CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester B, 2014/2015

Time allowed : Two hours

This paper has **SEVEN** pages (including this cover page).

A brief table of derivatives is attached on pages 6 and 7.

Instructions to candidates:

1. This paper has **NINE** questions.
2. Attempt **ALL** questions in Section A and B.
3. Each question in Section A carries 10 marks.
4. Each question in Section B carries 15 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Section A
Answer **ALL** questions in this section., Each question carries 10 marks.

Question 1

(a) Find, in radians, the general solution of the equation $4\sin^2 x = 1$. (6 marks)

(b) Starting from the formula $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$,

show that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$.

Deduce that $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$. (4 marks)

Question 2

Show from first principles that

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(3x^2) = 6x$, (5 marks)

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin 5x) = 5\cos 5x$. (5 marks)

(Hint: $\sin A - \sin B = 2\cos(\dfrac{A+B}{2})\sin(\dfrac{A-B}{2})$.)

Question 3

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following:

(a) $y = \sqrt{x^2+1} + \dfrac{2}{x+3}$; (3 marks)

(b) $y = \dfrac{\sin 3x}{1+\cos x}$; (3 marks)

(c) $\dfrac{2y}{x} = \log_e\left(x^2+y^2\right)$. (4 marks)

Your results may be left in an unsimplified form.

Question 4

(a) Evaluate the following limits, if they exist:

(i) $\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}$, (2 marks)

(ii) $\lim_{x\to\infty}\frac{3x^2-x+5}{2x^3+1}$. (2 marks)

(b) Let

$$f(x)=\begin{cases} x^2 & \text{for } x\le 1 \\ \sqrt{x} & \text{for } x>1 \end{cases} .$$ (6 marks)

Determine whether $f(x)$ is differentiable at $x=1$. Give your reason.

Question 5

(a) Differentiate with respect to x

(i) $2^{\sqrt{x}}$, (2 marks)

(ii) $e^{-x}\left(\frac{1+x+x^2}{1-x+x^2}\right)^{\frac{1}{2}}$. (2 marks)

(b) Show that the point P whose coordinates are $x=a\sec\theta$, $y=b\tan\theta$, lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for all real values of θ. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of θ.

(6 marks)

Question 6

(a) Express $\frac{7x+17}{(x-4)(2x+1)}$ in partial fractions. (4 marks)

(b) If $y=(\alpha x+\beta)^{-1}$, where α and β are non-zero constants, find the general formula for the nth derivative of y with respect to x. (3 marks)

(c) Using the result in parts (a) and (b), or otherwise, find the sixth derivative of $\frac{7x+17}{(x-4)(2x+1)}$ with respect to x. You need not simplify your answer. (3 marks)

Question 7

Let $y = 1 + \dfrac{x-3}{(x-1)^2}$ for $x \in \mathrm{R}\backslash\{1\}$.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for $x \neq 1$. (3 marks)

(b) Does it have a local maximum point or a local minimum point? Explain your answer. (4 marks)

(c) Does it have a point of inflexion? Explain your answer. (3 marks)

Section B

Answer **ALL** questions in this section. Each question carries 15 marks.

Question 8

(a) If $y = (1+x^2)^{-\frac{1}{2}} \sinh^{-1} x$, show that $(1+x^2)\dfrac{dy}{dx} + xy = 1$. -------- (*)

By repeated differentiation of both sides of equation (*), or otherwise, find the Maclaurin series of y in ascending powers of x, up to and including the term in x^5. (8 marks)

(b) For any positive integer n, the function $P_n(x)$ is defined by

$$P_n(x) = \frac{d^n}{dx^n}\left[(x^2-1)^n\right] \text{ for } x \in [-1,1].$$

(i) Find $P_1(x)$, $P_2(x)$ and $P_3(x)$.

(ii) Show that $z = P_n(x)$ satisfies the equation $(1-x^2)\dfrac{d^2z}{dx^2} - 2x\dfrac{dz}{dx} + n(n+1)z = 0$.

(Hint: Put $u = (x^2-1)^n$ then $(1-x^2)\dfrac{du}{dx} + 2nxu = 0$.) (7 marks)

Question 9

(a) Let $f(x)=\frac{1}{4}\left(x^2+4x-4\right)$.

The equation has a positive real root $x^*(\approx 0.5)$, which is to be computed by the iterative scheme $x_{k+1}=x_k-\dfrac{f(x_k)}{f'(x_k)}$, $k=0, 1, 2, 3, \cdots\cdots$.

Carry out two iterations to compute an approximation to x^* with an initial approximation $x_0=0.5$.

(6 marks)

(b) Show that $x^2+4x-4y-4=0$ is the equation of a parabola. Find the vertex, focus, and directrix and sketch its graph.

(9 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	**Derivative of y with respect to x**
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d\,f(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	**Derivative of** y **with respect to** x
$y = \cot^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$
$y = \sec^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \text{cosec}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \sinh u$	$\dfrac{dy}{dx} = \cosh u\dfrac{du}{dx}$
$y = \cosh u$	$\dfrac{dy}{dx} = \sinh u\dfrac{du}{dx}$
$y = \tanh u$	$\dfrac{dy}{dx} = \text{sech}^2 u\dfrac{du}{dx}$
$y = \coth u$	$\dfrac{dy}{dx} = -\text{cosech}^2 u\dfrac{du}{dx}$
$y = \text{sech}\, u$	$\dfrac{dy}{dx} = -\text{sech}\, u \tanh u\dfrac{du}{dx}$
$y = \text{cosech}\, u$	$\dfrac{dy}{dx} = -\text{cosech}\, u \coth u\dfrac{du}{dx}$
$y = \sinh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$
$y = \cosh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \tanh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$
$y = \coth^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$
$y = \text{sech}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$
$y = \text{cosech}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{\lvert u\rvert\sqrt{u^2+1}}\dfrac{du}{dx}$

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