MA1200 Calculus and Basic Linear Algebra I

Lecture Note 3

Polynomial, Rational function and Binomial theorem

More about Polynomials

In Chapter 2, we know that a polynomial, denoted by $P(x)$, is a function of the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is non-negative integers (called the *degree* of the polynomial), $a_n, a_{n-1}, \dots, a_0$ are real numbers (called coefficients of polynomial) and $a_n \neq 0$.

The (largest) domain of polynomial is $\mathbb{R}$.

Example:

- $f_1(x) = x^5 - \sqrt{3}x^2 + x - 1$ is a polynomial with degree 5.
- $f_2(x) = 2$ (constant function) is also a polynomial with degree 0.
- $f_3(x) = (\cos x)^5 + \cos x - 1$ is not polynomial.
- $f_4(x) = x^{\sqrt{2}} + x - 1$ is not polynomial ($\sqrt{2}$ is not integer).
- $f_5(x) = x^{-3} + x - 1 \left(= \frac{1}{x^3} + x - 1\right)$ is not polynomial (-3 is negative).

A special example of polynomial: Quadratic function

A quadratic function is a function of the form:

$$y = f(x) = ax^2 + bx + c$$

where $a(\neq 0)$, b and c are real numbers.

Standard Form of quadratic function:

In many applications, we will rewrite the $f(x) = ax^2 + bx + c$ as

$$f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(\underbrace{x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2}_{A^2+2AB+B^2=(A+B)^2} - \left(\frac{b}{2a}\right)^2\right) + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a^2}$$

$= a(x-h)^2 + k$ which is called standard form of quadratic function.

Graph of quadratic function

It is easier to investigate the properties of quadratic functions by studying its graph. To do this, we consider the standard form of quadratic function:

$$y = f(x) = a(x-h)^2 + k \ldots \ldots (*)$$

- **The graph of quadratic function is a *parabola* with vertex $(\boldsymbol{h}, \boldsymbol{k})$**

  One can rearrange the terms in $(*)$ and obtain

  $y = a(x-h)^2 + k$

  $\Rightarrow a(x-h)^2 = y - k$

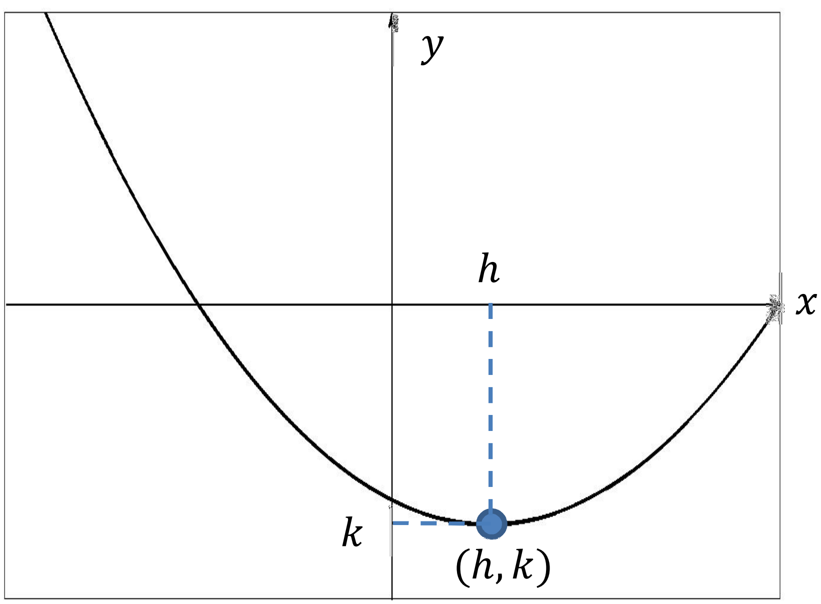
  $\Rightarrow (x-h)^2 = \dfrac{1}{a}(y-k)$

  $\Rightarrow \underbrace{(x-h)^2}_{(x-h)^2} = \underbrace{4\left(\dfrac{1}{4a}\right)}_{4a'}\underbrace{(y-k)}_{(y-k)}.$

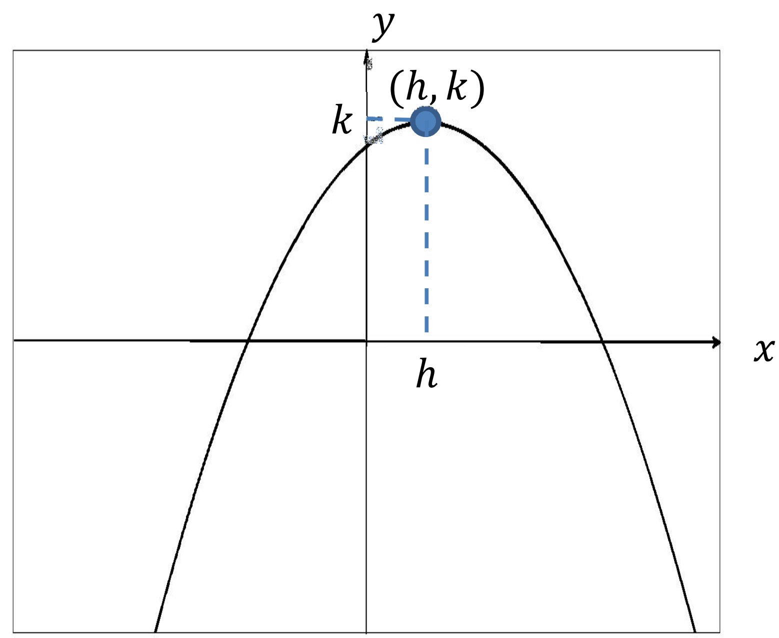
  Hence, the equation represents the parabola.

In addition, one can see that

- ✓ The vertex of parabola is (h,k)
- ✓ The parabola is symmetric about the vertical line $x=h$.
- ✓ The parabola is either U-shape (if $a'=\frac{1}{4a}>0\Rightarrow a>0$) or inverted U-shape (if $a'=\frac{1}{4a}<0\Rightarrow a<0$)



$a>0$ (U-shape)



$a<0$ (inverted U-shape)

Using the above figures, we can easily see that

- The *range* of the function is $[k, \infty)$ if $a > 0$ or $(-\infty, k]$ if $a < 0$.

- If $a > 0$, the *minimum value* of $f(x)$ is k and it is attained when $x = h$.

- If $a < 0$, the *maximum value* of $f(x)$ is k and it is attained when $x = h$.

In summary, we have

$$f(x) = a(x - h)^2 + k$$

Shape of the graph
$a > 0 \Rightarrow$ U-shape curve
$a < 0 \Rightarrow$ inverted U-shape

The value of x which the function attains its maximum/minimum.
[x-coordinate of vertex]

maximum (if $a < 0$) or minimum (if $a > 0$) of the function
[y-coordinate of vertex]

Example 1

A quadratic function is given by $f(x) = 3x^2 - 12x + 7$.

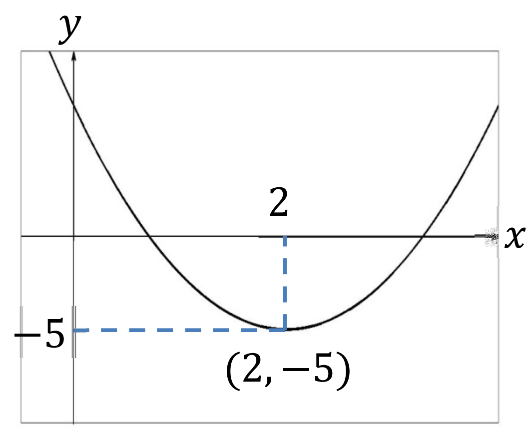
(a) Sketch the graph.

(b) What are the maximum value, minimum value and the range of this function?

☺Solution

(a) Using completing square techniques, we have

$$f(x) = 3x^2 - 12x + 7 = 3(x^2 - 4x) + 7$$
$$= 3\left(\underbrace{x^2 - 2(2)x + 2^2}_{a^2-2ab+b^2=(a-b)^2} - 2^2\right) + 7$$
$$= 3(x-2)^2 - 5.$$



(b) From the graph, we see

Range of function $= [-5, \infty)$

Maximum value $= \infty$ and minimum value $= -5$.

Finding zeros of the quadratic function

We would like to solve for the equation

$$f(x) = ax^2 + bx + c = 0, \qquad (a \neq 0).$$

One can use “completing square” technique and rewrite the equation (see P.3) as

$$ax^2 + bx + c = 0 \Rightarrow a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$= x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is called *quadratic formula*.

☺Note:

If $b^2 - 4ac > 0$, we have two distinct real solution.

If $b^2 - 4ac = 0$, we have one real solution.

If $b^2 - 4ac < 0$, we have no real solution (since $\sqrt{b^2 - 4ac}$ is

Example 2

Consider the equation $4x^2 + ax + 2 = 0$.

(a) If $a = 10$, solve the equation.

(b) Find the value of a such that the equation has real roots.

☺Solution:

(a) Take $a = 10$, then the equation becomes $4x^2 + 10x + 2 = 0$ and quadratic formula suggests that

$$x = \frac{-10 \pm \sqrt{10^2 - 4(4)(2)}}{2(4)} = \frac{-10 \pm \sqrt{68}}{8} = \frac{-5 \pm \sqrt{17}}{4}$$

(b) In general, the solution of $4x^2 + ax + 2 = 0$ is given by

$$x = \frac{-a \pm \sqrt{a^2 - 4(4)(2)}}{2(4)} = \frac{-a \pm \sqrt{a^2 - 32}}{8}.$$

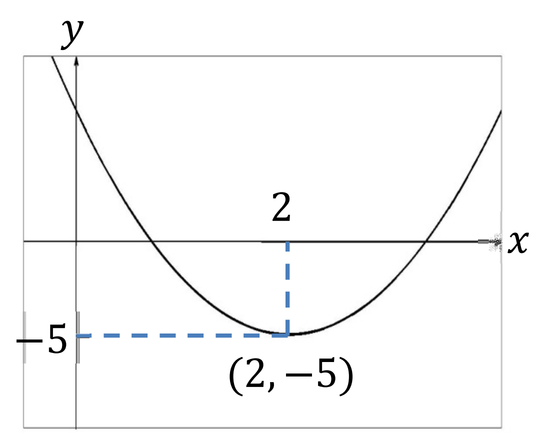
The equation has real roots only when $a^2 - 32 \geq 0 \Rightarrow a^2 \geq 32$ $\Rightarrow a \leq -\sqrt{32}$ or $a \geq \sqrt{32}$.

Example 3

We consider the quadratic function $f(x) = 3x^2 - 12x + 7$ in Example 1.

(a) What are the x-intersects of the functions?

(b) Is it possible to find x such that $f(x) = -10$?



☺Solution:

(a) We solve $f(x) = 0 \Rightarrow 3x^2 - 12x + 7 = 0$.

Using quadratic formula, we see

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(7)}}{2(3)} = \frac{12 \pm \sqrt{60}}{6} \approx 0.709 \;\; or \;\; 3.291.$$

(b) We solve $f(x) = -10 \Rightarrow 3x^2 - 12x + 17 = 0$.

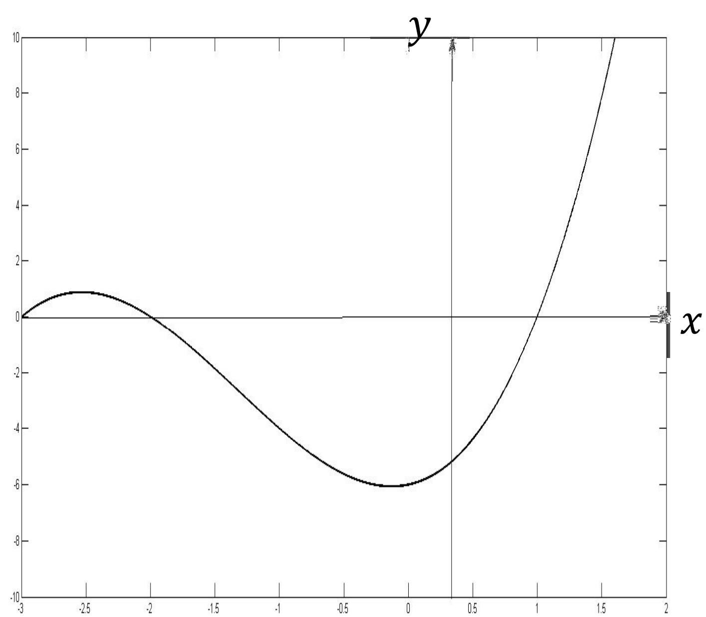
Using quadratic formula, we see

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(17)}}{2(3)} = \frac{12 \pm \sqrt{-60}}{6}$$

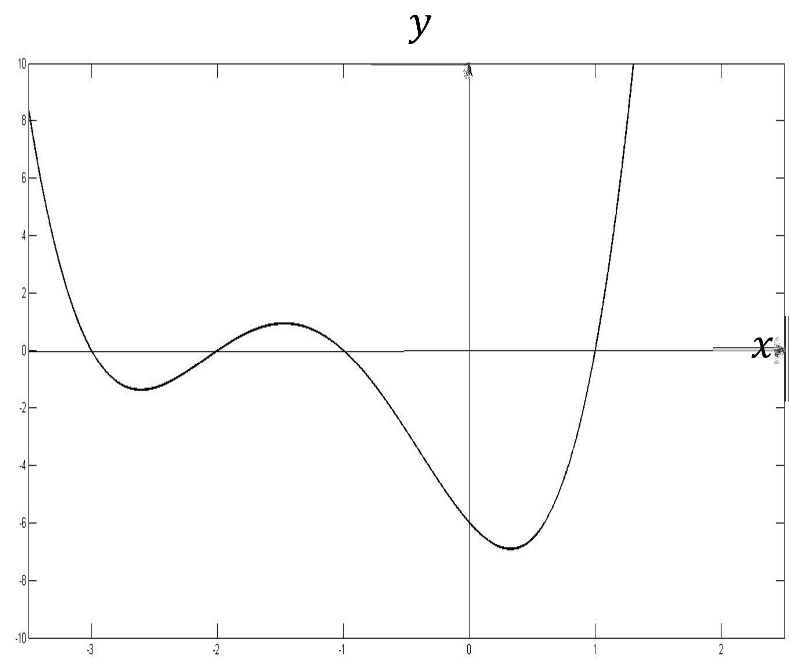
Since $\sqrt{-60}$ is not real, so there is no x such that $f(x) = -10$.

Polynomial: General Case

In general, it is hard to analyze the polynomial using its graph since the "shape" of their graphs are irregular in general.



$y = x^3 + 4x^2 + x - 6$



$y = x^4 + 5x^3 + 5x^2 - 5x - 6$

Working with polynomial

Recall that one way to study the properties of a given integer is to "factorize" the integer:

$$1500 = 2^2 \times 3 \times 5^3, \;\; 17929296 = 2^4 \times 3^3 \times 7^3 \times 11^2.$$

Using the same philosophy, one can study the polynomial by factorizing it into the product of simpler expressions. For example:

$$2x^3 - 7x^2 + 7x - 2 = (2x-1)(x-1)(x-2),$$

$$x^4 + 2x^3 - 2x^2 - 9x - 6 = (x^2+3x+3)(x-2)(x+1).$$

There are two purposes of factorizing polynomial:

- **Find the zeros of the polynomials**
  One can solve the equation
  $$2x^3 - 7x^2 + 7x - 2 = 0$$
  by factorizing the polynomial into the product of linear factors:

$$2x^3 - 7x^2 + 7x - 2 = 0 \Rightarrow (2x-1)(x-1)(x-2) = 0$$
$$\Rightarrow x = \frac{1}{2} \quad or \quad x = 1 \quad or \quad x = 2.$$

- **Rewrite some expressions into another form which is useful in computation (partial fraction technique)**

  We consider the following expression:
  $$\frac{5x+3}{x^3 - 2x^2 - 3x}$$
  Note that the denominator is polynomial and can be factorized into $x^3 - 2x^2 - 3x = x(x^2 - 2x - 3) = x(x-3)(x+1)$. Using some technique (called "partial fraction"), one can rewrite the expression as:
  $$\frac{5x+3}{x^3 - 2x^2 - 3x} = \frac{3}{2(x-3)} - \frac{1}{x} - \frac{1}{2(x+1)}.$$
  The decomposition on R.H.S. is often useful in differentiation (will be taught later) and integration (will be taught in MA1201).

Factorization of Polynomial

To factorize a polynomial, one can use “trial and error” method to test which linear factor $ax + b$ is factor of the polynomial.

The following two theorems, called *remainder theorem* and *factor theorem*, are useful in finding the linear factor of the polynomial.

Remainder Theorem

If a polynomial $f(x)$ is divided by $ax + b$, then the remainder is given by

$$r = f\left(-\frac{b}{a}\right)$$

Factor Theorem

The linear function $ax + b$ is a factor of $f(x) \Leftrightarrow r = f\left(-\frac{b}{a}\right) = 0.$

Example 4

We consider the polynomial $f(x) = 2x^3 - 5x^2 - 4x + 3$.

(a) Find the remainders when $f(x)$ is divided by (i) $2x + 1$ and (ii) $x - 3$ respectively.

(b) Using the result in (a), factorize the polynomial $f(x)$.

☺Solution:

(a) When $f(x)$ is divided by $2x + 1$, the remainder is

$$r_1 = f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 - 5\left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) + 3 = \frac{7}{2}.$$

When $f(x)$ is divided by $x - 3$, the remainder is

$$r_2 = f(3) = 2(3)^3 - 5(3)^2 - 4(3) + 3 = 0.$$

(b) From (a), we see that $x-3$ is a factor of $f(x)$. Then we factorize the polynomial using *long division*.

$$
\begin{aligned}
f(x) &= 2x^3 - 5x^2 - 4x + 3 \\
&= (x-3)(2x^2 + x - 1) \\
&= (x-3)(2x-1)(x+1).
\end{aligned}
$$

(*Note: The factorization in the last step can be done by "cross multiplication".)

$$
\begin{array}{r}
2x^2 \quad + x - 1 \\
x-3 \overline{\smash{\big)}\, 2x^3 - 5x^2 - 4x + 3} \\
\underline{2x^3 - 6x^2} \qquad\qquad \\
x^2 - 4x \qquad \\
\underline{x^2 - 3x} \qquad \\
-x + 3 \\
\underline{-x + 3}
\end{array}
$$

Example 5

Solve the polynomial equation

$$2x^3 - x^2 - 15x + 18 = 0.$$

☺Solution:

First of all, we need to factorize the polynomial on the L.H.S. One can find the factor of the polynomial using the following "trial and error" strategy:

x	-2	-1	0	1	2
$f(x)$	28	30	18	4	0

Since $f(2) = 0$, so $x - 2$ is the factor of $f(x)$ by factor theorem.

By long division, we can factorize the polynomial on L.H.S. as

$$2x^3 - x^2 - 15x + 18 = 0 \Rightarrow (x - 2)(2x^2 + 3x - 9) = 0$$

$$\Rightarrow (x - 2)(2x - 3)(x + 3) = 0 \Rightarrow x = 2 \;\; or \;\; x = \frac{3}{2} \;\; or \;\; x = -3.$$

Example 6

(a) Solve the polynomial equation

$$f(x) = x^6 - x^4 - 16x^2 + 16 = 0.$$

(b) Hence, factorize the polynomial $f(x)$.

☺Solution:

(a) Since only the terms x^6, x^4, x^2 and constant terms appear in the equation. One can simplify the equation by letting $y = x^2$ and the equation becomes

$$x^6 - x^4 - 16x^2 + 16 = 0 \Rightarrow \underbrace{y^3 - y^2 - 16y + 16}_{g(y)} = 0.$$

Using trial and error, we have

y	-2	-1	0	1	2
$g(y)$	36	30	16	0	-12

Since $g(1) = 0$, so we observe that $y - 1$ is the factor of $g(y)$.

Using long division, one can rewrite the equation as

$y^3 - y^2 - 16y + 16 = 0 \Rightarrow (y-1)(y^2-16) = 0$

$\Rightarrow (y-1)(y-4)(y+4) = 0 \ldots\ldots (*)$

$\Rightarrow y = 1 \ \ or \ \ y = 4 \ \ or \ \ y = -4$

$\Rightarrow x^2 = 1$ or $x^2 = 4$ or $x^2 = -4$ (rejected)

$\Rightarrow x = \pm 1$ or $x = \pm 2$.

(b) Using the result of (a) and the factor theorem, we know that the functions $x-1$, $x+1$, $x-2$ and $x+2$ are the factors of $f(x)$.

Using long division again (or use the equation $(*)$ with $y = x^2$), we get

$x^6 - x^4 - 16x^2 + 16$

$= (x^2-1)(x^2-4)(x^2+4)$

$= (x-1)(x+1)(x-2)(x+2)(x^2+4)$.

(*Note: Here, we cannot factorize the quadratic function x^2+4).

Rational Functions

Recall that a rational function is a quotient (or ratio) of two polynomials, i.e.

$$r(x) = \frac{f(x)}{g(x)}, \qquad g(x) \neq 0.$$

where $f(x)$ and $g(x)$ are polynomial.

- The following are some examples of rational functions:

$$h_1(x) = \frac{x^5 + 2x^3 - x + 1}{x^3 + 5x}, \qquad h_2(x) = \frac{1}{x^2 - 5x + 6},$$

$$h_3(x) = \frac{x + 4}{2x^7 - 3x - 1}.$$

- Since $r(x)$ is not defined when $g(x) = 0$, so the domain of $r(x)$ is all real numbers which $g(x) \neq 0$.

We say $h(x) = \frac{f(x)}{g(x)}$ is *proper rational function* if the degree of $f(x)$ is *less than* the degree of $g(x)$.

- $h_2(x) = \frac{\overbrace{1}^{degree\ 0}}{\underbrace{x^2-5x+6}_{degree\ 2}}, h_3(x) = \frac{\overbrace{x+4}^{degree\ 1}}{\underbrace{2x^7-3x-1}_{degree\ 7}}$ are proper rational functions.

We say $h(x) = \frac{f(x)}{g(x)}$ is *improper rational function* if the degree of $f(x)$ is *greater than or equal to* the degree of $g(x)$.

- $h_1(x) = \frac{\overbrace{x^5+2x^3-x+1}^{degree\ 5}}{\underbrace{x^3+5x}_{degree\ 3}}$ is improper rational function.

Using long division, an improper function can be written as a sum of polynomial in x and a proper rational function:

$$h_1(x) = \frac{x^5 + 2x^3 - x + 1}{x^3 + 5x} = \frac{(x^2 - 3)(x^3 + 5x) + 14x + 1}{x^3 + 5x} = (x^2 - 3) + \frac{14x + 1}{x^3 + 5x}.$$

Decomposition of Rational Function: Method of Partial Fraction

Given a rational function, say

$$f(x) = \frac{x+1}{x^4 - x^3 + x^2 - x}.$$

The present form may not be useful in some purposes (say calculating derivative or integral).

One can see that the denominator can be factorized into

$$x^4 - x^3 + x^2 - x = x(x-1)(x^2+1).$$

Then one can try to "decompose" $f(x)$ into the following form:

$$f(x) = \underbrace{\frac{x+1}{x^4 - x^3 + x^2 - x}}_{hard\ to\ use} = \frac{x+1}{x(x-1)(x^2+1)} = \underbrace{\frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2+1}}_{easy\ to\ use}.$$

This decomposition procedure is called *method of partial fraction*.

Procedure of method of partial fraction

Proper rational fraction

For simplicity, we first consider the case when the function is proper rational function. Roughly speaking, the entire procedure is divided into three steps:

1. Given a proper rational fraction $r(x) = \frac{f(x)}{g(x)}$, we factorize the denominator as much as possible.

2. We choose the decomposed form based on different form of factorization obtained in Step 1.

Scenario	Proposed decomposition
If $g(x)$ can be factorized into a product of distinct linear factor: $r(x) = \frac{f(x)}{(a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)}$	$r(x) = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots \frac{A_n}{a_nx + b_n}.$

If there is some repeated linear factors in the factorization of $g(x)$: $$r(x) = \frac{f(x)}{(ax+b)^n(cx+d)}$$	$$r(x) = \underbrace{\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}}_{for\ (ax+b)^n} + \underbrace{\frac{B}{cx+d}}_{for\ (cx+d)}$$
If there is quadratic factor (cannot be factorized, e.g. x^2+4) in the factorization of $g(x)$: $$r(x) = \frac{f(x)}{(ax^2+bx+c)(dx+e)}$$	$$r(x) = \frac{\overbrace{Ax+B}^{degree\ 1}}{\underbrace{ax^2+bx+c}_{degree\ 2}} + \frac{C}{dx+e}$$
If there is degree n polynomial in the factorization of $g(x)$: $$r(x) = \frac{f(x)}{(a_nx^n+\cdots+a_1x+a_0)(bx+c)}$$	$$r(x) = \frac{\overbrace{A_{n-1}x^{n-1}+\cdots+A_1x+A_0}^{degree\ n-1}}{\underbrace{a_nx^n+\cdots+a_1x+a_0}_{degree\ n}} + \frac{B}{bx+c}$$

3. Find the unknowns (say $A, B, C, A_1, A_2, \ldots, A_n$ etc.) in the proposed decomposition.

Example 7

Express the fraction $\frac{1}{x^2-4}$ into partial fraction.

☺Solution:

Step 1: (Factorize the denominator) Note that $\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$.

Step 2: (Choose the decomposition) Since all factors are linear and distinct, we consider the following decomposition:

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

Step 3: (Find the unknowns), one can rewrite the expression as

$$\frac{1}{(x-2)(x+2)} = \frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$$\Rightarrow A(x+2)+B(x-2) = 1 \ldots\ldots (*)$$

There are two different ways to find the unknowns:

Method 1: Comparing coefficients

One can express the $(*)$ as

$$\overbrace{(A+B)x+(2A-2B)}^{A(x+2)+B(x-2)}=\overbrace{0x+1}^{1}.$$

By comparing the coefficients in x-term and constant term between two expressions, we have

$$\begin{cases} A+B=0 \\ 2A-2B=1 \end{cases} \Rightarrow \begin{cases} A+B=0 \\ A-B=1/2 \end{cases} \Rightarrow A=\frac{1}{4}, B=-\frac{1}{4}$$

Method 2: Substitution & Elimination

Substitute $x=2$ (so that $x-2=0$) in $(*)$, we get

$$A(2+2)+\cancel{B(2-2)}=1 \Rightarrow 4A=1 \Rightarrow A=1/4$$

Substitute $x=-2$ (so that $x+2=0$) in $(*)$, we get

$$\cancel{A(-2+2)}+B(-2-2)=1 \Rightarrow -4B=1 \Rightarrow B=-1/4$$

$$\Rightarrow \frac{1}{x^2-4}=\frac{1/4}{x-2}+\frac{-1/4}{x+2}=\frac{1}{4(x-2)}-\frac{1}{4(x+2)}.$$

Example 8

Express $\frac{5x+3}{x^3-2x^2-3x}$ into partial fractions.

☺Solution:

Step 1: (Factorize the denominator) Note that

$$\frac{5x+3}{x^3-2x^2-3x}=\frac{5x+3}{x(x^2-2x-3)}=\frac{5x+3}{x(x-3)(x+1)}.$$

Step 2: (Choose the decomposition) Since all factors are linear and distinct, we propose the following decomposition again:

$$\frac{5x+3}{x^3-2x^2-3x}=\frac{5x+3}{x(x+1)(x-3)}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}.$$

Step 3: (Find the unknowns) We first rewrite the equation as

$$\frac{5x+3}{x(x+1)(x-3)}=\frac{A(x+1)(x-3)+Bx(x-3)+Cx(x+1)}{x(x+1)(x-3)}$$

$$\Rightarrow 5x + 3 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1).$$

To find the unknowns, we

- Substitute $x = 0$, we have
$$3 = A(1)(-3) + \cancel{B(0)(-3)} + \cancel{C(0)(1)} \Rightarrow -3A = 3 \Rightarrow A = -1.$$
- Substitute $x = -1$ (so that $x + 1 = 0$), we have
$$-2 = \cancel{A(0)(-4)} + B(-1)(-4) + \cancel{C(-1)(0)} \Rightarrow 4B = -2 \Rightarrow B = -\frac{1}{2}.$$
- Substitute $x = 3$ (so that $x - 3 = 0$), we have
$$18 = \cancel{A(4)(0)} + \cancel{B(3)(0)} + C(3)(4) \Rightarrow 12C = 18 \Rightarrow C = \frac{3}{2}.$$

So we conclude that

$$\frac{5x+3}{x^3 - 2x^2 - 3x} = \frac{-1}{x} + \frac{-\frac{1}{2}}{x+1} + \frac{\frac{3}{2}}{x-3} = -\frac{1}{x} - \frac{1}{2(x+1)} + \frac{3}{2(x-3)}.$$

Example 9

Express $\frac{3x^2-8x+13}{(x+3)(x-1)^2}$ into partial fraction.

☺Solution:

Step 1: The denominator is factorized already.

Step 2: (Choose the decomposition) Since there is a repeated factor $(x-1)^2$ in the factorization, we propose the following decomposition:

$$\frac{3x^2-8x+13}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Step 3: (Find the unknowns) One can express the equation as

$$\frac{3x^2-8x+13}{(x+3)(x-1)^2} = \frac{A(x-1)^2 + B(x-1)(x+3) + C(x+3)}{(x+3)(x-1)^2}$$

$$\Rightarrow 3x^2 - 8x + 13 = A(x-1)^2 + B(x-1)(x+3) + C(x+3)$$

To solve for unknown, we

- substitute $x = -3$ (so that $x + 3 = 0$), we have
$$A(-4)^2 + B(-4)(0) + C(0) = 3(-3)^2 - 8(-3) + 13 \Rightarrow 16A = 64$$
$$\Rightarrow A = 4.$$
- substitute $x = 1$ (so that $x - 1 = 0$), we have
$$A(0)^2 + B(0)(4) + C(4) = 3(1)^2 - 8(1) + 13 \Rightarrow 4C = 8 \Rightarrow C = 2.$$
- substitute $x = 0$ (or any number you like), we have
$$A(-1)^2 + B(-1)(3) + 3C = 3(0)^2 - 8(0) + 13 \Rightarrow \underbrace{4}_{A} - 3B + 3\underbrace{(2)}_{C} = 13$$
$$\Rightarrow B = -1.$$

So we have

$$\frac{3x^2 - 8x + 13}{(x+3)(x-1)^2} = \frac{4}{x+3} - \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

Remark on Example 9

Intuitively, one may try the following decomposition:

$$\frac{3x^2 - 8x + 13}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{(x-1)^2}.$$

One can rearrange the term and yields

$$A(x-1)^2 + B(x+3) = 3x^2 - 8x + 13.$$

Putting $x = -3$, one can find that $A(-4)^2 = 3(-3)^2 - 8(-3) + 13 \Rightarrow A = 4$.

Putting $x = 1$, one can find that $B(1+3) = 3(1)^2 - 8(1) + 13 \Rightarrow B = 2$.

However, if you substitute $A = 4$ and $B = 2$, one can see that

$$4(x-1)^2 + 2(x+3) = 3x^2 - 8x + 13 \Rightarrow 4x^2 - 6x + 10 = 3x^2 - 8x + 13.$$

The two sides cannot be equal, thus this decomposition is *infeasible*.

Example 10

Express $\frac{6x^2-3x+1}{(4x+1)(x^2+1)}$ into partial fraction.

☺Solution:

Step 2: (Choose decomposition) Since there is a quadratic factor x^2+1 in the denominator, so we propose the following decomposition:

$$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{A}{4x+1} + \underbrace{\frac{Bx+C}{x^2+1}}_{Not\ \frac{B}{x^2+1}!}.$$

Step 3: (Find the unknowns) Note that

$$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(4x+1)}{(4x+1)(x^2+1)}$$

$$\Rightarrow 6x^2-3x+1 = A(x^2+1)+(Bx+C)(4x+1).$$

To find the unknowns, we

- substitute $x = -\frac{1}{4}$ (so that $4x + 1 = 0$), then

$$6\left(-\frac{1}{4}\right)^2 - 3\left(-\frac{1}{4}\right) + 1 = A\left[\left(-\frac{1}{4}\right)^2 + 1\right] + \cancel{\left(B\left(-\frac{1}{4}\right) + C\right)(0)} \Rightarrow A = 2.$$

- To find the remaining two unknowns, we can substitute two values of x and obtain two equations governing B and C.

  Substitute $x = -1$ and $x = 1$ (also $A = 2$) into the equation, we get

$$\begin{cases} 6(-1)^2 - 3(-1) + 1 = \overbrace{2}^{a}(2) + (B(-1) + C)(-3) \\ 6(1)^2 - 3(1) + 1 = \underbrace{2}_{a}(2) + (B(1) + C)(5) \end{cases} \Rightarrow \begin{cases} -B + C = -2 \\ B - C = 0 \end{cases}.$$

  Solving the equations yield $B = 1$ and $C = -1$.

Thus we conclude that

$$\frac{6x^2 - 3x + 1}{(4x + 1)(x^2 + 1)} = \frac{2}{4x + 1} + \frac{x - 1}{x^2 + 1}.$$

Remark on Example 10

Again, some may try the following decomposition:

$$\frac{6x^2 - 3x + 1}{(4x+1)(x^2+1)} = \frac{A}{4x+1} + \frac{B}{x^2+1}.$$

One can rearrange the term and yields

$$A(x^2+1) + B(4x+1) = 6x^2 - 3x + 1.$$

Put $x = -\frac{1}{4}$, we get $A\left(1 + \left(-\frac{1}{4}\right)^2\right) = 6\left(-\frac{1}{4}\right)^2 - 3\left(-\frac{1}{4}\right) + 1 \Rightarrow A = 2$

Put $x = 0$, $A = 2$, we get $2 + B = 1 \Rightarrow B = -1$.

However, if we substitute them into the equation, we get

$$2(x^2+1) - (4x+1) = 6x^2 - 3x + 1 \Rightarrow 2x^2 - 4x + 1 = 6x^2 - 3x + 1.$$

They are not equal in general. Thus this decomposition is feasible.

Improper rational function

Suppose that the given rational function is improper (i.e. the degree of numerator is greater than that of the denominator), say

$$f(x)=\frac{\overbrace{x^4-2x+1}^{degree\ 4}}{\underbrace{x^2-5x+6}_{degree\ 2}}=\frac{x^4-2x+1}{(x-2)(x-3)}.$$

One cannot apply the method of partial fraction (used in proper rational function) since

$$\frac{x^4-2x+1}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}=\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\Rightarrow \underbrace{x^4-2x+1}_{degree\ 4}=\underbrace{(A+B)x+(-3A-2B)}_{degree\ 1}.$$

One can see that two sides cannot be equal. So this decomposition is not OK.

In order to apply the method of partial fraction, one can perform the long division and rewrite the fraction into a sum of polynomial and proper rational fraction.

$$\begin{array}{r@{}l}
 & x^2+5x+19 \\ \cline{2-2}
x^2-5x+6 \big) & x^4+0x^3+0x^2-2x+1 \\
 & \underline{x^4-5x^3+6x^2} \\
 & \quad 5x^3-6x^2-2x \\
 & \quad \underline{5x^3-25x^2+30x} \\
 & \qquad 19x^2-32x+1 \\
 & \qquad \underline{19x^2-95x+114} \\
 & \qquad\qquad 63x-113
\end{array}$$

$$\frac{x^4-2x+1}{x^2-5x+6}$$

$$=\frac{(x^2-5x+6)(x^2+5x+19)+(63x-113)}{x^2-5x+6}$$

$$=\underbrace{x^2+5x+19}_{\substack{polynomial\\(degree\ 2)}}+\underbrace{\frac{63x-113}{x^2-5x+6}}_{\substack{proper\ rational\\ function}}$$

$$=x^2+5x+19-\frac{13}{x-2}+\frac{76}{x-3}$$

- One can decompose the remaining proper rational function using the method described above.

Example 11

Express $f(x) = \frac{x^5+2x^3-x+1}{x^3+5x}$ into partial fraction.

☺Solution:

Using long division (divide $x^5 + 2x^3 - x + 1$ by $x^3 + 5x$), we can express the function as

$$\underbrace{\frac{x^5 + 2x^3 - x + 1}{x^3 + 5x}}_{\substack{improper\ rational \\ function}} = \frac{(x^2 - 3)(x^3 + 5x) + 14x + 1}{x^3 + 5x} = \underbrace{x^2 - 3}_{polynomial} + \underbrace{\frac{14x + 1}{x^3 + 5x}}_{\substack{proper \\ rational \\ function}}.$$

We concentrate on the decomposition of $\frac{14x+1}{x^3+5x}$.

Step 1: (Factorize the denominator) $x^3 + 5x = x(x^2 + 5)$.

Step 2: (Choose the decomposition)

We consider the following decomposition

$$\frac{14x+1}{x^3+5x} = \frac{14x+1}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5}.$$

Step 3: (Find the unknowns)

$$\frac{14x+1}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5} = \frac{A(x^2+5)+(Bx+C)x}{x(x^2+5)}$$

$$\Rightarrow 14x+1 = A(x^2+5)+(Bx+C)x.$$

- Put $x=0$, we have $A(5)+(C)(0) = 14(0)+1 \Rightarrow A = 1/5$.
- Put $x=1$ and $A=1/5$, we have

$$\frac{1}{5}(6)+(B+C)(1) = 14(1)+1 \Rightarrow B+C = \frac{69}{5}.$$

Put $x = -1$ and $A = 1/5$, we have

$$\frac{1}{5}(6) + (-B + C)(-1) = 14(-1) + 1 \Rightarrow B - C = -\frac{71}{5}.$$

Solve the equations, we get $B = -\frac{1}{5}$ and $C = 14$.

Therefore, we have the following decomposition

$$\frac{14x + 1}{x(x^2 + 5)} = \frac{\frac{1}{5}}{x} + \frac{-\frac{1}{5}x + 14}{x^2 + 5} = \frac{1}{5x} - \frac{x - 70}{5(x^2 + 5)}.$$

Thus $f(x)$ can be decomposed as

$$\frac{x^5 + 2x^3 - x + 1}{x^3 + 5x} = x^2 - 3 + \frac{1}{5x} - \frac{x - 70}{5(x^2 + 5)}.$$

Binomial Theorem

In some situations (will be seen when we study differentiation), one may need to compute the expression

$$\frac{(x+h)^n - x^n}{h} \quad for \ \ h \neq 0, n \in \mathbb{N}.$$

One can compute this directly by expanding the term $(x+h)^n$.

- When $n = 2,3,4$, one can compute $(x+h)^2$, $(x+h)^3$, $(x+h)^4$ by

  $$(x+h)^2 = (x+h)(x+h) = x^2 + 2hx + h^2$$

  $$\begin{aligned}(x+h)^3 &= (x+h)^2(x+h) = (x^2 + 2hx + h^2)(x+h) \\ &= x^3 + 3hx^2 + 3h^2x + h^3.\end{aligned}$$

  $$\begin{aligned}(x+h)^4 &= (x+h)^3(x+h) = (x^3 + 3hx^2 + 3h^2x + h^3)(x+h) \\ &= x^{+}+4hx^{3}+6hx+4h^{3}x+\mathrm{h}^{4}\end{aligned}$$

- The computation becomes very tedious when n is large.

The binomial theorem provides a shortcut in computing $(a+b)^n$

Binomial Theorem

Let n be a positive integer, then for any two real numbers a, b, we have

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r = C_0^n a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \cdots + C_n^n b^n.$$

where $C_r^n = \frac{n!}{r!(n-r)!}$ and $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1, 0! = 1.$

Remarks on the notations in binomial theorem

- $n!$ Is called *factorial* of n. For example:

$$4! = 4 \times 3 \times 2 \times 1 = 24, \qquad \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 7 \times 6 \times 5 = 210.$$

- C_r^n is called *combination* (number of possible combination obtained when we pick r objects from n objects without considering the drawing order).

Example 12

Calculate

(a) C_7^{10}, (b) $C_3^6 C_6^{12}$, (c) C_3^n, (d) C_{n-2}^n

☺Solution:

(a) $C_7^{10} = \dfrac{10!}{7!(10-7)!} = \dfrac{10\times9\times8\times\overbrace{7!}^{7\times6\times\ldots\times1}}{7!3!} = \dfrac{10\times9\times8}{3\times2\times1} = 120.$

(b) $C_3^6 C_6^{12} = \left(\dfrac{6!}{3!(6-3)!}\right)\left(\dfrac{12!}{6!(12-6)!}\right) = \left(\dfrac{6!12!}{3!3!6!6!}\right) = \dfrac{12!}{3!3!6!} =$

$\dfrac{12\times11\times10\times9\times8\times7\times\overbrace{6!}^{6\times5\times\ldots\times2\times1}}{(3\times2\times1)\times(3\times2\times1)\times6!} = \cdots = 18480.$

(c) $C_3^n = \dfrac{n!}{3!(n-3)!} = \dfrac{n\times(n-1)\times(n-2)\times\overbrace{(n-3)!}^{(n-3)\times\ldots\times3\times2\times1}}{3\times2\times1\times(n-3)!} = \dfrac{n(n-1)(n-2)}{6}$

(d) $C_{n-2}^n = \dfrac{n!}{(n-2)!\big(n-(n-2)\big)!} = \dfrac{n!}{(n-2)!2!} = \dfrac{n\times(n-1)\times\overbrace{(n-2)!}^{(n-2)\times\ldots\times3\times2\times1}}{(n-2)!\times2\times1} = \dfrac{n(n-1)}{2}.$

Example 13

Compute

$$(x+1)^5-(x-2)^4.$$

☺Solution:

One can expend each term using binomial theorem:

$$\begin{aligned}(x+1)^5 &= C_0^5x^5+C_1^5x^4(1)+C_2^5x^3(1)^2+C_3^5x^2(1)^3+C_4^5x(1)^4+C_5^5(1)^4x^0\\ &= x^5+5x^4+10x^3+10x^2+5x+1.\end{aligned}$$

$$\begin{aligned}(x-2)^4 &= C_0^4x^4+C_1^4x^{\cancel{4}3}(-2)+C_2^4x^2(-2)^2+C_3^4x^{\cancel{2}1}(-2)^3+C_4^4(-2)^4x^0\\ &= x^4-4x^3(2)+6x^2(4)-4x(8)+1(16)\\ &= x^4-8x^3+24x^2-32x+16.\end{aligned}$$

Therefore,

$$\begin{aligned}&(x+1)^5-(x-2)^4\\ &= x^5+(5-1)x^4+(10+8)x^3+(10-24)x^2+(5+32)x+(1-16)\\ &= x^5+4x^4+18x^3-14x^2+37x-15.\end{aligned}$$

Example 14

Find the coefficient of x^7 of each of the following expressions:

(a) $(3x-2)^9+(x-3)^7$

(b) $(2x+1)^5(x-1)^3$

☺Solution:

(a) Using Binomial theorem, we have

$$(3x-2)^9+(x-3)^7=\sum_{r=0}^{9} C_r^9(3x)^{9-r}(-2)^r+\sum_{r=0}^{7} C_r^7\, x^{7-r}(-3)^r$$

Since the question only requires the coefficients of x_7 only instead of the whole expressions, we may concentrate on the calculation of x^7-term

$$=\cdots+\left[\underbrace{C_2^9 3^7(-2)^2}_{1st\ term}+\underbrace{C_0^7(-3)^0}_{2nd\ term}\right]x^7+\cdots=\cdots+145x^7+\cdots$$

The coefficient of x^7 is 314929.

Using Binomial Theorem again, we have

$$
\begin{aligned}
&(2x+1)^5(x-1)^3 \\
&= \big(C_0^5(2x)^5(1)^0 + C_1^5(2x)^4(1)^1 + C_2^5(2x)^3(1)^2 + C_3^5(2x)^2(1)^3 \\
&\qquad + C_4^5(2x)(1)^4 + C_5^5(2x)^0(1)^5\big)(C_0^3x^3(-1)^0 + C_1^3x^2(-1)^1 \\
&\qquad + C_2^3x(-1)^2 + C_3^3x^0(-1)^3) \\
&= \big(2^5C_0^5x^5 + 2^4C_1^5x^4 + 2^3C_2^5x^3 + 2^2C_3^5x^2 + 2C_4^5x + C_5^5\big) \\
&\qquad \times (C_0^3x^3 - C_1^3x^2 + C_2^3x - C_3^3x^0) \\
&= \cdots + \big(2^5C_0^5x^5\big)(-C_1^3x^2) + \big(2^4C_1^5x^4\big)(C_0^3x^3) + \cdots \\
&= \cdots + \big(-2^5C_0^5C_1^3 + 2^4C_1^5C_0^3\big)x^7 + \cdots \\
&= \cdots + (-32\times 1\times 3 + 16\times 5\times 1)x^7 + \cdots \\
&= \cdots - 16x^7 + \cdots
\end{aligned}
$$

The coefficient of x^7 is 16.

Appendix A – Proof of Binomial Theorem

In this appendix, we will discuss the proof of binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r.$$

where n is positive integer.

Of course, one can show it using brute force by computing $(a+b)$, $(a+b)^2$, $(a+b)^3$, … . However, this appears to be not efficient.

Instead, we try to show the following two statements:

- Show that the statement is true for $n = 1$, i.e. $(a+b)^1 = \sum_{r=0}^{1} C_r^1 a^{1-r} b^r$.
- Show that if the statement is true for $n = k$, then the statement is also true for $n = k+1$, i.e.

$$\underbrace{(a+b)^k = \sum_{r=0}^{k} C_r^k a^{k-r} b^r}_{Given} \Rightarrow \underbrace{(a+b)^{k+1} = \sum_{r=0}^{k+1} C_r^{k+1} a^{k+1-r} b^r}_{Need\ to\ show}.$$

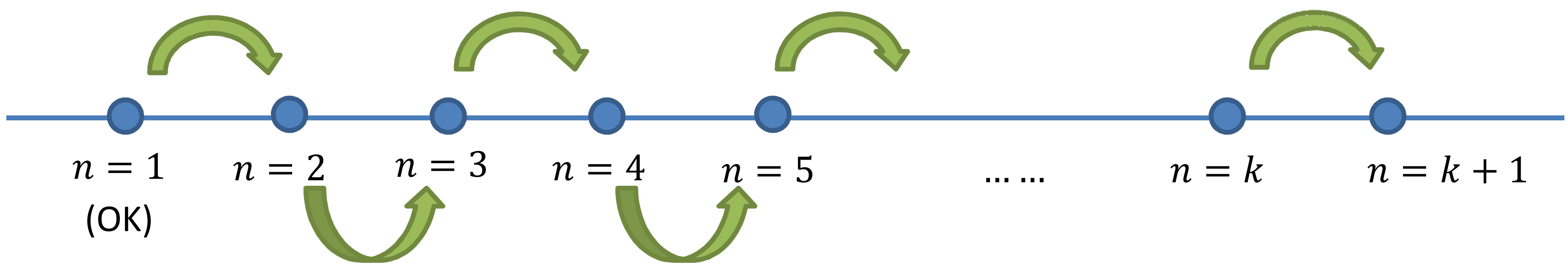
If the above statements are true, then

From the second statement (with $k = 1$), we conclude that the statement is true for $n = k + 1 = 2$.

From the second statement (with $k = 3$), we conclude that the statement is true for $n = k + 1 = 4$.

Since this process can be repeated infinitely, therefore this "chain effect" implies that the statement is OK for every positive integer n, i.e.

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r \text{ , } for\ n \in \mathbb{N}$$



From the second statement (with $k = 2$), we conclude that the statement is true for $n = k + 1 = 3$.

- Formally, this method is called *Mathematical Induction*.

Proof of Binomial Theorem

Step 1: The statement is true for $n = 1$

By direct calculation, one can see that

$$\overbrace{(a+b)^1}^{=a+b} = \sum_{r=0}^{1} C_r^1 a^{1-r} b^r = C_0^1 a^1 b^0 + C_1^1 a^0 b^1 = a + b.$$

Step 2: Assume the statement is true for $n = k$, show that the statement is also true for $n = k + 1$.

Given that $(a+b)^k = \sum_{r=0}^{k} C_r^k a^{k-r} b^r$, then we consider

$(a+b)^{k+1} = (a+b)(a+b)^k$

$$= (a+b)\left(\sum_{r=0}^{k} C_r^k a^{k-r} b^r\right) \quad (Given)$$

$$= (a+b)\left(C_0^k a^k b^0 + C_1^k a^{k-1} b + C_2^k a^{k-2} b + \cdots + C_k^k a^0 b^k\right)$$

$$= \left(C_0^k a^{k+1} b^0 + C_1^k a^k b^1 + C_2^k a^{k-1} b^2 + C_3^k a^{k-2} b^3 + \cdots + C_k^k a^1 b^k\right)$$

$$+\left(C_0^k a^k b^1 + C_1^k a^{k-1} b^2 + C_2^k a^{k-2} b^3 + \cdots + C_{k-1}^k a^1 b^k + C_k^k a^0 b^{k+1}\right)$$

$$= C_0^k a^{k+1} b^0 + \left(C_1^k + C_0^k\right) a^k b^1 + \left(C_2^k + C_1^k\right) a^{k-1} b^2 + \left(C_3^k + C_2^k\right) a^{k-2} b^3 + C_k^k a^0 b^{k+1}$$

By direct computation, one can find that

$$C_{r+1}^k + C_r^k = \frac{k!}{(r+1)!\left(k-(r+1)\right)!} + \frac{k!}{r!\,(k-r)!} = \frac{k!\,(k-r) + k!\,(r+1)}{(r+1)!\,(k-r)!}$$

$$= \frac{k!\,(k-r+r+1)}{(r+1)!\,(k-r)!} = \frac{k!\,(k+1)}{(r+1)!\,(k-r)!} = \frac{(k+1)!}{(r+1)!\left((k+1)-(r+1)\right)!} = C_{r+1}^{k+1}$$

where $r \geq 0$

Together with the fact that $C_0^k = C_0^{k+1} = 1$ *and* $C_k^k = C_{k+1}^{k+1} = 1$. We get

$(a+b)^{k+1}$

$$= C_0^k a^{k+1} b^0 + \left(C_1^k + C_0^k\right) a^k b^1 + \left(C_2^k + C_1^k\right) a^{k-1} b^2 + \left(C_3^k + C_2^k\right) a^{k-2} b^3 + C_k^k a^0 b^{k+1}$$

$$= \underbrace{C_0^{k+1}}_{C_0^k = C_0^{k+1} = 1} a^{k+1}b^0 + \underbrace{C_1^{k+1}}_{\substack{C_{r+1}^k + C_r^k = C_{r+1}^{k+1} \\ with\ r=0}} a^k b^1 + \underbrace{C_2^{k+1}}_{\substack{C_{r+1}^k + C_r^k = C_{r+1}^{k+1} \\ with\ r=1}} a^{k-1}b^2$$

$$+ \underbrace{C_3^{k+1}}_{\substack{C_{r+1}^k + C_r^k = C_{r+1}^{k+1} \\ with\ r=2}} a^{k-2}b^3 + \cdots + \underbrace{C_{k+1}^{k+1}}_{C_k^k = C_{k+1}^{k+1} = 1} a^0 b^{k+1}$$

$$= \sum_{r=0}^{k+1} C_r^{k+1} a^{k+1-r} b^r.$$

Hence, we have

$$(a+b)^{k+1} = \sum_{r=0}^{k+1} C_r^{k+1} a^{k+1-r} b^r.$$

So using the argument in P.45, we conclude that

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r, \qquad for\ n \in \mathbb{N}.$$

Appendix B – How to memorize the binomial theorem?

Recall the binomial theorem:

$$(a+b)^n = \sum_{r=0}^{n} C_r^n\, a^{n-r} b^r = C_0^n a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \cdots + C_n^n b^n$$

This theorem may be hard to remember in a first glance. In fact, one can write down this theorem easily by using the following three steps: (I take $(a+b)^5$ as example)

Step 1: Write down the terms $a^n, a^{n-1}, \ldots, a^2, a^1, a^0$ in descending order:

$$(a+b)^5 = (?)a^5(?) + (?)a^4(?) + (?)a^3(?) + (?)a^2(?) + (?)a^1(?) + (?)a^0(?).$$

Step 2: Write down the terms $b^0, b^1, b^2, \ldots, b^n$ in ascending order:

$$(a+b)^5 = (?)a^5b^0 + (?)a^4b + (?)a^3b^2 + (?)a^2b^3 + (?)ab^4 + (?)a^0b^5.$$

Step 3: Write down the coefficient $C_0^n, C_1^n, C_2^n, \ldots, C_n^n$ in ascending order:

$$(a+b)^5 = C_0^5a^5b^0 + C_1^5a^4b + C_2^5a^3b^2 + C_3^5a^2b^3 + C_4^5ab^4 + C_5^5a^0b^5.$$

Appendix C – Finding unknowns using comparing coefficients in Example 8,9,10

In this Appendix, we use another method (comparing coefficients) used in Example 7 to find the unknown A, B, C in Example 8, 9 and 10 respectively.

From Example 8

Recall that A, B and C satisfy the equation

$$5x + 3 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1)$$

$$\Rightarrow 0x^2 + 5x + 3 = (A+B+C)x^2 + (-2A-3B+C)x - 3A$$

By comparing coefficients, we have

$$\begin{cases} A+B+C = 0 \\ -2A-3B+C = 5. \\ -3A = 3 \end{cases}$$

From the 3rd equation, we get $A = -1$. Substitute it into the first equation and second equation

$$\begin{cases} -1+B+C = 0 \\ 2-3B+C = 5 \end{cases} \Rightarrow \cdots \Rightarrow B = -\frac{1}{2}, \qquad C = \frac{3}{2}.$$

From Example 9

Recall that A, B and C satisfy the equation

$$3x^2 - 8x + 13 = A(x-1)^2 + B(x-1)(x+3) + C(x+3)$$

$$\Rightarrow 3x^2 - 8x + 13 = (A+B)x^2 + (-2A+2B+C)x + (A-3B+3C)$$

By comparing coefficients, we have

$$\begin{cases} A+B=3 \\ -2A+2B+C=-8 \\ A-3B+3C=13 \end{cases}$$

From 1st equation, we have $A = 3 - B$. We substitute this into 2nd and 3rd equation, we get

$$\begin{cases} -2(3-B)+2B+C=-8 \\ (3-B)-3B+3C=13 \end{cases} \Rightarrow \begin{cases} 4B+C=-2 \\ -4B+3C=10 \end{cases} \Rightarrow C=2, \qquad B=-1.$$

Then $A = 3 - B = 3 - (-1) = 4$.

From Example 10

Recall that A, B and C satisfies

$$6x^2 - 3x + 1 = A(x^2 + 1) + (Bx + C)(4x + 1)$$

$$\Rightarrow 6x^2 - 3x + 1 = (A + 4B)x^2 + (B + 4C)x + (A + C)$$

By comparing coefficients, we have

$$\begin{cases} A + 4B = 6 \\ B + 4C = -3. \\ A + C = 1 \end{cases}$$

From 1st equation, we have $A = 6 - 4B$. Substitute this into 3rd equation and using 2nd equation, we have

$$\begin{cases} B + 4C = -3 \\ (6 - 4B) + C = 1 \end{cases} \Rightarrow \begin{cases} B + 4C = -3 \\ -4B + C = -5 \end{cases} \Rightarrow B = 1, \qquad C = -1.$$

So $A = 6 - 4(1) = 2$.