CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester B, 2015/2016

Time allowed : Three hours

This paper has **SEVEN** pages (including this cover page).

A brief table of derivatives is attached on pages 6 and 7.

Instructions to candidates:

1. This paper has **TEN** questions.
2. Attempt **ALL** questions.
3. Each question carries 10 marks.

---

*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Question 1

Consider the function defined by

$$f(x)=\begin{cases} x^2-1 & , \text{ if } x\le 0 \\ x-2 & , \text{ if } 0<x<1 \\ c & , \text{ if } x=1 \\ -x & , \text{ if } x>1 \end{cases}$$

(a) At what point does both right hand and left hand limits exist, but the limit does not exist? Give your reason.

(4 marks)

(b) Find the value of c for which $f(x)$ is continuous at $x=1$. Give your reason.

(6 marks)

Question 2

(a) Let $F(x)=1+\cos 2x$.
Find the largest possible domain and the largest possible range of $F(x)$

(4 marks)

(b) The function $G(x)$ is defined by $G(x)=x^2+2x-2$, $x\in[0,\infty)$
Find the inverse function $G^{-1}(x)$ and state its domain.

(6 marks)

Question 3

(a) If $y=(ax+b)^{-p}$, where a and p are positive integers, b is a constant, find the general formula for the nth derivative of y with respect to x.

(3 marks)

(b) Using Leibnitz' rule and the result of Question 3(a), or otherwise, find
$\frac{\mathrm{d}^6}{\mathrm{d}x^6}\left[(2x^3-17x^2+45x-30)(x-2)^{-4}\right]$.

You need not simplify your answer.

(7 marks)

Question 4

(a) Show that $\cos^2 x + \cos^2 2x = 1 + \cos x \cos 3x$.

(4 marks)

(Hint: $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$.)

(b) Hence, find the general solution of the trigonometric equation
$\cos^2 x + \cos^2 2x = 1 + \sin x \cos 3x$.

(6 marks)

Question 5

A curve has the parametric equations $x = 3t^2 + 2$, $y = -2t^3$, where t is the parameter of the curve.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t .

(2 marks)

(b) Show that the equation of the curve in Cartesian form is $4(x-2)^3 = 27y^2$.

(2 marks)

(c) Show that the tangent to the curve at the point when $t = 1$ is also a normal to the parabola $y^2 = 4x$ at the point $(1, 2)$.

(6 marks)

Question 6

Differentiate with respect to x :

(a) $5(2+3x)^{\frac{2}{3}}$; (3 marks)

(b) $\tan^{-1}(\sinh 2x)$; (3 marks)

(c) $\dfrac{\sqrt{x^2+1}\cos^2 x}{x^3 e^{5x}}$. (4 marks)

You need not simplify your answer.

Question 7

(a) Evaluate the limit $\lim_{x\to 0}\dfrac{1-2e^{x}+e^{2x}}{\cos x-2\cos 2x+\cos 3x}$. (5 marks)

(b) Let $g(x)=x^{\frac{4}{3}}$ for $x\in\mathrm{R}$.
Is $g(x)$ differentiable at $x=0$? Give your reason.

(5 marks)

Question 8

(a) Using the mean value theorem, show that $\dfrac{\pi}{6}+\dfrac{\sqrt{3}}{15}<\sin^{-1}(0.6)<\dfrac{\pi}{6}+\dfrac{1}{8}$.

(3 marks)

(Hint: Consider the function $f(x)=\sin^{-1}x$, for $x\in[0.5,0.6]$.)

(b) If $y=\sin^{-1}x$, $-1\le x\le 1$, show that $(1-x^{2})\dfrac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}-x\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$. ------- (*)

By repeated differentiation of equation (*) and use the Maclaurin series, or otherwise, show that the first three non-zero terms in the expansion of $\sin^{-1}x$ are

$x+\dfrac{x^{3}}{6}+\dfrac{3x^{5}}{40}+\ldots\ldots$. (5 marks)

(c) Using the result in part (b), find an approximation to the value of $\sin^{-1}(0.6)$, giving 5 decimal places in your answer.

(2 marks)

Question 9

(a) Show that the non-linear equation
$e^{x}+x^{3}+2x-3=0$ ------- (**)
has exactly one real root and that root lies between 0 and 1 .

(5 marks)

(b) Perform two iterations of Newton's method with an initial approximation, $x_0=0.5$ to compute an approximation to the root of equation (**).

(5 marks)

(Hint: Newton iterative scheme for the solution of $f(x)=0$ is
$x_{k+1}=x_k-\dfrac{f(x_k)}{f'(x_k)}$, $k=0,1,2,\ldots\ldots$.)

Question 10

(a) Show that the equation $5x^2 + 9y^2 - 10x + 54y + 41 = 0$ represents an ellipse whose centre is at the point $C(1, -3)$.

(3 marks)

(Hint: You may use the method of completing the square.)

(b) Find its eccentricity, equations of its directrices and the coordinates of its foci and vertices.

(5 marks)

(Hint: The eccentricity of the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b > 0$ is given by $b^2 = a^2(1 - e^2)$.)

(c) Sketch the graph of the ellipse.

(2 marks)

Short Table of Derivatives of $y=f(u)$ with respect to x, where u is a function of x

Functions, $y=f(u)$	Derivative of y with respect to x
$y=c$, where c is a constant.	$\frac{dy}{dx}=0$
$y=cu$, where c is a constant.	$\frac{dy}{dx}=c\frac{du}{dx}$
$y=u^p$, where p is a constant.	$\frac{dy}{dx}=pu^{p-1}\frac{du}{dx}$
$y=u+v$	$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$
$y=uv$	$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$
$y=\frac{u}{v}$	$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
$y=f(u)$, where u is a function of x.	$\frac{dy}{dx}=\frac{d f(u)}{du}\cdot\frac{du}{dx}$, the chain rule
$y=\log_a u$, $a>0$.	$\frac{dy}{dx}=\frac{1}{u}\log_a e\frac{du}{dx}$
$y=a^u$, $a>0$.	$\frac{dy}{dx}=a^u\log_e a\frac{du}{dx}$
$y=e^u$	$\frac{dy}{dx}=e^u\frac{du}{dx}$
$y=u^v$	$\frac{dy}{dx}=vu^{v-1}\frac{du}{dx}+u^v\log_e u\frac{dv}{dx}$
$y=\sin u$	$\frac{dy}{dx}=\cos u\frac{du}{dx}$
$y=\cos u$	$\frac{dy}{dx}=-\sin u\frac{du}{dx}$
$y=\tan u$	$\frac{dy}{dx}=\sec^2 u\frac{du}{dx}$
$y=\cot u$	$\frac{dy}{dx}=-\operatorname{cosec}^2 u\frac{du}{dx}$
$y=\sec u$	$\frac{dy}{dx}=\sec u\tan u\frac{du}{dx}$
$y=\operatorname{cosec} u$	$\frac{dy}{dx}=-\operatorname{cosec} u\cot u\frac{du}{dx}$
$y=\sin^{-1}u$	$\frac{dy}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y=\cos^{-1}u$	$\frac{dy}{dx}=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y=\tan^{-1}u$	$\frac{dy}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	**Derivative of** y **with respect to** x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u\frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u\frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \operatorname{sech}^2 u\frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\operatorname{cosech}^2 u\frac{du}{dx}$
$y = \operatorname{sech} u$	$\frac{dy}{dx} = -\operatorname{sech} u \tanh u\frac{du}{dx}$
$y = \operatorname{cosech} u$	$\frac{dy}{dx} = -\operatorname{cosech} u \coth u\frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

- END -