CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester B, 2012/2013

Time allowed : Two hours

This paper has **SIX** pages (including this cover page).

A brief table of derivatives is attached on pages 5 and 6.

Instructions to candidates:

1. This paper has **TEN** questions.
2. Attempt **ALL** questions in Section A and B.
3. Each question in Section A carries 9 marks.
4. Each question in Section B carries 15 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Section A
Answer **ALL** questions in this section.

Question 1

Find all values of x, such that $0 \le x \le 2\pi$, satisfying the equation

$$\frac{\sin x}{\cos 3x} = \frac{\sin 3x}{\cos x}.$$

(Hint: $\sin 2A = 2\sin A\cos A$, $\sin A - \sin B = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$.)

(9 marks)

Question 2

(a) Evaluate $\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^3 + x^2 - 5}$.

(4 marks)

(b) Find the value of k such that $\lim_{x\to 0} \frac{\sin 3x - 3x + kx^3}{x^3} = 1$.

(5 marks)

Question 3

Differentiate the following with respect to x:

(a) $\left(\frac{1-x}{1+x}\right)^{\frac{1}{3}}$,

(3 marks)

(b) $\log_e(1 + \cosh 2x)$,

(3 marks)

(c) $2^{x^{-1}}$.

(3 marks)

Question 4

(a) A curve has parametric equations $x = 3\cos t$, $y = 2\sin t$, where t is the parameter and $0 \le t \le 2\pi$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

(5 marks)

(b) Let $f(x) = |\tan x|$, for $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Determine whether $f(x)$ is differentiable at $x = 0$. Give your reason.

(4 marks)

Question 5

Let $g: \mathbf{R} \to \mathbf{R}$ and $g(x) = x^2 - 2x - 1$, for $x \in [1, \infty)$. Find

(a) the largest possible range of g,

(3 marks)

(b) the inverse function g^{-1} and state its domain.

(6 marks)

Question 6

(a) If $y = \dfrac{1}{ax+b}$, where a and b are constants and $a \neq 0$, find the general formula for the n th derivative of y with respect to x.

(3 marks)

(b) Resolve $\dfrac{2x+32}{(2x-1)(x+5)}$ into partial fractions.

(4 marks)

(c) Using the result in parts (a) and (b), or otherwise, find the fifth derivative of $\dfrac{2x+32}{(2x-1)(x+5)}$ with respect to x. You need not simplify your answer.

(2 marks)

Question 7

Let $P(0,-2)$ be a turning point of the curve $y = \dfrac{x^2 + px + q}{x+1}$.

(a) Find the values of p and q.

(3 marks)

(b) Show that $P(0,-2)$ is a local minimum point of the curve.

(3 marks)

(c) Show that y has no value between -6 and -2.

(3 marks)

Question 8

(a) Given that $\sinh\theta = \dfrac{1}{2}(e^{\theta} - e^{-\theta})$, show that $\sinh(3\theta) = 4\sinh^3\theta + 3\sinh\theta$.

(3 marks)

(b) Using the result in part (a), or otherwise, solve the cubic equation $4x^3 + 3x = 2$, giving your answers correct to 3 significant figures.

(6 marks)

Section B
Answer **ALL** questions in this section.

Question 9

(a) Show that the straight line $y = mx + \frac{a}{m}$, $m \neq 0$ touches the parabola $y^2 = 4ax$. Hence find the coordinates of the point of contact.

(4 marks)

(b) Using the result in part (a), or otherwise, find the equation of the tangent to the parabola $y^2 = 6x$ which is parallel to the line $y = -2x + 1$.

(5 marks)

(c) Draw a rough sketch the graph of the parabola $(y+2)^2 = 6(x-1)$. Find the coordinates of its focus and the equation of its directrix.

(6 marks)

Question 10

(a) For any non-negative integer n, the Hermite polynomial $H_n(x)$ is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2}), \text{ for } -\infty < x < \infty.$$

Show that $y = H_n(x)$ satisfies the equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2ny = 0$.

(Hint: Let $u = e^{-x^2}$. Show that $\frac{\mathrm{d}u}{\mathrm{d}x} = -2xu$.)

(8 marks)

(b) By the Maclaurin theorem, or otherwise, find the expansion of $\tan x$ in ascending powers of x as far as the term in x^5.

(7 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	**Derivative of y with respect to x**
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d f(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	**Derivative of** y **with respect to** x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u\frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u\frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \operatorname{sech}^2 u\frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\operatorname{cosech}^2 u\frac{du}{dx}$
$y = \operatorname{sech} u$	$\frac{dy}{dx} = -\operatorname{sech} u \tanh u\frac{du}{dx}$
$y = \operatorname{cosech} u$	$\frac{dy}{dx} = -\operatorname{cosech} u \coth u\frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

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