1. Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$.

Solution:

(a)

Let $A = \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$, then $A - \lambda I = \begin{pmatrix} 3-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix}$ and

$\det(A - \lambda I) = (3-\lambda)(4-\lambda) - 6 = 12 - 7\lambda + \lambda^2 - 6 = (\lambda - 6)(\lambda - 1)$.

Eigenvalues of A are 1, 6.

When $\lambda = 1$, $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 = -x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k\begin{pmatrix} -1 \\ 1 \end{pmatrix}, k \neq 0$.

When $\lambda = 6$, $\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 = \frac{2}{3}x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k\begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}, k \neq 0$.

(b)

Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$, then $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{pmatrix}$ and

$\det\ (A - \lambda I) = (5-\lambda)[(2-\lambda)(2-\lambda) - 1] = (5-\lambda)[4 - 4\lambda + \lambda^2 - 1] = (5-\lambda)(\lambda - 3)(\lambda - 1)$.

Eigenvalues of A are 1, 3, 5.

When $\lambda = 1$,

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_3 = 0, x_1 = -x_2$.

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, k \neq 0$

When $\lambda = 3$,

$$\begin{pmatrix}-1&1&1\\1&-1&1\\0&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Leftrightarrow\left(\begin{array}{ccc|c}-1&1&1&0\\1&-1&1&0\\0&0&2&0\end{array}\right)\sim\left(\begin{array}{ccc|c}-1&1&1&0\\0&0&2&0\\0&0&2&0\end{array}\right)\sim\left(\begin{array}{ccc|c}-1&1&1&0\\0&0&2&0\\0&0&0&0\end{array}\right)$$

$$\Rightarrow x_3=0, x_1=x_2\Rightarrow\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=k\begin{pmatrix}1\\1\\0\end{pmatrix}, k\neq 0$$

When $\lambda=5$,

$$\begin{pmatrix}-3&1&1\\1&-3&1\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Leftrightarrow\left(\begin{array}{ccc|c}1&-3&1&0\\-3&1&1&0\\0&0&0&0\end{array}\right)\sim\left(\begin{array}{ccc|c}1&-3&1&0\\0&-8&4&0\\0&0&0&0\end{array}\right)$$

$$\Rightarrow x_2=\frac{1}{2}x_3, x_1=3x_2-x_3=\frac{1}{2}x_3\Rightarrow\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=k\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\1\end{pmatrix}, k\neq 0$$

(c)

Let $A=\begin{pmatrix}1&0&\sqrt{2}\\0&2&0\\\sqrt{2}&0&0\end{pmatrix}$, then $A-\lambda I=\begin{pmatrix}1-\lambda&0&\sqrt{2}\\0&2-\lambda&0\\\sqrt{2}&0&-\lambda\end{pmatrix}$ and

$$\det\ (A-\lambda I)=(1-\lambda)(2-\lambda)(-\lambda)-2(2-\lambda)=(2-\lambda)\left(-\lambda+\lambda^2-2\right)=(2-\lambda)(\lambda-2)(\lambda+1)$$

Eigenvalues of A are - 1, 2.

When $\lambda=-1$,

$$\begin{pmatrix}2&0&\sqrt{2}\\0&3&0\\\sqrt{2}&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Leftrightarrow\left(\begin{array}{ccc|c}2&0&\sqrt{2}&0\\0&3&0&0\\\sqrt{2}&0&1&0\end{array}\right)\sim\left(\begin{array}{ccc|c}2&0&\sqrt{2}&0\\0&3&0&0\\0&0&0&0\end{array}\right)$$

$$\Rightarrow x_2=0, x_1=-\frac{\sqrt{2}}{2}x_3\Rightarrow\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=k\begin{pmatrix}-\frac{\sqrt{2}}{2}\\0\\1\end{pmatrix}, k\neq 0$$

When $\lambda=2$,

$$\begin{pmatrix}-1&0&\sqrt{2}\\0&0&0\\\sqrt{2}&0&-2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Leftrightarrow\left(\begin{array}{ccc|c}-1&0&\sqrt{2}&0\\0&0&0&0\\0&0&0&0\end{array}\right)\Rightarrow x_1=\sqrt{2}x_3$$

Let $x_2 = t,\ x_3 = k$, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \sqrt{2}k \\ t \\ k \end{pmatrix} = \begin{pmatrix} \sqrt{2}k \\ 0 \\ k \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = k\begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$,where $k^2 + t^2 \neq 0$

□

2. For the matrix in question 1(b), find a matrix P such that $P^{-1}AP = D$, a diagonal matrix with the eigenvalues of A as its elements. Check your solution by evaluating $P^{-1}AP$.

Solution:

Let $P = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$, then $AP = PD$. That is $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, where

$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Since the eigenvalues of $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ are distinct , columns of $P = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$ are linearly independent. It follows that P^{-1} exists and $AP = PD \Rightarrow P^{-1}AP = D$.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1 & 0 & 0 \\ 1 & 1 & 1/2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

We have $P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By matrix multiplication, we observe that

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

□

3. Find a 2×2 matrix A which has eigenvalue 1 with corresponding eigenvector $\begin{pmatrix}1\\0\end{pmatrix}$ and eigenvalue 3 with corresponding eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$.

Solution:

Let $P=\begin{pmatrix}1&1\\0&1\end{pmatrix}$. To evaluate P^{-1}, we have $\left(\begin{array}{cc|cc}1&1&1&0\\0&1&0&1\end{array}\right)\sim\left(\begin{array}{cc|cc}1&0&1&-1\\0&1&0&1\end{array}\right)$. Therefore $P^{-1}=\begin{pmatrix}1&-1\\0&1\end{pmatrix}$

$$AP=PD\Rightarrow A=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&3\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&3\\0&3\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&2\\0&3\end{pmatrix}$$

□

4. (a) Using Gaussian elimination, find a matrix X such that

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\-1&0&1&1\end{pmatrix}X=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

(b) With the help of the result of (a), find a 4×4 matrix A which has eigenvalues $-1,0,0,1$ with corresponding eigenvectors $\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\\1\end{pmatrix}$, which are <u>rows</u> of $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\-1&0&1&1\end{pmatrix}$.

Solution:

(a)

$$\left(\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&1&1&0&0&0&1&0\\-1&0&1&1&0&0&0&1\end{array}\right)\underset{r_4+r_1}{\sim}\left(\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&1&1&0&0&0&1&0\\0&0&1&1&1&0&0&1\end{array}\right)$$

$$\underset{r_3-r_2}{\sim}\left(\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&-1&1&0\\0&0&1&1&1&0&0&1\end{array}\right)\underset{r_4-r_3}{\sim}\left(\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&-1&1&0\\0&0&0&1&1&1&-1&1\end{array}\right)$$

Therefore,

$$X=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\1&1&-1&1\end{pmatrix}.$$

(b)

From (a) we have $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}$.

Therefore

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}^{T}\right]^{-1} = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}^{-1}\right]^{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}^{T} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$A\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

□

5. Given that $\xi = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{pmatrix}$.

(a) Find a and b.
(b) Find the eigenvalues and eigenvectors of A.
(c) Is A diagonalizable? Please give reasons.

Solution:

(a)

$$A\xi = \lambda\xi \Leftrightarrow \begin{pmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 \\ 5 + a - 3 \\ -1 + b + 2 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix} \Rightarrow \lambda = -1,\ a = -3,\ b = 0$$

(b)

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix},\ \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = -(\lambda^3 + 3\lambda^2 + 3\lambda + 1) = -(\lambda + 1)^3$$

For $\lambda = -1$, we have

$$\begin{pmatrix} 3 & -1 & 2 \\ 5 & -2 & 3 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 5 & -2 & 3 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & -1 & 2 & 0 \\ 5 & -2 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_3 = k$ then $x_2 = -k$, $x_1 = -k$ and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is one of the eigenvectors associated with $\lambda = -1$.

(c)
Only one eigenvector is found, A is not diagonalizable.

□

6. (a) Find the eigenvalues and corresponding eigenvectors of the symmetric matrix
$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and verify that the eigenvectors are mutually orthogonal.

(b) Let $B = A^5 - 5(A+3I)^{-1} + 3A^T$. Find the eigenvalues of B.

Solution:
(a)

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix},\ A - \lambda I = \begin{pmatrix} 4-\lambda & -2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}.$$

det $(A-\lambda I) = (3-\lambda)[(4-\lambda)(1-\lambda)-4] = (3-\lambda)[4-5\lambda+\lambda^2-4] = (3-\lambda)(\lambda-5)\lambda$

$\therefore$ eigenvalues of $A = 3,\ 5,\ 0$

When $\lambda = 3$

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{cases} x_1 = 2x_2 \\ 2x_1 = -2x_2 \end{cases} \Rightarrow x_1 = x_2 = 0,\ x_3 = k.$$

Eigenvectors: $k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \overrightarrow{x_1},\ k \neq 0$. We choose the eigenvector $\overrightarrow{x_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

When $\lambda = 0$

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_3 = 0,\ -2x_1 + x_2 = 0,\ x_1 = \frac{x_2}{2} = k.$$

Eigenvectors: $k\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \overrightarrow{x_2},\ k \neq 0$. We choose the eigenvector $\overrightarrow{x_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

When $\lambda = 5$

$$\begin{pmatrix} -1 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_3 = 0,\ \frac{-x_1}{2} = x_2 = k .$$

Eigenvectors: $k\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \overrightarrow{x_3},\ k \neq 0$. We choose the eigenvector $\overrightarrow{x_3} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

Observe that $\overrightarrow{x_1} \cdot \overrightarrow{x_2} = 0,\ \overrightarrow{x_2} \cdot \overrightarrow{x_3} = 0,\ \overrightarrow{x_1} \cdot \overrightarrow{x_3} = 0$.

(b)

Consider $B = A^5 - 5(A+3I)^{-1} + 3A^T$

Let $M = \left[\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}\right] = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.Then $AM = MD$,where $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Therefore $A = MDM^{-1}$. In addition, we have

$A^5 = MD^5M^{-1} = M\begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5^5 \end{pmatrix}M^{-1} \Leftrightarrow A^5M = M\begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5^5 \end{pmatrix}$. So A^5 has eigenvalues $3^5, 0, 5^5$ and their corresponding eigenvectors vectors are the corresponding columns of *M*. Similarly,

$$A + 3I = MDM^{-1} + M(3I)M^{-1} = M\left(D + 3I\right)M^{-1} = M\begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}M^{-1},$$

$$\left(A+3I\right)^{-1} = \left[M\begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}M^{-1}\right]^{-1} = \left(M^{-1}\right)^{-1}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}^{-1} M^{-1} = M\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix}M^{-1}$$

$$5\left(A+3I\right)^{-1} = M\begin{pmatrix} \frac{5}{6} & 0 & 0 \\ 0 & \frac{5}{3} & 0 \\ 0 & 0 & \frac{5}{8} \end{pmatrix}M^{-1},\ \ 3A^T = 3A = M(3D)M^{-1} = M\begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{pmatrix}M^{-1}$$

It follows that

$$B = A^5 - 5(A+3I)^{-1} + 3A^T$$

$$= M\left[\begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5^5 \end{pmatrix} - \begin{pmatrix} \frac{5}{6} & 0 & 0 \\ 0 & \frac{5}{3} & 0 \\ 0 & 0 & \frac{5}{8} \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{pmatrix}\right]M^{-1} = M\begin{pmatrix} 3^5 - \frac{5}{6} + 9 & 0 & 0 \\ 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 5^5 - \frac{5}{8} + 15 \end{pmatrix}M^{-1}$$

So

$$BM = M\begin{pmatrix} 3^5-\frac{5}{6}+9 & 0 & 0 \\ 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 5^5-\frac{5}{8}+15 \end{pmatrix}$$

Eigenvalues of B are :

$3^5-\frac{5}{6}+9=251.1666$, $-\frac{5}{3}=-1.6666$, $5^5-\frac{5}{8}+15=3139.375$

Remark: Let $D=P^{-1}AP$, where P is non-singular. We observe that:

$\det(D-\lambda I)=\det\left(P^{-1}AP-\lambda I\right)=\det\left(P^{-1}AP-\lambda P^{-1}IP\right)$

$=\det\left[P^{-1}(A-\lambda I)P\right]=\det\left(P^{-1}\right)\det(A-\lambda I)\det(P)=\det(A-\lambda I)$

Thus, A has the same eigenvalues as D. We also notice that the eigenvalus of D are just the main diagonal entries.

7. Consider $A=\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix}$

(a) Find rank A.

(b) Find a column vector $\vec{x}$ and a row vector $\vec{y}^{\mathrm{T}}$, where $\vec{x},\ \vec{y}\in R^5$ such that $A=\vec{x}\vec{y}^{\mathrm{T}}$.

(c) Show that 0 is an eigenvalue of A and find its corresponding independent eigenvectors.

(d) Does A have eigenvalues other than 0? If yes, find those eigenvalues and the corresponding independent eigenvectors.

(e) Find an invertible matrix P and a diagonal matrix D such that $AP=PD$.

(f) Find all eigenvalues of A^2+3I_5, where I_5 is the 5×5 unit matrix.

Solution:

(a)

Column 2 of A is a nonzero column and all other columns of A are scalar multiple of it.

So $\operatorname{rank} A=\operatorname{rank}\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix}=1$

(b)

$$\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 & 6 & 0.5 \end{pmatrix}$$

(c)

Since rank $A = 1 < 5$, the equation $\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ has nontrivial solutions and any nontrivial solutions of the homogeneous equations are eigenvectors of A corresponding to the eigenvalue 0.

$$\left(\begin{array}{ccccc|c} 8 & 2 & 4 & 12 & 1 & 0 \\ 4 & 1 & 2 & 6 & 0.5 & 0 \\ 12 & 3 & 6 & 18 & 1.5 & 0 \\ 6 & 1.5 & 3 & 9 & 0.75 & 0 \\ 18 & 4.5 & 9 & 27 & 2.25 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 8 & 2 & 4 & 12 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Consider $8x_1 + 2x_2 + 4x_3 + 12x_4 + x_5 = 0$.

Let $x_2 = a, x_3 = b, x_4 = c, x_5 = d$ then $x_1 = -\frac{1}{4}a - \frac{1}{2}b - \frac{3}{2}c - \frac{1}{8}d$. So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -0.25a - 0.5b - 1.5c - 0.125d \\ a \\ b \\ c \\ d \end{pmatrix} = a \begin{pmatrix} -0.25 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -0.5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1.5 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -0.125 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$. Then

$\begin{pmatrix} -0.25 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1.5 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.125 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are independent eigenvectors of A corresponding to the eigenvalue 0.

(d)

Observe that

$$\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 & 6 & 0.5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} (26.25) = 26.25 \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix}.$$

26.25 is the only nonzero eigenvalue of A and one of its corresponding eigenvector is $\begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix}$.

(e)

Let $P = \begin{pmatrix} -0.25 & -0.5 & -1.5 & -0.125 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 4.5 \end{pmatrix}$. Observe that columns of P are independent hence P is invertible.

See that

$$\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix}\begin{pmatrix} -0.25 & -0.5 & -1.5 & -0.125 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 4.5 \end{pmatrix}$$

$$= \begin{pmatrix} -0.25 & -0.5 & -1.5 & -0.125 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 4.5 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 26.25 \end{pmatrix}$$

(f)

Since P is invertible, we have

$$AP = PD \Rightarrow A = PDP^{-1} \Rightarrow A^2 + 3I_5 = PDP^{-1}PDP^{-1} + 3I_5 = PD^2P^{-1} + P3I_5P^{-1} = P\left(D^2 + 3I_5\right)P^{-1}$$

$$\Rightarrow \left(A^2 + 3I_5\right)P = P\left(D^2 + 3I_5\right)$$

, where

$$D^2 + 3I_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 26.25 \end{pmatrix}^2 + \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & (26.25)^2 + 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 692.0625 \end{pmatrix}$$

Observe that P is invertible , so the eigenvalues of $A^2 + 3I_5$ are 3 and 692.0625.

□

8. Suppose $AP = PD$, where A is 3×3, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} x & -4/5 & 0 \\ y & 3/5 & 0 \\ z & 0 & 1 \end{pmatrix}$ and P is invertible.

(a) Find the characteristic polynomial, $\det(A-\lambda I)$, of A.

(b) Find all eigenvalues of A.

(c) Suppose the first column of P, $\begin{pmatrix} x & y & z \end{pmatrix}^T$ with $x \geq 0$ is a unit vector and orthogonal to both $\begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, find $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(d) Compute $P^T P$ and then show that $P^T = P^{-1}$.

(e) Find A^n, $n \geq 0$.

Solution:

(a)

$$\det(A-\lambda I) = \det\left(PDP^{-1} - \lambda I\right) = \det\left(PDP^{-1} - P\lambda IP^{-1}\right) = \det\left[P(D-\lambda I)P^{-1}\right]$$

$$= \det P \det(D-\lambda I) \det P^{-1} = \det P \det(D-\lambda I)(\det P)^{-1} = \det(D-\lambda I) = \det\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}.$$

$$= -\lambda(1-\lambda)^2 = -\lambda\left(\lambda^2 - 2\lambda + 1\right) = -\lambda^3 + 2\lambda^2 - \lambda$$

(b)

$\det(A-\lambda I) = -\lambda(1-\lambda)^2 \Rightarrow \lambda = 0$ or $\lambda = 1$, so 0 and 1 are all the eigenvalues A has.

(c)

$$\begin{cases} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix} = 0 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \\ x^2 + y^2 + z^2 = 1 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} -\frac{4}{5}x + \frac{3}{5}y = 0 \\ z = 0 \\ x^2 + y^2 + z^2 = 1 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} y = \frac{4}{3}x \\ z = 0 \\ x^2 + y^2 + z^2 = 1 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} x^2 + \frac{16}{9}x^2 = 1 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} x^2 = \frac{9}{25} \\ x \geq 0 \end{cases} \Rightarrow x = \frac{3}{5}$$

$$\Rightarrow \begin{cases} x = \frac{3}{5} \\ y = \frac{4}{5} \\ z = 0 \end{cases}$$

(d)

Observe that $P^T P = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $P^T = PI = P^T\left(PP^{-1}\right) = \left(P^T P\right)P^{-1} = IP^{-1} = P^{-1}$.

(e)

$$A=\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{T}$$

$$=\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^n=\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 1^n \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & -\frac{4}{5} & 0 \\ 0 & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} \frac{16}{25} & -\frac{12}{25} & 0 \\ -\frac{12}{25} & \frac{9}{25} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\square$

9. Let A be a 3×3 matrix with eigenvalues 1, 2, 3.

(a) (i) Find the characteristic polynomial $|A-\lambda I|$ of A.

(ii) Determine $|A|$.

(iii) Is A invertible, why?

If A^{-1} of A exists, the adjoint adj A of A is defined as the matrix $\operatorname{adj} A \square |A| A^{-1}$.

Soppose $AM = M\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, where M is invertible.

(b) Find the eigenvalues of adj A.

Let $M=\begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $AM = M\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(c) (i) Compute MM^{T} and then find M^{-1}.

(ii) Find A^{-1}.

(iii) Find $\operatorname{adj} A$.

Solution:

(a)

(i)

$|A-\lambda I| = -(\lambda-1)(\lambda-2)(\lambda-3)$.

(ii)

$|A-\lambda I|_{\lambda=0} = |A| = -(-1)(-2)(-3) = 6$.

(iii)

Since $|A| = 6 \neq 0$, A is invertible.

(b)

$$AM = M\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow A = M\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} M^{-1}$$

$$\Rightarrow A^{-1} = \left(M^{-1}\right)^{-1}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} M^{-1} = M\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} M^{-1}$$

$$\Rightarrow \operatorname{adj} A = |A| A^{-1} \underset{|A|=6}{=} 6M\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} M^{-1} = M\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} M^{-1}$$

Since M is invertible, the eigenvalues of $\operatorname{adj} A$ are 3, 6, 2.

(c)

(i)

$$MM^{\mathrm{T}} = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \text{ So } M^{-1} = M^{\mathrm{T}} = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii)

$$A^{-1} = M\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} M^{-1} = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3/10 & -4/5 & 0 \\ 4/10 & 3/5 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}\begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 41/50 & -12/50 & 0 \\ -12/50 & 34/50 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

(iii)

$$\operatorname{adj} A = |A| A^{-1} = 6\begin{pmatrix} 41/50 & -12/50 & 0 \\ -12/50 & 34/50 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 123/25 & -36/25 & 0 \\ -36/25 & 102/25 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

-End-