \cdot \cos\theta$$

To maximize this value, we chose $\cos\theta = 1$, that is,

$$\vec{u} = \frac{\left(\frac{1}{9}, \frac{1}{3}\right)}{\left\|\left(\frac{1}{9}, \frac{1}{3}\right)\right\|} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) = \left(\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}\right)$$ 5'

5'

Q4:

Choose $l_1 = \{(0,y)\}$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0}{y^2} = 0$$ 4'

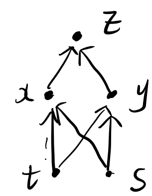
Choose $l_2 = \{(x,0)\}$, then,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{3x^2}{2x^2} = \frac{3}{2} \neq 0.$$ 4'

The limit DNS. 2'

Q5:

$$z = 2x^5 + y^5 \Rightarrow \begin{cases} \dfrac{\partial z}{\partial x} = 10x^4 \\ \dfrac{\partial z}{\partial y} = 5y^4 \end{cases}$$



$$\begin{cases} x = \sin t + \cos s \\ y = \cos t - \sin s \end{cases} \Rightarrow \begin{cases} \dfrac{\partial x}{\partial t} = \cos t \\ \dfrac{\partial y}{\partial t} = -\sin t \end{cases} \text{ and } \begin{cases} \dfrac{\partial x}{\partial s} = -\sin s \\ \dfrac{\partial y}{\partial s} = -\cos s \end{cases}$$

When $t = s = 0$, we have $x = 1$, $y = 1$ and $z = 3$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= 10x^4 \cos t + 5y^4(-\sin t) \qquad 4'$$

$$\frac{\partial^2 z}{\partial t \partial s} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial t}\right)$$

$$= 10\cos t \cdot 4x^3 \cdot x_s - 5\sin t \cdot 4y^3 \cdot y_s$$

$$= 10\cos t \cdot 4x^3 \cdot (-\sin s) - 5\sin t \cdot 4y^3 \cdot (-\cos s)$$

$$= -40x^3 \sin s \cos t + 20y^3 \sin t \cos s \qquad 4'$$

Therefore, $\dfrac{\partial z}{\partial t}(0,0) = 10$ $\quad 1'$

$\dfrac{\partial^2 z}{\partial t \partial s}(0,0) = 0$ $\quad 1'$