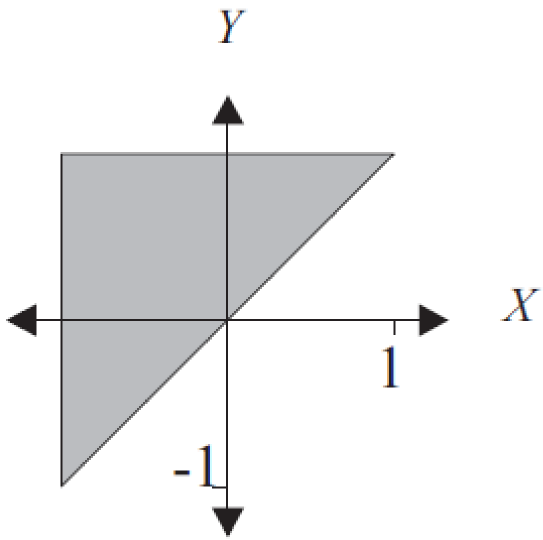
Tutorial 9

1. The joint probability density function (PDF) of random variables X and Y is given as:

$$P_{XY}(x,y)=\begin{cases}1/2, & -1\le x\le y\le 1\\ 0, & \text{otherwise}\end{cases}$$



(a) Compute $\mathbb{E}\{XY\}$.
(b) Compute $\mathbb{E}\{e^{X+Y}\}$.

2. Consider an observation x which is of the form:

$$x = A + n$$

where A is a constant to be estimated and $n \sim \mathcal{N}(0, \sigma^2)$. It is suggested to estimate A using $\hat{A}$:

$$\hat{A} = x$$

Compute the mean of the estimate $\mathbb{E}\{\hat{A}\}$ and mean square error (MSE) $\mathbb{E}\{(\hat{A} - A)^2\}$.

Suppose now the noise is changed to $n \sim \mathcal{U}(0, 1)$. Determine an unbiased estimate of A and then compute the corresponding MSE.

3. The joint PDF of random variables X and Y is given as:

$$P_{XY}(x,y) = ce^{-\frac{x^2}{8}-\frac{y^2}{18}}$$

where c is a constant.

(a) Are X and Y independent? Briefly explain your answer.
(b) Determine the marginal PDFs of X and Y.
(c) Find the value of c.

4. Consider the experiment of rolling a fair dice and let random variable X denote the outcome which is the face number. Find the conditional PMF of X given that we know the observed number is less than 5.

5. Consider tossing a coin and the probability of getting head is p. The coin is repeatedly tossed until two consecutive heads occur. Let X be the total number of coin tosses. Based on (2.6), it is suggested that the probability mass function (PMF) of X equal to:

$$p(r) = P(X = r) = (1 - p)^{r-2} p^2, \quad 2 \leq r < \infty$$

Do you agree? Explain your answer.

6. Consider tossing a coin and the probability of getting head is p. The coin is repeatedly tossed until two consecutive heads occur. Let X be the total number of coin tosses. Determine $\mathbb{E}\{X\}$.

Solution

1.(a)

$$\mathbb{E}\{XY\} = \int_{-1}^{1}\int_{x}^{1} \frac{xy}{2} dy dx = \int_{-1}^{1} \frac{x(1-x^2)}{4} dx = \frac{x^2}{8} - \frac{x^4}{16}\Big|_{-1}^{1} = 0$$

1.(b)

$$\mathbb{E}\{e^{X+Y}\} = \int_{-1}^{1}\int_{x}^{1} \frac{e^x e^y}{2} dy dx$$

$$= \int_{-1}^{1} \frac{e^x(e - e^x)}{2} dx$$

$$= \frac{e^{1+x}}{2} - \frac{e^{2x}}{4}\Big|_{-1}^{1}$$

$$= \frac{e^2}{4} + \frac{e^{-2}}{4} - \frac{1}{2}$$

2.

We can follow Example 3.18 to obtain the results:

$$\mathbb{E}\{\hat{A}\} = \mathbb{E}\{x\} = \mathbb{E}\{A+n\} = A + \mathbb{E}\{n\} = A + 0 = A$$

$$\mathrm{MSE}(\hat{A}) = \mathbb{E}\{(x-A)^2\} = \mathbb{E}\{(A+n-A)^2\} = \mathbb{E}\{n^2\} = \sigma^2$$

Alternatively, we notice that $\hat{A} = x$ is also a random variable, that is, $x \sim \mathcal{N}(A, \sigma^2)$:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-A)^2}$$

The PDF reaches the maximum value at $x = A$, implying that $\hat{A} = x$ is a reasonable choice to estimate A. From $x \sim \mathcal{N}(A, \sigma^2)$, we directly obtain $\mathbb{E}\{\hat{A}\} = \mathbb{E}\{x\} = A$ and $\mathrm{MSE}(\hat{A}) = \mathrm{var}(\hat{A}) = \mathrm{var}(x) = \sigma^2$.

For $n \sim \mathcal{U}(0,1)$, it has a mean of 0.5. Hence an unbiased estimate of A is:

$$\hat{A} = x - 0.5$$

We can easily check that

$$\mathbb{E}\{\hat{A}\} = \mathbb{E}\{x - 0.5\} = \mathbb{E}\{A + n - 0.5\} = A + \mathbb{E}\{n\} - 0.5 = A + 0.5 - 0.5 = A$$

$$\mathrm{MSE}(\hat{A}) = \mathbb{E}\{(x - 0.5 - A)^2\} = \mathbb{E}\{(A + n - 0.5 - A)^2\} = \mathbb{E}\{(n - 0.5)^2\}$$

If we write $m = n - 0.5$, it is clear that $m \sim \mathcal{U}(-0.5, 0.5)$. That is, the MSE is the second moment of m. Recalling Example 2.22, $\mathbb{E}\{m^2\}$ is:

$$\mathbb{E}\{m^2\} = \int_{-0.5}^{0.5} x^2 dx = \left.\frac{x^3}{3}\right|_{-0.5}^{0.5} = \frac{1}{12}$$

3.(a)

We observe that the joint PDF can be factorized as:

$$P_{XY}(x,y)=ce^{-\frac{x^2}{8}-\frac{y^2}{18}}=c_1e^{-\frac{x^2}{8}}\cdot c_2e^{-\frac{y^2}{18}},\quad c=c_1\cdot c_2$$

where

$$P_X(x)=c_1e^{-\frac{x^2}{8}},\quad P_Y(y)=c_2e^{-\frac{y^2}{18}}$$

As $P_{XY}(x,y)=P_X(x)P_Y(y)$, X and Y are independent.

3.(b)

The forms of $P_X(x)$ and $P_Y(y)$ correspond to Gaussian random variables, e.g.,

$$P_X(x)=\frac{1}{\sqrt{2\pi\sigma_x^2}}e^{-\frac{1}{2\sigma_x^2}(x-\mu_x)^2}$$

Equating

$$P_X(x) = c_1 e^{-\frac{x^2}{8}} = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{1}{2\sigma_x^2}(x-\mu_x)^2}$$

we easily obtain $\mu_x = 0$ and $\sigma_x = 2$.

Similarly, we get $\mu_y = 0$ and $\sigma_y = 3$.

The values of c_1 and c_2 are:

$$c_1 = \frac{1}{2\sqrt{2\pi}}, \quad c_2 = \frac{1}{3\sqrt{2\pi}} \Rightarrow P_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}x^2}, \quad P_Y(y) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{18}y^2}$$

3.(c)

$$c = c_1 \cdot c_2 = \frac{1}{12\pi}$$

4.

For a fair dice, the probability is the same for all face numbers:

$$P(X=1)=P(X=2)=\cdots=P(X=6)=\frac{1}{6}$$

On the other hand, the given information corresponds to an event, say, $A=\{X<5\}$ or $A=\{X=1,2,3,4\}$.

Hence we have:

$$P(A)=\frac{4}{6}=\frac{2}{3}$$

The conditional PMF is thus:

$$P_{X|A}(x)=\begin{cases}(1/6)/(4/6)=0.25, & x=1,2,3,4\\ 0, & \text{otherwise}\end{cases}$$

5.

The suggested PMF assumes that the first $(r-2)$th trials must correspond to tails only. However, non-consecutive heads can be included, and this probability is not included. Hence the PMF is not correct.

That is, we can see the possibilities include:

HH + THH + HTHH + TTHH + HTTHH + …

6.

Let $\mu = \mathbb{E}\{X\}$. We can follow Example 4.6 and let the events of having a head and tail be H and T, respectively. Clearly, $P(H) = p$ and $P(T) = 1 - p$.

We first condition on the result of the first coin toss:

$$\begin{aligned} \mathbb{E}\{X\} &= \mathbb{E}\{X|H\}P(H) + \mathbb{E}\{X|T\}P(T) \\ &= pE\{X|H\} + (\mu + 1)\cdot(1-p) \Rightarrow p\mu = pE\{X|H\} + (1-p) \end{aligned}$$

To find $\mathbb{E}\{X|H\}$, we need to condition on the result of the second coin toss:

$$\begin{aligned} \mathbb{E}\{X|H\} &= \mathbb{E}\{X|HH\}P(H) + \mathbb{E}\{X|HT\}P(T) \\ &= 2p + (2+\mu)(1-p) = 2 + (1-p)\mu \end{aligned}$$

Note that for $\mathbb{E}\{X|HT\}$, because the first two tosses are HT, we have wasted two coin tosses and we start over at the third toss, resulting in 2 and $\mu = \mathbb{E}\{X\}$.

As a result, we get:

$$\mu = p(2 + (1-p)\mu) + (1-p) \Rightarrow \mu = \mathbb{E}\{X\} = \frac{1+p}{p^2}$$