2 Linear Time-Invariant Systems

Major References:

- *Chapter 2, Signals and Systems by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall*
- *Chapter 2, Schaum's Outline of Signals and Systems, 2nd Edition, 2010, McGraw-Hill*

2.1 Convolution

2.1.1 Convolution Integral of CT Signal

1. **Definition**
   Convolution Integral of two continuous-time signals $x(t)$ and $y(t)$ is defined by

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau. \tag{2.1}$$

   Convolution $x(t) * y(t)$ represents the degree to which $x \& y$ overlap at t as y sweeps across the domain t.

   Step. 1) $y(\tau)$ is time-reversed, then shifted by t; $y(\tau) \to y(-\tau) \to y(t-\tau)$
   Step. 2) $x(\tau)$ and $y(t-\tau)$ are multiplied, then integrated over τ
   Step. 3) Convolution will remain zero as long as $x \& y$ do not overlap
   Step. 4) Sweep $y(t-\tau)$ from $t=-\infty$ to $t=\infty$ to produce the entire output

2. **Properties of the Convolution Integral**
   The convolution integral has the following properties. Refer [Schaum's text, Problem 2.1] for the proof.

   a) Commutative

$$x(t) * y(t) = y(t) * x(t)$$

   b) Associative

$$\{x(t) * y_1(t)\} * y_2(t) = x(t) * \{y_1(t) * y_2(t)\}$$

$$= \int_{-T/2}^{T/2} x(\tau) y(t-\tau) d\tau$$

   c) Distributive

$$x(t) * \{y_1(t) + y_2(t)\} = x(t) * y_1(t) + x(t) * y_2(t)$$

3. **Additional Properties**
   Refer [Schaum's text, Problem 2.2, 2.8] for the proof.

$$\int_0^T x(\tau)\, y(t-\tau)\, d\tau = \int_a^{a+T} x(\tau) y(t-\tau) d\tau$$

   a) $x(t) * \delta(t) = x(t)$
   b) $x(t) * \delta(t-t_0) = x(t-t_0)$
   c) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$
   d) $x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau) d\tau$
   e) If $x(t)$ and $y(t)$ are periodic signals with a common period T, the convolution in (2.1) does not converge. Instead, we define the *periodic convolution* $f(t) = x(t) \circledast y(t)$, where $f(t)$ is periodic with period T.

$$\begin{aligned} f(t) = x(t) \circledast y(t) &= \int_0^T x(\tau)\, y(t-\tau)\, d\tau \\ &= \int_a^{a+T} x(\tau)\, y(t-\tau)\, d\tau \quad \text{for arbitrary } a \end{aligned} \tag{2.2}$$

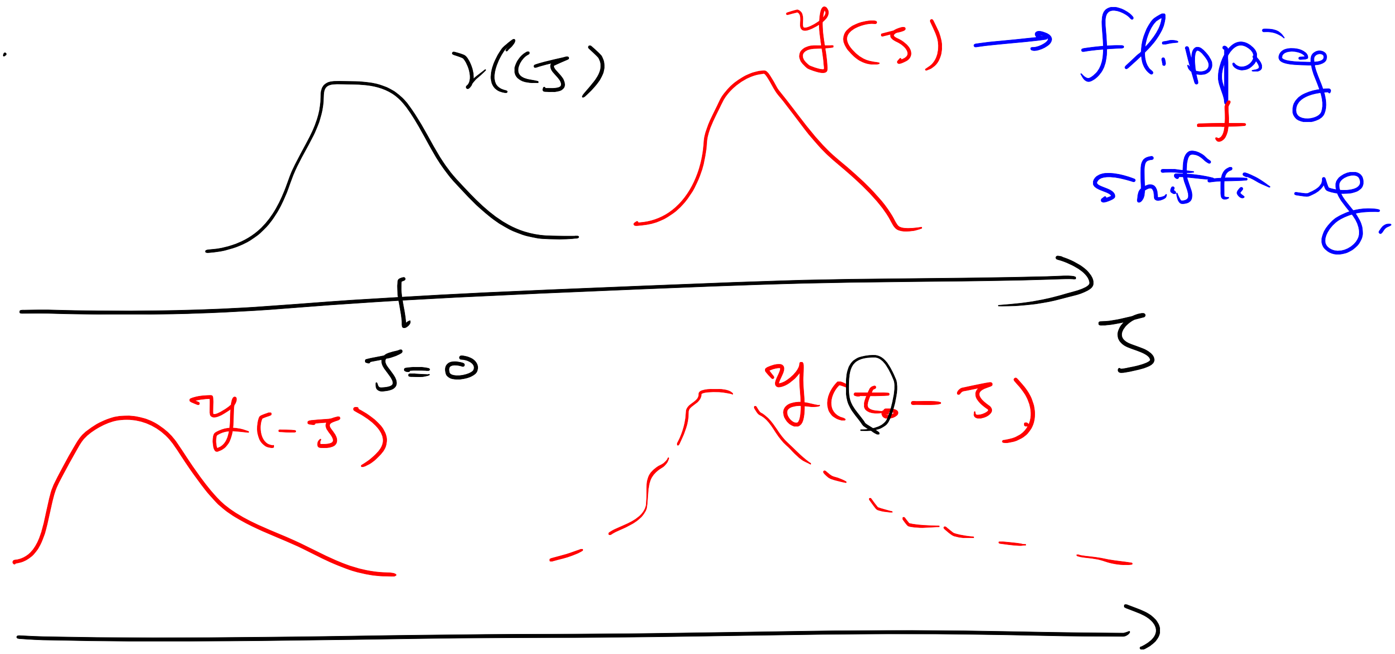
Convolution → CT / DT

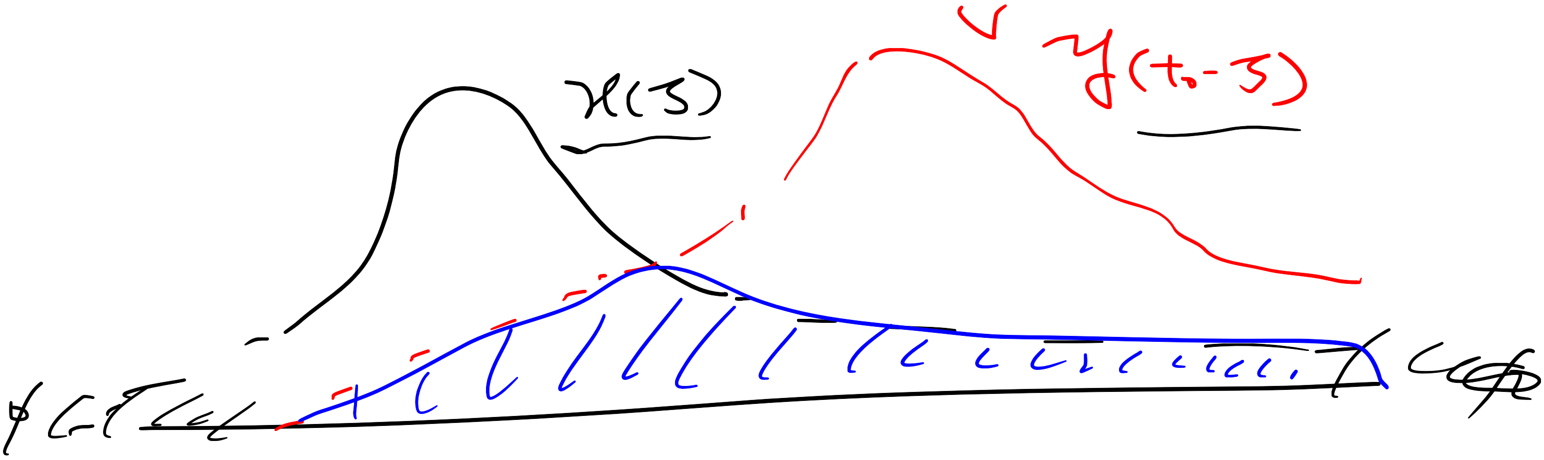
$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

if) $\boxed{t = t_0}$

$$z(t_0) = \int_{-\infty}^{\infty} x(\tau)\, \underbrace{y(t_0-\tau)}\, d\tau$$

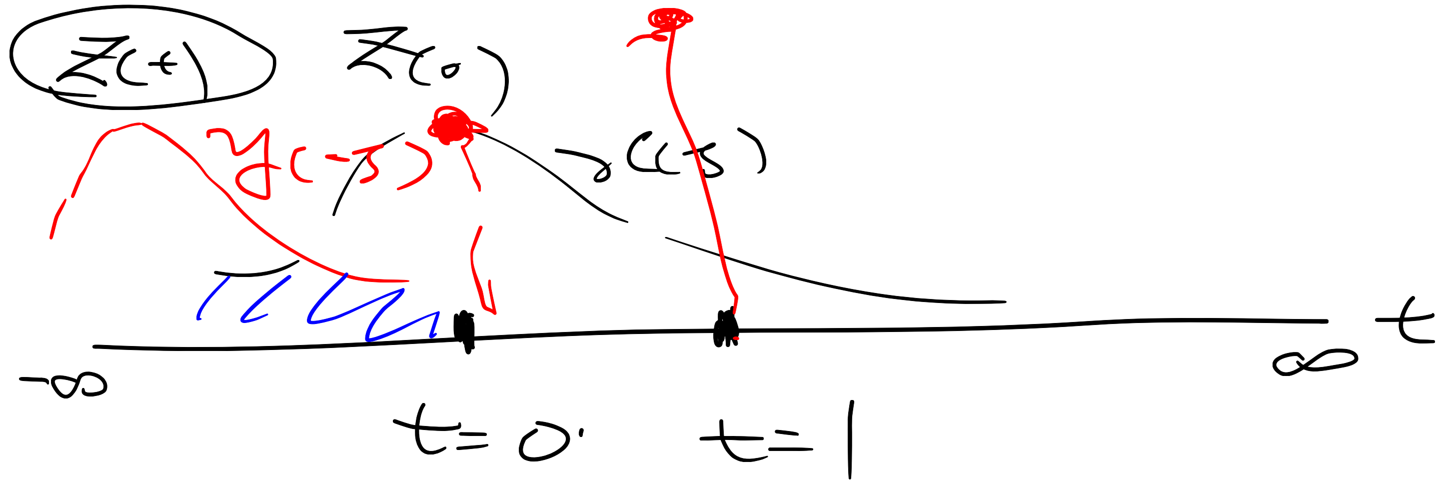
$z(0)$.

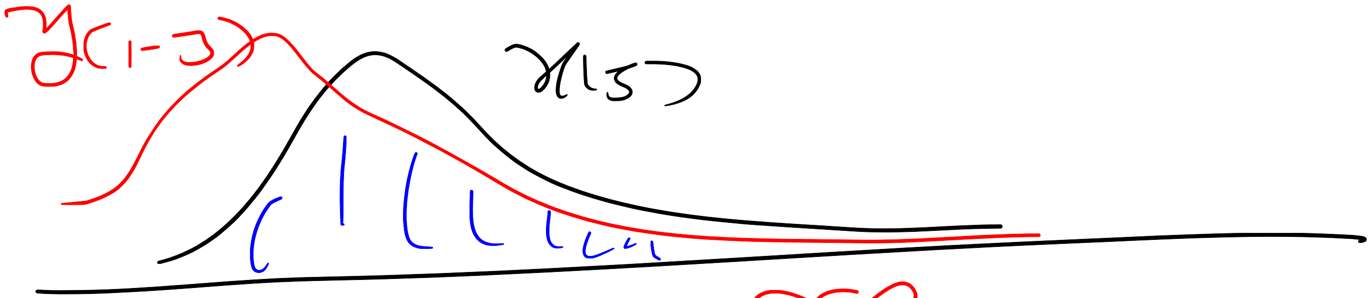




$$z(0) = \int_{-\infty}^{\infty} x(\tau)\, y(-\tau)\, d\tau$$

$$z(1) = \int_{-\infty}^{\infty} x(\tau)\, y(1-\tau)\, d\tau$$





$t = -\infty$

$$\int_{-\infty}^{\infty} x(\tau)\, y(-\infty-\tau)$$

$$= z(-\infty)$$

$t = \infty$

$$\int_{-\infty}^{\infty} x(\tau)\, y(\infty-\tau)$$

$$z(t) = \boxed{\int_{-\infty}^{\infty} x(\tau)\, y(\underbrace{t-\tau}_{= u})\, d\tau}$$

$$= \int_{-\infty}^{\infty} y(\tau)\, x(t-\tau)\, d\tau.$$

$$t - \tau = u$$

$$\tau = t - u$$

$$\boxed{d\tau = -du}$$

τ	u
$-\infty$	$t+\infty = \infty$
∞	$t - \infty$
	$= -\infty$

$$-\int_{\infty}^{-\infty} x(t-u)\, y(u)\, du$$

$$= \int_{-\infty}^{\infty} y(u)\, x(t-u)\, du$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$\|$$

$$y(t) * x(t) = \int_{-\infty}^{\infty} y(\tau)\, x(t-\tau)\, d\tau$$

$$\|$$

$$z(t).$$

Associative.

$$(x(t) * y(t)) * z(t)$$

$$= x(t) * (y(t) * z(t))$$

Distributive.

$$x(t) * (y_1(t) + y_2(t))$$

$$= (x(t) * y_1(t)) + (x(t) * y_2(t))$$

Properties

1) $\boxed{x(t) * \delta(t) = x(t)}$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$= x(t) * \delta(t)$$

2) $x(t) * \delta(t-t_0) = \boxed{x(t-t_0)}$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\boxed{\tau})\, d\tau$$

$$\int_{-\infty}^{\infty} x(\tau)\, \delta(t-t_0-\tau)\, d\tau$$

$$= x(t-t_0).$$

Ex 2-1)

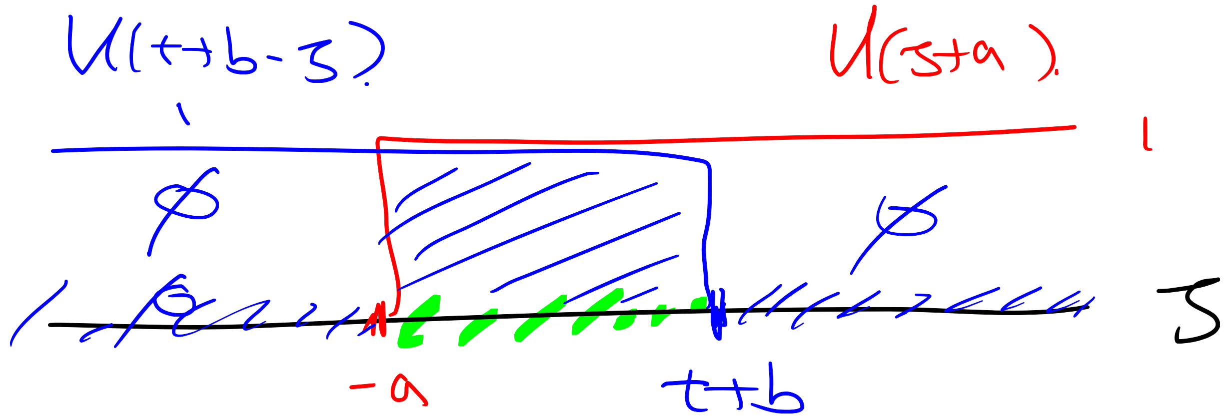
a) $$u(t+a) * u(t+b)$$

$$= \int_{-\infty}^{\infty} u(\tau+a)\, u(t+b-\tau)\, d\tau$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

1 if $\tau+a>0$, $\tau>-a$

1 if $t+b-\tau>0$; $\tau<t+b$

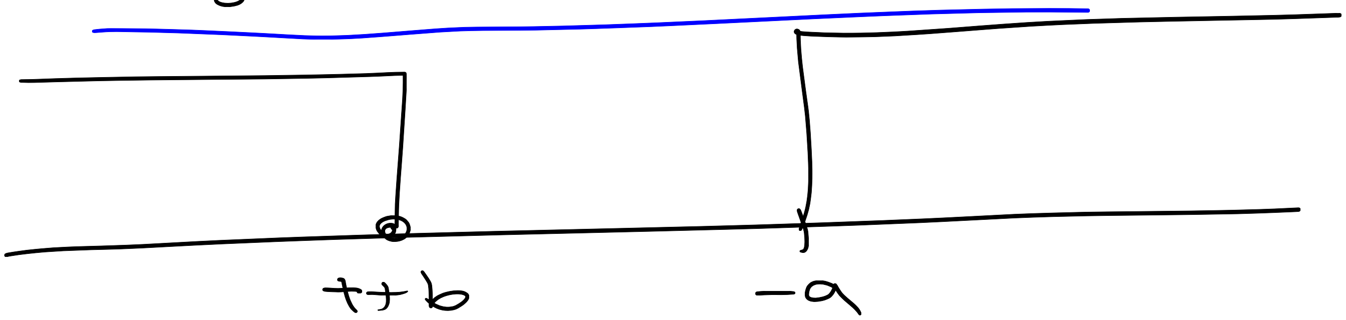


if $t+b > -a \iff \boxed{t+b+a > 0}$

$$U(\zeta+a)\,U(t+b-\zeta) = 1$$

if $t+b \le -a$ $\quad \boxed{t+b+a \le 0}$

$$U(\zeta+a)\,U(t+b-\zeta) = 0$$



$$\int_{-\infty}^{\infty} U(\zeta+a)\,U(t+b-\zeta)\,d\zeta$$

$$= \begin{cases} \text{if } \boxed{t+b+a>0} & \boxed{\int_{-a}^{t+b} 1\cdot d\zeta = t+b-(-a) = t+b+a} \\ \text{if } \boxed{t+b+a\le 0}, & \phi \end{cases}$$

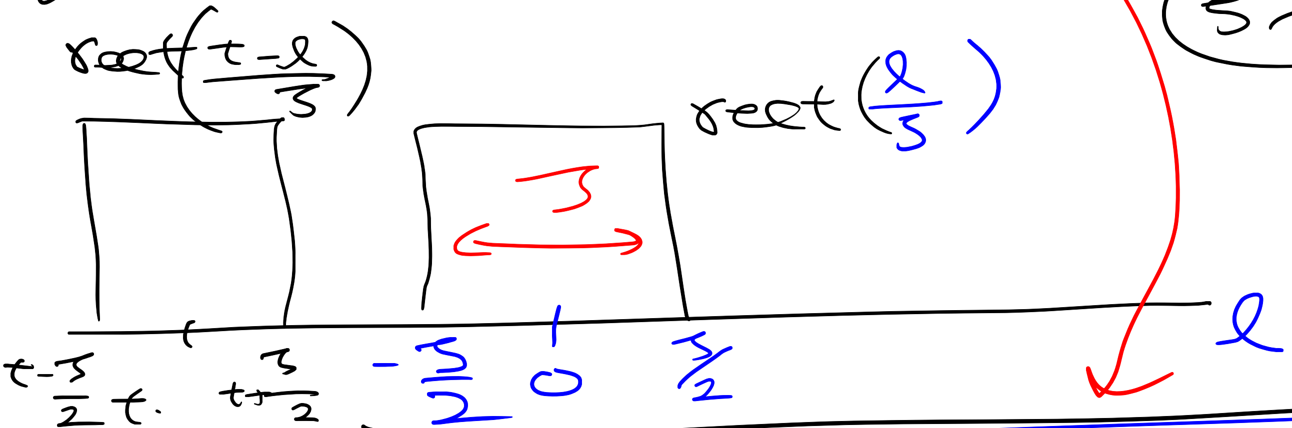
$$= u(t+b+a)(t+b+a)$$

$$u(t+a) * u(t+b)$$
$$= (t+a+b)\, u(\boxed{t+b+a})$$

Ex 2-1)

$$\mathrm{rect}\left(\frac{t}{5}\right) * \mathrm{rect}\left(\frac{t}{5}\right)$$

$5>0.$



$$\left(u\left(t+\frac{3}{2}\right) - u\left(t-\frac{3}{2}\right)\right) * \left(u\left(t+\frac{3}{2}\right) - u\left(t-\frac{3}{2}\right)\right)$$

$$= \overbrace{u\left(t+\tfrac{5}{2}\right)}^{t+a} * \overbrace{u\left(t+\tfrac{5}{2}\right)}^{t+b} \rightarrow (t+5)\,u(t+5)$$

$$- \underbrace{u\left(t+\tfrac{5}{2}\right)}_{t+a} * \underbrace{u\left(t-\tfrac{5}{2}\right)}_{t+b} \rightarrow -t\,u(t)$$

$$- \underbrace{u\left(t-\tfrac{5}{2}\right)}_{t+a} * \underbrace{u\left(t+\tfrac{5}{2}\right)}_{t+b} \rightarrow -t\,u(t)$$

$$+ u\left(t-\tfrac{5}{2}\right) * u\left(t-\tfrac{5}{2}\right) \rightarrow (t-5)\,u(t-5)$$

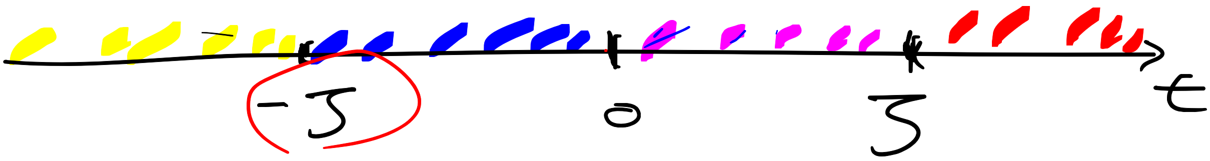
$a = -\frac{5}{2}$ $\quad b = -\frac{5}{2}$

$$(t+5)\,u(t+5) - 2t\,u(t) + (t-5)\,u(t-5)$$

Case 1) if $t < -5 \Rightarrow \phi$

Case 2) if $-5 < t < 0 \Rightarrow (t+5)$

Case 3) if $0 < t < 5 \Rightarrow (t+5)\cdot 1 - 2t\cdot 1$



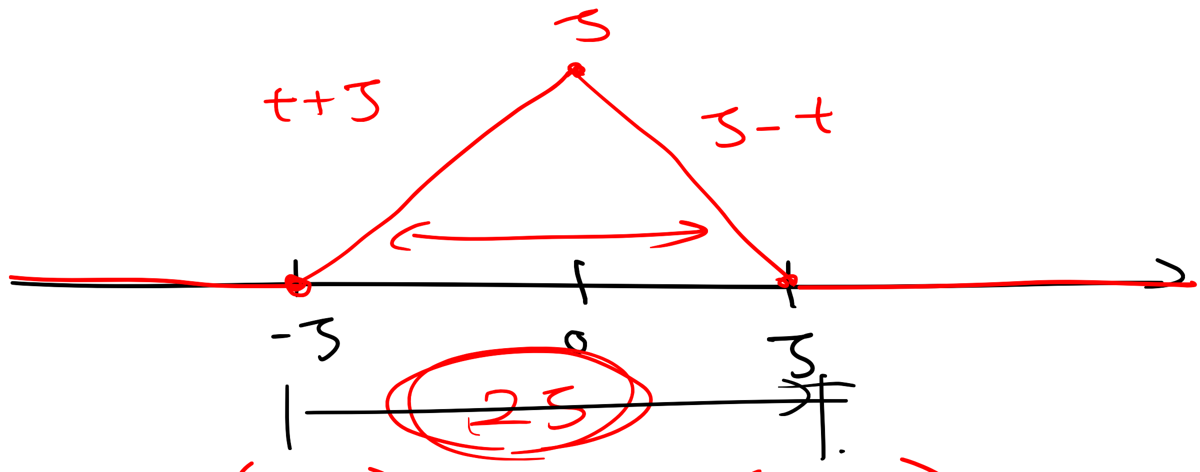
$\Rightarrow t+5-2t$

$\Rightarrow 5-t$

Case 4) if $t > 5 \Rightarrow$

$\Rightarrow (t+5) - 2t + (t-5)$

$= \phi$



$$\mathrm{rect}\left(\frac{t}{5}\right) * \mathrm{rect}\left(\frac{t}{5}\right).$$

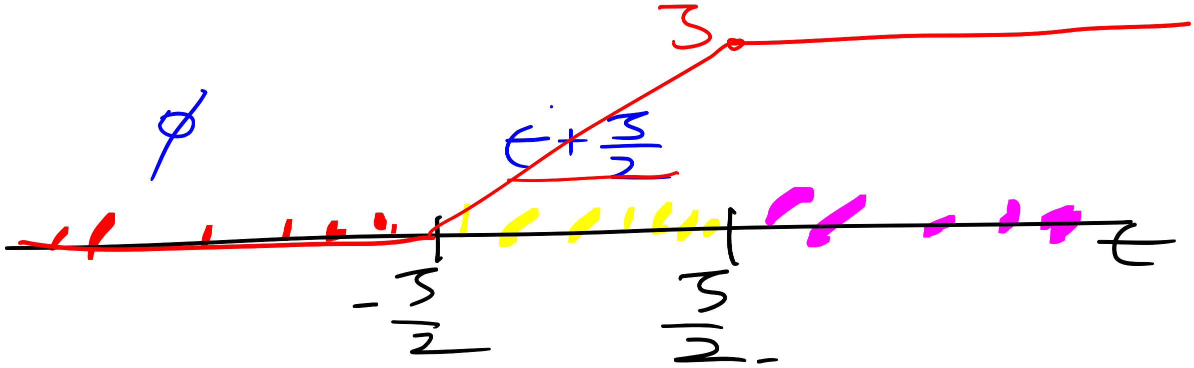
Ex 2-1)

3) $\boxed{\mathrm{rect}\left(\frac{t}{5}\right)} * u(t)$

$$\Rightarrow \left(u\left(t+\frac{5}{2}\right) - u\left(t-\frac{5}{2}\right)\right) * u(t).$$

$$\Rightarrow \underbrace{u\left(t+\frac{5}{2}\right) * u(t)}_{a=+\frac{5}{2} \quad b=0} - \underbrace{u\left(t-\frac{5}{2}\right) * u(t)}_{a=-\frac{5}{2} \quad b=0}$$

$$\boxed{\left(t+\frac{5}{2}\right) u\left(t+\frac{5}{2}\right)} - \boxed{\left(t-\frac{5}{2}\right) u\left(t-\frac{5}{2}\right)}$$



if $-\frac{5}{2} < t < \frac{5}{2}$, $t+\frac{5}{2}$

if $t > \frac{5}{2}$, $\cancel{t}+\frac{5}{2} - \left(\cancel{t}-\frac{5}{2}\right) = 5$

$$x(t) * \delta(t) = x(t)$$

$$\boxed{x(t) * u(t)} = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

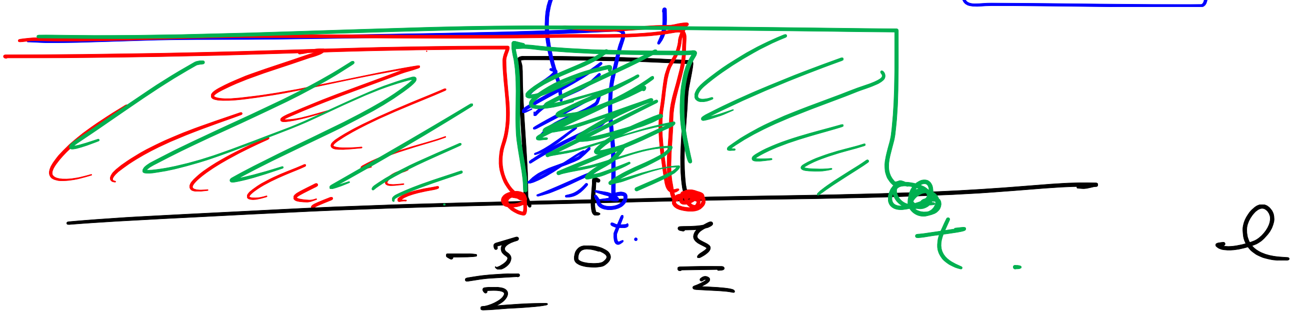
0 if $\tau > t$

$=1$ if $t-\tau > 0 \iff \tau < t$

$$= \boxed{\int_{-\infty}^{t} x(\tau)\, d\tau}$$

$$\mathrm{rect}\left(\frac{t}{\tau}\right) * u(t) = \int_{-\infty}^{t} \mathrm{rect}\left(\frac{l}{\tau}\right) dl$$

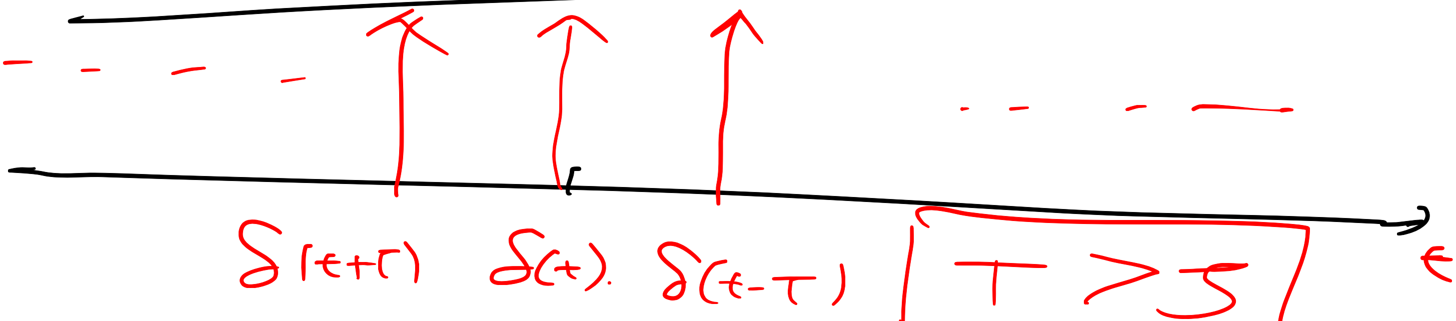
$$t - \left(-\frac{\tau}{2}\right) = \boxed{t + \frac{\tau}{2}}$$

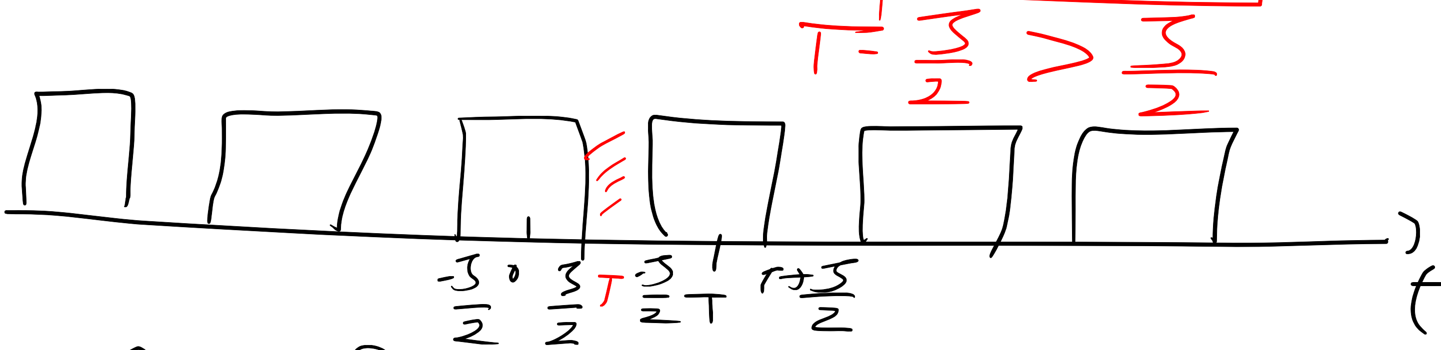


$$x(t) * u(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$$

$$\boxed{\mathrm{rect}\left(\frac{t}{5}\right) * \delta_T(t)}$$

$$\boxed{\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)}$$





$$\mathrm{rect}\left(\frac{t}{5}\right) * \sum_{n=-\infty}^{\infty} \delta(t-nT) \qquad nT = t_0$$

$$= \sum_{n=-\infty}^{\infty} \left[\underbrace{\mathrm{rect}\left(\frac{t}{5}\right)} * \underbrace{\delta(t-nT)} \right]$$

$$x(t) * \delta(t-t_0)$$

$$= x(t-t_0)$$

$$= \mathrm{rect}\left(\frac{t-nT}{5}\right)$$

$$= \boxed{\sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-nT}{5}\right)}$$