Pink

MA2001/MA2149/MA2170 **Mid-term Test** **Semester A, 2013/2014**

Name: ______________________________ Student ID: ________________

Course: MA2001 / MA2149 / MA2170 (Circle your course)

Answer ALL questions. (Full marks: 50)

Total Marks: ________

Q1: _____ Q2: _____

Q3: _____ Q4: _____

Question 1 (12 marks)

Suppose near the point $(x,y,u,v)=(1,1,1,1)$, we can solve $\begin{cases} xu+yvu^2=2 \\ xu^4+y^2v^3=2 \end{cases}$ uniquely for u and v as functions of x and y. Compute $\dfrac{\partial u}{\partial x}(1,1)$.

Question 2 (12 marks)

Find the quadratic surface approximation of $f(x, y) = \ln(x^2 + y^2)$ at $(-1, 0)$. Estimate $f(-0.9, 0.1)$ by the quadratic surface approximation.

Question 3 (10 marks)

Find the value I by changing the order of the integration in $I = \int_0^3 \int_0^{3x} (2 + x + xy)\,\mathrm{d}y\,\mathrm{d}x$.

Question 4 (16 marks)

Given a 3×3 real matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{pmatrix}$,

(a) find the eigenvalues of matrix A, and find the eigenvectors corresponding to each of these eigenvalues;

(b) show that there exists an invertible matrix P such that $P^{-1}AP$ gives a diagonal matrix D;

(c) calculate P^{-1} and $P^{-1}AP$.

[Hint: Use the three eigenvectors found in part (a) as the columns of P.]

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1. Question. Suppose that near the point $(x,y,u,v)=(1,1,1,1)$, we can solve $\begin{cases} xu+yvu^2=2 \\ xu^4+y^2v^3=2 \end{cases}$ uniquely for u and v as functions of x and y. Compute $\frac{\partial u}{\partial x}(1,1)$.

Solution. $\begin{cases} xu+yvu^2=2 \\ xu^4+y^2v^3=2 \end{cases} \Rightarrow \begin{cases} (u+xu_x)+y(v_xu^2+2uv\cdot u_x)=0 \\ (u^4+4xu^3\cdot u_x)+y^2\cdot 3v^2\cdot v_x=0 \end{cases}$ 6

$\Longrightarrow$ At $(x,y,u,v)=(1,1,1,1)$, we have $\begin{cases} 3u_x+v_x+1=0 \\ 4u_x+3v_x+1=0 \end{cases}$

$\Rightarrow \begin{cases} u_x=-\frac{2}{5} \\ v_x=\frac{1}{5} \end{cases} \Rightarrow \frac{\partial u}{\partial x}(1,1)=-\frac{2}{5}$ 6

2. Question. Find the quadratic surface approximation of $f(x,y)=\ln(x^2+y^2)$ at $(-1,0)$. Estimate $f(-0.9,0.1)$ by the quadratic surface approximation

Solution. $f(-1,0)=0$, $f_x=\frac{2x}{x^2+y^2} \Rightarrow f_x(-1,0)=-2$

$f_y=\frac{2y}{x^2+y^2} \Rightarrow f_y(-1,0)=0$

$f_{xx}=\frac{2(x^2+y^2)-2x\cdot 2x}{(x^2+y^2)^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2} \Rightarrow f_{xx}(-1,0)=-2.$

$f_{xy}=\frac{-2x\cdot 2y}{(x^2+y^2)^2}=\frac{-4xy}{(x^2+y^2)^2} \Rightarrow f_{xy}(-1,0)=0$

$f_{yy}=\frac{2x^2-2y^2}{(x^2+y^2)^2} \Rightarrow f_{yy}(-1,0)=2$ 6

$$\Rightarrow f(x,y)\approx P_2(x,y)=f(-1,0)+f_x(-1,0)(x+1)+f_y(-1,0)y$$
$$+\frac{1}{2}\left(f_{xx}(-1,0)(x+1)^2+2f_{xy}(-1,0)(x+1)y+f_{yy}(-1,0)y^2\right)$$
$$=-2(x+1)+\frac{1}{2}\left(-2(x+1)^2+2y^2\right)$$
$$=-2(x+1)-(x+1)^2+y^2$$
$$=-x^2-4x-3+y^2$$

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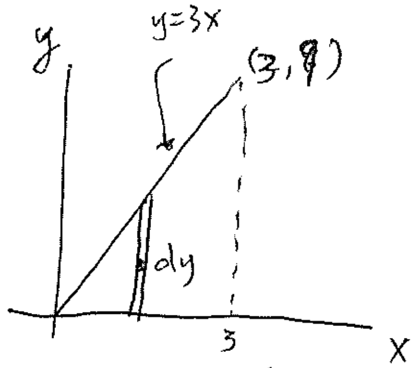
$$\Rightarrow f(-0.9,0.1)\approx P(-0.9,0.1)=-2\times 0.1-0.1^2+0.1^2=-0.2.$$

2

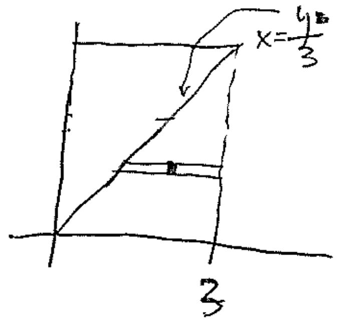
Q3 Find the value I by changing the order of the integration in

$$I = \int_{x=0}^{x=3} \int_{y=0}^{y=3x} (2 + x + xy)\, dy\, dx$$

Solutions



$$I = \int_{x=0}^{x=3} \int_{y=0}^{y=3x} (2+x+xy)\, dy\, dx$$



2 for figures

(5) $$= \int_{y=0}^{9} \int_{x=y/3}^{x=3} (2+x+xy)\, dx\, dy$$

$$= \int_{y=0}^{9} \left[2x + \frac{x^2}{2} + \frac{x^2 y}{2} \right]_{x=y/3}^{x=3} dy$$

$$= \int_{y=0}^{9} \left[2\times 3 + \frac{(3)^2}{2} + \frac{(3)^2 y}{2} \right] - \left[\frac{2y}{3} + \frac{(y/3)^2}{2} + \frac{(y/3)^2 y}{2} \right] dy$$

(3) $$= \int_{y=0}^{9} \left[\frac{21}{2} + \left(\underbrace{\frac{9y}{2} - \frac{2y}{3}}_{\frac{23y}{6}} \right) - \left(\frac{y^2}{18} + \frac{y^3}{18} \right) \right] dy$$

$$= \left[\frac{21}{2} y + \frac{23 y^2}{12} - \frac{y^3}{54} - \frac{y^4}{72} \right]_{y=0}^{9}$$

(2) $$= \left[\frac{21\times 9}{2} + \frac{23\times 9^2}{12} - \frac{9^3}{54} - \frac{9^4}{72} \right] = 145.125 = 145\tfrac{1}{8}$$

Q4

Mid-term test question. (16 marks)

Question: Given a 3×3 real matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{pmatrix}$.

a). find the eigenvalues of matrix A, and find the eigenvectors corresponding to each of these eigenvalues;

b). show that there exists an invertible matrix P such that $P^{-1}AP$ gives a diagonal matrix D;

c). Calculate P^{-1} and $P^{-1}AP$.

[Hint: Use the three eigenvectors found in part a) as the columns of P.

Sol^n. a).

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ -2 & 4-\lambda & 2 \\ -2 & 1 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)(3-\lambda)(6-\lambda).$$

Hence, we find that matrix A has two real eigenvalues, i.e. $\lambda_1 = 6$ and $\lambda_2 = \lambda_3 = 3$. (repeated).

For $\lambda = 6$: The augmented matrix can be written as:

$$\left(\begin{array}{ccc|c} 3-6 & 0 & 0 & 0 \\ -2 & 4-6 & 2 & 0 \\ -2 & 1 & 5-6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -2 & -2 & 2 & 0 \\ -2 & 1 & -1 & 0 \end{array}\right)$$

$$\overset{R_3-R_2}{\sim} \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$$

$$\overset{R_2+\frac{2}{3}R_3}{\sim} \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$$

$$\overset{R_1 - \frac{3}{2}R_2}{\sim} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right),$$

Hence, we have that $-2x_1 = 0 \Rightarrow x_1 = 0$

$3x_2 - 3x_3 = 0 \Rightarrow x_2 = x_3$.

We can then conclude that $E_6 = \begin{bmatrix} 0 & t & t \end{bmatrix}^T$, $\forall t \in \mathbb{R}$.

Let $t = 1$, $\vec{v}_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ is one of the eigenvectors of A with $\lambda = 6$.

For $\lambda = 3$: The augmented matrix can be written as:

$$\left(\begin{array}{ccc|c} 3-3 & 0 & 0 & 0 \\ -2 & 4-3 & 2 & 0 \\ -2 & 1 & 5-3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 \\ -2 & 1 & 2 & 0 \end{array}\right)$$

$$\overset{R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, we have that $-2x_1 + x_2 + 2x_3 = 0$,

We can then conclude that $E_3 = \begin{bmatrix} s & 2s - 2t & t \end{bmatrix}^T$, $\forall t \in \mathbb{R}$.

Let $s = 0$, $t = 1$, $\vec{v}_2 = \begin{bmatrix} 0 & -2 & 1 \end{bmatrix}^T$ is one of the eigenvectors of A with λ =

$s = 1$, $t = 0$, $\vec{v}_3 = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$ is one of the eigenvectors of A with λ =

b). We construct $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$, with $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ as its columns

By using the formula that $P^{-1} = \dfrac{(P_{jk})^T}{\det P}$, we have:

$$(P_{jk})^T = \begin{pmatrix} -2 & 1 & 2 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}, \quad \det P = 3.$$

$$\therefore \quad P^{-1} = \begin{pmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & -1/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix},$$ and hence we have:

$$P^{-1}AP = \begin{pmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & -1/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 & 4 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D.$$