Take Home Assignment MA2001 #1

For each of the following questions, write down your solution with details of steps. Marks will not given if only final answers are provided.

1. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$. (Hint: $3, 6$ are eigenvalues of A).

   Solution.

   (a) Eigenvalues: $|A - \lambda I| = -(\lambda - 3)(\lambda - 6)(\lambda - 9) = 0$. Hence eigenvalues are $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 9$.

   (b) Eigenvectors: (i) For $\lambda_1 = 3$, we have

$$\left(\begin{array}{ccc|c} 3 & 2 & -2 & 0 \\ 2 & 2 & 0 & 0 \\ -2 & 0 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 2 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

   By solving

$$3x_1 + 2x_2 - 2x_3 = 0, \quad x_2 + 2x_3 = 0, \quad x_3 = t,$$

   we have eigenvector $t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $t \neq 0$.

   (ii) For $\lambda = 6$, we have we have

$$\left(\begin{array}{ccc|c} 0 & 2 & -2 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

   By solving

$$2x_1 - x_2 = 0, \quad x_2 - x_3 = 0, \quad x_3 = t,$$

   we have eigenvector $t \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$, $t \neq 0$.

   (iii) For $\lambda = 9$, we have

$$\left(\begin{array}{ccc|c} -3 & 2 & -2 & 0 \\ 2 & -4 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 2 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$-3x_1 + 2x_2 - 2x_3 = 0, \quad 2x_2 + x_3 = 0, \quad x_3 = t,$$

we have eigenvector $t\begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$, $t \neq 0$.

2. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$.

Solution.

(a) Eigenvalues: $|A - \lambda I| = -(\lambda - 9)^3$. Hence eigenvalues are $\lambda = 9$.

(b) Eigenvectors: For $\lambda = 9$, we have

$$\left(\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 2 & -2 & -8 & 0 \\ 5 & 4 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$4x_1 + 5x_2 + 2x_3 = 0, \quad x_2 + 2x_3 = 0, \quad x_3 = t,$$

we have eigenvector $t\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $t \neq 0$.

3. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & -1 & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$, whose characteristic polynomial is $(\lambda + 1)^2(\lambda + 5)(\lambda - 3)$.

Solution.

(a) Eigenvalues: $|A - \lambda I| = (\lambda + 1)^2(\lambda + 5)(\lambda - 3) = 0$. Hence eigenvalues are $\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = -5$.

(b) Eigenvectors: (i) For $\lambda_1 = -1$, we have

$$\left(\begin{array}{cccc|c} 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -4 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

By solving

$$x_1 = t, \quad x_2 = s, \quad x_3 = 0, \quad x_4 = 0,$$

we have eigenvector $t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $t^2 + s^2 \neq 0$.

(ii) For $\lambda = 3$, we have

$$\left(\begin{array}{cccc|c} -4 & 0 & 12 & 0 & 0 \\ 0 & -4 & 0 & 12 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & 0 & -4 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$x_1 - 3x_3 = 0, \quad x_2 - 3x_4 = 0, \quad x_3 + x_4 = 0, \quad x_4 = t,$$

we have eigenvector $t \begin{pmatrix} -3 \\ 3 \\ -1 \\ 1 \end{pmatrix}$, $t \neq 0$.

(iii) For $\lambda = -5$, we have

$$\left(\begin{array}{cccc|c} 4 & 0 & 12 & 0 & 0 \\ 0 & 4 & 0 & 12 & 0 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$x_1 + 3x_3 = 0, \quad x_2 + 3x_4 = 0, \quad x_3 - x_4 = 0, \quad x_4 = t,$$

we have eigenvector $t \begin{pmatrix} -3 \\ -3 \\ 1 \\ 1 \end{pmatrix}$, $t \neq 0$.

4. It is given the symmetric matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

   (a) find the eigenvalues of A;

(b) find the eigenvectors corresponding to each of these eigenvalues;

(c) find an orthogonal matrix P such that $P^{\top}AP$ gives a diagonal matrix D and calculates P^{-1};

(d) Determine the eigenvalues of the matrix $B = A^5 + (A^2)^{\top}$.

Solution. (a) It is easy to show that $|A - \lambda I| = -(\lambda - 1)(\lambda - 2)(\lambda - 3)$. Hence the eigenvalues of A is $1, 2, 3$.

(b) (i) For $\lambda_1 = 1$, we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$x_1 + x_3 = 0, \quad x_2 = 0, \quad x_3 = t,$$

we have eigenvector $t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $t \neq 0$. Let $\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(ii) For $\lambda = 2$, we have we have

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

By solving

$$x_1 = 0, \quad x_2 = t, \quad x_3 = 0,$$

we have eigenvector $t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $t \neq 0$. Let $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(iii) For $\lambda = 3$, we have

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$x_1 - x_3 = 0, \quad x_2 = 0, \quad x_3 = t,$$

we have eigenvector $t\begin{pmatrix}1\\0\\1\end{pmatrix}$, $t \neq 0$. Let $\vec{v}_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$.

Finally, by

$$\vec{v}_1 \cdot \vec{v}_2 \times \vec{v}_3 = \begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -2 \neq 0,$$

we conclude that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent eigenvectors of A.

(c) Let $\vec{u}_1 = \frac{\vec{v}_1}{|\vec{v}_1|}$, $\vec{u}_2 = \frac{\vec{v}_2}{|\vec{v}_2|}$, $\vec{u}_3 = \frac{\vec{v}_3}{|\vec{v}_3|}$. We obtain an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. Define

$$P = [\vec{u}_1, \vec{u}_2, \vec{u}_3] = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Note that $PP^\top = P^\top P = I$. Hence P is an orthogonal matrix, i.e., $P^{-1} = P^\top$. Moreover, since $AP = PD$ with $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, we have $P^\top AP = D$ is a diagonal matrix.

(d) $B = A^5 + (A^2)^\top = A^5 + A^2 = P(D^5 + D^2)P^\top$. The eigenvalues of A are 1, 2, 3. Hence the eigenvalues of B are given by $1^5 + 1^2 = 2$, $2^5 + 2^2 = 32 + 4 = 36$, and $3^5 + 3^2 = 243 + 9 = 252$.

5. If A is a $n \times n$ matrix and $\{\lambda_1, \ldots, \lambda_k\}$ are its eigenvalues, show that the eigenvalues of $\alpha I + A$, where I is the identity matrix and α is a scalar, are $\{\lambda_1 + \alpha, \ldots, \lambda_k + \alpha\}$.

   Solution. Generally, if λ is an eigenvalue of A, there is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. Then

$$(\alpha I + A)\vec{v} = \alpha\vec{v} + A\vec{v} = \alpha\vec{v} + \lambda\vec{v} = (\alpha + \lambda)\vec{v},$$

   which implies that $\alpha + \lambda$ is an eigenvector of $\alpha I + A$. The conversation holds as well.

6. A quadratic form Q in the components $x_1, \ldots, x_n$ of a vector $\vec{x} = [x_1, \ldots, x_n]^\top$ with symmetric coefficient matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is defined to be

$$Q(\vec{x}) := \vec{x}^\top A\vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

Determine whether each of the following quadratic forms in two variables is positive or negative definite or semidefinite, or indefinite.

(a) $3x_1^2 + 8x_1x_2 - 3x_2^2$.

(b) $9x_1^2 + 6x_1x_2 + x_2^2$.

Solution.

(a) The coefficient matrix is given by

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

Diagonalize A we obtain

$$A = PDP^\top = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

Let

$$\vec{y} = P^\top \vec{x}.$$

Then

$$Q(x_1, x_2) = 3x_1^2 + 8x_1x_2 - 3x_2^2 = \vec{x}^\top A\vec{x} = \vec{x}^\top PDP^\top \vec{x} = \vec{y}^\top D\vec{y} = 10.$$

Hence its canonical form is given by

$$5y_1^2 - 5y_2^2 = 10 \quad \text{or} \quad y_1^2 - y_2^2 = 2,$$

which is a hyperbola. See below

(b) The coefficient matrix is given by

$$A = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$$

Diagonalize A we obtain

$$A = PDP^\top = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$$

Let

$$\vec{y} = P^\top \vec{x}.$$

Then

$$Q(x_1, x_2) = 9x_1^2 + 6x_1x_2 + x_2^2 = \vec{x}^\top A\vec{x} = \vec{x}^\top PDP^\top \vec{x} = \vec{y}^\top D\vec{y} = 10.$$

Hence its canonical form is given by

$$10y_2^2 = 10 \quad \text{or} \quad y_2 = \pm 1,$$

which is a pair of straight lines.

7. Determine the values of a for which the quadratic form $2x^2+2axy+2xz+y^2+z^2$ is positive definite.

Solution. Its matrix is

$$A = \begin{pmatrix} 2 & a & 1 \\ a & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The 1st, 2nd and 3rd order leading principal minors are 2, $\left|\begin{pmatrix} 2 & a \\ a & 1 \end{pmatrix}\right| = 2 - a^2$, and $|A| = 1 - a^2$, respectively. Thus the matrix is positive definite when all of them are positive and thus $-1 < a < 1$.

8. **Discovery Question**. Read "https://en.wikipedia.org/wiki/Gram-Schmidt process" to use the Gram-Schmidt process to find an orthogonal basis spanning the same space of $\mathbb{R}^n$ as the given of vectors:

   (a) $< 1, 4, 0 >, < 2, -5, 0 >$ in $\mathbb{R}^3$.

   (b) $< 0, 2, 1, -1 >, < 0, -1, 1, 6 >, < 0, 2, 2, 3 >$ in $\mathbb{R}^4$.

Solution.

(a) Let $\vec{x}_1 =< 1, 4, 0 >$, $\vec{x}_2 =< 2, -5, 0 >$. Then by Gram-Schmidt orthogonalization process, we have

$$\vec{v}_1 = \vec{x}_1 =< 1, 4, 0 > .$$

and

$$
\begin{aligned}
\vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= < 2, -5, 0 > - \frac{< 2, -5, 0 > \cdot < 1, 4, 0 >}{\| < 1, 4, 0 > \|^2} < 1, 4, 0 > \\
&= < 2, -5, 0 > + \frac{18}{17} < 1, 4, 0 > \\
&= \frac{13}{17} < 4, -1, 0 > .
\end{aligned}
$$

To form an orthogonal basis, one needs to find another orthogonal vector, eg: in this case, we choose $\vec{v}_3 =< 0, 0, 1 >$ which is orthogonal to $\vec{v}_1$ and $\vec{v}_2$.

(b) Let $\vec{x}_1 =< 0, 2, 1, -1 >$, $\vec{x}_2 =< 0, -1, 1, 6 >$, and $\vec{x}_3 =< 0, 2, 2, 3 >$. Then by Gram-Schmidt orthogonalization process, we have

$$\vec{v}_1 = \vec{x}_1 =< 0, 2, 1, -1 > .$$

and

$$
\begin{aligned}
\vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 \\
&= < 0, -1, 1, 6 > - \frac{< 0, -1, 1, 6 > \cdot < 0, 2, 1, -1 >}{\| < 0, 2, 1, -1 > \|^2} < 0, 2, 1, -1 > \\
&= < 0, -1, 1, 6 > + \frac{7}{6} < 0, 2, 1, -1 > \\
&= \frac{1}{6} < 0, 8, 13, 29 >,
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 \\
&= < 0, 2, 2, 3 > - \frac{< 0, 2, 2, 3 > \cdot < 0, 2, 1, -1 >}{\| < 0, 2, 1, -1 > \|^2} < 0, 2, 1, -1 > \\
&\quad - \frac{< 0, 2, 2, 3 > \cdot < 0, 8, 13, 29 >}{\| < 0, 8, 13, 29 > \|^2} < 0, 8, 13, 29 > \\
&= < 0, 2, 2, 3 > - \frac{1}{2} < 0, 2, 1, -1 > - \frac{43}{358} < 0, 8, 13, 29 > \\
&= \frac{1}{179} < 0, 7, -11, 3 > .
\end{aligned}
$$

To form an orthogonal basis, one needs to find another orthogonal vector, eg: in this case, we choose $\vec{v}_4 =< 1, 0, 0, 0 >$ which is orthogonal to $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.