CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester A, 2015/2016

Time allowed : Three hours

This paper has **SEVEN** pages (including this cover page).

A brief table of derivatives is attached on pages 6 and 7.

Instructions to candidates:

1. This paper has **TEN** questions.
2. Attempt **ALL** questions.
3. Each question carries 10 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Question 1

The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = e^x \ , \ x \in [0,\infty) \ ,$$
$$g(x) = -\log_e(x-3) \ , \ x \in (3,\infty) \ .$$

(a) Find the value of x for which $(f \circ g)(x) = 2$.

(3 marks)

(b) Find the inverse function $f^{-1}(x)$ and state its domain.

(4 marks)

(c) Sketch, in a single diagram, the graphs of the curves $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the two graphs.

(3 marks)

Question 2

(a) Find, in radians, the general solution of the equation $5\sin x - 12\cos x = 13$.

(5 marks)

(b) Given that $\sin 3\theta = \cos 2\theta$, $0° < \theta < 90°$, find the value of $\sin\theta$, giving your answer in its simplest surd form.

(5 marks)

[Hint: You may use the formulae

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.]

Question 3

(a) If $y = (ax+b)^{-p}$, where a and p are positive integers, b is a constant, find the general formula for the nth derivative of y with respect to x.

(3 marks)

(b) Express $\dfrac{2x^2+29x-29}{(x+3)(2x-1)^2}$ in partial fractions.

(4 marks)

(c) Using the results in parts (a) and (b), or otherwise, find the fourth derivative of $\dfrac{2x^2+29x-29}{(x+3)(2x-1)^2}$ with respect to x .

You need not simplify your answer. (3 marks)

Question 4

(a) Let $F(x) = (x - [x])^2$, $x \in \mathrm{R}$, where $[x]$ denotes the greatest integer not greater than x.

(i) Sketch the graph of $y = F(x)$ for $-3 \le x \le 3$.

(ii) Find the range of $F(x)$.

(iii) Is $F(x)$ a periodic function of x ?

(6 marks)

(b) Let $G(x)$ be a function of x, defined for all real values of x. Show that $G(x)$ can be expressed as the sum of an even and an odd function of x.

[Hint: You may use the identity $G(x) = \frac{1}{2}(G(x) + G(-x)) + \frac{1}{2}(G(x) - G(-x))$.]

(4 marks)

Question 5

(a) A curve has parametric equations

$x = 1 + t^{-1}$,

$y = t^3 e^{-t}$,

where t is the parameter and $t \neq 0$.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

(5 marks)

(b) Find $\frac{dy}{dx}$ when

(i) $y = \dfrac{e^{-2x}(x^2 - 1)^4}{(x^2 + 3)^2 \sqrt{(x^3 + 1)}}$,

(ii) $x^5 + y^5 = kx^2 y^2$, where k is a constant.

You need not simplify your answers.

(5 marks)

Question 6

Differentiate with respect to x :

(a) $(3x+2)^5 - 4(x^2-1)^3$; (2 marks)

(b) $e^{2x}\cos 3x$; (2 marks)

(c) $\log_e\left(x+\sqrt{x^2+1}\right)$; (2 marks)

(d) $\tan^{-1}\left(\dfrac{9+x^2}{9-x^2}\right)$; (2 marks)

(e) $\dfrac{\sinh^{-1}x}{\sqrt{1+x^2}}$. (2 marks)

Question 7

The function $h(x)$ is defined by $h(x)=\sin\left(\dfrac{1}{x}\right)$, $x\in\mathrm{R}$, $x\neq 0$.

(a) For any integer *n*, find the values of $h(x)$ when $x=\dfrac{1}{n\pi}$, $\dfrac{1}{2n\pi+\dfrac{\pi}{2}}$ and $\dfrac{1}{2n\pi-\dfrac{\pi}{2}}$. (3 marks)

(b) How do the curve, $y=h(x)$ behave as *x* approaches to zero. (2 marks)

(c) Does the limit $\lim\limits_{x\to 0}\dfrac{\sin\left(\dfrac{1}{x}\right)}{\sin\left(\dfrac{1}{x}\right)}$ exist? Why? (2 marks)

(d) Evaluate the limit $\lim\limits_{x\to 0^+}\left(x\sin\left(\dfrac{1}{x}\right)\right)$. (3 marks)

Question 8

Let the equation of the hyperbola H be $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$.

(a) Find the coordinates of the foci of H. (4 marks)

(b) Show that an equation of the normal to H at the point $P(4\sec\theta, 3\tan\theta)$ is $4x\sin\theta+3y=25\tan\theta$. (6 marks)

Question 9

(a) Starting from the formula $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$, show that, if the inverse function has its principal values, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$.

Deduce that $2\tan^{-1}\left(\dfrac{1}{3}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)=\dfrac{\pi}{4}$. (3 marks)

(b) If $y=\tan^{-1}x$, show that $(1+x^2)y^{(2)}+2xy^{(1)}=0$.
Deduce that

$(1+x^2)y^{(n+2)}+2(n+1)xy^{(n+1)}+n(n+1)y^{(n)}=0$, where $y^{(r)}$ denotes $\dfrac{d^r y}{dx^r}$.

Hence, or otherwise, find the expansion of $\tan^{-1}x$ in ascending powers of x as far as the term in x^5.

(5 marks)

(c) Using the results in parts (a) and (b), find an approximation to the value of π , giving 5 decimal places in your answer.

(2 marks)

Question 10

The curve C has equation $y=\dfrac{x^2-2x-2}{x+1}$.

(a) Find the largest possible domain and the range of the function y. (4 marks)

(b) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (3 marks)

(c) Show that $P(0,-2)$ is a stationary point of C. (1 mark)

(d) Determine whether this stationary point is a local maximum or a local minimum.

(2 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	Derivative of y with respect to x
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d\,f(u)}{du}\cdot\frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y=f(u)$	Derivative of y with respect to x
$y=\cot^{-1}u$	$\frac{dy}{dx}=-\frac{1}{1+u^2}\frac{du}{dx}$
$y=\sec^{-1}u$	$\frac{dy}{dx}=\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y=\operatorname{cosec}^{-1}u$	$\frac{dy}{dx}=-\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y=\sinh u$	$\frac{dy}{dx}=\cosh u\frac{du}{dx}$
$y=\cosh u$	$\frac{dy}{dx}=\sinh u\frac{du}{dx}$
$y=\tanh u$	$\frac{dy}{dx}=\operatorname{sech}^2 u\frac{du}{dx}$
$y=\coth u$	$\frac{dy}{dx}=-\operatorname{cosech}^2 u\frac{du}{dx}$
$y=\operatorname{sech} u$	$\frac{dy}{dx}=-\operatorname{sech} u\tanh u\frac{du}{dx}$
$y=\operatorname{cosech} u$	$\frac{dy}{dx}=-\operatorname{cosech} u\coth u\frac{du}{dx}$
$y=\sinh^{-1}u$	$\frac{dy}{dx}=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y=\cosh^{-1}u$	$\frac{dy}{dx}=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y=\tanh^{-1}u$	$\frac{dy}{dx}=\frac{1}{1-u^2}\frac{du}{dx}$
$y=\coth^{-1}u$	$\frac{dy}{dx}=\frac{1}{1-u^2}\frac{du}{dx}$
$y=\operatorname{sech}^{-1}u$	$\frac{dy}{dx}=-\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y=\operatorname{cosech}^{-1}u$	$\frac{dy}{dx}=-\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

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