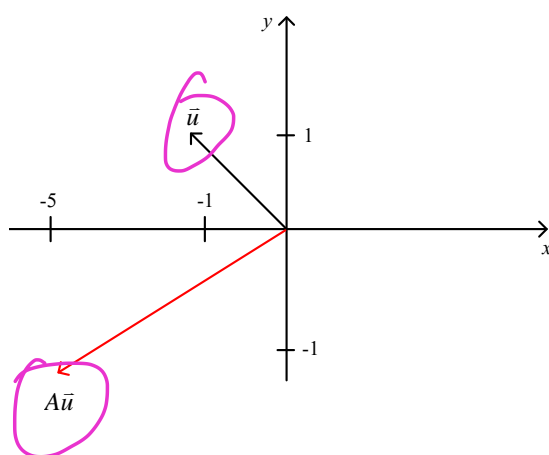


Consider

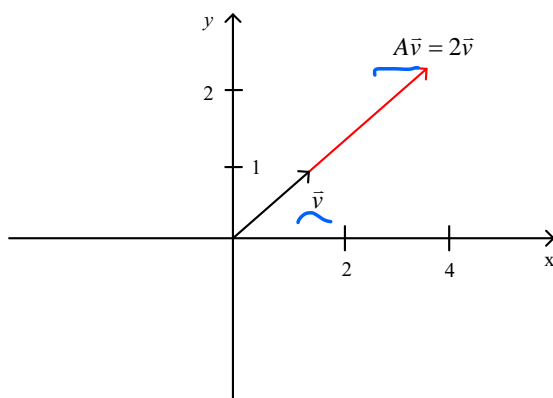
$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}, \quad \underbrace{\vec{u}}_{\text{blue}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \underbrace{\vec{v}}_{\text{pink}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$\underbrace{A\vec{u}}_{\text{pink}} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$



$$\underbrace{A\vec{v}}_{\text{pink}} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \underbrace{\vec{v}}_{\text{pink}} \quad \begin{matrix} \uparrow \\ \uparrow \end{matrix}$$



**Definition:** Given a  $n \times n$  matrix  $A$ , a nonzero vector  $\vec{x}$  is called eigenvector of  $A$  if for some scalar  $\lambda$ ,

$$\underline{A\vec{x} = \lambda\vec{x}}.$$

On the other hand, a scalar  $\lambda$  is called eigenvalue of  $A$  if there is a nontrivial solution such that

$$A\vec{x} = \lambda\vec{x}$$

and  $\vec{x}$  is called eigenvector corresponding to  $\lambda$ .

**Remark:** Any eigenvector is a nonzero vector.

**Example.**  $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ ,  $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ . Since

$$A\vec{v} = 2\vec{v},$$

$\vec{v}$  is an eigenvector of  $A$  corresponding to the eigenvalue 2.

**Question:** Are there other eigenvalues and eigenvectors for  $A$

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$A a \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \cdot a \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

all  $\left\{ k \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$  is of  $A$  corresponding to  $\lambda=2$

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is also an eigenvector of  $A$   <sup>$\lambda=1$</sup>  corr.

$\left\{ k \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$  is eigenspace of  $A$  corresponding to  $\lambda=1$

**Example** Show that  $\underbrace{7}$  is an eigenvalue of  $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$ .

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is eigenvector  
cor. to  $\lambda=7$ .

Sol.

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 7 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 7 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \left[ \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1=1 \\ x_2=1 \end{cases} \text{ has nontrivial sol.}$$

Idea:

$$\det \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \text{ not invertible}$$

$\underbrace{(-6) \cdot (-5) - 5 \cdot 6}$

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \Rightarrow A\vec{x} = \lambda I_n \vec{x} \\ &\Rightarrow A\vec{x} - \lambda I_n \vec{x} = \vec{0} \\ &\Rightarrow (A - \lambda I_n) \vec{x} = \vec{0} \end{aligned}$$

Finding  $\lambda$  to make  $A\vec{x} = \lambda\vec{x}$  has nontrivial solution is equal to make

$$\boxed{\det(A - \lambda I_n) = 0.}$$

**Definition:** For a  $n \times n$  matrix  $A$ ,  $\det(A - \lambda I_n)$  is a polynomial in  $\lambda$ , called as characteristic polynomial of  $A$ . And,  $\det(A - \lambda I_n) = 0$  is called the characteristic equation of  $A$ .

**Question:** Given a square matrix  $A$ , how to find its all eigenvalues and eigenvectors.

**Example:**  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}_{2 \times 2}$ . Find all eigenvalues and eigenvectors of  $A$ .

Sol:

$$\begin{aligned} A - \lambda I_2 &= \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \lambda & 4 - 0 \\ 2 - 0 & 3 - \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A \vec{x} &= \lambda \vec{x} \\ \Downarrow \\ (A - \lambda I_2) \vec{x} &= \vec{0} \end{aligned}$$

① characteristic poly

$$\begin{aligned} \det(A - \lambda I_2) &= \begin{vmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(3 - \lambda) - 2 \cdot 4 \end{aligned}$$

$$= \lambda^2 - 4\lambda + 3 - 8$$

$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

$$= (\lambda - 5)(\lambda + 1) = 0$$

$$\checkmark \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 1 \cdot 3 - 2 \cdot 4 = -5$$

$$\begin{matrix} 1 & 5 \\ 1 & -1 \end{matrix} \Rightarrow -1 + 5 = +4$$

② solve characteristic equation

$$\begin{aligned}\det(A - \lambda I_2) &= 0 \\ \implies (\lambda - 5)(\lambda + 1) &= 0 \\ \implies \lambda &= \underline{5}, \underline{-1}\end{aligned}$$

So  $A$  has two eigenvalues  $5, -1$ .

③ Solve  $(A - \lambda I_2)\vec{x} = \vec{0}$ , for each eigenvalue,  $A\vec{x} = \lambda\vec{x}$   
when  $\lambda = 5$ .

$$\begin{aligned}\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -4 & 4 & 0 \\ 2 & -2 & 0 \end{pmatrix} \xrightarrow[\substack{2R_2 + R_1 \\ -\frac{1}{4}R_1}]{\substack{R_1 \leftarrow R_1 + R_2 \\ R_2 \leftarrow R_2 - R_1}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

*reduced echelon.*

So  $x_1 - x_2 = 0 \implies x_1 = x_2$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{let } x_2=k} k \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad k \neq 0.$$

Hence,  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  is one eigenvector of  $A$  corresponding to 5.  $\left\{ k \begin{pmatrix} 1 \\ 1 \end{pmatrix} : k \neq 0 \right\}$  is  
the set of all eigenvectors of  $A$  corresponding to 5. eigenspace.

When  $\lambda = -1$ ,

$$\begin{aligned}(A - (-1)I_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \checkmark \\ \Rightarrow \left[ \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} .\end{aligned}$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 2 & 4 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$$

so  $x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$ .

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = k \begin{pmatrix} -2 \\ 1 \end{pmatrix} .$$

So,  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$  is one eigenvector of  $A$  corresponding to  $-1$ .  $\left\{ k \begin{pmatrix} -2 \\ 1 \end{pmatrix} : k \neq 0 \right\}$  is eigenspace of  $A$  corresponding to  $-1$ .

**Practice:** Find eigenvalues of  $A = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}$  and the corresponding eigenvectors.

$$\det \left[ \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] = \det \begin{pmatrix} 5-\lambda & 3 \\ -3 & -1-\lambda \end{pmatrix}$$

$$= (5-\lambda)(-1-\lambda) + 9 = \lambda^2 - 4\lambda - 5 + 9 = \lambda^2 - 4\lambda + 4$$

$$= (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$$

multiplicity of  $\lambda = 2$  is 2

$$\begin{pmatrix} 5-2 & 3 \\ -3 & -1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ c \begin{pmatrix} 1 \\ -1 \end{pmatrix}, c \neq 0 \right\}$$

Strategy to find eigenvalues and eigenvectors of  $A_{n \times n}$ .

①  $A - \lambda I_n$

② characteristic polynomial

$$\det(\underline{A - \lambda I_n})$$

③ characteristic equation

$$\underline{\det(A - \lambda I_n)} = 0$$

④ Solve  $(A - \lambda I_n)\vec{x} = \vec{0}$  for each eigenvalue.

**Example** For  $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ , find its eigenvalues and eigenvectors. 3x3

①

$$\underline{A - \lambda I_3} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ & \lambda \\ 0 & & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}$$

② characteristic poly

$$\det(A - \lambda I_3) = (1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 0$$

$$= (1-\lambda) [(2-\lambda)(1-\lambda) - 1] + (\lambda - 1) + 0$$

$$\Rightarrow (1-\lambda) [\lambda^2 - 3\lambda + 1] + (\lambda - 1)$$

$$\Rightarrow (\lambda - 1) \underbrace{[-\lambda^2 + 3\lambda]} = \underbrace{(\lambda - 1)\lambda(3 - \lambda)}.$$

$$\begin{aligned} \det(A - \lambda I_3) &= \begin{vmatrix} 1 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{vmatrix} \\ &= (1 - \lambda) \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 1 - \lambda \end{vmatrix} + (-1)^3(-1) \begin{vmatrix} -1 & -1 \\ 0 & 1 - \lambda \end{vmatrix} + 0 \begin{vmatrix} -1 & 2 - \lambda \\ 0 & -1 \end{vmatrix} \\ &= (1 - \lambda)(\lambda^2 - 3\lambda) = (1 - \lambda)\lambda(\lambda - 3) \end{aligned}$$

③ characteristic equation

$$\begin{aligned} \det(A - \lambda I_3) &= 0 \\ (1 - \lambda)\lambda(\lambda - 3) &= 0 \\ \implies \lambda &= 0, 1, \text{ or } 3. \end{aligned}$$

④ Solve  $(A - \lambda I_3)\vec{x} = \vec{0}$  for each eigenvalue.

when  $\lambda = 0$

$$\begin{aligned} \underline{A\vec{x} = \vec{0}} &\implies \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\implies \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 + R_2 \rightarrow R_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} x_3 \text{ free} \\ \downarrow \\ \rightarrow x_1 = x_3 \\ x_2 = x_3 \end{matrix} \\ &\implies \begin{cases} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \end{cases} \implies \begin{cases} x_1 = x_2 = x_3 \\ x_2 = x_3 \end{cases} \end{aligned}$$



$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = K \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, K \neq 0$$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  is an eigenvector of  $A$  corresponding to  $\lambda = 0$ .

when  $\lambda = 1$  ..

when  $\lambda = 3$  ..

**Example**  $A = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}_{2 \times 2}$

①

$$\begin{aligned} A - \lambda I_2 &= \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 - \lambda & -1 \\ 4 & 1 - \lambda \end{pmatrix} \end{aligned}$$

②

$$\begin{aligned} \det(A - \lambda I_2) &= \begin{vmatrix} 1 - \lambda & -1 \\ 4 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 + 4 \\ &= \lambda^2 - 2\lambda + 1 + 4 \\ &= \lambda^2 - 2\lambda + 5. \end{aligned}$$

③  $\lambda^2 - 2\lambda + 5 = 0$ .

Recall:  $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Rightarrow (\lambda - 1)^2 = -4$   
 $\Rightarrow \lambda - 1 = \pm \sqrt{-4} = \pm 2i$   
 $\Rightarrow \lambda = 1 \pm 2i$

If  $A$  is not invertible, then  $0$  must be an eigenvalue.  $\Leftrightarrow \det A = 0$

$i^2 = -1$

$$a = 1, b = -2, c = 5$$

$$\begin{aligned}\lambda &= \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} \\ &= \frac{2 \pm \sqrt{-16}}{2} \underline{i^2 = -1} \frac{2 \pm 4i}{2} \\ &= \underline{1 \pm 2i}\end{aligned}$$

So the eigenvalues of  $A$  are  $\underline{1 + 2i}$  and  $\underline{1 - 2i}$ .

④ Solve  $(A - \lambda I_2)\vec{x} = \vec{0}$

when  $\lambda = 1 - 2i$ .

$$\begin{aligned}\begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2i & -1 & 0 \\ 4 & 2i & 0 \end{pmatrix} \xrightarrow{\frac{1}{2i}r_1} \begin{pmatrix} 1 & \frac{-1}{2i} & 0 \\ 4 & 2i & 0 \end{pmatrix} \xrightarrow[\frac{1}{i} = -i]{-4 \times r_1 + r_2} \begin{pmatrix} 1 & \frac{-1}{2i} & 0 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}x_1 - \frac{1}{2i}x_2 &= 0 \\ \implies x_1 + \frac{1}{2}ix_2 &= 0 \\ \implies x_1 &= \left(-\frac{1}{2}i\right)x_2\end{aligned}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}ix_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -\frac{1}{2}i \\ 1 \end{pmatrix}$$

$\begin{pmatrix} -\frac{1}{2}i \\ 1 \end{pmatrix}$  is one eigenvector of  $A$  corresponding to  $\lambda = 1 - 2i$ .

when  $\lambda = 1 + 2i$ .

Recall: Given  $A_{n \times n}$ . eigenvalue  $\lambda$ , eigenvector  $\vec{x} \neq 0$ : such that

$$A\vec{x} = \lambda\vec{x}.$$

Key idea

$$A\vec{x} = \lambda\vec{x} \iff (A - \lambda I_n)\vec{x} = \vec{0}.$$

characteristic poly in  $\lambda$ :  $\det(A - \lambda I_n)$

characteristic equation:  $\det(A - \lambda I_n) = 0$

### Remarks:

① eigenvalue and eigenvector are only defined for square matrix.

② eigenvalue could be complex scalar.

$P_n(\lambda) = 0$  must have.  
 $n$  solutions in account multiple  
and complex.

eg.  $A = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$ ,  $\lambda = 1 \pm 2i$ .

③ If  $A_{n \times n}$  is a singular matrix ( $\det A = 0$ ), then 0 must be one eigenvalue of  $A$ .  
not invertible

④ For a  $n \times n$  matrix  $A$ , it has at most  $n$  distinct eigenvalues, at most  $n$  linearly independent eigenvectors.

⑤ For one eigenvalue, it has infinite many eigenvectors, but only finitely many linearly independent eigenvectors.

eg.  $A = \begin{pmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$A$  has two eigenvalues  $\lambda_1 = 0$ ,  $\lambda_2 = 4$ .

For  $\lambda_1 = 0$ ,

$$\begin{aligned}
 (\underline{A} - \underline{0I_3}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \checkmark \\
 \Rightarrow \begin{pmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
 \rightarrow \begin{pmatrix} 3 & 3 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

basic free free

$$\underline{x_1 + x_2 + x_3 = 0} \Rightarrow \underline{x_1} = \underline{-x_2 - x_3}.$$

$$\begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$\underline{\vec{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = \underline{x_2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \underline{x_3} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

So for  $\underline{\lambda = 0}$ ,  $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  are two linearly independent eigenvectors.

$$\begin{array}{l}
 \nearrow \\
 x_2 = 1 \\
 x_3 = 0
 \end{array}
 \quad
 \begin{array}{l}
 \nearrow \\
 x_2 = 0 \\
 x_3 = 1
 \end{array}$$

$$\left\{ a \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

eigenspace of  $A$  for  $\lambda=0$

**Properties of eigenvalue and eigenvectors:** Let  $A$  be a  $n \times n$  matrix,  $\lambda$  is an eigenvalue of  $A$ ,  $\vec{v}$  is an eigenvector corresponding for  $\lambda$ . Then:

- ①  $A^k$  has eigenvalue  $\lambda^k$  with eigenvector  $\vec{v}$ .  *$k$  integer.*
- ②  $kA$  has eigenvalue  $k\lambda$  with eigenvector  $\vec{v}$ . *for any  $k \in \mathbb{R}$ .*
- ③  $A^{-1}$  (if  $A$  is invertible) has eigenvalue  $\frac{1}{\lambda}$ , with eigenvector  $\vec{v}$ .  *$\Rightarrow \lambda \neq 0$*
- ④  $A^T$  has eigenvalue  $\lambda$  with different eigenvector  $\vec{v}$ .
- ⑤  $A + kI_n$  has eigenvalue  $\lambda + k$  with eigenvector  $\vec{v}$ .

Proof:

① take  $k=3$  as example.

$$\begin{aligned}
 & A \vec{v} = \lambda \vec{v} \quad \checkmark \\
 & \xRightarrow{\text{multiply by } A} A(A \vec{v}) = A \cdot (\lambda \vec{v}) \\
 & \xRightarrow{\text{multiply by } A} A^2 \vec{v} = \lambda A \vec{v} = \lambda \cdot (\lambda \vec{v}) = \lambda^2 \vec{v} \\
 & \xRightarrow{\text{multiply by } A} A^3 \vec{v} = \lambda^2 A \vec{v} = \lambda^3 \vec{v}
 \end{aligned}$$

③

$$\begin{aligned}
 & A \vec{v} = \lambda \vec{v} \\
 & \xRightarrow{\text{multiply by } A^{-1}} A^{-1} A \vec{v} = \lambda A^{-1} \vec{v} \\
 & \xRightarrow{\quad} I_n \vec{v} = \vec{v} = \lambda A^{-1} \vec{v} \\
 & \xRightarrow{\quad} \frac{1}{\lambda} \vec{v} = A^{-1} \vec{v} \\
 & \frac{1}{\lambda} \text{ is eigenvalue of } A^{-1}
 \end{aligned}$$

eg.  $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

**Definition:** An  $n \times n$  matrix  $A$  is said to be diagonalizable if there is a non-singular (invertible)  $n \times n$  matrix  $P$  such that  $A = PDP^{-1}$ , where  $D$  is a diagonal matrix.

$\Leftrightarrow \det(P) \neq 0$   
 $\Leftrightarrow$  columns of  $P$  is linear indep.

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

**Remarks:**

- ①  $A$  and  $P^{-1}AP = D$  have the same characteristic polynomial thus the same eigenvalues.  
 $\det(A - \lambda I_n) = \det(PDP^{-1} - \lambda I_n) = \det(PDP^{-1} - \lambda PP^{-1})$   
 $= \det P [D - \lambda I] P^{-1}$   
 $= \det P \cdot \det(D - \lambda I_n) \det P^{-1}$   
 $= \det(D - \lambda I_n)$
- ② The eigenvalues of  $D$  are just the main diagonal entries.
- ③ Not every square matrix is diagonalizable.
- ④  $A = PDP^{-1}$  equivalent  $AP = PD$  equivalent  $D = P^{-1}AP$ .

**Example:** Show that  $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is not diagonalizable.

if  $A$  is diagonalizable  
 $\det \left[ A - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] = \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = \underline{(1-\lambda)^2}$   $\lambda=1$

so  $D$  has to be  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

But for any non-singular  $P$ ,  $PDP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq A$

So  $A$  can not be diagonalizable.

**Example:** Show that  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$  is diagonalizable.

$$A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$\underbrace{\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}}_P \quad \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}}_D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$

$$AP = PD$$

$\Leftrightarrow [A\vec{v}_1, A\vec{v}_2] = [\lambda_1\vec{v}_1, \lambda_2\vec{v}_2]$

Proof:

For  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ , we have eigenvalues  $\lambda_1 = 5, \lambda_2 = -1$  (they are distinct).

For  $\lambda_1 = 5$ , one of the corresponding eigenvector is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

For  $\lambda_2 = -1$ , one of the corresponding eigenvector is  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ .

Let  $P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ , as the eigenvectors of  $A$  are linearly independent,

$$P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1} \text{ exists and } P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}.$$

$$\text{Also } P^{-1}AP = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

It concludes that  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$  is diagonalizable.