<u>Solution</u>

1.
We use R and B to denote the red and black balls, respectively.

(a)
The sample space is {BBB, RBB, BRB, BBR, RRB, RBR, BRR, RRR}.

(b)
The probability is $\frac{\binom{2}{2}\binom{7}{1}}{\binom{9}{3}} = \frac{1}{12}$

2.
(a)
The sample space is {0, 50, 60, 30, 110, 80, 90, 140, 120,170}.

(b)
P(0)= 1/16; P(50)=1/16; P(60)= 2/16; P(30)= 2/16; P(110)= 2/16; P(90)= 2/16; P(80)= 2/16; P(140)= 2/16; P(120)= 1/16; P(170)= 1/16;

(c)
You need to wait for 110s in the first and second signals. If you spend 140s on waiting for traffic lights, then you should encounter one red signal between the last two, that is, 50+60+0+30 and 50+60+30+0. Hence the probability is 2/16.

3.

$$P(B \mid A \cup \overline{B}) = \frac{P(A \cap B)}{P(A \cup \overline{B})} = \frac{P(A) - P(A \cap \overline{B})}{P(A) + P(\overline{B}) - P(A \cap \overline{B})} = \frac{0.8 - 0.5}{0.8 + 0.6 - 0.5} = \frac{1}{3}.$$

4.

$$\begin{aligned}
& P(A \mid B) = P(A \mid \overline{B}) \\
& \Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap \overline{B})}{P(\overline{B})} \\
& \Rightarrow P(A \cap B)P(\overline{B}) = P(A \cap \overline{B})P(B) \\
& \Rightarrow P(A \cap B)[1 - P(B)] = [P(A) - P(A \cap B)]P(B) \\
& \Rightarrow P(A \cap B) = P(A)P(B)
\end{aligned}$$

Therefore, events A and B are independent.

5.
We first determine the probability that all 20 people do not have the same birthday, and we denote this event as D.

Among 20 people, we can first consider one of them and there is no restriction for his birthday. However, when we pick the second one, his birthday cannot be identical to the first one, which results in a probability of 364/365. For the third one, the probability is 363/365, and so on. Then the probability that all 20 people do not have the same birthday is:

$$P(D) = 1 \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{346}{365} = 0.5886$$

The required probability is then:

$$1 - P(D) = 0.4114$$

For the second question, it is required that $P(D) < 0.05$. Let N be the number of people required. We establish the following inequality:

$$P(D) = 1 \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{365 - (N-1)}{365} < 0.05$$

Solving it gives $N = 47$.

6.
Yes. Peter should switch his choice to maximize the probability of getting the iPhone.

Without loss of generality, assume that Peter chooses the first door. There are 3 equally likely possibilities for what is behind the 3 doors, namely, PBB, BPB, and BBP, where P and B denote the prize and balloons, respectively. It is clear that if Peter does not switch his choice, he has the winning probability of 1/3 (PBB). But if he switches, the probability will be increased to 2/3 (BPB or BBP), noting that the host must open the door corresponding to balloons.

7.
Firstly, we may need to list out the possible numbers of dices:
1,2,3 $\Rightarrow P(3,3) = 3! = 6$ permutations.
1,1,4 $\Rightarrow P(3,3)/P(2,2) = 3!/2! = 3$ permutations. The denominator of 2! is to account for the overcounting due to the two equivalent “1”.
2,2,2 $\Rightarrow P(3,3)/P(3,3) = 3!/3! = 1$ permutations. The denominator of 3! is to account for the overcounting due to the three equivalent “2”.
Hence the total number of dice permutations is 10. The probability is $\frac{10}{6^3} = \frac{10}{216} = \frac{5}{108}$.

8.
Use Mathematical Induction, we obtain:

(1) When $N=1$, $C(1,0)+C(1,1)=2^1=2$, satisfied.

(2) Assuming that when $N=n$, we have $\sum_{k=0}^{n} C(n,k)=2^n$.

(3) When $N=n+1$,

$$
\begin{aligned}
&C(n+1,0)+C(n+1,1)+C(n+1,2)+\cdots+C(n+1,n)+C(n+1,n+1)\\
&=C(n+1,0)+[C(n,0)+C(n,1)]+[C(n,1)+C(n,2)]+\cdots+[C(n,n-1)+C(n,n)]+C(n+1,n+1)\\
&=(C(n,0)+C(n,1)+\cdots+C(n,n))+(C(n,0)+C(n,1)+\cdots+C(n,n))\\
&=2\times 2^n\\
&=2^{n+1}
\end{aligned}
$$

Based on (1) to (3), the equality is proved.

9.
The probability of selecting box B conditioned to continuously draw k red balls is

$$P(B\,|\,kR)=\frac{P(kR\,|\,B)P(B)}{P(kR\,|\,A)P(A)+P(kR\,|\,B)P(B)},$$

where $P(A)=P(B)=0.5$, $P(kR\,|\,A)=\left(\frac{2}{5}\right)^k$, $P(kR\,|\,B)=\left(\frac{2}{3}\right)^k$, then we have

$$P(B\,|\,kR)=\frac{P(kR\,|\,B)P(B)}{P(kR\,|\,A)P(A)+P(kR\,|\,B)P(B)}=\frac{5^k}{3^k+5^k}.$$

10.
Assigning $A=\{\text{study hard}\}$ and $B=\{\text{pass}\}$, then we have $P(A)=0.8$, $P(\overline{A})=0.2$, $P(B\,|\,A)=0.95$, $P(\overline{B}\,|\,\overline{A})=0.9$, then we can find

(a)

$$
\begin{aligned}
P(\overline{A}|B)&=\frac{P(\overline{A}\cap B)}{P(B)}=\frac{P(\overline{A})P(B|\overline{A})}{P(A)P(B|A)+P(\overline{A})P(B|\overline{A})}\\
&=\frac{0.2\times 0.1}{0.8\times 0.95+0.2\times 0.05}=\frac{1}{38.5}=0.026
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(A|\overline{B})&=\frac{P(A\cap\overline{B})}{P(\overline{B})}=\frac{P(A)P(\overline{B}|A)}{P(A)P(\overline{B}|A)+P(\overline{A})P(\overline{B}|\overline{A})}\\
&=\frac{0.8\times 0.1}{0.8\times 0.05+0.2\times 0.95}=\frac{4}{11.5}=0.3478
\end{aligned}
$$