EE3331 Probability Models in Information Engineering

Semester B 2021 – 2022

Test 2

12:00 p.m. – 1:30 p.m.

Answer **ALL SIX** questions:

Question 1

Suppose there are 11 identical boxes on a table and only one of them contains a diamond ring inside. You open the box(es) in any order, one by one, until the ring is found. Determine the expected number of box(es) to be opened. **(15 marks)**

Question 2

The joint probability distribution function (PDF) of two random variables X and Y has the form of:

$$P_{XY}(x,y) = \begin{cases} c, & x+y \leq 2,\ x \geq 0,\ y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the value of c. **(3 marks)**

(b) Determine the marginal PDFs of X and Y. **(6 marks)**

(c) Find $\mathbb{E}\{X\}$ and $\mathrm{var}(X)$. **(6 marks)**

(d) Find the correlation of X and Y. **(4 marks)**

(e) Determine the PDF of X conditioned on Y, denoted by $P_{X|Y}(x|y)$. **(3 marks)**

(f) Determine $P(X \leq Y)$. **(4 marks)**

(g) Determine $P(X+Y \leq 1)$. **(4 marks)**

Question 3

Let X be a random variable uniformly distributed on the interval $[-4, 1]$. A discrete random variable Y is generated based on X as:

$$Y = \begin{cases} 1, & \text{if } X \geq 0 \\ 0, & \text{if } X < 0 \end{cases}$$

Compute the probability mass function (PMF) of Y. **(10 marks)**

<u>Question 4</u>

X and Y are two independent uniform random variables, namely, $X \sim \mathcal{U}(0,2)$ and $Y \sim \mathcal{U}(-3,1)$. Let $Z = X + 2Y$.

(a) Compute $\mathbb{E}\{Z\}$. **(4 marks)**

(b) Compute $\mathbb{E}\{Z^2\}$. **(8 marks)**

(c) Compute $\mathbb{E}\{Z^3\}$. **(8 marks)**

<u>Question 5</u>

Consider an experiment of tossing a coin twice, and the probability of getting tail is p. Let X be the total number of tail(s) in the experiment, and let Y be the head number in the second toss. Compute the joint probability mass function (PMF) of X and Y, denoted by $P_{XY}(x,y)$. **(10 marks)**

<u>Question 6</u>

Let $X \sim \mathcal{N}(0, 0.5)$ be a Gaussian random variable. The probability density function (PDF) and cumulative distribution function (CDF) of X are denoted by $P_X(x)$ and $F_X(x)$, respectively. Based on X, a random variable Y is generated as $Y = X^2$.

(a) Write down the expressions of $P_X(x)$ and $F_X(x)$. **(3 marks)**

(b) Express the CDF of Y, denoted by $F_Y(y)$, in terms of $F_X(\cdot)$. **(4 marks)**

(c) By differentiating $F_Y(y)$, determine the PDF of Y, denoted by $P_Y(y)$.

Hint: $\frac{d}{dy}\left(\int_{-\infty}^{g(y)} f(u)du\right) = f(g(y)) \cdot \frac{dg(y)}{dy}$ where $f(\cdot)$ and $g(\cdot)$ are functions. **(8 marks)**