MA2001 Supplementary Notes on Eigenvalues and Eigenvectors

Some general properties of the eigenvalues

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a given matrix A.

i.e. $$A\overrightarrow{x_i} = \lambda_i \overrightarrow{x_i}, i = 1, 2, \ldots, n$$

Then,

(1) The matrix A^{k} has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and the same eigenvectors as A.

(2) The matrix kA has eigenvalues $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ and the same eigenvectors as A.

(3) The inverse A^{-1} (if exists) has eigenvalues $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ and the same eigenvectors as A.

(4) The matrix A^{T} has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ but NOT necessarily the same eigenvectors as A.

(5) The matrix $(A + kI)$ has eigenvalues $(\lambda_1 + k), (\lambda_2 + k), \ldots, (\lambda_n + k)$ and the same eigenvectors as A.

Proof of (1):

Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a given matrix A, we have

$$A\overline{x_i} = \lambda_i \overrightarrow{x_i}, \quad i = 1, 2, \ldots, n$$

$$A^k\overrightarrow{x_i} = A^{k-1}A\overrightarrow{x_i} = A^{k-1}\lambda_i\overrightarrow{x_i} = \lambda_i A^{k-2}A\overrightarrow{x_i} = \lambda_i A^{k-2}\lambda_i\overrightarrow{x_i} = \ldots = \lambda_i^k\overrightarrow{x_i}, \quad i = 1, 2, \ldots, n$$

$\therefore$ The matrix A^{k} has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ and the same eigenvectors as A.

Exercise:

Verify the properties $(2) - (5)$ and give an example for each of the properties above.