Take Home Assignment MA2001 #1

For each of the following questions, write down your solution with details of steps. Marks will not given if only final answers are provided.

1. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$. (Hint: $3, 6$ are eigenvalues of A).

2. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$.

3. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$, whose characteristic polynomial is $(\lambda+1)^2(\lambda+5)(\lambda-3)$.

4. It is given the symmetric matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.
   (a) find the eigenvalues of A;
   (b) find the eigenvectors corresponding to each of these eigenvalues;
   (c) find an orthogonal matrix P such that $P^\top AP$ gives a diagonal matrix D and calculates P^{-1};
   (d) Determine the eigenvalues of the matrix $B = A^5 + (A^2)^\top$.

5. If A is a $n \times n$ matrix and $\{\lambda_1, \ldots, \lambda_k\}$ are its eigenvalues, show that the eigenvalues of $\alpha I + A$, where I is the identity matrix and α is a scalar, are $\{\lambda_1 + \alpha, \ldots, \lambda_k + \alpha\}$.

6. A quadratic form Q in the components $x_1, \ldots, x_n$ of a vector $\vec{x} = [x_1, \ldots, x_n]^\top$ with symmetric coefficient matrix $A = (a_{ij})_{1 \le i,j \le n}$ is defined to be

$$Q(\vec{x}) := \vec{x}^\top A\vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

Determine whether each of the following quadratic forms in two variables is positive or negative definite or semidefinite, or indefinite.

(a) $3x_1^2 + 8x_1x_2 - 3x_2^2$.

(b) $9x_1^2 + 6x_1x_2 + x_2^2$.

7. Determine the values of a for which the quadratic form $2x^2 + 2axy + 2xz + y^2 + z^2$ is positive definite.

8. **Discovery Question**. Read "https://en.wikipedia.org/wiki/Gram-Schmidt process" to use the Gram-Schmidt process to find an orthogonal basis spanning the same space of $\mathbb{R}^n$ as the given of vectors:

(a) $< 1, 4, 0 >, < 2, -5, 0 >$ in $\mathbb{R}^3$.

(b) $< 0, 2, 1, -1 >, < 0, -1, 1, 6 >, < 0, 2, 2, 3 >$ in $\mathbb{R}^4$.