MA2001 Vector Differential Calculus

1. Line Integrals

1.1 Line Integral of The First Kind

The line integral of the first kind is

$$I = \int_C f(x,y,z)ds$$

where $f(x,y,z)$ is a scalar function and s is arc length along C. If C has the parametric representation $[x(t), y(t), z(t)],\ t_1 \leq t \leq t_2$, we have that $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ and hence

$$I = \int_{t_1}^{t_2} f[x(t), y(t), z(t)] \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Example

Find the mass of an ideal wire with density $\rho(x,y,z) = kz$ if it has the shape of the helix,

$$\begin{cases} x = 3\cos t \\ y = 3\sin t, 0 \leq t \leq \pi. \\ z = 4t \end{cases}$$

Solution:

Mass $= \int_C \rho ds = k\int_0^\pi 4t\sqrt{9\sin^2 t + 9\cos^2 t + 16}dt = k\int_0^\pi 20t dt = 10k\pi^2.$

□

Example

Let the line density of the curve $C: x^2 + y^2 = ax, a \geq 0$ be $f(x,y) = \sqrt{x^2+y^2}$.Find the mass of the curve $C: x^2 + y^2 = ax, a \geq 0$.

Solution:

$x^2 + y^2 = ax \Rightarrow 2x + 2y\frac{dy}{dx} = a \Rightarrow \frac{dy}{dx} = \frac{a-2x}{2y}$

$ds = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}dx = \sqrt{1 + \left(\frac{a-2x}{2y}\right)^2}dx = \sqrt{\frac{4y^2+a^2-4ax+4x^2}{4y^2}}dx = \sqrt{\frac{4ax+a^2-4ax}{4y^2}}dx = \frac{a}{2|y|}dx$

For the upper part of the circle $x^2 + y^2 = ax \Leftrightarrow \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}, a \geq 0$, we have

$ds = \frac{a}{2y}dx = \frac{a}{2\sqrt{ax-x^2}}dx, 0 \leq x \leq a$

mass of the line $= \int_C \sqrt{x^2+y^2}ds = 2\int_0^a \sqrt{ax}\frac{a}{2\sqrt{ax-x^2}}dx = 2\sqrt{a}\frac{a}{2}\int_0^a \sqrt{\frac{x}{ax-x^2}}dx = a\sqrt{a}\int_0^a \frac{-1}{\sqrt{a-x}}d(a-x)$
$= -\,a\sqrt{a}\frac{\sqrt{a-x}}{\frac{1}{2}}\bigg|_0^a = 2a^2$
(Note: the mass of the upper part and the lower part of the circle
$x^2+y^2=ax \Leftrightarrow \left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}, a\geq 0$ are equal.)
(Note: $\sqrt{ax}\frac{a}{2\sqrt{ax-x^2}}$ is not defined at $x=0$ and $x=a$ so the integral $\int_0^a \sqrt{ax}\frac{a}{2\sqrt{ax-x^2}}dx$ we get is in the sense of improper integral.)

□

Example

Let the line density of the curve$C:\begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0 \end{cases}$ be $f(x,y,z)=x^2$ Find the mass of the curve $C:\begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0 \end{cases}$.

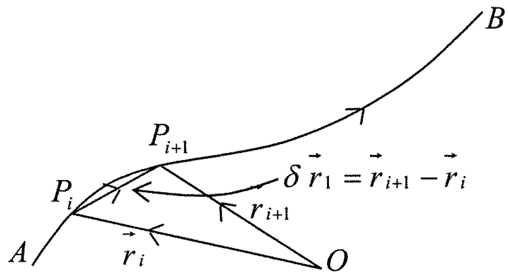
Solution:

We observe that $x^2+y^2+z^2=a^2$ is a sphere of radius a with center at (0,0,0) and $x+y+z=0$ is plane passing through (0,0,0). So, their intersection C is a circle with center at (0,0,0) and radius a. From the property of symmetry, we know that $\int_C x^2ds = \int_C y^2ds = \int_C Z^2ds$, that is, if $(p,q,r)\in C$ then $(-q,-p,-r)\in C$ (symmetry property). At (p, q, r) we have value p^2ds at $(p,q,r)\in C$ for the integral $\int_C x^2ds$ and at $(-q,-p,-r)$ we have value p^2ds at $(-q,-p,-r)$ too for the integral $\int_C y^2ds$. Since C is a close path and the orientation of the line integral of the first kind is insignificant, we have $\int_C x^2ds = \int_C y^2ds$. Thus $\int_C x^2ds = \frac{1}{3}\int_C \left(x^2+y^2+z^2\right)ds = \frac{1}{3}\int_C a^2ds = \frac{a^2 2\pi a}{3} = \frac{2\pi a^3}{3}$.

□

1.2 Line Integral of The Second Kind

Consider a vector field $\vec{F}=F_1(x,y,z)\vec{i}+F_2(x,y,z)\vec{j}+F_3(x,y,z)\vec{k}$ and a smooth curve C. Let A and B be two points on the curve C.



Divide AB into n small segments. Let the i^{th} segment be from $P_i\,(x_i,y_i,z_i)$ (Position vector r_i) to $P_{i+1}\,(x_{i+1},y_{i+1},z_{i+1})$ (position vector $\overrightarrow{r_{i+1}}=\vec{r}_i+\overrightarrow{\delta r_i}$). At P_i let the vector field assume the vector $\vec{F}_i=F_1\,(x_i,y_i,z_i)\,\vec{i}+F_2\,(x_i,y_i,z_i)\,\vec{j}+F_3\,(x_i,y_i,z_i)\,\vec{k}$. Let $P=\{\delta\vec{r}_i : 1\le i\le n\}$,
$|P|=\max\limits_{1\le i\le n}\{|\delta\vec{r}_i| : 1\le i\le n\}$.

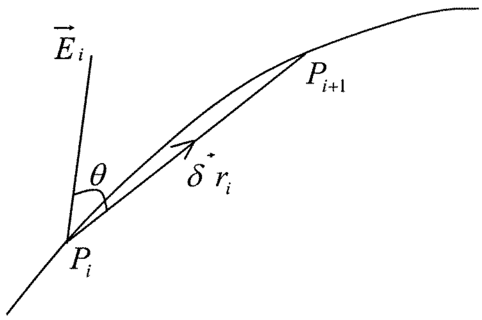
The line integral of the second kind of $\vec{F}$ along C is defined to be
$\int_C \vec{F}\cdot d\vec{r}=\int_{AB}\vec{F}\cdot d\vec{r}=\lim\limits_{|P|\to 0}\sum_{i=1}^{n}\vec{F}_i\cdot\delta\vec{r}_i$

Remark:

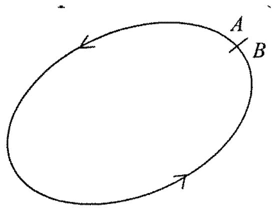
A major difference between Line Integral of The First Kind and Line Integral of The Second Kind is: For Line Integral of The Second Kind, the orientation of the integration along the curve is significant and $\int_C \vec{F}\cdot d\vec{r}=-\int_{-C}\vec{F}\cdot d\vec{r}$, however, for Line Integral of The First Kind , the orientation of the integration along the curve is in significant and we have $\int_C f ds=\int_{-C} f ds$.

Example

If $\vec{E}$ is an electric field and a unit charge moves from A to B along C, $\vec{E}_i\cdot\delta\vec{r}_i=\left|\vec{E}_i\right||\delta\vec{r}_i|\cos\theta$ represents the work done by the field in moving the charge from P_i to P_{i+1} (approximately). Hence $\int_{AB}\vec{E}\cdot d\vec{r}$ represents the total work done in moving the charge from A to B, that is, the potential difference between A and B.



The curve C is called the path of the integral. If the two end-points coincide (that is, $A = B$) we say we have a closed path and usually write $\oint_C \vec{F} \cdot d\vec{r}$



By convention we go anticlockwise around a closed path.

Usually a curve is given in parametric form $x = x(t), y = y(t), z = z(t), t_1 \leq t \leq t_2$ and we write $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, t_1 \leq t \leq t_2$, where t may be the arc length.

$\square$

Example

$\begin{cases} x(t) = a\cos t \\ y(t) = a\sin t \\ z = 1 \end{cases}$, where $0 \leq t \leq 2\pi$ describes the circle $x^2 + y^2 = a^2$ lying in the plane $z = 1$ in positive sense (anticlockwise). Note that t is not the arc length here.

We may then evaluate the line integral as

$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \oint_C \left[F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}\right] \cdot \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)$

$= \oint_C \left[F_1(x,y,z)\frac{dx}{dt} + F_2(x,y,z)\frac{dy}{dt} + F_3(x,y,z)\frac{dz}{dt}\right] dt$

$\square$

Example

Determine the work done by the force

$\vec{F} = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k} = (x^2-y)\,\vec{i} + (y^2-z)\,\vec{j} + (z^2-x)\,\vec{k}$ in moving a particle from $A(0,0,0)$ to $B(1,1,1)$ along (i) the straight line $C_1 : x=y=z$, (ii) the curve $C_2 : \begin{cases} y = x^2 \\ z = x^3 \end{cases}$.

Solution:

(i) Choose the parametric representation $\begin{cases} x = t \\ y = t, 0 \le t \le 1 \\ z = t \end{cases}$,then $\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 1 \\ \frac{dz}{dt} = 1 \end{cases}$.

$\int_{\mathrm{C}} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot \frac{d\vec{r}}{dt}dt$

$= \int_{C_1} \left(F_1[x(t),y(t),z(t)]\vec{i} + F_2[x(t),y(t),z(t)]\vec{j} + F_3[x(t),y(t),z(t)]\vec{k}\right)\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right) dt$

$= \int_{\mathrm{C}_1} \left(F_1[x(t),y(t),z(t)]\frac{dx}{dt} + F_2[x(t),y(t),z(t)]\frac{dy}{dt} + F_3[x(t),y(t),z(t)]\frac{dz}{dt}\right) dt$

$= \int_0^1 \left[(t^2-t)\times 1 + (t^2-t)\times 1 + (t^2-t)\times 1\right] dt = \int_0^1 3\,(t^2-t)\,dt = -\frac{1}{2}$

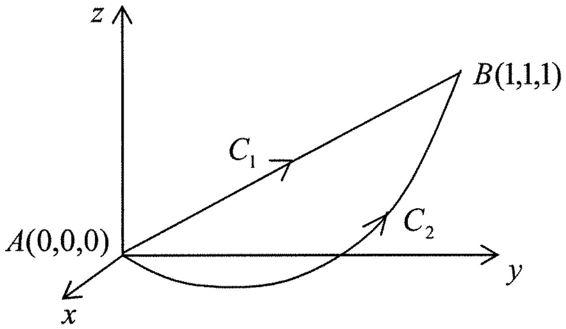
(ii) Choose $\begin{cases} x = t \\ y = t^2, 0 \le t \le 1 \\ z = t^3 \end{cases}$, then $\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 2t \\ \frac{dz}{dt} = 3t^2 \end{cases}$.

$\oint_{C_2} \vec{F}\cdot d\vec{r} = \oint_{C_2} \vec{F}\cdot\frac{dr}{dt}dt = \int_{C_2}\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)\cdot\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right) dt = \oint_{C_2}\left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt$

$= \int_0^1 \left[(t^2-t^2) + (t^4-t^3)\,2t + (t^6-t)\,3t^2\right] dt = \int_0^1 (3t^8 + 2t^5 - 2t^4 - 3t^3)\,dt = -\frac{29}{60}$

Note that the value of the integral in this case depends on the path taken (but not the parametric representation chosen).

If we had integrated from B to A the results would have been $\frac{1}{2}$ and $\frac{29}{60}$ respectively, that is, $\int_{AB} \vec{F}\cdot d\vec{r} = -\int_{BA} \vec{F}\cdot d\vec{r}$ along the same path.



Hence, if C is the close path from A to B along C_1 and then back to A again along C_2 , the result is

$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} = -\frac{1}{2} - \left(-\frac{29}{60}\right) = -\frac{1}{60}$

And this result is independent of the starting (= finishing) point.

□

An alternative notation for line integrals is

$\int_C \vec{F}\cdot d\vec{r} = \int_C \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k}) = \int_C F_1 dx + F_2 dy + F_3 dz$

<u>Remark:</u> f the parametric representation for C is $C : \begin{cases} x = x(t) \\ y = y(t), \quad t_0 \le t \le t_1 \\ z = z(t) \end{cases}$, then one of the parametric representation for $-C$ is $-C : \begin{cases} x = x\left(t_0 - t + t_1\right) \\ y = y\left(t_0 - t + t_1\right), \quad t_0 \le t \le t_1 \\ z = z\left(t_0 - t + t_1\right) \end{cases}$.

Example

Evaluate $\int_C \left(y^2\vec{i} + 2xy\vec{j}\right) \cdot (dx\vec{i} + dy\vec{j}) = \int_C y^2dx + 2xydy$ from the origin (0,0,0) to the point (4,2,0) along the paths:

(i) a straight line,

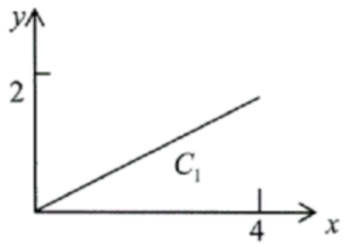
(ii) the parabola $y^2 = x$,

(iii) part of the x -axis and then the line $x = 4$.

This is equivalent to evaluating $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y^2\vec{i} + 2xy\vec{j}$.

Solution:

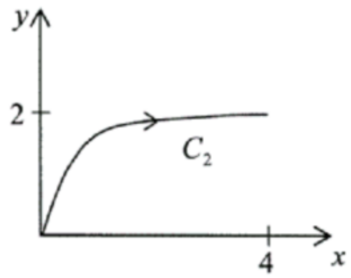
(i)



$C_1 : y = \frac{1}{2}x, 0 \leq x \leq 4$ then $\frac{dy}{dx} = \frac{1}{2}$ therefore $dy = \frac{1}{2}dx$

$\int_{C_1} y^2dx + 2xydy = \int_{C_1} \left(\frac{x}{2}\right)^2 dx + 2x\left(\frac{x}{2}\right)\frac{1}{2}dx = \int_0^4 \left(\frac{x^2}{4} + \frac{x^2}{2}\right) dx = \int_0^4 \frac{3x^2}{4}dx = \left.\frac{x^3}{4}\right|_0^4 = \frac{64}{4} = 16$

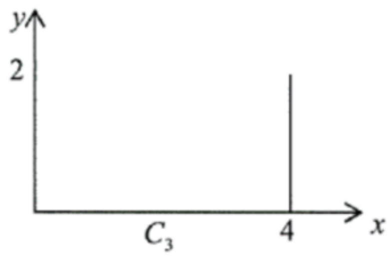
(ii)



$C_2 : y^2 = x, 0 \leq x \leq 4$ then $2ydy = dx$.

$\int_{C_2} y^2dx + 2xydy = \int_0^4 xdx + xdx = \int_0^4 2xdx = x^2|_0^4 = 16$

(iii)



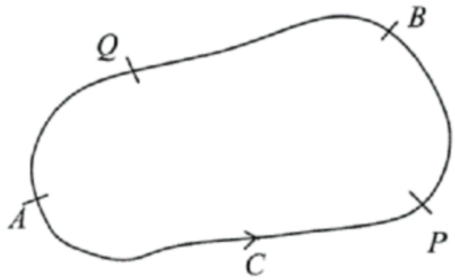
$C_3 = C_{31} + C_{32}$, where $C_{31} : \begin{cases} y = 0 \\ 0 \le x \le 4 \end{cases}$, $C_{32} : \begin{cases} x = 4 \\ 0 \le y \le 2 \end{cases}$. For $C_{31} : \begin{cases} y = 0 \\ 0 \le x \le 4 \end{cases}$ we have $dy = 0$, for $C_{32} : \begin{cases} x = 4 \\ 0 \le y \le 2 \end{cases}$ $dx = 0$.

$\int_{C_3} y^2dx + 2xydy = \int_{C_{31}+C_{32}} y^2dx + 2xydy = \int_{C_{31}} y^2dx + 2xydy + \int_{C_{32}} y^2dx + 2xydy = \int_{C_{32}} y^2dx + 2xydy$
$= \int_0^2 8ydy = 4y^2|_0^2 = 16$

□

Is this just a coincidence? Unlikely!

If a field $\vec{F}$ is such that $\int_{AB} \vec{F} \cdot d\vec{r}$ is independent of the path from A to B, then $\vec{F}$ is said to be a conservative field.



Consider $\oint_C \vec{F} \cdot d\vec{r}$ around any closed curve in a conservative field. Then
$\oint_C \vec{F}\cdot d\vec{r} = \int_{APBQA} \vec{F}\cdot d\vec{r} = \int_{APB} \vec{F}\cdot d\vec{r} + \int_{BQA} \vec{F}\cdot d\vec{r} = \int_{APB} \vec{F}\cdot d\vec{r} - \int_{-BQA} \vec{F}\cdot d\vec{r} = \int_{APB} \vec{F}\cdot d\vec{r} - \int_{AQB} \vec{F}\cdot d\vec{r} = 0$
in a conservative field.

If $\vec{F} = \nabla\varphi$, where φ is some scalar field, then

$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{AB} \nabla\varphi \cdot d\vec{r} = \int_{AB} \left(\frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k} \right) \cdot \left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \right) dt = \int_{AB} \left(\frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} \right) dt$
$= \int_{AB} \frac{d\varphi}{dt} dt = \varphi(B) - \varphi(A) \underset{\substack{A=(x_1,y_1,z_1)\\ B=(x_2,y_2,z_2)}}{=} \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1)$
, independent of the path from A to B. Hence if $\vec{F} = \nabla\varphi$, $\vec{F}$ is conservative.

Example

Show that $\vec{F}(x,y,z) = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k} = y^2\vec{i} + 2xy\vec{j} + 0\vec{k}$ is a conservative field and hence evaluate $\int_C \vec{F} \cdot d\vec{r}$ along any path C from (0,0,0) to (4,2,0).

Solution:

$\vec{F}(x,y,z)$ is conservative if $\vec{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$.Then (i) $y^2 = \frac{\partial\varphi}{\partial x}$, (ii) $2xy = \frac{\partial\varphi}{\partial y}$ and (iii) $0 = \frac{\partial\varphi}{\partial z}$.
Integrating (i) we have $\varphi(x,y,z) = y^2x + f(y,z)$, where $f(y,z)$ – an arbitrary function of y,z.
Then $\frac{\partial\varphi}{\partial y} = 2yx + \frac{\partial f}{\partial y}$. Matching with (ii) shows that $\frac{\partial f}{\partial y} = 0$.
So $f = g(z)$ and $\varphi(x,y,z) = y^2x + g(z)$.From (iii), $0 = \frac{\partial\varphi}{\partial z} = g'(z)$ and $g(z) = C$.
It follows that $\varphi(x,y,z) = y^2x + C$. Hence $\int_C \vec{F} \cdot d\vec{r} = \varphi(4,2,0) - \varphi(0,0,0) = 16$.

$\square$

Note that $\oint_C \vec{F} \cdot d\vec{r}$ is the circulation of $\vec{F}$ around C and so an alternative definition of curl $\vec{F}$ is
curl $\vec{F} = \left(\lim_{A\to 0} \frac{\oint_{C_1} \vec{F}\cdot d\vec{r}}{A_1} \right) \vec{i} + \left(\lim_{A_2\to 0} \frac{\oint_{C_2} \vec{F}\cdot d\vec{r}}{A_2} \right) \vec{j} + \left(\lim_{A\to 0} \frac{\oint_{C_3} \vec{F}\cdot d\vec{r}}{A_3} \right) \vec{k}$, where C_1, C_2 and C_3 are curves parallel to the yz-plane, xz-plane and xy-plane respectively around a point (x,y,z) and A_1, A_2 and A_3 are the areas enclosed by these three curves respectively. But $\oint_C \vec{F} \cdot d\vec{r} = 0$ if $\vec{F}$ is conservative, and hence curl $\vec{F}$ if $\vec{F}$ is conservative.

In fact it is also true that if curl $\vec{F} = \overrightarrow{0}$ everywhere, then $\vec{F}$ is conservative (and $\vec{F}$ may be written as $\nabla\varphi$) and this is sometimes taken as an alternative definition of a conservative field.

This gives us a very convenient test for deciding whether $\vec{F}$ is conservative or not. We see if curl $\vec{F}$ is zero vector or not.

Hence the following are all equivalent:

<u>Theorem</u>

Let $\vec{F}$ be a vector field defined over the whole space.

(i) $\vec{F}$ is conservative.

(ii) $\int_{AB} \vec{F} \cdot d\vec{r}$ is independent of the path.

(iii) $\oint_C \vec{F} \cdot d\vec{r} = 0$ around any closed path (circulation is zero).

(iv) $\vec{F} = \nabla\varphi$ for some scalar field φ.

(v) $\int_{AB} \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$.

(vi) $curl\vec{F} = \overrightarrow{0}$ ($\vec{F}$ is irrotational).

<u>Example</u>

The electric field $\vec{E}$ and potential φ due to a point charge at the origin are $\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$ and $\varphi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$ where $\vec{E} = -\nabla\varphi = \nabla(-\varphi)$. Hence $\vec{E}$ is a conservative field. This is consistent with the work done in moving a unit charge from A to B being the potential difference $\varphi(A) - \varphi(B)$ between the points and independent of the path taken between them.

$\square$

<u>Example</u>

Show that $\vec{F} = yz\vec{i} + (xz + 2y)\vec{j} + xy\vec{k}$ is a conservative field and find a scalar function $f(x, y, z)$ such that $\vec{F} = \nabla f$.

Proof:

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz+2y & xy \end{vmatrix} = \left[\frac{\partial(xy)}{\partial y} - \frac{\partial(xz+2y)}{\partial z}\right]\vec{i} + \left[\frac{\partial(yz)}{\partial z} - \frac{\partial(xy)}{\partial x}\right]\vec{j} + \left[\frac{\partial(xz+2y)}{\partial x} - \frac{\partial(yz)}{\partial y}\right]\vec{k} = \overrightarrow{0}.$$

Thus $\vec{F}(x,y,z)$ is conservative and a scalar function f exists such that $\vec{F} = \nabla f$.

Let $\vec{F} = yz\vec{i} + (xz+2y)\vec{j} + xy\vec{k} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$, then. (i) $\frac{\partial f}{\partial x} = yz$,(ii) $\frac{\partial f}{\partial y} = xz+2y$, (iii)$\frac{\partial f}{\partial z} = xy$.

Integrate (i) we have $f(x,y,z) = xyz + g(y,z)$.Match with (ii) we have $\frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} = xz + 2y$. Hence, $g(y,z) = y^2 + h(z)$ and $f(x,y,z) = xyz + y^2 + h(z)$. Match with (iii) $\frac{\partial f}{\partial z} = xy + h'(z) = xy$.

Hence $h'(z) = 0$, therefore $h(z) = c$, c is a constant. And $f(x,y,z) = xyz + y^2 + c$.

□

2. Surface integrals

2.1 Surface Integral of The First Kind

Consider integral of the form $\iint_S g(x,y,z)dS$, where $g(x,y,z)$ is a scalar field, which is known as the surface integral of the first kind, and can also, arise in its own right.

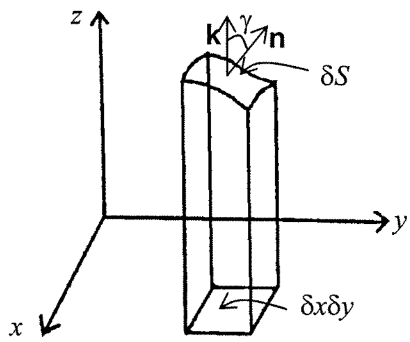
Example

(i) If $g(x,y,z)=1$, then $\iint_S 1dS$ is the surface area of S.

(ii) If $g(x,y,z)$ =electrostatic charge density, then $\iint_S 1dS$ =total charge on S.

□

To evaluate this integral suppose we project S onto one of the coordinate planes where the surface integral becomes a double integral. Suppose the surface is given implicitly by $\phi(x,y,z)=0$ and we decide to project S onto the xy-plane, where its projection is the region σ_{xy}. Let γ be the angle between the normal n to δS and $\vec{k}$, that is, $\cos\gamma=\vec{n}\cdot\vec{k}$. Now $\delta x\delta y=\delta S|\cos\gamma|=\delta S|\vec{n}\cdot\vec{k}|$, therefore $\delta S=\frac{\delta x\delta y}{|\vec{n}\cdot\vec{k}|}$ and in the limit, we have $\iint_S g(x,y,z)dS=\iint_S g(x,y,z)\frac{1}{|\vec{n}\cdot\vec{k}|}dxdy=\iint_{\sigma_{xy}} g(x,y,z(x,y))\frac{\sqrt{\varphi_x^2+\varphi_y^2+\varphi_z^2}}{|\varphi_z|}dxdy$.



If the surface is given explicitly by $z = \varphi(x, y)$, then $dS = \frac{\sqrt{\phi_x^2+\phi_y^2+1}}{|1|}dxdy = \sqrt{1+\phi_x^2+\phi_y^2}dxdy$ and $\iint_S g(x, y, z)dS = \iint_{\sigma_{xy}} g(x, y, z)\sqrt{1+\phi_x^2+\phi_y^2}dxdy$.

Example

Compute $\iint_S ydS$ where S is the part of $z = z(x, y) = x + y^2$ whose projection onto xy-plane is the region $\sigma_{xy} : 0 \le x \le 1, 0 \le y \le 2$.

Solution:

The projection of S onto xy-plane is the region $\sigma_{xy} : 0 \le x \le 1, 0 \le y \le 2$.

So $dS = \sqrt{1+z_x^2+z_y^2}dxdy = \sqrt{1+1+(2y)^2}dxdy = \sqrt{2+4y^2}dxdy$.

Therefore

$\iint_S ydS = \iint_{\sigma_{xy}} y\sqrt{2+4y^2}dxdy = \int_0^1 \left(\int_0^2 y\sqrt{2+4y^2}dy\right) dx = \int_0^1 \left[\int_0^2 \frac{1}{8}\sqrt{2+4y^2}d\left(2+4y^2\right)\right] dx$
$= \int_0^1 \frac{1}{8}\left(\frac{2}{3}\right)\left(2+4y^2\right)^{\frac{3}{2}}\Big|_0^2 dx = \int_0^1 \frac{1}{12}\left(18^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) dx = \frac{1}{12}\left(18^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = \frac{1}{12}[27(2\sqrt{2}) - 2\sqrt{2}] = \frac{52\sqrt{2}}{12} = \frac{13\sqrt{2}}{3}$

□

Example

Compute $\iint_S (x + y + z)dS$ over the surface $S : x^2 + y^2 + z^2 = a^2, z \ge 0$.

Solution:

Since we are going to project the open surface $S : x^2 + y^2 + z^2 = a^2, z \ge 0$ onto xy-plane, we will use the formula $\iint_S g(x, y, z)dS = \iint_{\sigma_{xy}} g(x, y, z)\frac{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}{|\phi_z|}dxdy$.

The projection of $x^2 + y^2 + z^2 = a^2, z \ge 0$ onto xy-plane is σ_{xy}, which is $x^2 + y^2 \le a^2$.

Let $\varphi(x, y, z) = x^2 + y^2 + z^2 - a^2$, then the surface S is just the equation : $\begin{cases} \phi(x, y, z) = 0 \\ z \ge 0 \end{cases}$, that is, $x^2 + y^2 + z^2 = a, z > 0$ and this equation defines z as an implicit function of x, y. Also, $\frac{\partial\phi}{\partial x} = 2x, \frac{\partial\phi}{\partial y} = 2y, \frac{\partial\phi}{\partial z} = 2z$. And

$dS = \frac{\sqrt{\phi_X^2+\phi_Y^2+\phi_Z^2}}{|\phi_z|}dxdy = \frac{\sqrt{4x^2+4y^2+4z^2}}{2z}dxdy = \frac{\sqrt{4a^2}}{2z}dxdy = \frac{2a}{2z}dxdy = \frac{a}{z}dxdy$
$= \frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$

Or you can figure out in the following way, of course, the following method requires more being able in mathematics:

$dS = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy = \sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}dxdy = \frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$

So

$\iint_S (x,y,z)dS = \iint_{\sigma_{xy}} \left(x+y+\sqrt{a^2-x^2-y^2}\right)\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$
$= \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(x+y+\sqrt{a^2-x^2-y^2}\right)\frac{a}{\sqrt{a^2-x^2-y^2}}dy\right]dx$
$= \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{a(x+y)dy}{\sqrt{a^2-x^2-y^2}}+ady\right]dx = \int_{-a}^{a}\left[\begin{array}{c}\sqrt{a^2-x^2}\\ -\sqrt{a^2-x^2}\end{array}\frac{axdy-\frac{a}{2}d\left(a^2-x^2-y^2\right)}{\sqrt{a^2-x^2-y^2}}+ady\right]dx$
$= \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{axdy}{\sqrt{a^2-x^2-y^2}}\right]dx - \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{\frac{a}{2}d\left(a^2-x^2-y^2\right)}{\sqrt{a^2-x^2-y^2}}\right]dx + \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}ady\right]dx$

Observe that

$\int_{-a}^{a}\left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{axdy}{\sqrt{a^2-x^2-y^2}}\right)dx \underset{\substack{l=\sqrt{a^2-x^2}\\ \Rightarrow l^2=a^2-x^2}}{=} \int_{-a}^{a}ax\left(\int_{-l}^{l}\frac{dy}{\sqrt{l^2-y^2}}\right)dx$

$\underset{\int\frac{dy}{\sqrt{l^2-y^2}}=a\arcsin\frac{y}{|l|}}{=} \int_{-a}^{a}ax\left(\arcsin\frac{y}{\sqrt{a^2-x^2}}\right)\mid\begin{array}{c}\sqrt{a^2-x^2}\\ -\sqrt{a^2-x^2}\end{array}dx$
$= \int_{-a}^{a}ax\left(\arcsin\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}-\arcsin\frac{-\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right)dx = \int_{-a}^{a}ax[\arcsin 1-\arcsin(-1)]dx$
$= \int_{-a}^{a}ax\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]dx = \int_{-a}^{a}a\pi xdx \underset{a\pi x\ is\ odd}{=} 0$
$\int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\frac{\frac{a}{2}d\left(a^2-x^2-y^2\right)}{\sqrt{a^2-x^2-y^2}}\right]dx = \int_{-a}^{a}\left(a\sqrt{a^2-x^2-y^2}\right)\Big|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx$
$= \int_{a\pi x}^{a}\left[a\sqrt{a^2-x^2-\left(\sqrt{a^2-x^2}\right)^2}-a\sqrt{a^2-x^2-\left(-\sqrt{a^2-x^2}\right)^2}\right]dx = \int_{-a}^{a}0dx = 0$

$\int_{-a}^{a}\left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} a dy\right)dx = \int_{-a}^{a} 2a\sqrt{a^2-x^2}dx \underset{2a\sqrt{a^2-x^2}}{=}$ is even $4a\int_0^a \sqrt{a^2-x^2}dx$

$\overline{\int \sqrt{a^2-x^2}dx=\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{|a|}\right)} \left[2a\left(x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{|a|}\right)\right]\Big|_0^a$

$\underset{\text{Assume } a>0}{=} 2a^3\arcsin\frac{a}{a} - a^3\arcsin\frac{0}{a} = 2a^3\times\frac{\pi}{2} = \pi a^3$

Therefore,

$\iint_S (x+y+z)dS = \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{axdy}{\sqrt{a^2-x^2-y^2}}\right]dx - \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{\frac{a}{2}d\left(a^2-x^2-y^2\right)}{\sqrt{a^2-x^2-y^2}}\right]dx + \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} a dy\right]dx$
$= \pi a^3$

Or

$\iint_S (x+y+z)dS = \iint_{\sigma_{xy}}\left(x+y+\sqrt{a^2-x^2-y^2}\right)\frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$

$\underset{\substack{x=r\cos\theta\\ y=r\sin\theta\\ dxdy=rd\theta dr\\ Assume\ a>0}}{=} \int_0^a\left[\int_0^{2\pi}\left(r\cos\theta + r\sin\theta + \sqrt{a^2-r^2}\right)\frac{a}{\sqrt{a^2-r^2}}rd\theta\right]dr = \int_0^a\left(\frac{ra\sin\theta}{\sqrt{a^2-r^2}} - \frac{ra\cos\theta}{\sqrt{a^2-r^2}} + ra\theta\right)\Big|_0^{2\pi}dr$

$= \int_0^a 2\pi ardr = \left(\pi ar^2\right)\big|_0^a = \pi a^3$

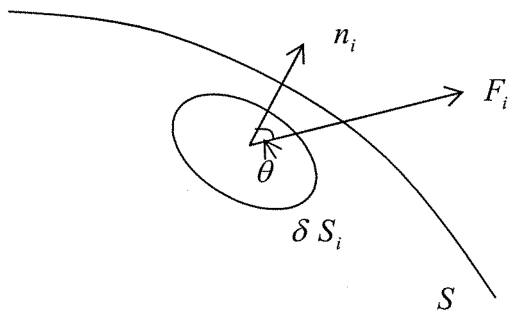
$\square$

2.2 Surface Integral of The Second Kind

Consider a vector field $\vec{F}$ and a surface S in the field.Divide the surface into N small area elements δS_i (Approximately flat). Define $\delta\vec{S_i}$ as the vector whose magnitude is δS_i and whose direction is normal to the element of surface δS_i (usually outwards if S is closed). Then $\delta\overrightarrow{S_i} = \delta S_i\overrightarrow{n_i}$, where $\overrightarrow{n_i}$ is a unit vector normal to the region δS_i. Let $\vec{F_i}$ be the vector field evaluated at some point on δS_i.

Then $\vec{F_i}\cdot\delta\vec{S_i} = \vec{F_i}\cdot\delta S_i\vec{n_i} = \left|\vec{F_i}\right|\delta S_i\cos\theta$ represents the flux of $\vec{F}$ through the element of surface δS_i . We define the total flux of $\vec{F}$ through S to be the surface integral of the second kind

$\iint_S \vec{F}\cdot d\vec{S} = \lim_{N\to\infty}\sum_{i=1}^{N}\vec{F_i}\cdot\delta\vec{S_i}$



Example

(i) If $\vec{v}$ is the velocity field of a fluid, $\iint_S \vec{v}\cdot d\vec{S}$ represents the total volume of fluid crossing S in a unit time. Replacing $\vec{v}$ by $\rho\vec{v}$ ($\rho =$ density) gives mass flux.

(ii) If $\vec{J}$ is an electric current vector, $\iint_S \vec{J}\cdot d\vec{S}$ represents the rate at which electric charge crosses S.

(iii) If $\vec{E}$ is an electric field vector, $\iint_S \vec{E}\cdot d\vec{S}$ represents the electric flux through S.

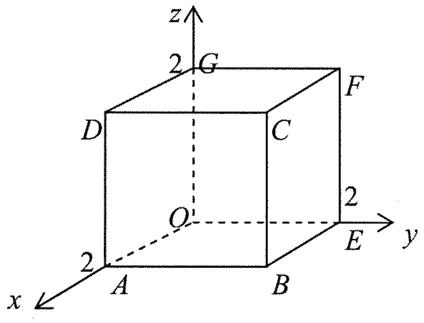
(iv) If $\vec{q}$ is a heat conduction vector, $\iint_S \vec{q}\cdot d\vec{S}$ represents the rate at which heat flows through S.

□

Example

Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}=x^2\vec{i}+xy\vec{j}+z^2\vec{k}$ and S is the surface of the cube bounded by $x=0,x=2,y=0,y=2,z=0,z=2.$

Solution:



We observe that the surface S of the cube is a close surface. A close surface S is a surface which divides the space into two parts, one has finite volume and the other has infinite volume. For a close surface S the normal to S is always assumed to be the outer normal which is directed from the part with finite volume to the part with infinite volume.

If a surface S is not a close surface then it is called an open surface.

We integrate over each face in turn and sum the results.

For the face $ABCD$:

$dS=dydz$, the outward normal $\vec{n}=\vec{i},x=2.$

$\iint_{ABCD}\vec{F}\cdot d\vec{S}=\iint_{ABCD}\left(x^2\vec{i}+xy\vec{j}+z^2\vec{k}\right)\cdot\vec{i}dS=\iint_{ABCD}x^2dS=\int_0^2\left[\int_0^2 2^2dy\right]dz=16$

For the face $BEFC$:

$dS=dydz$, the outward normal $\vec{n}=\vec{j},y=2.$

$\iint_{BEFC}\vec{F}\cdot d\vec{S}=\iint_{BEFC}\left(x^2\vec{i}+xy\vec{j}+z^2\vec{k}\right)\cdot\vec{j}dS=\iint_{BEFC}xydS=\int_0^2\left[\int_0^2 2xdz\right]dx=\int_0^2 4xdx=8$

Similarly, $\iint_{CFGD}\vec{F}\cdot d\vec{S}=16,\iint_{OEFG}\vec{F}\cdot d\vec{S}=\iint_{OADG}\vec{F}\cdot d\vec{S}=\iint_{OABE}\vec{F}\cdot d\vec{S}=0.$

Hence $\iint_S\vec{F}\cdot d\vec{S}=40.$

□

If part of S is not parallel to one of the coordinate planes, we proceed as follows:

If the surface S is written as $\phi(x,y,z) = constant$ then $\nabla\varphi = \pm grad\varphi$ is normal to the surface and hence the unit normals are

$$\vec{n} = \pm\frac{\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}}{\sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2}}$$

If the surface is given in the form $z = \varphi(x,y)$, define $\Phi(x,y,z) = z - \phi(x,y)$, then $\Phi(x,y,z) = z - \phi(x,y) = 0 \Leftrightarrow z = \phi(x,y)$, the unit normals are

$$\vec{n} = \pm\frac{\vec{i}\frac{\partial\Phi}{\partial x}+\vec{j}\frac{\partial\Phi}{\partial y}+\vec{k}\frac{\partial\Phi}{\partial z}}{\sqrt{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2}} \underset{\substack{\frac{\partial\Phi}{\partial x}=-\frac{\partial\varphi}{\partial x}\\ \frac{\partial\Phi}{\partial y}=-\frac{\partial\varphi}{\partial y}\\ \frac{\partial\Phi}{\partial z}=1}}{=} \pm\frac{-\vec{i}\frac{\partial\varphi}{\partial x}-\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}}{\sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+1}}$$

Let $C:\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, -\delta \le t \le \delta$ be a curve in the surface $S:\phi(x,y,z) = 0$, that is, $\phi[x(t),y(t),z(t)] = 0, -\delta \le t \le \delta$, and $\vec{r}(0) = x(0)\vec{i} + y(0)\vec{j} + z(0)\vec{k} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$. We would like to show that $\pm\nabla\varphi\,(x_0,y_0,z_0)$ is perpendicular to the tangent to $C:\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, -\delta \le t \le \delta$ at (x_0,y_0,z_0), that is, $\pm\nabla\varphi\,(x_0,y_0,z_0)$ is perpendicular to the tangent plane T to S at (x_0,y_0,z_0), that is, $\pm\nabla\varphi\,(x_0,y_0,z_0)$ is perpendicular to S at (x_0,y_0,z_0).

$\varphi[x(t),y(t),z(t)] = 0, -\delta < t < \delta$ where $\delta > 0 \Rightarrow \frac{d}{dt}\varphi[x(t),y(t),z(t)] = 0$

$$\Rightarrow \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt} = 0 \Rightarrow \frac{\partial\varphi}{\partial x\Big|\begin{smallmatrix}x=x_0\\y=y_0\\z=z_0\end{smallmatrix}}\frac{dx}{dt|_{t=0}} + \frac{\partial\varphi}{\partial y\Big|\begin{smallmatrix}x=x_0\\y=y_0\\z=z_0\end{smallmatrix}}\frac{dy}{dt|_{t=0}} + \frac{\partial\varphi}{\partial z\Big|\begin{smallmatrix}x=x_0\\y=y_0\\z=z_0\end{smallmatrix}}\frac{dz}{dt|_{t=0}} = 0$$

$$\Rightarrow \pm\nabla\varphi\,(x_0,y_0,z_0)\cdot\left(\frac{dx}{dt}_{|t=0}\vec{i} + \frac{dy}{dt}_{|t=0}\vec{j} + \frac{dz}{dt}_{|t=0}\vec{k}\right) = 0$$

There are two normals to the open surface at each point which are of the same magnitude but are of opposite directions. The ways to decide the required normal are explained through examples:

Example

Consider the surface S:$\varphi(x, y, z) = x^2 + y^2 + z^2 = a^2, z > 0$ (the upper sphere).

At the point (x,y,z), there are two unit vectors (normals) which are perpendicular to the surface S: $\phi(x, y, z) = x^2 + y^2 + z^2 = a^2$, where $z > 0$. They are the vectors

$\frac{\pm\nabla\phi}{|\nabla\phi|} = \frac{\pm\left(\phi_x\vec{i}+\phi_y\vec{j}+\phi_z\vec{k}\right)}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}} = \pm\left(\frac{\phi_x}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}\vec{i} + \frac{\phi_y}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}\vec{j} + \frac{\phi_z}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}\vec{k}\right) = \frac{\pm\left(2x\vec{i}+2y_y+2z\vec{k}\right)}{2a}$

$= \pm\left(\frac{x}{a}\vec{i} + \frac{y}{a}\vec{j} + \frac{z}{a}\vec{k}\right) = \pm[(\cos\alpha)\vec{i} + (\cos\beta)\vec{j} + (\cos\gamma)\vec{k}]$

,where α is the angle between $\frac{\nabla\phi}{|\nabla\phi|}$ and the positive direction of x-axis (or the angle between $\nabla\phi$ and the positive direction of x-axis) , β is the angle between $\frac{\nabla\phi}{|\nabla\phi|}$ and the positive direction of y-axis and γ is the angle between $\frac{\nabla\phi}{|\nabla\phi|}$ and the positive direction of z-axis.

So cosine of the angle between the normals and the positive direction of x-axis is

$\pm\cos\alpha = \pm\frac{\phi_x}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}$

So cosine of the angle between the normals and the positive direction of y-axis is

$\pm\cos\beta = \pm\frac{\phi_Y}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}$

So cosine of the angle between the normals and the positive direction of z-axis is

$\pm\cos\gamma = \pm\frac{\phi_Z}{\sqrt{\phi_x^2+\phi_y^2+\phi_z^2}}$

We observe that the outward normal at the point (x, y, z) to the upper sphere $\begin{cases} x^2 + y^2 + z^2 = a^2 \\ z > 0 \end{cases}$ makes an acute angle θ with the positive direction of z-axis, so $\cos\theta > 0$, therefore the normal we need is $\frac{x}{a}\vec{i} + \frac{y}{a}\vec{j} + \frac{z}{a}\vec{k}$.

As for the inward normal at the point (x, y, z) to the upper sphere $\begin{cases} x^2+y^2+z^2=a^2 \\ z>0 \end{cases}$ makes an acute angle θ makes an obtuse angle θ with the positive direction of z-axis, so $\cos\theta < 0$, therefore the normal we need is $-\left(\frac{x}{a}\vec{i}+\frac{y}{a}\vec{j}+\frac{z}{a}\vec{k}\right)$

Similarly, the outward normal at the point (x, y, z) to the lower sphere $\begin{cases} x^2+y^2+z^2=a^2 \\ z<0 \end{cases}$ makes an obtuse angle θ with the positive direction of z-axis, so $\cos\theta < 0$, therefore the normal at the point (x, y, z) to the lower sphere $\begin{cases} x^2+y^2+z^2=a^2 \\ z<0 \end{cases}$ we need is $\frac{x}{a}\vec{i}+\frac{y}{a}\vec{j}+\frac{z}{a}\vec{k}$ (note: z is negative).

As for the inward normal at the point (x, y, z) to the lower sphere $\begin{cases} x^2+y^2+z^2=a^2 \\ z<0 \end{cases}$ makes an acute angle θ with the positive direction of z-axis, so $\cos\theta > 0$, therefore the normal at the point (x, y, z) to the lower sphere $\begin{cases} x^2+y^2+z^2=a^2 \\ z<0 \end{cases}$ we need is $-\left(\frac{x}{a}\vec{i}+\frac{y}{a}\vec{j}+\frac{z}{a}\vec{k}\right)$ (note: z is negative).

$\square$

We call the normal to a surface S: which makes an acute angle with the positive direction of z-axis an upper normal. Also we call the normal to a surface S: which makes an obtuse angle with the positive direction of z-axis a lower normal.

If the surface S given is an explicit function $z=\varphi(x,y)$, then the normals to surface S at the point $(x,y,z)=(x,y,\varphi(x,y))$ is: $\Phi(x,y,z)=z-\phi(x,y)=0 \Rightarrow \frac{\pm\left(\Phi_x\vec{i}+\Phi_y\vec{j}+\Phi_z\vec{k}\right)}{\sqrt{\Phi_x^2+\Phi_y^2+\Phi_z^2}} = \frac{\pm\left(-\phi_x\vec{i}-\phi_y\vec{j}+\vec{k}\right)}{\sqrt{\phi_x^2+\phi_y^2+1}}$

Once the normal to the open surface has been decided, we have

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}dS = \iint_S \vec{F}\cdot\frac{\pm\nabla\varphi}{|\nabla\varphi|}dS$$

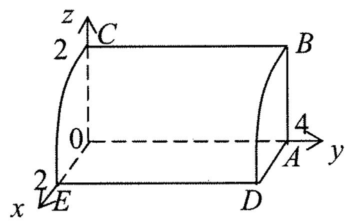
Now we are going to make a few concluding remarks on the ways of computing the surface integral of second kind, that is, compute $\iint_S \vec{F}(x,y,z)\cdot d\vec{S}$, where $\vec{F}(x,y,z)$ is a vector field defined on S. Suppose the surface is given implicitly by $\varphi(x,y,z)=0$ and we are going to project S onto xy-plane with the projection, the region σ_{xy}. Assume we have chosen the normal n to S at the point (x,y,z) by considering the cosine of the included angle between the normal and the positive direction of z-axis. Then $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \vec{n}dS = \iint_S \vec{F}\cdot \frac{\pm\nabla\varphi}{|\nabla\varphi|}dS = \iint_{\sigma_{xy}} \vec{F}\cdot \frac{\pm\nabla\varphi}{|\nabla\varphi|}\frac{|\nabla\varphi|}{|\varphi_z|}dxdy = \iint_{\sigma_{xy}} \vec{F}\cdot(\pm\nabla\varphi)\frac{1}{|\varphi_z|}dxdy$ Whether the positive sign or the negative sign is chosen is depending on the problem raised.

In addition, if the surface is given explicitly by $z=\varphi(x,y)=0$ nd still we are going to project S onto xy-plane with the projection, the region σ_{xy}, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \vec{n}dS = \iint_S \vec{F}\cdot \frac{\pm\left(-\phi_x\vec{i}-\phi_y\vec{j}+\vec{k}\right)}{\sqrt{\phi_x^2+\phi_y^2+1}}dS =$$
$$\iint_{\sigma_{xy}} \vec{F}\cdot \frac{\pm\left(-\phi_x\vec{i}-\phi_y\vec{j}+\vec{k}\right)}{\sqrt{\phi_x^2+\phi_y^2+1}}\frac{\sqrt{\phi_x^2+\phi_y^2+1}}{|1|}dxdy = \iint_{\sigma_{xy}} \vec{F}\cdot\left[\pm\left(-\phi_x\vec{i}-\phi_y\vec{j}+\vec{k}\right)\right]dxdy$$

Example

Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where S is the closed surface of the region bounded by the cylinder $x^2+z^2=4$ and the planes $x=0, y=0, z=0$ and $y=4$, and $\vec{F}=z\vec{i}+z\vec{j}-y\vec{k}$.



Solution:

Consider first the integral over the curved surface $BCED$ which is the level surface $\varphi = x^2 + z^2 = 4$. The curved surface is projected on xy-plane with the projection σ_{xy} which is the rectangle $OADE$ bounded by $y = 0, y = 4, x = 0, x = 2. \varphi = x^2 + z^2 \Rightarrow \nabla\varphi = 2x\vec{i} + 2z\vec{k}, \varphi_z = 2z$

$\iint_{BCED} \vec{F} \cdot d\vec{S} = \iint_{AOED} (z\vec{i} + z\vec{j} - y\vec{k}) \cdot (2x\vec{i} + 2z\vec{k}) \frac{1}{2z} dxdy = \int_0^2 \left[\int_0^4 (x - y) dy \right] dx = -8$

The other surfaces are all planes and give the results

$\iint_{OABC} \vec{F} \cdot d\vec{S} = -8, \iint_{OADE} \vec{F} \cdot d\vec{S} = 16, \iint_{OEC} \vec{F} \cdot d\vec{S} = -\frac{8}{3},$

$\iint_{ABD} \vec{F} \cdot d\vec{S} = \frac{8}{3}$ with sum $\iint_S \vec{F} \cdot d\vec{S} = 0$

□

Example

Calculate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x, y, z) = -2\vec{i} - 2\vec{j} - 2\vec{k}$, the open surface S is the part of the plane $x + z = 1$ which is enclosed by the cylinder $x^2 + y^2 = 1$ and the normal to S is pointing to xy-plane.

Solution:

Let $\phi(x, y, z) = x + z$, then $\nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k} = \vec{i} + \vec{k}$ and $\frac{\partial\varphi}{\partial x} = 1, \frac{\partial\varphi}{\partial y} = 0, \frac{\partial\varphi}{\partial z} = 1.$

Since the normal to S makes an obtuse angle with the positive direction of z- axis, we choose $\vec{n} = \frac{\nabla\varphi}{-|\nabla\varphi|}$ or the direction $-\nabla\varphi = -\vec{i} - \vec{k}$.

And we are going to project S onto the xy- plane and the projection, say, σ_{xy}. The following formula will be used:

$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\sigma_{xy}} \vec{F} \cdot \frac{-\nabla\varphi}{|\varphi_z|} dxdy = \iint_{\sigma_{xy}} -\vec{F} \cdot \nabla\varphi dxdy$ (note that $|\varphi_z| = 1$)

$= \iint_{\sigma_{xy}} -\vec{F} \cdot \nabla \varphi dxdy = \iint_{\sigma_{xy}} (2\vec{i} + 2\vec{j} + 2\vec{k}) \cdot (\vec{i} + \vec{k}) dxdy = \iint_{x^2+y^2 \le 1} 4dxdy = \int_0^1 \left[\int_0^{2\pi} 4r d\theta \right] dr = 4\pi$

□

Example

Find $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = x^2 y\vec{i} - 3xy^2\vec{j} + 4y^3\vec{k}$ and S is the part of $z = z(x,y) = x^2 + y^2 - 9$ whose projection onto xy-plane is the region σ_{xy}:

$0 \le x \le 2, 0 \le y \le 1$. The normal to S at (x,y,z) we choose is the lower normal, that is the normal which makes an obtuse angle with the positive direction of z-axis.

Solution:

The direction of the normal is the same as the direction of $-(-2x\vec{i} - 2y\vec{j} + \vec{k})$

since the normal to S at (x,y,z) makes an obtuse angle with the positive direction of z-axis.

$\iint_S \left(x^2 y\vec{i} - 3xy^2\vec{j} + 4y^3\vec{k} \right) \cdot d\vec{S} = \iint_{\sigma_{xy}} \left(x^2 y\vec{i} - 3xy^2\vec{j} + 4y^3\vec{k} \right) \cdot (2x\vec{i} + 2y\vec{j} - \vec{k}) dxdy$

$= \iint_{\sigma_{xy}} (2x^3 y - 6xy^3 - 4y^3)\, dxdy = \int_0^2 \left[\int_0^1 (2x^3 y - 6xy^3 - 4y^3)\, dy \right] dx = \int_0^2 \left(x^3 y^2 - \frac{3}{2} xy^4 - y^4 \right)\Big|_0^1 dx$

$= \int_0^2 \left(x^3 - \frac{3}{2}x - 1 \right) dx = \left(\frac{x^4}{4} - \frac{3x^2}{4} - x \right)\Big|_0^2 = 4 - 3 - 2 = -1$

□

Example

Find $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = (y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}$, S is the close surface which is bounded by $x^2 + y^2 = z^2 \& z = h$ where $h > 0$.

Solution:

For a close surface if the direction of the normal is not specifically mentioned, then it is understood that the normal is assumed to be outward.

S consists of two parts, say, $S_1 : x^2 + y^2 = z^2$, where $0 \le z \le h \& S_2 : \begin{cases} x^2 + y^2 \le h^2 \\ z = h \end{cases}$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$

For $\iint_{S_1} \vec{F} \cdot d\vec{S}$, there are two directions which are perpendicular to S_1, they are $\pm$ gradient, i.e. $\pm\nabla\phi$:
$\phi(x,y,z) = z^2 - x^2 - y^2 \Rightarrow \pm\nabla\phi = \pm\left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right) = \pm(-2x\vec{i} - 2y\vec{j} + 2z\vec{k})$

Since the question requires the direction which makes an obtuse angle with the positive direction of z-axis,

$-\nabla\phi = -\left(\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}\right) = -(-2x\vec{i} - 2y\vec{j} + 2z\vec{k}) = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$ is the direction we want.

We project $S_1 : x^2 + y^2 = z^2$, where $0 \le z \le h$ onto xy-plane, we have σ_{xy} which is $\sigma_{xy} : x^2 + y^2 \le h^2$.

Then

$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_{\sigma_{xy}} \vec{F} \cdot \frac{-\nabla\phi}{|\phi z|} dxdy = \iint_{\sigma_{xy}} [(y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}] \cdot \frac{(2x\vec{i}+2y\vec{j}-2z\vec{k})}{|2z|} dxdy$
$= \iint_{\sigma_{xy}} \left(\frac{x\left(y-\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} + \frac{y\left(\sqrt{x^2+y^2}-x\right)}{\sqrt{x^2+y^2}} - (x-y) \right) dxdy$
$= \int_0^{2\pi} \left[\int_0^h \left(\frac{r\cos\theta(r\sin\theta - r)}{r} + \frac{r\sin\theta(r - r\cos\theta)}{r} - r\cos\theta + r\sin\theta \right) rdr \right] d\theta$

$= \int_0^{2\pi} \left[\int_0^h (r\cos\theta\sin\theta - r\cos\theta + r\sin\theta - r\cos\theta\sin\theta - r\cos\theta + r\sin\theta) rdr \right] d\theta$
$= \int_0^{2\pi} \left[\int_0^h (-2r^2\cos\theta + 2r^2\sin\theta)\, dr \right] d\theta = \int_0^{2\pi} \left(-\frac{2}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta \right)\Big|_0^h d\theta$
$= \int_0^{2\pi} \frac{2}{3}h^3(-\cos\theta + \sin\theta) d\theta = \frac{2}{3}h^3(-\sin\theta - \cos\theta)\Big|_0^{2\pi} = 0$

For $\iint_{S_2} \vec{F} \cdot d\vec{S}, S_2$ is $\begin{cases} x^2 + y^2 \le h^2 \\ z = h \end{cases}$ whose projection of onto xy-plane is $\sigma_{xy} : x^2 + y^2 \le h$.

Since the outward normal to S_2 makes 0 radian with the positive direction of z-axis so k is chosen.

The projection of S_2 onto xy-plane, is $\sigma_{xy} : x^2 + y^2 \le h^2$.

So

$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{\sigma_{xy}} \vec{F} \cdot \vec{k} dxdy = \iint_{\sigma_{xy}} (x-y) dxdy = \int_0^{2\pi} \left[\int_0^h (r\cos\theta - r\sin\theta) rdr \right] d\theta$
$= \int_0^{2\pi} (\cos\theta - \sin\theta) \frac{1}{3}r^3\Big|_0^h d\theta = \int_0^{2\pi} \frac{1}{3}h^3(\cos\theta - \sin\theta) d\theta = 0$

Finally we have $\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = 0$ □

3. Surface Given Parametrically by Three Equations (Optional)

Let a surface S be given in space. The surface S can be represented parametrically by equations $x = x(u,v), y = y(u,v), z = z(u,v)$, where u and v vary in a region R_{uv} of the uv-plane.

Let $\vec{r} = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, then the surface area element dS is given by $dS = |\overrightarrow{r_u}du \times \overrightarrow{r_v}dv|$, where $\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}, \vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$.

Observe that

$|\vec{r}_u du \times \vec{r}_v dv| = \sqrt{(\vec{r}_u du \times \vec{r}_v dv) \cdot (\overrightarrow{r_u}du \times \overrightarrow{r_v}dv)} = \sqrt{(\overrightarrow{r_u} \times \overrightarrow{r_v}) \cdot (\overrightarrow{r_u} \times \overrightarrow{r_v})}dudv = \sqrt{|\overrightarrow{r_u} \times \vec{r}_v|^2}dudv$. And $|\vec{r}_u \times \vec{r}_v|^2 = |\vec{r}_u|^2 |\vec{r}_v|^2 \sin^2$ (the angle between $\overrightarrow{r_u}$ and $\vec{r}_v$) $= |\vec{r}_u|^2 |\vec{r}_v|^2 [1 - \cos^2$ (the angle between $\overrightarrow{r_u}$ and $\overrightarrow{r_v}$) $= |\vec{r}_u|^2 |\vec{r}_v|^2 - |\vec{r}_u|^2 |\vec{r}_v|^2 \cos^2$ (the angle between $\vec{r}_u$ and $\vec{r}_v$) $= |\vec{r}_u|^2 |\vec{r}_v|^2 - (\vec{r}_u \cdot \vec{r}_v)^2$

$= \left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2\right] \left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] - \left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)^2$

$\Rightarrow \sqrt{|\vec{r}_u \times \vec{r}_v|^2}dudv = \sqrt{EG - F^2}dudv$, where $E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$, $F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$

The area of the surface then is given by $S = \sum dS = \iint_{R_{uv}} \sqrt{EG - F^2}dudv$.

Example

Find the area of the part of the upper hemisphere $\begin{cases} x^2 + y^2 + z^2 = 4 \\ z \geq 0 \end{cases}$ which is enclosed by the cylinder $x^2 + y^2 = 1$.

Solution:

Using spherical coordinates, the upper hemisphere $\begin{cases} x^2 + y^2 + z^2 = 4 \\ z \geq 0 \end{cases}$ can be represented parametrically by

$x = 2\sin\varphi\cos\theta, y = 2\sin\varphi\sin\theta, z = 2\cos\varphi$ where $0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{2}$.

The part of the upper hemisphere $\begin{cases} x^2+y^2+z^2=4 \\ z \geq 0 \end{cases}$ which is enclosed by the cylinder $x^2+y^2=1$ can be represented parametrically by $x = 2\sin\varphi\cos\theta, y = 2\sin\varphi\sin\theta, z = 2\cos\varphi$, where

$0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \frac{\pi}{6}$.

And $\frac{\pi}{6}$ is found by the following way:

$\begin{cases} x^2+y^2+z^2=4 \\ z \geq 0 \\ x^2+y^2=1 \end{cases} \Rightarrow z^2 = 3 \Rightarrow z = \sqrt{3}$. Therefore $z = 2\cos\varphi = \sqrt{3} \Rightarrow \cos\varphi = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{\pi}{6}$

$E = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = 4\cos^2\varphi\cos^2\theta + 4\cos^2\varphi\sin^2\theta + 4\sin^2\varphi = 4$

$G = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 4\sin^2\varphi\sin^2\theta + 4\sin^2\varphi\cos^2\theta = 4\sin^2\varphi$

$F = \frac{\partial x}{\partial \varphi}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \varphi}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \varphi}\frac{\partial z}{\partial \theta} = 4\sin\varphi\cos\theta\cos\varphi\sin\theta - 4\sin\varphi\sin\theta\cos\varphi\cos\theta = 0$

So

$S = \iint_{R_{\varphi\theta}} \sqrt{EG-F^2}d\varphi d\theta = \int_0^{\frac{\pi}{6}} \left[\int_0^{2\pi} \sqrt{16\sin^2\varphi}d\theta\right] d\varphi$

$= 2\pi \int_0^{\frac{\pi}{6}} \sqrt{16\sin^2\varphi}d\varphi = 2\pi \int_0^{\frac{\pi}{6}} 4\sin\varphi d\varphi = (8-4\sqrt{3})\pi$

Let the surface S be represented parametrically by equations $x = x(u,v), y = y(u,v), z = z(u,v)$, where u and v vary in a region R_{uv} of the uv-plane, the surface integral of the first kind for $f(x,y,z)$ which is defined on S is given by $\iint_S f(x,y,z)dS = \iint_{R_{ln}} f(x(u,v), y(u,v), z(u,v))\sqrt{EG-F^2}dudv$, where

$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2, F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$

Example

Let the density $\rho(x,y,z)$ of the upper hemisphere $\begin{cases} x^2+y^2+z^2=4 \\ z \geq 0 \end{cases}$ at (x, y, z) is

$\rho(x,y,z)=\sqrt{x^2+y^2}$.

Find the mass of the part of the upper hemisphere $\begin{cases} x^2+y^2+z^2=4 \\ z \geq 0 \end{cases}$ which is enclosed by the cylinder $x^2+y^2=1$.

Solution:

Mass $= \iint_{R_{\varphi\theta}} f(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}d\varphi d\theta$

$= \int_0^{\frac{\pi}{6}} \left[\int_0^{2\pi} \sqrt{4\sin^2\varphi\cos^2\theta+4\sin^2\varphi\sin^2\theta}\sqrt{16\sin^2\varphi}d\theta\right]d\varphi = \int_0^{\frac{\pi}{6}} \left[\int_0^{2\pi} \sqrt{4\sin^2\varphi}\sqrt{16\sin^2\varphi}d\theta\right]d\varphi$

$= 16\pi\int_0^{\frac{\pi}{6}} \sin^2\varphi d\varphi = 16\pi\int_0^{\frac{\pi}{6}} \frac{1-\cos 2\varphi}{2}d\varphi = 16\pi\left(\frac{\varphi}{2}-\frac{\sin 2\varphi}{4}\right) \mid \frac{\pi}{6} = 16\pi\left(\frac{\pi}{12}-\frac{\sqrt{3}}{8}\right)$

□

Suppose the surface S is represented parametrically by equations $x=x(u,v), y=y(u,v), z=z(u,v)$, where u and v vary in a region R_{uv} of the uv-plane.

Then the unit normals to S at (x,y,z) is :

$\vec{n} = \pm\frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|} = \pm\frac{\left[\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)\vec{i}+\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right)\vec{j}+\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\vec{k}\right]}{\sqrt{|\vec{r}_u|^2|\vec{r}_v|^2-(\vec{r}_u\cdot\vec{r}_v)^2}}$

$= \pm\frac{\frac{\partial(y,z)}{\partial(u,v)}\vec{i}+\frac{\partial(z,x)}{\partial(u,v)}\vec{j}+\frac{\partial(x,y)}{\partial(u,v)}\vec{k}}{\sqrt{\left[\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2\right]\left[\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]-\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)^2}}$

$= \pm\frac{\frac{\partial(y,z)}{\partial(u,v)}\vec{i}+\frac{\partial(z,x)}{\partial(u,v)}\vec{j}+\frac{\partial(x,y)}{\partial(u,v)}\vec{k}}{\sqrt{EG-F^2}}$

The normal chosen is decided by the included angle between the normal and the positive direction of some axis. For instance, if the normal is making an acute angle with the positive direction of z-axis , then we have $\pm\frac{\partial(x,y)}{\partial(u,v)} > 0$ and whether the + sign or − sign chosen is depending on the inequality, say if $\frac{\partial(x,y)}{\partial(u,v)}$ is negative, then $\vec{n} = -\left(\frac{\frac{\partial(y,z)}{\partial(u,v)}\vec{i}+\frac{\partial(z,x)}{\partial(u,v)}\vec{j}+\frac{\partial(x,y)}{\partial(u,v)}\vec{k}}{\sqrt{EG-F^2}}\right)$.

Example

Let $\vec{F} = \vec{i}$. Find $\iint_S \vec{F}\cdot\vec{n}dS$ where S is the portion of the sphere $x^2+y^2+z^2=4$ enclosed by the planes $z=1, z=-\sqrt{3}$, $\begin{cases} y=0 \\ x\geq 0\end{cases}$ and $\begin{cases} x-y=0 \\ x\geq 0\end{cases}$, $\vec{n}$ is the unit normal to S pointing to the origin.

Solution:

Using spherical coordinates, S , the portion of the sphere $x^2+y^2+z^2=4$ enclosed by the planes $z=1$, $z=-\sqrt{3}$, $\begin{cases} y=0 \\ x\geq 0\end{cases}$ and $\begin{cases} x-y=0 \\ x\geq 0\end{cases}$ be represented parametrically by

$x = 2\sin\varphi\cos\theta, y = 2\sin\varphi\sin\theta, z = 2\cos\varphi$ where $0\leq\theta\leq\frac{\pi}{4}, \frac{\pi}{3}\leq\varphi\leq\frac{5\pi}{6}$.

$\iint_S \vec{F}\cdot\vec{n}dS = \iint_{R_{p\theta}} \vec{F}[x(\varphi,\theta),y(\varphi,\theta),z(\varphi,\theta)]\cdot\pm\frac{\frac{\partial(y,z)}{\partial(\varphi,\theta)}\vec{i}+\frac{\partial(z,x)}{\partial(\varphi,\theta)}\vec{j}+\frac{\partial(x,y)}{\partial(\varphi,\theta)}\vec{k}}{\sqrt{EG-F^2}}\sqrt{EG-F^2}d\varphi d\theta$

$= \iint_{R_{\rho\theta}} \vec{F}[x(\varphi,\theta),y(\varphi,\theta),z(\varphi,\theta)]\cdot\pm\left(\frac{\partial(y,z)}{\partial(\varphi,\theta)}\vec{i}+\frac{\partial(z,x)}{\partial(\varphi,\theta)}\vec{j}+\frac{\partial(x,y)}{\partial(\varphi,\theta)}\vec{k}\right)d\varphi d\theta$

$$\begin{cases} x = 2\sin\varphi\cos\theta \\ y = 2\sin\varphi\sin\theta \\ z = 2\cos\varphi \end{cases} \Rightarrow \begin{cases} \frac{\partial x}{\partial\varphi} = 2\cos\varphi\cos\theta, \frac{\partial x}{\partial\theta} = -2\sin\varphi\sin\theta \\ \frac{\partial y}{\partial\varphi} = 2\cos\varphi\sin\theta, \frac{\partial y}{\partial\theta} = 2\sin\varphi\cos\theta \\ \frac{\partial z}{\partial\varphi} = -2\sin\varphi, \frac{\partial z}{\partial\theta} = 0 \end{cases}$$

$\frac{\partial(y,z)}{\partial(\varphi,\theta)} = \begin{vmatrix} 2\cos\varphi\sin\theta & 2\sin\varphi\cos\theta \\ -2\sin\varphi & 0 \end{vmatrix} = 4\sin^2\varphi\cos\theta,$

$\frac{\partial(z,x)}{\partial(\varphi,\theta)} = \begin{vmatrix} -2\sin\varphi & 0 \\ 2\cos\varphi\cos\theta & -2\sin\varphi\sin\theta \end{vmatrix} = 4\sin^2\varphi\sin\theta,$

$\frac{\partial(x,y)}{\partial(\varphi,\theta)} = \begin{vmatrix} 2\cos\varphi\cos\theta & 2\sin\varphi\sin\theta \\ 2\cos\varphi\sin\theta & 2\sin\varphi\cos\theta \end{vmatrix} = 4\sin\varphi\cos\varphi\cos^2\theta - 4\sin\varphi\cos\varphi\sin^2\theta = 4\sin\varphi\cos\varphi.$

Observe that the normal $\vec{n}$ to S makes an obtuse angle with $\vec{i}$ and

$\frac{\partial(y,z)}{\partial(\varphi,\theta)} = \begin{vmatrix} 2\cos\varphi\sin\theta & 2\sin\varphi\cos\theta \\ -2\sin\varphi & 0 \end{vmatrix} = 4\sin^2\varphi\cos\theta > 0$ for $0 < \theta < \frac{\pi}{4}, \frac{\pi}{3} < \varphi < \frac{5\pi}{6}$

therefore, choose $-\left[\frac{\partial(y,z)}{\partial(\varphi,\theta)}\vec{i} + \frac{\partial(z,x)}{\partial(\varphi,\theta)}\vec{j} + \frac{\partial(x,y)}{\partial(\varphi,\theta)}\vec{k}\right]$ as the normal to S.

As a result,

$\iint_S \vec{F}\cdot\vec{n}dS = \iint_{R_{\rho\theta}} \vec{F}[x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta)] \cdot \left(-\left[\frac{\partial(y,z)}{\partial(\varphi,\theta)}\vec{i} + \frac{\partial(z,x)}{\partial(\varphi,\theta)}\vec{j} + \frac{\partial(x,y)}{\partial(\varphi,\theta)}\vec{k}\right]\right) d\varphi d\theta$

$\iint_{R_{\rho\theta}} \vec{F}[x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta)] \cdot \left(-\left[\frac{\partial(y,z)}{\partial(\varphi,\theta)}\vec{i} + \frac{\partial(z,x)}{\partial(\varphi,\theta)}\vec{j} + \frac{\partial(x,y)}{\partial(\varphi,\theta)}\vec{k}\right]\right) d\varphi d\theta$

$= \int_0^{\frac{\pi}{4}} \left(\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \vec{i}\cdot\left[-\left(4\sin^2\varphi\cos\theta\vec{i} + 4\sin^2\varphi\sin\theta\vec{j} + 4\sin\varphi\cos\varphi\vec{k}\right)\right] d\varphi\right) d\theta$

$= \int_0^{\frac{\pi}{4}} \left(\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} -4\sin^2\varphi\cos\theta d\varphi\right) d\theta = \int_0^{\frac{\pi}{4}} \left(\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} 2(\cos 2\varphi - 1)\cos\theta d\varphi\right) d\theta = \int_0^{\frac{\pi}{4}} (\sin 2\varphi - 2\varphi)\Big|_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \cos\theta d\theta$

$= \int_0^{\frac{\pi}{4}} \left(\sin\frac{5\pi}{3} - \frac{5\pi}{3} - \sin\frac{2\pi}{3} + \frac{2\pi}{3}\right)\cos\theta d\theta = \left(\sin\frac{5\pi}{3} - \frac{5\pi}{3} - \sin\frac{2\pi}{3} + \frac{2\pi}{3}\right)\sin\theta\Big|_0^{\frac{\pi}{4}}$

$= \left(\sin\frac{5\pi}{3} - \frac{5\pi}{3} - \sin\frac{2\pi}{3} + \frac{2\pi}{3}\right)\frac{1}{\sqrt{2}} = \frac{-\sqrt{3}-\pi}{\sqrt{2}}$

$\square$

4. Divergence Theorem (Gauss's Theorem)

The net outward flux of a vector field $\vec{v}$ through a closed surface S is $\iint_S \vec{v}\cdot d\vec{S}$. If we take a point P and surround it by a surface S enclosing a volume V we have

Average flux per unit volume $= \frac{1}{V}\iint_S \vec{v}\cdot d\vec{S}$

Now let $V \to 0$ so that the volume shrinks to the point P. The limiting value of the average flux per unit volume is then the divergence of $\vec{v}$ at P. This gives an alternative definition of div $\vec{v}$

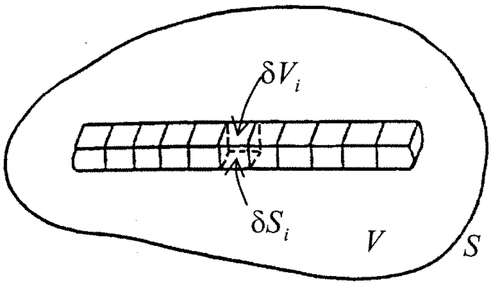
div $\vec{v} = \lim_{V\to 0}\left(\frac{1}{V}\iint_S \vec{v}\cdot d\vec{S}\right)$

Divergence Theorem (Gauss's Theorem):

For any closed surface S, enclosing a region V in a vector field $\vec{F}$, $\iiint_V div\vec{F}dxdydz = \iint_S \vec{F}\cdot d\vec{S}$, that is, the total divergence of $\vec{F}$ from a volume $V =$ total outward flux of $\vec{F}$ through S.

Proof:

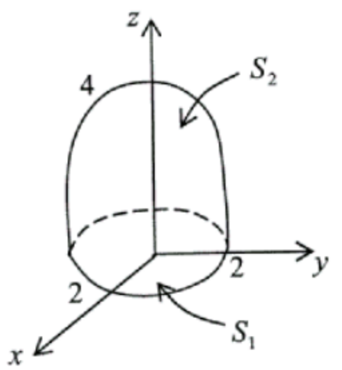
Divide the volume V into N small blocks δV_i with surface δS_i.



For each small volume, We have approximately div $\vec{F} = \frac{1}{\delta V_i}\iint_{\delta S_i}\vec{F}\cdot d\vec{S}$ or $div\vec{F}\delta V_i = \iint_{\delta S_i}\vec{F}\cdot d\vec{S}$, that is, div $\vec{F}\delta V_i =$ total outward flux of $\vec{F}$ across δS_i. Summing over all blocks and letting $N \to \infty$ as $\delta V_i \to 0$ we have $LHS = \iiint_V div\vec{F}dxdydz$ For the RHS we observe that flux out of a face of one block equals the flux into the face of an adjacent block, unless that face is part of the surface S. Hence for all internal faces the flux cancels, leaving $RHS = \iint_S \vec{F}\cdot d\vec{S}$ and the theorem is proved.

Example

Verify the divergence theorem for $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$ and the region bounded by $z = 4 - x^2 - y^2$ and the plane $z = 0$.



Solution:

div $\vec{F} = 3$

$\iiint_V div\vec{F}dxdydz = \iint_{S_1}\left(\int_0^{4-x^2-y^2} 3dz\right) = \iint_{S_1}(3z)\Big|_0^{4-x^2-y^2} = \iint_{S_1} 3\left(4-x^2-y^2\right)dxdy$

$\underset{\substack{x=r\cos\theta\\ y=r\sin\theta}}{=} \int_0^{2\pi}\left[\int_0^2 3\left(4-r^2\right)rdr\right]d\theta = 3\int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_0^2 d\theta = 3\int_0^{2\pi} 4d\theta = 24\pi$

Now consider $\iint_S \vec{F}\cdot d\vec{S}$. For the circular "base" we have

$\iint_{S_1}(x\vec{i}+y\vec{j}+z\vec{k})\cdot(-\vec{k})dxdy = \iint_{S_1} -zdxdy = 0$ as $z=0$ on S_1

For the curved surface $\varphi(x,y,z) = z+x^2+y^2 = 4, \nabla\varphi = 2x\vec{i}+2y\vec{j}+\vec{k}, \varphi_z = 1$

$\iint_{S_2}\vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\leq 4}(x\vec{i}+y\vec{j}+z\vec{k})\cdot(2x\vec{i}+2y\vec{j}+\vec{k})\frac{1}{1}dxdy = \iint_{x^2+y^2\leq 4}\left(2x^2+2y^2+z\right)dxdy$

$= \iint_{x^2+y^2\leq 4}\left(2x^2+2y^2+4-x^2-y^2\right)dxdy \underset{\substack{x=r\cos\theta\\ y=r\sin\theta}}{=} \int_0^2\left[\int_0^{2\pi}\left(4+r^2\right)rd\theta\right]dr = 2\pi\left(2r^2+\frac{r^4}{4}\right)\Big|_0^2 = 24\pi$

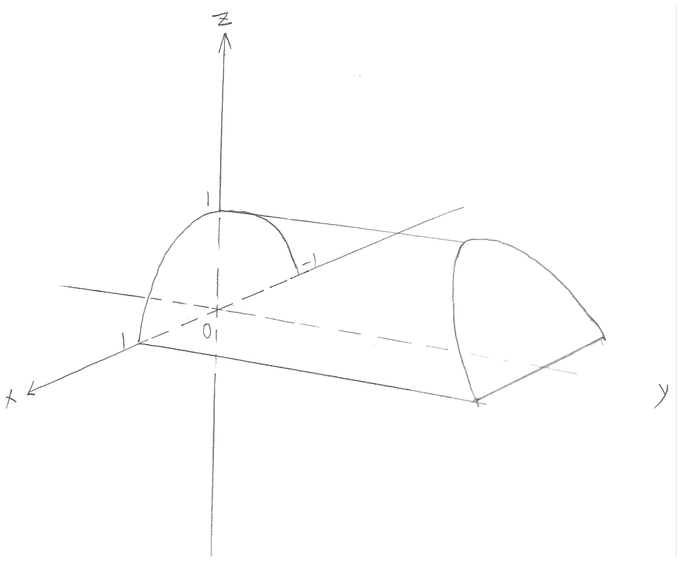
Hence $\iint_S \vec{F}\cdot d\vec{S} = \iiint_V div\vec{F}dxdydz$

□

Example

Calculate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = xy\vec{i}+\left(y^2+e^{xz^2}\right)\vec{j}+\sin(xy)\vec{k}$, S is the close surface which is bounded by $z=1-x^2, z=0, y=0$ and $y+z=2$

Solution:



It is difficult to compute $\iint_S \vec{F} \cdot d\vec{S}$ directly, by Divergence theorem we compute
$\iiint_V \operatorname{div} \vec{F} dxdydz = \iiint_V \left[\frac{\partial(xy)}{\partial x} + \frac{\partial\left(y^2+e^{xz^2}\right)}{\partial y} + \frac{\partial(\sin xy)}{\partial z}\right] dxdydz$instead, where V is the solid enclosed by S.

The projection of V onto xz-plane is σ_{xz} which is bounded by $z = 0, z = 1 - x^2$. So
$\iiint_V div\vec{F}dxdydz = \iiint_V 3ydxdydz = \iint_{\sigma_{xz}} \left[\int_0^{2-z} 3ydy\right] dxdz = \int_{-1}^{1} \int_0^{1-x^2} \left[\int_0^{2-z} 3ydy\right] dz\Big) dx =$
$\int_{-1}^{1} \left(\int_0^{1-x^2} \frac{3}{2}(2-z)^2 dz\right) dx$
$= \int_{-1}^{1} - \frac{1}{2}(2-z)^3\Big|_0^{1-x^2} dx = -\frac{1}{2}\int_{-1}^{1} \left[(x^2+1)^3 - 8\right] dx = -\frac{1}{2}\int_{-1}^{1} [x^6 + 3x^4 + 3x^2 - 7]\, dx =$
$-\int_0^1 [x^6 + 3x^4 + 3x^2 - 7]\, dx$
$\left(-\frac{x^7}{7} - \frac{3x^5}{5} - x^3 + 7x\right)\Big|_0^1 = -\frac{1}{7} - \frac{3}{5} - 1 + 7 = \frac{184}{35}$

If you prefer to project V on xy-plane, $\iiint_V div\vec{F}dxdydz$ should be as follows:

We observe that the surface $z = 1 - x^2$ and the plane $y + z = 2$ intersect at the line $\begin{cases} z = 1 - x^2 \\ y + z = 2 \end{cases}$ and the projection of the line on xy-plane is $y + 1 - x^2 = 2 \Leftrightarrow y - x^2 = 1$. The projection of V onto xy-plane is σ_{xy} which consists of two parts, say, $\sigma_{xy} = \overline{\sigma_{xy}} + \overline{\overline{\sigma_{xy}}}$, $\overline{\sigma_{xy}}$ is bounded by
$x = -1, x = 1, y = 0, y - x^2 = 1$ and $\overline{\overline{\sigma_{xy}}}$ is bounded by $x = -1, x = 1, y - x^2 = 1, y = 2$. And over $\overline{\sigma_{xy}}$ he upper surface of the solid is $z = 1 - x^2$ and the lower surface is $z = 0$. Over $\overline{\overline{\sigma_{xy}}}$ he upper surface of the solid is $y + z = 2$ and the lower surface is $z = 0$. Thus

$\iiint_V 3ydxdydz = \iint_{\overline{\sigma_{xy}}} \left[\int_0^{1-x^2} 3ydz\right] dxdy + \int_{\overline{\overline{\sigma_{xy}}}} \left[\int_0^{2-y} 3ydz\right] dxdy$
$= \int_{-1}^{1} \left[\int_0^{1+x^2} \left(\int_0^{1-x^2} 3ydz\right) dy\right] dx + \int_{-1}^{1} \left[\int_{1+x^2}^{2} \left(\int_0^{2-y} 3ydz\right) dy\right] dx$
$= \int_{-1}^{1} \left[\int_0^{1+x^2} 3y\left(1-x^2\right) dy\right] dx + \int_{-1}^{1} \left[\int_{1+x^2}^{2} 3y(2-y)dy\right] dx$
$= \int_{-1}^{1} \left(\frac{3}{2}\left(1+x^2\right)^2\left(1-x^2\right)\right) dx + \int_{-1}^{1} \left(3y^2-y^3\right)\Big|_{1+x^2}^{2} dx$
$= \int_{-1}^{1} \frac{3}{2}\left(1+x^2\right)^2\left(1-x^2\right) dx + \int_{-1}^{1} \left(4-3\left(1+x^2\right)^2+\left(1+x^2\right)^3\right) dx$
$= \int_0^1 3\left(1+2x^2+x^4\right)\left(1-x^2\right) dx + 2\int_0^1 \left(4-3\left(1+2x^2+x^4\right)+\left(1+3x^2+3x^4+x^6\right)\right) dx$
$= \int_0^1 3\left(1+x^2-x^4-x^6\right) dx + 2\int_0^1 \left(2-3x^2+x^6\right) dx = \int_0^1 \left(7-3x^2-3x^4-x^6\right) dx$
$= \left(7x-x^3-\frac{3}{5}x^5-\frac{1}{7}x^7\right)\Big|_0^1 = 7-1-\frac{3}{5}-\frac{1}{7} = \frac{184}{35}$

□

Example– Gauss's Law

Gauss's law for an electric field states that the total displacement flux (of $\varepsilon_0\vec{E}$) through a closed surface equals the total charge within the volume enclosed by the surface, that is,
$\iint_S \varepsilon_0\vec{E}\cdot d\vec{S} = \iiint_V \rho dV$, where E is the electric intensity and ρ is the charge density.
Proof: Applying the divergence theorem to the LHS gives $\iint_S \varepsilon_0\vec{E}\cdot d\vec{S} = \iiint_V \varepsilon_0 \operatorname{div}\vec{E}dV = \iiint_V \rho dV$. Since the region V is arbitrary, it must be true that $\varepsilon_0 \operatorname{div}\vec{E} = \rho$, that is, $\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$– one of Mxwell's equations of electromagnetism. Since $\vec{E} = -\nabla\varphi$ where φ is the electrostatic potential,
$-\operatorname{div}\operatorname{grad}\varphi = -\frac{\rho}{\varepsilon_0}$ or $\nabla\cdot\nabla\varphi = \nabla^2\varphi = -\frac{\rho}{\varepsilon_0}, \nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, which is Poisson's equation. In a region of no charge ($\rho = 0$) the electrostatic potential satisfies Laplace's equation $\nabla^2\varphi = 0$. These are important equations in electromagnetic field theory.

□

5. Stokes' Theorem

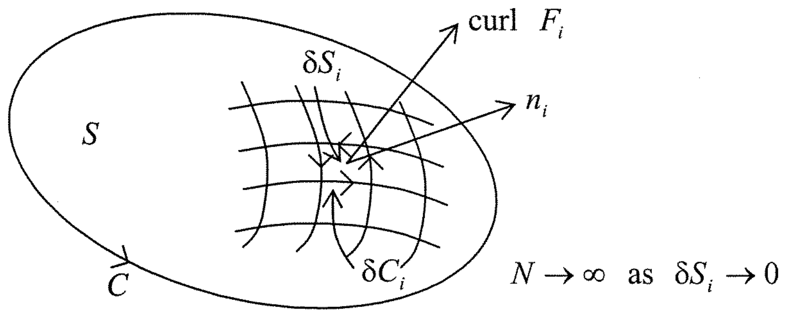
Stokes' Theorem:

For any open surface S, having a closed curve C as its edge, in a vector field $\iint_S curl\vec{F}\cdot d\vec{S}=\oint_C \vec{F}\cdot d\vec{r}$ that is, the flux of curl $\vec{F}$ through an open surface S= the circulation around the boundary curve C.

Proof:

Divide the surface S into N small patches δS_i with edge δC_i and normal $\vec{n}_i$. For each small area δS_i we have approximately curl $\vec{F}_i\cdot\vec{n}_i=\frac{1}{\delta S_i}\oint_{\delta C_i}\vec{F}\cdot d\vec{r}$ or curl $\vec{F}_i\cdot\vec{n}_i\delta S_i=\oint_{\delta C_i}\vec{F}\cdot\overrightarrow{dr}$.

Summing over all patches and letting $N\to\infty$ as $\delta S_i\to 0$ we have LHS $=\iint_S$ curl $\vec{F}\cdot d\vec{S}$. For the RHS we observe that the flow along an edge of δC_i is equal and opposite to the flow along that edge of an adjacent element, unless that edge is part of the boundary C. Hence for all internal edges the flow cancels leaving RHS $=\oint_C\vec{F}\cdot d\vec{r}$ and the theorem is proved.



Example

Verify Stokes' theorem for $\vec{F}=2xy\vec{i}+x^2\vec{j}-x\vec{k}$ where S is the hemisphere $x^2+y^2+z^2=1, y\geq 0$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy & x^2 & -x\end{vmatrix}=\vec{j}$$

Solution:

We project S onto the xz-plane where S_{xz} is the disc $x^2+z^2\leq 1$. We choose the outward normal which makes an acute angle with j. Thus for $\varphi(x,y,z)=x^2+y^2+z^2=1$ we choose

$\nabla\varphi=2x\vec{i}+2y\vec{j}+2z\vec{k}, \varphi_y=2y$

$\iint_S curl\vec{F}\cdot d\vec{S} = \iint_{x^2+z^2\le 1}\vec{j}\cdot\frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2y}dxdz = \iint_{x^2+z^2\le 1}1dxdz = \pi$

C is the circle $x^2+z^2=1$ in the xz-plane. According to Stokes' Theorem, looking from the positive direction of y axis the direction of the line integral along the circle $x^2+z^2=1$ is anticlockwise. Let the parametric equations of C be $z=\cos t, x=\sin t, y=0, 0\le t<2\pi$

$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left(2xy\vec{i}+x^2\vec{j}-x\vec{k}\right)\cdot\left(\frac{dx}{dt}\vec{i}+\frac{dy}{dt}\vec{j}+\frac{dz}{dt}\vec{k}\right)dt = \int_0^{2\pi}\sin^2 t dt = \pi$

Hence the theorem is verified.

Note the positive direction of C taken in the zx-plane, relative to the normal of S,

□

<u>Example</u>

Compute $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(y-z)\vec{i}+(z-x)\vec{j}+(x-y)\vec{k}$ and C is the ellipse $\begin{cases} x^2+y^2=a^2 \\ \frac{x}{a}+\frac{z}{h}=1, \end{cases}$, where $a, h>0$

The direction of the path C is anticlockwise when looking from the positive direction of x-axis.

Solution:

For Stokes' Theorem, the index finger of the right hand represents the direction of the line integral along the curve, the middle finger of the right hand points to the interior of the open surface which is enclosed by the curve, that is, the boundary of the open surface and the thumb finger of the right hand represents the normal to the surface which is closed the boundary. Notice the three vectors form a so called positive triple.

According to Stokes' Theorem we instead compute the corresponding surface integral rather than the line integral. By Stokes' Theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_S curl\vec{F}\cdot d\vec{S}$, where S is the open surface with C as its boundary.

$$\text{curl}\,\vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{pmatrix} = -2\vec{i}-2\vec{j}-2\vec{k}$$

The normal to S : $\phi(x,y,z)=\frac{x}{a}+\frac{z}{h} \Rightarrow \pm\nabla\phi(x,y,z) = \pm\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial z}\vec{k}\right) = \pm\left(\frac{1}{a}\vec{i}+\frac{1}{h}\vec{k}\right)$

Since the direction of the line integral along the path C is anticlockwise when looking from the positive direction of x-axis and we are going to project the surface onto xy-plane according to Stokes' Theorem, we have to choose $+\nabla\phi$, that is, $\frac{1}{a}\vec{i}+\frac{1}{h}\vec{k}$. So

$\oint_C \vec{F}\cdot d\vec{r} = \iint_S curl\vec{F}\cdot d\vec{S} = \iint_S(-2\vec{i}-2\vec{j}-2\vec{k})\cdot d\vec{S} = \iint_{x^2+y^2\leq a^2}(-2\vec{i}-2\vec{j}-2\vec{k})\cdot \frac{\left(\frac{1}{a}\vec{i}+\frac{1}{h}\vec{k}\right)}{\left|\frac{1}{h}\right|}dxdy$

$= \iint_{x^2+y^2\leq a^2}(-2\vec{i}-2\vec{j}-2\vec{k})\cdot\left(\frac{h}{a}\vec{i}+\vec{k}\right)dxdy = \iint_{x^2+y^2\leq a^2}\left(-\frac{2h}{a}-2\right)dxdy = -2\iint_{x^2+y^2\leq a^2}\frac{h+a}{a}dxdy$

$= -2\frac{(h+a)a^2\pi}{a} = -2\pi a(h+a)$

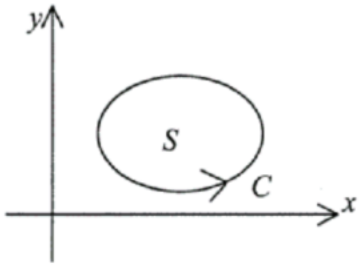
□

Example – Green's Theorem in the Plane

If the region S lies completely in the xy-plane, and $\vec{F}=F_1\vec{i}+F_2\vec{j}$, then

$curl\vec{F} = \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\vec{k}, dS = \vec{k}dxdy$ and we have

$\iint_S \text{curl}\,\vec{E}\cdot d\vec{S} = \iint_S\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\vec{k}\cdot dxdy\vec{k} = \iint_S\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy = \oint_C \vec{F}\cdot d\vec{r} = \oint_C\left(F_1\vec{i}+F_2\vec{j}\right)\cdot(dx\vec{i}+dy\vec{j})$



giving $\iint_S\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)dxdy = \oint_C F_1dx+F_2dy$, which is known as Green's Theorem in the Plane.

□

Example

Faraday's Law for a time-varying electric field states that the tangential component of $\vec{E}$, the electric intensity, measured around a closed curve C equals the negative rate of change with respect to time of the total magnetic flux through any surface bounded by C, that is, $\oint_C \vec{E}\cdot d\vec{r} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot d\vec{S}$, where $\vec{B}$ is the magnetic flux density.

Solution:

Applying Stoke's theorem to the LHS gives $\oint_C \vec{E} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{E} \cdot d\vec{S}$

Hence, $\iint_S curl\vec{E} \cdot d\vec{S} = \iint_S -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$ and since the surface S is arbitrary we have $\text{curl}\, \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

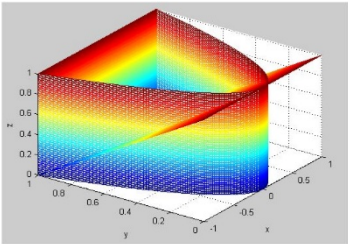
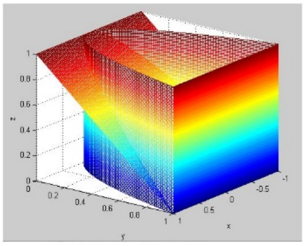
This is another of Maxwell's equations.

Example

(a) Let $\vec{F} = (y^2 - z^2)\, \vec{i} + (z^2 - x^2)\, \vec{j} + (x^2 - y^2)\, \vec{k}$.

Find a vector field $\vec{H} = h_1(x, y, z)\vec{i} + h_2(x, y, z)\vec{j} + h_3(x, y, z)\vec{k}$ with $h_3(x, y, z) \equiv 0$ such that $\vec{F} = \nabla \times \vec{H}$.

(b) Compute $\oint_C \vec{H} \cdot d\vec{r}$ where C is the path which is the intersection of the plane $y + z = 1$ and the faces of a surface formed by $\begin{cases} y = x^2 \\ -1 \leq x \leq 1 \end{cases}$ and $\begin{cases} y = 1 \\ -1 \leq x \leq 1 \end{cases}$. The direction of the line integral $\oint_C \vec{H} \cdot d\vec{r}$ along the path C is clockwise when observing at the point (0,2,0).

Solution:

(a)

$\vec{F} = (y^2 - z^2)\, \vec{i} + (z^2 - x^2)\, \vec{j} + (x^2 - y^2)\, \vec{k}$. Observe that $\nabla \cdot \vec{F} = 0$.

$$\nabla \times \vec{H} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ h_1 & h_2 & h_3 \end{pmatrix} = \left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right) \vec{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \right) \vec{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right) \vec{k}$$

$\vec{F} = \nabla \times \vec{H}$ and $h_3 \equiv 0 \Rightarrow (y^2 - z^2)\, \vec{i} + (z^2 - x^2)\, \vec{j} + (x^2 - y^2)\, \vec{k} = -\frac{\partial h_2}{\partial z}\vec{i} + \frac{\partial h_1}{\partial z}\vec{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right) \vec{k}$

That is, $\begin{cases} y^2 - z^2 = -\frac{\partial h_2}{\partial z} \\ z^2 - x^2 = \frac{\partial h_1}{\partial z} \\ x^2 - y^2 = \frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \end{cases}$

$z^2 - x^2 = \frac{\partial h_1}{\partial z} \Rightarrow h_1 = \frac{z^3}{3} - x^2 z, \quad y^2 - z^2 = -\frac{\partial h_2}{\partial z} \Rightarrow h_2 = \frac{z^3}{2} - y^2 z + C(x, y).$

$x^2 - y^2 = \frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} = \frac{\partial C}{\partial x} \Rightarrow C(x, y) = \frac{x^3}{3} - xy^2$

Thus, $h_2 = \frac{z^3}{2} - y^2 z + \frac{x^3}{3} - xy^2$

So, $\vec{H} = \left(\frac{z^3}{3} - x^2 z\right)\vec{i} + \left(\frac{z^3}{2} + \frac{x^3}{3} - y^2 z - xy^2\right)\vec{j}$. (Notice that $\vec{H}$ is <u>not unique</u>.)

(b)

According to the Stokes' theorem, we have $\oint_C \vec{H} \cdot d\vec{r} = \iint_S \nabla \times \vec{H} \cdot d\vec{S} = \iint_S \vec{F} \cdot d\vec{S}$, where S is the surface lying in $y + z = 1$ with C as its boundary. According to the right hand rule, the normal to S makes an obtuse angle with $\vec{k}$.

$\phi(x, y, z) = y + z \Rightarrow \pm\nabla\phi = \pm(\vec{j} + \vec{k})$.We choose $-\nabla\phi = -(\vec{j} + \vec{k})$ and observe that $|\phi_z| = 1$. The projection of S onto xy-plane is the region in xy-plane bounded by $y = x^2, y = 1$.

$\oint_C \vec{H} \cdot d\vec{r} = \iint_S \nabla \times \vec{H} \cdot d\vec{S} = \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_{\sigma_{vy}} \vec{F} \cdot \frac{-\nabla\phi}{|\phi_z|} dxdy$

$= \iint_{\sigma_{xy}} \left[(y^2 - z^2)\vec{i} + (z^2 - x^2)\vec{j} + (x^2 - y^2)\vec{k}\right] \cdot \frac{-(\vec{j}+\vec{k})}{1} dxdy = \iint_{\sigma_{xy}} (x^2 - z^2 + y^2 - x^2)\, dxdy$

$= \iint_{\sigma_{xy}} (y^2 - z^2)\, dxdy = \iint_{\sigma_{xy}} [y^2 - (1 - y)^2]\, dxdy = \iint_{\sigma_{xy}} (2y - 1) dxdy = \int_{-1}^{1} \left[\int_{x^2}^{1} (2y - 1) dy\right] dx$

$= \int_{-1}^{1} (y^2 - y)\Big|_{x^2}^{1} dx$

$= \int_{-1}^{1} (x^2 - x^4)\, dx = \int_0^1 2\,(x^2 - x^4)\, dx = \left(\frac{2x^3}{3} - \frac{2x^5}{5}\right)\Big|_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$