MA1200 Calculus and Basic Linear Algebra I
Chapter 2 Sets and Functions

1 Set Notation

A set is a collection of distinct objects called *elements* or *members* of that set. For example, $A = \{1, 2, 3, 4, 5\}$ is a set and a list of all its elements is given. In general, we use the notation $\{x|x$ processes certain properties$\}$to denote a set of objects that share some common properties. Also, if e is an element of a set A, we write $e \in A$ (read as *e belongs to A*).

Illustration

Let V be the set of all vowels of the English alphabets, then

$V = \{a, e, i, o, u\}$

u is an element of the set V. However, p is NOT an element of the set V.

We can use $u \in V$ (read as *u belongs to V*) to show that e is an element of V, and use $p \notin V$ to show that p is NOT an element of V.

Some notations of the sets commonly used in Mathematics:

Z the set of all integers (the set that contains all integers), i.e. **Z** = $\{0, \pm 1, \pm 2, \ldots\}$

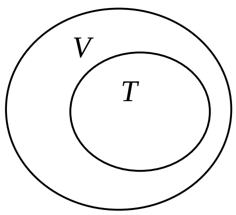
R the set of all real numbers (the set that contains all real numbers)

ϕ (called a *null set* or *empty set*) a set that contains no element

Two sets are equal if they contain the same elements.
e.g. If $A = \{3, 5, 7\}$ and $B = \{3, 5, 7\}$, then we can write $A = B$.

The relationships among sets can be conveniently illustrated by *Venn diagrams*.

e.g. Let $V = \{a, e, i, o, u\}$ and $T = \{a, u\}$. The figure on the right shows the Venn diagram:



Subset

Given two sets A and B, we say that A is a *subset* of B (denoted by $A \subset B$) if all elements of A belong to B. In the above case, T is a subset of V and therefore we can write $T \subset V$. For example, we can write **Z** $\subset$ **R** to indicate that the set of all integers is a subset of the set of all real numbers.

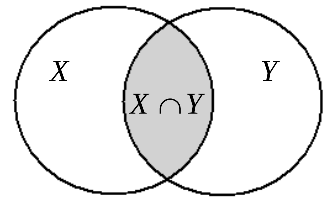
Operations of Sets

It is often necessary to combine two or more sets to form new sets. This is done by *set operations*.

(a) Intersection

The *intersection* of two sets X and Y is a set whose elements belong to <u>both</u> X and Y. It is denoted by $X \cap Y$.

e.g. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $Y = \{5, 6, 7, 8, 9, 10, 11, 12\}$. We see that the elements 5, 6, 7, 8 belong to both X and Y. Therefore, $X \cap Y = \{5, 6, 7, 8\}$. The figure on the right shows the Venn diagram:
The shaded part represents $X \cap Y$.

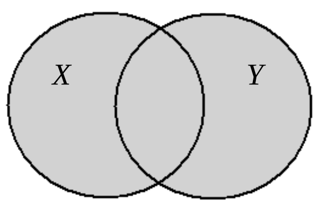


e.g. For the sets $A = \{1, 2, 3, 4\}$ and $B = \{9, 10, 11, 12\}$, no object belongs to both A and B. Therefore, $A \cap B = \phi$ (empty set).

(b) Union

The *union* of two sets X and Y is a set whose elements belong to <u>either</u> X or Y or both of them. It is denoted by $X \cup Y$.

e.g. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $Y = \{5, 6, 7, 8, 9, 10, 11, 12\}$.
Then $X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.



The shaded part represents $X \cup Y$.

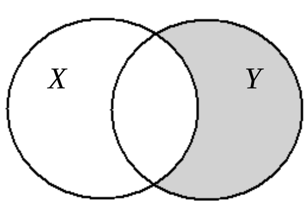
Question: Let $A = \{2, 4, 6, 8\}$ and $B = \{-3, 6, 8, 12.4\}$. Write the set described by each of the following. List all the elements in the set.

(i) $A \cup B$ (ii) $A \cap B$ (iii) $B \cap \mathbf{Z}$ (iv) $B \cap \mathbf{R}$

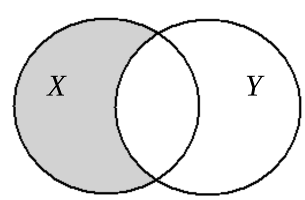
(c) Complements

The *complement* of X with respect to Y is a set whose elements belong to Y but not belong to X. It is denoted by $Y \setminus X$.

e.g. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $Y = \{5, 6, 7, 8, 9, 10, 11, 12\}$.
Then $Y \setminus X = \{9, 10, 11, 12\}$ and $X \setminus Y = \{1, 2, 3, 4\}$.



The shaded part represents $Y \setminus X$.



The shaded part represents $X \setminus Y$.

2 Intervals

We can also use the notation $\{x \mid x$ processes certain properties$\}$ to denote a set of objects that share some common properties. Sets with infinitely many elements are often denoted by this method.

e.g. $\{x \mid x$ is the outcome of throwing a die$\}$ is the set $\{1, 2, 3, 4, 5, 6\}$.
$\{x \mid x$ is a prime number$\}$ is the set that contains all prime numbers.
$\{x \mid x > 0$ and x is divisible by 3$\}$ is the set $\{3, 6, 9, 12, 15, 18, \ldots\}$.
$\{x \mid x = 3m$ and $m \in \mathbf{Z}\}$ is the set that contains all multiples of 3.
$\mathbf{Z} = \{x \mid x$ is an integer$\}$
$\{x \mid x$ is a real number and $3 < x < 7\}$ is the set of real numbers which are smaller than 7 and greater than 3.

As mentioned, we use the symbol $\mathbf{R}$ to denote the set which contains exactly all the real numbers. Also, the following symbols are frequently used to describe the corresponding subsets of real numbers (a, b are two distinct real numbers):

$$(a,b) = \{x \in \mathbf{R} \mid a < x < b\}$$
$$[a,b) = \{x \in \mathbf{R} \mid a \le x < b\}$$
$$[a,b] = \{x \in \mathbf{R} \mid a \le x \le b\}$$
$$[a,\infty) = \{x \in \mathbf{R} \mid x \ge a\}$$
$$(-\infty,a) = \{x \in \mathbf{R} \mid x < a\}$$

(The other subsets like $(a,b], (a,\infty), (-\infty,a], (-\infty,\infty)$ are defined similarly). These sets are usually called *intervals*. In our discussion, most of the sets we consider are intervals.

Example 1

Use set notations to represent each of the following sets.

(a) The set of integers which are smaller than -6 and greater than -13.
(b) The set of integers which are greater than 2 and smaller than 30.

Solutions

(a) $\{-12, -11, -10, -9, -8, -7\}$ or $\{x \mid x \in \mathbf{Z}$ and $-13 < x < -6\}$ or $\{x \in \mathbf{Z} \mid -13 < x < -6\}$
(b) $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$ or $\{x \mid x \in \mathbf{Z}$ and $2 < x < 30\}$ or $\{x \in \mathbf{Z} \mid 2 < x < 30\}$

Example 2

Use bounded intervals to represent each of the following sets.

(a) The set of real numbers which are greater than -3 and are smaller than or equal to 6.
(b) The set of real numbers which are smaller than -6.
(c) $[2, 8] \cap (3, 10)$
(d) $[2, 8] \cup (3, 10)$

Solutions

(a) $(-3, 6]$ (it represents the set $\{x \mid x \in \mathbf{R}$ and $-3 < x \le 6\}$, i.e. $\{x \in \mathbf{R} \mid -3 < x \le 6\}$.)
(b) $(-\infty, -6)$ (it represents the set $\{x \mid x \in \mathbf{R}$ and $x < -6\}$, i.e. $\{x \in \mathbf{R} \mid x < -6\}$.)
(c) $(3, 8]$
(d) $[2, 10)$

Question: Use bounded intervals to represent each of the following sets.

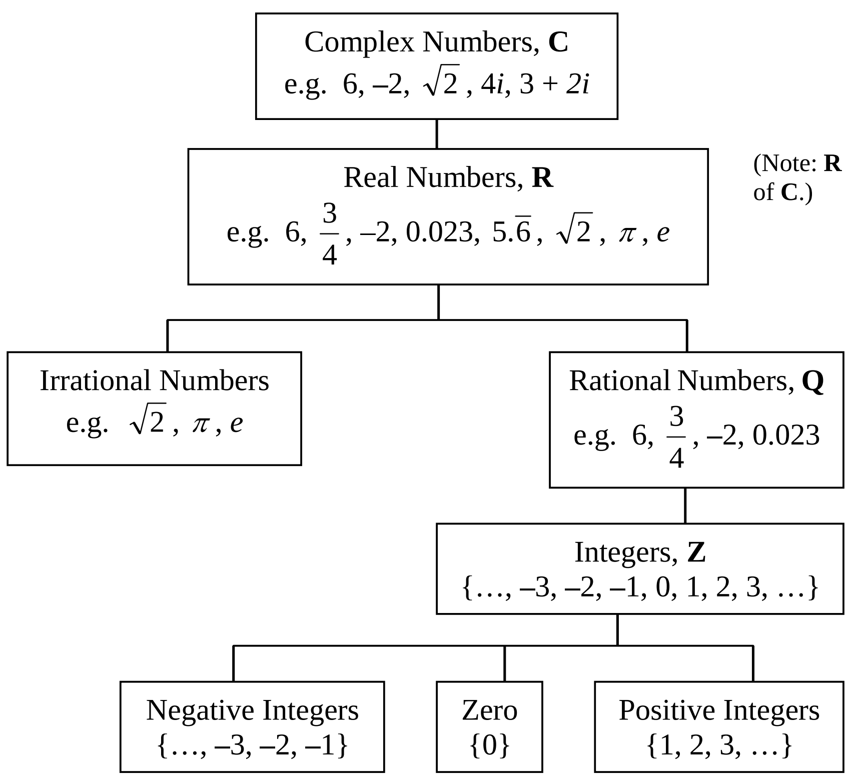
(i) $[-2, 3] \cup (3, \infty)$ (ii) $(-\infty, 6] \cap (3, \infty)$ (iii) $(-\infty, 6] \cup (3, \infty)$

The Number System

The following presents the description of some types of real numbers:

- *Natural Numbers* are the numbers 1, 2, 3, ...
- All natural numbers, together with 0, -1, -2, ... forms the set of *integers*. {1, 2, 3, 4, …} are the set of positive integers (also called the set of natural numbers) and {…, -3, -2, -1} are the set of negative integers. 0 is neither positive nor negative.
- *Rational numbers* are numbers that can be represented in the form $\frac{p}{q}$, where q is non-zero and p and q are both integers. In particular, all integers are rational (pick q =1). Other examples of rational numbers are: $\frac{3}{2}, -\frac{5}{7}, -\frac{343}{11}$.
- *Irrational numbers* are real numbers with non-repeating decimals. Examples are:
  e = 2.71828… (this special number is called the natural number)
  $\pi = 3.14159....$ (pi, the ratio of a circle's perimeter to its diameter)
  $\sqrt{2} = 1.4142...$

The following tree diagram represents the relations between different types of numbers:

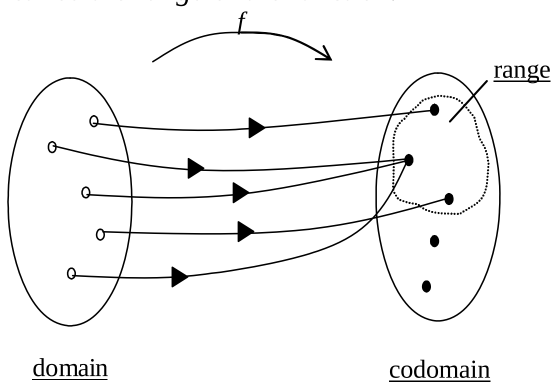


(Note: **R** is a proper subset of **C**.)

3 **Functions of a single real variable** (p.157 – p.158, p.173 – p.176, p.193 – p.197)

A. Definition of a Function

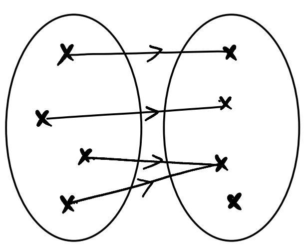
A function *f* is a rule of correspondence that associates with each object *x* in one set *A* (called the *domain* of *f*) a unique (exactly one) value $f(x)$ from a second set *B* (called the *codomain* of *f*). The set of values so obtained is called the *range* of the function.



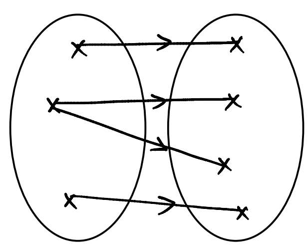
It is customary to write *f* as: $f : A \to B$

In this course, we will mainly study those functions whose domains and codomains are subsets of **R**, the set of real numbers. Moreover, when the rule for a function is given by an equation of the form $y = f(x)$ (for example: $y = x^2 + 1$), *x* is often called the *independent variable* and *y* the *dependent variable.*

The following diagram shows the interpretation:



A well-defined function



Not a function

The following equations define *y* as a function of *x* (where *x* is any real numbers):

$y = x^2 - 5x + 1$, $y = -x^3 + x^2 - 3x + 2$, $x + y = 4$

$y = \sin x$, $y = \cos x$, $y = \tan x$ (These are examples of *Trigonometric Functions*.)

$y = e^x$, $y = 10^x$, $y = \ln x$ (for $x > 0$), $y = \log x$ (for $x > 0$) (More properties of these functions will be discussed later.)

The following equations do <u>not</u> define *y* as a function of *x* (where *x* is any real numbers). (Why?)

$x + y^2 = 4$, $x^2 + y^2 = 9$

Question: If *x* is any real numbers, does $y = \sqrt{x}$ define *y* as a function of *x*?
How about if *x* is any non-negative real numbers?

Sometimes, we write $f(x) = x^2 - 5x + 1$ instead of $y = x^2 - 5x + 1$ to better indicate the relation that the value of *y* depends on the inputting value of *x*.

<u>Example 3</u>

It is given the function $f(x) = ax^2 + 3x + c$, where *a* and *c* are constants. If $f(0) = 5$ and $f(3) = 32$, find the values of *a* and *c*.

Solution

$f(0) = 5 \qquad a(0)^2 + 3(0) + c = 5 \qquad \therefore\ c = 5$

$f(3) = 32 \qquad a(3)^2 + 3(3) + 5 = 32 \qquad \therefore\ a = 2$

<u>Example 4</u>

Determine the largest possible domain and the largest possible range for each of the following functions.

(a) $f(x) = x^2$ (b) $f(x) = 25 - x$ (c) $f(x) = \sqrt{x+4}$ (d) $f(x) = 3 + \dfrac{1}{x-5}$

Solutions

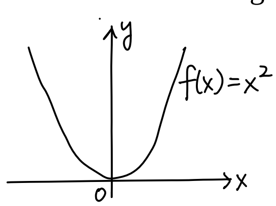
(a) For the function $f(x) = x^2$, it is well-defined for any real numbers *x*. Therefore, the largest possible domain is the set of all real numbers, i.e. **R**.
For any *x*, $x^2 \geq 0$. Therefore, the largest possible range is the set of all non-negative real numbers, i.e. $[0, \infty)$.

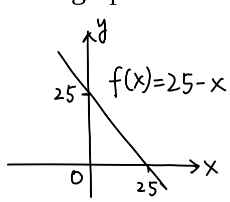
(b) For the function $f(x) = 25 - x$, it is well-defined for any real numbers *x*. Therefore, the largest possible domain is the set of all real numbers, i.e. **R**.
For any *x*, $25 - x$ is a real number which can be any number in **R**. Therefore, the largest possible range is **R**.

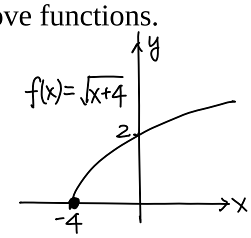
(c) For the function $f(x) = \sqrt{x+4}$, it is well-defined as long as $x + 4 \geq 0$, i.e. $x \geq -4$.
$\therefore$ The largest possible domain is $[-4, \infty)$.
For any *x*, $\sqrt{x+4}$ is a non-negative real number.
$\therefore$ The largest possible range is $[0, \infty)$.

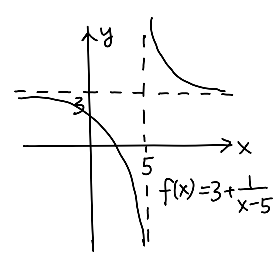
(d) For the function $f(x) = 3 + \dfrac{1}{x-5}$, it is well-defined as long as $x - 5 \neq 0$, i.e. $x \neq 5$.
$\therefore$ The largest possible domain is **R**$\backslash\{5\}$.
For any *x*, $3 + \dfrac{1}{x-5}$ is a real number EXCEPT 3.
$\therefore$ The largest possible range is **R**$\backslash\{3\}$.

Remark: The following show the graphs of the above functions.









Question:

Determine the largest possible domain and largest possible range for the function $f(x) = 5 + \sin x$.

B. Operations on functions

Consider the functions f and g with formulas $f(x)=\dfrac{x^2-1}{2}$ and $g(x)=x^9-x+1$.

We can make a new function of $f+g$, where

$$(f+g)(x)=f(x)+g(x)=\frac{x^2-1}{2}+x^9-x+1.$$

Clearly, x must be a number which belongs to both the domains of f and g. Similarly, we can define the functions $f-g$, fg and f/g as follows:

$$(f-g)(x)=f(x)-g(x)$$

$$(fg)(x)=f(x)\times g(x)$$

$$(f/g)(x)=\frac{f(x)}{g(x)} \text{ (defined only for those } x \text{ with } g(x)\neq 0\text{)}$$

C. Composition of functions

Let $f:A\to B,\ g:B\to C$ be two functions. We define the *composite* of g with f by

$$(g\circ f)(x)=g(f(x)).$$

Note that the domain of this function $g\circ f$ is A (and its codomain is C). In general $g\circ f$ and $f\circ g$ (if both are defined) are two different functions.

Example 5

Let $f:\mathbf{R}\to\mathbf{R},\ f(x)=x^2$ and $g:\mathbf{R}\to\mathbf{R},\ g(x)=x+1$. Find (a) $f\circ g$, (b) $g\circ f$.

Solution

(a) $f\circ g:\mathbf{R}\to\mathbf{R}$ is $(f\circ g)(x)=f(g(x))=f(x+1)=(x+1)^2$.

(b) $g\circ f:\mathbf{R}\to\mathbf{R}$ is $(g\circ f)(x)=g(f(x))=g(x^2)=x^2+1$. Note that $g\circ f\neq f\circ g$.

4 Elementary Functions

In this section we will introduce different types of functions that are frequently used.

A. Some typical examples of functions (p.169, p.182 – p.187, p.206 – p.209)

A function of the form $f(x)=k$ where k is a fixed real number, is called a *constant function.*

The function $f(x)=x$ is called the *identity function*.

A function $f(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_0$ (where the $a'_i\,s$ are real numbers and n is a non-negative integer) is called a *polynomial function.* If $a_n \neq 0$, n is the *degree* of the polynomial function. In particular,

$$f(x)=ax+b \quad (a\neq 0) \text{ is called a linear function}$$

$$f(x)=ax^2+bx+c \quad (a\neq 0) \text{ is called a quadratic function.}$$

Quotients of polynomial functions are called *rational functions.* That is, f is a rational function if it is of the form

$$f(x)=\frac{a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x+a_0}{b_m x^m + b_{m-1}x^{m-1}+\cdots+b_1 x+b_0},$$

where $a'_i\,s$ and $b'_j\,s$ are real numbers and both n and m are non-negative integers.

The trigonometric functions, inverse trigonometric functions, exponential functions and logarithmic functions and their properties will be discussed later. The following trigonometric identities will be discussed later.

Odd-Even Identities

$$\sin(-x)=-\sin x$$
$$\cos(-x)=\cos x$$
$$\tan(-x)=-\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2}-x\right)=\cos x$$
$$\cos\left(\frac{\pi}{2}-x\right)=\sin x$$
$$\tan\left(\frac{\pi}{2}-x\right)=\cot x$$

Pythagorean Identities

$$\sin^2 x+\cos^2 x=1$$
$$1+\tan^2 x=\sec^2 x$$
$$1+\cot^2 x=\csc^2 x$$

Addition Identities

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$
$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

Double-Angle Identities

$$\sin(2x) = 2\sin x\cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Half-Angle Identities

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

Sum Identities

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Product Identities

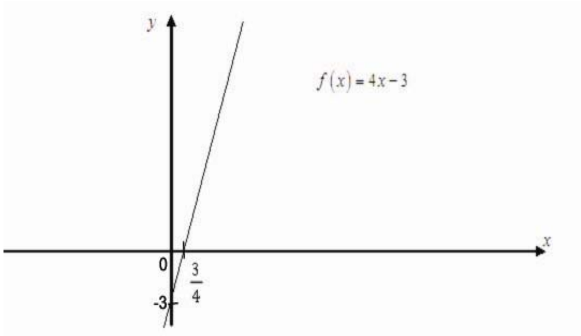
$$\sin x\sin y = -\frac{1}{2}\left[\cos(x+y) - \cos(x-y)\right]$$

$$\cos x\cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$\sin x\cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

B. Even and Odd Functions (p.178 – p.179, p.188 – p.189)

When both the domain and codomain of a function consist of real numbers, we can picture the function by drawing its graph on a coordinate plane. For example, the graph of the function $f(x) = 4x - 3$ is shown below:



We can often predict the symmetries of the graph of a function by inspecting the formula for the function.

If $f(-x) = f(x)$, then the graph is symmetric with respect to *y*-axis. Such a function is called an *even function*.

If $f(-x) = -f(x)$, the graph is symmetric with respect to the origin. We call such a function an *odd function*.

Example 6

Determine whether each of the functions are odd or even or neither of them.

(a) $h(x) = x^2$ (b) $f(x) = x^3$ (c) $g(x) = \dfrac{x^4 + x^2 + 1}{x^2}$ (d) $m(x) = x^5 + x^4 + 5$

Solutions

(a) $h(-x) = (-x)^2 = x^2 = h(x)$ $\quad\therefore$ It is an even function.

(b) $f(-x) = (-x)^3 = -x^3 = -f(x)$ $\quad\therefore$ It is an odd function.

(c) $g(-x) = \dfrac{(-x)^4 + (-x)^2 + 1}{(-x)^2} = \dfrac{x^4 + x^2 + 1}{x^2} = g(x)$ $\quad\therefore$ It is an even function.

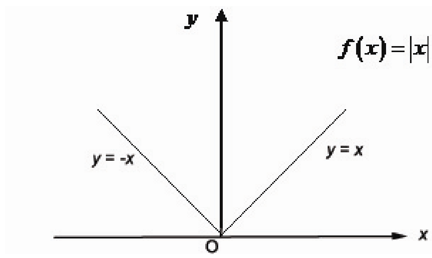
(d) $m(-x) = (-x)^5 + (-x)^4 + 5 = -x^5 + x^4 + 5$ which is neither $m(x)$ nor $-m(x)$

$\therefore$ It is neither an odd nor even function.

The *absolute value function*, defined by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is an example of even function.



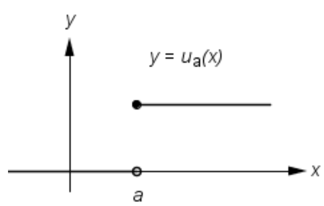
The absolute value function is also an example of **piecewise-defined function**, i.e. a function that is described by using different formulas on different parts of its domain.

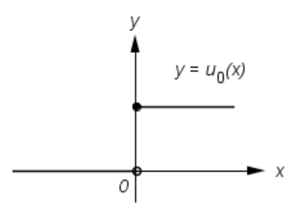
The *unit step function* at $x = a$ (where $a \geq 0$), is the function defined as

$$u_a(x) = \begin{cases} 0 & \text{if } x < a, \\ 1 & \text{if } x \geq a. \end{cases}$$

In particular, when a = 0, we write $u_0(x)$ as $u(x)$.

The unit step function is also an example of piecewise-defined function.





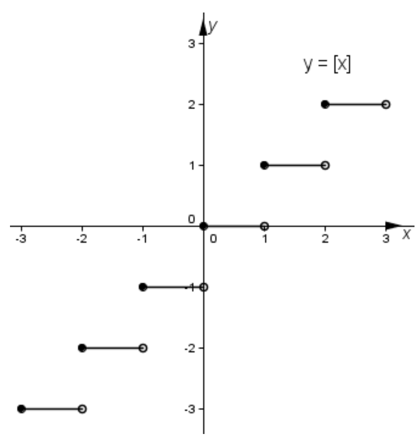
The *greatest integer function* is the function

$f(x) = [x] =$ the greatest integer smaller than or equal to *x*.

For example, $f(3.1) = [3.1] = 3$, $f(2) = [2] = 2$,

$f(-3.1) = [-3.1] = -4$.

The greatest integer function is sometimes denoted as $f(x) = \lfloor x \rfloor$.



Question: What is the graph of the *least integer function* defined as

$f(x) = \lceil x \rceil =$ the least integer greater than or equal to *x*?

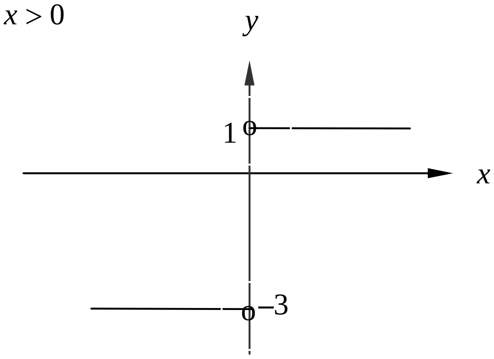
Example 7

Sketch the graph of $y = \dfrac{2x}{|x|} - 1$ for $x \neq 0$.

Solution

Note that $y = \begin{cases} \dfrac{2x}{-x} - 1, & x < 0 \\ \dfrac{2x}{x} - 1, & x > 0 \end{cases}$, i.e. $y = \begin{cases} -3, & x < 0 \\ 1, & x > 0 \end{cases}$

The graph of $y = \dfrac{2x}{|x|} - 1$ is as follows:



C. Periodic functions and Increasing/Decreasing Functions

A function $f(x)$ is *periodic* with *period* T if $f(x+T) = f(x)$ for any x which is contained in the domain of *f*. If we look at the graph of such *f*, we will find that the graph of *f* 'repeat' itself periodically.

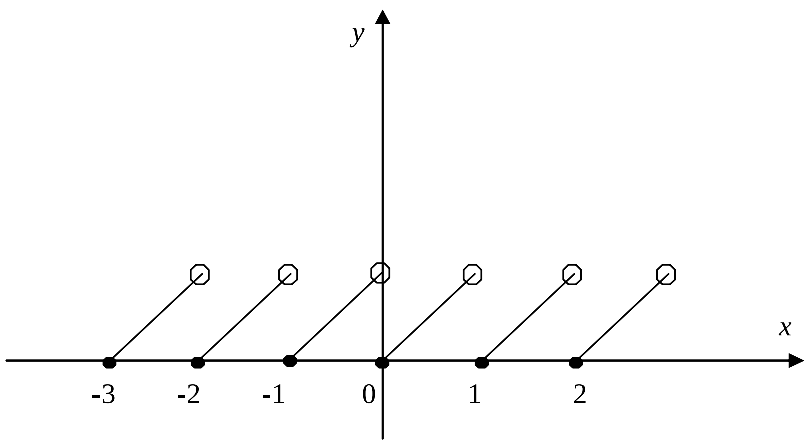
Illustration

The function $\sin x$ is periodic with period 2π since $\sin(x+2\pi) = \sin x$.

The function $f(x) = x - [x]$ is periodic with period 1. It is because

$$f(x+1) = (x+1) - [x+1] = x+1-[x]-1 = x-[x] = f(x)$$

for any $x \in \mathbf{R}$. The graph of $f(x)$ is shown below:



A function f is said to be *monotonic increasing* (resp. *monotonic decreasing*) if the following condition is satisfied:

$$f(x_1) \geq f(x_2) \text{ whenever } x_1 > x_2$$
$$[\text{resp. } f(x_1) \leq f(x_2) \text{ whenever } x_1 > x_2]$$

Furthermore, if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$, we call this f a *strictly increasing function*. Of course, we can define *strictly decreasing functions* similarly.

<u>Illustration</u>

$f(x) = x^3$ is strictly increasing over **R**. For any two given real numbers x_1, x_2 with $x_1 > x_2$, we have

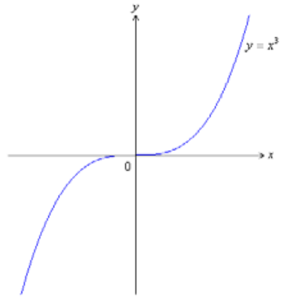
$$f(x_1) - f(x_2) = x_1^3 - x_2^3 = (x_1 - x_2)\left(x_1^2 + x_1 x_2 + x_2^2\right) = \frac{1}{2}(x_1 - x_2)\left[(x_1 + x_2)^2 + x_1^2 + x_2^2\right] > 0$$

(since $x_1 > x_2$). That is, $f(x_1) > f(x_2)$ whenever $x_1 > x_2$.

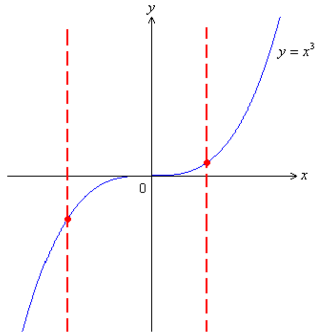
Before ending this section, we want to point out there are functions like
$e^x, \sin^{-1} x$, $\log x, \cosh x \cdots$ etc.

5 Inverse Functions

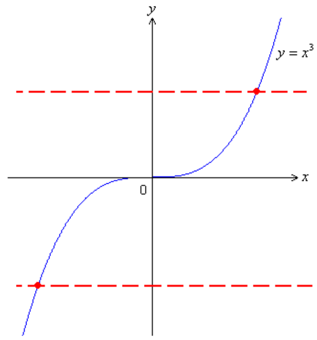
Consider the function $f(x) = x^3$.



$f(x)$ has only one value for each value of x in its domain (that is $\mathbf{R}$). In geometric terms, any vertical line meets the graph of f at only one point as shown below.



For this function $f(x) = x^3$, any horizontal line also meets the graph at only one point, as shown in the following graph.



This means that different values of x always give different values to $f(x)$. Such a function is said to be one-to-one.

Definition:
A function f is *one-to-one* if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 belong to the domain of f and $x_1 \neq x_2$. Equivalently, if

$$f(x_1) = f(x_2) \qquad \Rightarrow \qquad x_1 = x_2.$$

Illustration
$f(x) = x^3$ is one-to-one.
The equation $y = x^3$ has a unique solution x for every given value of y in the range of f.

$$x = y^{\frac{1}{3}}$$

This equation defines x as a function of y. We call this new function the inverse of f and denote it f^{-1}. Thus,

$$f^{-1}(y) = y^{\frac{1}{3}}.$$

Whenever a function f is one-to-one, for any number y in its range there will always exist a single number x in its domain such that $y = f(x)$. Since x is determined uniquely by y, it is a function of y. We write

$$x = f^{-1}(y)$$

and call f^{-1} the *inverse* of f.

We usually like to write functions with the independent variable x rather than y, so we reverse the roles of x and y.

Theorem:
A function f has an inverse if and only if it is one-to-one.

In practice, the inverse function f^{-1} is calculated from f by the following procedure:

1) check whether the function $y = f(x)$ is one-to-one,
2) solve x in terms of y (if possible),
3) rewrite the independent variable as x and the dependent variable as y.

Illustration
Given $f : \mathbf{R} \to \mathbf{R}$ where $f(x) = 2x - 3$.
The function $f(x)$ is one-to-one.
$y = 2x - 3$
$\Rightarrow x = \dfrac{y+3}{2}$
$\therefore$ The inverse function is $f^{-1}(x) = \dfrac{x+3}{2}$.

Illustration
Let $f : \mathbf{R} \to [0, \infty)$ and $f(x) = x^2$. Then f has no inverse function since it is not one-to-one as it as observed that $f(x)$ takes the same value twice for $x \neq 0$. For example, $f(-1) = 1 = f(1)$.

Illustration
Let $g : [0, \infty) \to [0, \infty)$ where $g(x) = x^2$. Then g is one-to-one, so it has an inverse $g^{-1}(x) = \sqrt{x}$.

Note: Do not confuse the -1 in f^{-1} with an exponent. The inverse f^{-1} is NOT the reciprocal $\dfrac{1}{f}$, which can be written as $(f(x))^{-1}$.

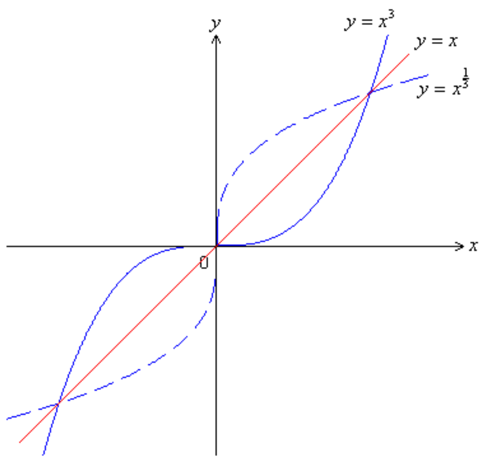
$$\therefore \boxed{f^{-1}(x) \neq (f(x))^{-1}.}$$

Properties of inverse functions

1. $y = f^{-1}(x) \Leftrightarrow x = f(y)$.
2. The domain of f^{-1} is the range of f.
3. The range of f^{-1} is the domain of f.
4. $f^{-1}\big(f(x)\big) = x$ for all x in the domain of f.
5. $f\big(f^{-1}(x)\big) = x$ for all x in the domain of f^{-1}.
6. $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f (i.e. the inverse of f^{-1} is f.)
7. The graph of f^{-1} is the reflection of the graph f in the line $y = x$.

E.g. $f(x) = x^3$

$f^{-1}(x) = x^{\frac{1}{3}}$



Some common examples of

1. Let $f : \mathbf{R} \to [0, \infty)$ and $f(x) = 10^x$. Then the inverse of f is $f^{-1}(x) = \log_{10} x$.
2. Let $g : \mathbf{R} \to [0, \infty)$ and $g(x) = e^x$. Then the inverse of g is $g^{-1}(x) = \ln x$.
3. Let $h : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ and $h(x) = \sin x$. Then the inverse of h is $h^{-1}(x) = \sin^{-1} x$.
4. Let $f : [0, \pi] \to [-1, 1]$ and $f(x) = \cos x$. Then the inverse of f is $f^{-1}(x) = \cos^{-1} x$.
5. Let $g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbf{R}$ and $g(x) = \tan x$. Then the inverse of g is $g^{-1}(x) = \tan^{-1} x$.