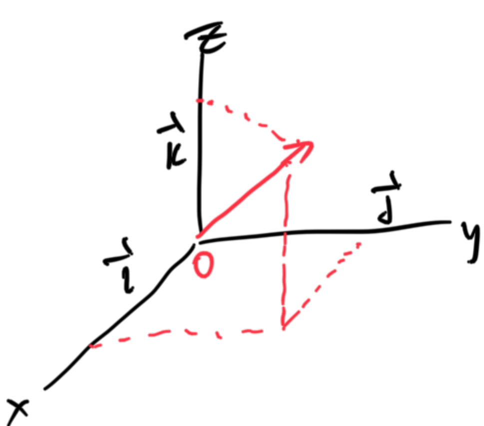
Preliminary

Vectors

Scalar	Vector
- magnitude	- magnitude and direction
ie. tempt, speed, etc	$\vec{i}, \vec{j}, \vec{k}$ ← notation
	ie. velocity, acceleration, force.



position vector $\overrightarrow{OA}$

displacement vector $\overrightarrow{AB}$

Vector form of a line segment.

if. $\vec{r_0}$ vector in 2-space or 3-space w/ its initial point at the origin, then the line that passes through the terminal point of $\vec{r_0}$ and is parallel to the vector $\vec{v}$ can be expressed in the vector form

$$\vec{r} = \vec{r_0} + t\vec{v}$$

$$\vec{r} = \vec{r_0}(1-t)\vec{r_0} + t\vec{r_1}$$

Vector Algebra.

① Addition

② [illegible] law

(2) associative law

$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + C$

(3) commutative law $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

(4) multiplication by a scalar

(5) Distributive law

$k(\vec{B} + \vec{C}) = k\vec{B} + k\vec{C}$

(6) Unit vectors $\vec{i}, \vec{j}, \vec{k}$

$\vec{A} = \langle A_1, A_2, A_3 \rangle = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

$\vec{i} = \langle 1, 0, 0 \rangle \quad \vec{j} = \langle 1, 0, 1 \rangle, \quad \vec{k} = \langle 0, 0, 1 \rangle$

** unit vector $\vec{u}$ of $\vec{v}$: $\hat{u} = \dfrac{\hat{v}}{|\hat{v}|}$

(7) Dot product (scalar)

$\vec{A} \cdot \vec{B} = |\hat{A}||\hat{B}| \cos\theta_{AB}$

$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

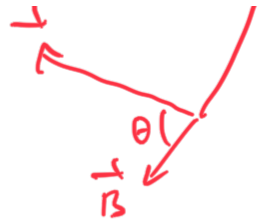
$\vec{A} \cdot (\vec{B} \cdot \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

$\vec{A} \cdot \vec{B} = A_1B_1 + A_2B_2 + A_3B_3$

(8) cross product (vector)

$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta_{AB}$

$\vec{A} \times \vec{B}$



$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A} \neq \vec{B}\times\vec{A}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

<u>Introduction</u> vector function – domain: set of real numbers
Range : vectors.

$\vec{r}(t) = \langle \underline{f(t)}, \underline{g(t)}, \underline{h(t)} \rangle$ — component functions of $\vec{r}$
t - time (Independent variable)

** If Domain are intervals of real numbers then the vector functions represent a space curve.

If domain are regions in the plane, the vector functions represent surfaces in space.

... moving through space during a

Ex consider a particle moving through space ... time interval

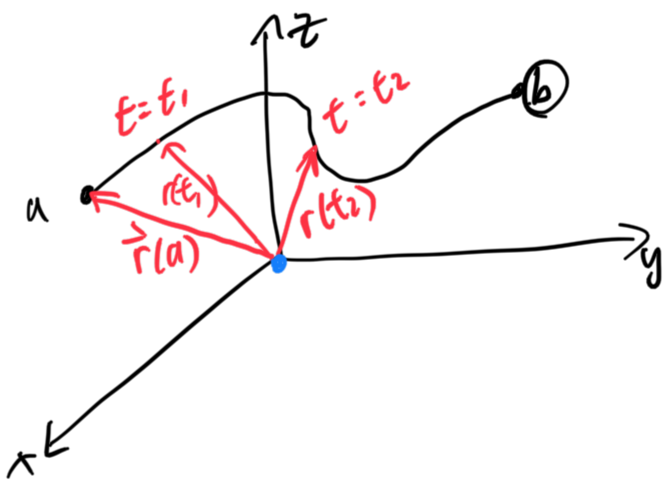
$x = f(t) \quad y = g(t) \quad z = h(t) \quad t \in I$

the points of $(x, y, z) = (f(t), g(t), h(t))$ make up the curve in space - particle's path.

Ex $x = f(t) \quad y = g(t) \quad z = h(t)$ ← parameterize the curve

curve in vector form

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ ← particle's position vector



$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$a \le t \le b$

Theorem

let $\vec{F}$ and $\vec{G}$ be vector functions of the real variable t and let $f(t)$ be a scalar function

① $(\vec{F} + \vec{G})(t) = \vec{F}(t) + \vec{G}(t)$

② $(f\vec{F})(t) = f(t)\,\vec{F}(t)$

③ $(\vec{F} \times \vec{G})(t) = \vec{F}(t) \times \vec{G}(t)$ ← cross product

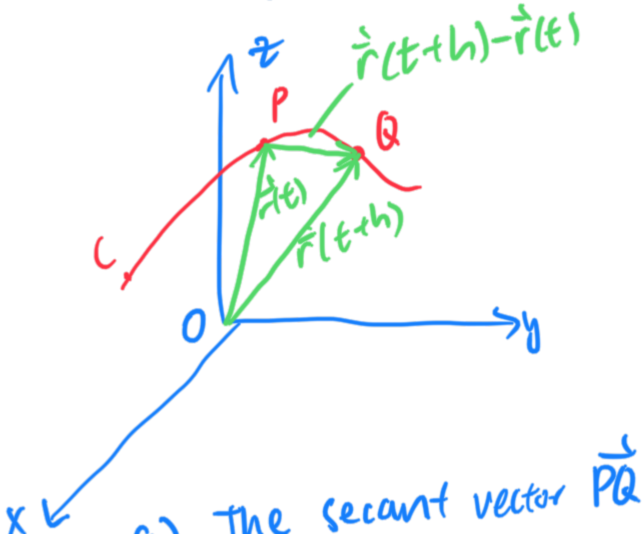
④ $(\vec{F}\cdot\vec{G})(t) = \vec{F}(t)\cdot\vec{G}(t)$ ← dot product.

Vector derivatives

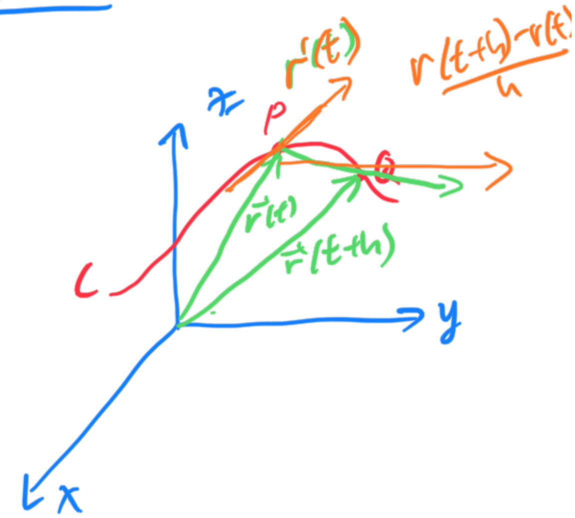
$$\frac{d\vec{F}}{dt} = \vec{F}'(t) = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t}$$

where $\vec{F}(t) = \langle f(t), g(t), h(t) \rangle$

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$



a) The secant vector $\vec{PQ}$



b) tangent vector $r'(t)$

Tangent line to C at P is defined to be the line through P parallel to the tangent vector $r'(t)$

(unit Tangent vector) → $\vec{T}(t) = \dfrac{r'(t)}{|r'(t)|}$

Theorem

The vector function

$\vec{F}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is differentiable whenever the component function $f(t)$, $g(t)$, and $h(t)$ are all differentiable.

i.e.

$$\vec{F}'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

Ex. (a) Find derivative of $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \sin 2t\,\vec{k}$

(b) Find unit tangent vector at the point where $t=0$

(a) $\vec{r}'(t) = 3t^2\vec{i} + (1-t)e^{-t}\vec{j} + 2\cos 2t\,\vec{k}$

(b) $T(0) = \dfrac{r'(0)}{|r'(0)|} = \quad r'(0) = \dfrac{0\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{0^2+1^2+2^2}} = \dfrac{1}{\sqrt{5}}\vec{j} + \dfrac{2}{\sqrt{5}}\vec{k}$

remark: same idea as last year tangent line / Given point

Higher Vector Derivatives

$\vec{F}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ differentiating successively !!

$\vec{F}'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$

$\vec{F}''(t) = f''(t)\vec{i} + g''(t)\vec{j} + h''(t)\vec{k}$

$\vdots$

$r''(t)$ let's say we have $r(t) = \langle 2\cos t, \sin t, t\rangle$

$r'(t) = \langle -2\sin t, \cos t, 1\rangle$

$r''(t) = \langle -2\cos t, -\sin t, 0\rangle$

Differentiation rule.

$\vec{F}$ and $\vec{G}$ are differentiable vector functions

... is a scalar and f is a real valued and c is a ...

function. Then

① $\frac{d}{dt}[\vec{F}(t)+\vec{G}(t)] = \vec{F}'(t)+\vec{G}'(t)$

② $\frac{d}{dt}[c\vec{F}(t)] = c\vec{F}'(t)$

③ $\frac{d}{dt}[f(t)\vec{F}(t)] = f'(t)\vec{F}(t) + f(t)\vec{F}'(t)$ ← times

④ $\frac{d}{dt}[\vec{F}(t)\cdot\vec{G}(t)] = \vec{F}'(t)\cdot\vec{G}(t) + \vec{F}(t)\cdot\vec{G}'(t)$ ← dot

⑤ $\frac{d}{dt}[\vec{F}(t)\times\vec{G}(t)] = \vec{F}'(t)\times\vec{G}(t) + \vec{F}(t)\times\vec{G}'(t)$ ← cross

⑥ $\frac{d}{dt}[\vec{F}(f(t))] = f'(t)\vec{F}'(f(t))$ ← chain

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partial derivatives of multivariable vector function

suppose $\vec{R}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is differentiable

functions of n variables, $t_1, t_2, t_3 \dots t_n$

Then the partial derivative of $\vec{R}(t)$

$$\frac{\partial}{\partial t_1}\vec{R}(t) = \frac{\partial f}{\partial t_1}\vec{i} + \frac{\partial g}{\partial t_1}\vec{j} + \frac{\partial h}{\partial t_1}\vec{k}$$

$$\vdots$$

$$\frac{\partial}{\partial t_n}\vec{R}(t) = \frac{\partial f}{\partial t_n}\vec{i} + \frac{\partial g}{\partial t_n}\vec{j} + \frac{\partial h}{\partial t_n}\vec{k}$$ ← general solution

$$\frac{\partial^2 \vec{R}(t)}{\partial t_1 \partial t_m} = \frac{\partial^2 f}{\partial t_1 \partial t_m}\vec{i} + \frac{\partial^2 g}{\partial t_1 \partial t_m}\vec{j} + \frac{\partial^2 h}{\partial t_1 \partial t_m}\vec{k}$$

Insert Eg.

Vector Integral

let $\vec{F}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

f, g and h are continuous functions for

$a \le t \le b$.

① Definite Integral $\vec{F}(t)$

$$\int_a^b \vec{F}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\vec{i} + \left[\int_a^b g(t)\,dt\right]\vec{j} + \left[\int_a^b h(t)\,dt\right]\vec{k}$$

② Indefinite integral of $\vec{F}(t)$ is the vector function

$$\int \vec{F}(t)\,dt = \left[\int f(t)\,dt\right]\vec{i} + \left[\int g(t)\,dt\right]\vec{j} + \left[\int h(t)\,dt\right]\vec{k}$$

$$\int_a^b r(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} r(t_i)\Delta t$$

$$= \lim_{n\to\infty}\left[\left(\sum_{i=1}^{n} f(t_i)\Delta t\right)\vec{i} + \left(\sum_{i=1}^{n} g(t_i)\Delta t\right)\vec{j} + \left(\sum_{i=1}^{n} h(t_i)\Delta t\right)\vec{k}\right]$$

Application Fundamental theorem of Calculus

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

note: $R'(t) = \underline{\vec{r}(t)}$

Ex. $r(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$

$$\int \vec{r}(t)\,dt = \left(\int 2\cos t\,dt\right)\vec{i} + \left(\int \sin t\,dt\right)\vec{j} + \left(\int 2t\,dt\right)\vec{k}$$

$$= 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k} + C$$

$$\int_0^{\pi/2} \vec{r}(t)\,dt = 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k}\Big|_0^{\pi/2}$$

$$= 2\sin\left(\frac{\pi}{2}\right)\vec{i} - \cos\left(\frac{\pi}{2}\right)\vec{j} + \left(\frac{\pi}{2}\right)^2\vec{k} -$$

$$2\sin(0)\vec{i} - \cos(0)\vec{j} + 0^2\vec{k}$$

$$= 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}$$