This is a open-book exam and the exam schedule is November 16th, Tuesday, 9:00 AM - 10:50 AM (two hour). If you need more space, please feel free to attach additional papers. Once you're finished, sign your name and student ID at the top of each page. Also, make sure to sign the following honor pledge.

Honor Pledge

Please review the following honor code, then sign your name and write down the date.

1. I pledge that the answers in this exam are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,
   (a) I will not plagiarize (copy without citation) from any source;
   (b) I will not communicate or attempt to communicate with any other person during the exam;
   (c) neither will I give or attempt to give assistance to another student taking the exam; and
   (d) I will use only approved devices (e.g., calculators) and/or approved device models.
2. I understand that any act of academic dishonesty can lead to disciplinary action.

____
Signature

____
Name & Student ID

____
Date

1. (20 points) (a) Write down the complex and trigonometric Fourier Series (FS) representation of the following impulse train signal $\delta_1(t)$.

$$\delta_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$$

(b) Determine the complex FS representation of the following signal with $T_0 = 2\pi$.

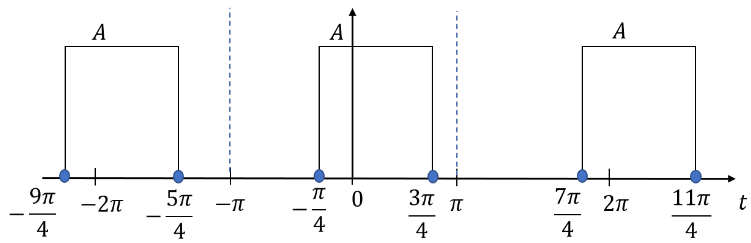


Figure 1

(c) Derive the Fourier Transform (FT) of the following signal with $B > 0$.

$$4B \cdot \mathcal{F}\left[\mathrm{Sinc}^2(2Bt) \cdot \cos(2\pi f_0 t)\right] = ?$$

2. (20 points) Consider a period signal $x(t)$ defined by

$$x(t) = \frac{3t}{2}, \quad 0 < t < 2 \quad \text{and} \quad x(t+2) = x(t).$$

(a) Determine the complex and trigonometric FS representation of the periodic signal $x(t)$.

(b) Use the trigonometric FS representation to find the sum of the following infinite series.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

(c) Use the Parseval's theorem to find the sum of the following infinite series.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

Test 2. Solution)

Q1. a) $\left(\begin{array}{c}\text{Since} \\ T_0 = 1 \\ \omega_0 = 2\pi\end{array}\right)$, $\Big($ $C_0 = 1 = C_k$ ← (No derivation required)

$a_0 = 2$, $a_k = 2Re(C_k) = 2$, $b_k = -2Im(C_k) = 0$

$$\Rightarrow \delta_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{k=-\infty}^{\infty} e^{j2\pi k t} = 1 + 2\sum_{k=1}^{\infty} \cos(2\pi k t)$$

b) $C_0 = \dfrac{A}{2\pi}\displaystyle\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} dt = \dfrac{A}{2}$, $\omega_0 = \dfrac{2\pi}{2\pi} = 1$

The given signal $x(t)$ is time-shifted version of Ex 3-2) a) in the lecture note.

$x(t) = x_a(t + \frac{\pi}{4})$ where FS d_k of $x_a(t)$ is derived as

$$d_k = \begin{cases} 0 & \text{for even } k = 2m \\ \dfrac{A}{j\pi(2m+1)} & \text{for odd } k = 2m+1 \end{cases}$$

Hence, the FS coefficient C_k of $x(t)$ is given by

$$C_k = e^{jk\frac{\pi}{4}} d_k = \frac{A e^{jk\frac{\pi}{4}}}{j2\pi k}\left[1 - e^{-jk\pi}\right] = \frac{A e^{jk\frac{\pi}{4}}}{j2\pi k}\left(1 - (-1)^k\right)$$

$$= \begin{cases} \emptyset & \text{for even } k = 2m \\ \dfrac{A e^{jk\frac{\pi}{4}}}{j\pi(2m+1)} & \text{for odd } k = 2m+1 \end{cases}, \quad C_0 = \frac{A}{2}$$

$C_k = \dfrac{A}{j2\pi k}\left[e^{jk\frac{\pi}{4}} - e^{-jk\frac{3\pi}{4}}\right]$ or $\dfrac{Ae^{-jk\frac{\pi}{4}}}{2}\,\mathrm{Sinc}\left(\dfrac{k}{2}\right)$ are all ok.

Test 2. Solution)

Q1. c) $\mathcal{F}\left(\text{sinc}^2(2Bt)\right) = \frac{1}{2B}\,\text{tri}\left(\frac{f}{2B}\right)$

$$\mathcal{F}\left(\cos(2\pi f_0 t)\,\text{sinc}^2(2Bt)\right) = \frac{1}{4B}\left[\text{tri}\left(\frac{f-f_0}{2B}\right) + \text{tri}\left(\frac{f+f_0}{2B}\right)\right]$$

Hence $\mathcal{F}\left[4B\,\text{sinc}^2(2Bt)\cos(2\pi f_0 t)\right]$

$$= \left[\text{tri}\left(\frac{f-f_0}{2B}\right) + \text{tri}\left(\frac{f+f_0}{2B}\right)\right]$$

Q 2. a) $T_0 = 2$. $\omega_0 = \frac{2\pi}{T_0} = \pi$

$$C_0 = \frac{1}{2}\int_0^2 \frac{3t}{2}\,dt = \frac{3}{4}\cdot\frac{t^2}{2}\Big|_0^2 = \frac{3}{2}$$

$$C_k = \frac{1}{2}\int_0^2 \frac{3t}{2}e^{-jk\pi t}\,dt = \frac{3}{4}\int_0^2 t\,e^{-jk\pi t}\,dt$$

$$= \frac{3}{4}\left[\left(\frac{t}{-jk\pi} - \frac{1}{(-jk\pi)^2}\right)e^{-jk\pi t}\Big|_0^2\right]$$

$$= \frac{3}{4}\left[\left(\frac{2}{-jk\pi} - \cancel{\frac{1}{(-jk\pi)^2}}\right)e^{-j2\pi k} - \left(-\cancel{\frac{1}{(-jk\pi)^2}}\right)\right] \quad \text{where } e^{-j2\pi k} = 1$$

$$= \frac{3j}{2k\pi} \Rightarrow \begin{array}{l} a_0 = 2C_0 = 3 \\ a_k = 2\,\text{Re}(C_k) = 0 \end{array}, \quad b_k = -2\,\text{Im}(C_k) = -\frac{3}{k\pi}$$

Test 2. Solution)

Q 2. a)

$$x(t) = \frac{3}{2} + \left(\frac{3j}{2\pi}\right) \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k} e^{jk\pi t}$$

$$= \frac{3}{2} + \left(-\frac{3}{\pi}\right) \sum_{k=1}^{\infty} \frac{1}{k} \sin(k \cdot \pi t) \quad \text{for} \quad 0 < t < 2$$

b) Based on the previous result,

$$\frac{3t}{2} - \frac{3}{2} = -\frac{3}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\pi t) \quad — \ (1)$$

let's assume $t = \frac{1}{2}$, then (1) becomes

$$\sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{k\pi}{2}\right) = \frac{\pi}{4}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Hence, the sum of the infinite series is $\frac{\pi}{4}$.

c) $$P = \frac{1}{2} \int_0^2 |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

(left-hand side) $\Rightarrow \frac{1}{2} \cdot \frac{9}{4} \int_0^2 t^2 dt = \frac{9}{8} \cdot \frac{1}{3} t^3 \Big|_0^2 = 3$.

(Right-hand side) $\Rightarrow \frac{9}{4} + 2 \times \frac{9}{4\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$

Negative k and Positive k are symmetric

3. (20 points) Use time differentiation property to find the FT of the following signal $x(t)$

$$x(t) = \begin{cases} t & \text{for} \quad 0 \leq t < 1, \\ 1 & \text{for} \quad 1 \leq t < 2, \\ 3 - t & \text{for} \quad 2 \leq t < 3, \\ 0 & \text{for} \quad t < 0 \text{ or } t > 3. \end{cases}$$

4. (20 points) (a) What is the Nyquist sampling rate of the following signal?

$$x(t) = \cos(150\pi t) + 2\sin(300\pi t) - 400\cos(600\pi t).$$

(b) Consider a filter with the following impulse response $h(t)$. For the given filter, calculate the 3-dB bandwidth f_{3dB} and the equivalent bandwidth f_{eq}.

$$h(t) = \text{Sinc}^2(t).$$

5. (20 points) Consider a continuous-time LTI system described by the following equation.

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12y(t) = \frac{dx(t)}{dt} + 2x(t)$$

(a) Derive the frequency response $H(f)$ and the corresponding impulse response $h(t)$.

(b) Derive the system output $y(t)$ and $Y(f)$ for the input signal $x(t) = e^{-2t}u(t)$.

Test 2. Solution

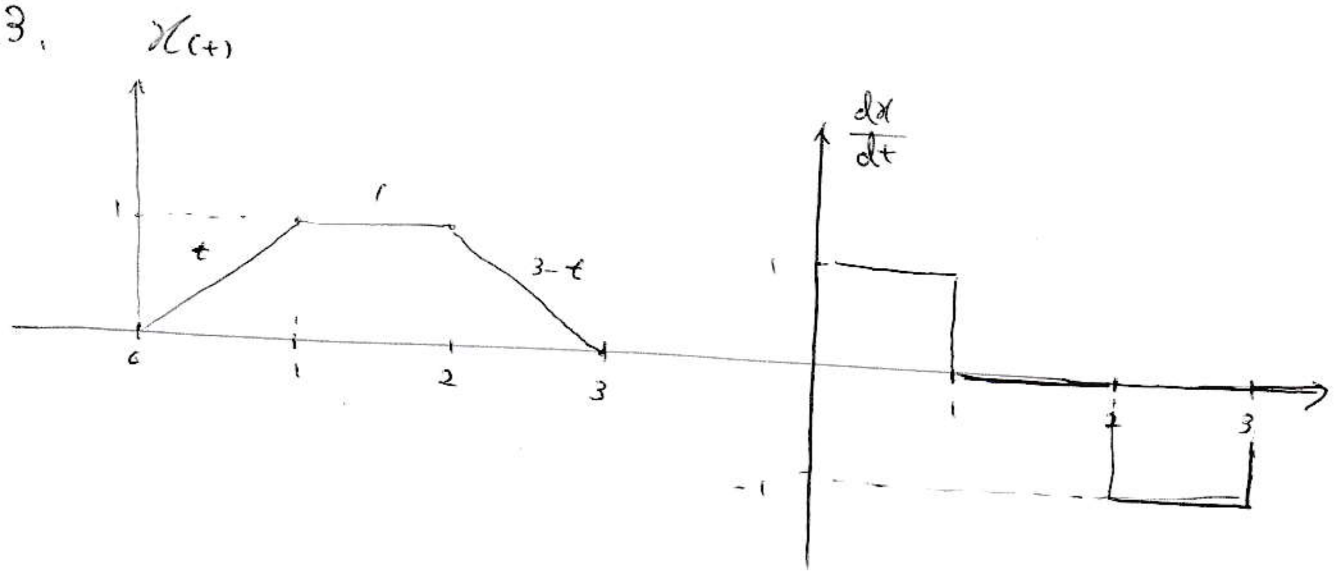
Q 2. c) Due to the Parseval's theorem.

$$3 = \frac{9}{4}\left[1 + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\right]$$

$\Rightarrow$ Hence, the sum of the infinite series is

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

Q3.



$$\mathcal{F}\left(\frac{dx}{dt}\right) = \int_0^1 e^{-j2\pi ft}dt - \int_2^3 e^{-j2\pi ft}dt$$

$$= \frac{1}{j2\pi f}\left(1 - e^{-j2\pi f}\right) + \frac{1}{j2\pi f}\left(e^{-j6\pi f} - e^{-j4\pi f}\right)$$

Hence $$\mathcal{F}(x(t)) = \frac{1}{4(\pi f)^2}\left[e^{-j2\pi f} - 1 - e^{-j6\pi f} + e^{-j4\pi f}\right]$$

Test 2. Solution)

Q4. a) Since the BW of $\cos(150\pi t)$, $\sin(300\pi t)$, $\cos(600\pi t)$

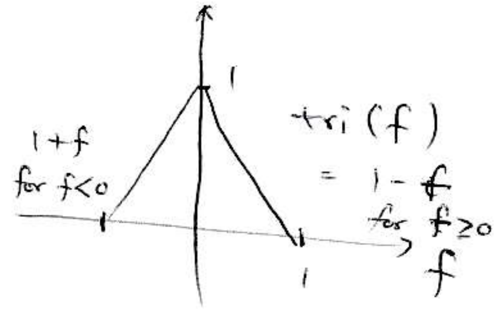
are $f_1 = 75$, $f_2 = 150$, $f_3 = 300$, respectively,

the BW of signal $x(t)$ is $f_M = 300\ Hz$.

Hence, the Nyquist sampling rate is

$2f_M = 600\ Hz$.

b) $\mathcal{F}(\text{sinc}^2(t)) = \text{tri}(f)$



3dB BW: $|H(0)| = 1$ and $|H(f_{3dB})| = \dfrac{|H(0)|}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$= 1 - |f|$

$f_{3dB} = 1 - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}-1}{\sqrt{2}}$

Equivalent BW: $f_{eq} = \int_0^1 |H(f)|^2 df = \int_0^1 f^2 df = \dfrac{1}{3}$

Hence $\begin{cases} f_{3dB} = 1 - \dfrac{1}{\sqrt{2}} \text{ or } \dfrac{\sqrt{2}-1}{\sqrt{2}} \\ f_{eq} = \dfrac{1}{3} \end{cases}$

Test 2. Solution)

Q5. a)

$$\{(j2\pi f)^2 + 7(j2\pi f) + 12\} Y(f) = (j2\pi f + 2) X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{(j2\pi f + 2)}{(j2\pi f)^2 + 7(j2\pi f) + 12} = \frac{(j2\pi f + 2)}{(j2\pi f + 4)(j2\pi f + 3)}$$

$$= \frac{\alpha_1}{j2\pi f + 4} + \frac{\alpha_2}{j2\pi f + 3}$$

and after Partial Frational Expansion

$\alpha_1 = 2$, $\alpha_2 = -1$

$\mathcal{F}^{-1}$

$$h(t) = (2e^{-4t} - e^{-3t}) u(t)$$

b) If $x(t) = e^{-2t} u(t)$, then $X(f) = \frac{1}{j2\pi f + 2}$ and

$$Y(f) = H(f) X(f)$$

$$= \frac{j2\pi f + 2}{(j2\pi f + 4)(j2\pi f + 3)} \times \frac{1}{j2\pi f + 2}$$

$$= \frac{-1}{j2\pi f + 4} + \frac{1}{j2\pi f + 3}$$

$\mathcal{F}^{-1}$

$$y(t) = (-e^{-4t} + e^{-3t}) u(t)$$