1. Find the eigenvalues and eigenvectors of the following matrices:

(a) $\begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$.

2. For the matrix in question 1(b), find a matrix P such that $P^{-1}AP = D$, a diagonal matrix with the eigenvalues of A as its elements. Check your solution by evaluating $P^{-1}AP$.

3. Find a 2×2 matrix A which has eigenvalue 1 with corresponding eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and eigenvalue 3 with corresponding eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4. (a) Using Gaussian elimination, find a matrix X such that $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) With the help of the result of (a), find a 4×4 matrix A which has eigenvalues $-1, 0, 0, 1$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, which are <u>rows</u> of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$.

5. Given that $\xi = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{pmatrix}$.

(a) Find a and b.
(b) Find the eigenvalues and eigenvectors of A.
(c) Is A diagonalizable? Please give reasons.

6. (a) Find the eigenvalues and corresponding eigenvectors of the symmetric matrix $A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and verify that the eigenvectors are mutually orthogonal.

(b) Let $B = A^5 - 5(A+3I)^{-1} + 3A^T$. Find the eigenvalues of B.

7. Consider $A = \begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix}$

(a) Find rank A.
(b) Find a column vector $\vec{x}$ and a row vector $\vec{y}^{T}$, where $\vec{x}, \vec{y} \in R^5$ such that $A = \vec{x}\vec{y}^{T}$.

(c) Show that 0 is an eigenvalue of A and find its corresponding independent eigenvectors.

(d) Does A have eigenvalues other than 0? If yes, find those eigenvalues and the corresponding independent eigenvectors.

(e) Find an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(f) Find all eigenvalues of $A^2 + 3I_5$, where I_5 is the 5×5 unit matrix.

8. Suppose $AP = PD$, where A is 3×3, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} x & -4/5 & 0 \\ y & 3/5 & 0 \\ z & 0 & 1 \end{pmatrix}$ and P is invertible.

(a) Find the characteristic polynomial, $\det(A - \lambda I)$, of A.

(b) Find all eigenvalues of A.

(c) Suppose the first column of P, $\begin{pmatrix} x & y & z \end{pmatrix}^T$ with $x \geq 0$ is a unit vector and orthogonal to both $\begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, find $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(d) Compute $P^T P$ and then show that $P^T = P^{-1}$.

(e) Find A^n, $n \geq 0$.

9. Let A be a 3×3 matrix with eigenvalues 1, 2, 3.

(a) (i) Find the characteristic polynomial $|A - \lambda I|$ of A.

(ii) Determine $|A|$.

(iii) Is A invertible, why?

If A^{-1} of A exists, the adjoint $\operatorname{adj} A$ of A is defined as the matrix $\operatorname{adj} A = |A| A^{-1}$.

Soppose $AM = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, where M is invertible.

(b) Find the eigenvalues of $\operatorname{adj} A$.

Let $M = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $AM = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(c) (i) Compute MM^{T} and then find M^{-1}.

(ii) Find A^{-1}.

(iii) Find $\operatorname{adj} A$.

-End-