<u>**Solution**</u>

1.
To facilitate the probability computation, we assign X as the value of twice the absolute difference, and we have $X \in \{0, 2, 4, 6, 8, 10\}$. The sample space contains 36 outcomes and the PMF of each value of X is determined as:

$$p(0) = \frac{6}{36} = \frac{1}{6}, \quad p(2) = \frac{10}{36} = \frac{5}{18}, \quad p(4) = \frac{8}{36} = \frac{2}{9}, \quad p(6) = \frac{6}{36} = \frac{1}{6}, \quad p(8) = \frac{4}{36} = \frac{1}{9},$$
$$p(10) = \frac{2}{36} = \frac{1}{18}$$

The required probability is:

$$P(X > 5) = p(6) + p(8) + p(10) = \frac{1}{3}$$

2.
Define this event as A. The possible combination includes 5+5 or 6+4 or 4+6.

The probability of A in one trial is:

$$\mathrm{P(A)} = 3/36 = 1/12$$

Hence, the required probability is:

$$\mathrm{P} = (11/12)^4(1/12) = 0.0588$$

3.
We have $n = 100$ and $p = 0.5$, assuming that a head corresponds to a success. The required probability is the sum of two PMFs of the binomial RV:

$$\begin{aligned}
&p(49) + p(50) + p(51) \\
&= C(100, 49)0.5^{49}(1 - 0.5)^{100-49} + C(100, 50)0.5^{50}(1 - 0.5)^{100-50} + C(100, 51)0.5^{51}(1 - 0.5)^{100-51} \\
&= (C(100, 49) + C(100, 50) + C(100, 51))0.5^{100} = 0.2356
\end{aligned}$$

Using Poisson approximation, we have $\lambda = np = 50$, the probability is:

$$p(49) + p(50) + p(51) = e^{-50}\frac{\lambda^{49}}{49!} + e^{-50}\frac{\lambda^{50}}{50!} + e^{-50}\frac{\lambda^{51}}{51!} = e^{-50}\left(\frac{\lambda^{49}}{49!} + \frac{\lambda^{50}}{50!} + \frac{\lambda^{51}}{51!}\right) = 0.1679$$

The approximation is not accurate enough. It is because the approximation will be accurate for large n and small p. However, in this case, $p = 0.5$ is not a small value.

4.(a)
Using the fact that the sum of all PMFs should be equal to 1, we have:

$$\frac{1}{\alpha^2}\left[1+\frac{1}{\alpha}+\frac{1}{\alpha^2}+\cdots\right]\frac{1}{\alpha^2}\left[\frac{1}{1-1/\alpha}\right]=\frac{1}{6}\Rightarrow \alpha^2-\alpha-6=0\Rightarrow \alpha=3$$

Note that α should be positive because PMF is nonnegative.

4.(b)
Let $F(r)=P(X\leq r)$. We have

$$F(1)=\frac{5}{6}$$

For $r\geq 2$, we

$$F(r)=\frac{5}{6}+\frac{1}{3^2}+\cdots+\frac{1}{3^r}=\frac{5}{6}+\frac{1}{6}\left[1-\frac{1}{3^{r-1}}\right]$$

Combining the results yields:

$$F(r)=\begin{cases}0, & r<1\\ \frac{5}{6}+\frac{1}{6}\left[1-\frac{1}{3^{r-1}}\right], & r\geq 1\end{cases}$$

5.
We can convert to standard Gaussian RV for computation as follows.

$X\sim N(175,25)$ which implies $\frac{X-175}{5}\sim N(0,1)$

$$\mathrm{P(X>x)=1-P(X\leq x)=1-}\phi(\tfrac{\mathrm{x}-175}{5})<0.05$$

Hence $\phi(\frac{x-175}{5})\geq 0.95$, resulting in $\frac{x-175}{5}\geq 1.65$.

Finally, the door height is 183.25m.

Same answer can also be obtained without performing the standardization.

6.(a)

When Peter arrives at the platform on or before 8:32:00, he takes the train at 8:32:00. When Peter arrives at the platform after 8:32:00, he takes the train at 8:34:00.

Hence there are 2 cases:

(i) The waiting time is equal or less than 60s, when Peter arrives at [8:31,8:32] or [8:33,8:34].

(ii) The waiting time is between 60s and 120s, when Peter arrives at [8:32,8:33] or [8:33,8:34].

As Peter's arrival time is uniformly distributed between [8:31,8:34], the probability of case (i) is 2/3 while that of case (ii) is 1/3. Together with the fact that the area of the PDF should be equal to 1, we have:

$$\mathrm{P(X = x)} = \begin{cases} \frac{1}{90}, & \text{if } 0 < \mathrm{x} \leq 60 \\ \frac{1}{180}, & \text{if } 60 < \mathrm{x} \leq 120 \end{cases}$$

6.(b)

$$\mathbb{E}\{X\} = \int_{-\infty}^{\infty} xp(x)dx = \int_{0}^{60} \frac{1}{90}xdx + \int_{60}^{120} \frac{1}{180}xdx = 50$$

Hence the mean waiting time is 50s.

7.(a)

$$E(X) = \tfrac{1}{5}(-2 + -1 + 0 + 1 + 2) = 0$$

7.(b)

$$E(Z) = \tfrac{1}{5}(4 + 2 + 0 + 2 + 4) = 2.4$$

8.

It is clear that $p = 0.01$ is the probability of getting the diamond ring while the failure probability is 0.99. It can be modelled as a geometric distribution such that the diamond ring is selected at the rth draw is:

$$p(r) = P(X = r) = (1-p)^{r-1}p, \quad 1 \leq r < \infty$$

Now the required probability is:

$$\sum_{r=21}^{\infty} p(r) = \sum_{r=21}^{\infty} (1-p)^{r-1}p = (1-p)^{20}p\left[1 + (1-p) + (1-p)^2 + \cdots +\right] = (1-p)^{20} = 0.99^{20} = 0.8179$$

9.
Let $\mu = \mathbb{E}\{X\}$ and the PMF of X be $p(x)$ with range $R_x = \{x_i\}$. The variance is

$$\mathrm{var}(X) = \sum_{x_i} (x_i - \mu)^2 p(x_i)$$

As PMF is nonnegative or $p(x_i) \geq 0$, $\mathrm{var}(X) = 0$ implies that $x_i - \mu = 0$ for all x_i or all x_i are constants having the same value μ. Hence X must be a constant with value μ.

In this case, there is no uncertainty in X for a zero spread.

10.(a)
Differentiating the CDF yields the PDF:

$$p(x) = \begin{cases} 0.5, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbb{E}\{X\} = \int_{-\infty}^{\infty} xp(x)dx = \int_{-1}^{1} 0.5xdx = 0$$

10.(b)

$$\mathbb{E}\{X^2\} = \int_{-\infty}^{\infty} x^2 p(x)dx = \int_{-1}^{1} 0.5x^2 dx = \frac{1}{3}$$