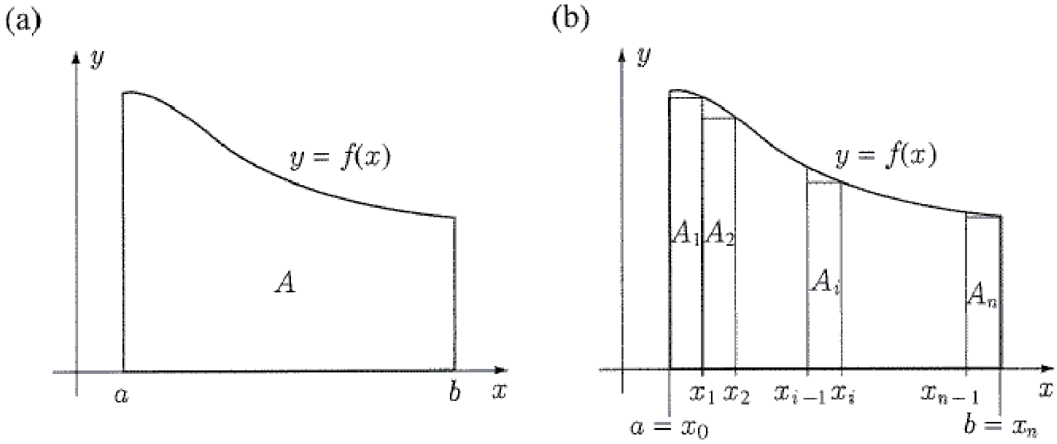
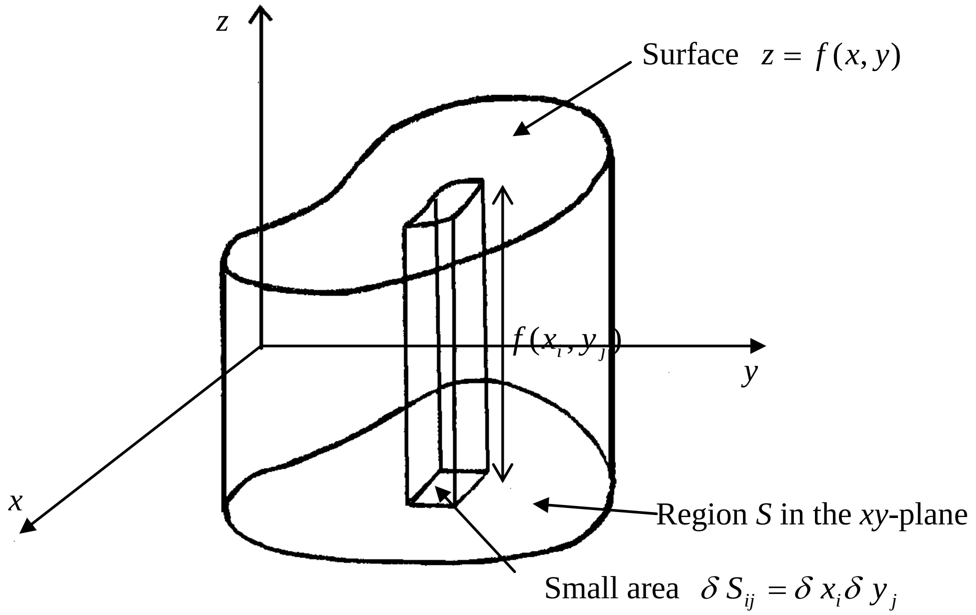
MA2001 Multiple Integrals

1 Double Integrals

For a function $f(x)$ of one variable defined over $a \le x \le b$, let $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$ and $\delta x_i = x_i - x_{i-1}$. Let $P = \{\delta x_i : 1 \le i \le n\}$, $|P| = \max\limits_{1 \le i \le n} \{\delta x_i : 1 \le i \le n\}$. We define the definite integral of *f* from *a* to *b* as $\int_a^b f(x)\,dx = \lim\limits_{|P| \to 0} \sum_{i=1}^{n} f(x_i)\delta x_i$, which corresponds geometrically, if $f(x)$ is positive, to the area between the graph, the *x*-axis and the lines $x = a$ and $x = b$.



A function $f(x, y)$ defines a surface and, if $f(x, y)$ is positive, we may analogously determine the volume enclosed by this surface and a cylinder erected on a region *S* of the *xy*-plane.



Divide *S* into *n* small area elements such that the element has area $\delta S_{ij} = \delta x_i \delta y_j$ and contains the point (x_i, y_j). Then $f(x_i, y_j)\delta x_i \delta y_j$ is the volume of the column with base $\delta S_{ij} = \delta x_i \delta y_j$ and height $f(x_i, y_j)$.

Let $P = \{\delta S_{ij} = \delta x_i \delta y_j : 1 \le i \le m, 1 \le j \le n\}$, $|P| = \max\limits_{1 \le i \le m, 1 \le j \le n} \{\delta S_{ij} = \delta x_i \delta y_j : 1 \le i \le m, 1 \le j \le n\}$.

We define the <u>double integral</u> of $f(x,y)$ over the region S to be

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{\text{all}\,\delta x_i,\,\delta y_j\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\delta x_i\delta y_j.$$

The direct evaluation of the limit in the definition of double integral is impractical. We will calculate the double integral by means of the <u>iterated integrals</u>.

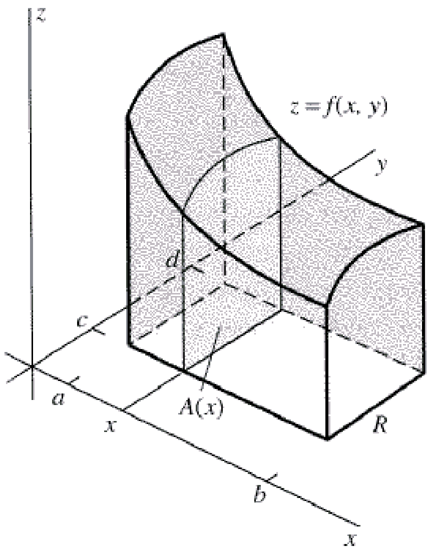
<u>Theorem</u> (Rectangular regions)
Let $f(x,y)$ be continuous on the region S: $a \le x \le b,\ c \le y \le d$.

Let $A(x) = \int_c^d f(x,y)dy$, which is the area of the cross section at x (Figure A), then

$$\iint_S f(x,y)dxdy = \int_a^b A(x)dx = \int_a^b\left[\int_c^d f(x,y)dy\right]dx.$$

Similarly, we also have $\iint_S f(x,y)dxdy = \int_c^d\left[\int_a^b f(x,y)dx\right]dy$.

Figure A



<u>Example 1</u>
Evaluate $\iint_S (2xy+y^2)dxdy$ where S is the rectangle $1\le x\le 2,\ 0\le y\le 1$.

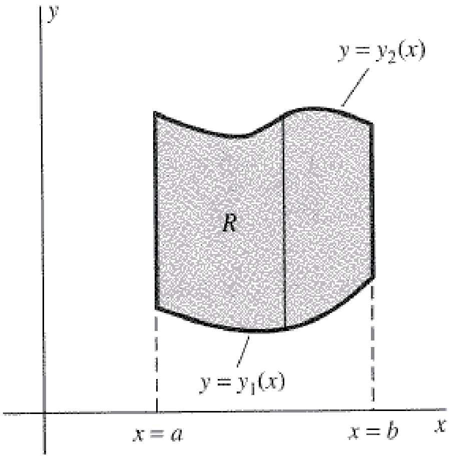
Solution:

$$\int_0^1\left[\int_1^2(2xy+y^2)dx\right]dy = \int_0^1 (x^2y+xy^2)\Big|_1^2 dy = \int_0^1(4y+2y^2-y-y^2)dy = \left(\frac{3y^2}{2}+\frac{y^3}{3}\right)\Big|_0^1 = \frac{11}{6}$$ or

$$\int_1^2\left[\int_0^1(2xy+y^2)dy\right]dx = \int_1^2\left(xy^2+\frac{y^3}{3}\right)\Big|_0^1 dx = \int_1^2\left(x+\frac{1}{3}\right)dx = \left(\frac{x^2}{2}+\frac{x}{3}\right)\Big|_1^2 = \frac{4}{2}+\frac{2}{3}-\frac{1}{2}-\frac{1}{3} = \frac{11}{6}.$$ □

Example 2 (Non-rectangular regions)

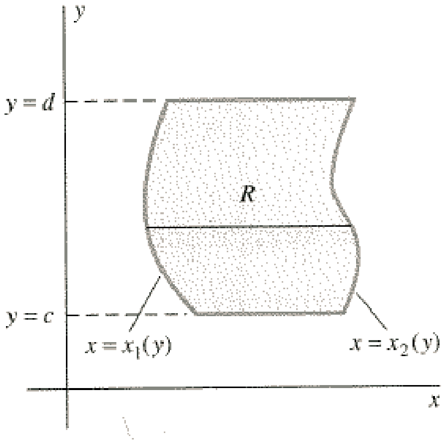
A vertically simple region R



$$a \le x \le b,\ \ y_1(x) \le y \le y_2(x)$$

$$\int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy \right] dx$$

A horizontally simple region R



$$c \le y \le d,\ \ x_1(y) \le x \le x_2(y)$$

$$\int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx \right] dy$$

□

Example 3

$\iint\limits_S 1dxdy =$ area of the region S if $f(x,y) \equiv 1$.

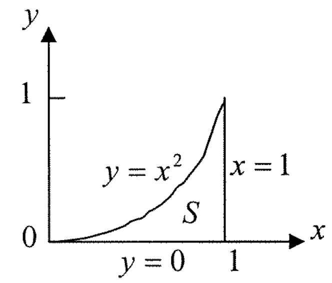
□

Example 4

Evaluate $\iint\limits_S xy^2 dxdy$ where S is the region bounded by $y = 0,\ \ y = x^2$ and $x = 1$.

Solution:

A sketch of the region S is important, but it is not necessary to be able to visualize the surface $f(x, y)$.



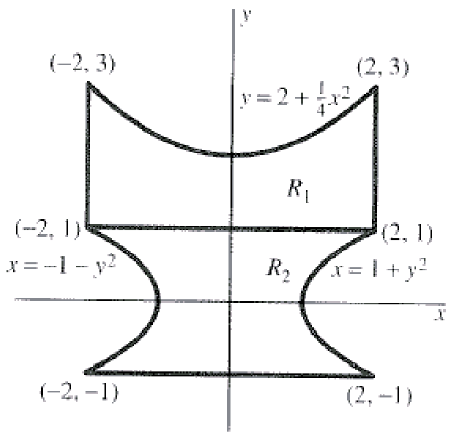
$$\iint\limits_S xy^2 dxdy = \int_0^1 \left(\int_0^{x^2} xy^2 dy \right) dx = \int_0^1 \left(\frac{xy^3}{3} \right) \Bigg|_0^{x^2} dx = \int_0^1 \frac{x^7}{3} dx = \frac{x^8}{24} \Bigg|_0^1 = \frac{1}{24} \quad \text{or}$$

$$\iint\limits_S xy^2 dxdy = \int_0^1 \left(\int_{\sqrt{y}}^1 xy^2 dx \right) dy = \int_0^1 \left(\frac{y^2 x^2}{2} \right) \Bigg|_{\sqrt{y}}^1 dy = \int_0^1 \left(\frac{y^2}{2} - \frac{y^3}{2} \right) dy = \left(\frac{y^3}{6} - \frac{y^4}{8} \right) \Bigg|_0^1 = \frac{1}{24}.$$

□

Example 5

The nonsimple region R is the union of the nonoverlapping simple regions R_1, R_2



Solution:

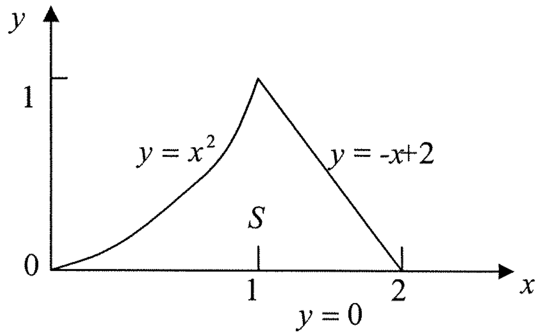
$$\iint\limits_R f(x, y) dxdy = \iint\limits_{R_1} f(x, y) dxdy + \iint\limits_{R_2} f(x, y) dxdy = \int_{-2}^{2} \left[\int_1^{2 + \frac{x^2}{4}} f(x, y) dy \right] dx + \int_{-1}^{1} \left[\int_{-1-y^2}^{1+y^2} f(x, y) dx \right] dy$$

□

In some cases, one order of integration may be easier than the other.

Example 6

Consider an integrable function $f(x,y)$, which is defined on a region of xy - plane shown as follows:



We observe that $\iint_S f(x,y)dxdy = \int_0^1 \left[\int_{\sqrt{y}}^{2-y} f(x,y)\,dx \right] dy$ looks like easier than

$\iint_S f(x,y)dxdy = \int_0^1 \left[\int_0^{x^2} f(x,y)\,dy \right] dx + \int_1^2 \left[\int_0^{2-x} f(x,y)\,dy \right] dx$.

Example 7 (Optional)

Let $$f(x,y) = \begin{cases} \dfrac{1}{y^2} & \text{if } 0 < x < y < 1 \\ -\dfrac{1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that the double integral $\iint_Q f(x,y)dxdy$, where $Q = [0,1] \times [0,1]$ can not exist by pointing out

$\int_0^1 \left[\int_0^1 f(x,y)\,dy \right] dx \neq \int_0^1 \left[\int_0^1 f(x,y)\,dx \right] dy$.

Solution:

$$\int_0^1 \left[\int_0^1 f(x,y)\,dy \right] dx = \int_0^1 \left[\int_0^x f(x,y)\,dy + \int_x^1 f(x,y)\,dy \right] dx = \int_0^1 \left(-\int_0^x \frac{1}{x^2}\,dy + \int_x^1 \frac{1}{y^2}\,dy \right) dx = \int_0^1 -1\,dx = -1.$$

$$\int_0^1 \left[\int_0^1 f(x,y)\,dx \right] dy = \int_0^1 \left[\int_0^y f(x,y)\,dx + \int_y^1 f(x,y)\,dx \right] dy = \int_0^1 \left(\int_0^y \frac{1}{y^2}\,dx - \int_y^1 \frac{1}{x^2}\,dx \right) dy = \int_0^1 1\,dx = 1.$$

Hence,

$$\int_0^1 \left[\int_0^1 f(x,y)\,dy \right] dx \neq \int_0^1 \left[\int_0^1 f(x,y)\,dx \right] dy$$

As the two iterated integrals exist but are unequal, the double integral $\iint_Q f(x,y)dxdy$ cannot exist over $Q=[0,1]\times[0,1]$.

□

Example 8

Change the order of the integration in $\int_0^1\left[\int_x^{\sqrt{x}} f(x,y)\,dy\right]dx$.

Solution:

The line $y=x$ and the parabola $y=\sqrt{x}$ cut at $(0,0)$ and $(1,1)$. The domain of integration is the area bounded by $y=x$ and $y=\sqrt{x}$ (the same as $x=y^2$).

So $\int_0^1\left[\int_x^{\sqrt{x}} f(x,y)\,dy\right]dx=\int_0^1\left[\int_{y^2}^{y} f(x,y)\,dx\right]dy$.

□

Example 9

Change the order of the integration in $\int_0^1\left[\int_{-y}^{y} f(x,y)\,dx\right]dy$

Solution:

Let R be the region of integration R is given by the inequalities $-y\le x\le y$ and $0\le y\le 1$.
Therefore, R is bounded by $y=x$ and $y=-x$ between $y=0,\ y=1$.
To find limits for integration in the reverse order, we imagine a vertical line passing from bottom to the top through the region. From $x=-1$ to $x=0$, it enters at $y=-x$ and leaves at $y=1$. Then from $x=0$ to $x=1$, it enters at $y=x$ and leaves at $y=1$. So

$$\int_0^1\left[\int_{-y}^{y} f(x,y)\,dx\right]dy=\iint_R f(x,y)\,dxdy=\int_{-1}^{0}\left[\int_{-x}^{1} f(x,y)\,dy\right]dx+\int_0^1\left[\int_x^1 f(x,y)\,dy\right]dx$$

□

2 Change of Variable in Double Integrals

For the definite integral $I = \int_a^b f(x)\,dx$, we know that the change of variable $x = x(u)$ produces

$$I = \int_\alpha^\beta f\left[x(u)\right]\frac{dx}{du}du,$$

where $x(\alpha) = a,\ x(\beta) = b$.

For the double integral $I = \iint_S f(x,y)dxdy$, the change of variable $x = x(u,v),\ y = y(u,v)$ gives

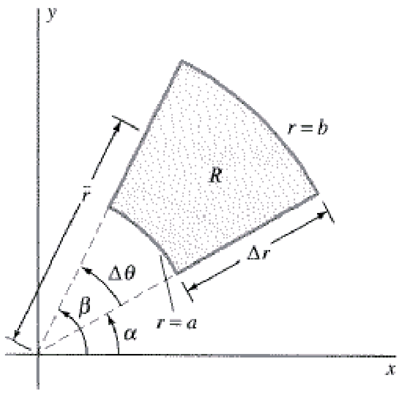
$$I = \iint_{S^*} f\left[x(u,v), y(u,v)\right]\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv,$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$ is called the Jacobian of the transformation. S^* is the region in the uv-plane corresponding to the region S in the xy-plane injectively (one to one) and J must be of one sign in S^* .

In particular, when using polar coordinates, we have

$$x = r\cos\theta,\ y = r\sin\theta$$

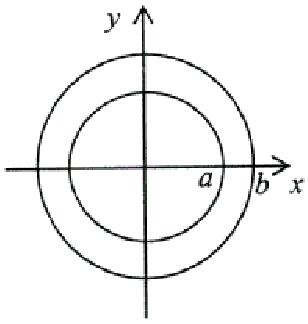
For polar coordinates, the area of a small polar rectangle R is approximately $\bar{r}\Delta r\Delta\theta$, (Note $\delta \equiv \Delta$).



$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$ i.e. $dxdy \to rdrd\theta$, where $rdrd\theta$ corresponds to an element of area in the $r\theta$- plane.

Example 10

Find the moment of inertia of a hollow circular cylinder of inner radius a, outer radius b, height h and constant density ρ about the axis of the cylinder. The formula for the moment of inertia of the hollow circular cylinder is given as $MI = \iint_S \rho h\left(x^2+y^2\right)dxdy$.



Solution:

Using polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$.

Under $x = r\cos\theta,\ y = r\sin\theta$, the hollow circular ring S of inner radius a, outer radius b is transformed as a rectangle R in the $r\theta$- plane, which is bounded by $\theta = 0,\ \theta = 2\pi$, $r = a,\ r = b$.

The region of integration is now a rectangle R in the $r\theta$- plane, $R:\begin{cases} a \le r \le b \\ 0 \le \theta \le 2\pi \end{cases}$.

$J = r \qquad |J| = r \qquad$ i.e. $dxdy \to rdrd\theta$

$$MI = \iint_S \rho h\left(x^2+y^2\right)dxdy = \iint_R \rho hr^2|J|drd\theta = \iint_R \rho hr^2 rdrd\theta$$

$$= \int_0^{2\pi}\left[\int_a^b \rho hr^3dr\right]d\theta = \rho h\int_0^{2\pi}\left[\int_a^b r^3dr\right]d\theta = \rho h\left(\int_a^b r^3dr\right)\left(\int_0^{2\pi} 1d\theta\right)$$

$$= \rho h\left(\frac{r^4}{4}\bigg|_a^b\right)\left(\theta\bigg|_0^{2\pi}\right) = \rho h2\pi\frac{b^4-a^4}{4} = \rho h\pi\frac{\left(a^2+b^2\right)\left(a^2-b^2\right)}{2} = \frac{M}{2}\left(a^2+b^2\right)$$

, where mass $= M = \rho h\pi\left(b^2-a^2\right)$

Example 11

Evaluate the integral $\iint_Q e^{\frac{y-x}{y+x}}dxdy$, where Q is bounded by $x+y=2, x=0, y=0$.

(Hint: Use the change of variables $u = y-x$ and $v = y+x$.)

Solution:

By using the change of variables $u = y-x$ and $v = y+x$ and solving for x and y, we find $x = \frac{v-u}{2}, y = \frac{v+u}{2}$.

Under the transformations $u = y - x$ and $v = y + x$, the lines $x + y = 2, x = 0, y = 0$ map onto the lines $v = 2, u - v = 0, u + v = 0$, respectively. Points inside Q are carried into points inside the triangular region T bounded by $v = 2, u - v = 0, u + v = 0$.

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\frac{1}{2} \quad \text{or} \quad J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}} = \frac{1}{-2}$$

Therefore, $\iint\limits_Q e^{\frac{y-x}{y+x}} dxdy = \iint\limits_T e^{\frac{u}{v}} \left|-\frac{1}{2}\right| dudv = \frac{1}{2}\int_0^2 \left(\int_{-v}^{v} e^{\frac{u}{v}} du \right) dv = \frac{1}{2}\int_0^2 v\left(e - \frac{1}{e}\right) dv = e - \frac{1}{e}$.

□

Example 12

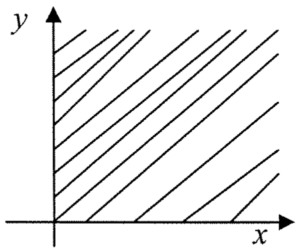
By considering $I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right)$, evaluate $I = \int_0^\infty e^{-x^2} dx$.

Solution:

$$I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = \int_0^\infty \left(\int_0^\infty e^{-x^2} dx\right) e^{-y^2} dy = \int_0^\infty e^{-y^2} \left(\int_0^\infty e^{-x^2} dx\right) dy = \int_0^\infty \left(\int_0^\infty e^{-y^2} e^{-x^2} dx\right) dy$$

$$= \int_0^\infty \left[\int_0^\infty e^{-\left(x^2+y^2\right)} dx\right] dy$$

$\int_0^\infty \left[\int_0^\infty e^{-\left(x^2+y^2\right)} dx\right] dy$ is equivalent to integrating the function $f(x,y) = e^{-\left(x^2+y^2\right)}$ over the first quadrant of xy- plane.



Under the transformations $x = r\cos\theta, y = r\sin\theta$, the first quadrant R of xy- plane is corresponding to the region $\overline{R}$, which is bounded by $r = 0, \theta = 0, \theta = \frac{\pi}{2}$ in $r\theta$ -plane. (We observe that under $x = r\cos\theta, y = r\sin\theta$ the quarter circle $x^2 + y^2 = r_0^2, x, y \geq 0$ to $r = r_0$ corresponds the line segment $\begin{cases} r = r_0 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$ in $r\theta$ -plane). So

$$I^2 = \int_0^\infty \left[\int_0^\infty e^{-\left(x^2+y^2\right)} dx\right] dy = \iint\limits_R e^{-\left(x^2+y^2\right)} dxdy = \iint\limits_{\overline{R}} e^{-r^2} |J| drd\theta = \iint\limits_{\overline{R}} e^{-r^2} rdrd\theta = \int_0^\infty \left(\int_0^{\frac{\pi}{2}} e^{-r^2} rd\theta\right) dr$$, where

$$|J| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r .$$

And

$$\int_0^\infty \left(\int_0^{\frac{\pi}{2}} e^{-r^2} r d\theta \right) dr = \int_0^\infty \frac{\pi}{2} e^{-r^2} r dr \underset{\substack{\text{Let } x=r^2 \\ r=0 \to x=0 \\ r=\infty \to x=\infty \\ dx=2rdr}}{=} \int_0^\infty \frac{\pi}{2} e^{-x} \frac{dx}{2} = \frac{\pi}{4} \int_0^\infty e^{-x} dx = -\frac{\pi}{4} e^{-x} \Big|_0^\infty = -\frac{\pi}{4} \frac{1}{e^x} \Big|_0^\infty$$

$$= -\frac{\pi}{4}\left(\frac{1}{e^\infty} - 1\right) = -\frac{\pi}{4}(0-1) = \frac{\pi}{4}$$

Finally, we have $I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = \frac{\pi}{4} \Rightarrow \int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$

$\square$

3 Triple Integrals

Suppose that a scalar field $f(x,y,z)$ is defined at all points (x_i, y_i, z_i) within a region V of three-dimensional space. Divide V into N subregions such that the ith subregion has volume $\delta V_{ijk} = \delta x_i \delta y_j \delta z_k$ and contains a point (x_i, y_j, z_k).

Let $P = \{\delta V_{ijk} = \delta x_i \delta y_j \delta z_k : 1 \le i \le N_1, 1 \le j \le N_2, 1 \le k \le N_3\}$,

$|P| = \max\limits_{\substack{1\le i\le N_1, \\ 1\le j\le N_2, \\ 1\le k\le N_3}} \{\delta V_{ijk} = \delta x_i \delta y_j \delta z_k : 1 \le i \le N_1, 1 \le j \le N_2, 1 \le k \le N_3\}$.

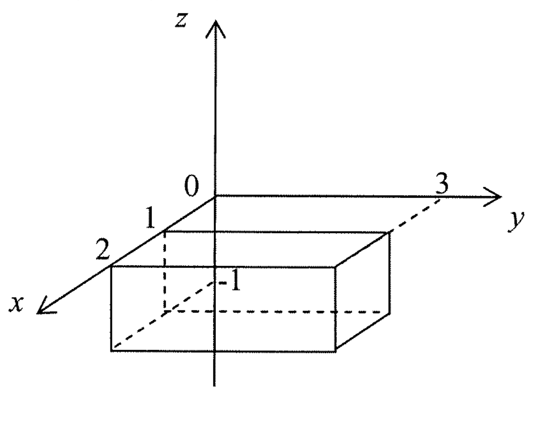
We define the triple integral of $f(x,y,z)$ over the region V to be

$$\iiint_V f(x,y,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \lim_{\text{all } \delta x_i, \delta y_j, \delta z_k \to 0} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{k=1}^{N_3} f(x_i, y_j, z_k) \delta x_i \delta y_j \delta z_k .$$

As expected, we can evaluate triple integrals by iterated single integration.

<u>Example 13</u> (Rectangular block)

The density of a rectangular blocks V bounded by the planes $x=1,\ x=2,\ y=0,\ y=3,\ z=-1, z=0$ is given by the scalar function $\rho(x,y,z) = x(y+1) - z$. Find the mass of the block.



Solution:

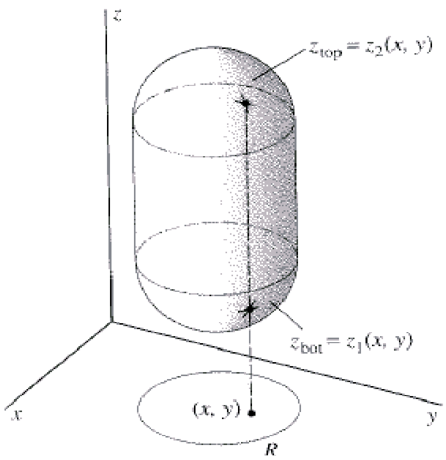
$$\text{Mass} = \iiint_V \left[x(y+1)-z\right]dxdydz = \int_1^2\left[\int_0^3\left[\int_{-1}^0\left[x(y+1)-z\right]dz\right]dy\right]dx = \int_1^2\left[\int_0^3\left[x(y+1)z-\frac{z^2}{2}\right]\Bigg|_{-1}^{0}dy\right]$$

$$= \int_1^2\left[\int_0^3\left[x(y+1)+\frac{1}{2}\right]dy\right]dx = \int_1^2\left(\frac{xy^2}{2}+xy+\frac{1}{2}y\right)\Bigg|_0^3 dx = \int_1^2\left(\frac{9x}{2}+3x+\frac{3}{2}\right)dx = \left(\frac{15}{4}x^2+\frac{3}{2}x\right)\Bigg|_1^2 = \frac{51}{4}$$

Non-rectangular block:

Suppose that the solid T with piecewise smooth boundary is z-simple: Each line parallel to the z-axis intersects T (if at all) in a single line segment. In effect, this means that T can be described by the inequalities $z_1(x,y)\le z\le z_2(x,y)$ with $(x,y)\in R$, where R is the vertical projection of T onto the xy-plane (Figure B). Then $\iiint_T f(x,y,z)\,dxdydz = \iint_R\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right]dxdy$.

Figure B



Similarly, we may integrate first with respect to either x or y if the space region T is either x-simple or y-simple. Such situations appear in Figure C and Figure D, respectively.

Figure C

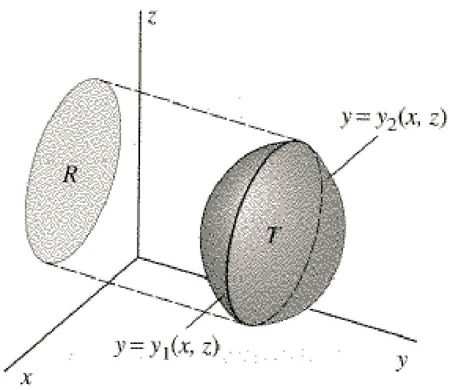
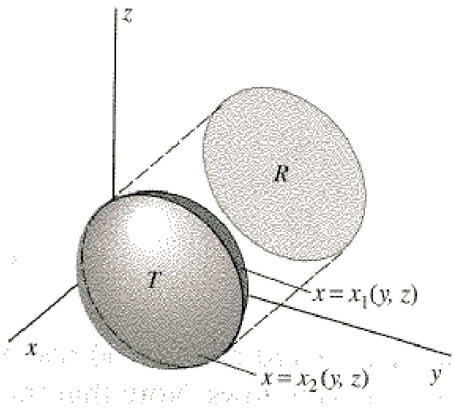


Figure D



Examples 14

(a) If $f(x,y,z) \equiv 1$, then $\iiint\limits_V 1\,dxdydz$ = **volume** of the region *V*.

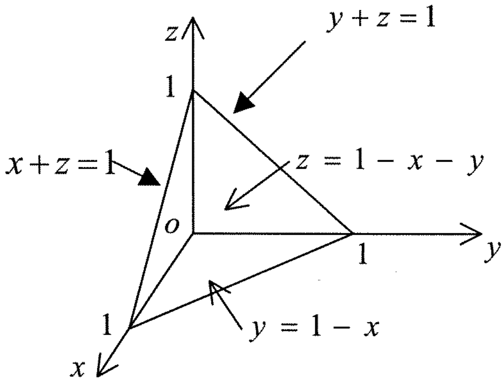
(b) If the scalar function $\rho(x, y, z)$ gives the density at a point (x, y, z) of the region *V*, then
$\iiint\limits_V \rho(x, y, z)\,dxdydz$ = **mass** of the region *V*.

(c) If the scalar function $\rho(x, y, z)$ gives the charge density at a point (x, y, z) of the region *V*, then
$\iiint\limits_V \rho(x, y, z)\,dxdydz$ = **total charge** within the region *V*.

□

Example 15

Evaluate $\iiint\limits_V \dfrac{1}{(x+y+2z+1)^3}\,dxdydz$ where *V* is the region enclosed by the planes
$x=0,\ y=0,\ z=0,\ x+y+z=1$.



Solution:

The projection of *V* on *xy*-plane is σ_{xy} which is bounded by $x=0, y=0, y=1-x$.

$$\iint\limits_{\sigma_{xy}}\left[\int_0^{1-x-y}\frac{1}{(x+y+2z+1)^3}\,dz\right]dxdy=\int_0^1\left[\int_0^{1-x}\left[\int_0^{1-x-y}\frac{1}{(x+y+2z+1)^3}\,dz\right]dy\right]dx$$

$$=\int_0^1\left(\int_0^{1-x}\left[\frac{-1}{4(x+y+2z+1)^2}\right]\Bigg|_0^{1-x-y}dy\right)dx=\int_0^1\left(\int_0^{1-x}\left[\frac{1}{4(x+y+1)^2}-\frac{1}{4(3-x-y)^2}\right]dy\right)dx$$

$$=\int_0^1\left[\frac{-1}{4(x+y+1)}-\frac{1}{4(3-x-y)}\right]\Bigg|_0^{1-x}dx=\int_0^1\left[\frac{-1}{8}-\frac{1}{8}+\frac{1}{4(x+1)}+\frac{1}{4(3-x)}\right]dx$$

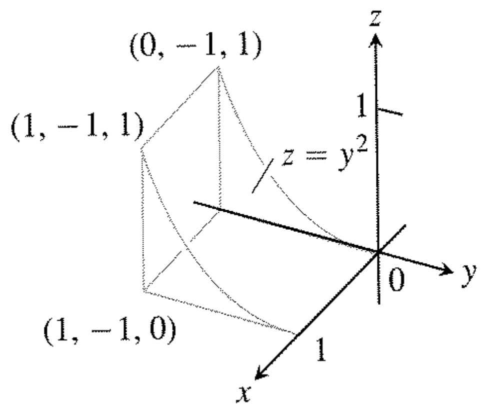
$$=\left[-\frac{x}{4}+\frac{\log(x+1)}{4}-\frac{\log(3-x)}{4}\right]\Bigg|_0^1=-\frac{1}{4}+\frac{\log 2}{4}-\frac{\log 2}{4}-\left(-\frac{\log 3}{4}\right)=\frac{\log 3}{4}-\frac{1}{4}$$

□

Example 16

An iterated integral like $\int_0^1\left[\int_{-1}^0\left(\int_0^{y^2} f(x,y,z)\,dz\right)dy\right]dx$ is called an iterated integral with order $dzdydx$.

With the aid of the following figure, change the order of the iterated integral $\int_0^1\left[\int_{-1}^0\left(\int_0^{y^2} f(x,y,z)\,dz\right)dy\right]dx$ to an equivalent iterated integral with order $dydzdx$.



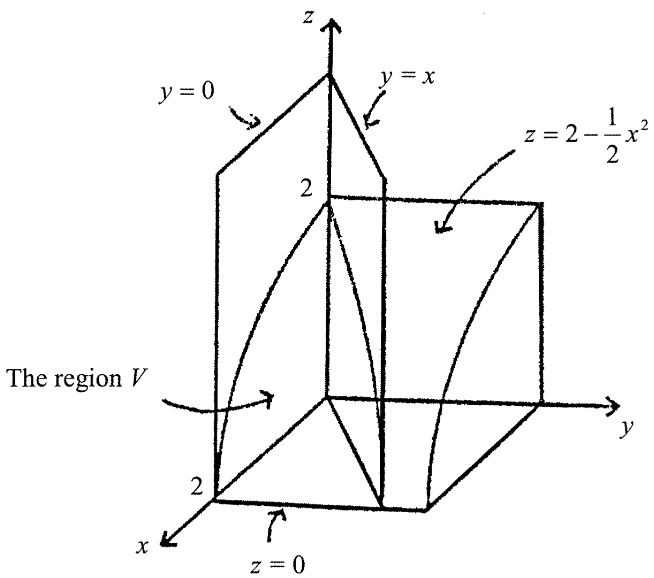
Solution:

$$\int_0^1\left[\int_{-1}^0\left(\int_0^{y^2} f(x,y,z)dz\right)dy\right]dx=\int_0^1\left(\iint_{S_{yz}} f(x,y,z)dydz\right)dx=\int_0^1\left[\int_0^1\left(\int_{-1}^{-\sqrt{z}} f(x,y,z)dy\right)dz\right]dx$$

□

Example 17

Evaluate $\iiint_V 2xyz\,dxdydz$ where V is the region bounded by the parabolic cylinder $z=2-\frac{1}{2}x^2$ and the planes $x=0$, $y=x$ and $y=0$, $z=0$.



Solution:

Projecting V onto the xz-plane gives σ_{xz} which is bounded by $z = 2 - \frac{1}{2}x^2$, $x = 0,\ z = 0$. Thus,

$$\iiint_V 2xyzdydxdz = \iint_{\sigma_{xz}} \left[\int_0^x 2xyzdy \right] dxdz = \iint_{\sigma_{xz}} \left(xy^2 z \right) \Big|_0^x dxdz = \iint_{\sigma_{xz}} x^3 zdxdz$$

$$= \int_0^2 \left(\int_0^{2-\frac{1}{2}x^2} x^3 zdz \right) dx = \int_0^2 \left(\frac{x^3 z^2}{2} \right) \Bigg|_0^{2-\frac{1}{2}x^2} dx = \int_0^2 \frac{x^3 \left(2 - \frac{1}{2}x^2 \right)^2}{2} dx = \frac{4}{3}$$

Alternatively, projecting V onto the xy-plane gives σ_{xy} which is bounded, $y = 0$, $y = x$, $x = 2$. Thus,

$$\iiint_V 2xyzdydxdydz = \iint_{\sigma_{xy}} \left(\int_0^{2-\frac{1}{2}x^2} 2xyzdz \right) dxdy = \int_0^2 \left[\int_0^x \left(\int_0^{2-\frac{1}{2}x^2} 2xyzdz \right) dy \right] dx = \int_0^2 \left(\int_0^x xyz^2 \Big|_{z=0}^{z=2-\frac{1}{2}x^2} dy \right) dx =$$

$$\int_0^2 \int_0^x xy \left(2 - \frac{1}{2}x^2 \right)^2 dydx = \int_0^2 \frac{xy^2 \left(2 - \frac{1}{2}x^2 \right)^2}{2} \Bigg|_{y=0}^{y=x} dx = \frac{1}{2} \int_0^2 x^3 \left(2 - \frac{1}{2}x^2 \right)^2 dx = \frac{4}{3}.$$

.

<u>Example 18</u>

Evaluate $\iiint_V xyzdxdydz$, where V is the region enclosed by $x^2 + y^2 + z^2 = 1$ and $x \ge 0,\ z \ge 0,\ y \ge 0$ and $x = 0,\ y = 0,\ z = 0$.

Solution:

The projection of V onto xy-plane is σ_{xy} which is bounded by $x^2 + y^2 = 1$ where $x \ge 0,\ y \ge 0$ and $x = 0,\ y = 0$. So $\iiint_V xyzdxdydz = \iint_{\sigma_{xy}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyzdz \right) dxdy = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyzdz \right) dy \right] dx$

Then

$$\iiint_V xyz\,dxdydz = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz \right) dy \right] dx = \int_0^1 \left(\int_0^{\sqrt{1-x^2}} xy \frac{z^2}{2} \Big|_0^{\sqrt{1-x^2-y^2}} dy \right) dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(xy \frac{1-x^2-y^2}{2} \right) dy \right] dx = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\frac{xy - x^3y - xy^3}{2} \right) dy \right] dx$$

$$= \int_0^1 \left[\frac{\left(x-x^3\right)y^2}{4} - \frac{xy^4}{8} \right] \Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \frac{\left(x-x^3\right)\left(1-x^2\right)}{4} - \frac{x\left(1-x^2\right)^2}{8} dx$$

$$= \int_0^1 \frac{\left(1-x^2\right)}{4} \left[x - x^3 - \frac{x\left(1-x^2\right)}{2} \right] dx = \int_0^1 \frac{\left(1-x^2\right)}{4} \left(\frac{x-x^3}{2} \right) dx = \int_0^1 \frac{x - 2x^3 + x^5}{8} dx$$

$$= \left(\frac{x^2}{16} - \frac{x^4}{16} + \frac{x^6}{48} \right) \Big|_0^1 = \frac{1}{48}$$

$\square$

<u>Illustration</u>

Find the volume V of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$.

Solution:

$$V = \iiint_R 1\,dxdydz = \int_0^2 \left[\int_0^6 \left(\int_0^{4-x^2} 1\,dz \right) dy \right] dx = \int_0^2 \left[\int_0^6 \left(4 - x^2\right) dy \right] dx = \int_0^2 \left(4 - x^2\right) y \Big|_0^6 dx = \int_0^2 \left(24 - 6x^2\right) dx = 32.$$

$\square$

4 Change of Variable in Triple Integrals

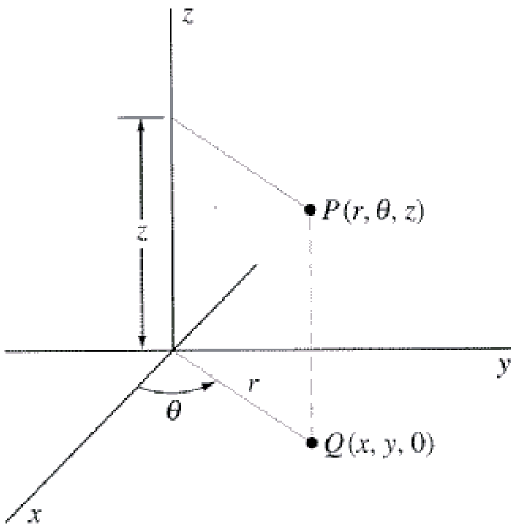
If $I=\iiint\limits_V f\left(x,y,z\right)dxdydz$, the change of variable $x=x\left(u,v,w\right),\ y=\left(u,v,w\right),\ z=z\left(u,v,w\right)$, gives

$$I=\iiint\limits_{V^*} f\left(x(u,v,w),y(u,v,w),z(u,v,w)\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw ,$$

where $J=\dfrac{\partial\left(x,y,z\right)}{\partial\left(u,v,w\right)}=\det\begin{pmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w}\\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w}\end{pmatrix}$ is the Jacobian of the transformation. V^* is the region in *uvw*-space corresponding to the region *V* in *xyz*-space injectively (one to one) and *J* must be of one sign in V^*.

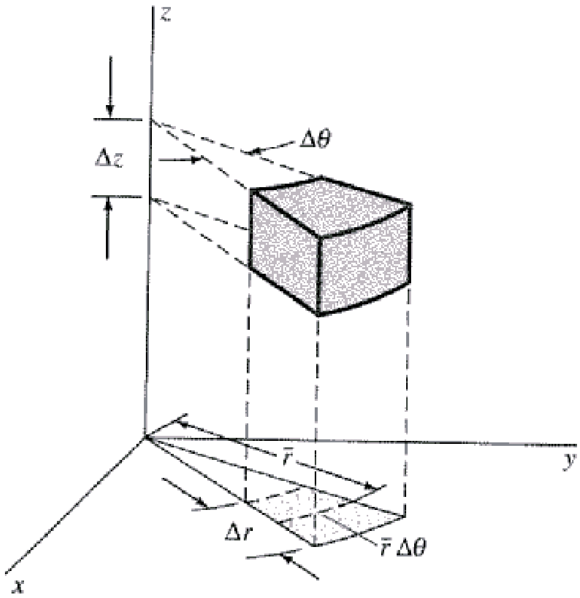
The most popular alternative coordinate systems to Cartesian coordinates are cylindrical polar coordinates and spherical polar coordinates.

Cylindrical Polar Coordinates:



$$x=r\cos\theta,\quad y=r\sin\theta,\quad z=z,\quad r\ge 0,\ \ 0\le\theta<2\pi,\ \ -\infty<z<\infty$$

$$r=\text{constant}-\text{cylinder},\quad \theta=\text{constant}-\text{plane},\quad z=\text{constant}-\text{plane}$$



The volume of the cylindrical block is approximately $\bar{r}\Delta z\Delta r\Delta\theta$. (Note $\delta \equiv \Delta$)

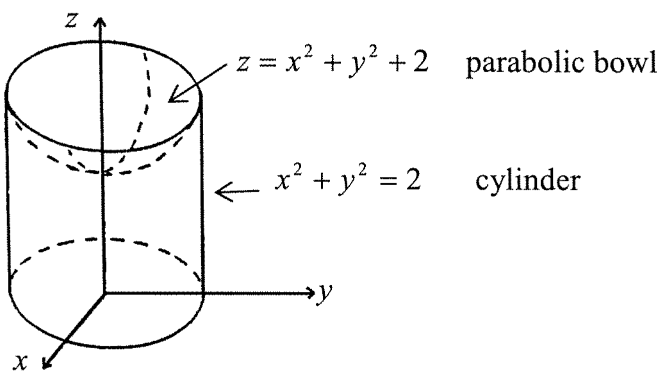
$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$dxdydz \to rdrd\theta dz\,,$$

where $rdrd\theta dz$ corresponds to an element of volume in cylindrical polar coordinates.

Example 19

Find the volume V between the surfaces $x^2+y^2=2,\ z=x^2+y^2+2$ and the plane $z=0$.



Solution:

Cartesian coordinates:

We observe that the projection of *V* onto *xy*-plane is $x^2 + y^2 \le 2$. So

$$\iiint\limits_V 1dxdydz = \iint\limits_{x^2+y^2\le 2}\left(\int_0^{x^2+y^2+2} 1dz\right)dxdy = \int_{-\sqrt{2}}^{\sqrt{2}}\left[\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\left(\int_0^{x^2+y^2+2} 1dz\right)dy\right]dx$$

Cylindrical polar coordinates:

Under the cylindrical transformations $x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$, the volume *V* is transformed into the solid V^*: $0 \le \theta \le 2\pi,\ 0 \le r \le \sqrt{2},\ 0 \le z \le x^2 + y^2 + 2$ in $r\theta z$ - space. The projection of V^* onto $r\theta$ - plane is $0 \le \theta \le 2\pi,\ 0 \le r \le \sqrt{2}$. Thus

$$\iiint\limits_V 1dxdydz = \iiint\limits_{V'} rdrd\theta dz = \iint\limits_{0\le\theta\le 2\pi,\ 0\le r\le\sqrt{2}}\left(\int_0^{r^2+2} rdz\right)drd\theta = \int_0^{\sqrt{2}}\left[\int_0^{2\pi}\left(\int_0^{r^2+2} rdz\right)d\theta\right]dr = \int_0^{\sqrt{2}}\left[\int_0^{2\pi}\left(r^3+2r\right)d\theta\right]dr$$

$$= \int_0^{\sqrt{2}} 2\pi\left(r^3+2r\right)dr = 6\pi$$

□

<u>Example 20</u>

Using cylindrical polar coordinates, evaluate $\iiint\limits_V \left(x^2+y^2\right)dxdydz$, where *V* is the solid enclosed by $x^2 + y^2 = 2z,\ z = 2$

Solution:

Under the cylindrical transformation $x = r\cos\theta,\ y = r\sin\theta,\ z = z$, the solid *V* is transformed into V^*: $0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ \dfrac{x^2+y^2}{2} = \dfrac{r^2}{2} \le z \le 2$. The projection of V^* onto $r\theta$ - plane is $0 \le \theta \le 2\pi,\ 0 \le r \le 2$.

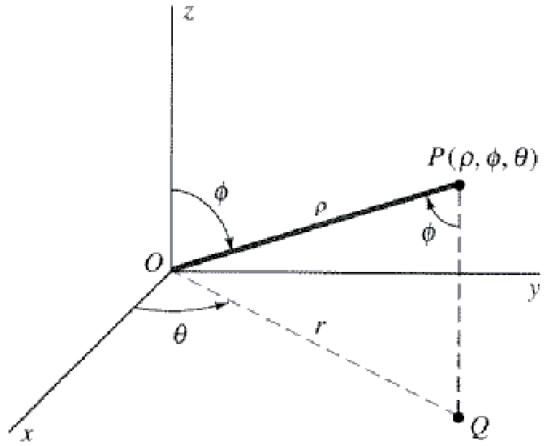
So

$$\iiint\limits_V \left(x^2+y^2\right)dxdydz = \iiint\limits_{V'} r^2 rdrd\theta dz = \iint\limits_{0\le\theta\le 2\pi,\ 0\le r\le 2}\left(\int_{\frac{r^2}{2}}^{2} r^3dz\right)drd\theta = \int_0^2\left[\int_0^{2\pi}\left(\int_{\frac{r^2}{2}}^{2} r^2rdz\right)d\theta\right]dr$$

$$= \int_0^2\left[\int_0^{2\pi} r^3\left(2-\frac{r^2}{2}\right)d\theta\right]dr = \int_0^2 2\pi r^3\left(2-\frac{r^2}{2}\right)dr = \left(\pi r^4 - \frac{\pi r^6}{6}\right)\Big|_0^2 = 16\pi - \frac{64\pi}{6} = \frac{32\pi}{6} = \frac{16\pi}{3}$$

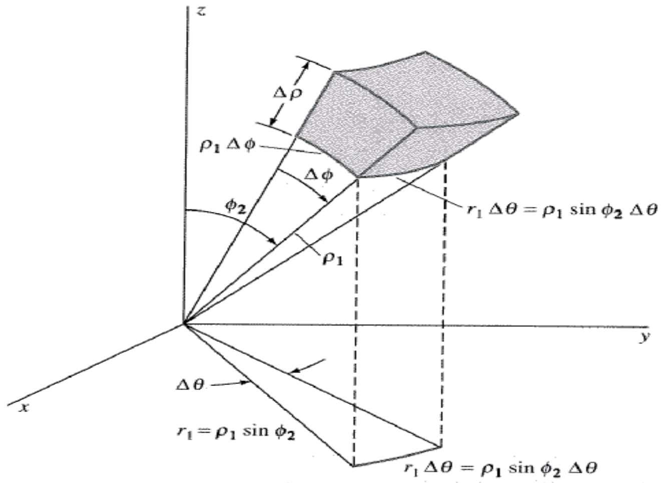
□

Spherical Polar Coordinates:



$$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi,\ \ \rho\ge 0,\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi$$

In particular: $\rho=$ constant $-$ sphere, $\phi=$ constant $-$ cone, $\theta=$ constant $-$ plane



The volume of the spherical block is approximately $\rho^2\sin\phi\Delta\rho\Delta\phi\Delta\theta$.

$$J=\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}=\det\begin{pmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta}\end{pmatrix}=\det\begin{pmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{pmatrix}=\rho^2\sin\phi$$

$$dxdydz\to\rho^2\sin\phi d\rho d\phi d\theta$$

$\rho^2\sin\phi drd\phi d\theta$ corresponds to an element of volume in spherical polar coordinates.

Example 21

Find an equation for the sphere $x^2+y^2+(z-1)^2=1$ in spherical coordinate.

Solution:

$$x^2+y^2+(z-1)^2=1 \underset{\substack{x=\rho\sin\phi\cos\theta \\ y=\rho\sin\phi\sin\theta \\ z=\rho\cos\phi}}{\Rightarrow} \rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+(\rho\cos\phi-1)^2=1$$

$$\Rightarrow \rho^2\sin^2\phi+\rho^2\cos^2\phi-2\rho\cos\phi=0 \Rightarrow \rho^2=2\rho\cos\phi \Rightarrow \rho=2\cos\phi$$

Example 22

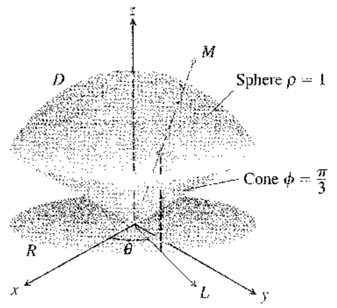
Find the volume of the "ice cream cone" D cut from the solid sphere $\rho\le 1$ by the cone $\phi=\frac{\pi}{3}$.

Solution:

The volume is $V=\iiint\limits_D \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$.

To find the limits of integration for evaluating the integral, we take the following steps.

Step 1**:** *A sketch.*
We sketch D and its projection R on the xy-plane (Figure A).



Step 2: *The $\rho-$limits of integration.*
We draw a ray M from the origin through D making an angle ϕ with the positive z-axis. We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho=0$ and leaves at $\rho=1$.

Step 3: *The ϕ-limits of integration.*

The cone $\phi=\frac{\pi}{3}$ makes an angle of $\frac{\pi}{3}$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi=0$ to $\phi=\frac{\pi}{3}$.

Step 4: *The* θ*-limits of integration.*

The ray *L* sweeps over *R* as θ runs from 0 to 2π.

The volume is

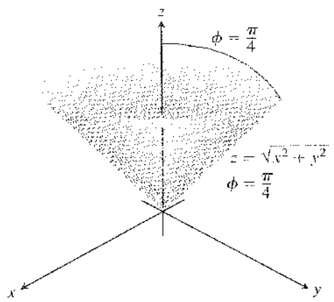
$$V=\iiint_D \rho^2\sin\phi\, d\rho\, d\phi\, d\theta=\int_0^{2\pi}\left[\int_0^{\frac{\pi}{3}}\left(\int_0^{1}\rho^2\sin\phi\, d\rho\right)d\phi\right]d\theta=\int_0^{2\pi}\left[\int_0^{\frac{\pi}{3}}\left(\frac{\rho^3\sin\phi}{3}\right)\Bigg|_0^1 d\phi\right]d\theta$$

$$=\int_0^{2\pi}\left(\int_0^{\frac{\pi}{3}}\frac{\sin\phi}{3}d\phi\right)d\theta=\int_0^{2\pi}\left(-\frac{\cos\phi}{3}\right)\Bigg|_0^{\frac{\pi}{3}}d\theta=\int_0^{2\pi}\left(-\frac{1}{6}+\frac{1}{3}\right)d\theta=\frac{2\pi}{6}=\frac{\pi}{3}$$

□

Example 23

With the aid of the following, convert $\int_{-1}^{1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(\int_{\sqrt{x^2+y^2}}^{1}1dz\right)dy\right]dx$ to an integral using spherical coordinates with order $d\theta d\rho d\phi$ and then evaluate the obtained integral.



Solution:

For $\int_{-1}^{1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(\int_{\sqrt{x^2+y^2}}^{1}1dz\right)dy\right]dx$, the integration is done in the solid *V* which is bound above by $z=1$ and bound below by $z=\sqrt{x^2+y^2}$.

Under the spherical transformations $\begin{cases}x=\rho\sin\phi\cos\theta\\ y=\rho\sin\phi\sin\theta\\ z=\rho\cos\phi\end{cases}$, V is transformed into $V^*\begin{cases}0\le\theta\le 2\pi\\ 0\le\phi\le\dfrac{\pi}{4}\\ 0\le\rho\le\sec\phi\end{cases}$.

$$\int_{-1}^{1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(\int_{\sqrt{x^2+y^2}}^{1} 1dz\right)dy\right]dx=\int_{0}^{\frac{\pi}{4}}\left[\int_{0}^{\sec\phi}\left(\int_{0}^{2\pi}\left|\rho^2\sin\phi\right|d\theta\right)d\rho\right]d\phi=\int_{0}^{\frac{\pi}{4}}\left[\int_{0}^{\sec\phi}\left(\int_{0}^{2\pi}\rho^2\sin\phi d\theta\right)d\rho\right]d\phi$$

$$=\int_{0}^{\frac{\pi}{4}}\left(\int_{0}^{\sec\phi}2\pi\rho^2\sin\phi d\rho\right)d\phi=\int_{0}^{\frac{\pi}{4}}\left(\frac{2\pi\rho^3\sin\phi}{3}\right)\Bigg|_{0}^{\sec\phi}d\phi=\int_{0}^{\frac{\pi}{4}}\frac{2\pi\sec^3\phi\sin\phi}{3}d\phi=\frac{2\pi}{3}\int_{0}^{\frac{\pi}{4}}\frac{\sin\phi}{\cos^3\phi}d\phi$$

$$=-\frac{2\pi}{3}\int_{0}^{\frac{\pi}{4}}\cos^{-3}\phi d\left(\cos\phi\right)=-\frac{2\pi\cos^{-2}\phi}{-6}\Bigg|_{0}^{\frac{\pi}{4}}=\frac{\pi}{3\cos^2\phi}\Bigg|_{0}^{\frac{\pi}{4}}=\frac{\pi}{3}\left(\frac{1}{\cos^2\frac{\pi}{4}}-\frac{1}{\cos^2 0}\right)=\frac{\pi}{3}\left(\frac{1}{\frac{1}{2}}-\frac{1}{1}\right)=\frac{\pi}{3}$$

□

<u>Example 24</u>

In a sample model of the charge distribution around the positively charged (Q) nucleus of the hydrogen atom the charge density in the electron cloud at a distance ρ from the atom is $f\left(\rho\right)=\frac{-Q}{\pi\ a^3}e^{-\frac{2\rho}{a}}$ where a is the Bohr radius. Determine the total charge in the electron cloud.

Solution:

Under the spherical transformation $x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi$, R^3 is transformed into $V^*:\quad 0\le\theta\le 2\pi,\ 0\le\phi\le\pi,\ 0\le\rho<\infty$. The total charge q is given by

$$q=\iiint_{R^3}\left(\frac{-Q}{\pi\ a^3}e^{-\frac{2\sqrt{x^2+y^2+z^2}}{a}}\right)dxdydz=\iiint_{V'}\frac{-Q}{\pi\ a^3}e^{-\frac{2\rho}{a}}\rho^2\sin\phi d\theta d\phi d\rho$$

$$=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}\left(\int_{0}^{\infty}\frac{-Q}{\pi\ a^3}e^{-\frac{2\rho}{a}}\rho^2\sin\phi d\rho\right)d\theta d\phi=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}\left[\int_{0}^{\infty}\frac{Q}{2\pi\ a^2}\sin\phi\rho^2 d\left(e^{-\frac{2\rho}{a}}\right)\right]d\theta d\phi$$

$$=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}\frac{Q}{2\pi\ a^2}\sin\phi\left[\rho^2 e^{-\frac{2\rho}{a}}\Bigg|_{0}^{\infty}-\int_{0}^{\infty}e^{-\frac{2\rho}{a}}d\left(\rho^2\right)\right]d\theta d\phi$$

$$=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}\frac{Q}{2\pi\ a^2}\sin\phi\left(-\int_{0}^{\infty}2e^{-\frac{2\rho}{a}}\rho d\rho\right)d\theta d\phi=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}\frac{Q}{2\pi\ a}\sin\phi\left[\int_{0}^{\infty}\rho d\left(e^{-\frac{2\rho}{a}}\right)\right]d\theta d\phi$$

$$=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}\frac{Q}{2\pi\ a}\sin\phi\left(\rho e^{-\frac{2\rho}{a}}\Bigg|_{0}^{\infty}-\int_{0}^{\infty}e^{-\frac{2\rho}{a}}d\rho\right)d\theta d\phi=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}\frac{Q}{2\pi\ a}\sin\phi\left(-\int_{0}^{\infty}e^{-\frac{2\rho}{a}}d\rho\right)d\theta d\phi$$

$$=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}\frac{Q}{4\pi}\sin\phi\left(e^{-\frac{2\rho}{a}}\Bigg|_{0}^{\infty}\right)d\theta d\phi=\iint_{0\le\theta\le 2\pi,\ 0\le\phi\le\pi}-\frac{Q}{4\pi}\sin\phi d\theta d\phi=\int_{0}^{\pi}\left(\int_{0}^{2\pi}\frac{-Q}{4\pi}\sin\phi d\theta\right)d\phi$$

$$=\int_{0}^{\pi}\frac{-Q}{2}\sin\phi d\phi=\frac{-Q}{2}\cos\phi\Bigg|_{0}^{\pi}=-Q$$

Note: To evaluate $\rho^2 e^{\ a}\Big|_{\rho=0}^{\rho=\infty}$, we apply l'Hôpital's rule twice, i.e.,

$$\rho^2 e^{-\frac{2r}{a}}\Big|_{\rho=0}^{\rho=\infty}=\lim_{\rho\to\infty}\frac{\rho^2}{e^{\frac{2\rho}{a}}}=\lim_{\rho\to\infty}\frac{2\rho}{\frac{2}{a}e^{\frac{2\rho}{a}}}=\lim_{\rho\to\infty}\frac{2}{\frac{4}{a^2}e^{\frac{2\rho}{a}}}=0.$$

□