Take Home Assignment MA2001 #2

For each of the following questions, write down your solution with details of steps. Marks will not be given if only final answers are provided.

1. Suppose $3z^3 - 2yz + x^2 = 2$ determines the function $z = z(x,y)$ as a function of x, y locally at $(x,y,z) = (1,1,1)$.

   (a) Find the linear approximation of z at $(x,y,z) = (1,1,1)$.

   (b) Find the quadratic surface approximation of z at $(x,y,z) = (1,1,1)$.

   Solution. (a) At $(x,y) = (1,1)$, we have $z(1,1) = 1$. By taking the differentiation with respect to x and y on both side of $3z^3 - 2yz + x^2 = 2$, we get

$$\begin{cases} 9z^2 z_x - 2yz_x + 2x = 0, \\ 9z^2 z_y - 2yz_y - 2z = 0, \end{cases}$$

   we have

$$\begin{cases} z_x = \frac{-2x}{9z^2-2y}, \\ z_y = \frac{2z}{9z^2-2y}. \end{cases} \tag{1}$$

   At $(x,y,z) = (1,1,1)$, we get the evaluation of (1) as

$$z_x(1,1) = -\frac{2}{7}, \quad z_y(1,1) = \frac{2}{7}.$$

   Hence, the linear approximation of z at $(x,y,z) = (1,1,1)$ is given by

$$z(x,y) \approx z(1,1) + z_x(1,1)(x-1) + z_y(1,1)(y-1) = 1 - \frac{2}{7}(x-1) + \frac{2}{7}(y-1) = 1 - \frac{2}{7}x + \frac{2}{7}y.$$

   (b) By (1), we have

$$\begin{cases} z_{xx} = \frac{-2(9z^2-2y)+2x(18zz_x)}{(9z^2-2y)^2}, \\ z_{xy} = \frac{2x(18zz_y-2)}{(9z^2-2y)^2}, \\ z_{yx} = \frac{2z_x(9z^2-2y)-2z(18zz_x)}{(9z^2-2y)^2}, \\ z_{yy} = \frac{2z_y(9z^2-2y)-2z(18zz_y-2)}{(9z^2-2y)^2}. \end{cases}$$

   Evaluating at $(x,y,z) = (1,1,1)$, we obtain

$$z_{xx}(1,1) = -\frac{170}{343}, z_{xy}(1,1) = z_{yx}(1,1) = \frac{44}{343}, z_{yy}(1,1) = -\frac{16}{343}.$$

Consequently, the quadratic surface approximation of z at $(x, y, z) = (1, 1, 1)$ is

$$\begin{aligned}
z(x,y) &\approx z(1,1) + z_x(1,1)(x-1) + z_y(1,1)(y-1) \\
&\quad + \frac{1}{2!}[z_{xx}(1,1)(x-1)^2 + 2z_{xy}(1,1)(x-1)(y-1) + z_{yy}(1,1)(y-1)^2] \\
&= 1 - \frac{2}{7}x + \frac{2}{7}y + \frac{1}{2}\left[-\frac{170}{343}(x-1)^2 + \frac{88}{343}(x-1)(y-1) - \frac{16}{343}(y-1)^2\right] \\
&= \frac{6}{7} + \frac{4}{49}x + \frac{10}{49}y - \frac{85}{343}x^2 + \frac{44}{343}xy - \frac{8}{343}y^2.
\end{aligned}$$

2. It is given that $f(x, y) = e^{2x}\sin(2y)$.

   (a) Use Taylor's formula to find a linear approximation of $f(x, y)$ at the origin.

   (b) Estimate the error in the linear approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

Solution. By Taylor's formula,

$$f(x,y) = e^{2x}\sin(2y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + R_2(x,y),$$

where

$$R_2(x,y) = \frac{1}{2!}(x^2 f_{xx}(cx,cy) + 2xy f_{xy}(cx,cy) + f_{yy}(cx,cy)y^2), \quad 0 < c < 1.$$

(a) We have

$$\begin{aligned}
f(0,0) &= 0, \\
f_x(0,0) &= 2e^{2x}\sin(2y)\Big|_{x=0,y=0} = 0, \\
f_y(0,0) &= 2e^{2x}\cos(2y)\Big|_{x=0,y=0} = 2.
\end{aligned}$$

Hence, the linear approximation of f is

$$f(x,y) \approx p_1(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = 2y.$$

(b) We have

$$f_{xx}(x,y) = 4e^{2x}\sin(2y), \quad f_{xy}(x,y) = 4e^{2x}\cos(2y), \quad f_{yy}(x,y) = -4e^{2x}\sin(2y).$$

Hence, by $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$ for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
&|f(x,y) - p_1(x,y)| \\
=&|R_2(x,y)| \\
=&\left|\frac{1}{2!}\left[4x^2e^{2cx}\sin(2cy) + 8xye^{2cx}\cos(2cy) - 4y^2e^{2cx}\sin(2cy)\right]\right| \\
\leq&\frac{1}{2}\left[4|x|^2|e^{2cx}||\sin(2cy)| + 8|x||y||e^{2cx}||\cos(2cy)| + 4|y|^2|e^{2cx}||\sin(2cy)|\right] \\
\leq&\frac{1}{2}(4|x|^2|e^{2cx}| + 8|x||y||e^{2cx}| + 4|y|^2|e^{2cx}|) \\
\leq&2(|x|^2 + 2|x||y| + |y|^2)\cdot e^{2cx} \\
\leq&2(|x| + |y|)^2e^{0.2} \\
\leq&2(0.2)^2e^{0.2} = 0.08e^{0.2}.
\end{aligned}$$

3. Find the stationary points of the function $f(x,y) = xye^{-2(x^2+y^2)}$ and determine their nature.

   Solution. From,

$$f_x = y(1-4x^2)e^{-2(x^2+y^2)} = 0, \quad f_y = x(1-4y^2)e^{-2(x^2+y^2)} = 0.$$

   which is equivalent to solving $y(1-4x^2) = 0$ and $x(1-4y^2) = 0$. We get

$$\begin{cases} y = 0 \text{ or } x = \pm\frac{1}{2}, \\ x = 0 \text{ or } y = \pm\frac{1}{2}. \end{cases}$$

   Hence, stationary points are $(0,0), (\frac{1}{2},\frac{1}{2}), (\frac{1}{2},-\frac{1}{2}), (-\frac{1}{2},\frac{1}{2}), (-\frac{1}{2},-\frac{1}{2})$.

   Note that

$$\begin{aligned}
f_{xx} &= 4xy(4x^2-3)e^{-2(x^2+y^2)}, \\
f_{yy} &= 4xy(4y^2-3)e^{-2(x^2+y^2)}, \\
f_{xy} &= (1-4x^2)(1-4y^2)e^{-2(x^2+y^2)}.
\end{aligned}$$

   Then,

$$D = f_{xx}f_{yy} - f_{xy}^2 = e^{-4(x^2+y^2)}[16x^2y^2(4x^2-3)(4y^2-3) - (1-4x^2)^2(1-4y^2)^2].$$

   We have Table 1 showing the nature of the stationary points.

stationary point	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$	Nature
$(0,0)$	0	0	1	$-1 < 0$	saddle point
$(\frac{1}{2}, \frac{1}{2})$	$\frac{-2}{e} < 0$	$\frac{-2}{e}$	0	$\frac{4}{e^2} > 0$	local max.
$(\frac{1}{2}, -\frac{1}{2})$	$\frac{2}{e} > 0$	$\frac{2}{e}$	0	$\frac{4}{e^2} > 0$	local min.
$(-\frac{1}{2}, \frac{1}{2})$	$\frac{2}{e} > 0$	$\frac{2}{e}$	0	$\frac{4}{e^2} > 0$	local min.
$(-\frac{1}{2}, -\frac{1}{2})$	$\frac{-2}{e} < 0$	$\frac{-2}{e}$	0	$\frac{4}{e^2} > 0$	local max.

Table 1: Table for Q3

4. Let $f(x,y) = x^2 - xy + y^2 - y$. Find the directions $\vec{u}$ and the values of $D_{\vec{u}}f(1,-1)$ for which

   (a) $D_{\vec{u}}f(1,-1)$ is the largest;

   (b) $D_{\vec{u}}f(1,-1)$ is the smallest;

   (c) $D_{\vec{u}}f(1,-1) = 0$;

   (d) $D_{\vec{u}}f(1,-1) = 4$;

   (e) $D_{\vec{u}}f(1,-1) = -3$.

Solution. Let $\vec{u} = (a,b)^\top$ be a unit vector; i.e., $a^2 + b^2 = 1$. Since

$$\nabla f = (f_x, f_y)^\top = (2x - y, 2y - x - 1)^\top.$$

Hence

$$\nabla f(1,-1) = (3,-4)^\top.$$

Therefore,

$$D_{\vec{u}}f(1,-1) = \nabla f(1,-1) \cdot \vec{u}$$

Note that the gradient ∇f points to the direction where the function changes the most.

(a) When $\vec{u}$ has the same direction as ∇f, $D_{\vec{u}}f$ is the largest. Hence

$$\vec{u} = \frac{\nabla f(1,-1)}{|\nabla f(1,-1)|} = (\tfrac{3}{5}, -\tfrac{4}{5})^\top$$

and

$$D_{\vec{u}}f(1,-1) = |\nabla f(1,1)| = 5.$$

(b) When $\vec{u}$ has the opposite direction as ∇f, $D_{\vec{u}}f$ is the smallest. Hence

$$\vec{u} = -\frac{\nabla f(1,-1)}{|\nabla f(1,-1)|} = (-\tfrac{3}{5}, \tfrac{4}{5})^\top.$$

and

$$D_{\vec{u}}f(1,-1) = -|\nabla f(1,1)| = -5.$$

(c) When $\vec{u}$ is orthogonal to ∇f, $D_{\vec{u}}f = 0$. Hence

$$\vec{u} \perp \nabla f(1,-1) \Longrightarrow \vec{u} = \pm(\tfrac{4}{5}, \tfrac{3}{5})^\top.$$

(d) $D_{\vec{u}}f = 4 = \nabla f(1,-1) \cdot \vec{u}$ implies $3a - 4b = 4$. Together with $a^2 + b^2 = 1$, we get

$$\vec{u} = (0,-1)^\top \text{ or } \vec{u} = (\frac{24}{25}, -\frac{7}{25})^\top.$$

(e) $D_{\vec{u}}f = -3 = \nabla f(1,-1) \cdot \vec{u}$ implies $3a - 4b = -3$. Together with $a^2 + b^2 = 1$, we get

$$\vec{u} = (-1,0)^\top \text{ or } \vec{u} = (\frac{7}{25}, \frac{24}{25})^\top.$$

5. **Discovery Question**. (Read Lecture Note Chapter 2 Page 46). Consider a point $P(x_0, y_0)$ and a parabola $y = ax^2 + bx + c$. The value

$$[y_0 - (ax_0^2 + bx_0 + c)]^2$$

is called the square of the vertical displacement of the data point $P(x_0, y_0)$ from the parabola $y = ax^2 + bx + c$. Determine the parabola $y = ax^2 + bx + c$ such that the sum S of the squares of the vertical displacements of the data points

$$\begin{aligned} P_1(x_1, y_1) &= P_1(1, 1.2), \\ P_2(x_2, y_2) &= P_2(-1, 1.4), \\ P_3(x_3, y_3) &= P_3(2, 4.2), \\ P_4(x_4, y_4) &= P_4(-2, 4.4) \end{aligned}$$

from the parabola is the smallest. In other words, determine the values of a, b, c such that S is the smallest where

$$S(a,b,c) = \sum_{i=1}^{4} (y_i - ax_i^2 - bx_i - c)^2.$$

Hint: Solve the question by the following steps:

(a) Find the only stationary point (a_0, b_0, c_0) of S.

(b) Use the Taylor series for triple variables, you can expand S at (a_0, b_0, c_0) as

$$\begin{aligned} S(a,b,c) &= S(a_0,b_0,c_0) \\ &+ [S_a(a_0,b_0,c_0)(a-a_0) + S_b(a_0,b_0,c_0)(b-b_0) + S_c(a_0,b_0,c_0)(c-c_0)] \\ &+ \frac{1}{2}\left[S_{aa}(a_0,b_0,c_0)(a-a_0)^2 + S_{bb}(a_0,b_0,c_0)(b-b_0)^2 + S_{cc}(a_0,b_0,c_0)(c-c_0)^2\right. \\ &+ 2S_{ab}(a_0,b_0,c_0)(a-a_0)(b-b_0) \\ &+ 2S_{ac}(a_0,b_0,c_0)(a-a_0)(c-c_0) \\ &+ \left. 2S_{bc}(a_0,b_0,c_0)(b-b_0)(c-c_0)\right] \end{aligned}$$

Rewrite the above $S(a,b,c) - S(a_0,b_0,c_0)$ in terms of the **quadratic form**:

$$S(a,b,c) - S(a_0,b_0,c_0) = \frac{1}{2}\begin{bmatrix} a-a_0 & b-b_0 & c-c_0 \end{bmatrix} \begin{bmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ab} & S_{bb} & S_{bc} \\ S_{ac} & S_{bc} & S_{cc} \end{bmatrix} \begin{bmatrix} a-a_0 \\ b-b_0 \\ c-c_0 \end{bmatrix}.$$

(c) Show that the matrix

$$A = \begin{bmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ab} & S_{bb} & S_{bc} \\ S_{ac} & S_{bc} & S_{cc} \end{bmatrix}$$

is **positive definite**. Then

$$S(a,b,c) - S(a_0,b_0,c_0) \geq 0$$

for all a, b, c. Therefor proving that (a_0, b_0, c_0) is a global minimum. There are many ways to show that a (symmetric) matrix is positive definite. Please refer to page 346 of the reference book [Advanced Engineering Mathematics (10th ed.) by Erwin Kreyszig, Wiley 2011].

Solution.

(a) Note that

$$\begin{aligned} S_a &= 68\,a + 20\,c - 74, \\ S_b &= 20\,b + \frac{6}{5}, \\ S_c &= 20\,a + 8\,c - \frac{112}{5}. \end{aligned}$$

Solve $S_a = 0, S_b = 0, S_c = 0$, we obtain

$$a_0 = 1, \quad b_0 = -\frac{3}{50}, \quad c_0 = \frac{3}{10}$$

and

$$S(a_0, b_0, c_0) = \frac{1}{250}.$$

(b) By computation, we have

$$S_{aa} = 68, S_{bb} = 20, S_{cc} = 8, S_{ac} = 20, S_{ab} = S_{bc} = 0.$$

Hence, by the Taylor's series for three variables, we can express S as

$$S(a,b,c) = S(a_0,b_0,c_0) + \frac{1}{2}[68(a-a_0)^2 + 20(b-b_0)^2 + 8(c-c_0)^2 + 40(a-a_0)(c-c_0)]$$

which can be rewritten as a quadratic form:

$$S(a,b,c) - S(1, -\frac{3}{50}, \frac{3}{10}) = \frac{1}{2}\begin{bmatrix} a-1 & b+\frac{3}{50} & c-\frac{3}{10} \end{bmatrix} \begin{bmatrix} 68 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 8 \end{bmatrix} \begin{bmatrix} a-1 \\ b+\frac{3}{50} \\ c-\frac{3}{10} \end{bmatrix}.$$

(c) Let

$$A = \begin{bmatrix} 68 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 8 \end{bmatrix}.$$

Then, we have the principle minors of A as follows.

$$\begin{aligned} & 68 > 0, \\ & \begin{vmatrix} 68 & 0 \\ 0 & 20 \end{vmatrix} = 68 \times 20 = 1360 > 0, \\ & \begin{vmatrix} 68 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 8 \end{vmatrix} = 2880 > 0. \end{aligned}$$

Note that A is symmetric. Therefore, A is positive definite and hence $\vec{x}^\top A\vec{x} \geq 0$ for any vector $\vec{x} \in \mathbb{R}^3$. Consequently,

$$S(a, b, c) - S(1, -\frac{3}{50}, \frac{3}{10}) \geq 0 \text{ for all } a, b, c,$$

and hence S attains its global minimum $S(a_0, b_0, c_0) = \frac{1}{250}$ at $(a_0, b_0, c_0) = (1, -\frac{3}{50}, \frac{3}{10})$.