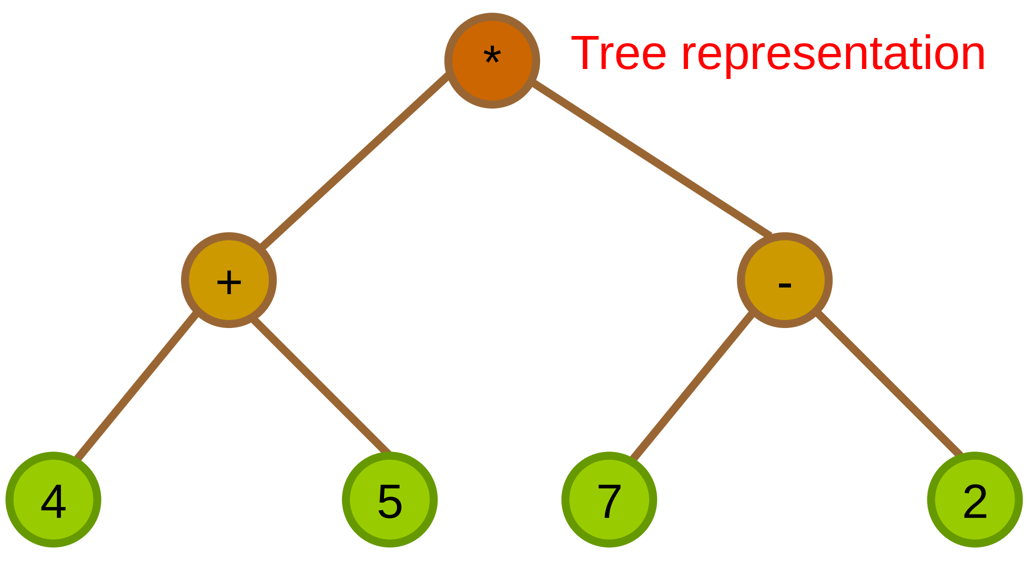
EE2331 Data Structures and Algorithms

Trees

Remember?

- How does a computer evaluate mathematical expressions?
  - e.g. (4 + 5) * (7 - 2)
  - Use **postfix expressions** (4 5 + 7 2 - *)
- May we transform it to tree representation?



Stacks, Queues vs. Trees

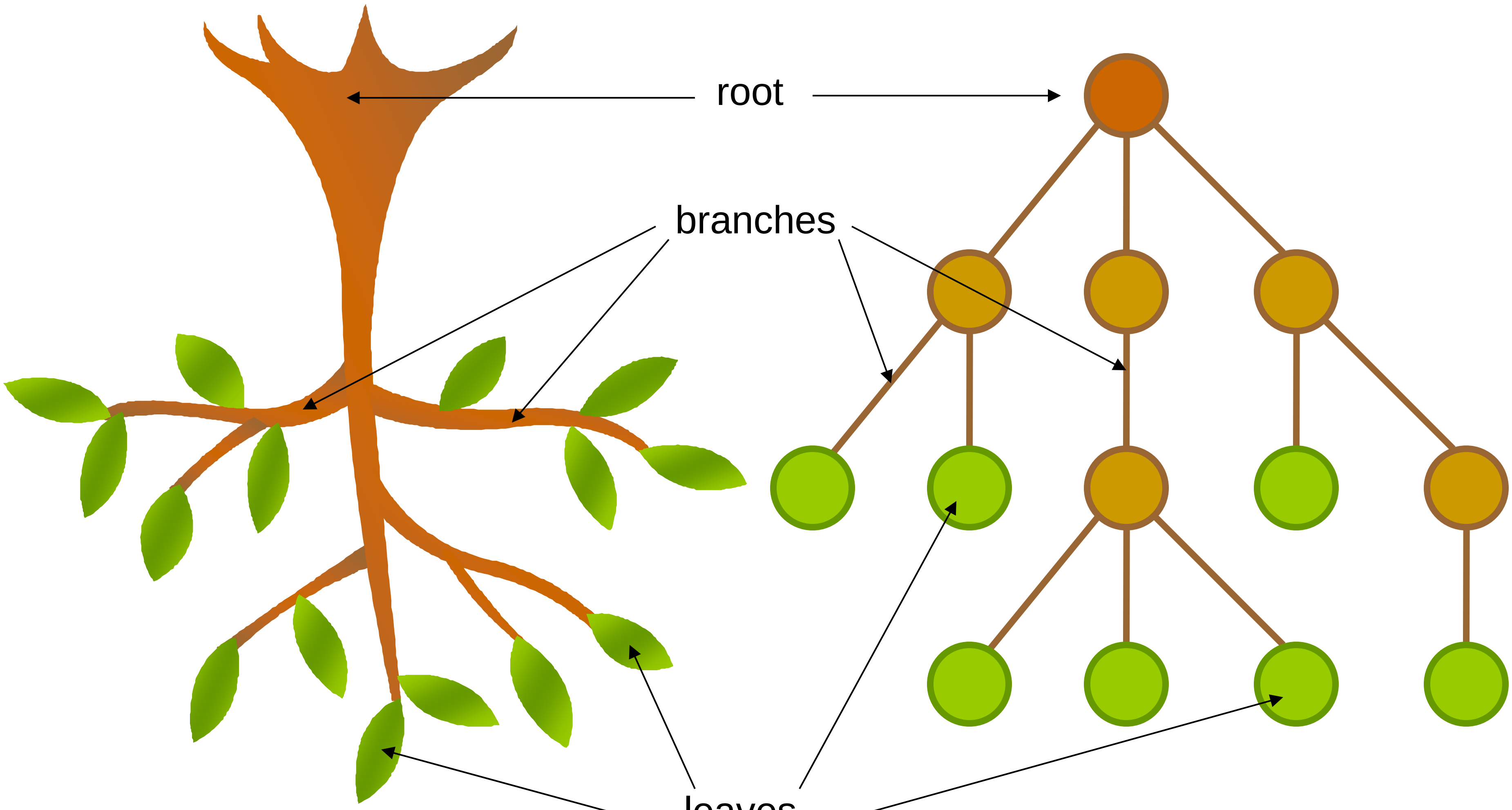
- Data structures discussed so far are linear
  - One preceding/succeeding element
  - e.g. linked lists, stacks, queues

- Tree is a non-linear linked data structure
  - Multiple succeeding elements
  - Tree structure is recursively defined, so tree operations often involve **recursion** and **linked list**

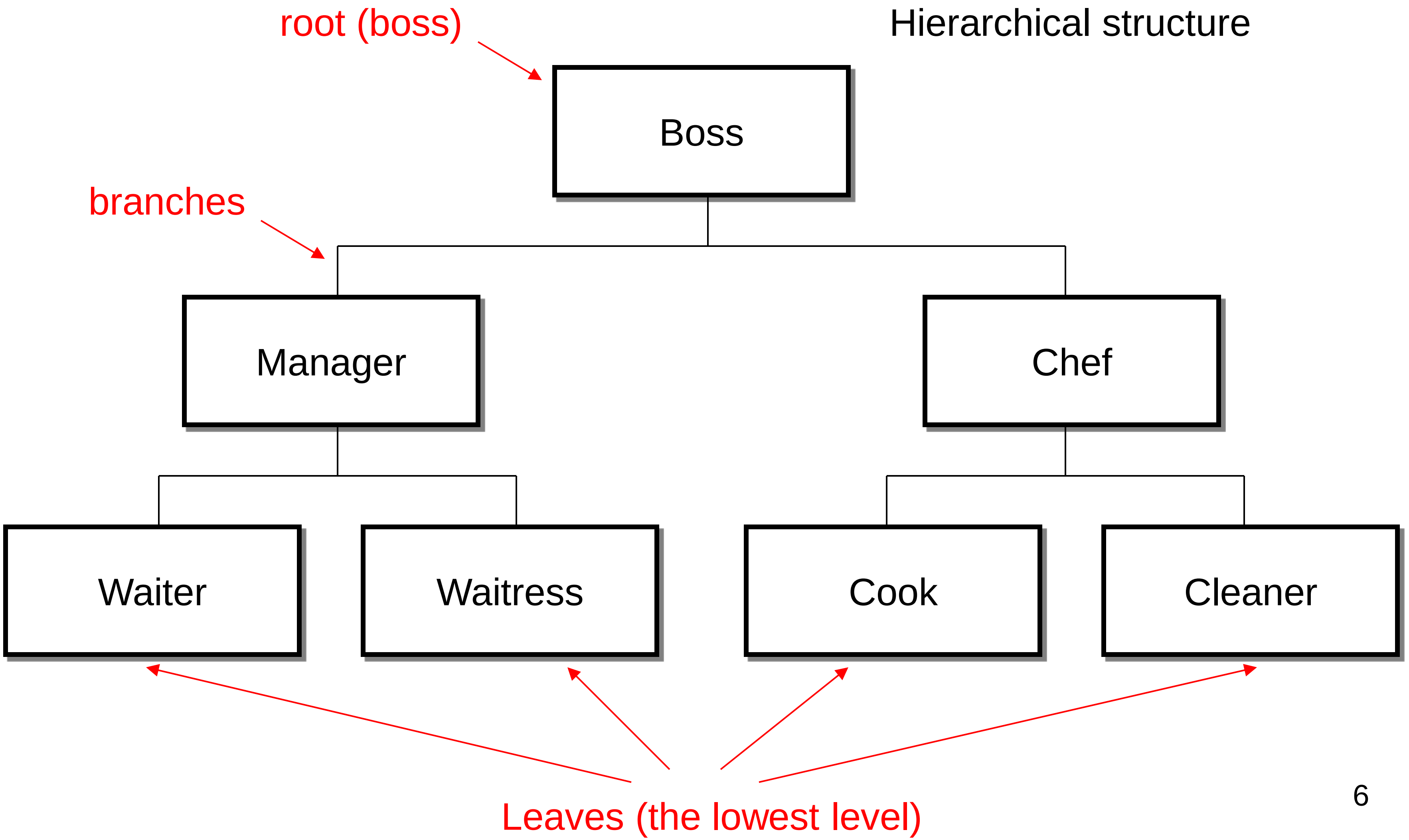
Outline

- Terminology
- Representation
- Binary Trees
- Implementations with Array and Linked List
- Common Operations of Binary Tree
- Trees Traversal
  - Preorder, inorder, postorder, level order
- Relationship between Trees, Stacks and Queues
- Reconstruction of Binary Trees
- Special Binary Trees
  - Binary Search Trees
  - Heap Trees
- Applications
- General Trees and Other tree structures

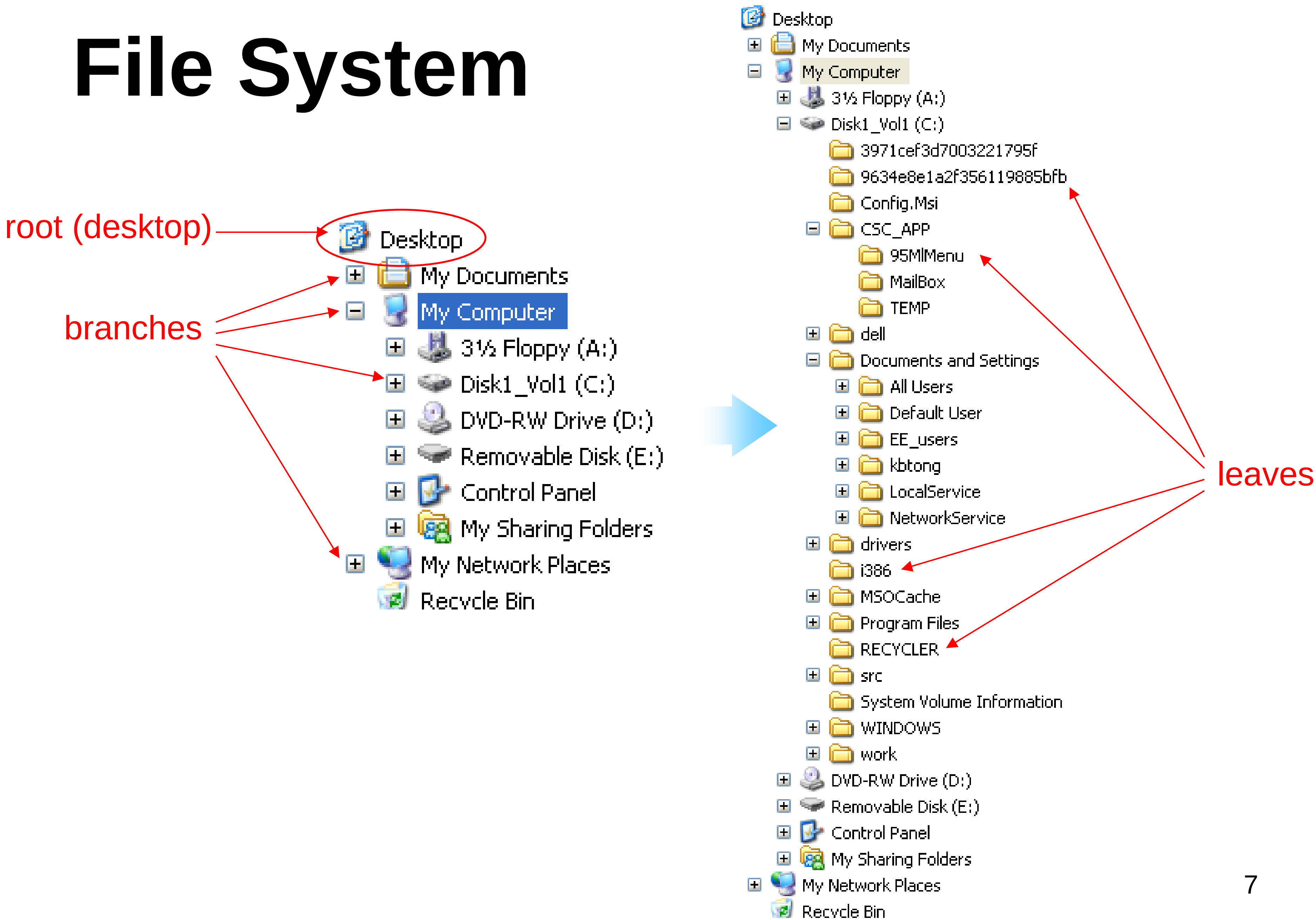
An Inverted Tree



Tree Example: Restaurant

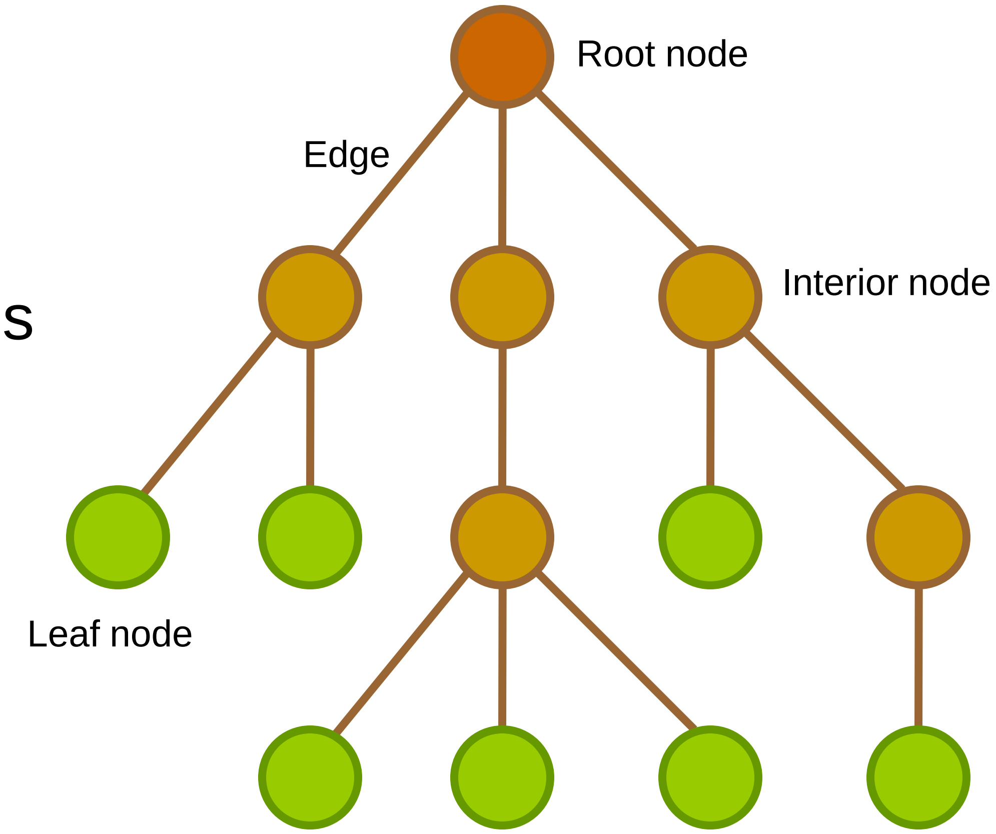


File System


root (desktop)
branches
Desktop
My Documents
My Computer
3½ Floppy (A:)
Disk1_Vol1 (C:)
DVD-RW Drive (D:)
Removable Disk (E:)
Control Panel
My Sharing Folders
My Network Places
Recycle Bin
Desktop
My Documents
My Computer
3½ Floppy (A:)
Disk1_Vol1 (C:)
3971cef3d7003221795f
9634e8e1a2f356119885bfb
Config.Msi
CSC_APP
95MlMenu
MailBox
TEMP
dell
Documents and Settings
All Users
Default User
EE_users
kbtong
LocalService
NetworkService
drivers
i386
MSOCache
Program Files
RECYCLER
src
System Volume Information
WINDOWS
work
DVD-RW Drive (D:)
Removable Disk (E:)
Control Panel
My Sharing Folders
My Network Places
Recycle Bin
leaves

Composition of a Tree

- Types of tree node
  - Root node (the top node in a tree)
  - Interior nodes (nodes with at least one child)
  - Leaf nodes (nodes with no children)

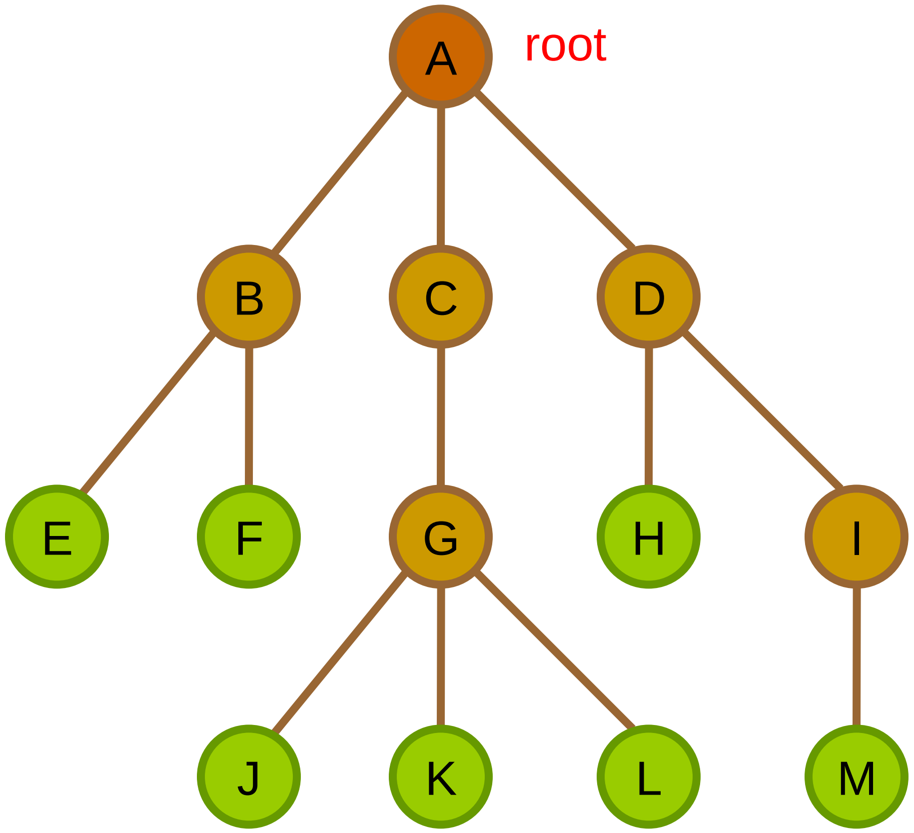


Property of Trees

- Nodes represent information (data)
- Branches represent links between the nodes
- If the total number of nodes (i.e. **root node**, **interior nodes** and **leaf nodes**) is n, how many branches in the tree?
  - Number of branches is $n - 1$

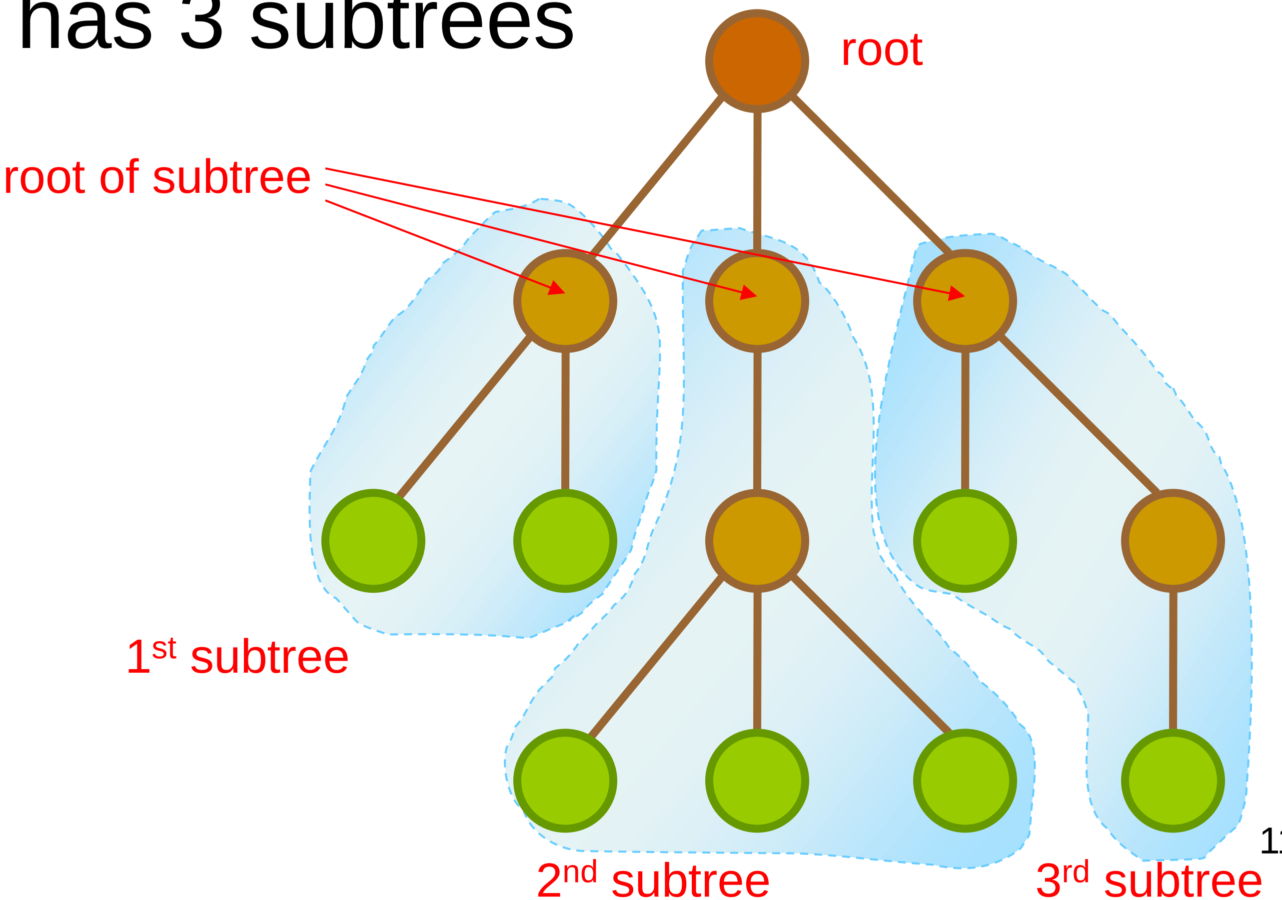
Parent, Children & Sibling

- This tree has 13 nodes
- Node A has 3 children
  - Nodes B, C and D
- Node A is the parent of
  - B, C and D
- Node G is the parent of
  - Nodes J, K and L
- Node G is the child of C
- J, K and L are sibling nodes (share the same parent)



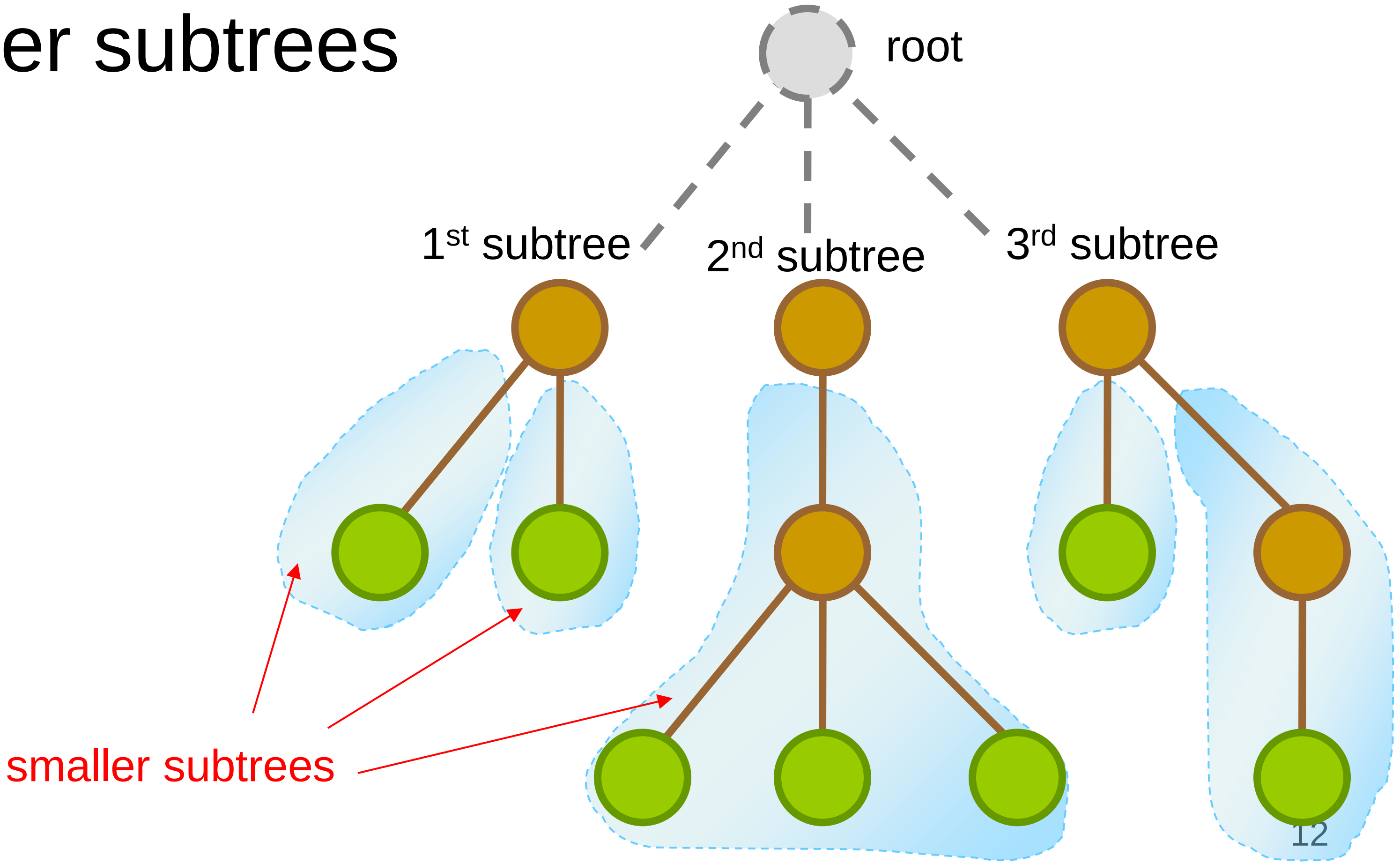
Subtrees

- A tree is composed of several subtrees
- e.g. root has 3 subtrees



Smaller subtrees

- A subtree can be further broken down into smaller subtrees

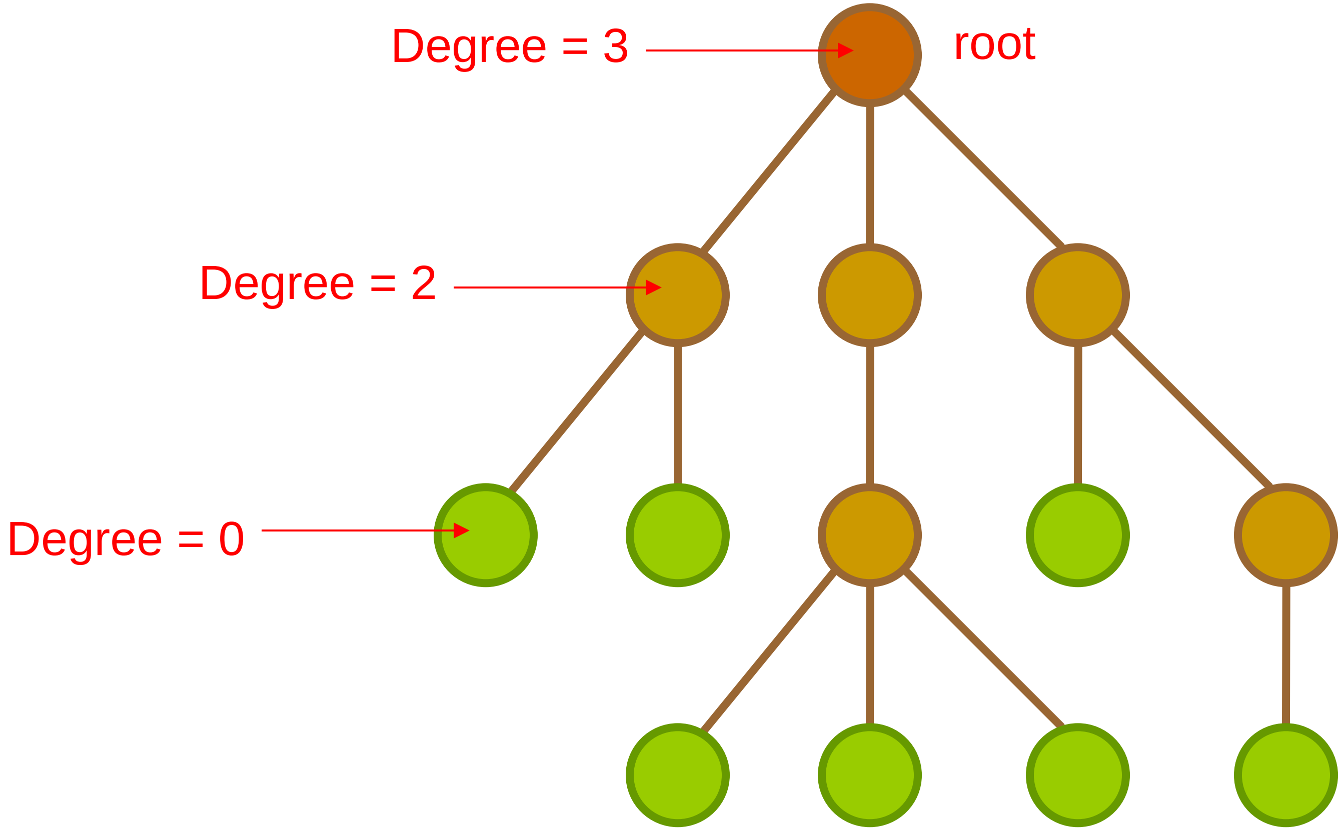


Ancestor and Descendant

- A simple path is a sequence of nodes $n_1, n_2, \ldots, n_k$ such that the nodes are all distinct and there is an edge between each pair of nodes $(n_1, n_2), (n_2, n_3), \ldots, (n_{k-1}, n_k)$
- The nodes along the simple path from the root to node x are the ancestors of x
- The descendants of a node x are the nodes in the subtrees of x
- Length of a path = no. of branches on the path

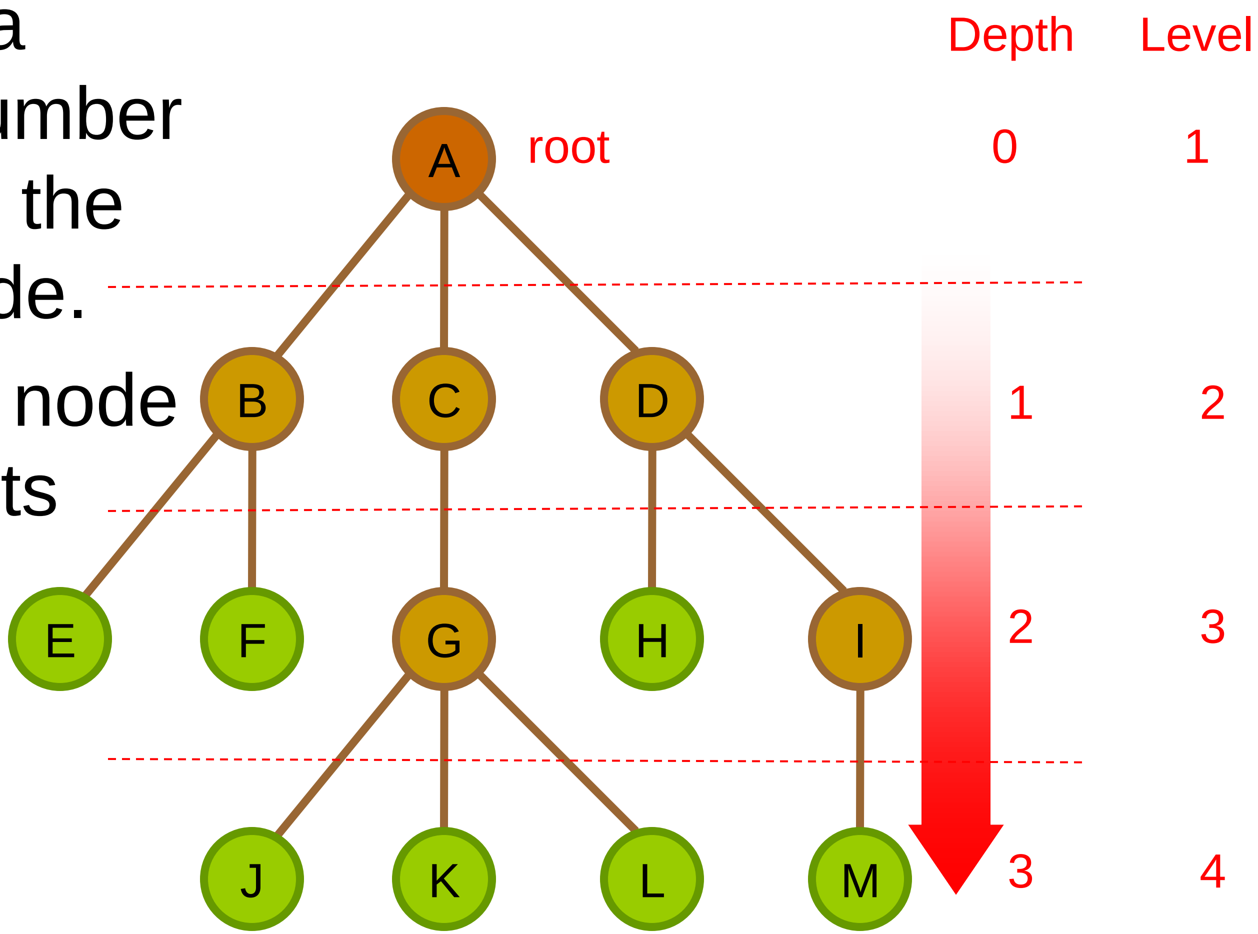
Degree

- The number of subtrees of a node



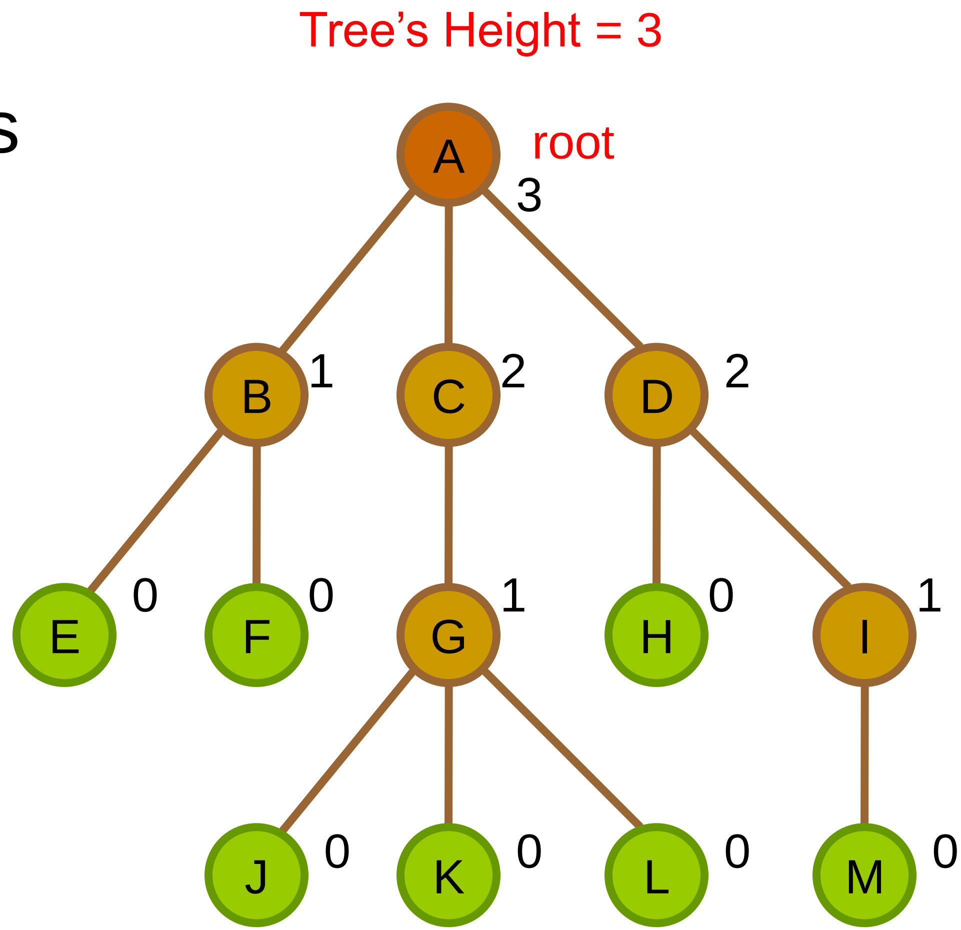
Depth and Level

- The depth of a node is the number of edges from the <u>root</u> to the node.
- The level of a node is defined by its depth plus 1.



Height

- The height of a node is the number of edges from the node to the <u>deepest leaf</u>.
- The height of a tree is a height of the root.



Class Exercise

- A node has no parent is called ______________________________
- A node has no children is called ____________________________
- A node has both parent and children is called ________________

- If a tree has 5 branches, how many nodes does this tree contain? ____________________

- What is the degree of a leaf node? ___________________________
- Can a node has more than one parent? _____________________
- Is there a unique path from root to every node? ______________

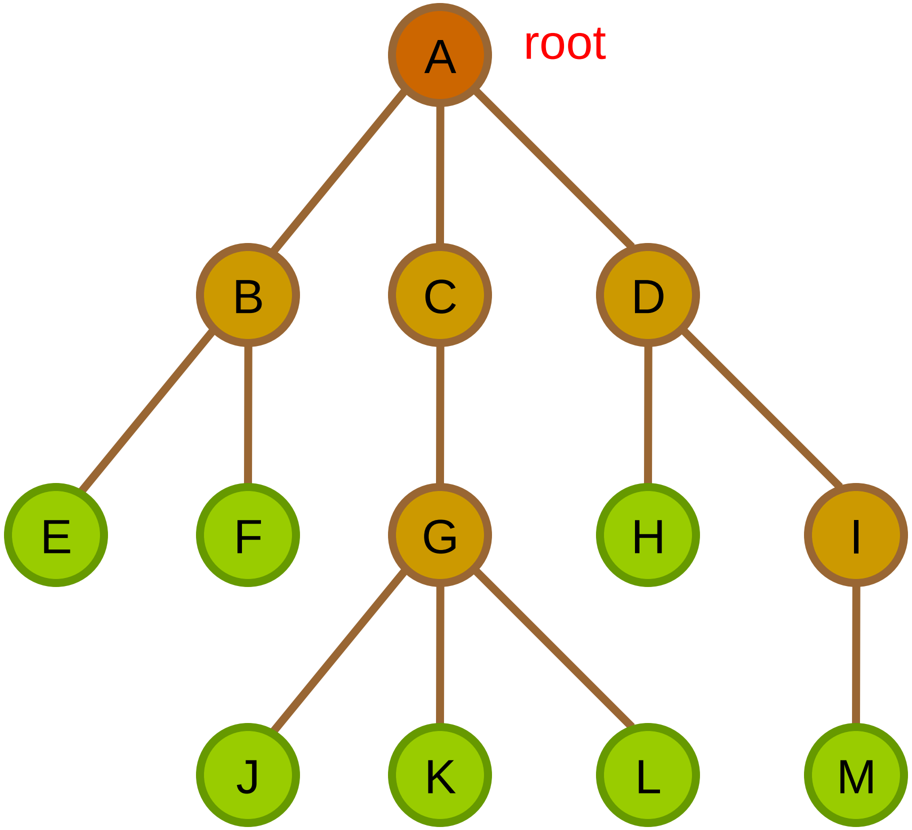
Tree Representation

Representation of Trees

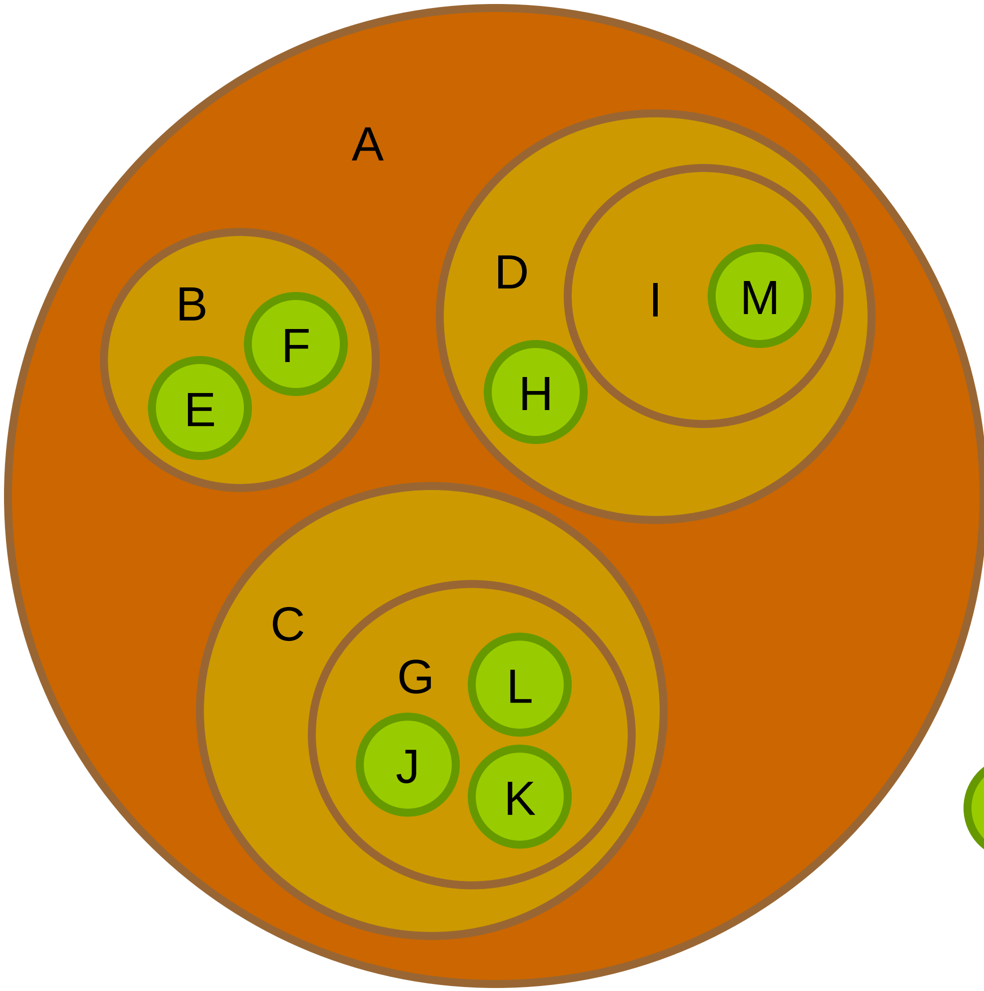
- List representation
- Set representation
- Indentation

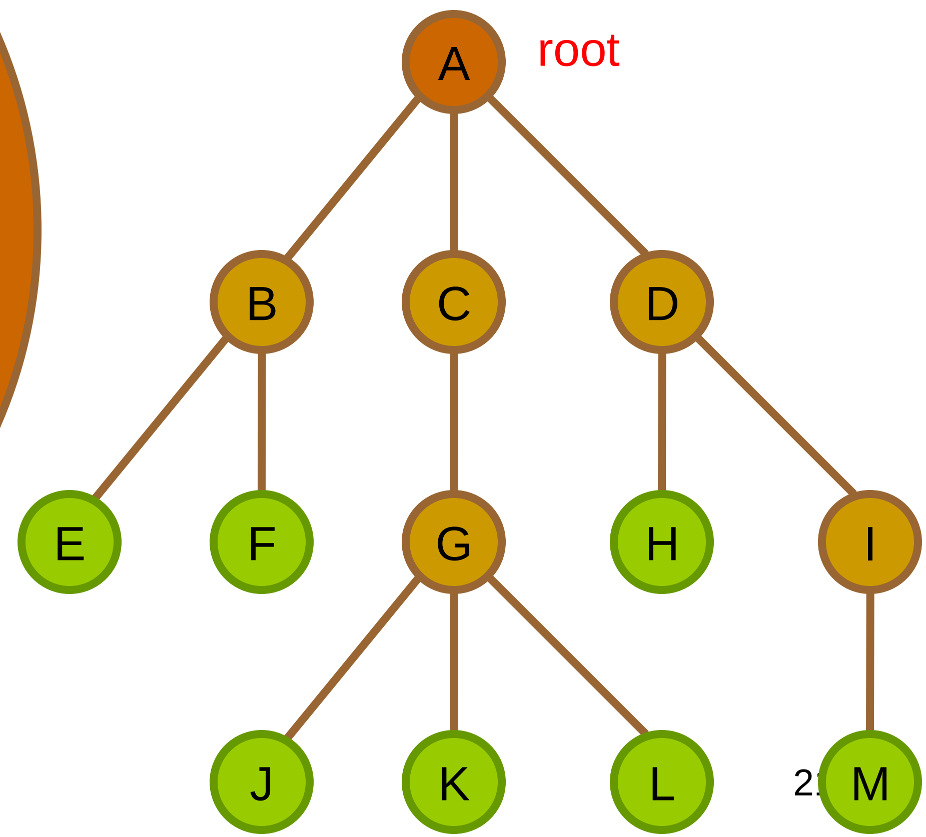
List Representation

- The tree can be represented by this list
  - (A(B(E, F), C(G(J, K, L), D(H, I(M)))))

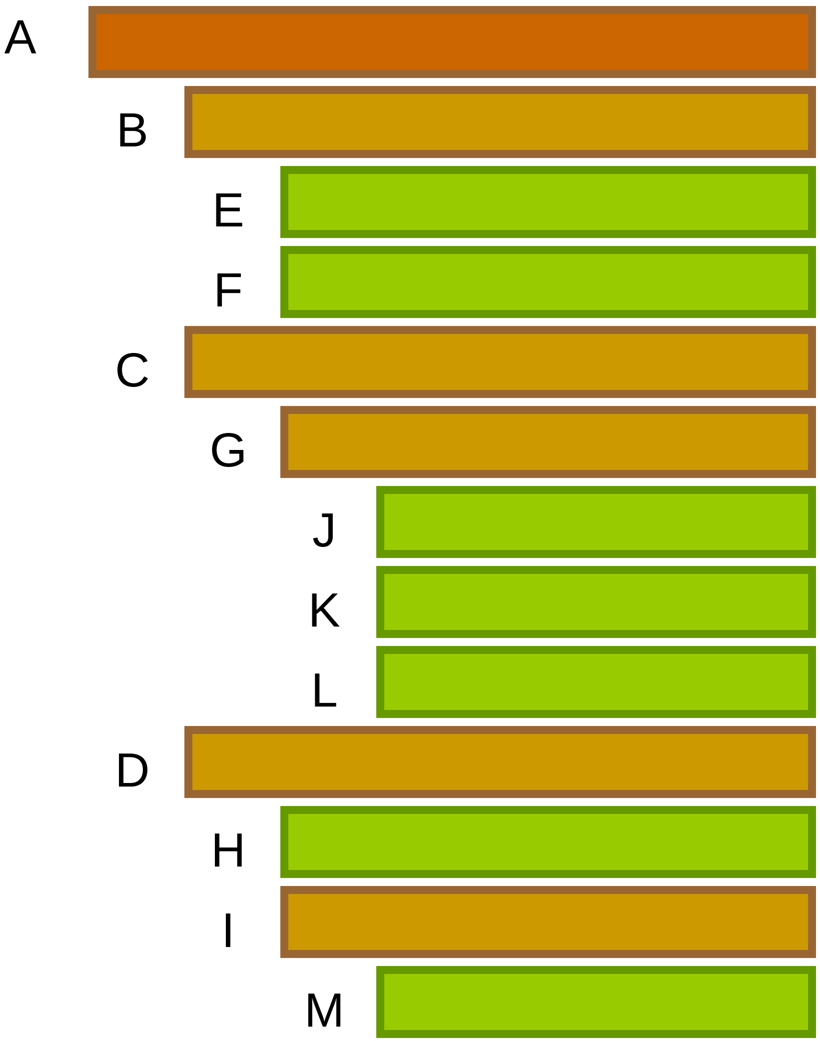


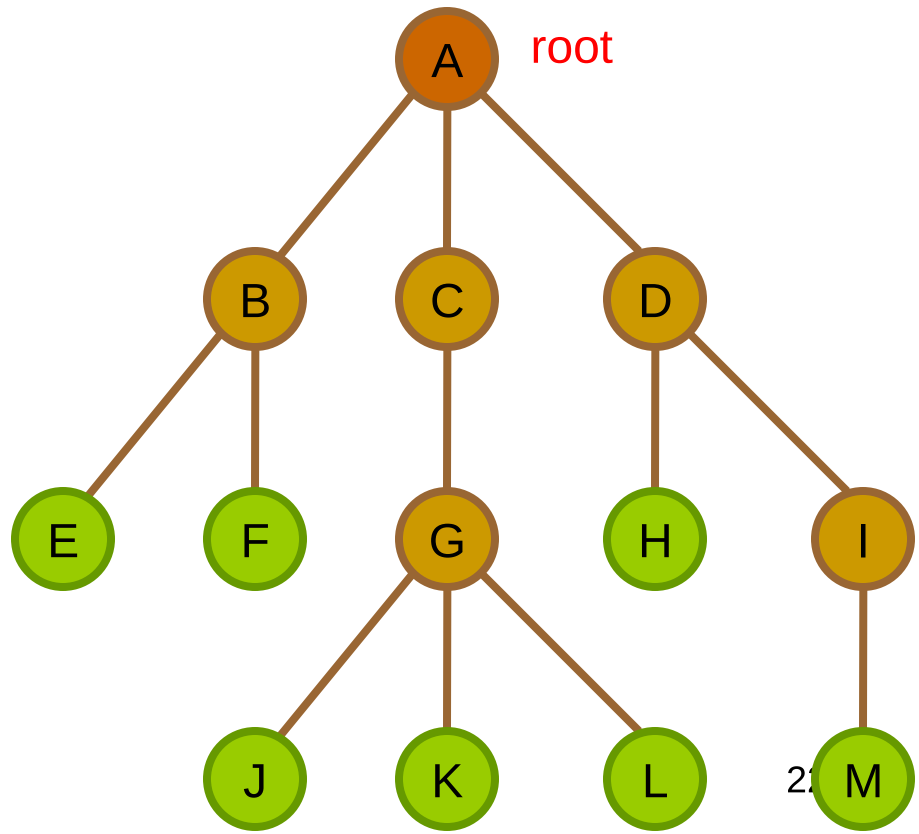
Set Representation





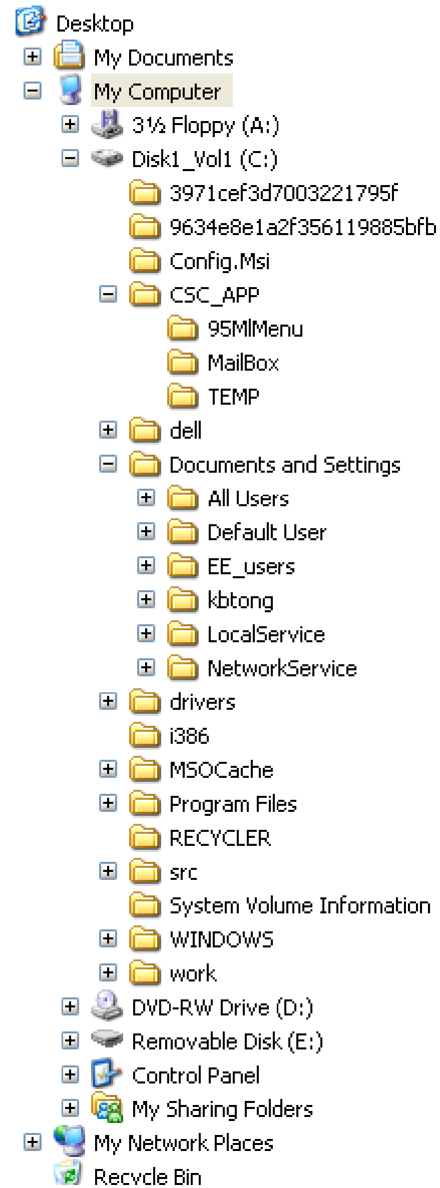
Indentation Representation





They Are Also Indentation

- A
  - B
    - E
    - F
  - C
    - G
      - J
      - K
      - L
  - D
    - H
    - I
      - M

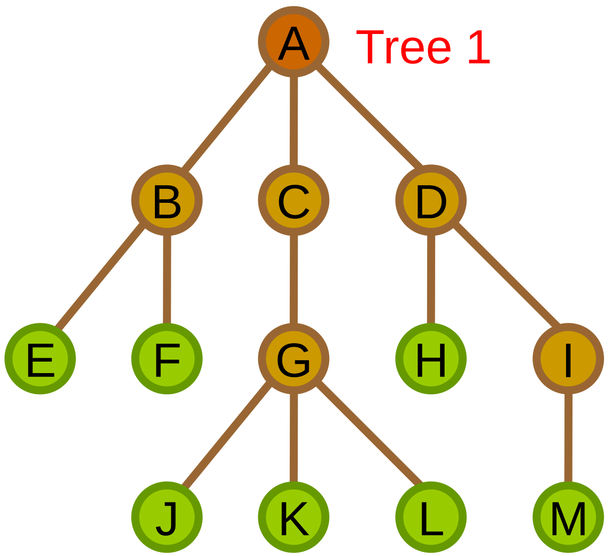


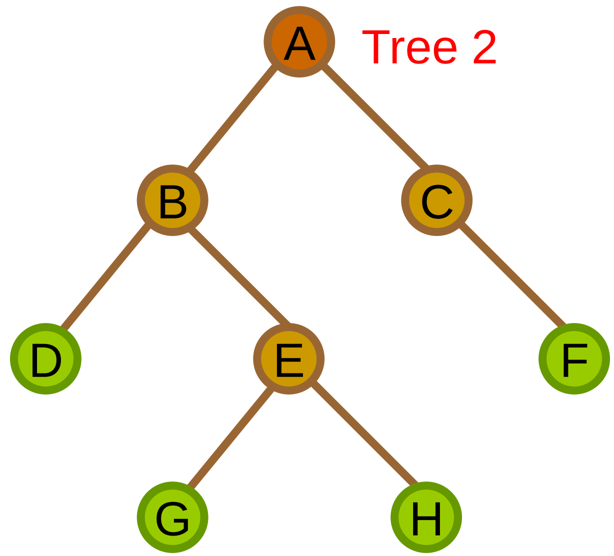
Binary Tree

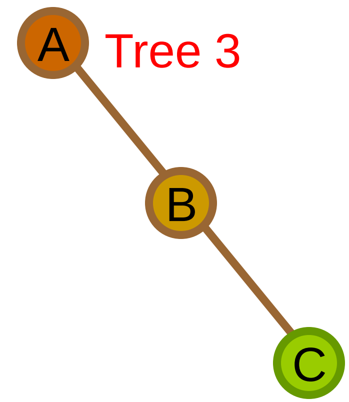
Binary Trees

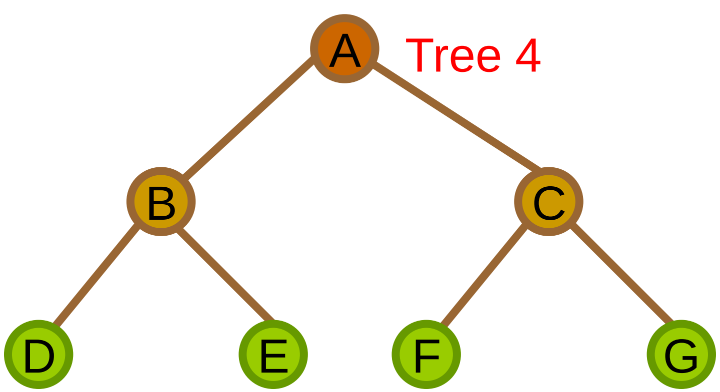
- A special kind of tree
- Simple design
- Fixed max. degree of each node
  - Easier to represent with fixed data structure
- Each node has at most 2 children
  - i.e. the node in binary tree should have either no children (leaf node), 1 child or 2 children
- Easy

Are They Binary Tree? Why?







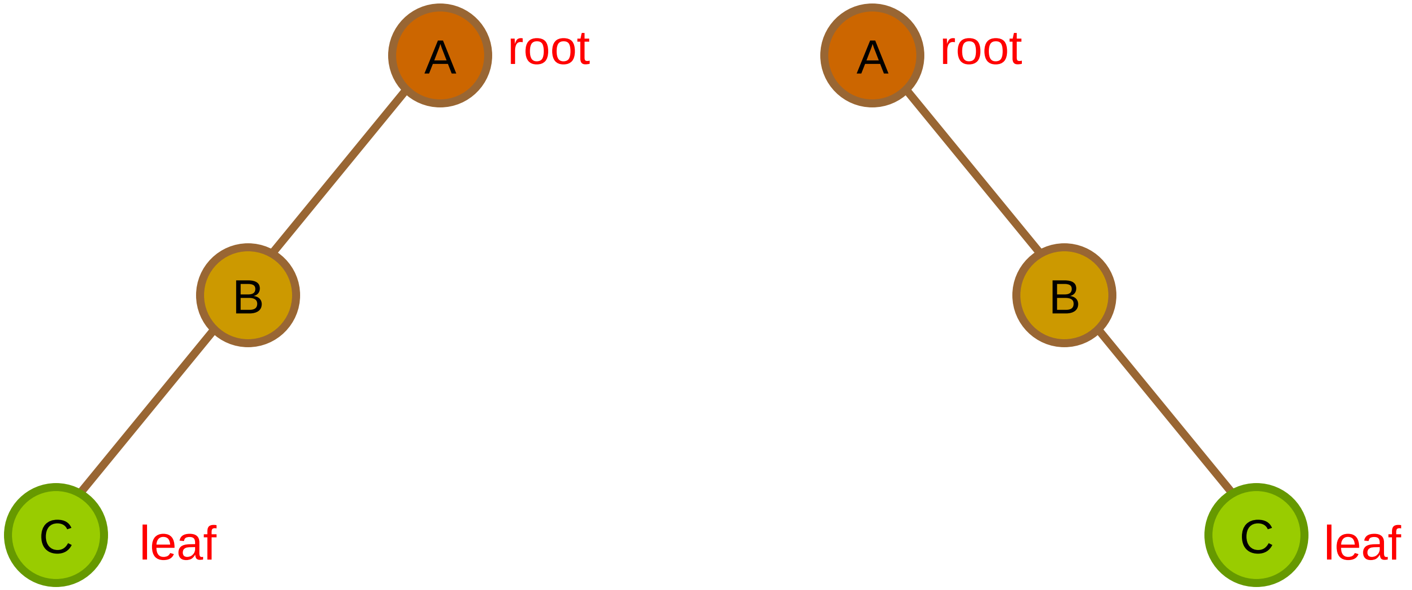


Tree 5
(root only)

Tree 6 (empty
tree)

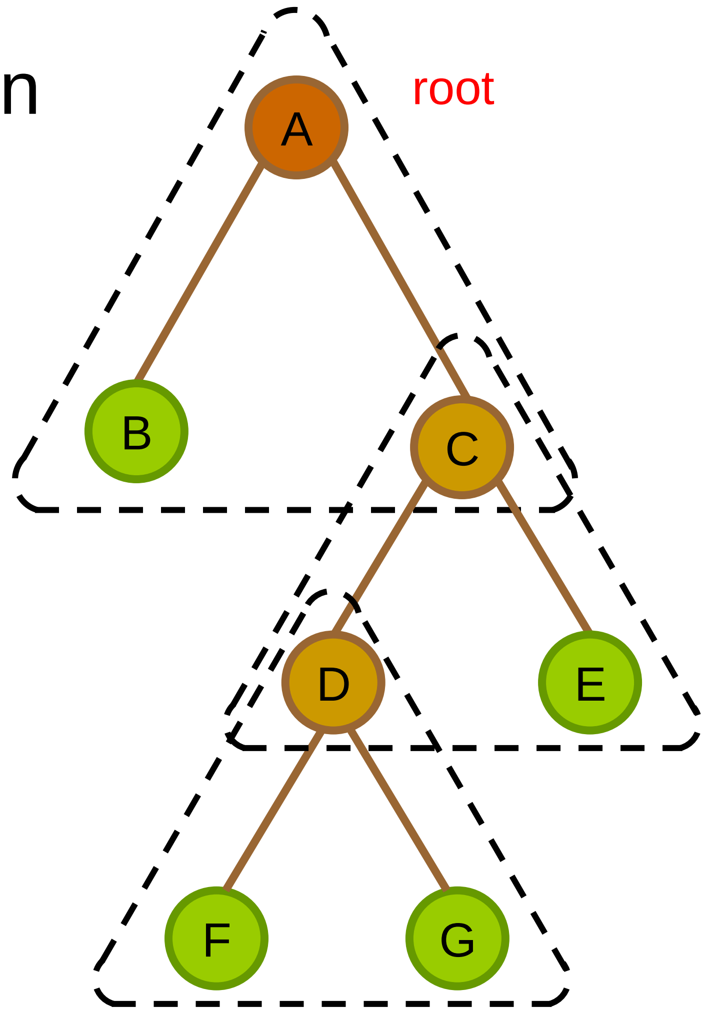
Types of Binary Tree

- **Skewed tree**
- All nodes are either on the left hand side or right hand side



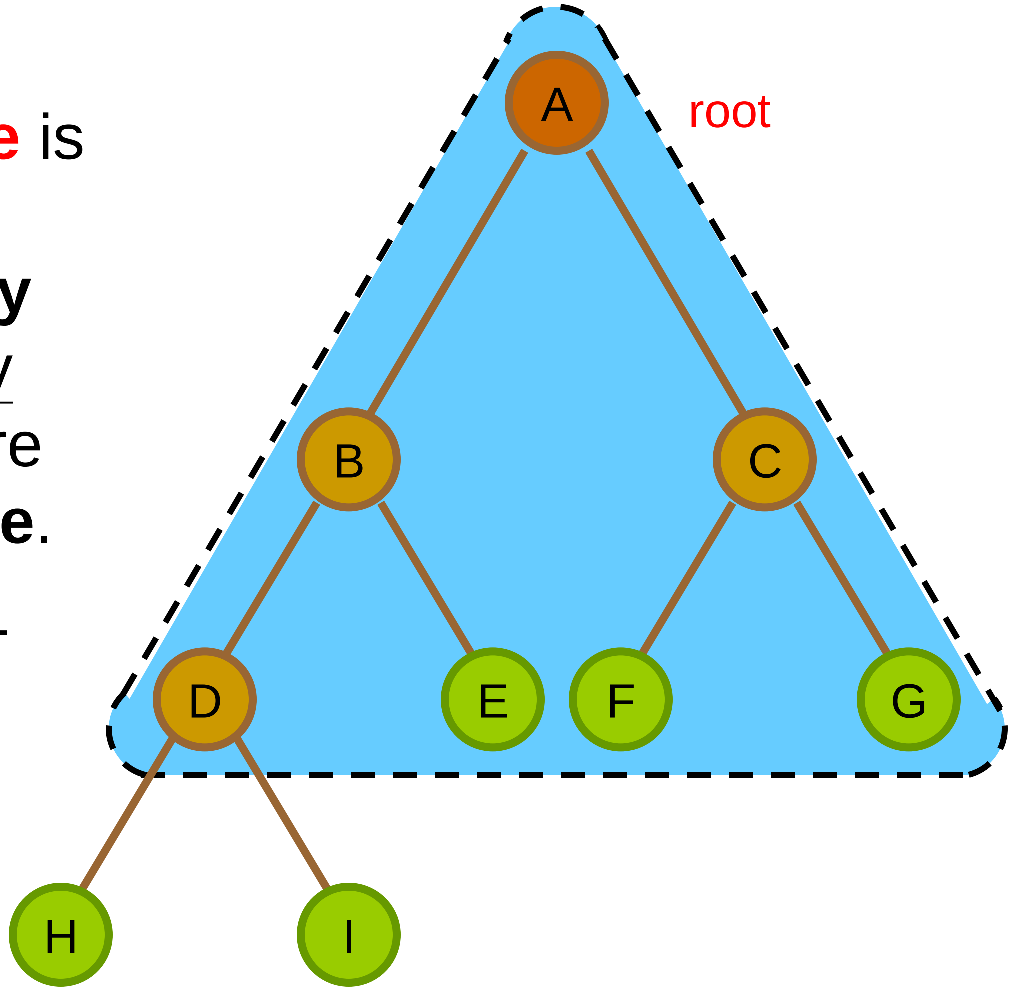
Types of Binary Tree

- **Full binary tree** is a tree in which every node in the tree has either 0 or 2 children.



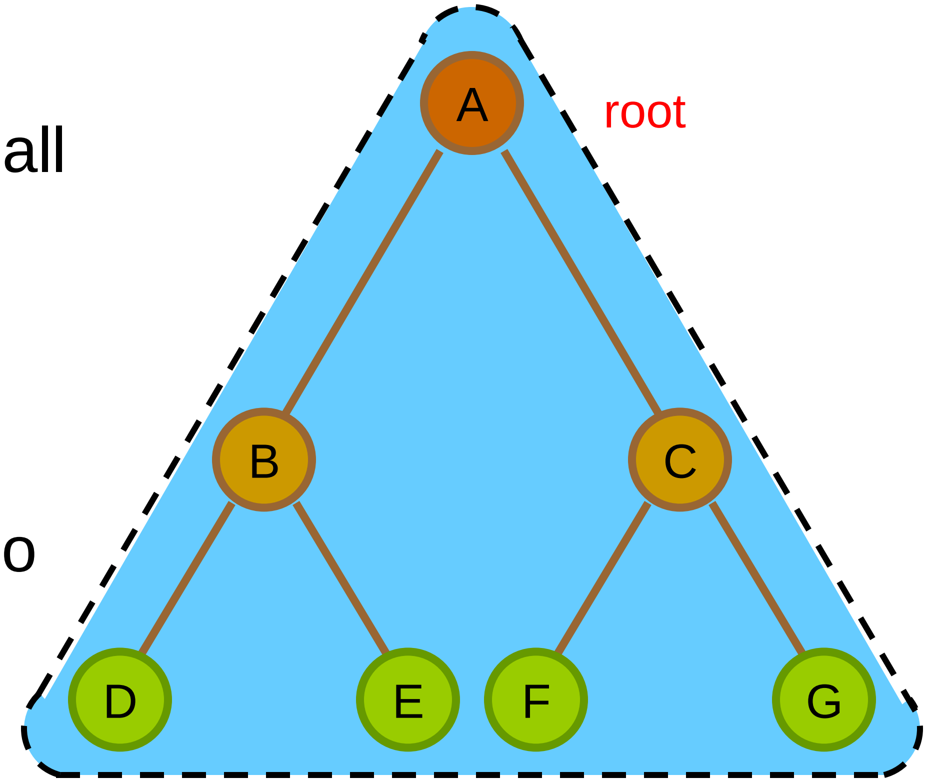
Types of Binary Tree

- **Complete binary tree** is a tree in which every level, **except possibly the last**, is completely filled, and all nodes are **as far left as possible**.
- It can have between 1 and 2^{m-1} nodes at the last level m.



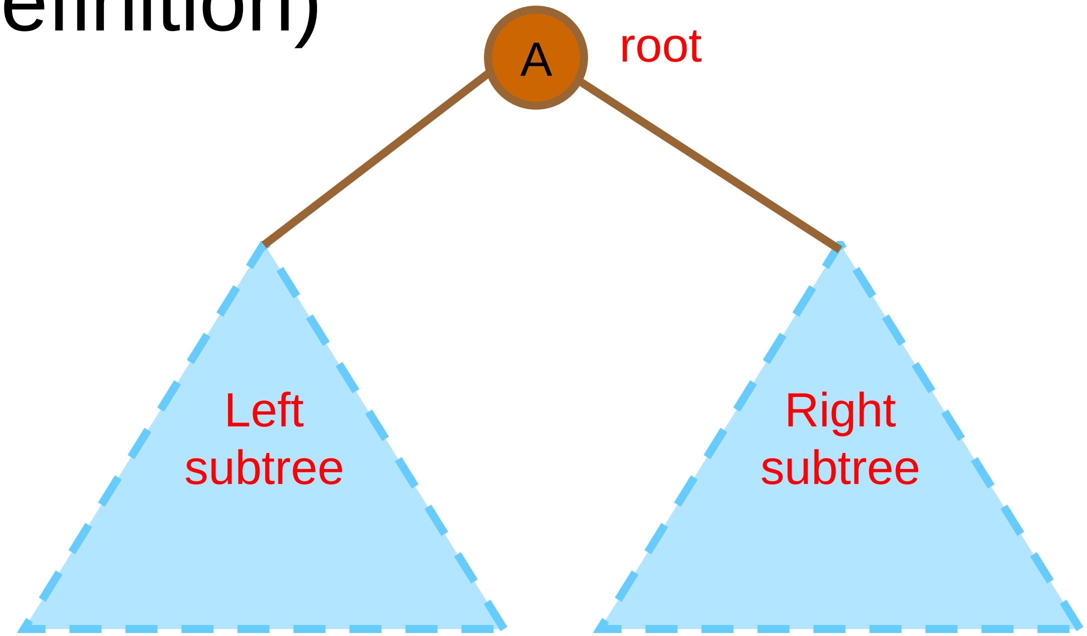
Types of Binary Tree

- **Perfect binary tree** is a binary tree in which all interior nodes have two children *and* all leaves have the same *depth* or same *level*.
- Perfect binary tree is also full binary tree and complete binary tree.



Formation of Binary Trees

- It contains 3 parts, namely
  - root node, left subtree, right subtree
- For each subtree, it has 3 parts again (recursive definition)



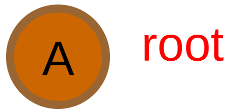
Properties of Binary Trees

- Maximum no. of nodes on level m is 2^{m-1}
- Maximum no. of nodes is $2^{h+1} - 1$, where h is the height of the tree
- For a non-empty binary tree, if n_0 is the total no. of leaf nodes and n_2 is total no. of degree 2 nodes, then $n_0 = n_2 + 1$

Counting Binary Trees

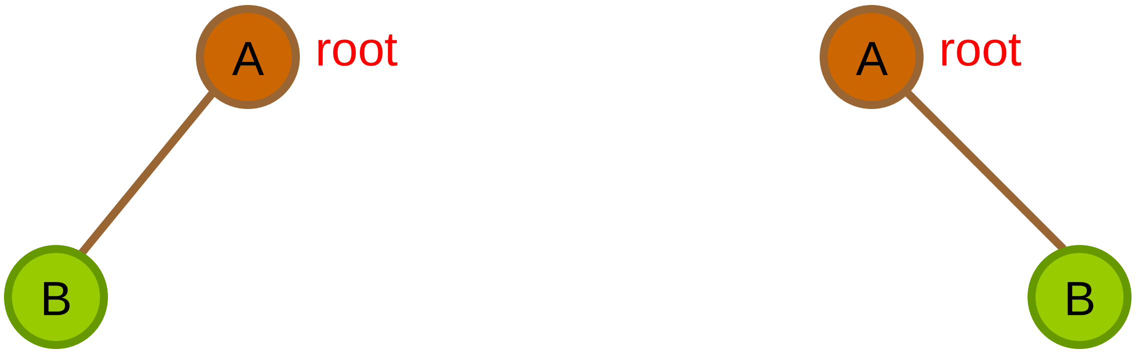
- How many different combination of a tree can have if it has *n* nodes?

- For *n* = 1, only one combination



Counting Binary Trees

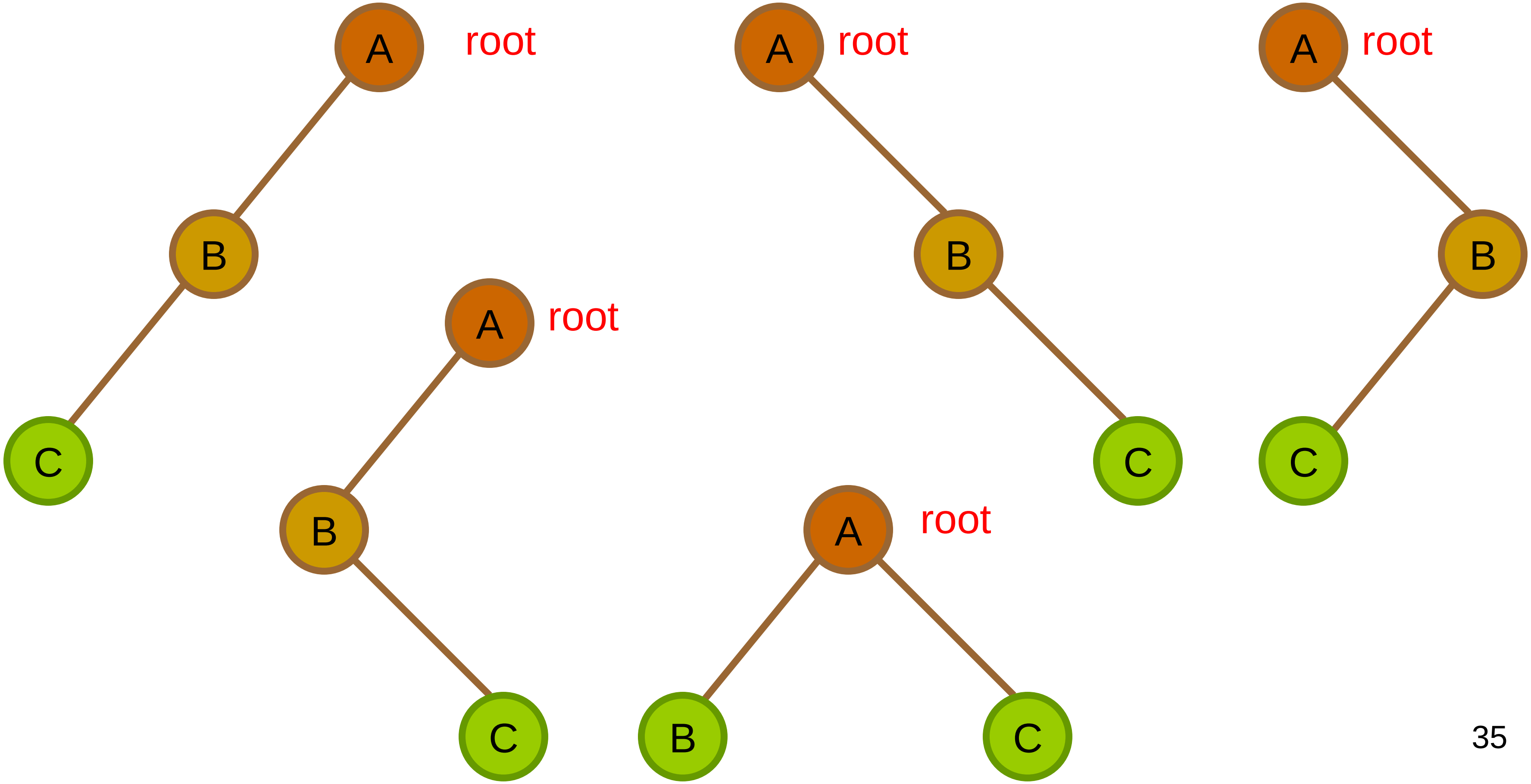
- For n = 2, two combinations



Note: they are different!

Counting Binary Trees

- For n = 3, five combinations

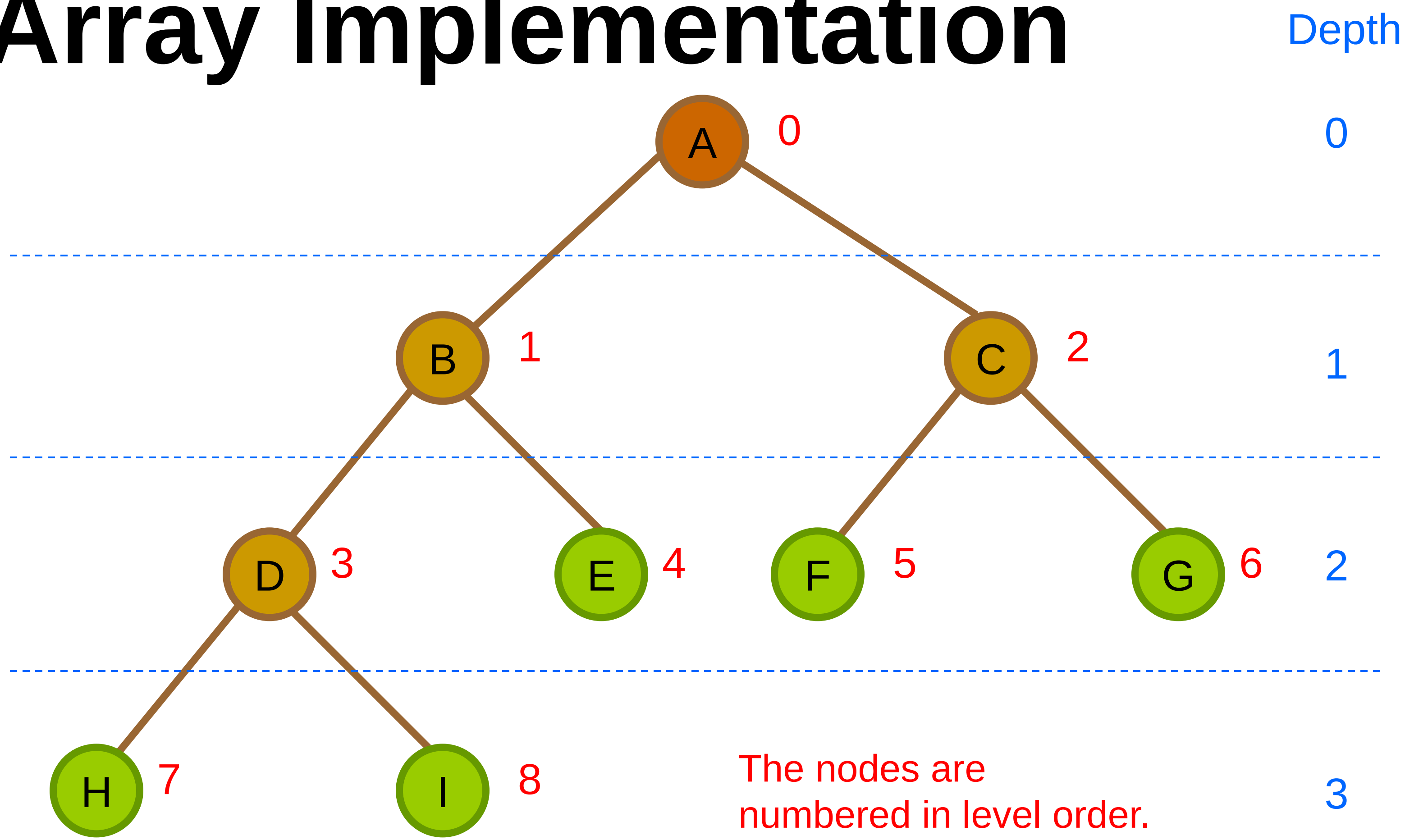


Counting Binary Trees

- For n = 4, 14 combinations
  - Try yourself
- For $n = k$, $\dfrac{1}{k+1} \times \dfrac{(2k)!}{k!k!}$ combinations

Array Implementation

Array Implementation

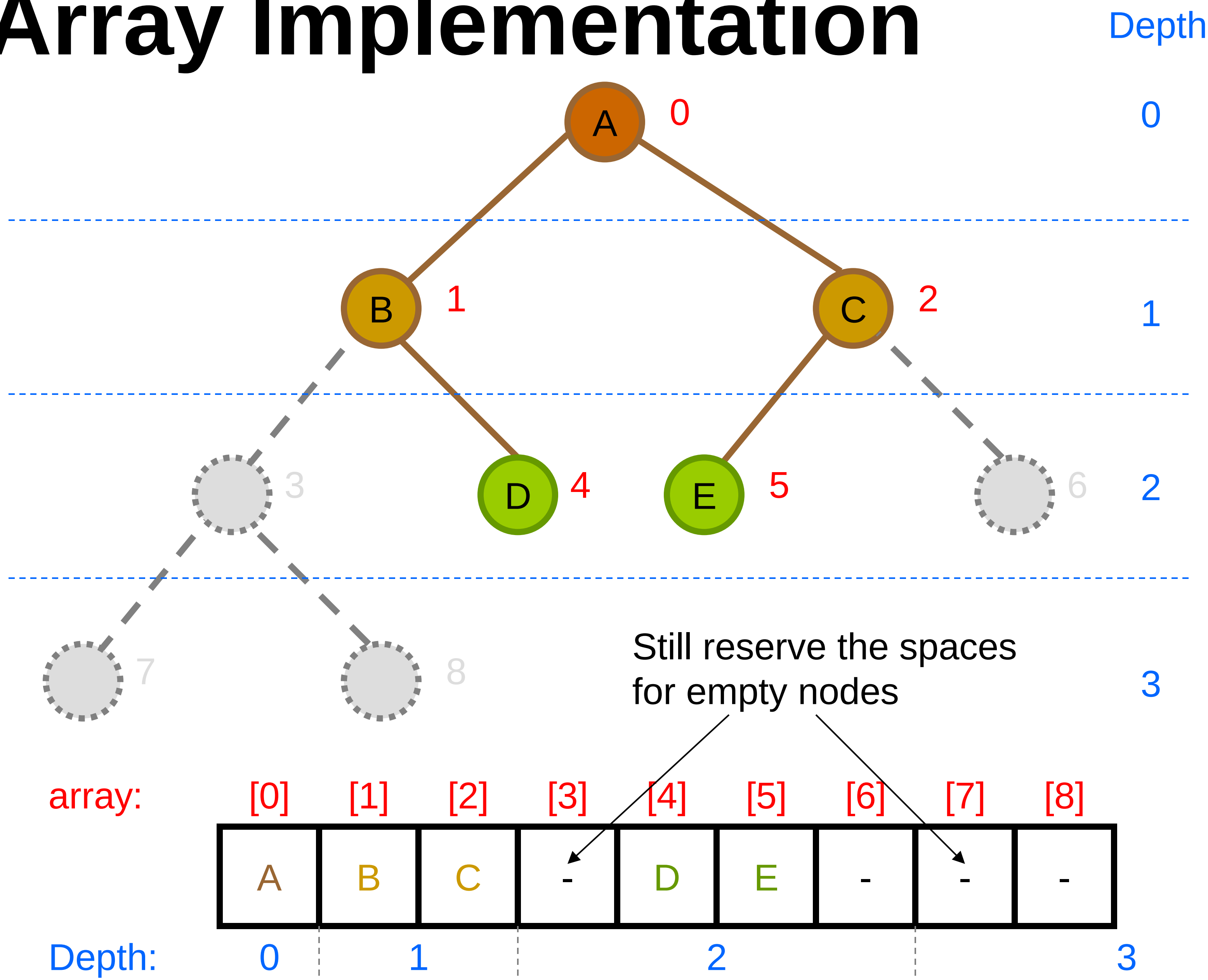


array:

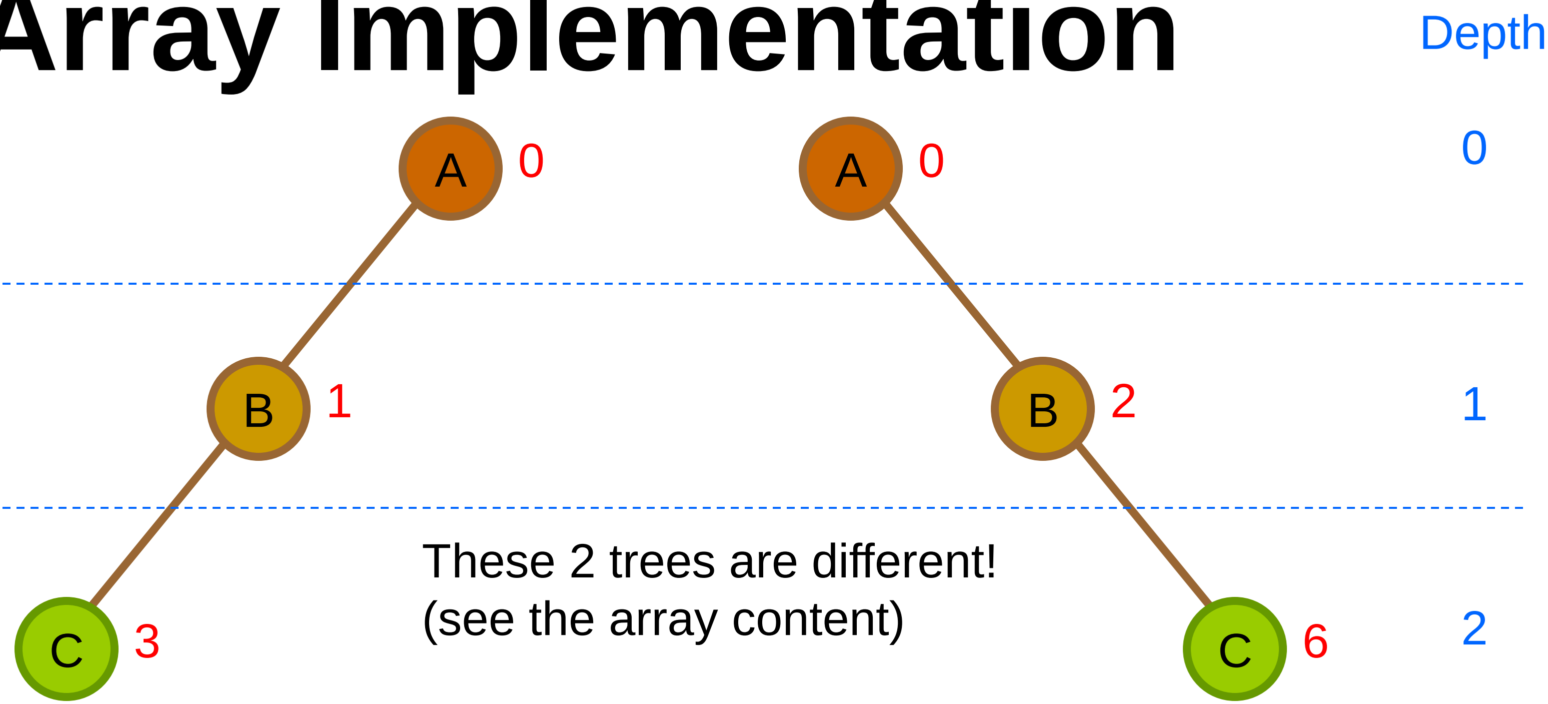
[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
A	B	C	D	E	F	G	H	I

Depth: 0 | 1 | 2 | 3

Array Implementation



Array Implementation



These 2 trees are different!
(see the array content)

array:

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
A	B	-	C	-	-	-	-	-

array:

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
A	-	B	-	-	-	C	-	-

A lot of unused space for non-complete binary tree

Indicating Unused Nodes

Method 1:

array:

A	B	-	C	-	-	-	-	-

Assign a special or invalid value (e.g. -1, '\0')

Method 2:

array:

A	B	?	C	?	?	?	?	?

additional array:

1	1	0	1	0	0	0	0	0

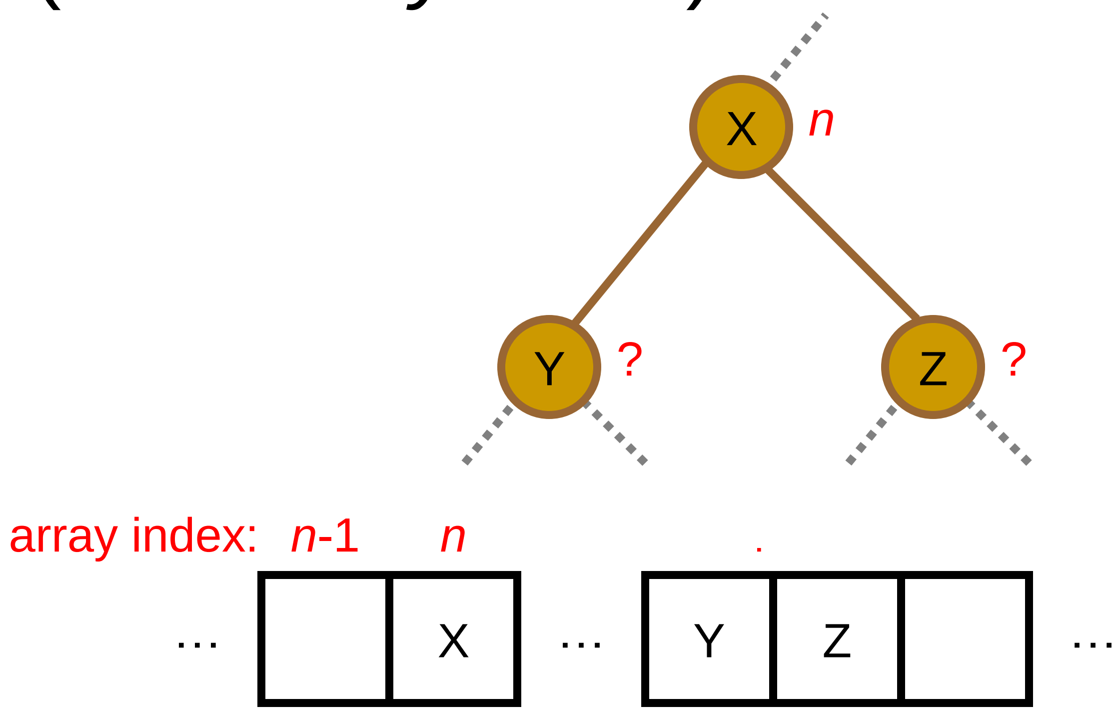
Create another boolean array to indicate the unused node

Memory Efficiency

- For **complete** **binary tree**, array implementation is a very good approach
  - Simple
  - Utilize the memory very well
- But for other binary tree
  - Much memory has been wasted

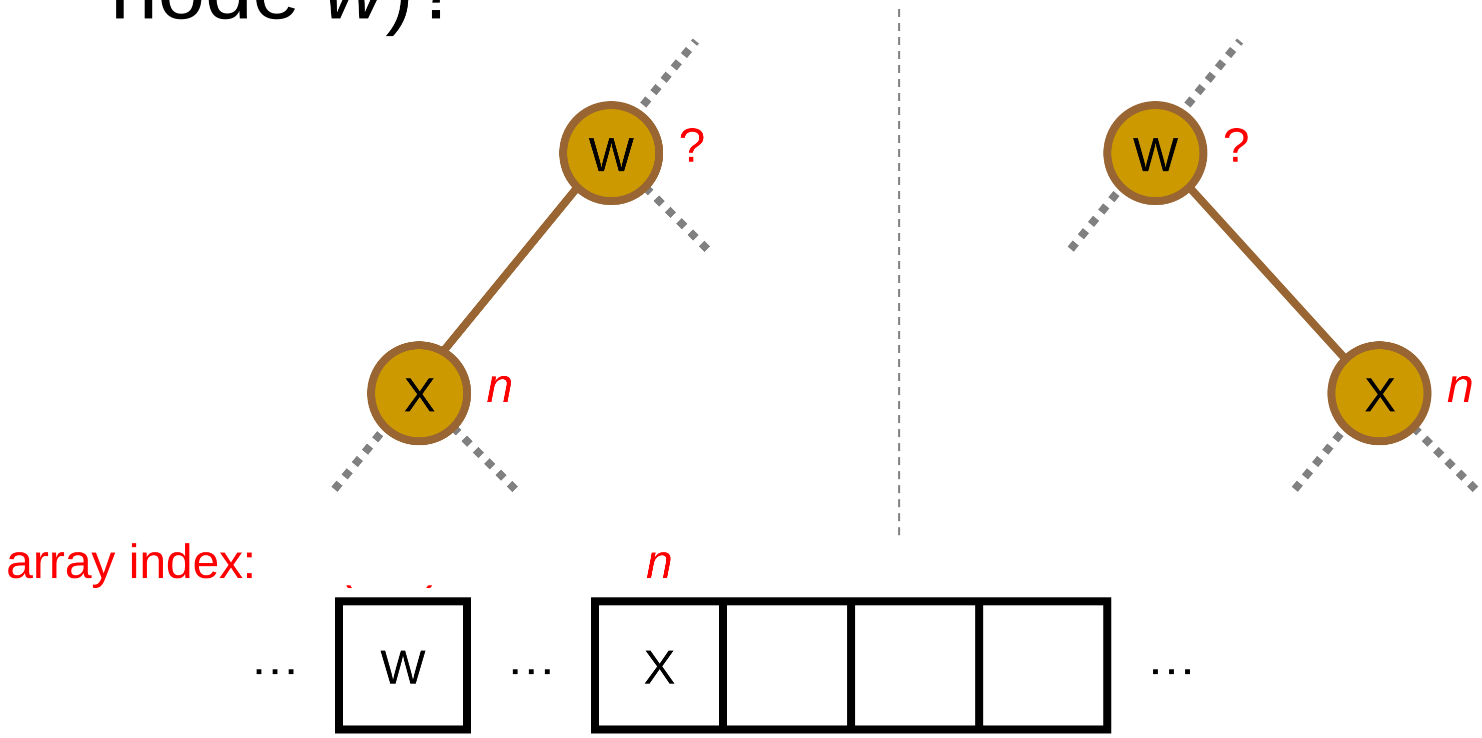
Determine the Index of Children

- If the array index of node *x* is *n*, what are the array indexes of the children of node *x* (i.e. node *y* and *z*)?



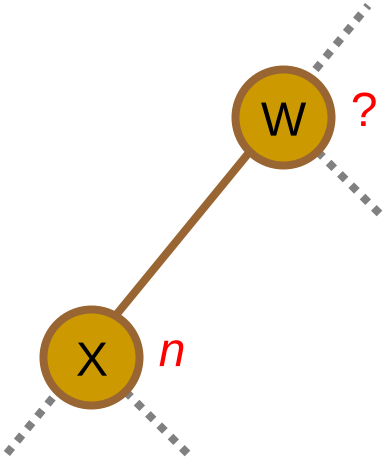
Determine the Index of Parent

- If the array index of node *x* is *n*, what is the array index of the parent of node *x* (i.e. node *w*)?



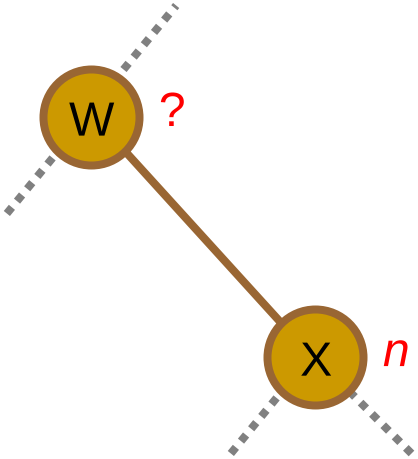
Left or Right Child?

- If the array index of node *x* is *n*, how to determine if node *x* is the left child or right child?



left child if *n* is odd

if (n % 2 == 1) { /* left */ }



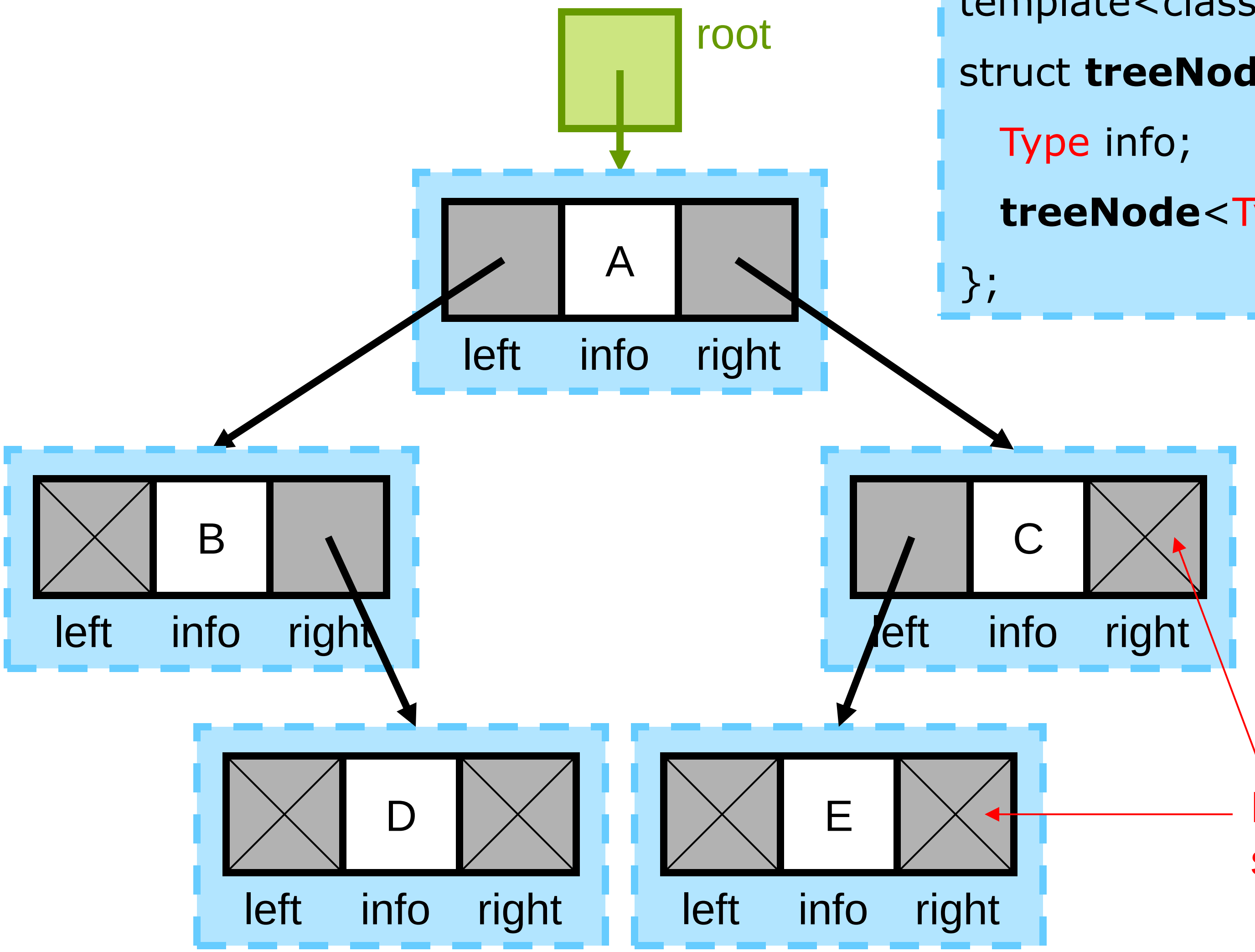
right child if *n* is even

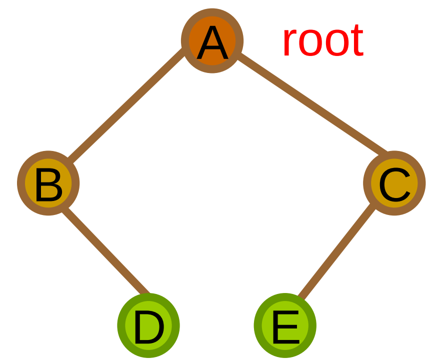
if (n % 2 == 0) { /* right */ }

Linked List Implementation

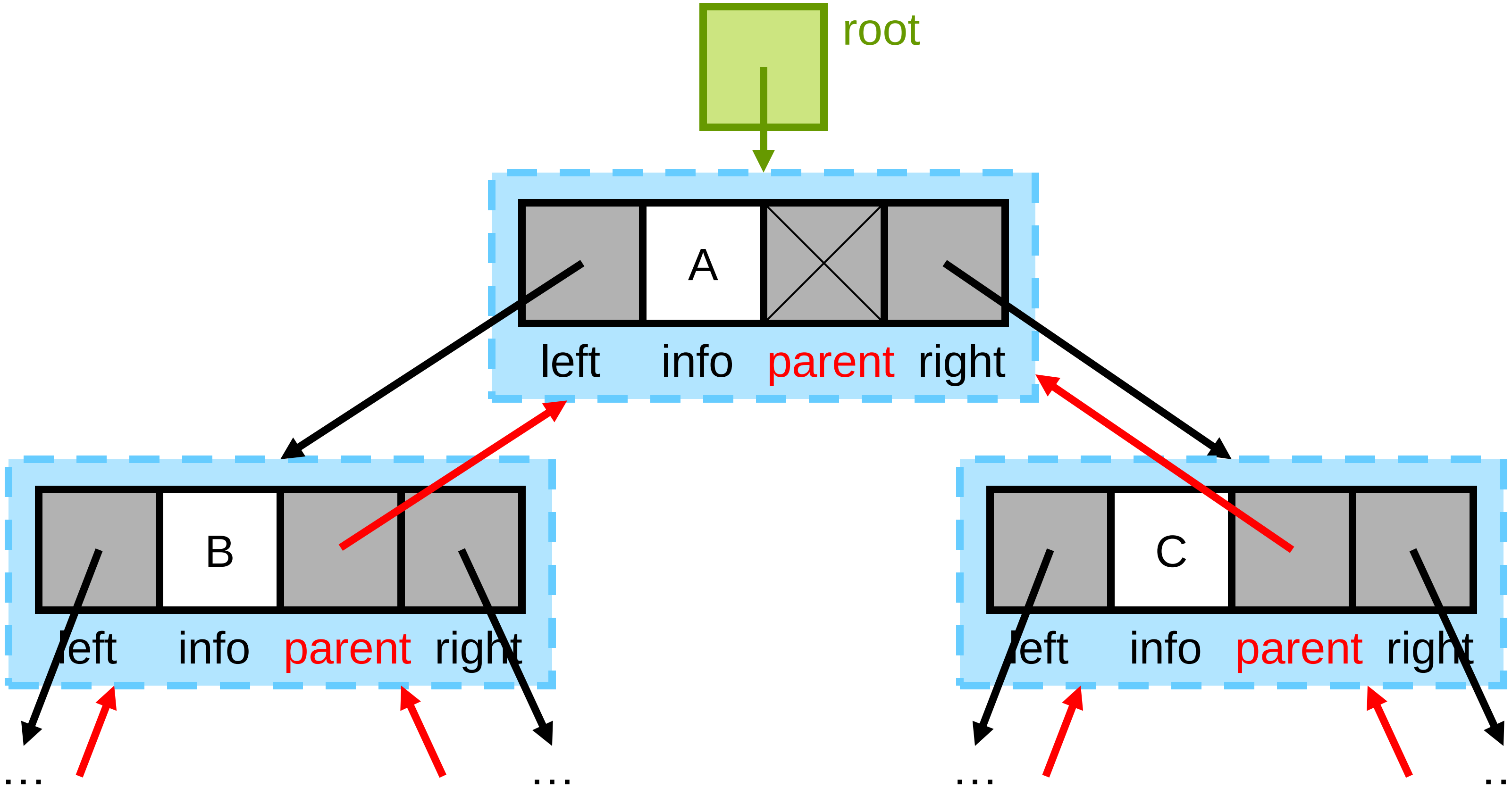
Linked List Implementation

```
template<class Type>
struct treeNode {
   Type info;
   treeNode<Type> *left, *right;
};
```





Possible Variations



Each node has 3 references:

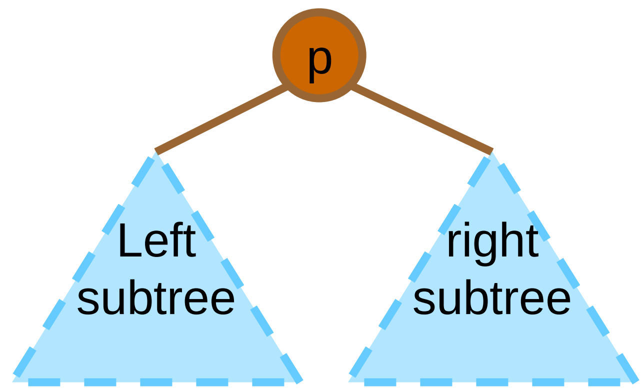
left, right and parent

Common Operations

Compute the Height

```
template<class Type>
int height(treeNode<Type> *tree)
{
   if (tree == NULL)
      return -1;         // some definitions of empty tree's height = 0

   if ((tree->left == NULL) && (tree->right == NULL))
      return 0;

   int HL = height(tree->left);
   int HR = height(tree->right);

   if (HL > HR)
      return 1+HL;
   else
      return 1+HR;
}
```

Height of root =
max(1 + h_{left} , 1 + h_{right})



Height of left subtree = h_{left}

Height of right subtree = h_{right}

Count No. of Nodes / Leaves

```
template<class Type>
int nodeCount(treeNode<Type> *tree) {
  if (tree == NULL)
    return 0;
  return 1 + nodeCount(tree->left) + nodeCount(tree->right));
}
```

```
template<class Type>
int leavesCount(treeNode<Type> *tree) {
  if (tree == NULL)
    return 0;
  else if ((tree->left == NULL) && (tree->right == NULL))
    return 1;
  else
    return leavesCount(tree->left) + leavesCount(tree->right);
}
```

Copy Binary Tree

```
template<class Type>
void copyTree(treeNode<Type>*& copiedTree, treeNode<Type> *other) {
   if (other == NULL)
      copiedTree = NULL;
   else {
      copiedTree = new treeNode<Type>;
      copiedTree->info = other->info;
      copyTree(copiedTree->left, other->left);     // copy left subtree
      copyTree(copiedTree->right, other->right);  // copy right subtree
   }
}
```

Copy Binary Tree (Alternative)

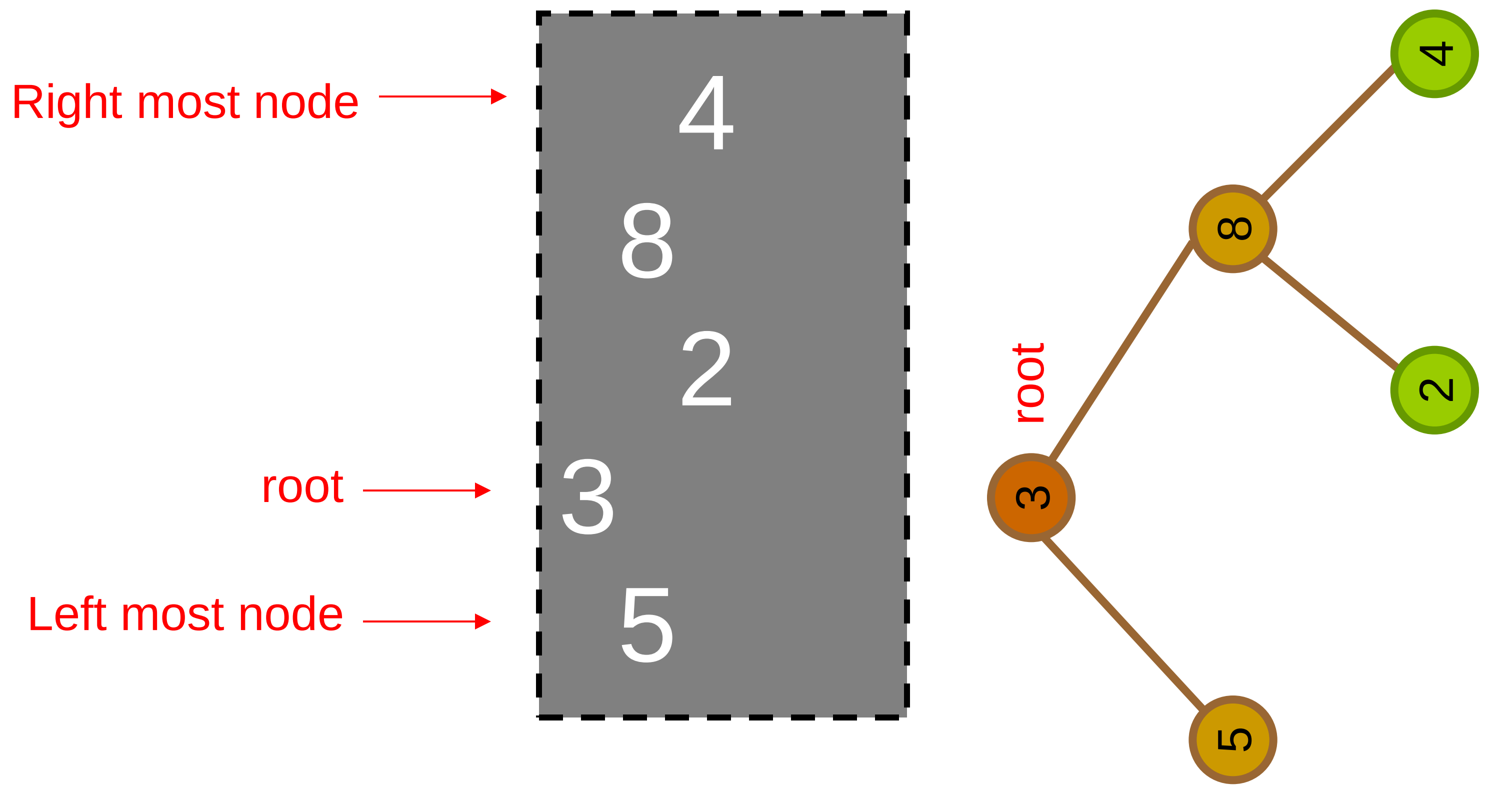
```
template<class Type>
treeNode<Type>* copyTree_2(treeNode<Type> *other)
{
   if (other == NULL)
      return NULL;

   treeNode<Type> *p = new treeNode<Type>;
   p->info = other->info;
   p->left = copyTree_2(other->left);
   p->right = copyTree_2(other->right);

   return p;
}
```

Compare Two Binary Tree

- The two binary trees are identical iff
  - Their root nodes are equal;
  - their left subtrees are equal and;
  - their right subtrees are equal.

```
template<class Type>
bool equal(treeNode<Type> *tree1, treeNode<Type> *tree2) {



}
```

Printing a Binary Tree



Printing a Binary Tree

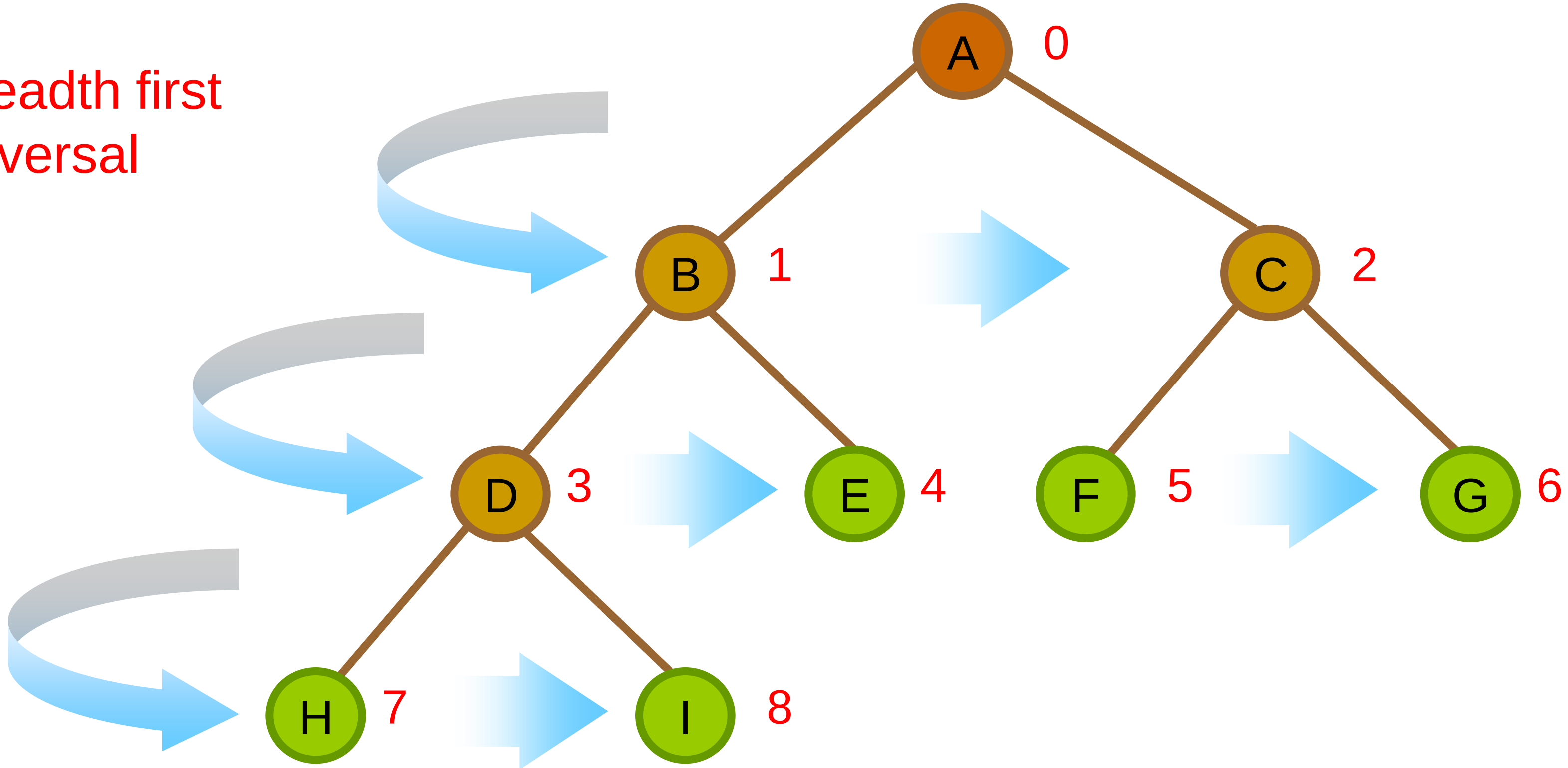
- Print the right subtree first

```
#include <iomanip>          //setw(), set width

template<class Type>
void printTree(treeNode<Type> *p, int indent) {
    if (p != NULL) {
        //print right subtree, root, and then left subtree
        printTree(p->right, indent+3);
        cout << setw(indent) << p->info << endl;
        printTree(p->left, indent+3);
    }
}
```

Four Basic Traversal Orders

- Describe the way to visit every nodes of the entire tree
- **Level order**
  - visit the nodes from left to right, level by level starting from the root

Breadth first traversal



Four Basic Traversal Orders

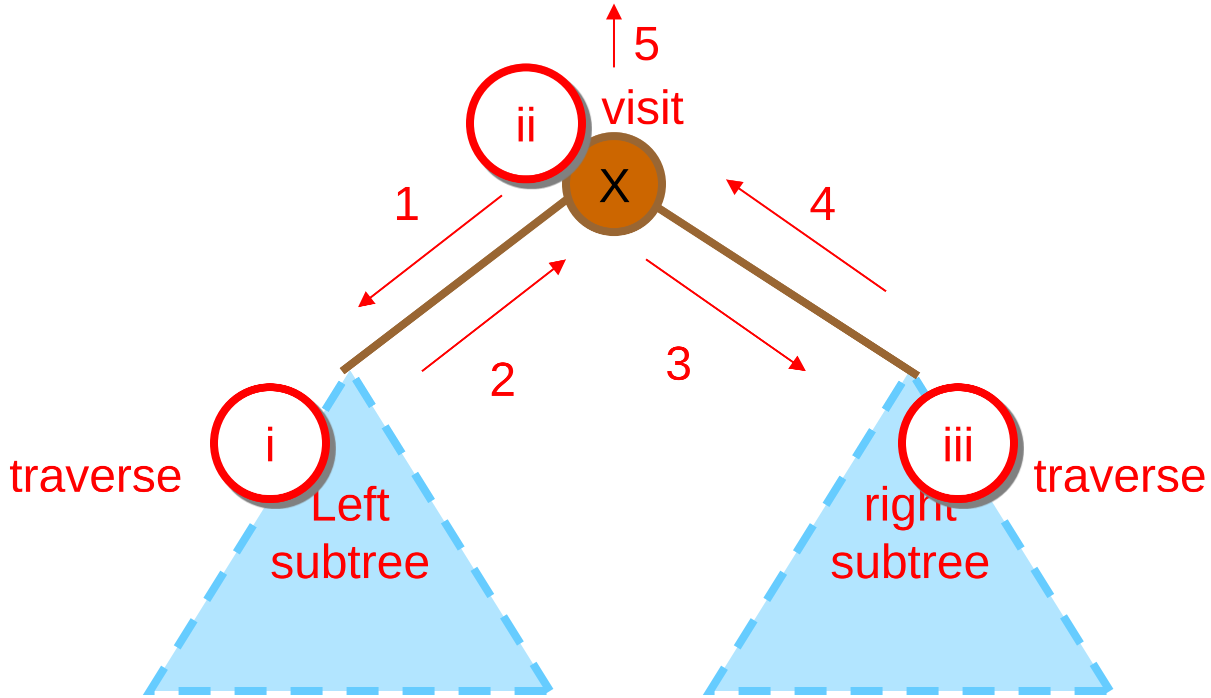
- **Preorder**
  - visit the root (V)
  - visit the left subtree in preorder (L)
  - visit the right subtree in preorder (R)
- **Inorder**
  - visit the left subtree in inorder (L)
  - visit the root (V)
  - visit the right subtree in inorder (R)
- **Postorder**
  - visit the left subtree in postorder (L)
  - visit the right subtree in postorder (R)
  - visit the root (V)

Depth first traversal

- Which kind of traversal does backtracking use?

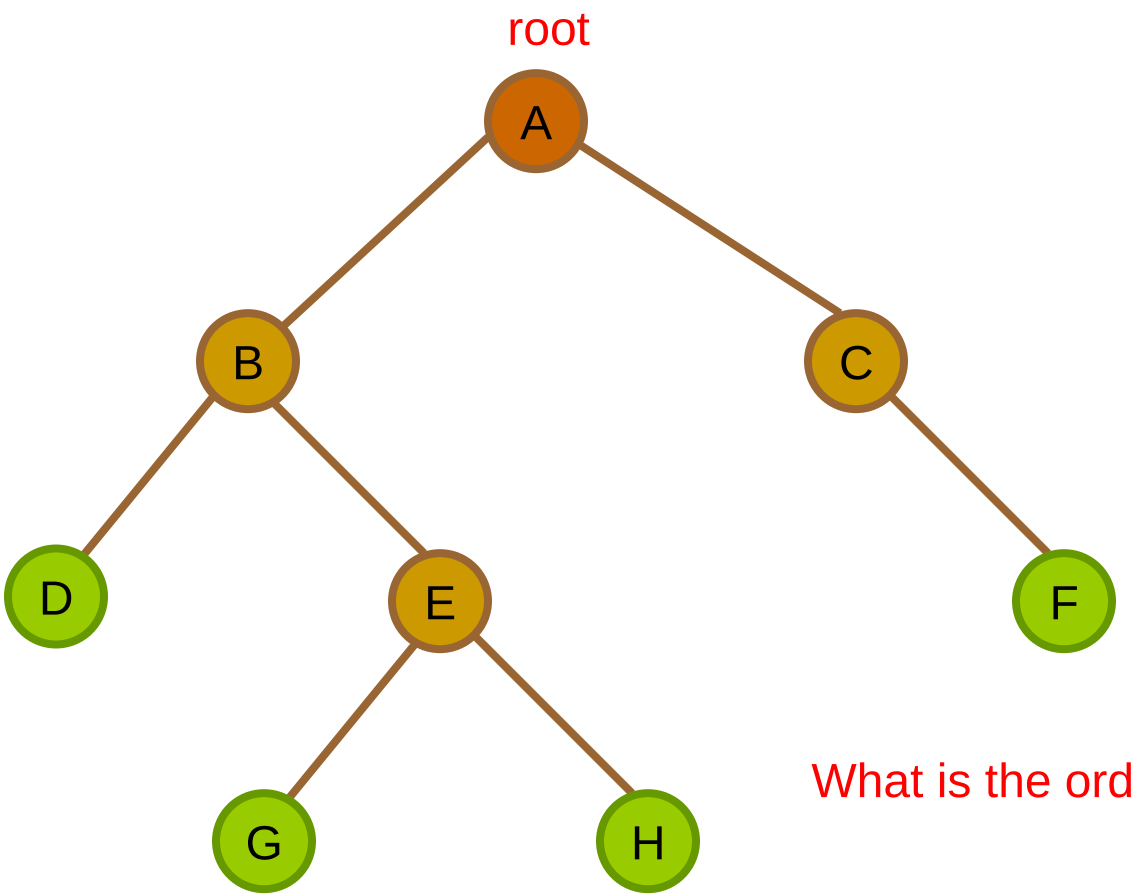
Example: LVR

- Step i) go to left subtree (**recursion**)
- Step ii) visit node x
- Step iii) go to right subtree (**recursion**)



Inorder Traversal

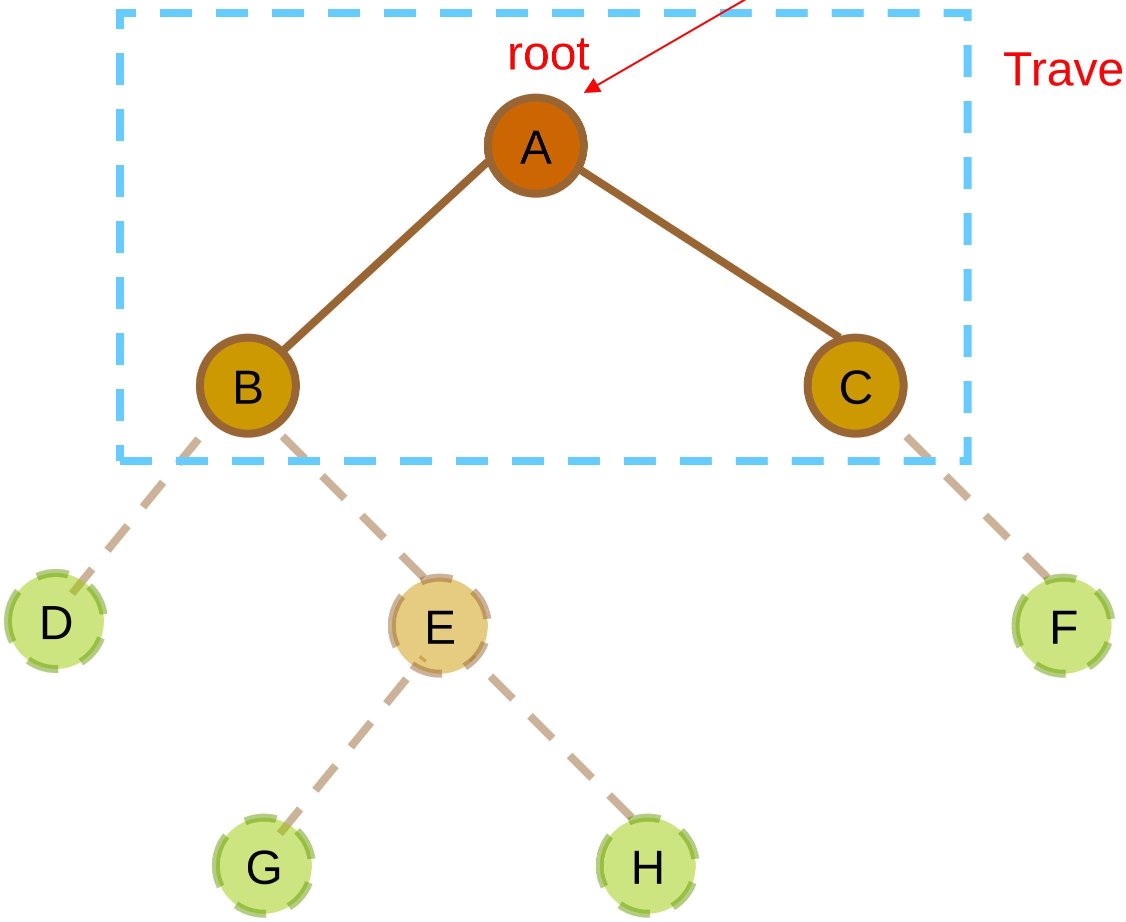
- LVR



What is the order of traversal?

Inorder Traversal

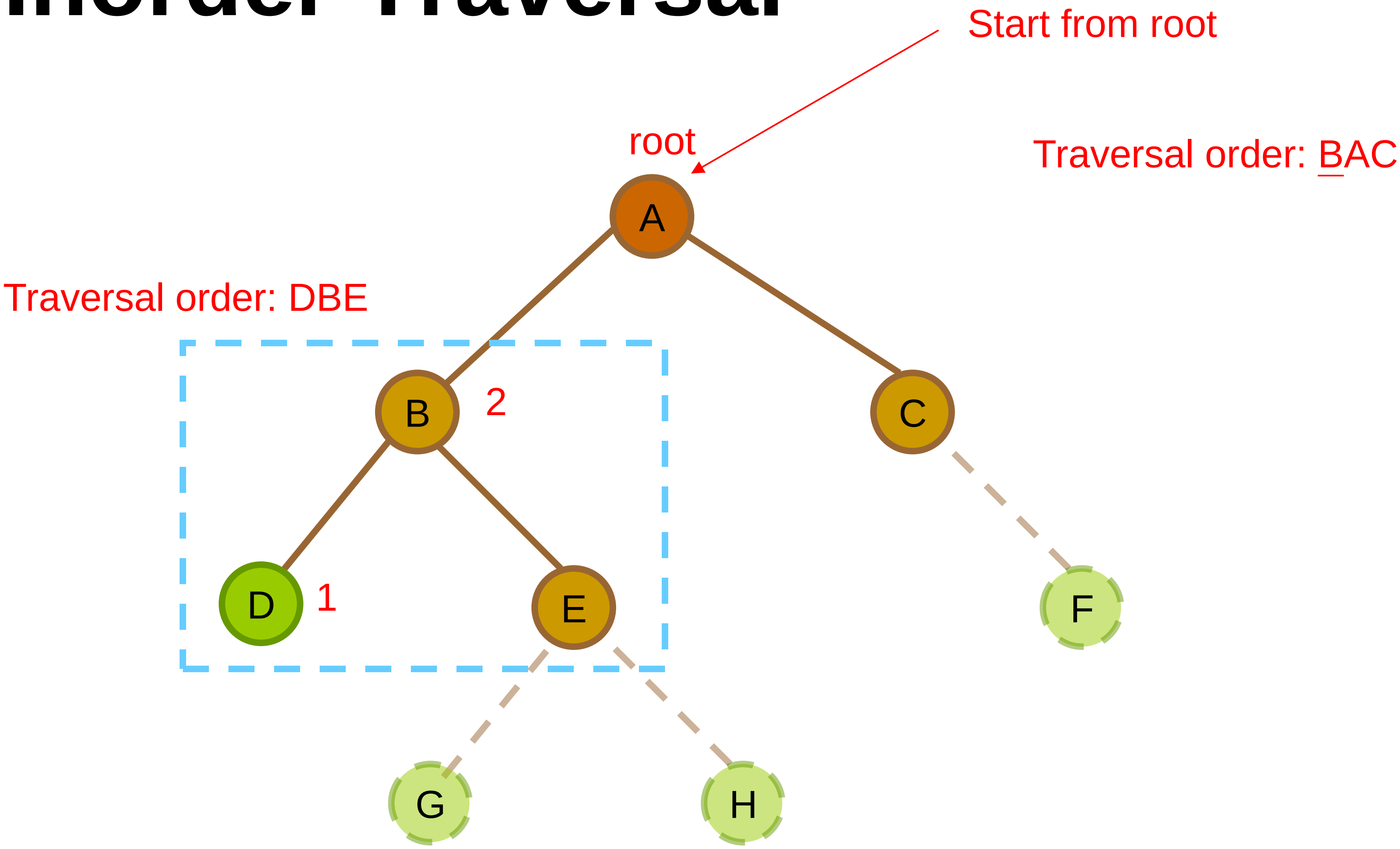
Start from root



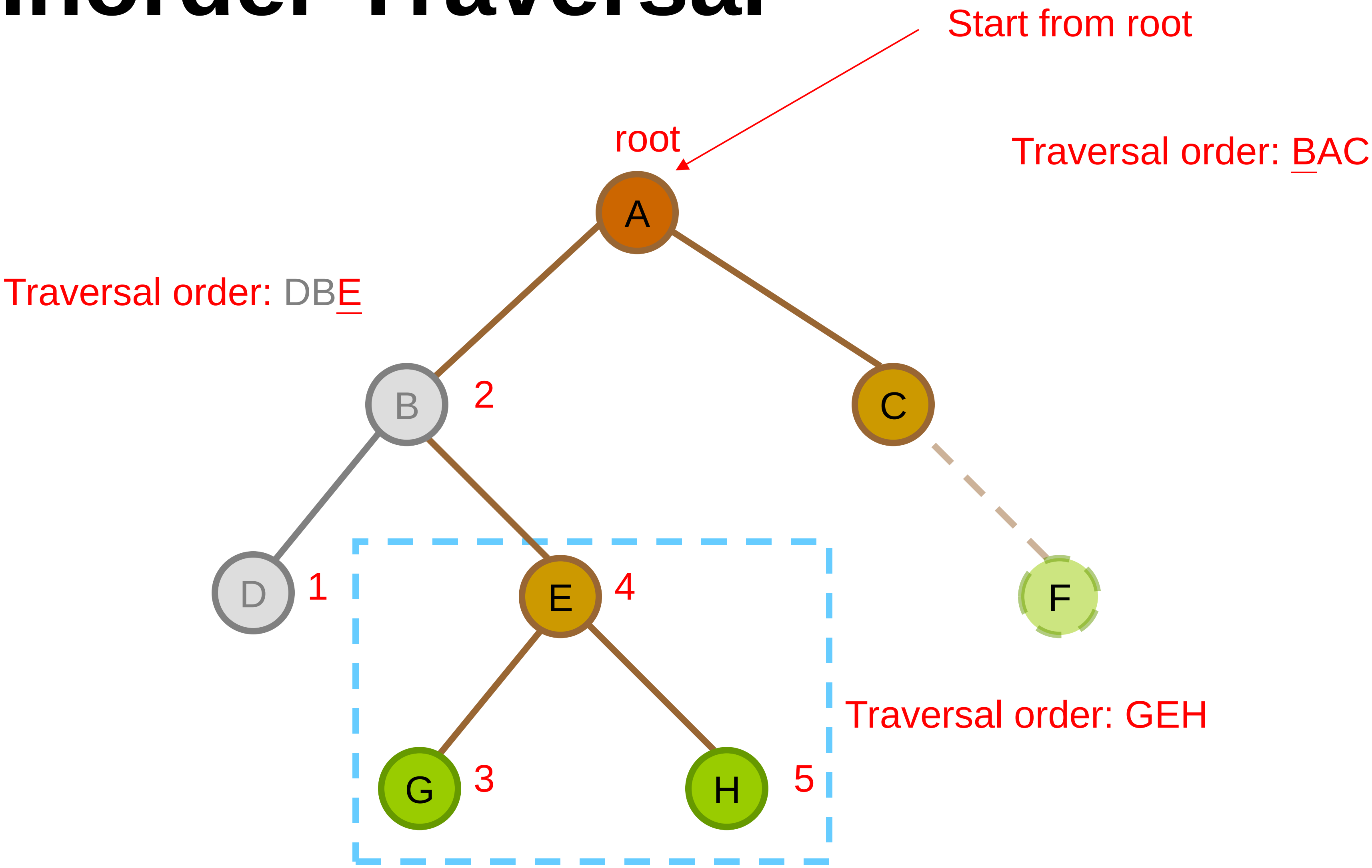
Traversal order: BAC

But B is the root of a subtree

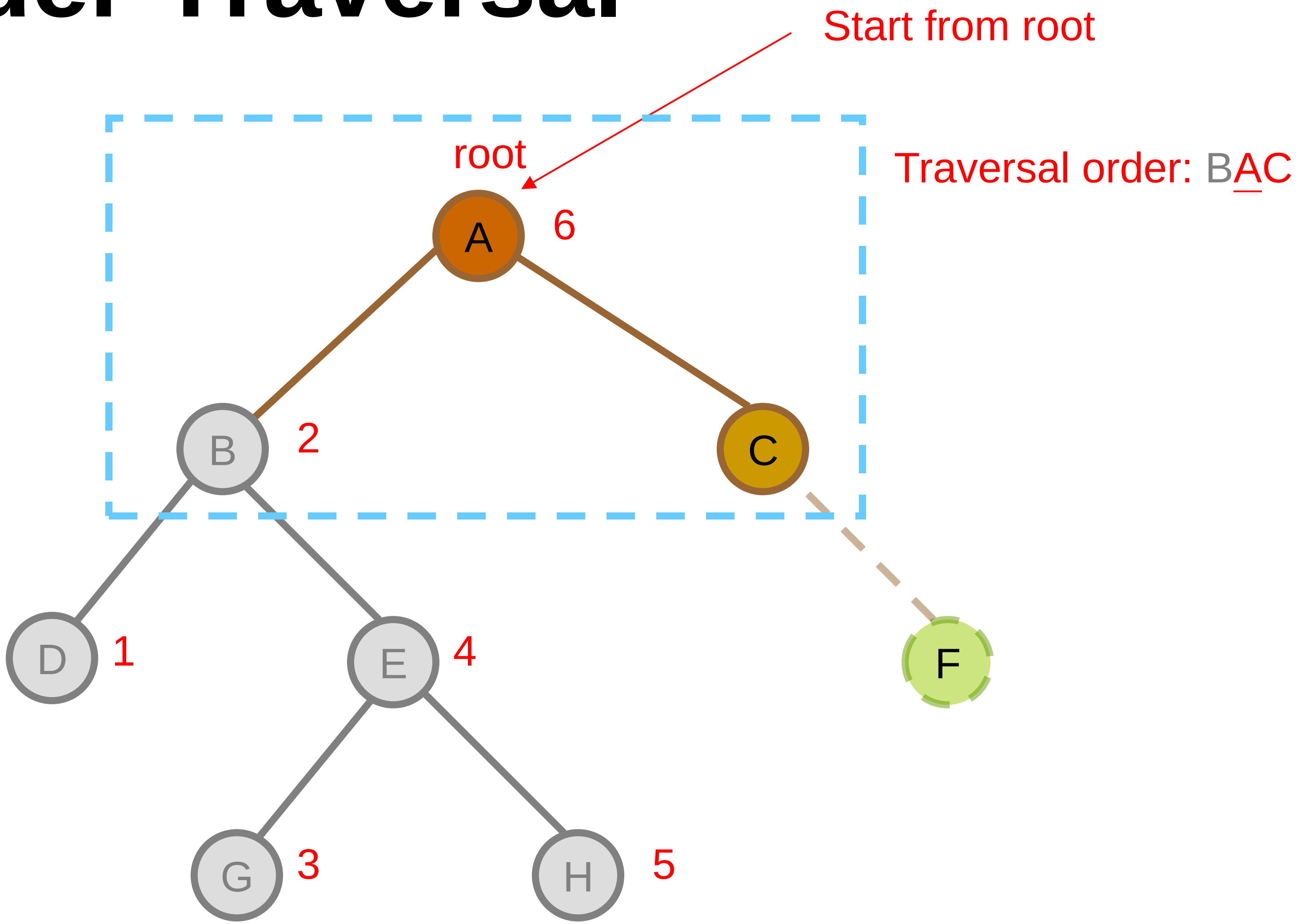
Inorder Traversal



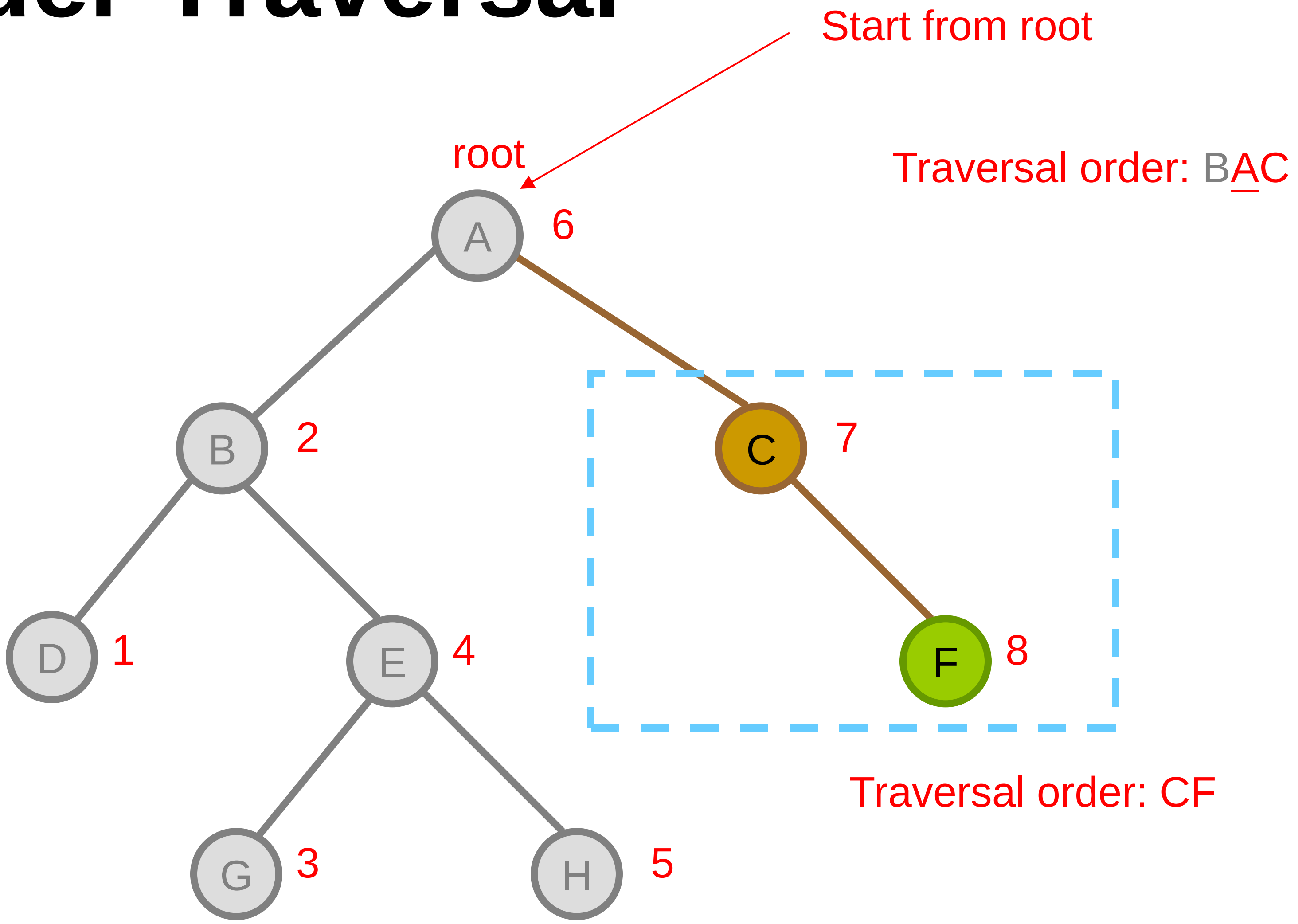
Inorder Traversal



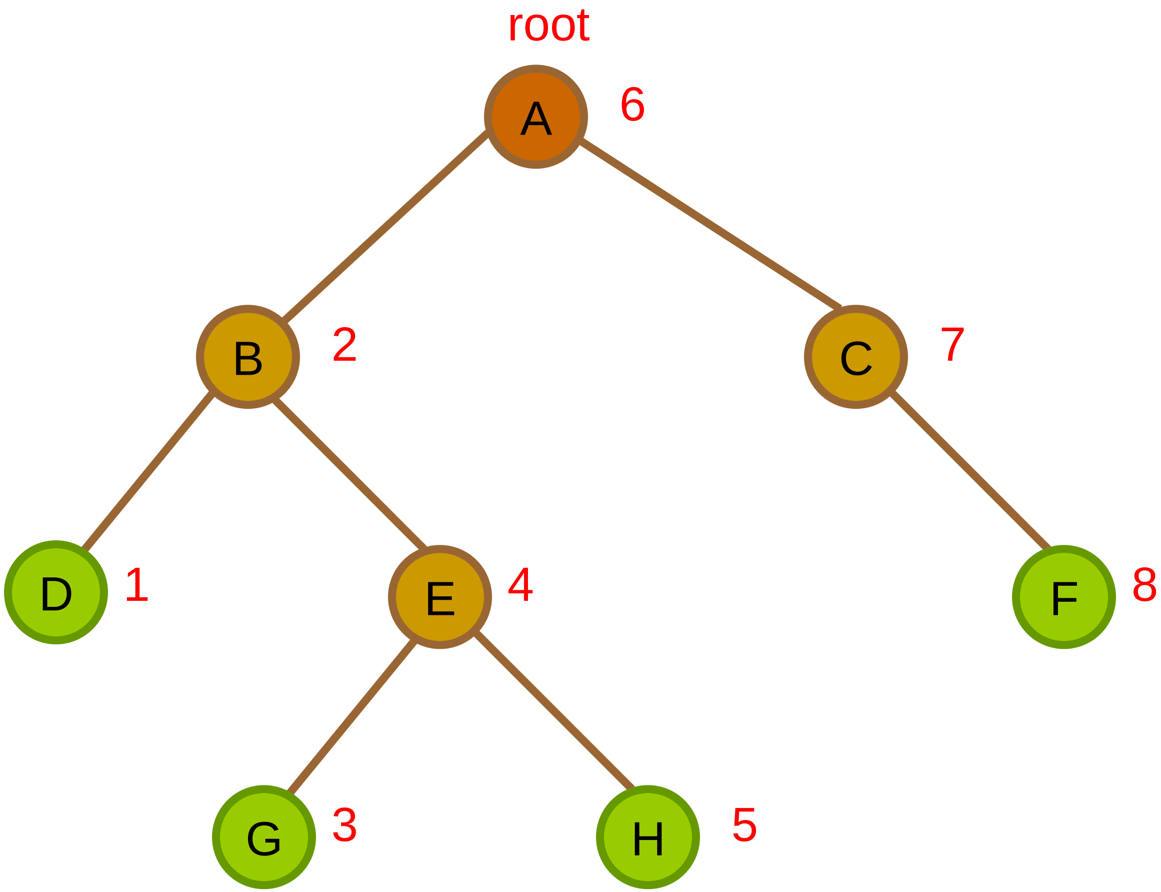
Inorder Traversal



Inorder Traversal



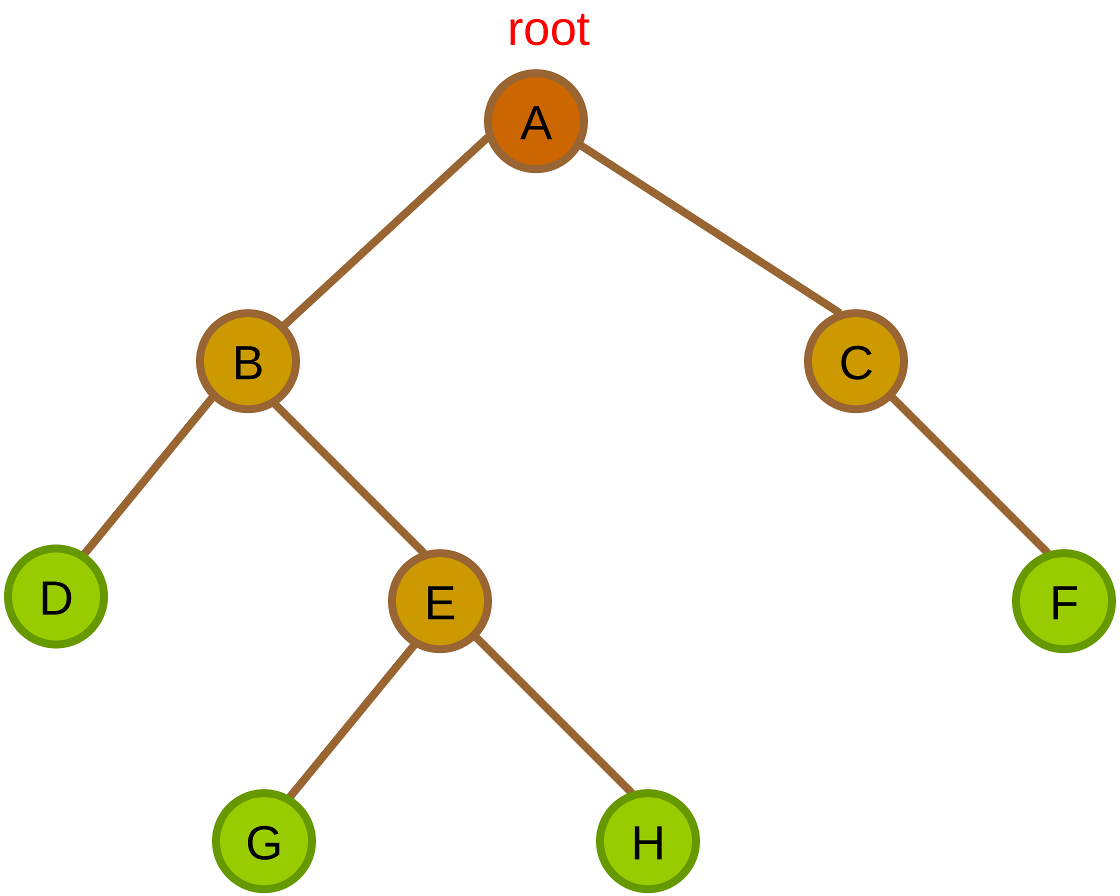
Inorder Traversal



The final sequence: DBGEHACF

Postorder Traversal

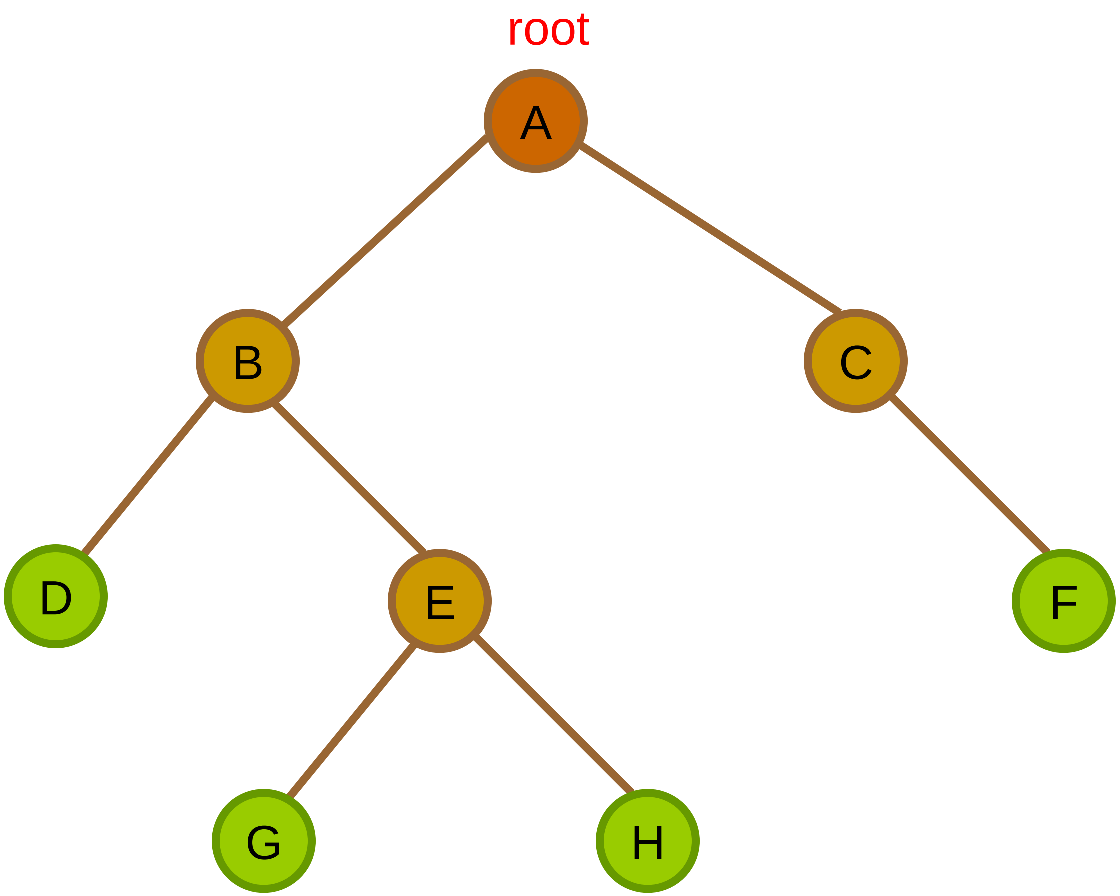
- LRV



The final sequence: ______________________

Preorder Traversal

- VLR



The final sequence: ______________________

Preorder Traversal

```
template<class Type>
void preorder(treeNode<Type> *p)
{
   if (p != NULL)
   {
      cout << p->info << " ";         //visit the node
      preorder(p->left);              // visit left subtree
      preorder(p->right);             // visit right subtree
   }
}
```

Go to right subtree (i.e. p->right) by recursion

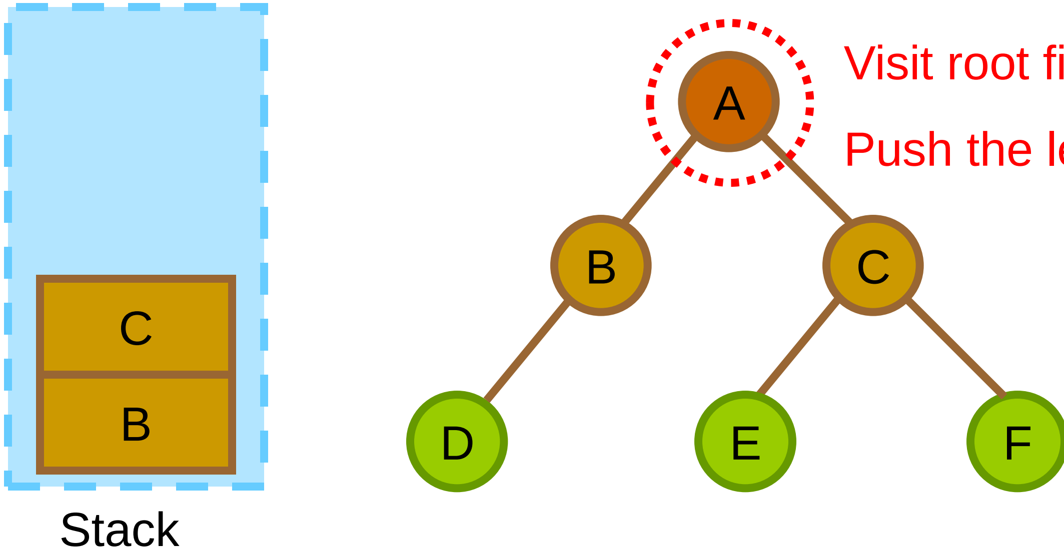
Inorder & Postorder Traversal

```
template<class Type>
void inorder(treeNode<Type> *p) {
   if (p != NULL) {
      inorder(p->left);
      cout << p->info << " ";        //visit the node
      inorder(p->right);
   }
}

template<class Type>
void postorder(treeNode<Type> *p) {



}
```

Non-Recursive Tree Operations

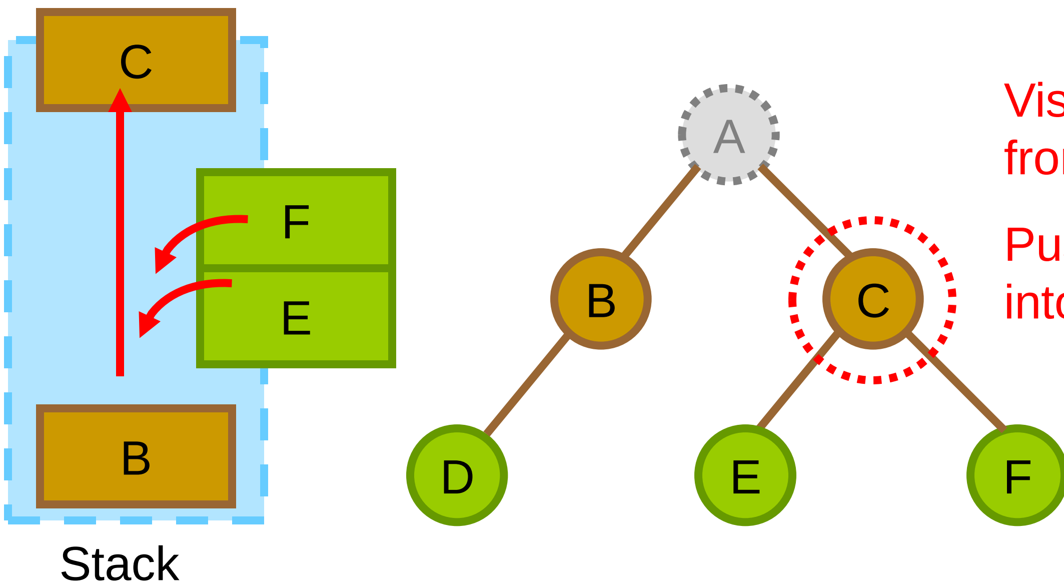
- Recursion algorithm intrinsically uses the internal Call Stack to buffer tree nodes for further processing
- We may also explicitly use stack and queue for this purpose and implement tree operations with iterative approach
- Use stack for depth first traversal / search
  - Inorder / preorder / postorder traversal are depth first traversal
  - Traversals are go along the left subtree or right subtree until meeting the leaf nodes
- Use queue for breadth first traversal / search
  - Breadth first traversal is along the levels
- Now rewrite the function
  - Count number of nodes (i.e. the size of tree)
  - How to do it in a non-recursive way?

Count Nodes (Iterative Approach)



Visit root first, count + 1

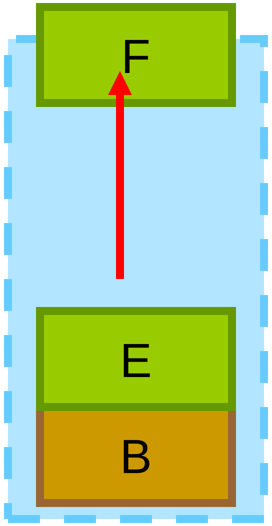
Push the left child and right child into stack!



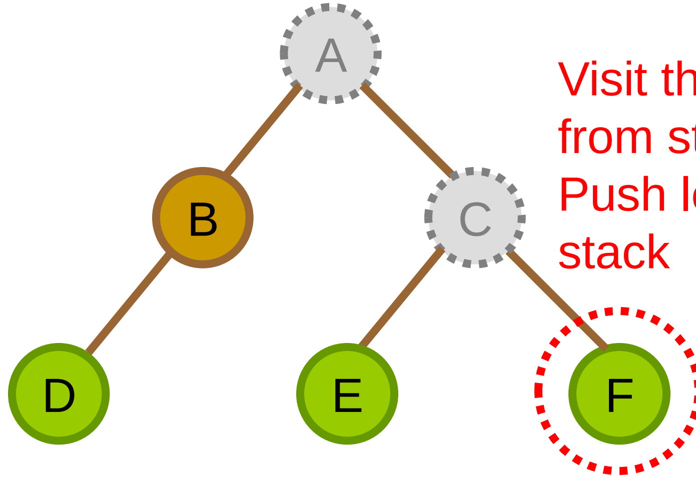
Visit the next node (the top element from stack), count + 1

Push left child and right child of C into stack

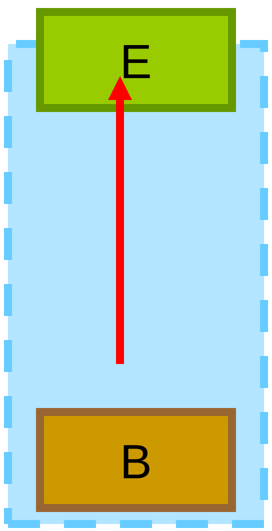
Count Nodes (Iterative Approach)



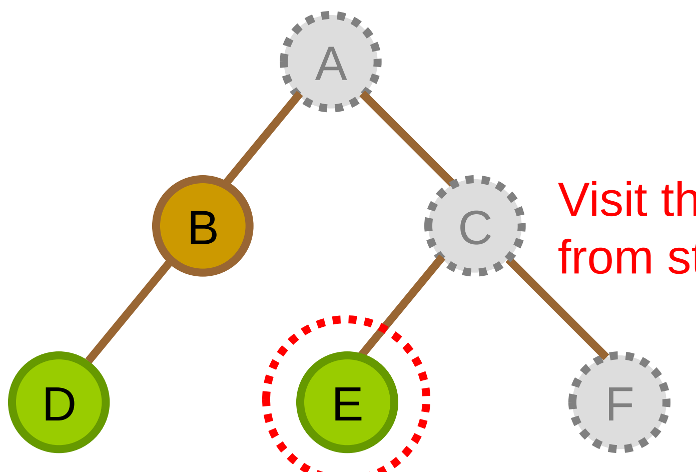
Stack



Visit the next node (the top element from stack), count + 1
Push left child and right child of F into stack

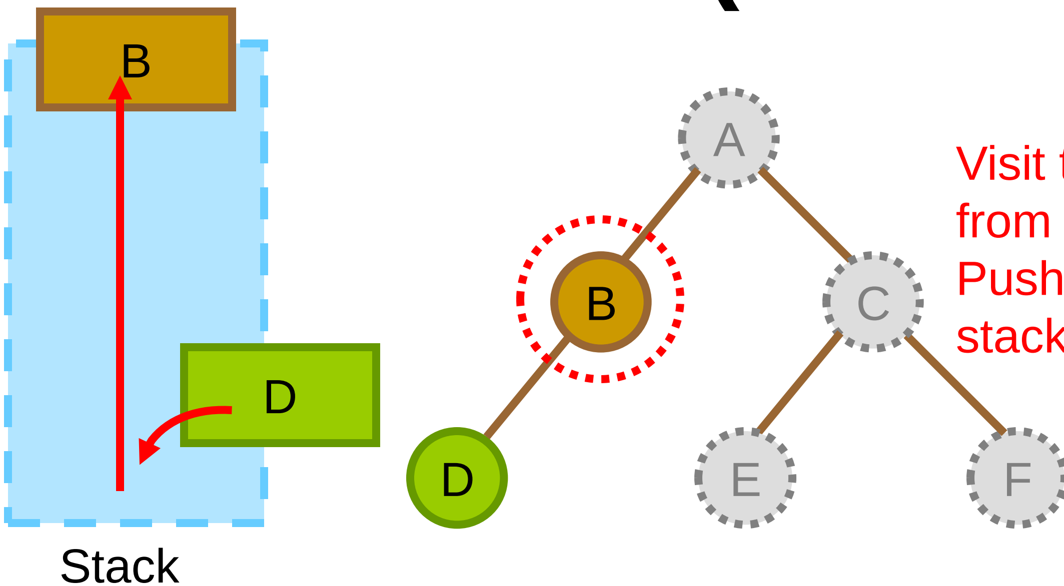


Stack

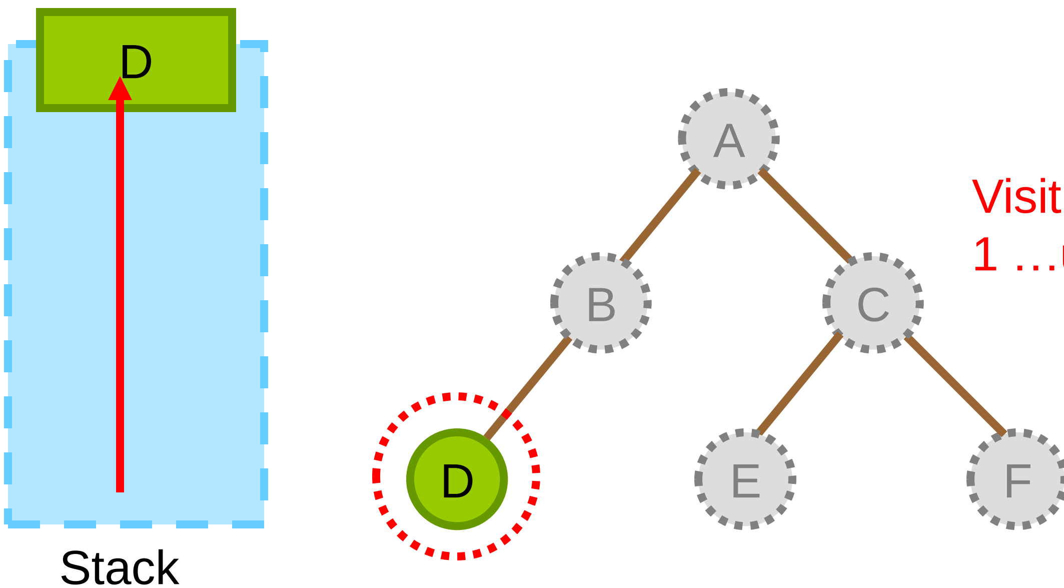


Visit the next node (the top element from stack), count + 1

Count Nodes (Iterative Approach)



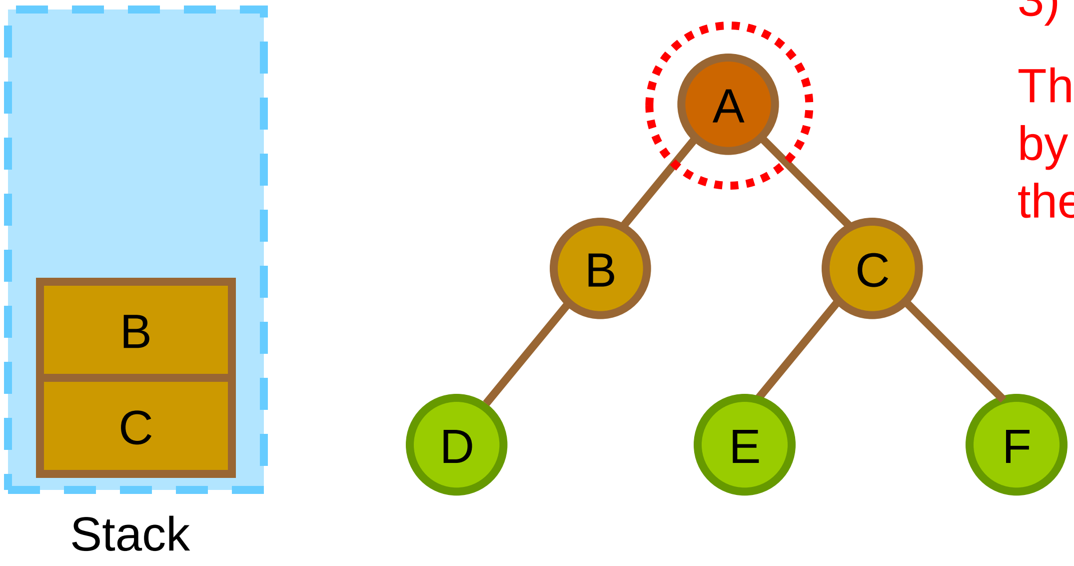
Visit the next node (the top element from stack), count + 1
Push left child and right child of B into stack



Visit the next node, count + 1 …until the stack is empty

Non-Recursive Tree Traversal

- Preorder traversal using stack



1) Visit root first

2) Push right child

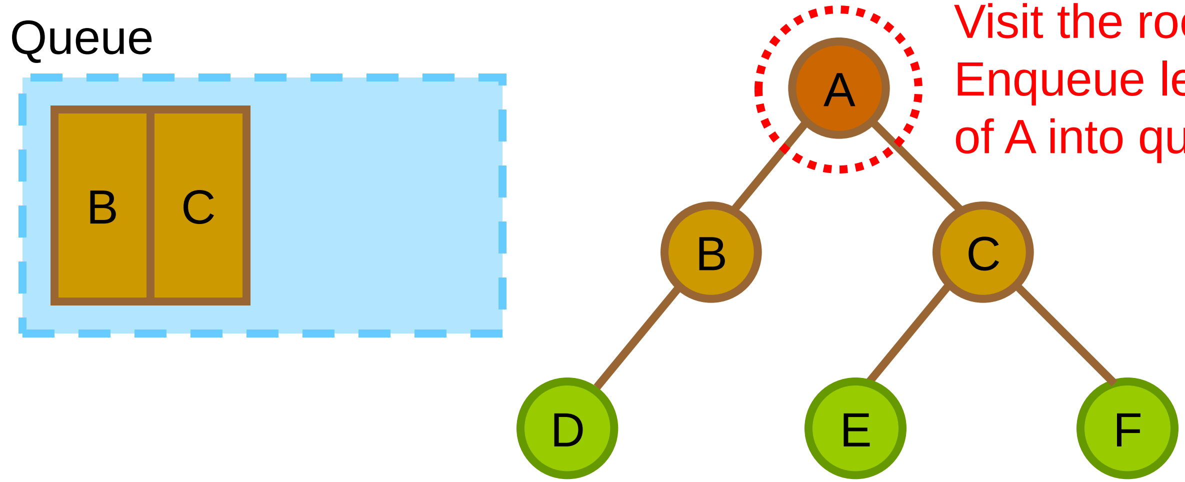
3) Push left child

Then iterate the next node by popping from stack until the stack is empty

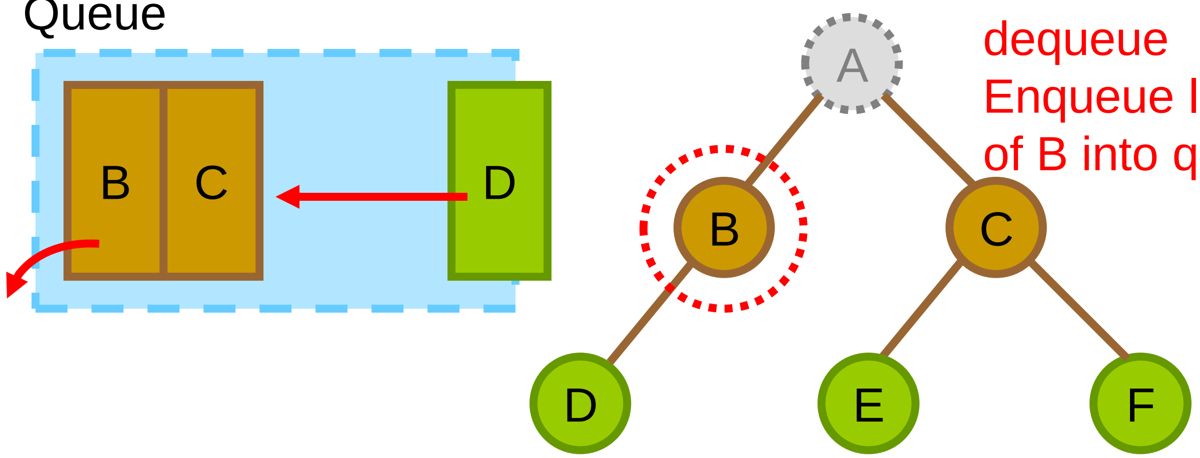
Non-Recursive Inorder Traversal Using Stack

```
#include <stack>

template<class Type>
void traverseLeft(treeNode<Type> *p, stack<treeNode<Type>*>& S) {
   while (p != NULL) {
      S.push(p);
      p = p->left;
   }
}

template<class Type>
void inorder_2(treeNode<Type> *tree) {
   Stack<treeNode<Type>*> S;         // store pointer only
   traverseLeft(tree, S);            // reach leftmost node

   while (!S.empty()) {              //there are nodes not yet visited
      treeNode<Type>* p = S.top();
      S.pop();
      cout << p->info << " ";
      traverseLeft(p->right, S);
   }
}
```

Breadth First Traversal



Visit the root node
Enqueue left child and right child of A into queue



Visit the next node by calling dequeue
Enqueue left child and right child of B into queue

Level Order Traversal Using Queue

```
#include <queue>

template<class Type>
void levelTrav(treeNode<Type> *tree) {
   queue<treeNode<Type>*> Q;         // store pointer only

   if (tree != NULL)
      Q.push(tree);

   while (!Q.empty()) {              //there are nodes not yet visited
      treeNode<Type>* p = Q.front();
      Q.pop();

      cout << p->info << " ";

      if (p->left != NULL)
         Q.push(p->left);

      if (p->right != NULL)
         Q.push(p->right);
   }
}
```

Reconstruction of Binary Tree

Class Exercise

- Can you draw the binary tree if the **postorder** and **inorder** traversal of the tree are HJBFGDECA and HBJAFDGCE respectively?

Reconstruction of Binary Tree

- The structure of a binary tree can be obtained if either preorder or postorder plus inorder traversal sequences are given
- Preorder + postorder
  - Fail to reconstruct the binary tree
- Only **inorder + preorder**, or **inorder + postorder** can provide sufficient information to reconstruct a binary tree

The Reconstruction Algorithm

- Step 1) Determine the root node, left and right subtrees
  - From **postorder**, the last node is the root
    - e.g. node A
  - Then from **inorder**, the nodes on the left hand side of node A belongs to the left subtree of node A, nodes on the right hand side belongs to its right subtree
- Step 2) Consider the traversal sequence of the subtrees, and determine its root, left and right subtrees recursively

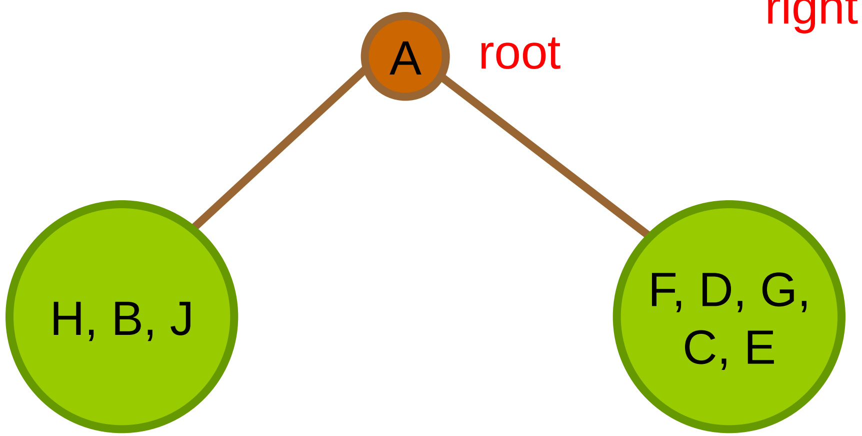
First Determine the Root

root node

- Postorder: H J B F G D E C A
- Inorder: <u>H B J</u> A <u>F D G C E</u>

left subtree

right subtree



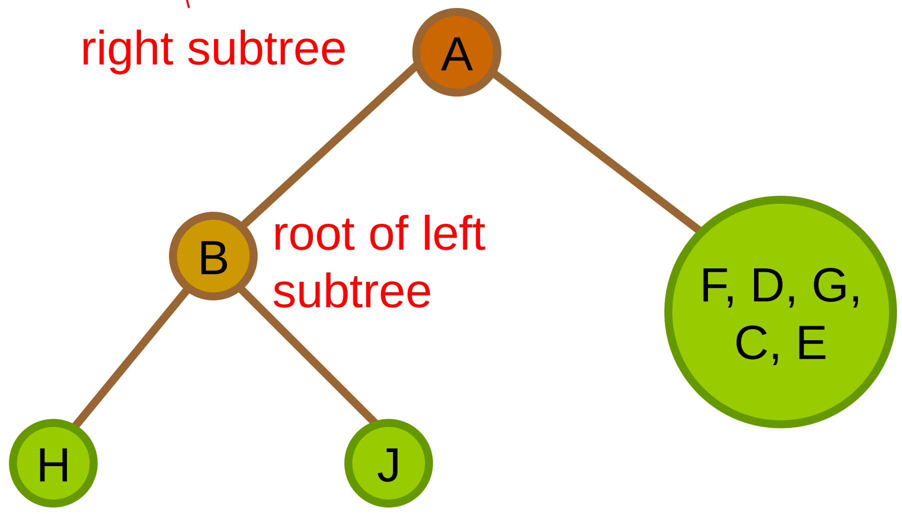
Consider Left Subtree of Root

root node of the subtree

- Postorder: H J B F G D E C ~~A~~
- Inorder: H B J ~~A~~ F D G C E

left subtree

right subtree



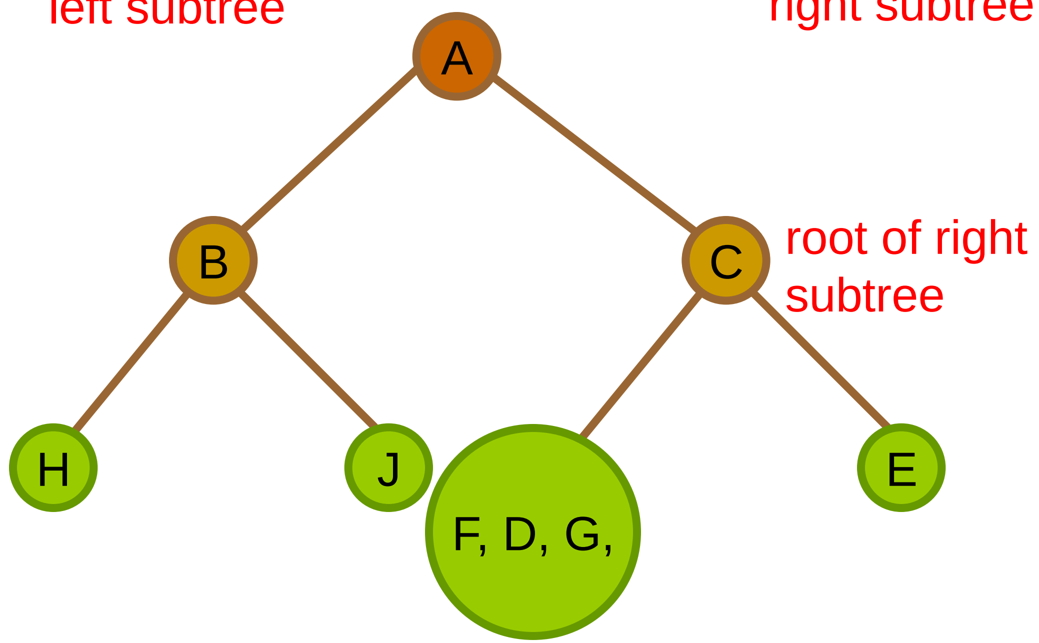
Consider Right Subtree of Root

root node of the subtree

- Postorder: ~~H~~ ~~J~~ ~~B~~ F G D E C ~~A~~
- Inorder: ~~H~~ ~~B~~ ~~J~~ ~~A~~ F D G C E

left subtree

right subtree



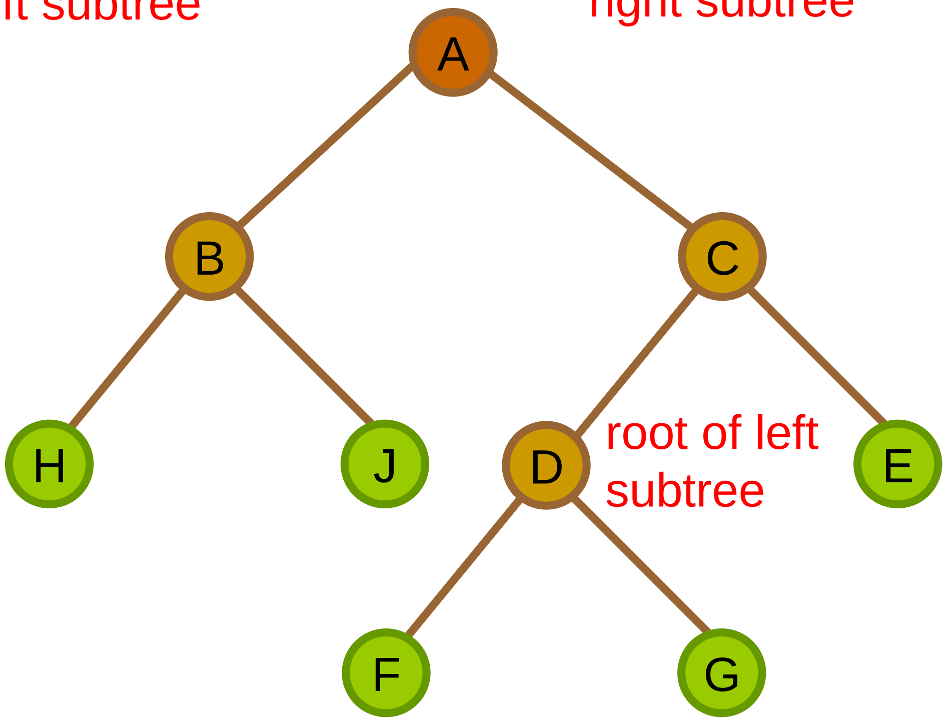
Consider Left Subtree of C

- Postorder: ~~H~~ ~~J~~ ~~B~~ F G D ~~E~~ ~~C~~ ~~A~~
- Inorder: ~~H~~ ~~B~~ ~~J~~ ~~A~~ F D G ~~C~~ ~~E~~

root node of the subtree

left subtree

right subtree



Summary

- For postorder, the **last node** is the root
- For preorder, the **first node** is the root
- For inorder, the nodes on the **left hand side** of last node of postorder (or first node of preorder) belongs to the left subtree, nodes on the **right hand side** belongs to its right subtree
- Apply this principle recursively in left/right subtrees

Binary Tree Implementation

Ex7_1

- **To avoid the tedious details, only the implementation of <u>some selected member functions</u> will be given.**

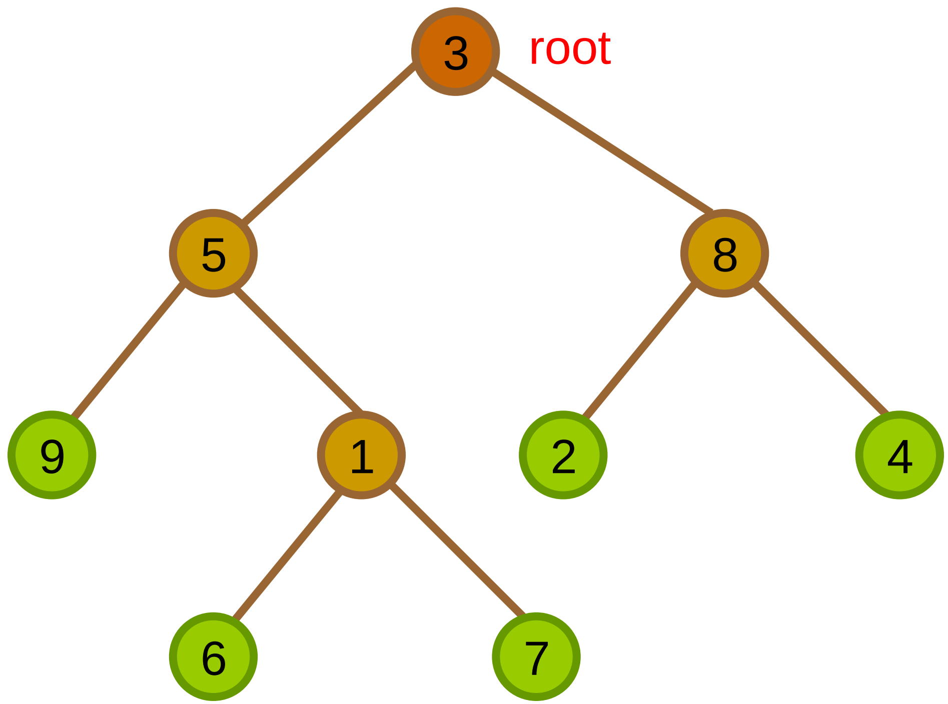
- A binary tree is a container (i.e. it is used to hold a collection of items).
- We need to provide one or more types of iterator such that the external user can use it to traverse the elements in the tree one at a time.
- The implementation of the iterator class given below only serves to illustrate the conceptual idea.
- Different implementation methods are used in the C++ STL.

Binary Search Tree (BST)

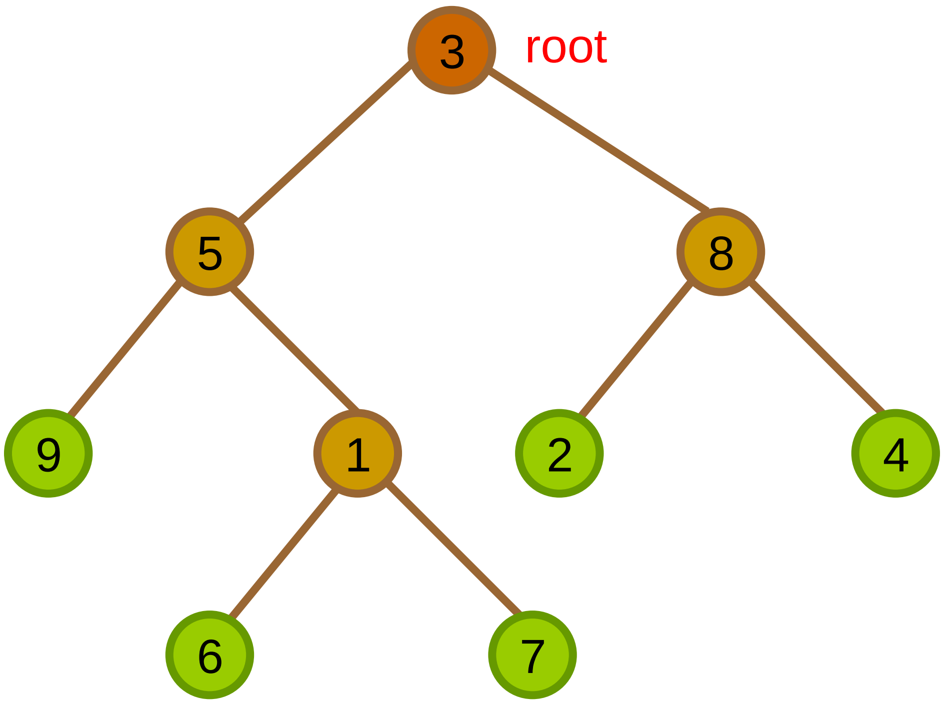
and its operations

How to Search a Tree?

- Suppose we have a binary tree like this
- Each node contains an integer data
- How do you find the node that contain value = *k*?
- Can you determine the max./min. node value?



How to Search a Tree?



array:

3	5	8	9	1	2	4	-	-	6	7

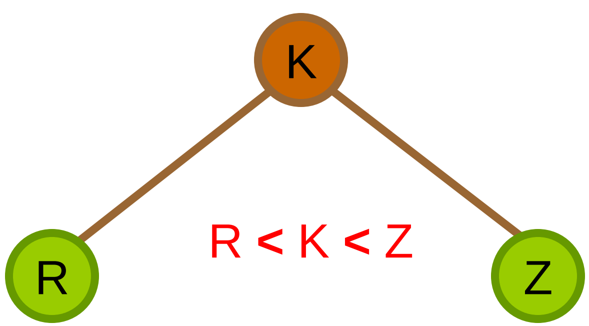
for-loop

In order to find the max. node, min. node or a node equal to particular value, you have to visit the entire tree once

How about linked list implemented tree?

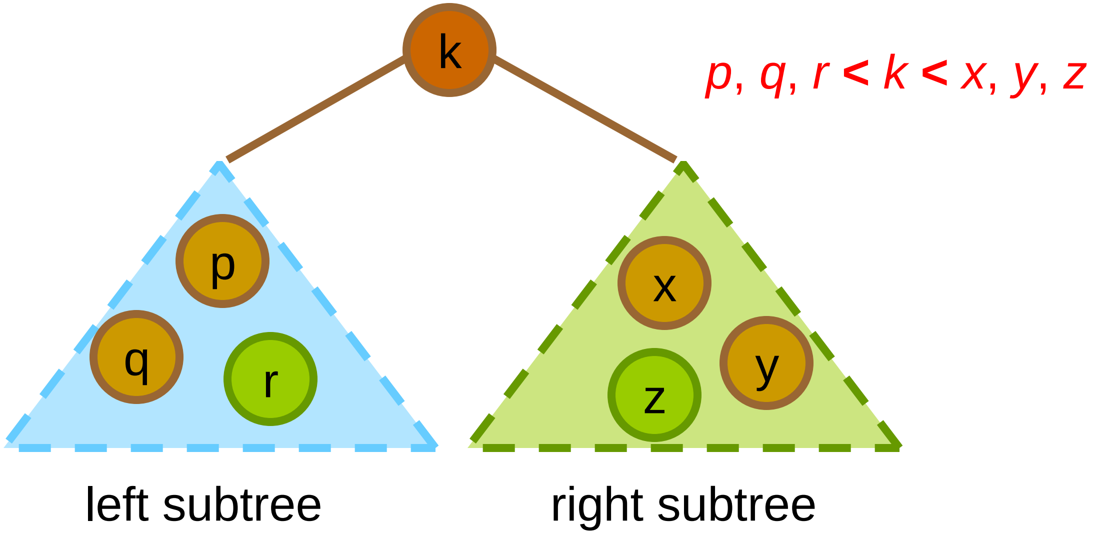
Pre-sort Tree

- How about if the tree is pre-sorted in some sense
- The value stored at a node is greater than the value stored at its left child, but less than the value stored at its right child
- This arrangement of nodes allow us to make decision of a searching going along its left or right path.

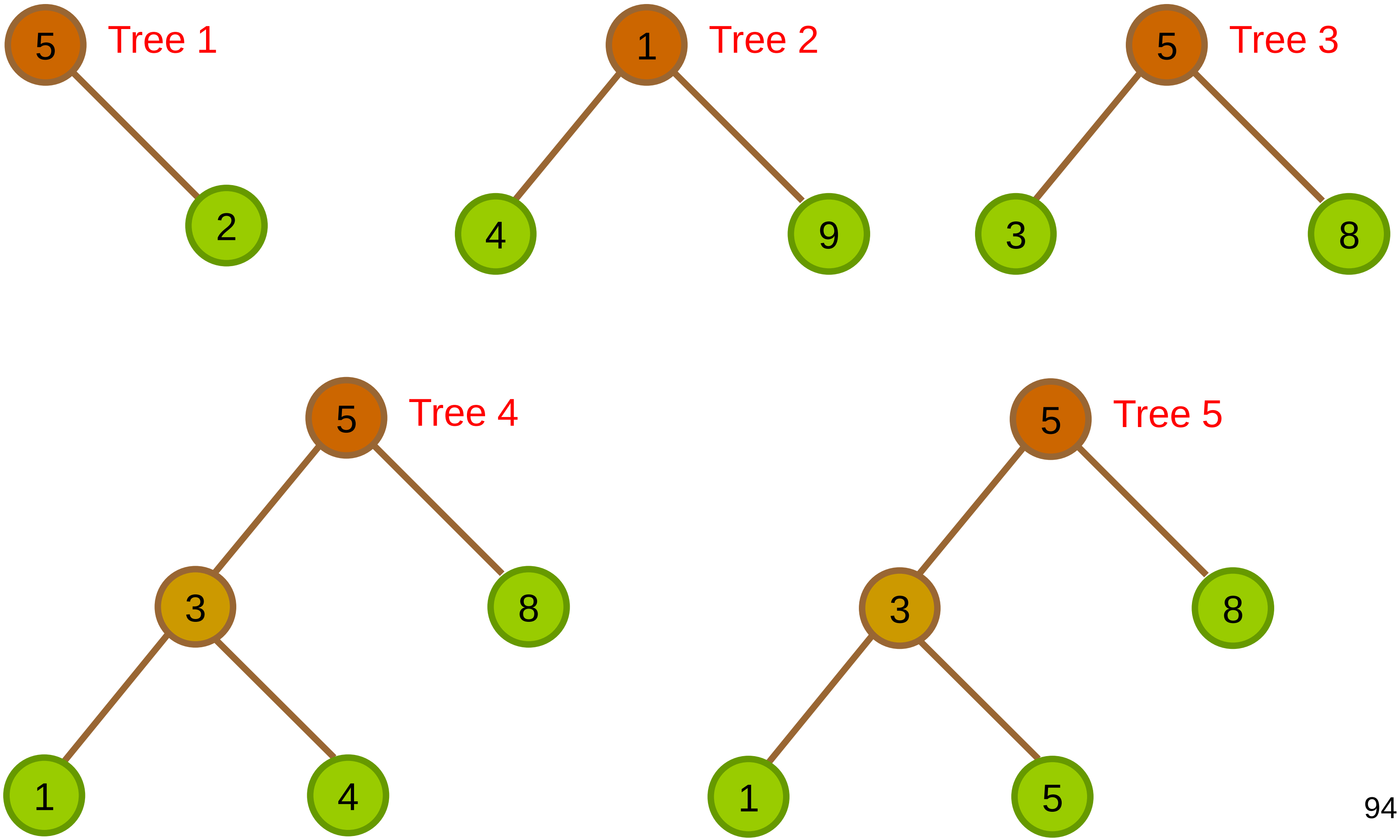


Binary Search Tree (BST)

- A binary search tree is a binary tree. It may be empty. If it is not empty, then it satisfies the following properties:
- Every element has a key field and no two elements in the BST have the same key, i.e. all keys are distinct. (Example, student ID is a key field in the student record.)
- The keys (if any) in the **left subtree** are smaller than the key in the root.
- The keys (if any) in the **right subtree** are larger than the key in the root.
- The left and right subtrees are also BST (recursively applied).

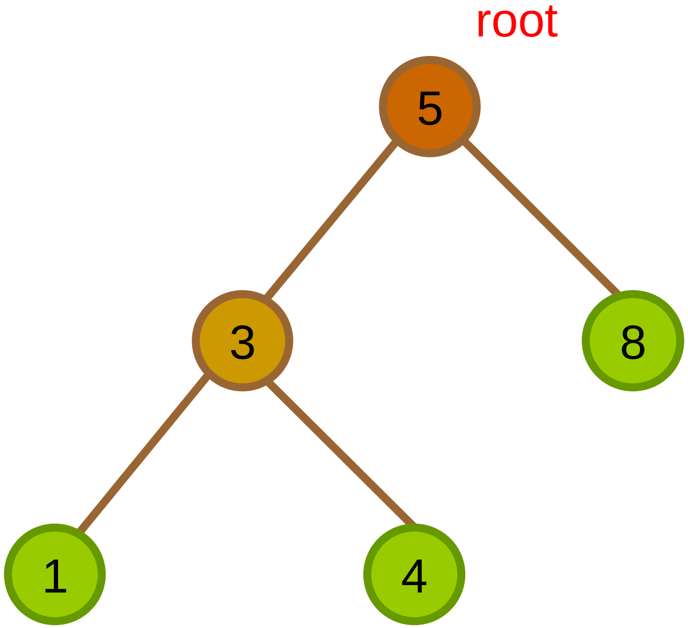


Exercise: Are They BST?



Find a Node in BST

- How to find a node with value = *k*?



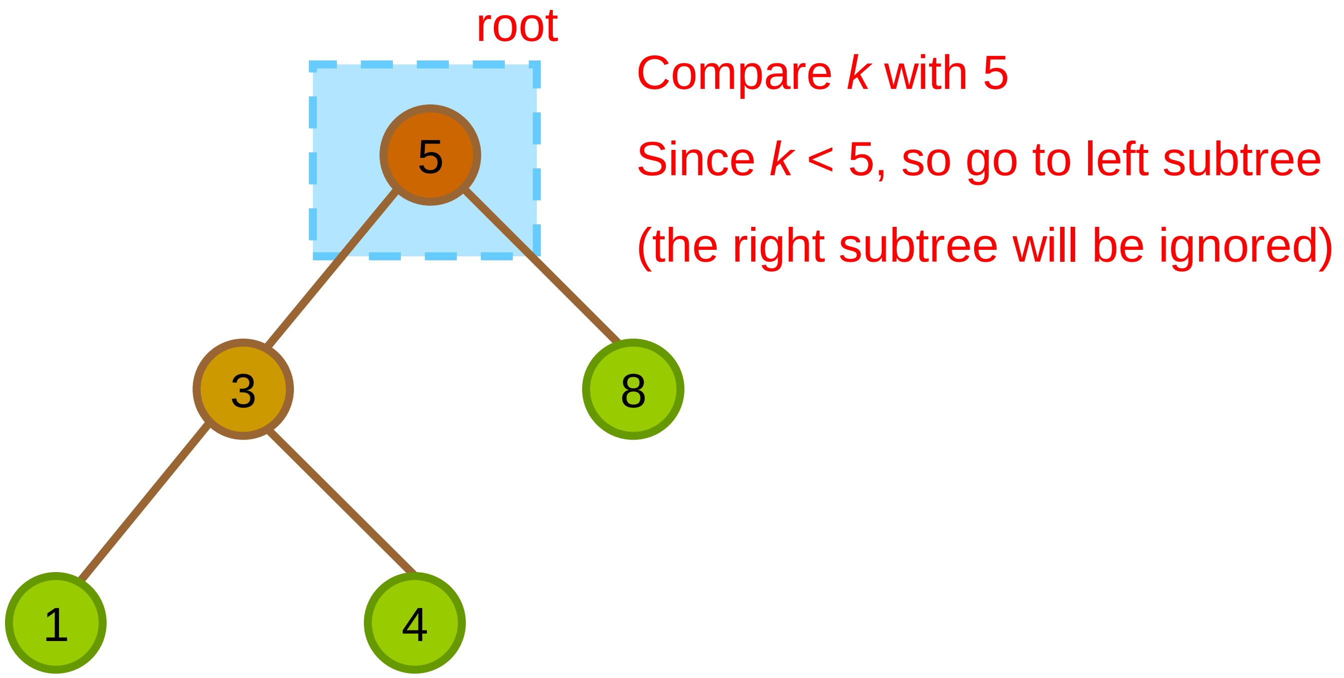
Find a node in BST

- Compare *k* with the value of root
- If value of root == *k*, the answer is root!
- If value of root > *k*, go to the left subtree
- If value of root < *k*, go to the right subtree

- Continue to compare recursively until it meets a leaf node

Find a Node in BST

- e.g. $k = 1$



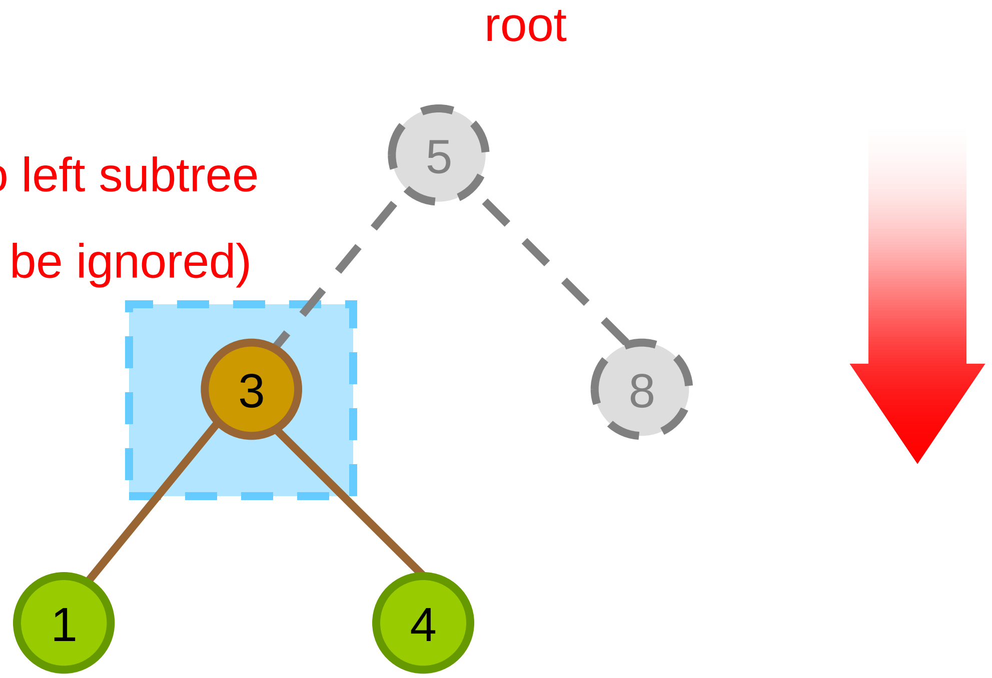
Find a node in BST

- e.g. k = 1

Compare *k* with 3

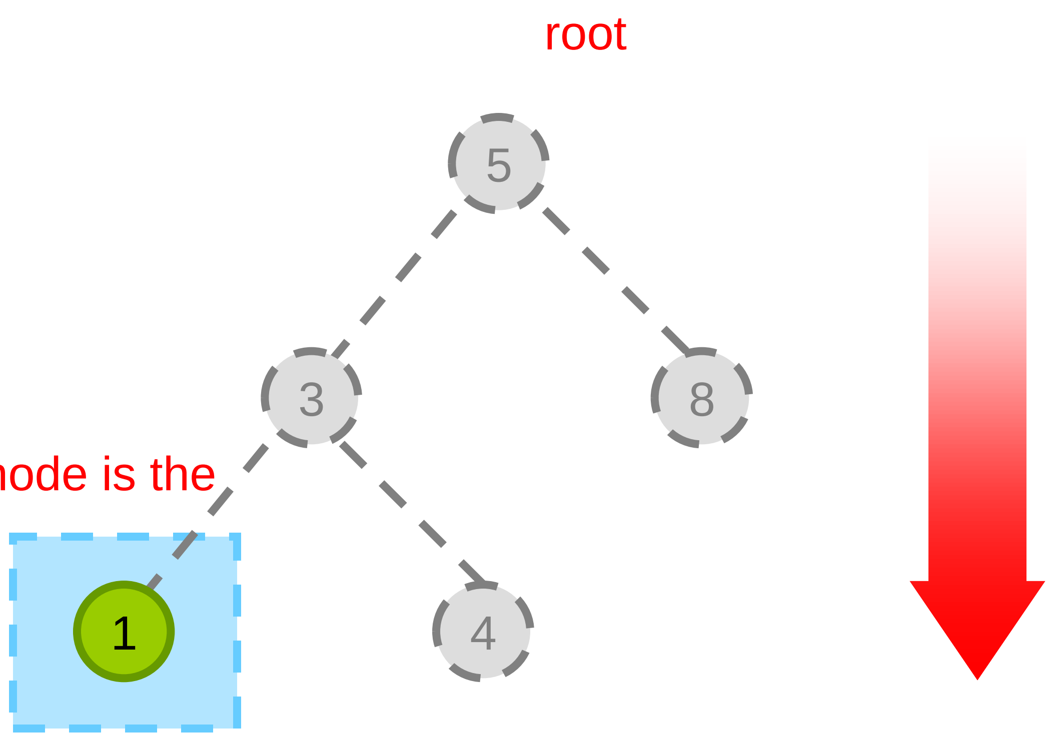
Since *k* < 3, so go to left subtree

(its right subtree will be ignored)



Find a node in BST

- e.g. $k = 1$



Compare k with 1

Since k == 1, this node is the solution!

How about if we want to find a node with value equal to 2?

Time Complexity

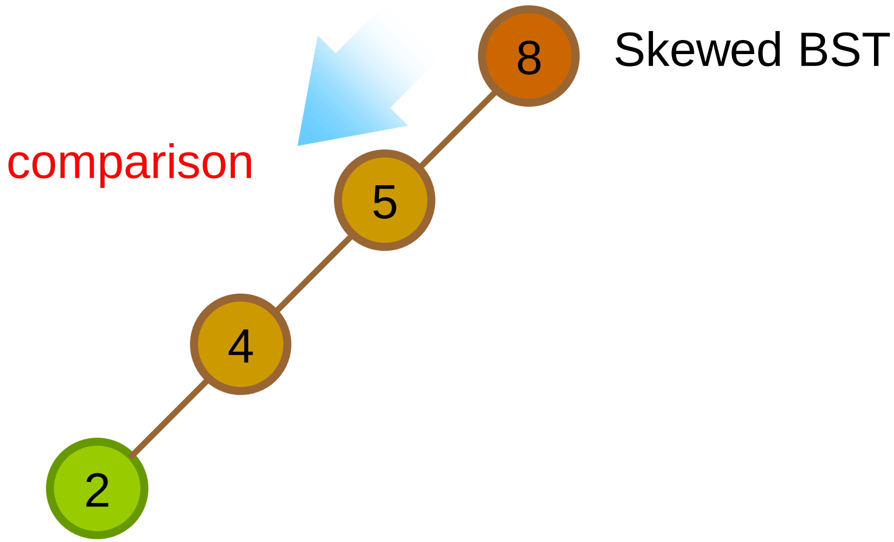
- What’s the time complexity of the **find** function?
  - Time complexity is proportional to the no. of comparison
  - The max. no. of comparison = no. of levels of the tree

Complete BST

- If it is a complete BST
  - After each comparison, either left subtree or right subtree will be **ignored**
  - About half nodes do not require to consider after <u>each comparison</u>
  - The depth of the tree is *floor($\log_2 n$)*
- **Average case**: O($\log_2 n$), where n is the total no. of nodes

Skewed BST

- If it is a skewed BST
  - The depth of the tree is *n-1*
- **Worst case**: O(*n*)



Conclusion: it is very important to maintain a complete BST

Non-Recursive Search BST

- If search() is a public member function of class BST, it should return an iterator that refers to the node containing x instead of a node pointer to x so as to prevent exposing internal structure.

```
// implemented as a member function of BST class
template<class Type>
treeNode<Type>* search(const Type& x) {

  treeNode<Type> *p = root;       // point to root node of the tree
  while (p != NULL && x != p->info) {    // compare key field
     if (x < p->info)
       p = p->left;
     else
       p = p->right;
  }
  return p;
}
```

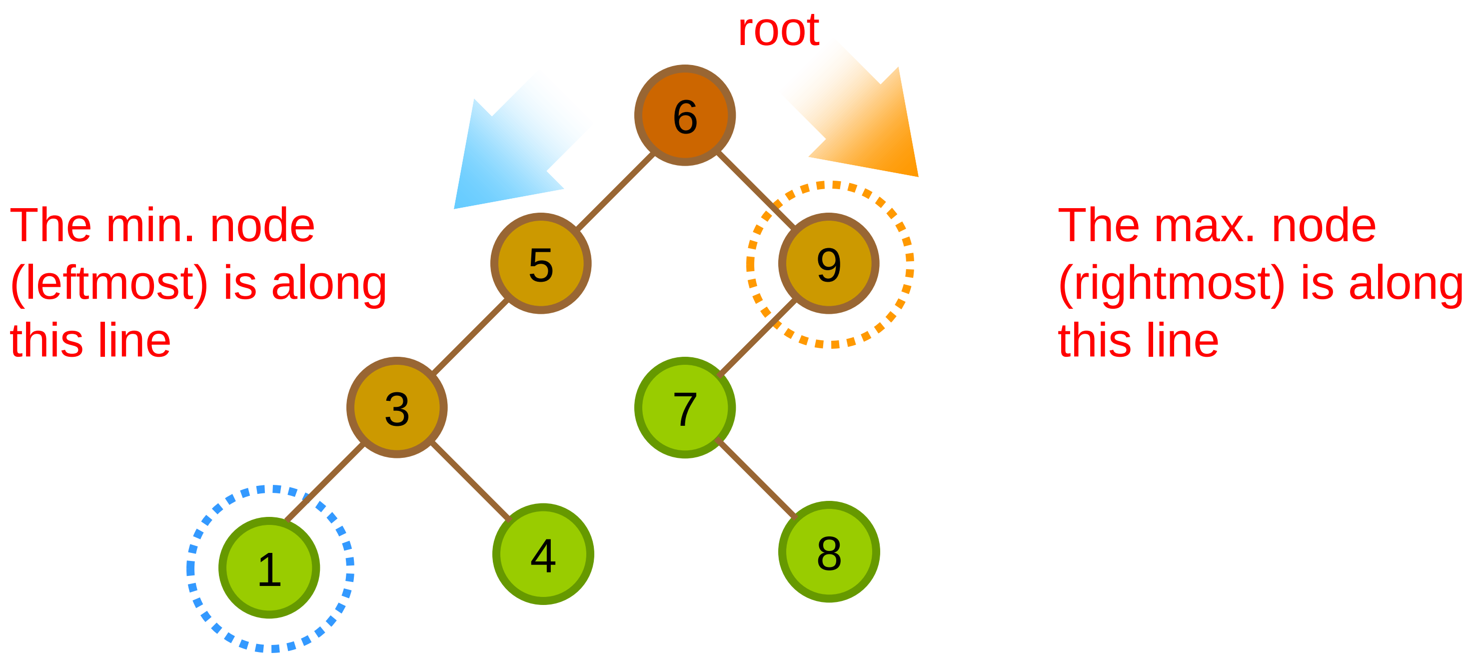
Recursive Search BST

- If this is implemented as a member function, it should be defined as a private function. Note that public member functions should not require any private member variables as input parameters.

```
template<class Type>
treeNode<Type>* search(treeNode<Type> *p, const Type& x) {
  if (p == NULL)
    return NULL;

  if (x == p->info)
    return p;

  if (x < p->info)
    return search(p->left, x);
  else
    return search(p->right, x);
}
```

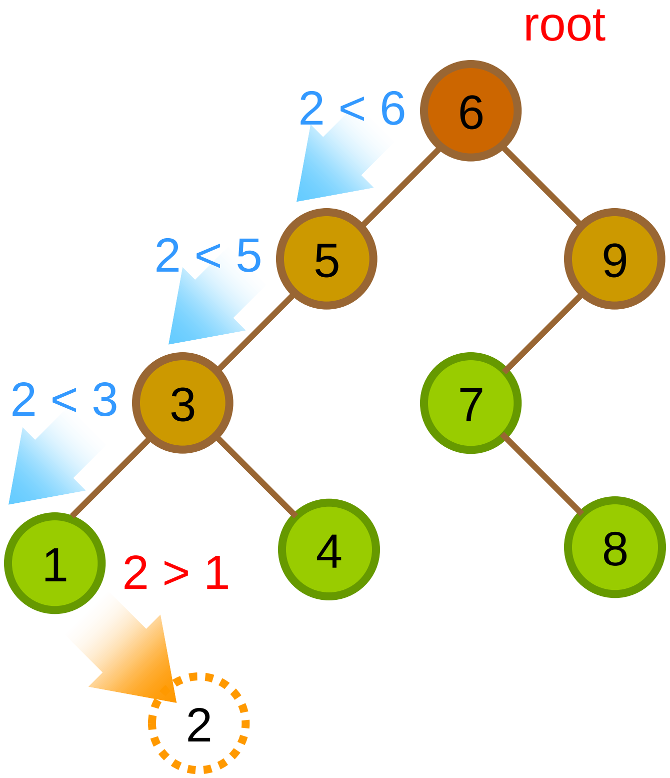
Class Exercise:
Min & Max Node of BST

- Exercise: write the code to find the min and max node (using recursion/iteration)



Insert a Node In BST

- How to insert a node in BST?
  - e.g. insert(2)
- Two major steps:
  - Verify if the new element is not exist in the BST
  - Determine the point of insertion



Order of Inserting Elements

- Does the order of inserting elements into a BST matter?
  - Yes, certain orders could produce very unbalanced trees
  - e.g. compare the resultant tree if inserting the elements in these order:
    - 1) 5, 3, 8, 1, 4 and
    - 2) 1, 3, 4, 5, 8
- Unbalanced trees are not desirable because search time increases

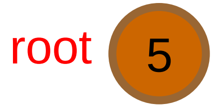
Insert Order: 5, 3, 8, 1, 4

Step 1

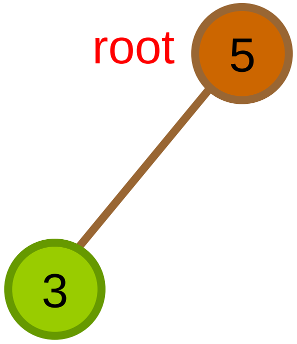
root

(empty tree)

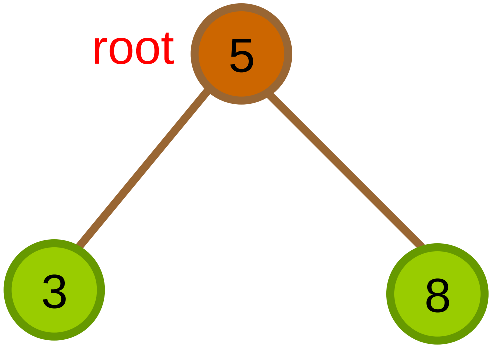
Step 2



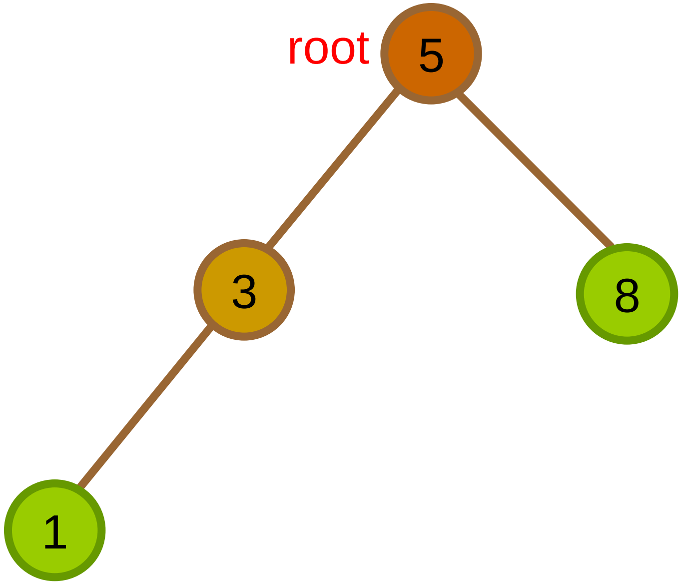
Step 3



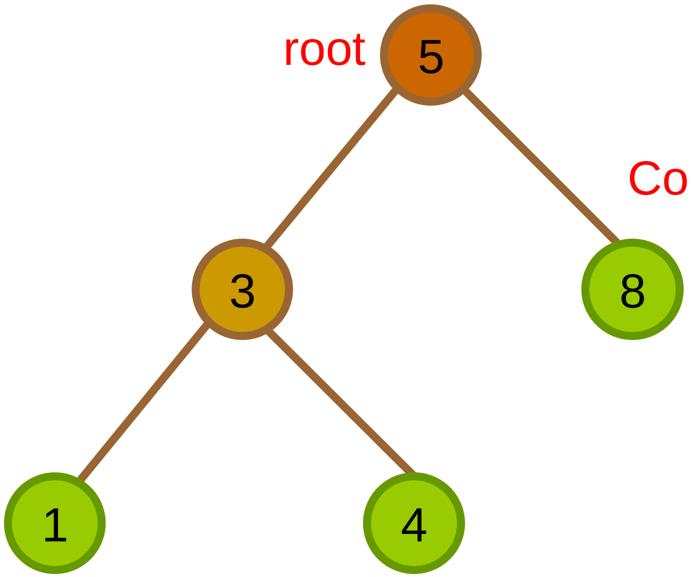
Step 4



Step 4

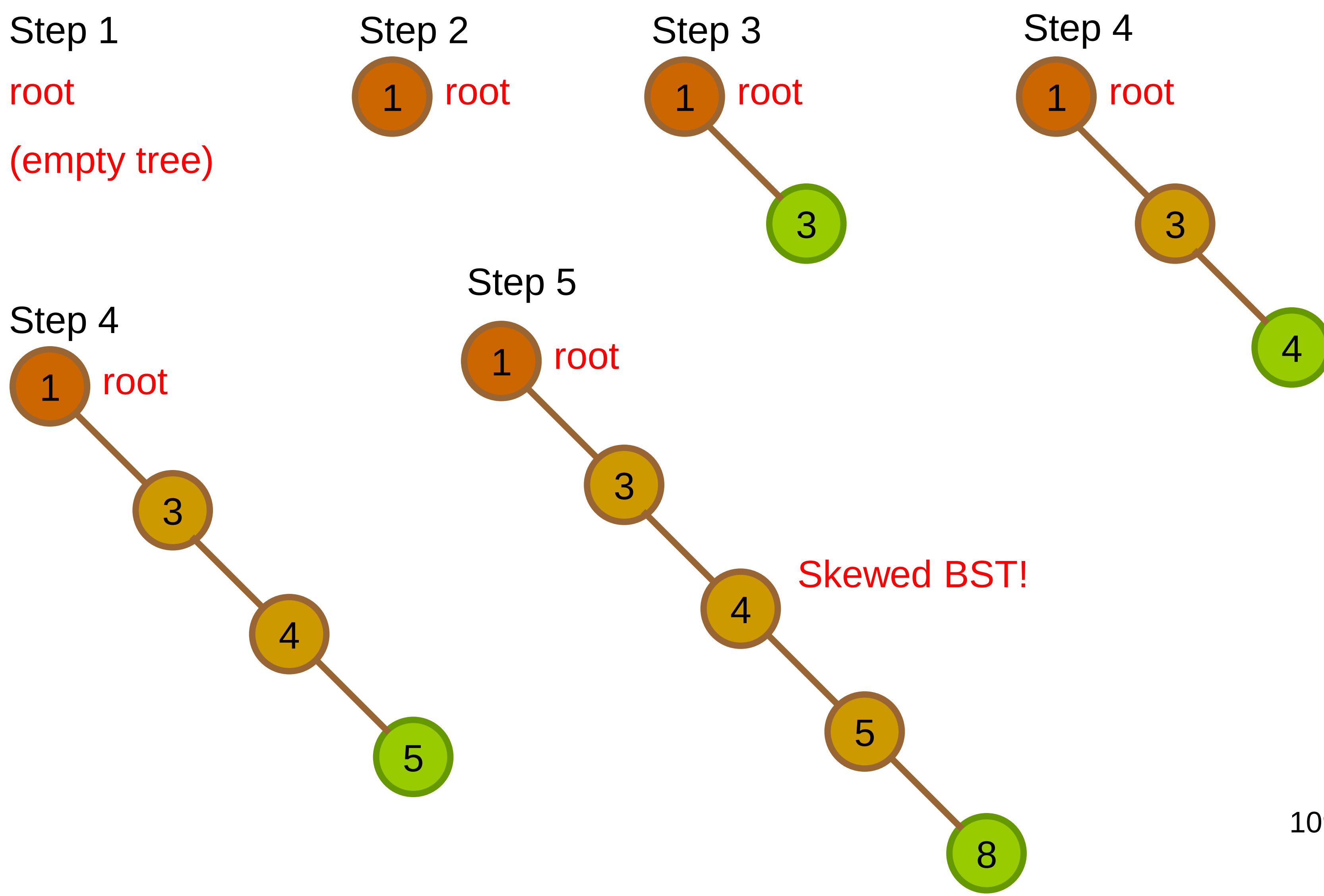


Step 5



Complete BST!

Insert order: 1, 3, 4, 5, 8



Insert Node to BST

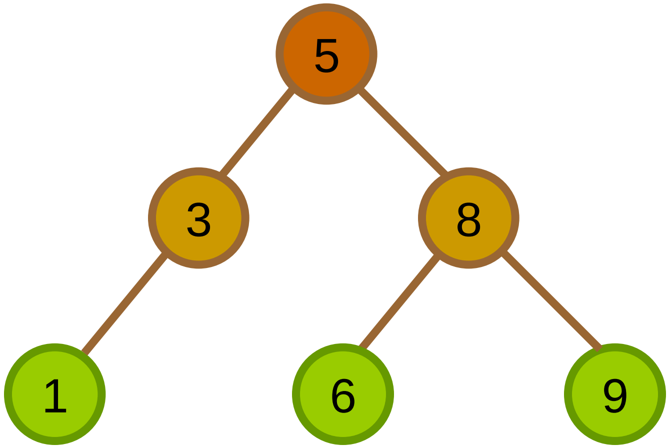
- The insertion function returns the pointer to the newly inserted node or the node with the given key value.

```
template<class Type>
treeNode<Type>* insert(const Type& x) {
  treeNode<Type> *p, *q;

  q = NULL;  // parent of p
  p = root;  // point to root
  while (p != NULL) {
    //element already exists
    if (x == p->info)
          return p;
    q = p;
    if (x < p->info)
          p = p->left;
    else
          p = p->right;
  }
```

```
  treeNode<Type> *v = new treeNode<Type>;
  v->info = x;
  v->left = v->right = NULL;

  if (q == NULL)
     root = v;
  else if (x < q->info)
     q->left = v;
  else
     q->right = v;

  return v;
}
```

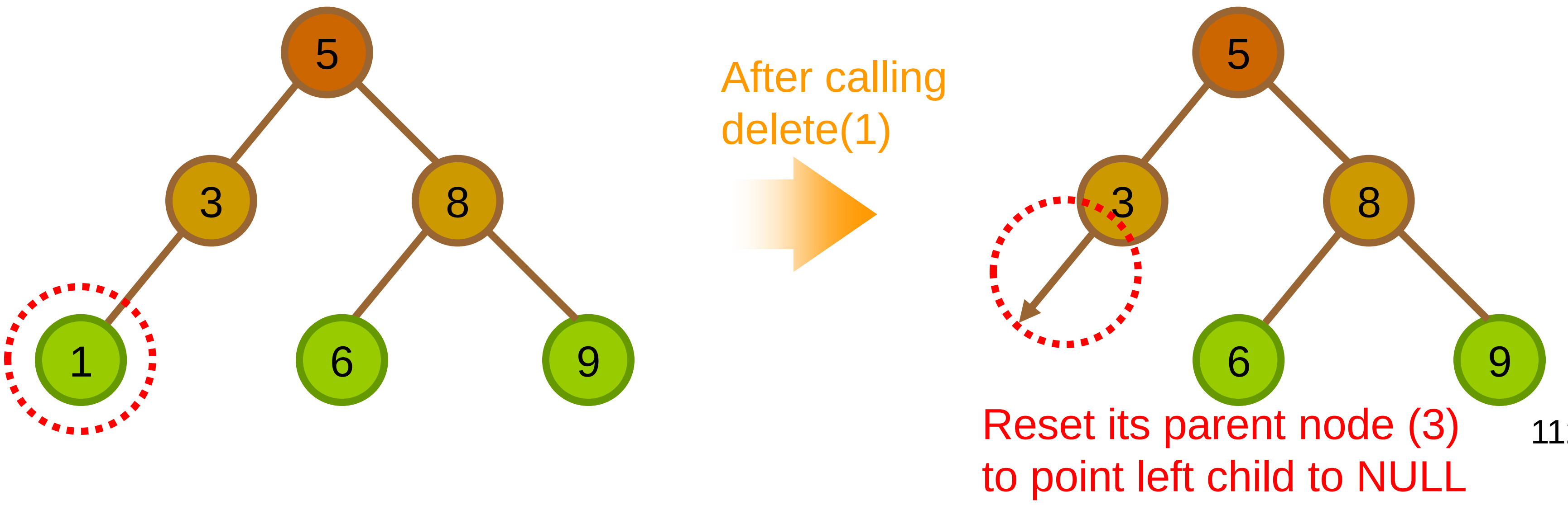
Delete a Node in BST

- The property of BST must be **preserved** after deletion
- We have to consider 3 different cases
  - The node to be deleted is:
  - 1) A leaf node (e.g. node 1)
  - 2) A node has only one child (e.g. node 3)
  - 3) A node has two children (e.g. node 5)



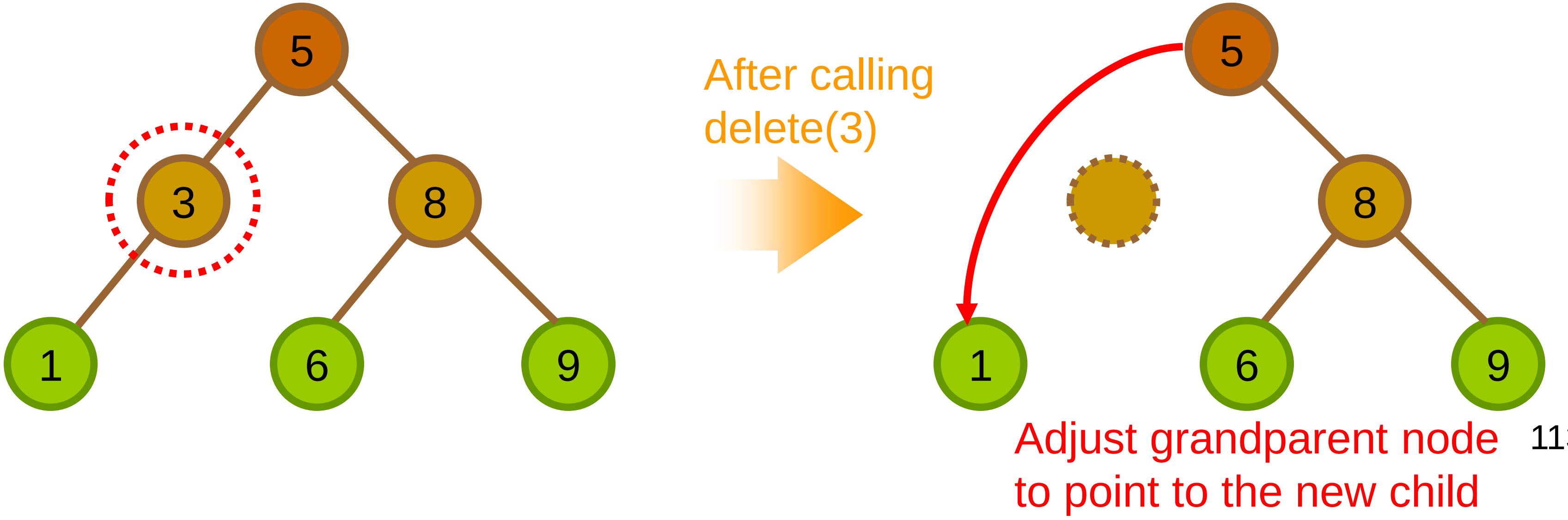
Delete Node: Case 1

- **Degree 0 Node (leaf node)**
  - Just delete it
  - Then reset the reference of its parent node



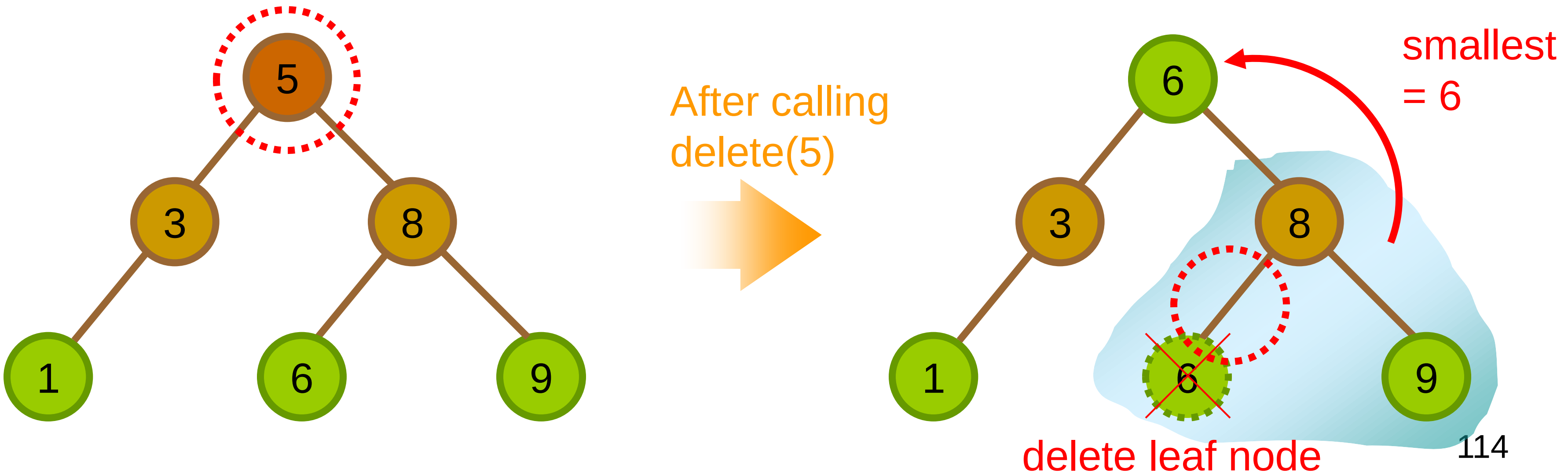
Delete Node: Case 2

- **Degree 1 Node (with 1 child)**
  - Before deletion, adjust the pointer of parent to point to the grandson
  - Then simply delete it



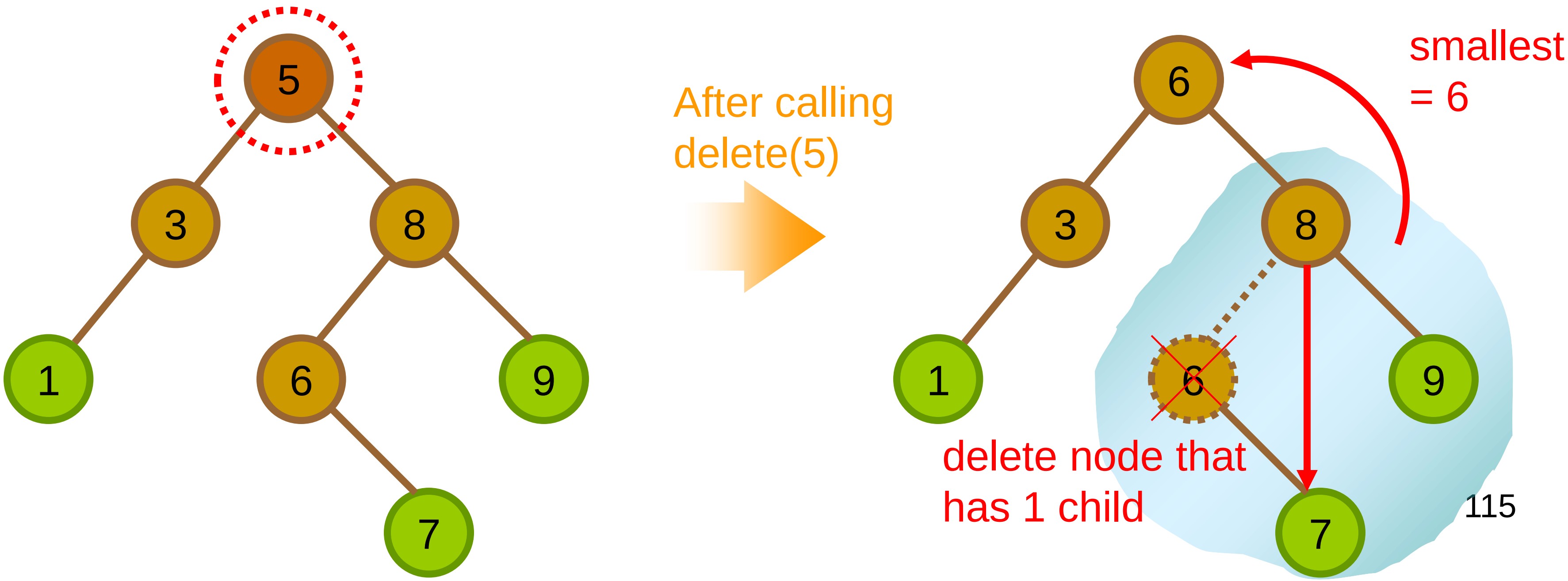
Delete Node: Case 3

- **Degree 2 Node (with 2 children)**
  - Replace the deleted node with its inorder successor (biggest node in left subtree) or inorder predecessor (smallest node in right subtree)



Delete Node: Case 3

- If the inorder successor or predecessor has a child, delete it in turn with the same steps in case 2.



Heap

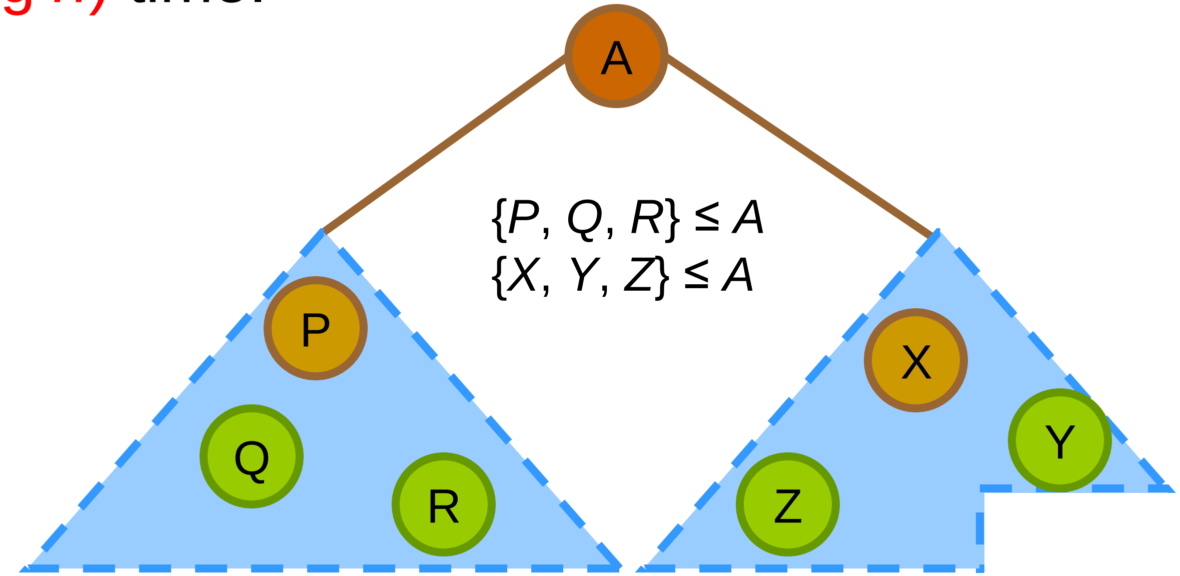
What is a Heap?

Ex7_2

- A max tree is a tree in which the key value in each node is no smaller than the key values in its children (if any).
- Similarly, a min tree is a tree in which the key value in each node is no bigger than the key values in its children (if any).
- A max heap (descending heap) is a complete binary tree that is also a max tree.
- A min heap (ascending heap) is a complete binary tree that is also a min tree.

Using Heap as Priority Queue

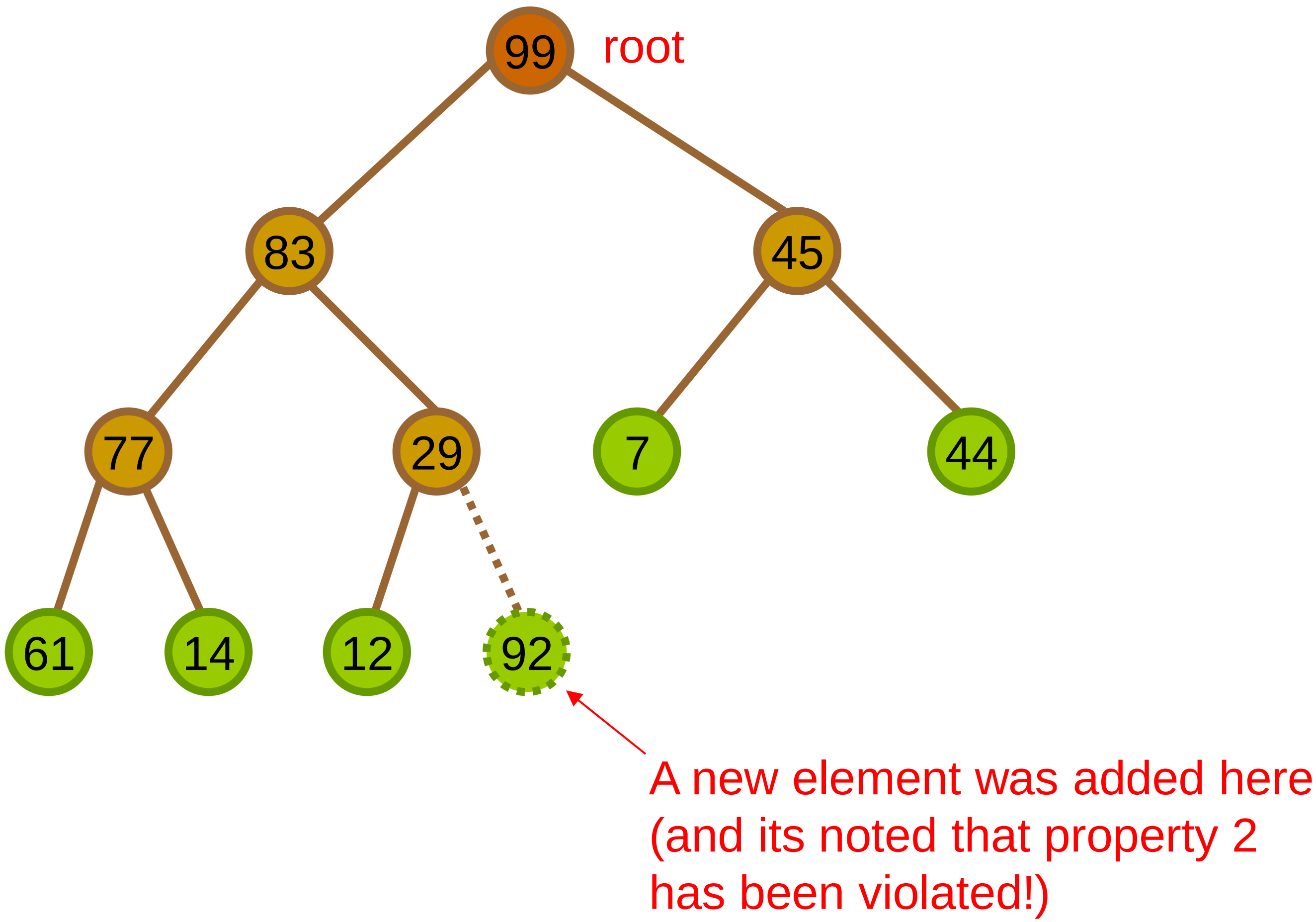
- In Priority Queue, the element to be deleted (dequeue) is the one with the highest priority.
- Max Heap always has the largest element in root
- Both the insert and delete operations on a heap require $O(\log n)$ time.



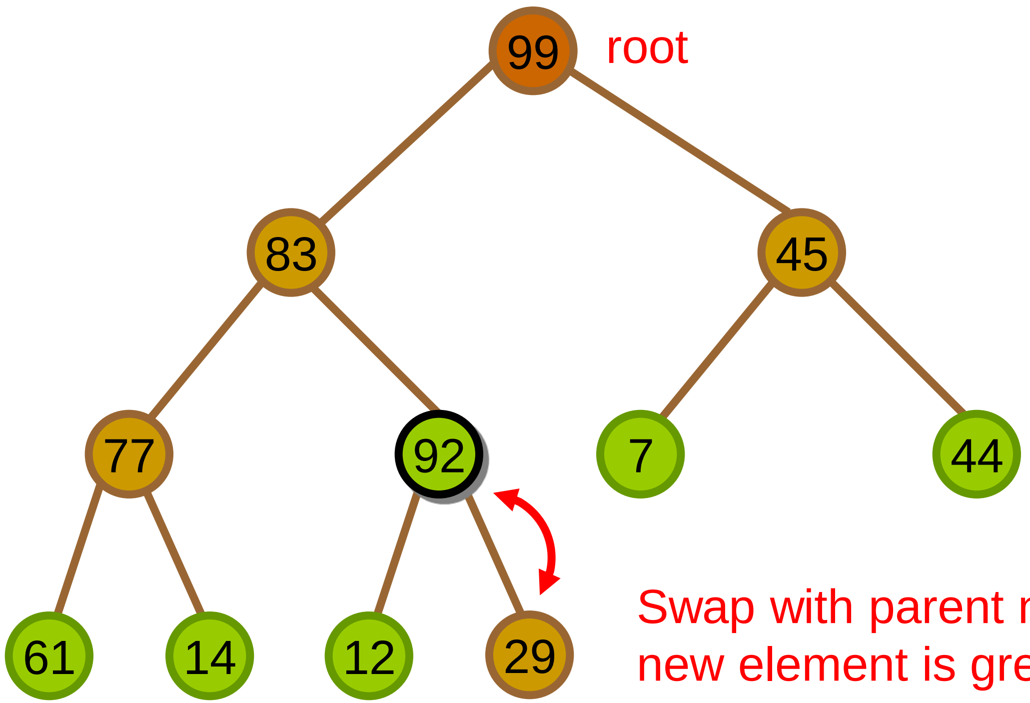
Insert Node Into Heap

- Step 1) Insert the new element in the next bottom leftmost place
- Step 2) **Percolate up**
  - Swap with its parent node recursively until it satisfy the property of heap

Example: Percolate Up

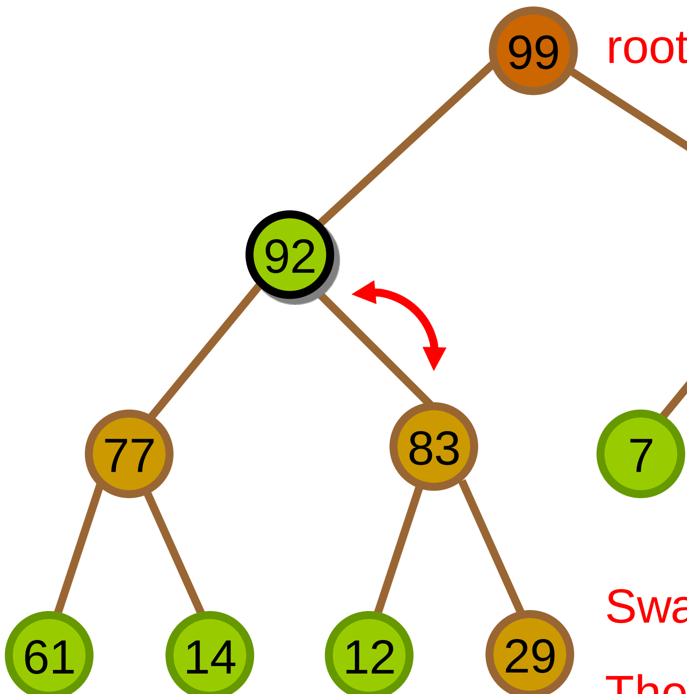


Example: Percolate Up



Swap with parent node if the new element is greater

Example: Percolate Up



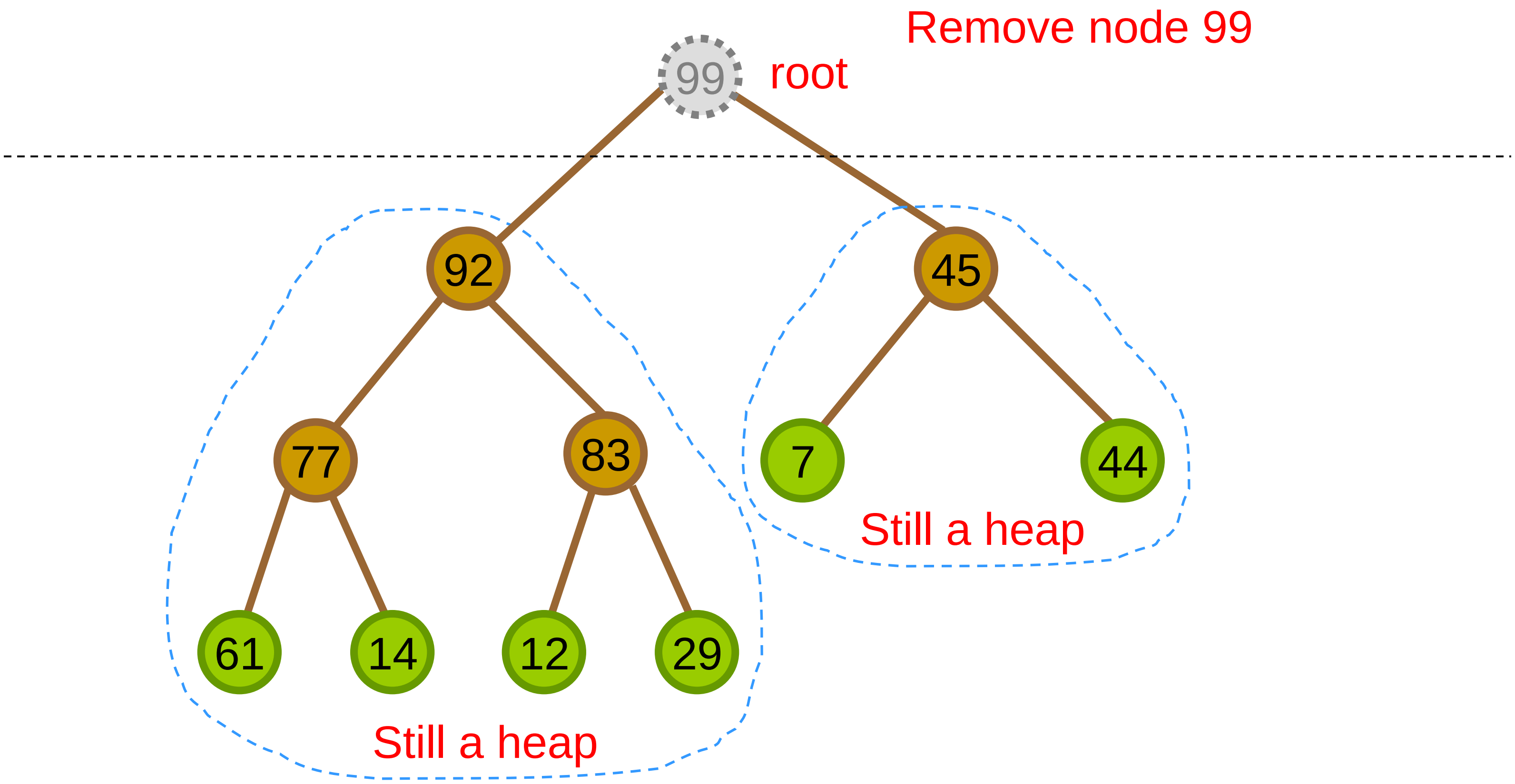
Swap again until property 2 is preserved!

The new node was propagated upward!

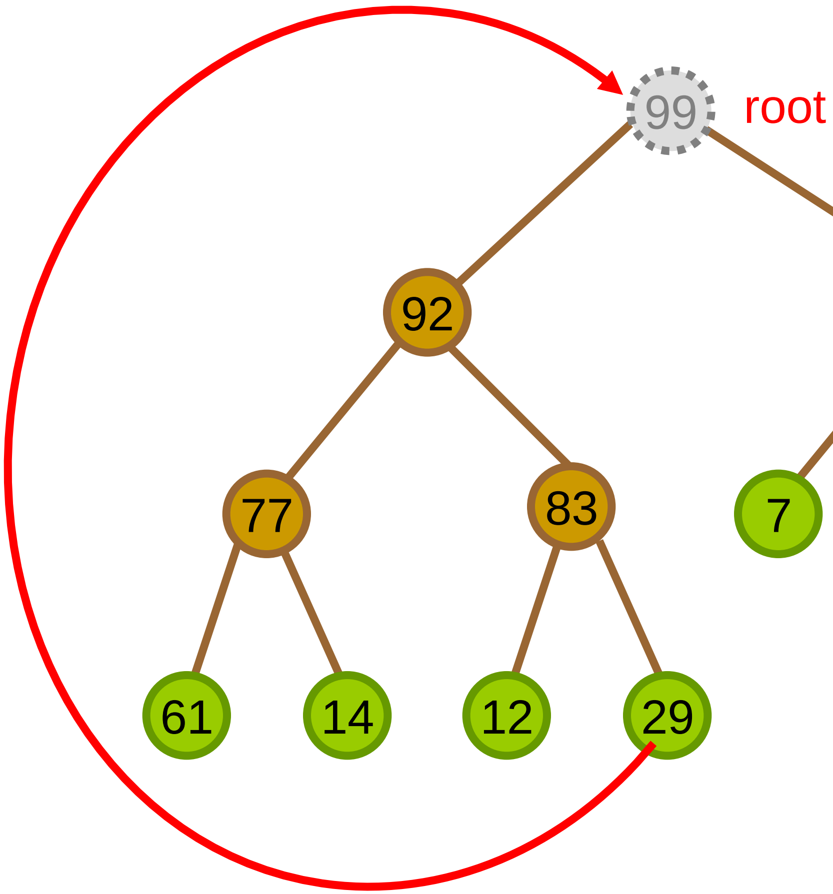
Remove Node From Heap

- In heap, nodes are always remove the root position (the largest element)
- Step 1) Replace the root node with the bottom rightmost element
- Step 2) **Percolate down**
  - Swap with the its greater child node recursively until it satisfies the heap property

Example: Percolate Down



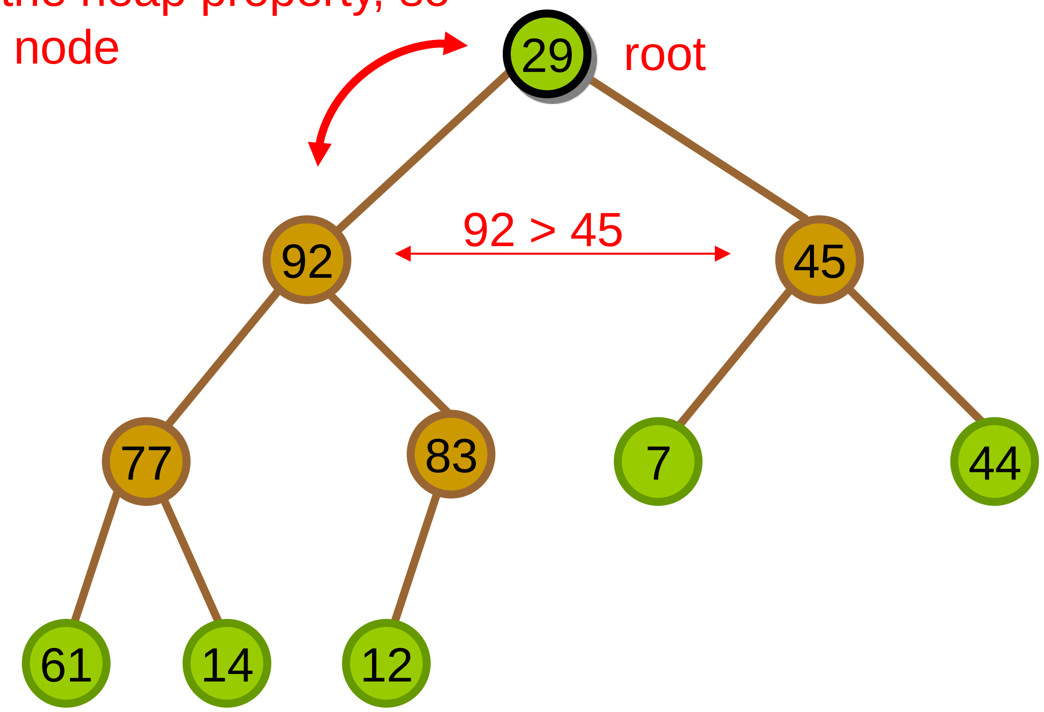
Example: Percolate Down



move node 29 to replace root

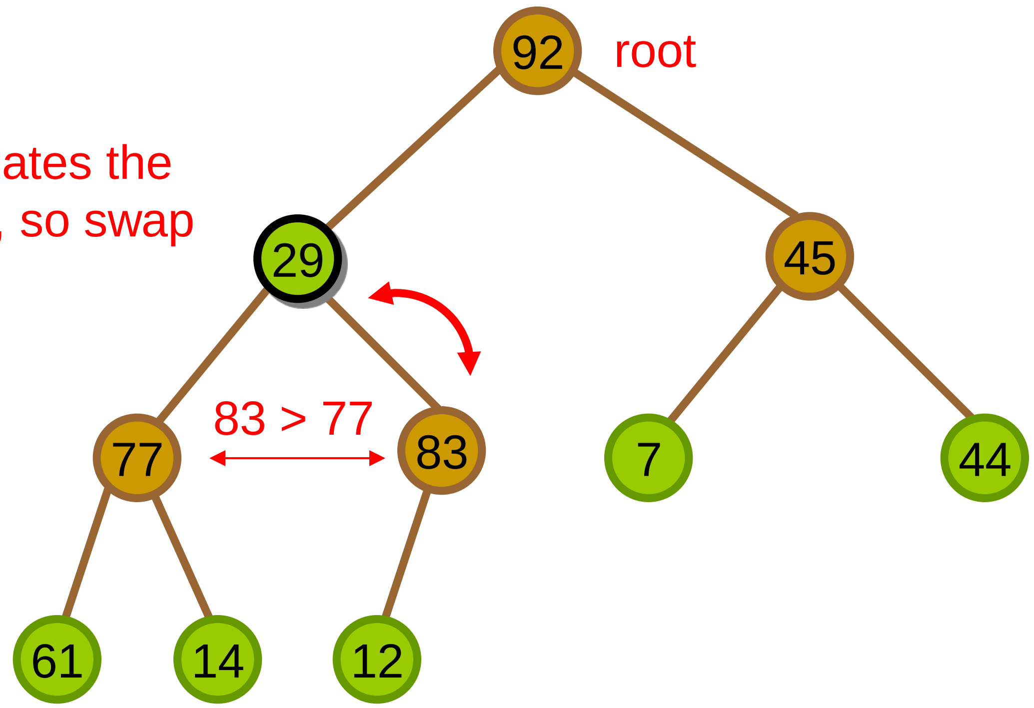
Example: Percolate Down

Now it violates the heap property, so
swap with child node



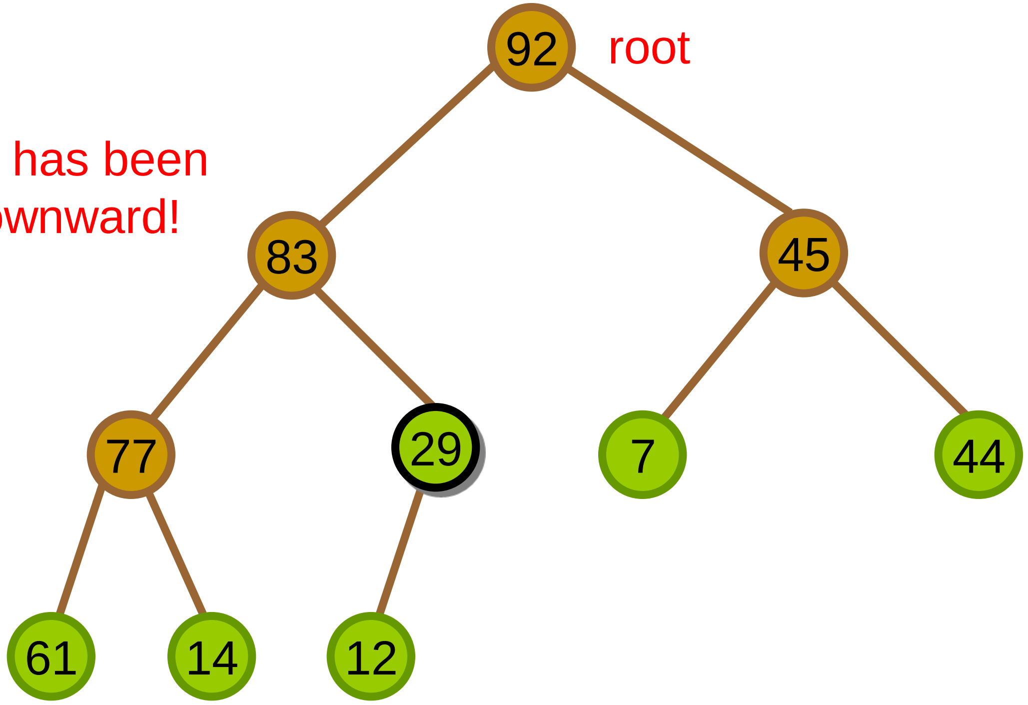
Example: Percolate Down

Now it still violates the heap property, so swap again



Example: Percolate Down

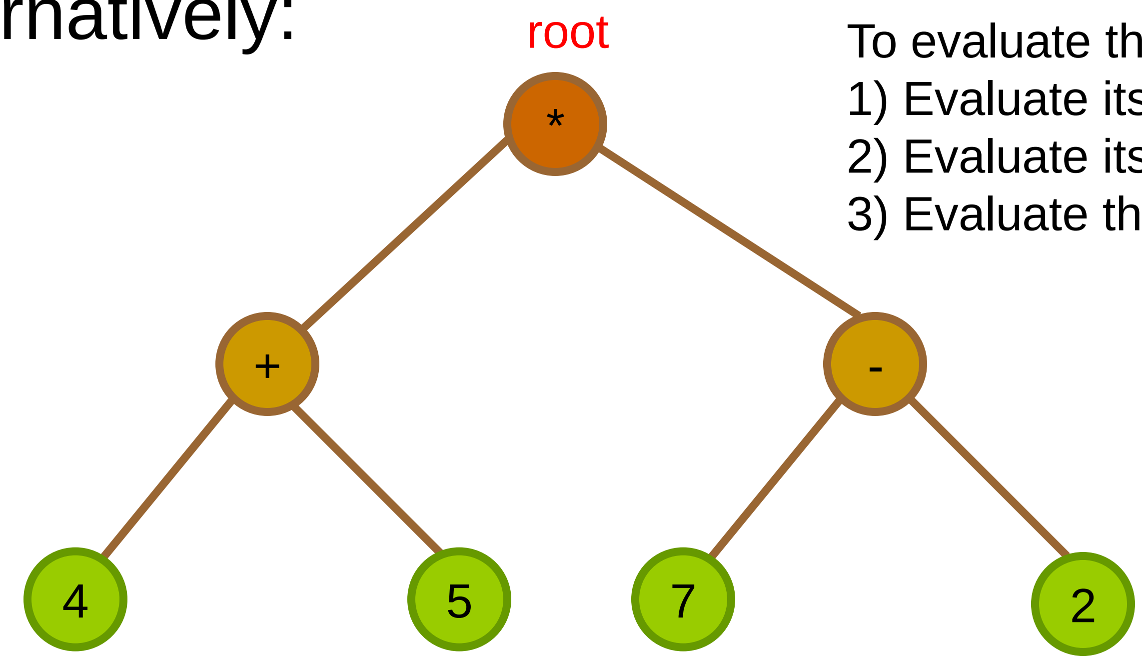
The new node has been propagated downward!



Applications

1st Applications: Infix & Postfix

- We learnt to use stack to convert infix to postfix, alternatively:



To evaluate the tree:
1) Evaluate its left subtree,
2) Evaluate its right subtree,
3) Evaluate the root

Inorder (LVR): (4 + 5) * (7 – 2) ← = infix!

Postorder (LRV): 4 5 + 7 2 - * ← = postfix!

Evaluate Arithmetic Expression Using a Binary Tree

```
#define operand 0
#define operator 1

struct infoRecord {
  char dataType;
  union {         //all members in union share the same physical space in memory
    char opr;
    double val;
  };
};

// Precondition: the expression tree is nonempty and has no
// syntax error. The algorithm is based on postorder traversal.
double evalExprTree(treeNode<infoRecord> *tree) {
  if (tree->info.dataType == operand)
    return tree->info.val;
  else {
    double d1 = evalExprTree(tree->left);
    double d2= evalExprTree(tree->right);
    char symb = tree->info.opr;
    return evaluate(symb, d1, d2);   // compute the result, not shown here
  }
}
```

2nd Application: Huffman Tree

- To encode and decode a message using shorter length
- e.g. the original message is “ABCDDAAA”
- We use 00 to represent A, 01 to represent B, 10 to represent C, 11 to represent D
- The message can be encoded as “0001101111000000” (16 bits)

A B C D D A A A

Not Optimal

- But we found that the previous encoding method is not optimal
- Since character A repeated many times
- It is not wise to use the same no. of bits to represent as other characters
- Variable length codeword
  - **frequently** appeared character should use **fewer** bits!

New Encoding Scheme

- 0 to represent A
- 100 to represent B
- 101 to represent C
- 11 to represent D

Code	Symbol
0	‘A’
100	‘B’
101	‘C’
11	‘D’

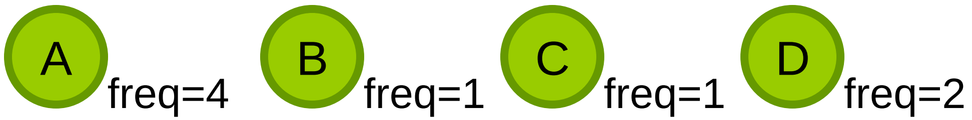
- The message can be encoded as “01001011111000” (14 bits only!)

A B C D D A A A

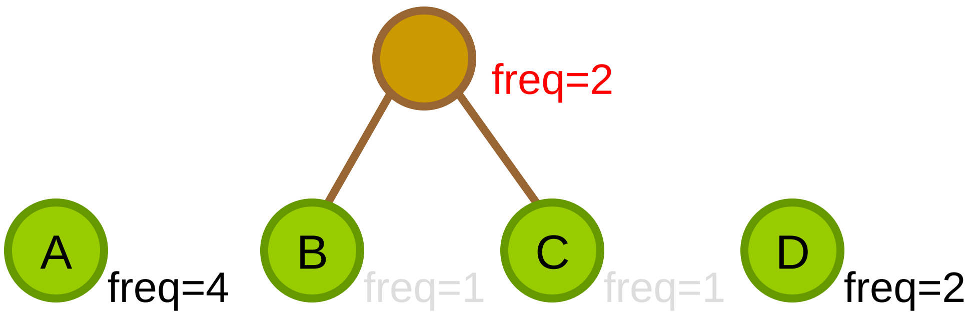
How to Determine the Code Table?

- Solution: Huffman Tree
- The original message is “ABCDDAAA”
- Count the frequency of each character
  - A: 4
  - B: 1
  - C: 1
  - D: 2
- Build the Huffman tree by **recursively** grouping the smallest two nodes together

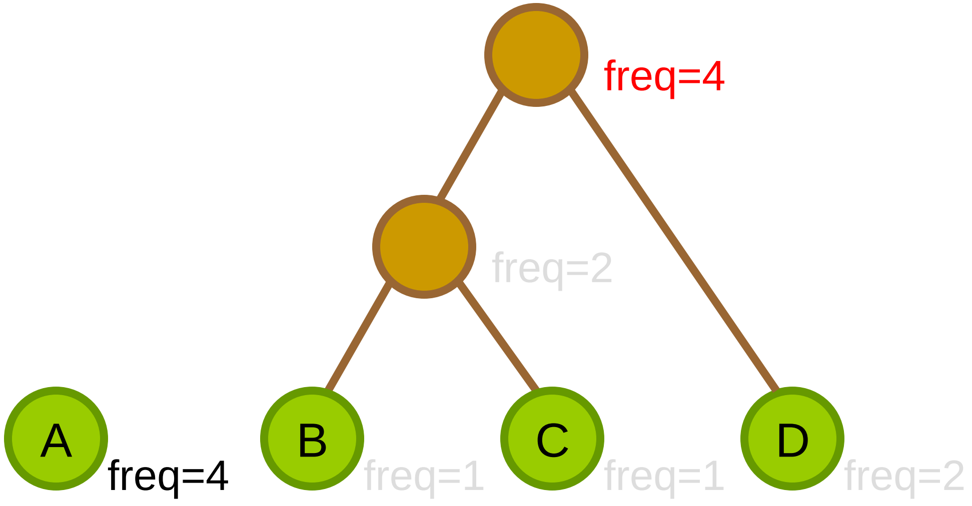
Combine Two Nodes Whose Frequency are Smallest



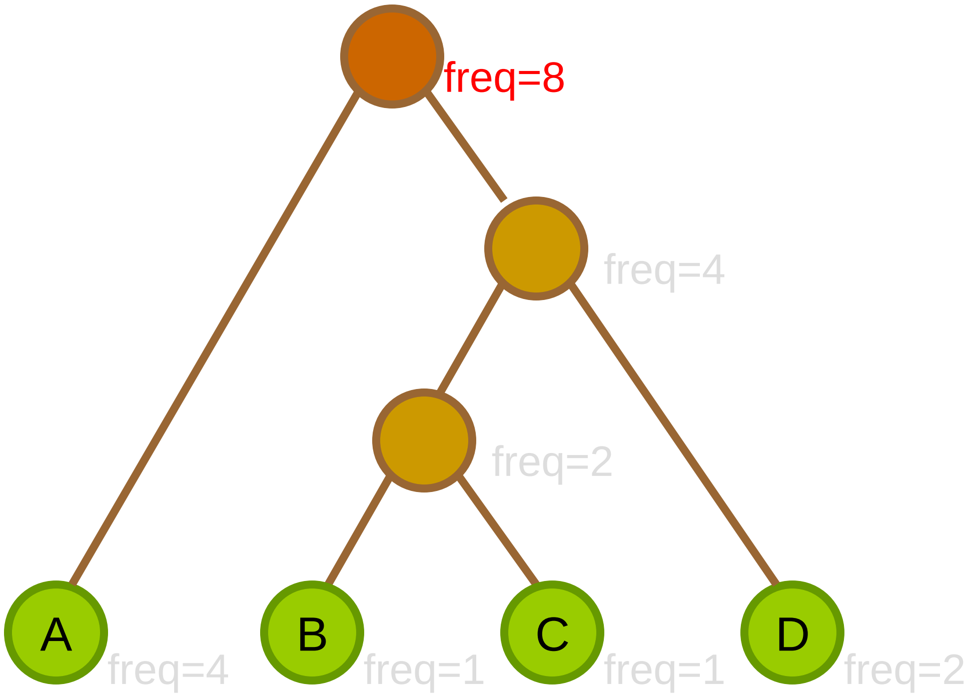
Combine and Update the Frequency



Combine Again

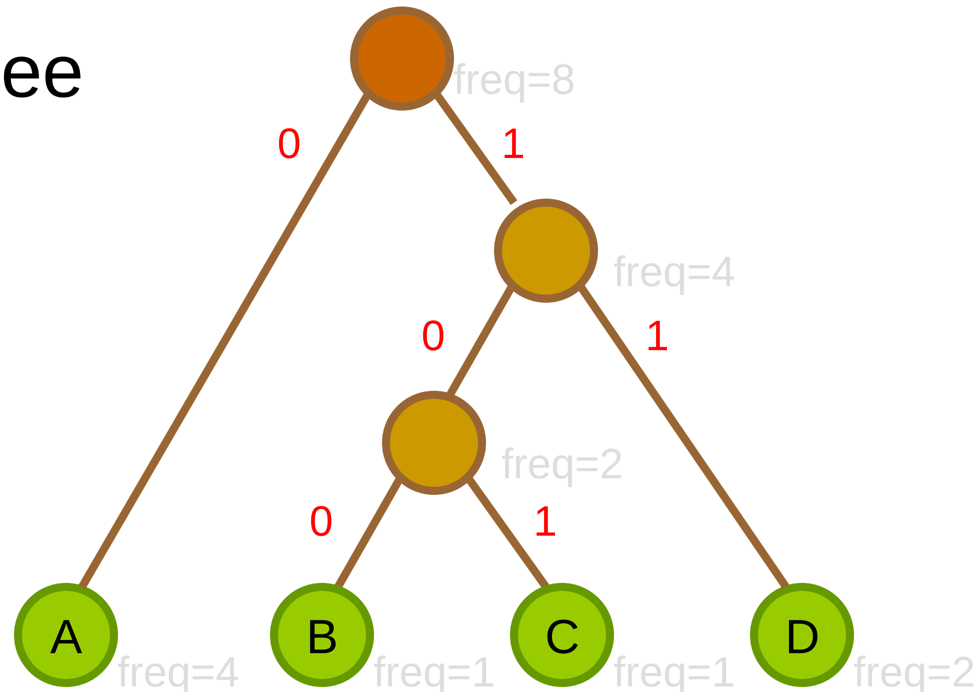


Combine Again Until…



Assign Values On Each Subtree

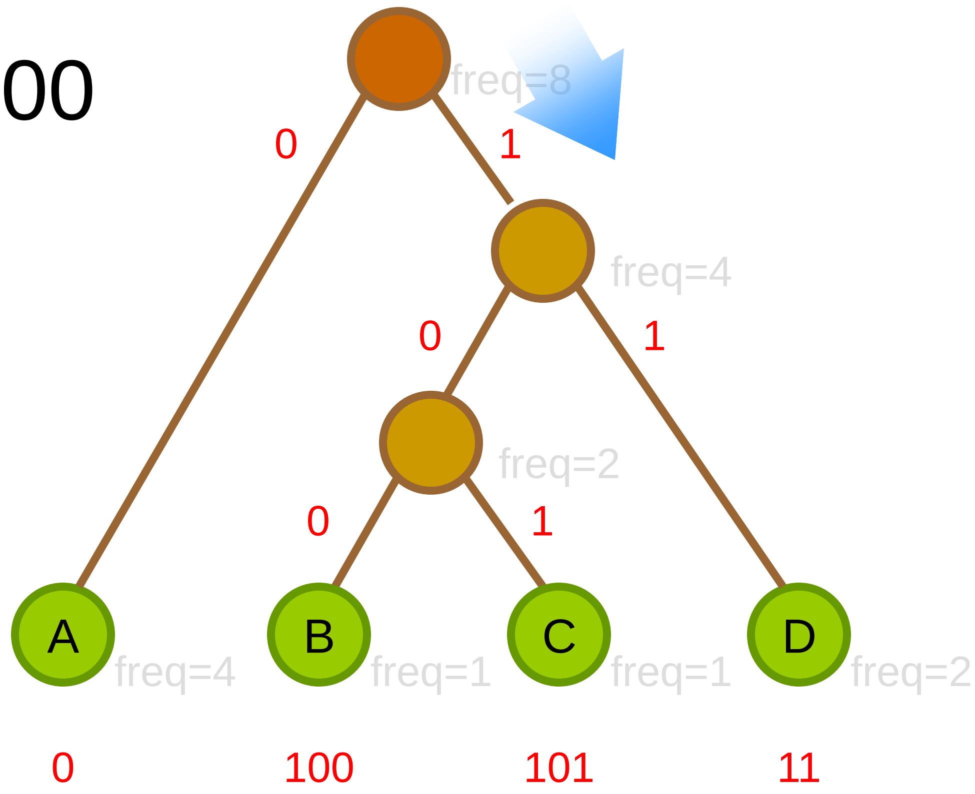
- By convention:
  - 0: left subtree
  - 1: right subtree



The final Huffman code: 0 100 101 11

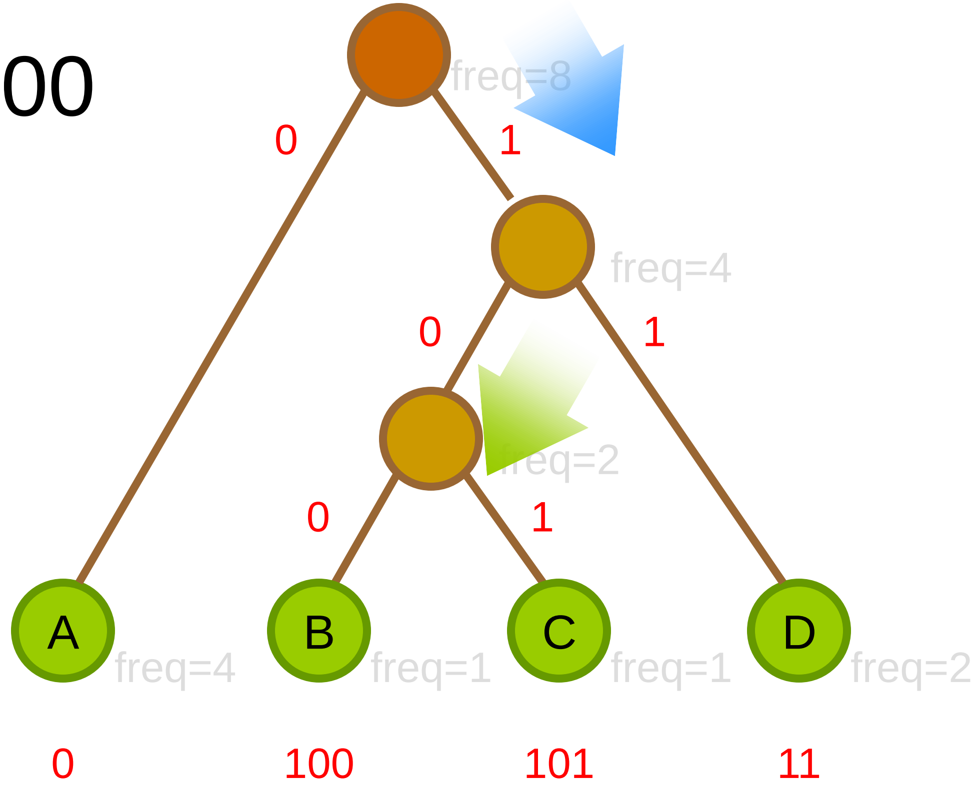
How to Decode This Message?

- 10111010001011100



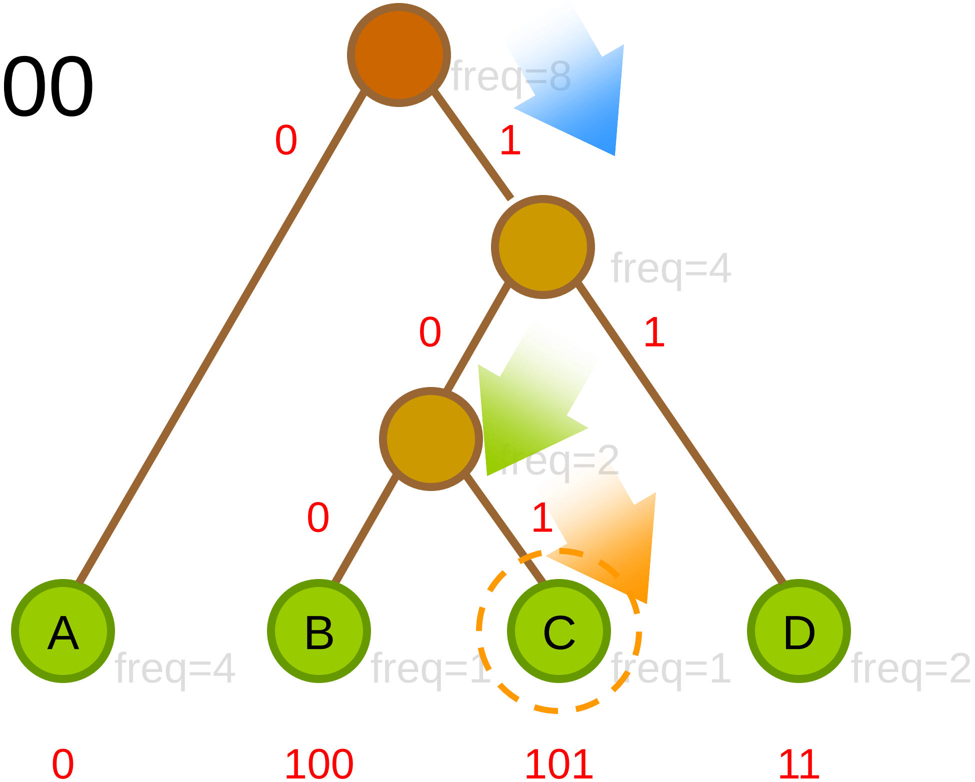
Traverse The Tree Node by Node

- 10111010001011100



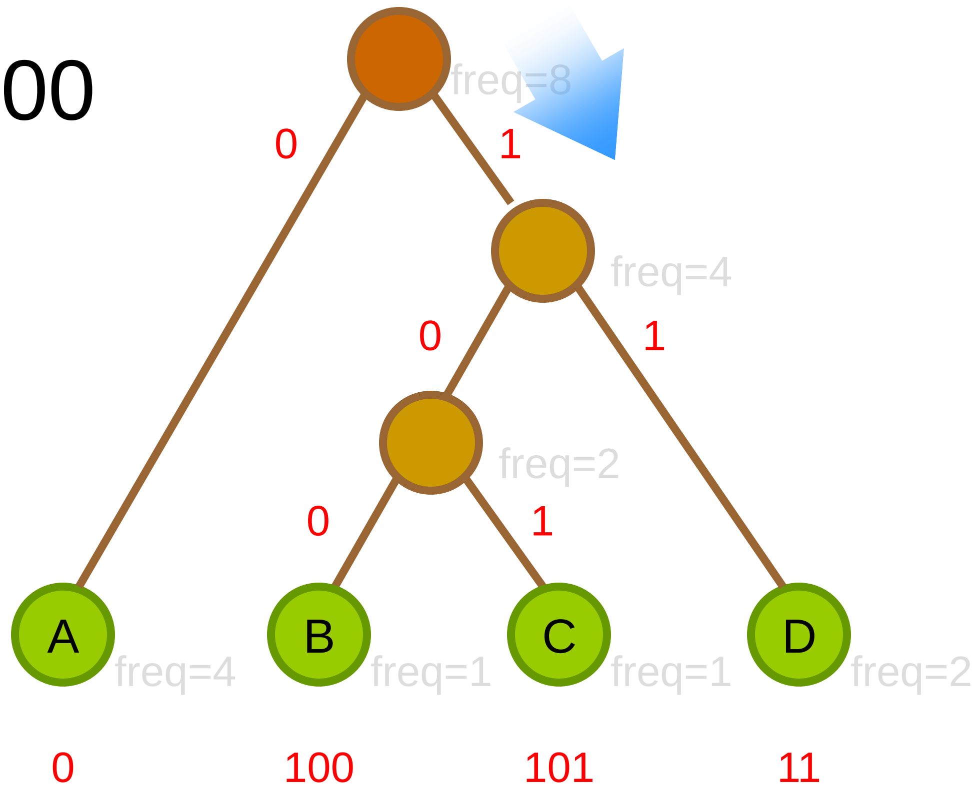
Until a Leaf Has Been Reached

- 10111010001011100
- C



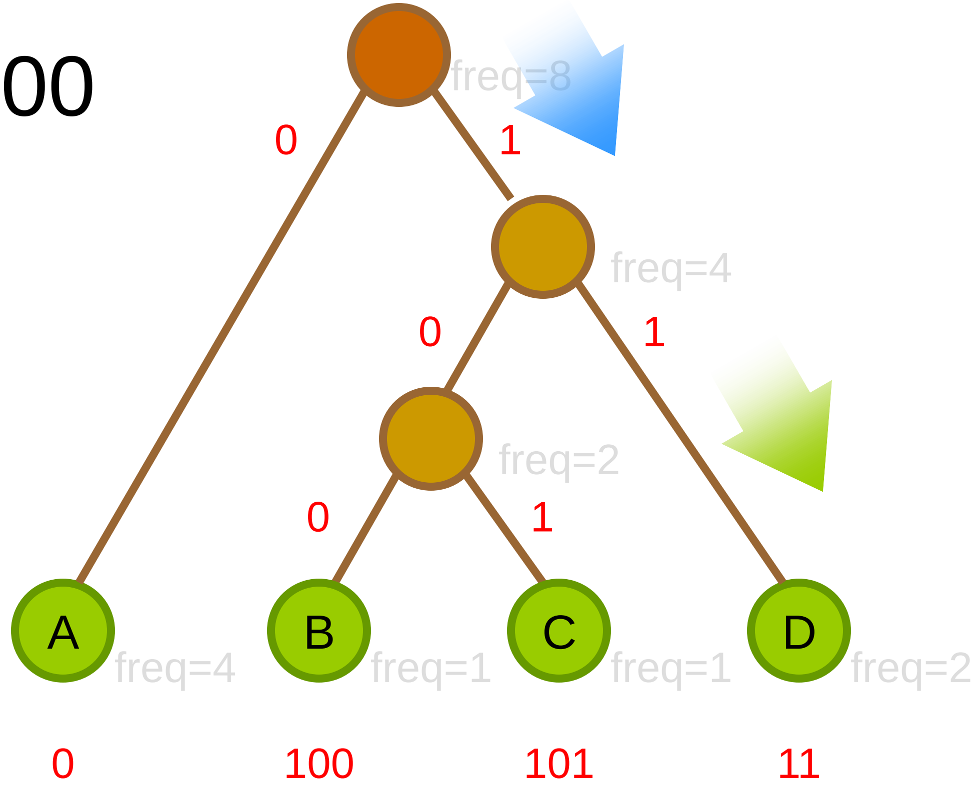
Restart From Root Again

- 1011110100001011100
- C



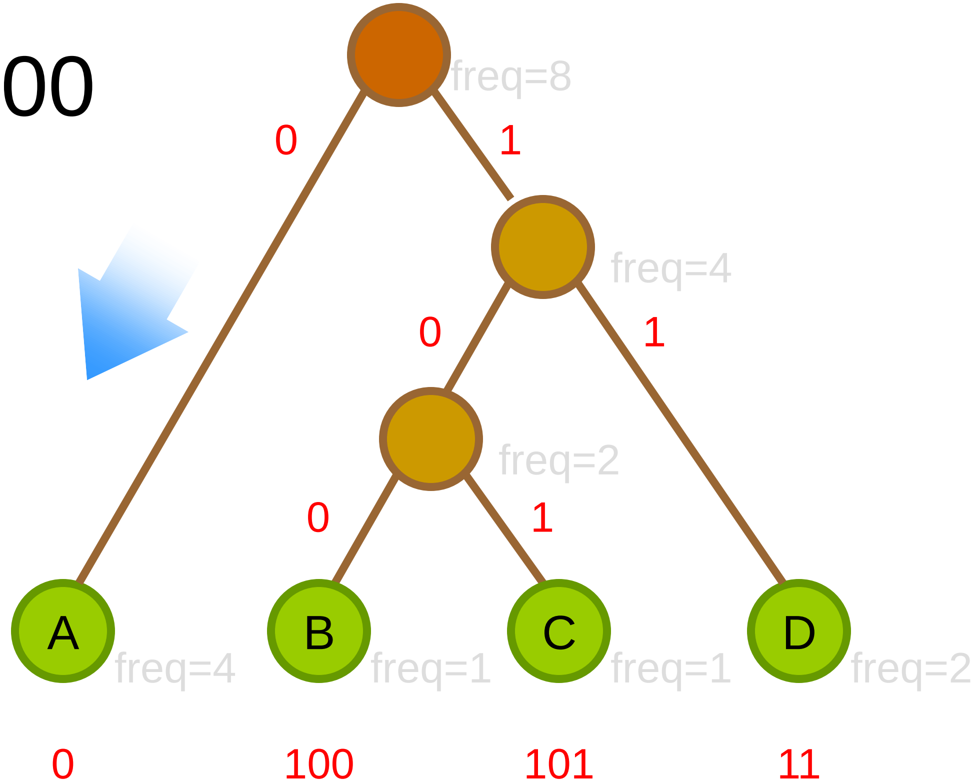
Reach Another Leaf Node

- 10111010001011100
- CD



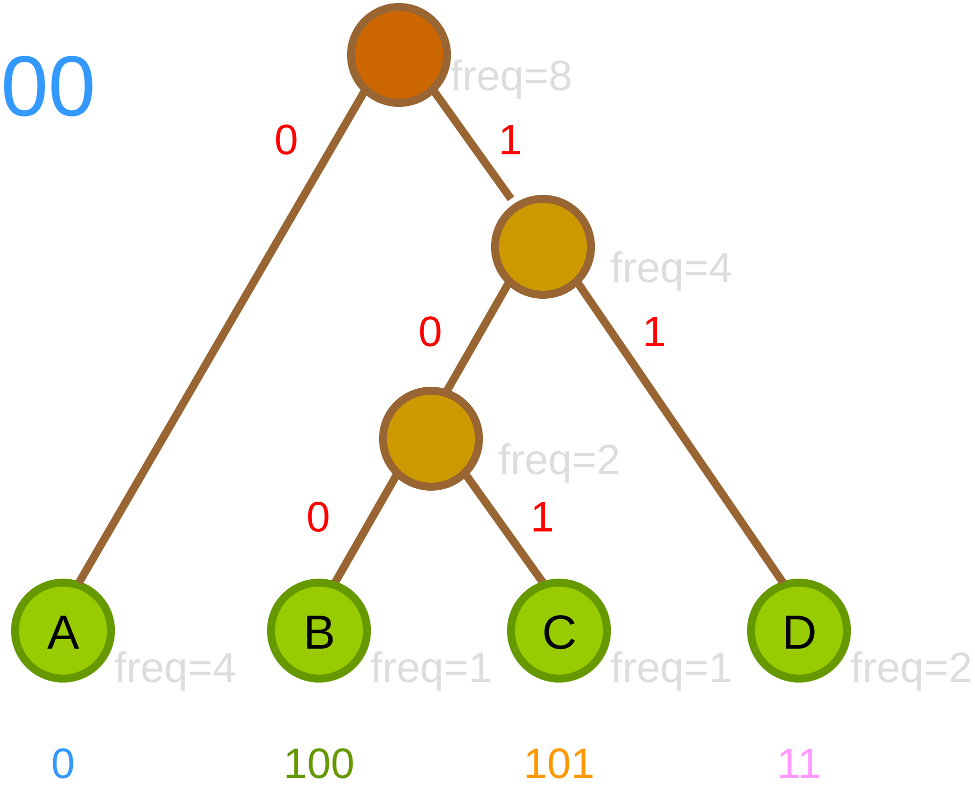
Decode the Remaining by Similar Method

- 101110100001011100
- CDA



Finally Obtain the Decoded Message

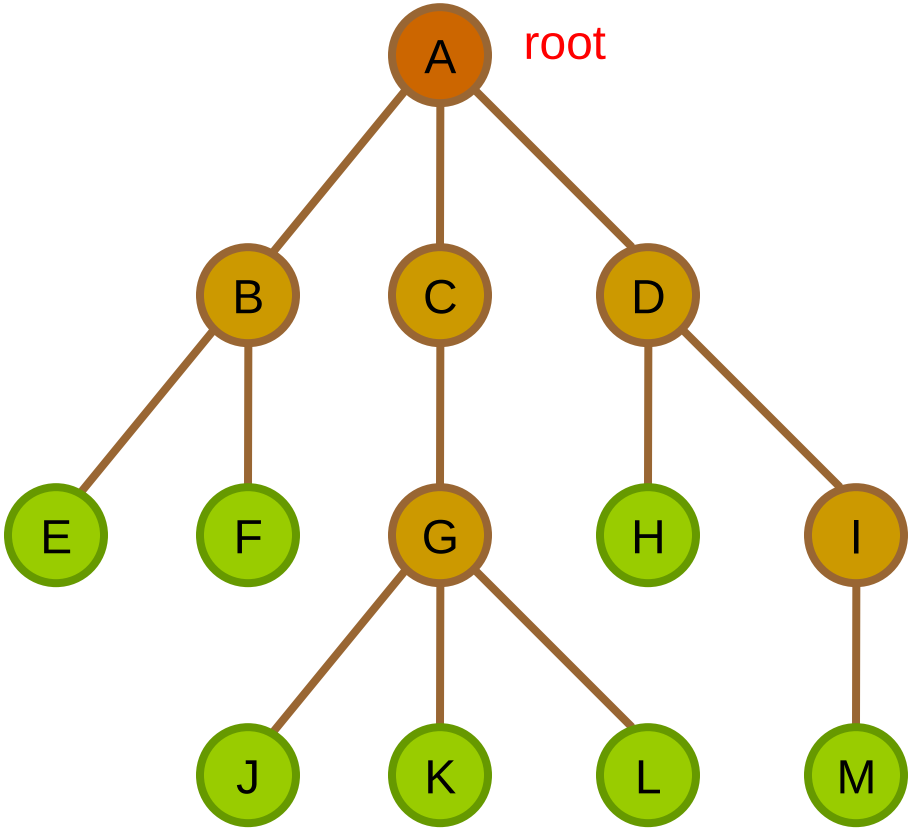
- 10111101000101111100
- CDABACDAA



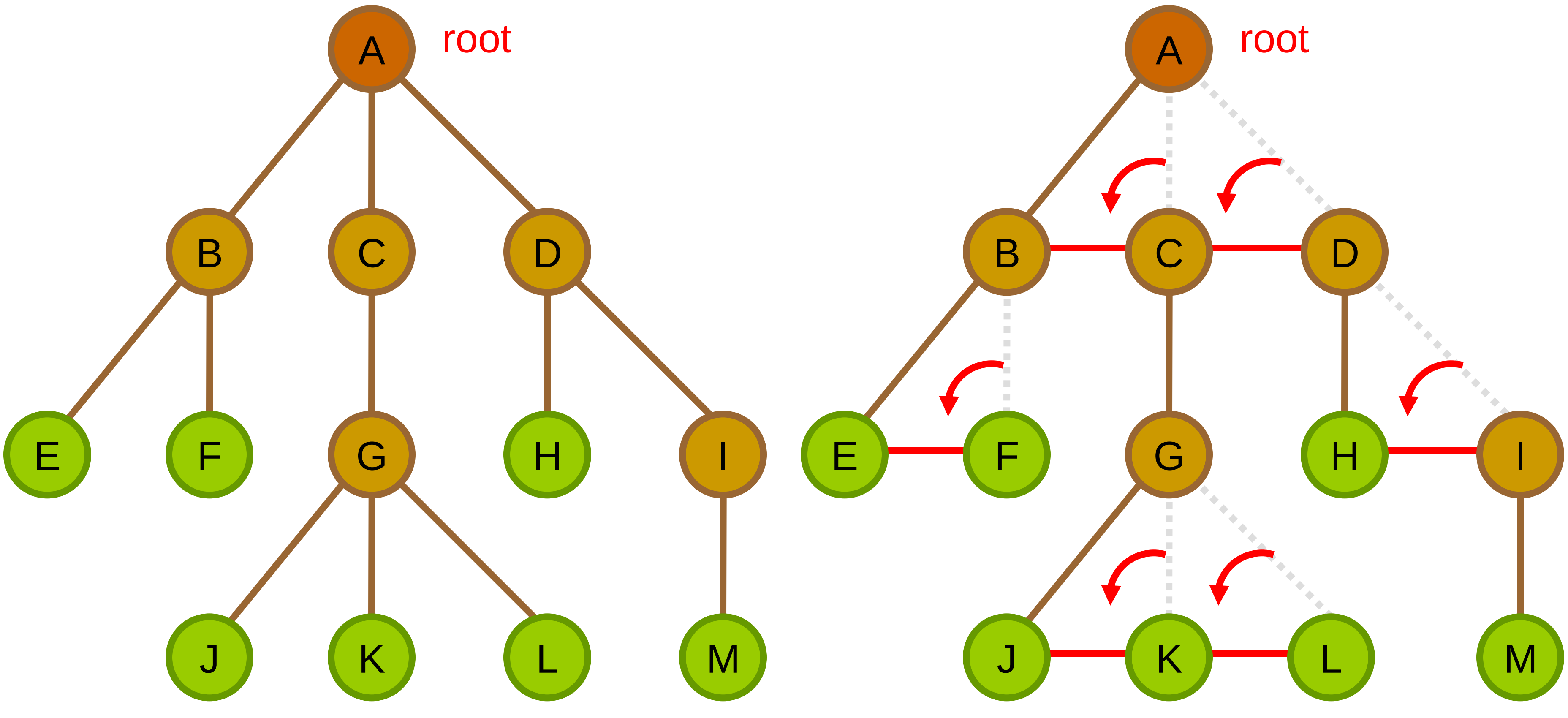
General Tree to Binary Tree Conversion

General tree

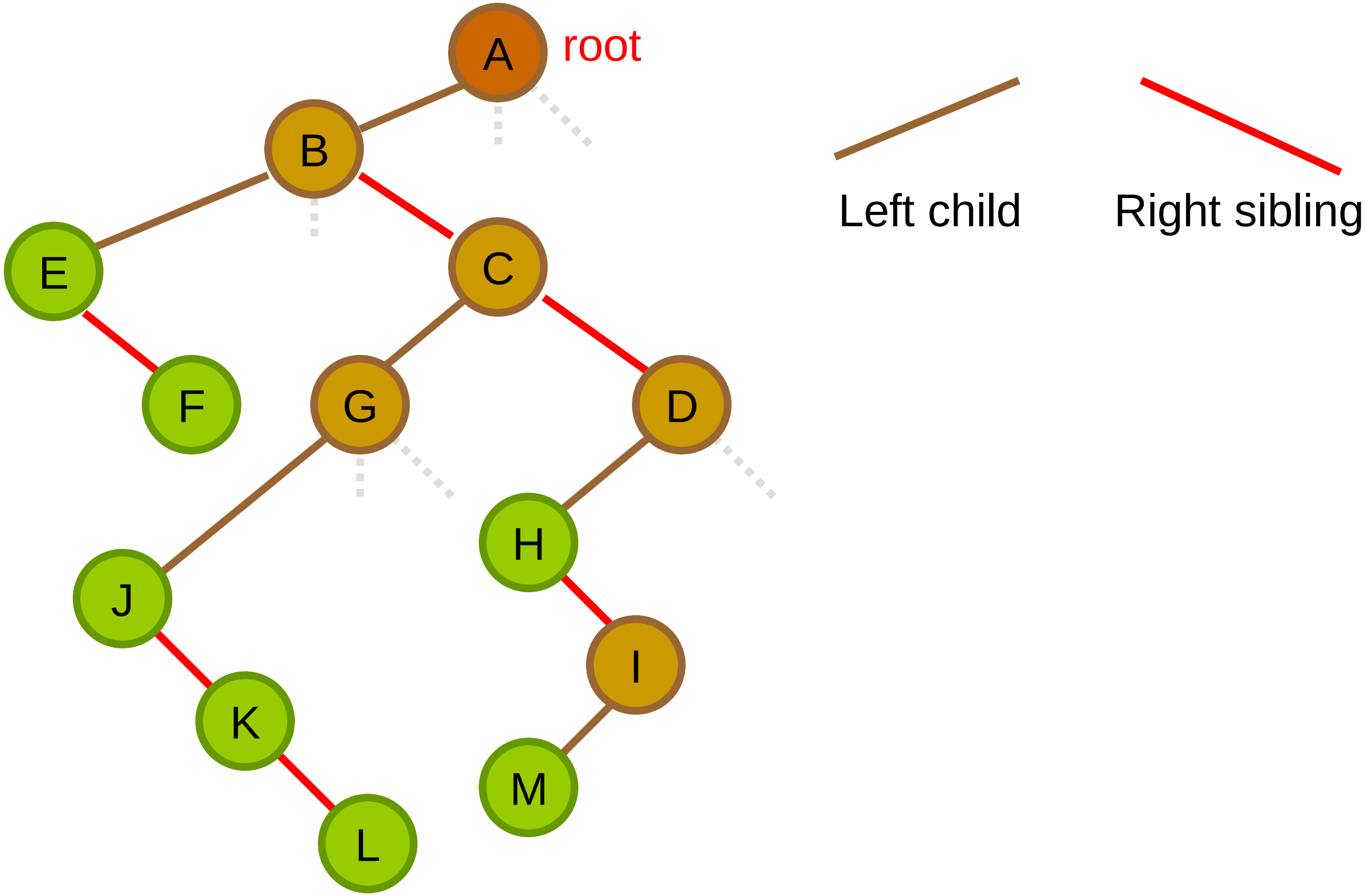
- We go back to the very beginning problem
- How to represent a general tree using binary tree?
  - Left Child-Right Sibling Representation



Left Child-Right Sibling



Left Child-Right Sibling



Count the No. of Leaf Nodes

```
template<class Type>
int countLeaf(treeNode<Type> *p) {
   // p is a general tree represented as a binary tree
   int count;

   if (p == NULL)            // tree is empty
      return 0;

   if (p->left == NULL)      // root has no subtree
      return 1;

   // root has 1 or more subtree.
   // no. of leaf nodes = sum of leaf nodes in the subtrees of the root
   count = 0;
   p = p->left;
   while (p != NULL) {       //for each subtree
      count += countLeaf(p);
      p = p->right;          //move on to the next subtree
   }
   return count;
}
```

Determine the Height

```
template<class Type>
int height(treeNode<Type> *p)  {
   // p is a general tree represented as a binary tree
   int h, t;

   if (p == NULL)
      return -1;   // leaf node’s height is 0

   h = 0;
   p = p->left;
   while (p != NULL) {
      t = height(p);
      if (t > h)
         h = t;
      p = p->right;
   }

   // h = max height of all subtrees
   return h+1;
}
```