yeowo

MA2001/MA2149/MA2170 **Mid-term Test** **Semester A, 2013/2014**

Name: ______________________ Student ID: ______________

Course: MA2001 / MA2149 / MA2170 (Circle your course)

Answer ALL questions. (Full marks: 50)

Total Marks: ________

Q1: ______ Q2: ______

Q3: ______ Q4: ______

Question 1 (12 marks)

Suppose near the point $(x, y, u, v) = (1,1,1,1)$, we can solve $\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$ uniquely for u and v as functions of x and y. Compute $\dfrac{\partial u}{\partial x}(1,1)$.

Question 2 (12 marks)

Find the quadratic surface approximation of $f(x,y)=\ln(x^2+y)$ at $(-1,0)$. Estimate $f(-0.9,0.1)$ by the quadratic surface approximation.

Question 3 (10 marks)

Find the value I by changing the order of the integration in $I = \int_0^2 \int_0^{2x} (1 + x + xy)\, \mathrm{d}y\, \mathrm{d}x$.

Question 4 (16 marks)

Given a 3×3 real matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{pmatrix}$,

(a) find the eigenvalues of matrix A, and find the eigenvectors corresponding to each of these eigenvalues;

(b) show that there exists an invertible matrix P such that $P^{-1}AP$ gives a diagonal matrix D;

(c) calculate P^{-1} and $P^{-1}AP$.

[Hint: Use the three eigenvectors found in part (a) as the columns of P.]

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1. <u>Question</u>. Suppose near the point $(x,y,u,v)=(1,1,1,1)$, we can
(10 marks) Solve $\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$ uniquely for u and v as functions ✓
of x and y. Compute $\frac{\partial u}{\partial x}(1,1)$

<u>Solution</u>: $\begin{cases} xu+yvu^2=2 \\ xu^3+y^2v^4=2 \end{cases} \Rightarrow \begin{cases} (u+xu_x)+y(v_xu^2+2uv\cdot u_x)=0 \\ (u^3+3xu^2u_x)+y^2\,4v^3v_x=0 \end{cases}$ 6

$\Rightarrow$ at $\begin{cases}(x,y)=(1,1) \\ (u,v)=(1,1)\end{cases}$, we have $\begin{cases} 3u_x+v_x+1=0 \\ 3u_x+4v_x+1=0 \end{cases} \Rightarrow \begin{cases} u_x=-\frac{1}{3} \\ v_x=0 \end{cases}$.

Therefore $\frac{\partial u}{\partial x}(1,1)=-\frac{1}{3}$. 6

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2. <u>Question</u>. Find the quadratic surface approximation of
(10 marks) $f(x,y)=\ln(x^2+y)$ at $(-1,0)$. Estimate $f(-0.9,0.1)$
by the quadratic surface approximation.

<u>Solution</u>. $f(-1,0)=0$, $f_x=\frac{2x}{x^2+y} \Rightarrow f_x(-1,0)=-2$

$f_y=\frac{1}{x^2+y} \Rightarrow f_y(-1,0)=1$

$f_{xx}=\frac{2(x^2+y)-2x(2x)}{(x^2+y)^2}=\frac{2y-2x^2}{(x^2+y)^2} \Rightarrow f_{xx}(-1,0)=-2$ 6

$f_{yy}=\frac{-1}{(x^2+y)^2} \Rightarrow f_{yy}(-1,0)=-1$

$f_{xy}=\frac{-2x}{(x^2+y)^2} \Rightarrow f_{xy}(-1,0)=2$

$\Rightarrow f(x,y)\approx P_2(x,y)=f(-1,0)+f_x(-1,0)(x+1)+f_y(-1,0)y+\frac{1}{2}\left[f_{xx}(-1,0)(x+1)^2+2f_{xy}(-1,0)(x+1)y + f_{yy}(-1,0)y^2\right]$

$=0-2(x+1)+y+\frac{1}{2}\left(-2(x+1)^2+4(x+1)y-y^2\right)$

$=-2x-2+y-(x+1)^2+2(x+1)y-\frac{1}{2}y^2=-x^2-\frac{1}{2}y^2+2xy-4x+3y-3$

~~$-x^2+\frac{1}{2}y^2-2xy-4x-3$~~

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$\Rightarrow f(-0.9,0.1)\approx P_2(-0.9,0.1)$ ~~$=-2\times0.1+0.1+(-0.1^2+4\times0.1^2-0.1^2)$~~ ~~$-0.1$~~

$=-2\times0.1+0.1+\frac{1}{2}\times(-2\times0.1^2+4\times0.1^2-0.1^2)$

$=-0.1+0.5\times0.01$ 2

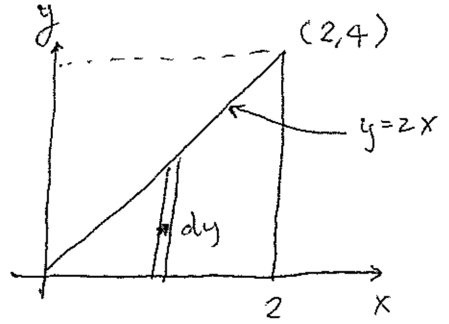
$=-0.095$

Q3 Find the value I by changing the order of the integration in

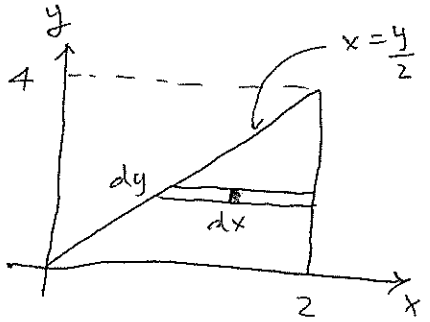
$$I = \int_{x=0}^{x=2} \int_{y=0}^{y=2x} (1 + x + xy)\, dy\, dx .$$

(10 marks)

Solutions:



$$I = \int_{y=0}^{4} \int_{x=y/2}^{x=2} (1+x+xy)\, dx\, dy$$



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(5) $$= \int_0^4 \left[\left(x + \frac{x^2}{2} + \frac{x^2 y}{2} \right) \right]_{x=y/2}^{x=2} dy$$

$$= \int_0^4 \left(\left[2 + \frac{4}{2} + \frac{4y}{2} \right] - \left[\frac{y}{2} + \frac{y^2}{8} + \frac{y^3}{8} \right] \right) dy$$

(3) $$= \int_0^4 \left(4 + \frac{3y}{2} - \frac{y^2}{8} - \frac{y^3}{8} \right) dy$$

$$= \left[4y + \frac{3y^2}{4} - \frac{y^3}{24} - \frac{y^4}{32} \right]_0^4$$

(2) $$= 16 + 12 - \frac{64}{24} - \frac{4^4}{32} = 17\tfrac{1}{3}$$

Q4.

Mid-term test question. (15 marks)

Question: Given a 3×3 real matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{pmatrix}$.

a). find the eigenvalues of matrix A, and find the eigenvectors corresponding to each of these eigenvalues;

b). show that there exists an invertible matrix P such that $P^{-1}AP$ gives a diagonal matrix D;

c). Calculate P^{-1} and $P^{-1}AP$.

[Hint: Use the three eigenvectors found in part a) as the columns of P.

Soln. a).

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ -2 & 4-\lambda & 2 \\ -2 & 1 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)(3-\lambda)(6-\lambda).$$

Hence, we find that matrix A has two real eigenvalues, i.e. $\lambda_1 = 6$ and $\lambda_2 = \lambda_3 = 3$. (repeated).

For $\lambda = 6$: The augmented matrix can be written as:

$$\left(\begin{array}{ccc|c} 3-6 & 0 & 0 & 0 \\ -2 & 4-6 & 2 & 0 \\ -2 & 1 & 5-6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -2 & -2 & 2 & 0 \\ -2 & 1 & -1 & 0 \end{array}\right)$$

$$\overset{R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$$

$$\overset{R_2 + \frac{2}{3}R_3}{\sim} \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$$

$$\overset{R_1 - \frac{3}{2}R_2}{\sim} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right),$$

Hence, we have that $-2x_1 = 0 \Rightarrow x_1 = 0$

$3x_2 - 3x_3 = 0 \Rightarrow x_2 = x_3$.

We can then conclude that $E_6 = [0 \quad t \quad t]^T, \forall t \in \mathbb{R}$.

Let $t = 1$, $\vec{v_1} = [0 \quad 1 \quad 1]^T$ is one of the eigenvectors of A with $\lambda = 6$.

For $\lambda = 3$: The augmented matrix can be written as:

$$\left(\begin{array}{ccc|c} 3-3 & 0 & 0 & 0 \\ -2 & 4-3 & 2 & 0 \\ -2 & 1 & 5-3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 \\ -2 & 1 & 2 & 0 \end{array}\right)$$

$$\overset{R_3 - R_2}{\sim} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, we have that $-2x_1 + x_2 + 2x_3 = 0$,

We can then conclude that $E_3 = [s \quad 2s - 2t \quad t]^T, \forall t \in \mathbb{R}$.

Let $s = 0, t = 1$, $\vec{V_2} = [0 \quad -2 \quad 1]^T$ is one of the eigenvectors of A with $\lambda =$

$s = 1, t = 0$, $\vec{V_3} = [1 \quad 2 \quad 0]^T$ is one of the eigenvectors of A with $\lambda =$

b). We construct $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$, with $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ as its columns

By using the formula that $P^{-1} = \dfrac{(P_{jk})^T}{\det P}$, we have:

$$(P_{jk})^T = \begin{pmatrix} -2 & 1 & 2 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}, \quad \det P = 3.$$

$$\therefore P^{-1} = \begin{pmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & -1/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and hence we have:}$$

$$P^{-1}AP = \begin{pmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & -1/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 & 4 \\ 2 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D.$$