

$$X(s) = \frac{(b_0 s^M + b_1 s^{M-1} + \dots + b_M) N(s)}{(a_0 s^N + a_1 s^{N-1} + \dots + a_N s^0) D(s)}$$

$$(M < N) = \frac{\prod_{i=1}^N (s - s_{p_i})}{D(s)}$$

Partial Fraction Expansion

- Assuming the poles are simple, the partial fraction expansion of rational $X(s)$ with $M < N$ is given by

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^M + b_1 s^{M-1} + \dots + b_M}{(s - s_{p_1})(s - s_{p_2}) \dots (s - s_{p_N})}$$

$$= \frac{A_1}{s - s_{p_1}} + \frac{A_2}{s - s_{p_2}} + \dots + \frac{A_N}{s - s_{p_N}}$$

- The coefficients $\{A_i\}$ are computed as

$$A_i = \left. (s - s_{p_i}) X(s) \right|_{s=s_{p_i}}$$

$M < N$

- For $M < N$, $X(s)$ is a proper rational function.

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s_{p_i} : roots of $D(s)$, $1 \leq i \leq N$

Example

- Find the partial-fraction expansion of

$$X(s) = \frac{s+0.5}{(s+1)(s+2)}$$

- Partial-fraction expansion.

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

where

$$A_1 = (s+1)X(s)|_{s=-1} = (s+1) \frac{s+0.5}{(s+1)(s+2)} \Big|_{s=-1} = \frac{s+0.5}{(s+2)} \Big|_{s=-1} = \frac{-0.5}{1} = -0.5$$

$$A_2 = (s+2)X(s)|_{s=-2} = (s+2) \frac{s+0.5}{(s+1)(s+2)} \Big|_{s=-2} = \frac{s+0.5}{(s+1)} \Big|_{s=-2} = \frac{-1.5}{-1} = 1.5$$

We have
$$X(s) = \frac{-0.5}{s+1} + \frac{1.5}{s+2}$$

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$$\underbrace{X(s)}_{(s-s_{p1})^x} = \frac{N(s)}{\underbrace{\prod_{i=1}^n (s-s_{p_i})}_{(s-s_{p1})^x}} = \left(\frac{A_1}{s-s_{p1}} + \frac{A_2}{s-s_{p2}} + \dots + \frac{A_N}{s-s_{pN}} \right)$$

$$A_1 = \frac{(s-s_{p1}) X(s)}{1} \quad \text{①}$$

$$\Rightarrow \frac{N(s)}{\prod_{i=2}^n (s-s_{p_i})} \quad \text{②}$$

$s = s_{p1}$

if $s = s_{p1}$

$$\Rightarrow A_1 + \underbrace{(s-s_{p1})}_{=0} \left\{ \frac{A_2}{s-s_{p2}} + \dots + \frac{A_N}{s-s_{pN}} \right\}$$

$$X(s) = \left(\frac{s+0.5}{(s+1)(s+2)} \right) \times \underline{(s+2)}$$

$$= \frac{C_1}{\boxed{s+1}} + \frac{C_2}{s+2} = \frac{1.5}{s+2} - \frac{0.5}{s+1}$$

$$C_1 = \underline{(s+1)} X(s) = \frac{s+0.5}{s+2} \Bigg|_{s=-1}$$

$$= \frac{-0.5}{1} = -0.5$$

$$C_2 = \underline{(s+2)} X(s) = \frac{(s+0.5)}{(s+1)} \Bigg|_{s=-2}$$

$$= \frac{-1.5}{-1} = 1.5$$

$$(2s-5) + \frac{15}{(s+2)} - \frac{2}{(s+1)}$$

Partial Fraction Expansion Cont.

- If $M > N$, we need to use long division to express the rational function as the sum of quotient plus a proper rational function.
- Example of long division:

$$X(s) = \frac{2s^3 + s^2 + 2s + 1}{s^2 + 3s + 2} = (2s - 5) + \frac{13s + 11}{(s+2)(s+1)}$$

Use the long division to convert the rational function to a proper one:

$$\begin{array}{r} 2s - 5 \\ s^2 + 3s + 2 \overline{) 2s^3 + s^2 + 2s + 1} \\ \underline{2s^3 + 6s^2 + 4s} \\ -5s^2 - 2s + 1 \\ \underline{-5s^2 - 15s - 10} \\ 13s + 11 \end{array} \quad -26 + 11$$

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$$\frac{13s+11}{(s+2)(s+1)} = \frac{a_1}{s+2} + \frac{a_2}{s+1} \quad \left| \begin{array}{l} a_1 = \frac{13s+11}{s+1} \Big|_{s=-2} = 15 \\ a_2 = \frac{13s+11}{s+2} \Big|_{s=-1} = -2 \end{array} \right.$$

Cont.

After long division, $X(s) = 2s - 5 + \frac{13s+11}{s^2+3s+2} = 2s - 5 + \frac{13s+11}{(s+1)(s+2)}$

Taking partial fraction expansion of the second term,

$$X(s) = 2s - 5 + \frac{A_1}{s+1} + \frac{A_2}{s+2},$$

where

$$A_1 = (s+1)X(s)|_{s=-1} = \frac{13s+11}{s+2} \Big|_{s=-1} = -2,$$

$$A_2 = (s+2)X(s)|_{s=-2} = \frac{13s+11}{s+1} \Big|_{s=-2} = 15.$$

Finally, the partial fraction expansion of $X(s)$ is

$$X(s) = 2s - 5 + \frac{-2}{s+1} + \frac{15}{s+2}$$

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$$\begin{aligned} &= \frac{-2}{1} \\ &= -2 \end{aligned}$$