

1. If $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{bmatrix} a & 2 & -2 \\ 2 & b & 0 \\ -2 & 0 & 7 \end{bmatrix}$, find the value for a, b .

Sol: $A\vec{v} = \lambda\vec{v}$ if \vec{v} is an eigenvector of A .

$$\Rightarrow \text{left} = \begin{pmatrix} a & 2 & -2 \\ 2 & b & 0 \\ -2 & 0 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a - 4 - 2 \\ 4 - 2b + 0 \\ -4 + 0 + 7 \end{pmatrix} \checkmark$$

$$\text{Right} = \begin{pmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2a - 6 = 2\lambda \\ 4 - 2b = -2\lambda \\ -4 + 7 = \lambda \end{cases} \Rightarrow \begin{cases} 2a - 6 = 2 \cdot 3 \\ 4 - 2b = -2 \cdot 3 \\ \lambda = 3 \end{cases} \Rightarrow \begin{cases} a = 6 \\ b = 5 \\ \lambda = 3 \end{cases}$$

To find all eigenvalues & eigenvectors of A .

$$\det(A - \lambda I) = 0$$

$$= \det \begin{pmatrix} 6-\lambda & 2 & -2 \\ 2 & 4-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{pmatrix} = -2 \begin{vmatrix} 2 & -2 \\ 5-\lambda & 0 \end{vmatrix} + (7-\lambda) \begin{vmatrix} 6-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= -2(0 + 2(5-\lambda)) + (7-\lambda)((6-\lambda)(5-\lambda) - 4)$$

$$= -\lambda^3 + 18\lambda^2 - 99\lambda + 9 \cdot 18$$

$$= (\lambda - 3)(-\lambda^2 + 15\lambda - 54)$$

$$= (\lambda - 3)(-(\lambda - 6)(\lambda - 9)) \rightarrow \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 6 \\ \lambda_3 = 9 \end{cases}$$

$$\begin{array}{r}
 \lambda \rightarrow \quad \overline{\begin{array}{r} -\lambda^2 + 15\lambda - 54 \\ -\lambda^3 + 18\lambda^2 - 99\lambda + 162 \\ -\lambda^3 + 3\lambda^2 \\ \hline 15\lambda^2 - 99\lambda \\ 15\lambda^2 - 45\lambda \\ \hline -54\lambda + 162 \\ -54\lambda + 162 \\ \hline 0 \end{array}}
 \end{array}$$

A is orthogonally diagonalizable. $A = \underline{P} \underline{D} \underline{P}^T$.

$$\tilde{P} = \begin{pmatrix} 2 & \frac{1}{2} & -1 \\ -2 & 1 & -\frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & & \\ & 6 & \\ & & 9 \end{pmatrix}$$

$$A = \tilde{P} D \tilde{P}^T$$

$$\sqrt{1 + (\frac{1}{2})^2 + 1} = \sqrt{\frac{9}{4}}$$

orthogonal matrix.

$$\underline{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$A = P D P^T$$

$$P^T = P^{-1}$$

2. It is given the symmetric matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

- (a) find the eigenvalues of A ;
 (b) find the eigenvectors corresponding to each of these eigenvalues;
 (c) find an orthogonal matrix P such that $P^T A P$ gives a diagonal matrix D and calculates P^{-1} ;
 (d) Determine the eigenvalues of the matrix $B = A^5 + (A^2)^T$.

$$A = P D P^T$$

$$B = A^5 + (A^2)^T$$

$$= (P D P^T)^5 + \left((P D P^T)^2 \right)^T$$

$$= (P D P^T P D P^T P D P^T P D P^T P D P^T) + (P D P^T P D P^T)^T$$

$$= P D^5 P^T + (P D^2 P^T)^T$$

$$= P D^5 P^T + P D^2 P^T$$

$$= P (D^5 + D^2) P^T$$

$$= P \left[\begin{pmatrix} 1^5 & & \\ & 2^5 & \\ & & 3^5 \end{pmatrix} + \begin{pmatrix} 1^2 & & \\ & 2^2 & \\ & & 3^2 \end{pmatrix} \right] P^T$$

$$= P \begin{bmatrix} 2 & & \\ & 12 & \\ & & 3^5 + 3^2 \\ & & & 2 \end{bmatrix} P^T$$

eigenvalues of B are $2, 12, 3^5 + 3^2$.

- $$Q(\vec{x}) := \vec{x}^\top A \vec{x} = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j.$$

(a) $3x_1^2 + 8x_1x_2 - 3x_2^2$.

(b) $9x_1^2 + 6x_1x_2 + x_2^2$.

(c) $4x_1^2 + 12x_1x_2 + 13x_2^2$.

positive
semidefinite

$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

eigenvalues

$LPM : \frac{LP}{LP}$

echelon form

$\det A_2 = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0$
 \Rightarrow negative.

- ① $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & a \\ 1 & a & 2 \end{pmatrix} \in$

$$Q(x, y, z) = (x, y, z) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

② $LP M_1 = a_{11} = 1$ > 0

$$LPM_2 = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{1} > 0$$

$$= x^2 + 2xz + y^2 + \dots$$

$$= a_{13}xz + a_{31}xz$$

$$LPM_3 = \det A = 1 \cdot \begin{vmatrix} 1 & a \\ a & z \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & a \end{vmatrix}$$

$$= 2 - a^2 + (-1) = \underline{1 - a^2} > 0$$

$$\Rightarrow -1 < a < 1$$



$$a^2 < 1$$



$$|a| < 1 \Rightarrow -1 < a < 1$$

5. Find the limit of f as $(x, y) \rightarrow (0, 0)$ or show that the limit does not exist.

$$f(x, y) = \frac{2x}{x^2 + x + y^2}$$

$$l_1 = \{(x, 0), \quad x > 0\} \quad x\text{-axis}$$

$$l_2 = \{(0, y), \quad y > 0\} \quad y\text{-axis}$$

$$\lim_{(x,y) \xrightarrow{l_1} (0,0)} \frac{2x}{x^2 + x + y^2} = \lim_{x \rightarrow 0^+} \frac{2x}{x^2 + x + 0} = \lim_{x \rightarrow 0^+} \frac{2}{x+1} = 2$$

$$\lim_{(x,y) \xrightarrow{l_2} (0,0)} \frac{2x}{x^2 + x + y^2} = \lim_{y \rightarrow 0^+} \frac{2 \cdot 0}{0^2 + 0 + y^2} = 0$$

6. Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases} \Rightarrow \text{one of inputs is } 0.$$

- (a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.
 (b) Prove that f is not continuous at the origin.
 (c) Show that both partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ exist at the origin.

$$a) \quad \lim_{(x,y) \xrightarrow{l_1} (0,0)} f(x, y) = \lim_{\substack{x=y \\ x \rightarrow 0 \\ xy \neq 0, x \neq 0}} f(x, x) = \lim_{x \rightarrow 0} 0 = 0$$

$$b) \quad \underline{f(0,0) = 1} \neq \lim_{(x,y) \rightarrow (0,0)} f(x, y) \Rightarrow f \text{ is not continuous.}$$

$$c) \quad \frac{\partial f}{\partial x}(0,0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0,0)}{x} = \lim_{x \rightarrow 0} \frac{1 - 1}{x} = 0$$

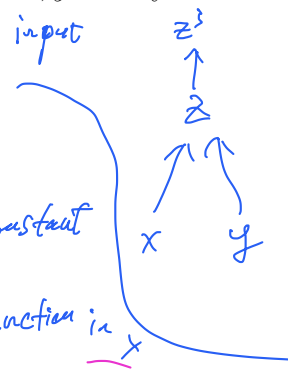
7. It is given that $f(x, y) = x \cos y + ye^x$. Find all the first and second order partial derivatives of f ,

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

8. Suppose $2z^3 - 2yz + x^2 = 1$ determines the function $z = z(x, y)$ as a function of x, y locally at $(x, y, z) = (1, 1, 1)$.

- (a) Find the linear approximation of z at $(x, y, z) = (1, 1, 1)$.
 (b) Find the quadratic surface approximation of z at $(x, y, z) = (1, 1, 1)$.

Take $\frac{\partial}{\partial x}$ for both sides, treating y as constant
 treating z as function in x



$$\frac{\partial}{\partial x} (2z^3 - 2yz + x^2) = \frac{\partial}{\partial x} 1$$

$$\Rightarrow 2 \cdot 3z^2 \cdot \frac{\partial z}{\partial x} - 2y \frac{\partial z}{\partial x} + 2x = 0$$

$$\begin{matrix} x=1 \\ y=1 \\ z=1 \end{matrix} \Rightarrow 6 \frac{\partial z}{\partial x} - 2 \frac{\partial z}{\partial x} + 2 = 0 \Rightarrow \frac{\partial z}{\partial x}(1, 1) = -\frac{2}{4}$$

Take $\frac{\partial}{\partial y}$ for both sides, treating x as constant
 ... z as function in y .

$$\frac{\partial}{\partial y} (2z^3 - 2yz + x^2) = \frac{\partial}{\partial y} 1$$

$$\Rightarrow 2 \cdot 3z^2 \cdot \frac{\partial z}{\partial y} - 2 \left(\frac{\partial z}{\partial y} \cdot z + y \cdot \frac{\partial z}{\partial y} \right) + 0 = 0$$

$$\begin{matrix} x=1 \\ y=1 \\ z=1 \end{matrix} \Rightarrow 6 \frac{\partial z}{\partial y} - 2 \cdot 1 - 2 \cdot 1 \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y}(1, 1) = \frac{2}{4}$$

$$\begin{aligned} L(x, y) &= z(1, 1) + \frac{\partial z}{\partial x}(x-1) + \frac{\partial z}{\partial y}(y-1) \\ &= 1 + \frac{-1}{2}(x-1) + \frac{1}{2}(y-1) \end{aligned}$$

9. It is given that $f(x, y) = e^{2x} \sin 2y$.
- (a) Use Taylor's formula to find a linear approximation of $f(x, y)$ at the origin.
 - (b) Estimate the error in the linear approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

10. Find the stationary points of the function $f(x, y) = xye^{-(x^2+2y^2)}$ and determine their nature.

Solution. From,

$$f_x = y(1 - 2x^2)e^{-x^2-2y^2} = 0, \quad f_y = x(1 - 4y^2)e^{-x^2-2y^2} = 0.$$

which is equivalent to solving $y(1 - 2x^2) = 0$ and $x(1 - 4y^2) = 0$. We get

$$f_y = \frac{1}{\sqrt{2}}(1-0) \neq 0$$

$$\begin{cases} y = 0 \text{ or } x = \pm \frac{1}{\sqrt{2}}, \\ x = 0 \text{ or } y = \pm \frac{1}{2}. \end{cases}$$

Hence, stationary points are $(0, 0), (\frac{1}{\sqrt{2}}, \frac{1}{2}), (\frac{1}{\sqrt{2}}, -\frac{1}{2}), (-\frac{1}{\sqrt{2}}, \frac{1}{2}), (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$.

Note that

$$\begin{aligned} f_{xx} &= 2xy(2x^2 - 3)e^{-x^2-2y^2}, \\ f_{yy} &= 4xy(4y^2 - 3)e^{-x^2-2y^2}, \\ f_{xy} &= (1 - 2x^2)(1 - 4y^2)e^{-x^2-2y^2}. \end{aligned}$$

Then,

$$D = f_{xx}f_{yy} - f_{xy}^2 = e^{-2x^2-4y^2}[8x^2y^2(2x^2 - 3)(4y^2 - 3) - (1 - 2x^2)^2(1 - 4y^2)^2].$$

We have Table 1 showing the nature of the stationary points.

point	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$	Nature
$(0, 0)$	0	0	1	$-1 < 0$	saddle point
$(\frac{1}{\sqrt{2}}, \frac{1}{2})$	$\frac{-\sqrt{2}}{e} < 0$	$\frac{-2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local max.
$(\frac{1}{\sqrt{2}}, -\frac{1}{2})$	$\frac{\sqrt{2}}{e} > 0$	$\frac{2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local min.
$(-\frac{1}{\sqrt{2}}, \frac{1}{2})$	$\frac{\sqrt{2}}{e} > 0$	$\frac{2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local min.
$(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$	$\frac{-\sqrt{2}}{e} < 0$	$\frac{-2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local max.

Table 1: Table for Q3

11. Let $f(x, y) = x^2 - xy + y^2 - y$. Find the directions \vec{u} and the values of $D_{\vec{u}}f(1, -1)$ for which

- (a) $D_{\vec{u}}f(1, -1)$ is the largest; \Leftrightarrow along \vec{u} f increases most rapidly $\Leftrightarrow \vec{u} \parallel \nabla f$.
 (b) $D_{\vec{u}}f(1, -1)$ is the smallest; \Leftrightarrow along \vec{u} f decreases most rapidly $\Leftrightarrow \vec{u} = \frac{\nabla f(1, -1)}{\|\nabla f(1, -1)\|}$.
 (c) $D_{\vec{u}}f(1, -1) = 0$;
 (d) $D_{\vec{u}}f(1, -1) = 4$;
 (e) $D_{\vec{u}}f(1, -1) = -3$.
- $\Leftrightarrow \vec{u} = -\frac{\nabla f(1, -1)}{\|\nabla f(1, -1)\|} = \frac{(-3, 4)}{5} = \frac{(3, -4)}{5}$

$$D_{\vec{u}}f(1, -1) = \nabla f(1, -1) \cdot \frac{\vec{u}}{\|\vec{u}\|}$$

inner product.

$$\begin{aligned}
 \nabla f(1, -1) &= (f_x, f_y) \Big|_{\substack{x=1 \\ y=-1}} \\
 &= (2x - y, -x + 2y - 1) \Big|_{\substack{x=1 \\ y=-1}} \\
 &= (3, -4)
 \end{aligned}$$

c). $\vec{u} = \frac{(-4, 3)}{5} \perp \nabla f$

12. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$.

13. If $\begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector of $A = \begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & c & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$. Determine the value for c and find eigenvalues and eigenvectors of A .