EE3210 Signals & Systems
Mid-term Exam 2 (20 points in total) Name: ______________________________

This is a open-book exam. Submission due date is **12 pm, noon, April 7th, 2020**. Late submission will not be accepted. If you need more space, please feel free to attach additional papers. Once you're finished, scan and upload it to Canvas course website.

Honor Pledge

Please review the following honor code, then sign your name and write down the date.

1. I pledge that the answers in this exam are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,
   (a) I will not plagiarize (copy without citation) from any source;
   (b) I will not communicate or attempt to communicate with any other person during the exam;
   (c) neither will I give or attempt to give assistance to another student taking the exam; and
   (d) I will use only approved devices (e.g., calculators) and/or approved device models.
2. I understand that any act of academic dishonesty can lead to disciplinary action.

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Signature

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Date

1. (4 points) Derive the Fourier Transform (or Inverse FT) of the given signals.

(1 pts) (a)

$$\mathcal{F}\left\{\text{sinc}^2(t) \cdot \cos(2\pi f_c t)\right\}, \quad \text{where } f_c \gg 1$$

Since $\mathcal{F}(\text{Sinc}^2(t)) = \text{tri}(f)$ and $\mathcal{F}(\cos(2\pi f_c t)) = \frac{1}{2}(\delta(f-f_c) + \delta(f+f_c))$,

$$\mathcal{F}\{\text{Sinc}^2(t) \cos(2\pi f_c t)\} = \frac{1}{2}\left[\text{tri}(f-f_c) + \text{tri}(f+f_c)\right]$$

//

(1 pts) (b)

$$\mathcal{F}^{-1}\left\{\frac{1}{2 - f^2 + j3f}\right\}$$

$$\mathcal{F}^{-1}\left\{\frac{1}{2-f^2+j3f}\right\} = 2\pi\left[e^{-2\pi t} - e^{-4\pi t}\right] u(t)$$

//

(1 pts) (c)

$$\mathcal{F}\left\{\text{rect}\left(\frac{t}{\tau}\right) * \text{rect}\left(\frac{t}{\tau}\right) * \text{rect}\left(\frac{t}{\tau}\right) * \text{rect}\left(\frac{t}{\tau}\right)\right\}$$

Since $\mathcal{F}(\text{rect}(\frac{t}{\tau})) = \tau\,\text{Sinc}(f\tau)$,

$$\mathcal{F}\left\{\text{rect}\left(\tfrac{t}{\tau}\right) * \text{rect}\left(\tfrac{t}{\tau}\right) * \text{rect}\left(\tfrac{t}{\tau}\right) * \text{rect}\left(\tfrac{t}{\tau}\right)\right\} = \tau^4\,\text{Sinc}^4(f\tau)$$

//

(1 pts) (d)

$$\mathcal{F}^{-1}\left\{\frac{1}{\alpha^2 + j4\pi f\alpha + 4\pi^2(1-f^2)}\right\}$$

Since $\dfrac{1}{\alpha^2 + j4\pi f\alpha + 4\pi^2(1-f^2)} = \dfrac{1}{(\alpha+j2\pi f)^2 + 4\pi^2}$

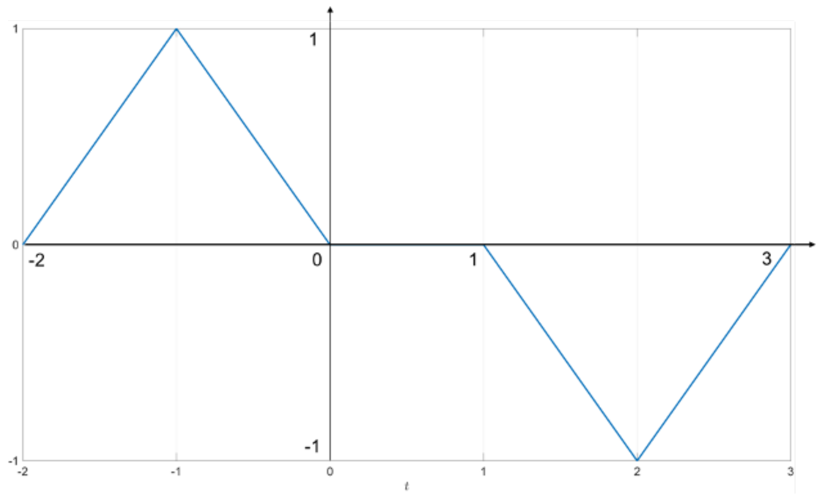
$$\mathcal{F}^{-1}\left\{\frac{1}{\alpha^2+j4\pi f\alpha + 4\pi^2(1-f^2)}\right\} = \frac{1}{2\pi} e^{-\alpha t} \sin(2\pi t)\, u(t)$$

//

2. (4 points) Derive the Fourier Transform of the following signals.

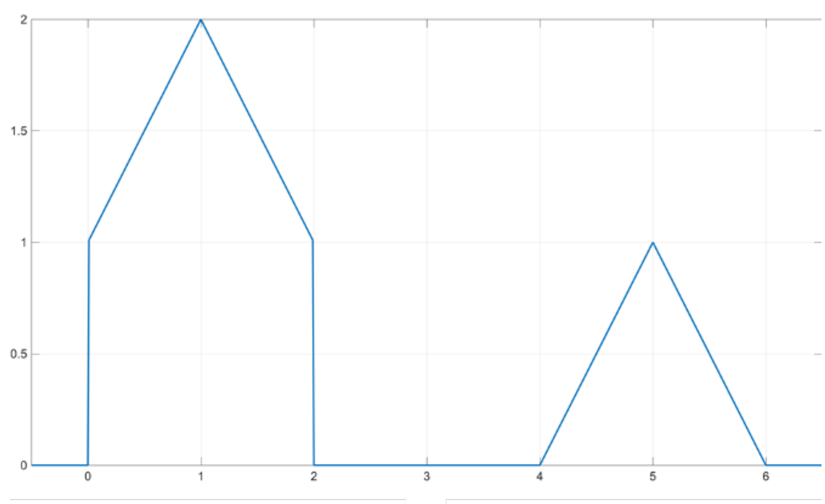
(2pts) (a)

$$x(t) = \text{tri}\,(t+1) - \text{tri}\,(t-2)\,, \quad \text{where } \text{tri}(t) = \begin{cases} 1-|t|, & |t|<1 \\ 0, & \text{otherwise} \end{cases}$$



(2pts) (b)

$$y(t) = y_1\,(t-1) + \text{tri}\,(t-5)\,, \quad \text{where } y_1(t) = \text{tri}(t) + \text{rect}\left(\frac{t}{2}\right)$$



(Answer Page for Question 2)

2-(a) Since $\mathcal{F}(tri(t)) = Sinc^2(f)$ and $\mathcal{F}(x(t-t_0)) = e^{-j2\pi f t_0} X(f)$,

$$X(f) = \mathcal{F}(x(t)) = \mathcal{F}(tri(t+1) - tri(t-2))$$
$$= Sinc^2(f)\left[e^{j2\pi f} - e^{-j4\pi f}\right]$$

//

2-(b)

$$\mathcal{F}(y_1(t)) = \mathcal{F}\left(tri(t) + rect\left(\frac{t}{2}\right)\right)$$
$$= Sinc^2(f) + 2\,Sinc(2f) = Y_1(f)$$

$$Y(f) = \mathcal{F}(y(t)) = \mathcal{F}(y_1(t-1) + tri(t-5))$$
$$= e^{-j2\pi f} Y_1(f) + e^{-j10\pi f} Sinc^2(f)$$
$$= \left[2\,Sinc(2f) + Sinc^2(f)\right] e^{-j2\pi f} + Sinc^2(f)\, e^{-j10\pi f}$$

//

3. (4 points) Consider a continuous LTI system where the input and output are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 4x(t)$$

(2pts) (a) Find the impulse response $h(t)$ and frequency response $H(f)$ of this system

After FT, $\Rightarrow \left[(j2\pi f)^2 + 4(j2\pi f) + 3\right] Y(f) = \left[j2\pi f + 4\right] X(f)$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{j2\pi f + 4}{(j2\pi f + 3)(j2\pi f + 1)} = \frac{3/2}{1 + j2\pi f} + \frac{-1/2}{3 + j2\pi f} \quad //$$

$$h(t) = \frac{3}{2} e^{-t} u(t) - \frac{1}{2} e^{-3t} u(t) \quad //$$

(2pts) (b) Find the system output $y(t)$ for an input signal $x(t) = e^{-2t}u(t) - 4e^{-3t}u(t)$

$$X(f) = \mathcal{F}(x(t)) = \frac{1}{2 + j2\pi f} - \frac{4}{3 + j2\pi f} = \frac{-(5 + 3 \times j2\pi f)}{(2 + j2\pi f)(3 + j2\pi f)}$$

$$Y(f) = H(f)\, X(f) = -\frac{(4 + j2\pi f)(5 + 3 \cdot j2\pi f)}{(1 + j2\pi f)(2 + j2\pi f)(3 + j2\pi f)^2}$$

$$= \frac{-3/2}{(x+1)} + \frac{-2}{(x+2)} + \frac{2}{(x+3)^2} + \frac{7/2}{(x+3)}$$

$$\Rightarrow y(t) = \mathcal{F}^{-1}(Y(f)) = -\frac{3}{2} e^{-t} u(t) - 2 e^{-2t} u(t) + 2t e^{-3t} u(t) + \frac{7}{2} e^{-3t} u(t)$$

$$= \left[-\frac{3}{2} e^{-t} - 2 e^{-2t} + \left(2t + \frac{7}{2}\right) e^{-3t}\right] u(t) \quad //$$

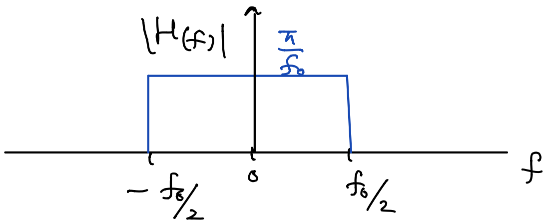
4. (4 points) Consider the following filters with impulse response $h(t)$. For each filter, derive the 3-dB bandwidth f_{3dB} and 70 percent energy containment bandwidth $f_{70\%}$, respectively.

(2pts) a)

$$h(t) = \frac{\sin(\pi f_0 t)}{f_0 t}, \quad f_0 > 0$$

(for (a), any answer regarding f_{3dB} is OK)

$$h(t) = \frac{\sin(\pi f_0 t)}{f_0 t} = \pi \, \mathrm{Sinc}(f_0 t) \rightarrow H(f) = \frac{\pi}{f_0} \, \mathrm{rect}\left(\frac{f}{f_0}\right)$$



$|H(f)|$ at $f = \pm f_0/2$ is defined as $\frac{\pi}{2f_0}$

✓ 3 dB bandwidth occurs at $|H(f)| = \frac{\pi}{f_0\sqrt{2}}$, which can not be defined in (a).

✓ $f_{70\%}$ can be derived as follows

$$\int_{-f_{70\%}}^{f_{70\%}} |H(f)|^2 df = (0.7) \int_{-\infty}^{\infty} |H(f)|^2 df \Rightarrow f_{70\%} = \frac{f_0}{2} \cdot (0.7)$$

(2pts) b)

$$h(t) = \frac{1}{t^2 + f_0^2}, \quad f_0 > 0$$

$$h(t) = \frac{1}{t^2 + f_0^2} \longrightarrow H(f) = \frac{\pi}{f_0} e^{-2\pi f_0 |f|}$$

→ 3dB BW : $f_{3dB} \Rightarrow |H(f)| = \frac{|H(0)|}{\sqrt{2}} = \frac{\pi}{\sqrt{2} \cdot f_0}$

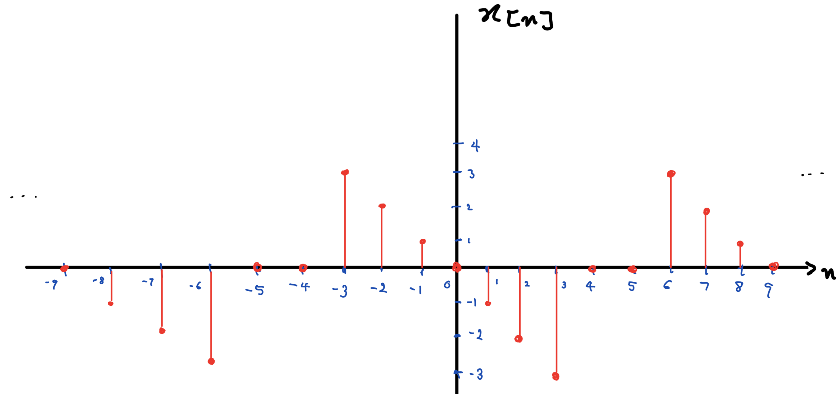
$$\Rightarrow f_{3dB} = \frac{\ln 2}{4\pi f_0}$$

→ 70% BW : $f_{70\%} \Rightarrow 2 \cdot \left(\frac{\pi}{f_0}\right)^2 \int_0^{f_{70\%}} e^{-4\pi f_0 f} df = 0.7 \times 2 \left(\frac{\pi}{f_0}\right)^2 \int_0^{\infty} e^{-4\pi f_0 f} df$

$$\Rightarrow f_{70\%} = -\frac{\ln(0.3)}{4\pi f_0}$$

5. (4 points) (a) Find the discrete-time Fourier series of the sequence $x[n]$ as plotted below

(2pts)



Period is $N = 9$, $\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{9}$

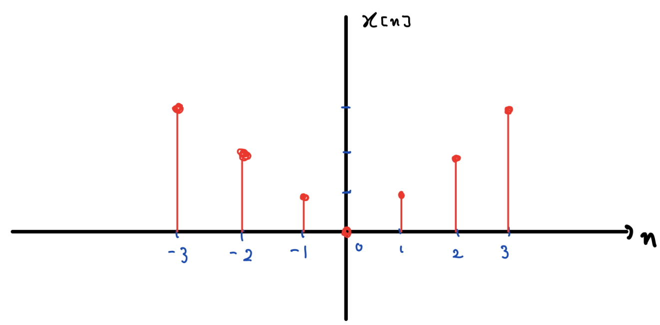
$$C_k = \frac{1}{9}\sum_{n=-4}^{4} x[n] e^{-jk\Omega_0 n} = \frac{1}{9}\Big[3e^{j3k\Omega_0} - 3e^{-j3k\Omega_0} + 2e^{j2k\Omega_0}$$

$$- 2e^{-j2k\Omega_0} + e^{jk\Omega_0} - e^{-jk\Omega_0}\Big]$$

$$= \frac{2j}{9}\left[\sin\left(\frac{2\pi k}{9}\right) + 2\sin\left(\frac{4\pi k}{9}\right) + 3\sin\left(\frac{6\pi k}{9}\right)\right] //$$

(b) Find the discrete-time Fourier transform of the sequence $x[n]$ as shown below

(2pts)



$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n} = \sum_{n=-3}^{3} x[n] e^{-j2\pi f n}$$

$$= \Big[3e^{j6\pi f} + 3e^{-j6\pi f} + 2e^{j4\pi f} + 2e^{-j4\pi f} + e^{j2\pi f} + e^{-j2\pi f} \Big]$$

$$= 2\left[\cos(2\pi f) + 2\cos(4\pi f) + 3\cos(6\pi f)\right] //$$