1 Introduction

Quantities of interest in real problems usually depend on more than one variable.

Examples

(a) The height of the ground at map coordinates (x, y) might be expressed mathematically as

Height $= h(x, y)$.

(b) The temperature in a lecture room varies with position and time,Temperature $= T(x, y, z, t)$.

(c) The volume of a rectangular box of length x, width y and height z is $V(x, y, z) = xyz$.

(d) The magnitude of the electric potential $u(x, y, z)$ due to a uniformly charged sphere with total charge Q is $u(x, y, z) = \frac{Q}{4\pi\varepsilon_0(x^2+y^2+z^2)}, x^2 + y^2 + z^2 > a^2$, where a is the radius of the sphere.

$\square$

These are all real-valued functions of real variables. In general we have

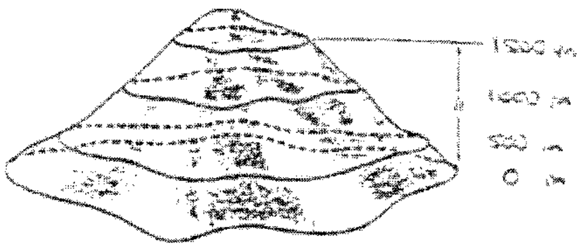
$z = f(x_1, x_2, \cdots, x_n) \equiv f\left(\vec{x}^T\right)$, where $x_1, x_2, \cdots, x_n$ are the independent variables and z is the dependent variable. $f\left(\vec{x}^T\right)$ is a scalar field.

It is possible to visualize a function of two variables (a surface in 3-D space) by drawing its graph. One way is to draw a contour map. A level curve of $f(x, y)$ is a set of points (x, y) for which $f(x, y) = c$, where c is a constant.

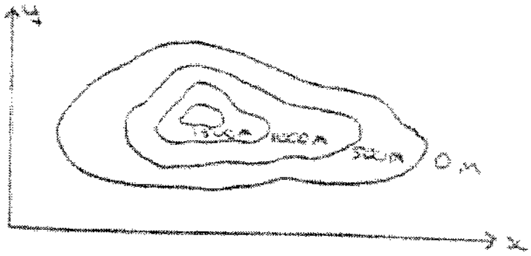
A collection of level curves then forms a contour map (as in the familiar contour maps produced by map makers).

Examples

(a)

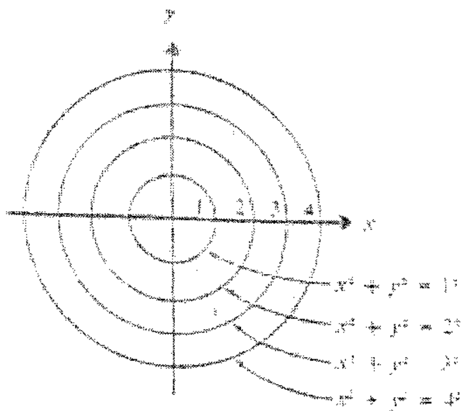


Level curves of a function are defined in the same manner as one obtains contour lines for a topographical map.



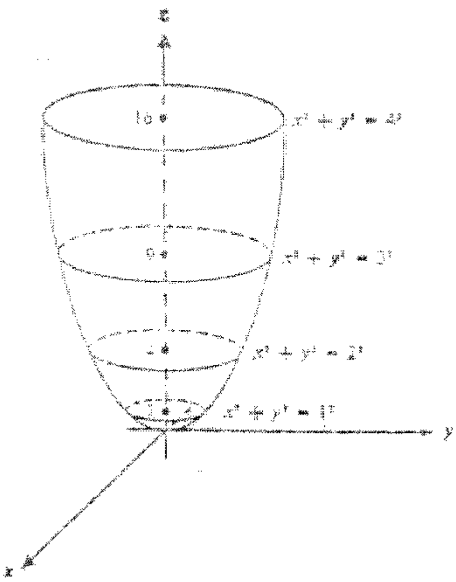
Contour map

(b)



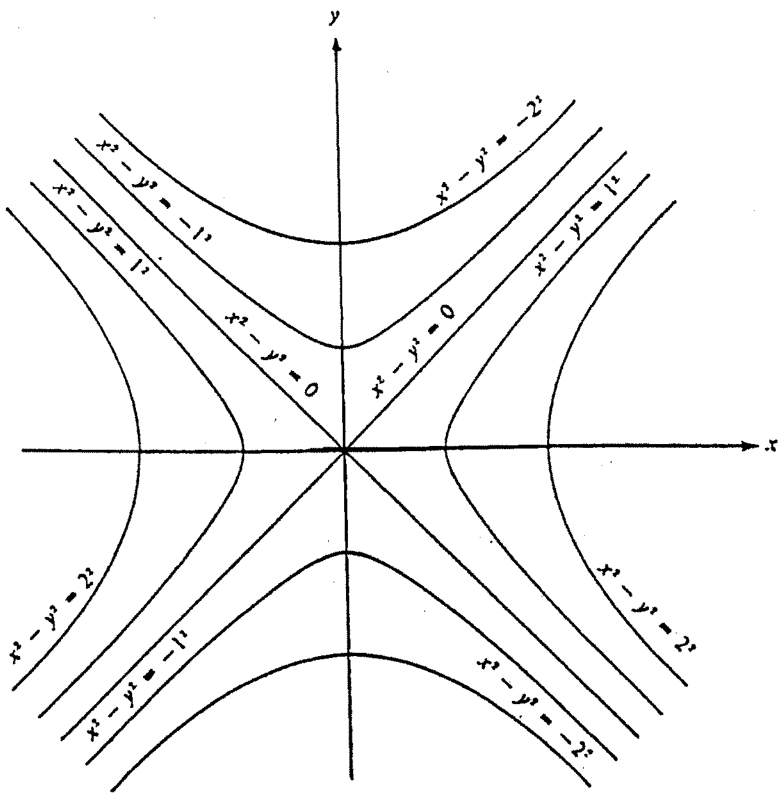
Contour map

Same level curves for the function $f(x, y) = x^2 + y^2$

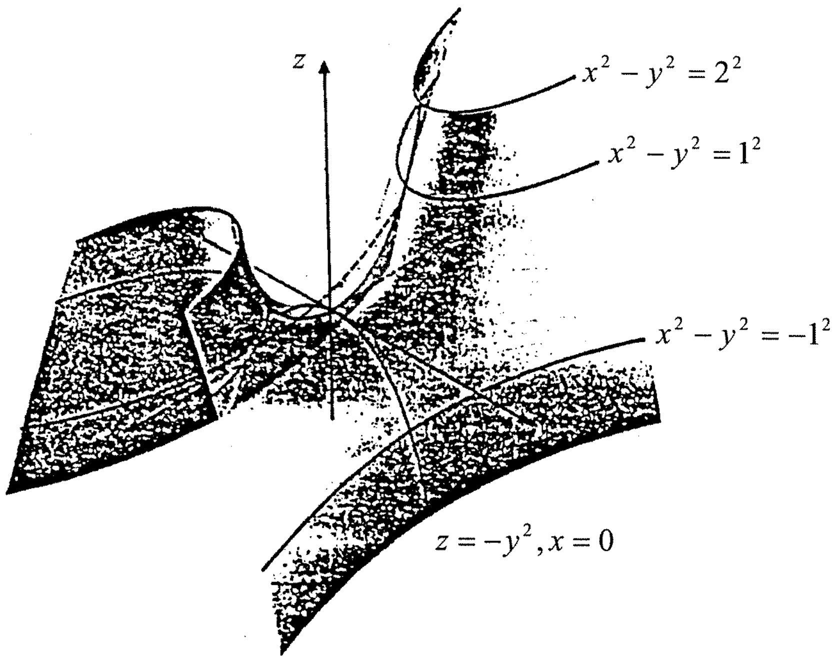


Level curves shown on the graph of $f(x,y) = x^2 + y^2$

(c)



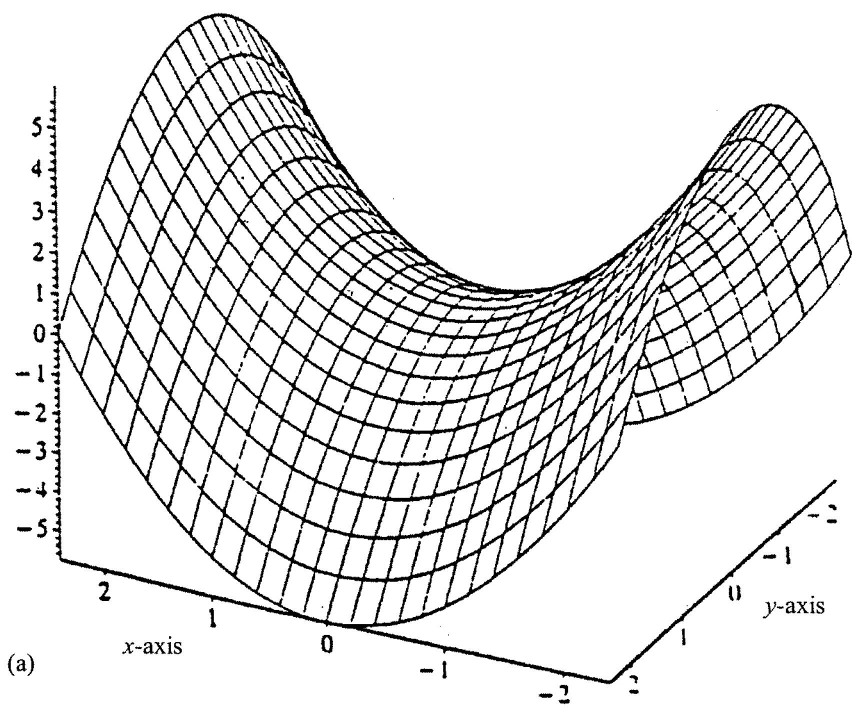
Level curves for the function $f(x,y) = x^2 - y^2$

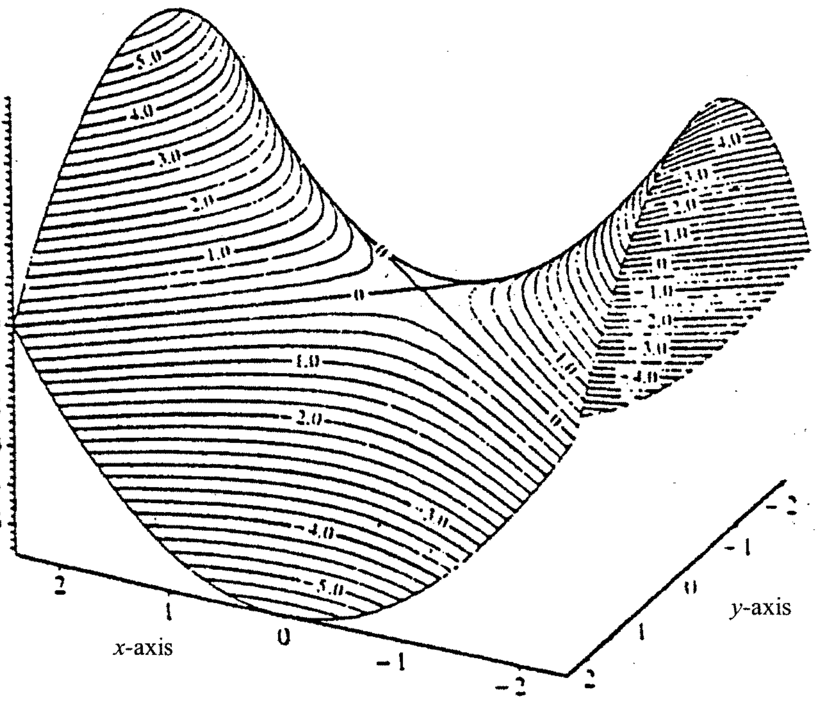


Some level curves on the graph of $f(x, y) = x^2 - y^2$

Computer generated graphs usually show lines for which x is constant and lines for which y is constant in order to give a picture of the surface.

<u>Examples</u>





(b)

Part (a) shows a computer-generated graph of $z = x^2 - y^2$. Part (b) shows this graph with level curves lifted to it.

$\square$

2 Limit, Continuity and Partial Derivative

The concepts of limit, continuity and derivative naturally generalize from the one variable situation to two or more variables.

A function $f(x, y)$ is said to possess a limit f_0 at the point (x_0, y_0) if for any $\varepsilon > 0$ it is possible to find a $\delta > 0$ such that whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, that is, (x, y) is in disc of radius δ centered at (x_0, y_0), we have $|f(x, y) - f_0| < \varepsilon$. We write $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f_0$ or $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} f(x, y) = f_0$. Intuitively, the above says that when (x, y) is approaching (x_0, y_0), $f(x, y)$ is approaching f_0 then the function $f(x, y)$ is said to possess a limit at the point (x_0, y_0).

Note that the limit exists only if, $f(x, y)$ is approaching f_0 independently of the manner in which (x, y) is approaching (x_0, y_0).

Example

$$\lim_{(x,y)\to(1,2)} (x^2 + 3y^2 + xy) = 1^2 + 3 \times 2^2 + 1 \times 2 = 15$$

□

Example

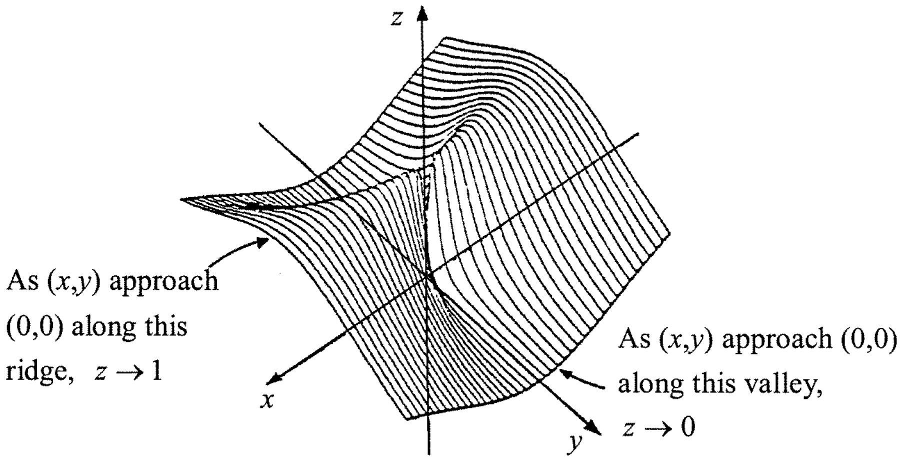
Show that $f(x, y) = \frac{x^2}{x^2+y^2}$ has no limit at the origin.

Solution:

As $(x, y) \to (0, 0)$ along the line $y = 0$, $\quad f(x, 0) = \frac{x^2}{x^2} = 1$ and the limiting value is 1.

As $(x, y) \to (0, 0)$ along the line $y = 0$, $\quad f(0, y)$ and the limiting value is 0.

Hence $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist.



$z = \frac{x^2}{x^2+y^2}, -1 \leq x \leq 1, -1 \leq y \leq 1$

□

Example

Show $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} = 0$

Solution:

Now,

$0 \leq \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} \leq \frac{x^2+y^2}{(x^2+y^2)^{\frac{1}{2}}} = (x^2+y^2)^{\frac{1}{2}}.$

We observe that $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)^{\frac{1}{2}} = 0$. Therefore, $\lim\limits_{(x,y)\to(0,0)} \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} = 0.$

□

A function $f(x,y)$ is said to be continuous at the point (x_0, y_0) if and only if it possesses a limit at (x_0, y_0) and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

Examples

(a) $f(x,y) = x^2 + 3y^2 + xy$ is continuous everywhere.

Solution:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} (x^2+3y^2+xy) = x_0^2 + 3y_0^2 + x_0 y_0.$$

(b) $f(x,y)=\frac{x^2}{x^2+y^2}$ is continuous everywhere except at the origin where $f(x,y)$ is not defined.

Solution:

If then $(x_0,y_0)\neq(0,0)$ is defined at (x_0,y_0) and $f(x_0,y_0)=\frac{x_0^2}{x_0^2+y_0^2}$. In addition,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=\lim_{(x,y)\to(x_0,y_0)}\frac{x^2}{x^2+y^2}=\frac{x_0^2}{x_0^2+y_0^2}=f(x_0,y_0).$$

(c) $f(x,y)=\begin{cases}\frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} & (x,y)\neq(0,0)\\ A & (x,y)=(0,0)\end{cases}$ is continuous everywhere iff $A=0$.

Solution:

Since $\lim_{(x,y)\to(0,0)}\frac{x^2}{(x^2+y^2)^{\frac{1}{2}}}=0$, $f(x,y)=\begin{cases}\frac{x^2}{(x^2+y^2)^{\frac{1}{2}}} & (x,y)\neq(0,0)\\ A & (x,y)=(0,0)\end{cases}$ is continuous everywhere iff $A=0$.

□

Example

Give an example of a function $f(x,y)$ such that $f(x,y)$ has the same limit as $f(0,0)$ when (x,y) approaches $(0,0)$ along any straight line passing through the origin, however, the function is not continuous at $(0,0)$.

Solution:

Consider the function:

$$f(x,y)=\begin{cases}\frac{2xy^2}{x^2+y^4} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}.$$

Suppose (x,y) is approaching $(0,0)$ along the line $x=t\cos\theta, y=t\sin\theta$ where θ is an angle made counterclockwise from the positive direction of x-axis to the line $x=t\cos\theta, y=t\sin\theta$ and t is any real number.

Then if $\cos\theta \neq 0$

$$\lim_{\substack{(x,y)\to(0,0) \\ x=t\cos\theta \\ y=t\sin\theta}} f(x,y) = \lim_{t\to 0} f(t\cos\theta, t\sin\theta) = \lim_{t\to 0} \frac{2t^3\cos\theta\sin^2\theta}{t^2\cos^2\theta + t^4\sin^4\theta} = \lim_{t\to 0} \frac{\frac{2t^3\cos\theta\sin^2\theta}{t^2}}{\frac{t^2\cos^2\theta + t^4\sin^4\theta}{t^2}}$$

$$= \lim_{t\to 0} \frac{2t\cos\theta\sin^2\theta}{\cos^2\theta + t^2\sin^4\theta} = \frac{0}{\cos^2\theta} = 0 = f(0,0)$$

If $\cos\theta = 0$, then this is the case that (x,y) is approaching $(0,0)$ along y-axis.

It is obvious that $\lim_{\substack{(x,y)\to(0,0) \\ x=0}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} 0 = 0 = f(0,0)$.

However, if is (x,y) approaching $(0,0)$ along the curve $x = t^2, y = t$ where t is any real number, we have $\lim_{\substack{(x,y)\to(0,0) \\ x=t^2 \\ y=t}} f(x,y) = \lim_{t\to 0} f(t^2, t) = \lim_{t\to 0} \frac{2t^4}{t^4+t^4} = 1 \neq 0 = f(0,0)$.

□

3 Partial Differentiation

Recall that for a function $f(x)$ of one variable

$$\frac{df}{dx}(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \text{ or } \frac{df}{dx}(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

Now consider the function $f(x,y)$ of two variables. If y is held constant, say $y = y_0$, then $f(x, y_0)$ becomes a function of one variable, x and we may define its derivative at x_0,

the partial derivative of f with respect to x at the point (x_0, y_0) as

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h} \text{ or } \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x\to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

Similarly, the partial derivative of f with respect to y at the point (x_0, y_0) as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k} \text{ or } \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y\to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

Note:

(a) A partial derivative is just an ordinary derivative with respect to the variable which is not held constant.

(b) For $z = f(x,y)$, alternative notations are:

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \frac{\partial z}{\partial x}(x,y) = z_x(x,y)$$

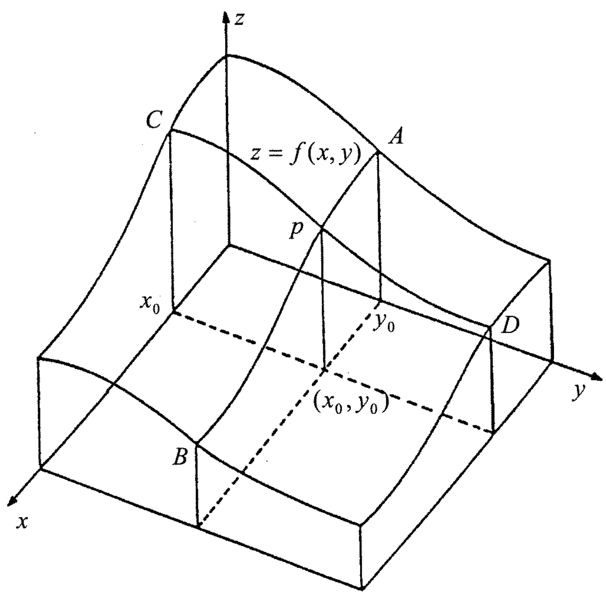


Figure 1: $AB-$ curve of $f(x, y_0)$, $CD-$ curve of $f(x_0, y)$

and similarly for $\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \frac{\partial z}{\partial y}(x,y) = z_y(x,y)$.

(c) The definition of a partial derivative only requires the idea of a limit of a function of a single variable.

Example

Find the first partial derivatives of $f(x,y) = x^2y + y^3 \sin x$.

Solution:

$f(x,y) = x^2y + y^3 \sin x \Rightarrow \frac{\partial f}{\partial x} = 2xy + y^3 \cos x, \frac{\partial f}{\partial y} = x^2 + 3y^2 \sin x$

□

Example

Evaluate $\frac{\partial f}{\partial x}(1,2)$ for $f(x,y) = x^2y + y^3 \sin x$

Solution:

Method 1:

Consider $f(x,2) = 2x^2 + 8 \sin x$. Then

$\frac{\partial f}{\partial x}(1,2) = \frac{d}{dx}\left(2x^2 + 8 \sin x\right)\Big|_{x=1} = (4x + 8 \cos x)|_{x=1} = 4 + 8 \cos 1.$

Method 2:

$f(x,y) = x^2y + y^3 \sin x \Rightarrow \frac{\partial f}{\partial x} = 2xy + y^3 \cos x$. Then $\frac{\partial f}{\partial x}(1,2) = (2xy + y^3 \cos x)|_{\substack{x=1 \\ y=2}} = 4 + 8\cos 1.$

$\square$

Example

Find the first partial derivatives of $f(x,y,z) = e^{2z} \cos xy$.

Solution:

$f(x,y,z) = e^{2z} \cos xy \Rightarrow \frac{\partial f}{\partial x} = -e^{2z} y \sin xy, \frac{\partial f}{\partial y} = -e^{2z} x \sin xy, \frac{\partial f}{\partial z} = 2e^{2z} \cos xy.$

$\square$

We may differentiate the first partial derivatives of a function again to obtain

second partial derivatives.

$f_{xx} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), f_{yy} = \frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$

Higher order partial derivatives may be similarly defined.

Example

Let $f(x,y) = \log(x^2+y^2)^{\frac{1}{2}}$, find its partial derivatives up to order two.

Solution:

$f(x,y) = \log(x^2+y^2)^{\frac{1}{2}} \Rightarrow f_x = \frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}, f_y = \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$

$\Rightarrow f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$, f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}, f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2}$

$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2}$

$\square$

Note that in the above example $f_{xy} = f_{yx}$, which is the case for most functions that are useful in engineering and science. However, it is not always true.

Example

$$f(x,y)=\begin{cases}\frac{xy\left(x^2-y^2\right)}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ \\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

(a)If $(x,y)\neq(0,0)$, find $f_{xy}(x,y), f_{yx}(x,y)$ and verify that they are equal.

(b)Show that $f_x(0,0)=f_y(0,0)$.

(c)Show that $f_{xy}(0,0), f_{yx}(0,0)$ exist but they are not equal.

Solution:

(a)

There is a theorem saying that if $f_{xy}(x,y)\&f_{yx}(x,y)$ exist at (x_0,y_0) and nearby and

$f_{xy}(x,y)\&f_{yx}(x,y)$ both are continuous at (x_0,y_0) then $f_{xy}\left(x_0,y_0\right)=f_{yx}\left(x_0,y_0\right)$.

Now, if $(x,y)\neq(0,0)$

$$f_{xy}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left[\frac{xy\left(x^2-y^2\right)}{x^2+y^2}\right]\right)=\frac{\partial}{\partial y}\left[\frac{4x^2y^3+x^4y-y^5}{\left(x^2+y^2\right)^2}\right]=\frac{-y^6-9x^2y^4+9x^4y^2+x^6}{\left(x^2+y^2\right)^3}$$

$$f_{yx}(x,y)=\frac{\partial}{\partial x}\left[\frac{-4y^2x^3-y^4x+x^5}{\left(x^2+y^2\right)^2}\right]=\frac{-y^6-9x^2y^4+9x^4y^2+x^6}{\left(x^2+y^2\right)^3}$$

So $f_{xy}(x,y)=f_{yx}(x,y)$

(b)

$$f_x(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to0}\frac{0-0}{x}=0,\quad f_y(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}=\lim_{x\to0}\frac{0-0}{y}=0$$

(c)

If $y\neq0$, $f_x(0,y)=\lim\limits_{x\to0}\frac{f(x,y)-f(0,y)}{x}=\lim\limits_{x\to0}\frac{\frac{xy\left(x^2-y^2\right)}{x^2+y^2}-0}{x}=\lim\limits_{x\to0}\frac{y\left(x^2-y^2\right)}{x^2+y^2}=-y$

$$\text{Thus, } f_x(0,y)=\begin{cases}-y & \text{if } y\neq0\\ \\ 0 & \text{if } y=0\end{cases}=-y,\forall y$$

$$f_{xy}(0,0)=\lim_{y\to0}\frac{f_x(0,y)-f_x(0,0)}{y}=\lim_{y\to0}\frac{-y-0}{y}=-1$$

Similarly,

If $x\neq0$, $f_y(x,0)=\lim\limits_{y\to0}\frac{f(x,y)-f(x,0)}{y}=\lim\limits_{y\to0}\frac{\frac{xy\left(x^2-y^2\right)}{x^2+y^2}-0}{y}=\lim\limits_{y\to0}\frac{x\left(x^2-y^2\right)}{x^2+y^2}=x$

Thus, $f_Y(x,0)=x$

$$f_{yx}(0,0)=\lim_{x\to0}\frac{f_Y(x,0)-f_Y(0,0)}{x}=\lim_{x\to0}\frac{x-0}{x}=1$$

We find that $f_{xy}(0,0)\neq f_{yx}(0,0)$. Why does this happen?

For $f_{xy}(x,y) = \begin{cases} \frac{-y^6-9x^2y^4+9x^4y^2+x^6}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0) \\ -1 & \text{if } (x,y) = (0,0) \end{cases}$

,we observe that along $y = 0$, $\lim\limits_{(x,y)\to(0,0)} f_{xy}(x,y) = \lim\limits_{x\to 0} f_{xy}(x,0) = 1$.

However, along $x = 0$, $\lim\limits_{(x,y)\to(0,0)} f_{xy}(x,y) = \lim\limits_{y\to 0} f_{xy}(0,y) = -1$.

It concludes that $f_{xy}(x,y)$ is not continuous at $(0,0)$.

$\square$

4 The Chain Rule

For functions of one variable, if $y = f(x)$ and $x = x(t)$, then $y = f[x(t)]$ and $\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$.

If $f = f(u)$ and $u = u(x, y)$ then $\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y}$.

Example

If $f = f(u) = \log u$ and $u = (x^2 + y^2)^{\frac{1}{2}}$, evaluate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

Solution:

$\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{1}{u} \times \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \times 2x = \frac{1}{(x^2+y^2)^{\frac{1}{2}}} \times \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} \times 2x = \frac{x}{x^2+y^2}$

And $\frac{\partial f}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

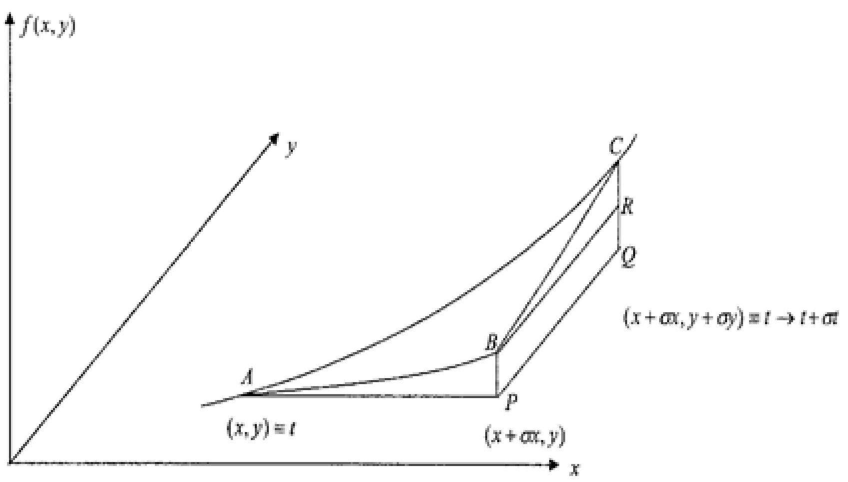
$\square$

For a function of two variables we have

Theorem

If $z = f(x, y)$ and $x = x(t), y = y(t)$ then $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

Proof:



$\frac{dz}{dt} = \lim\limits_{\delta t \to 0} \frac{z(t+\delta t) - z(t)}{\delta t} = \lim_{\delta t \to 0} \frac{f[x(t+\delta t), y(t+\delta t)] - f[x(t), y(t)]}{\delta t}$.

Now as t change to $t + \delta t$, x and y change by δx and δy respectively, where, $\delta x = x(t + \delta t) - x(t)$.

Thus the change in f may be expressed as

$f[x(t + \delta t), y(t + \delta t)] - f[x(t), y(t)] = f(x + \delta x, y + \delta y) - f(x, y) = QC = QR + RC = PB + RC$

$= [f(x + \delta x, y) - f(x, y)] + [f(x + \delta x, y + \delta y) - f(x + \delta x, y)]$

, and

$\frac{dz}{dt} = \lim\limits_{\delta t \to 0} \frac{z(t+\delta t)-z(t)}{\delta t} = \lim\limits_{\delta t \to 0} \frac{f[x(t+\delta t),y(t+\delta t)]-f[x(t),y(t)]}{\delta t}$

$= \lim\limits_{\delta t \to 0} \frac{f(x+\delta x,y)-f(x,y)+f(x+\delta x,y+\delta y)-f(x+\delta x,y)}{\delta t}$

$= \lim\limits_{\delta t \to 0} \left[\frac{f(x+\delta x,y)-f(x,y)}{\delta x} \frac{\delta x}{\delta t} + \frac{f(x+\delta x,y+\delta x)-f(x+\delta x,y)}{\delta y} \frac{\delta y}{\delta t} \right]$

$= \lim\limits_{\delta x \to 0} \frac{f(x+\delta x,y)-f(x,y)}{\delta x} \lim\limits_{\delta t \to 0} \frac{\delta x}{\delta t} + \lim\limits_{\delta y \to 0} \frac{f(x+\delta x,y+\delta x)-f(x+\delta x,y)}{\delta y} \lim\limits_{\delta t \to 0} \frac{\delta y}{\delta t}$

$= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$

<u>Example</u>

Let $z = f(x, y) = x^2 + 2xy$ and $x = \sin t$ and $y = \cos t$, find $\frac{dz}{dt}$.

Solution:

$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (2x + 2y) \cos t + 2x(-\sin t) = (2 \sin t + 2 \cos t) \cos t + 2 \sin t(-\sin t)$

$= \sin 2t + 2 \cos 2t$

This may be confirmed by direct substitution:

$z = f(\sin t, \cos t) = \sin^2 t + 2 \sin t \cos t \Rightarrow \frac{dz}{dt} = 2 \sin t \cos t - 2 \sin^2 t + 2 \cos^2 t = \sin 2t + 2 \cos 2t$

$\square$

A special case is when $z = f(x, y)$ and $y = y(x)$, that is, $x = x$ and hence z is a function of x only.

Then $\frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$.

Example

$z = f(x,y) = \tan^{-1}\left(\frac{x}{y}\right), y = \sin x$, find $\frac{dz}{dx}$

Solution:

$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\cos x = \frac{\sin x}{x^2+\sin^2 x} - \frac{x\cos x}{x^2+\sin^2 x} = \frac{\sin x - x\cos x}{x^2+\sin^2 x}$

□

Example (Chain rule involving second derivatives)

Let $z = f(x,y), x = x(t), y = y(t)$, evaluate $\frac{d^2z}{dt^2}$.

Solution:

$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right) = \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}$.

We observe that $\frac{\partial f}{\partial x}$ is depending on x and y directly, depending indirectly on t through x and y respectively. To evaluate $\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)$ we need the Chain Rule and we have

$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dt} = \frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y\partial x}\frac{dy}{dt}$

For the same reason, $\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt} = \frac{\partial^2 f}{\partial x\partial y}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}$

Then it follows that

$\frac{d^2z}{dt^2} = \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}$

$= \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dt}\right]\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}\right]\frac{dy}{dt}$

$= \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial^2 f}{\partial y\partial x}\frac{dy}{dt}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial^2 f}{\partial x\partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2$

□

Example

$z = f(x, y), x = t^2, y = t$. Suppose

$\frac{\partial f}{\partial x}(1,1) = \frac{\partial f}{\partial y}(1,1) = \frac{\partial^2 f}{\partial x^2}(1,1) = \frac{\partial^2 f}{\partial y^2}(1,1) = e$, $\frac{\partial^2 f}{\partial x \partial y}(1,1) = \frac{\partial^2 f}{\partial y \partial x}(1,1) = 2e$. Evaluate $\frac{d^2 z}{dt^2}$ for $t = 1$

Solution:

$x = t^2 \Rightarrow \frac{dx}{dt} = 2t, \frac{d^2x}{dt^2} = 2, y = t \Rightarrow \frac{dy}{dt} = 1, \frac{d^2y}{dt^2} = 0$. If $t = 1$, then $x = 1, y = 1$.

So

$$\left.\frac{d^2 z}{dt^2}\right|_{t=1} = \frac{\partial f}{\partial x}(1,1)\left.\frac{d^2 x}{dt^2}\right|_{t=1} + \frac{\partial^2 f}{\partial x^2}(1,1)\left.\left(\frac{dx}{dt}\right)^2\right|_{t=1} + \frac{\partial^2 f}{\partial y \partial x}(1,1)\left.\frac{dy}{dt}\right|_{t=1}\left.\frac{dx}{dt}\right|_{t=1}$$
$$+ \frac{\partial f}{\partial y}(1,1)\left.\frac{d^2 y}{dt^2}\right|_{t=1} + \frac{\partial^2 f}{\partial x \partial y}(1,1)\left.\frac{dx}{dt}\right|_{t=1}\left.\frac{dy}{dt}\right|_{t=1} + \frac{\partial^2 f}{\partial y^2}(1,1)\left.\left(\frac{dy}{dt}\right)^2\right|_{t=1}$$
$$= e \times 2 + e \times 4 + 2e \times 1 \times 2 + e \times 0 + 2e \times 2 \times 1 + e \times 1 = 15e$$

□

Theorem (Change of Variable)

If $z = f(x, y)$ and $x = x(s, t), y = y(s, t)$, find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$.

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

□

Example

If $z = f(x, y) = e^{2xy}$ and there is the change of variables $x = r\cos\theta, y = r\sin\theta$, find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$.

Solution:

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = 2ye^{2xy}\cos\theta + 2xe^{2xy}\sin\theta = 2re^{r^2\sin 2\theta}\sin 2\theta \text{ and}$$
$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = 2ye^{2xy}(-r\sin\theta) + 2xe^{2xy}r\cos\theta$$
$$= 2r^2e^{r^2\sin 2\theta}\cos 2\theta$$

Again these results may be checked by substitution and direct partial differentiation.

□

Example (Chain rule involving second partial derivatives)

Let $z = f(x, y), x = x(s, t), y = y(s, t)$, evaluate $\frac{\partial^2 z}{\partial t^2}$.

Solution:

$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right)$

$= \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)$

We observe that both $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are depending on x and y directly, depending indirectly on t through x and y respectively. To evaluate $\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y}\right)$ we need the Chain Rule and we have

$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial t}$

$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial t} = \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial t}$

It follows that

$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)$

$= \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial t^2} + \left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial t^2}$

$= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial t^2}$

$= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial f}{\partial y}\frac{\partial^2 y}{\partial t^2}$

□

Example

Consider $z = f(x, y)$, suppose $x = r\cos\theta, y = r\sin\theta$. Show that

$\frac{\partial^2 z}{\partial r^2} = \cos^2\theta\frac{\partial^2 f}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 f}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 f}{\partial y^2}$

Solution:

$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$

$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(\cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)$

$= \cos\theta\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right)$

$= \cos^2\theta\frac{\partial^2}{\partial x^2}f(x,y) + 2\sin\theta\cos\theta\frac{\partial^2}{\partial x\partial y}f(x,y) + \sin^2\theta\frac{\partial^2}{\partial y^2}f(x,y)$

$\square$

Example

Show that in polar coordinates (r,θ) in R^2, Laplace's equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ takes the form $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0$

Solution:

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$

$x = r\cos\theta, y = r\sin\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta, \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta$

Thus,

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \ \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta$

Therefore,

$$\begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}^{-1}\begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{pmatrix}$$

So $\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\cdots(*)$, $\quad \frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\cdots(**)$

From $(**)$, we learn that $\frac{\partial(f)}{\partial y} = \sin\theta\frac{\partial(f)}{\partial r} + \frac{\cos\theta}{r}\frac{\partial(f)}{\partial \theta}$. Notice that the f in the left side of $(**)$ is a function of x, y, however, the f in the right side of $(**)$ after x, y being replaced by $r\cos\theta$ and $r\sin\theta$, respectively, is a function of r, θ. Let $\frac{\partial u}{\partial y} = f$, therefore, we have

In addition, we notice that $\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$

Therefore,

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$
$= \sin\theta\left(\sin\theta\frac{\partial^2 u}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\cos\theta}{r^2}\frac{\partial u}{\partial \theta}\right) + \frac{\cos\theta}{r}\left(\cos\theta\frac{\partial u}{\partial r} + \sin\theta\frac{\partial^2 u}{\partial\theta\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r}\right)$

$\cdots(1)$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)$$
$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$
$$= \cos\theta\left(\cos\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}\right) - \frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 u}{\partial\theta\partial r} - \sin\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial \theta^2} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$\cdots(2)$

Finally, (1)+(2) we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0$

□

5 Implicit Functions

Let $f(x,y)$ be a function of two independent variables and suppose $f_x(x,y) \& f_y(x,y)$ exist and are continuous at (x_0, y_0) and nearby, in addition, $f(x_0, y_0) = 0 \& f_y(x_0, y_0) \neq 0$. Then about (x_0, y_0) the equation $f(x,y) = 0$ determines an mplicit function $y = y(x)$ about x_0 with $y_0 = y(x_0)$ and $y = y(x)$ being differentiable at x_0.

The above is also applicable to system of equations:

The system of equations $\begin{cases} f_1(x_1, \cdots, x_n, y_1, \cdots, y_m) = 0 \\ \vdots \\ f_m(x_1, \cdots, x_n, y_1, \cdots, y_m) = 0 \end{cases}$ of $x_1, \cdots, x_n$ and the partial derivatives of $y_1, \cdots, y_m$ with respect to $x_i, 1 \leq i \leq n$ can be found.

Example

The equation $x^3 - xy^2 + yz^2 - z^3 = 5$ determine z as a function of x, y. Find $\frac{\partial z}{\partial x}$.

Solution:

Differentiating $x^3 - xy^2 + yz^2 - z^3 = 5$ with respect to x gives

$3x^2 - y^2 + 2zy\frac{\partial z}{\partial x} - 3z^2\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial x} = \frac{y^2 - 3x^2}{2zy - 3z^2}$ if $2zy - 3z^2 \neq 0$

□

Example

Find the gradient of the tangent $\frac{dy}{dx}$ at a point (x,y) of the conic

$$f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Solution:

y is defined as an implicit function of x by the equation

$z = f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

In this case $\frac{dz}{dx} = 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$ and we have $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ whenever $\frac{\partial f}{\partial y} \neq 0$; $\frac{dy}{dx} = -\frac{2ax + 2hy + 2g}{2hx + 2by + 2f}$ whenever $2hx + 2by + 2f \neq 0$.

Example

If $z = e^x \cos y$, while x and y are implicit functions of t defined by the equations

$x^3 + e^x - t^2 - t = 1, yt^2 + y^2 t - t + y = 0$ then find $\frac{dz}{dt}$ for $t = 0$.

Solution:

Differentiate $z = e^x \cos y$ with respect to t, we have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$ The equation

$x^3 + e^x - t^2 - t = 1$ defines a function $x = x(t)$ implicitly.

Differentiate both sides of $x^3 + e^x - t^2 - t = 1$ with respect to t. Then $3x^2\frac{dx}{dt} + e^x\frac{dx}{dt} - 2t - 1 = 0$.

And we have $\frac{dx}{dt} = \frac{2t+1}{3x^2+e^x}$, if $3x^2 + e^x \neq 0$.

We observe that for $x^3 + e^x - t^2 - t = 1, yt^2 + y^2 t - t + y = 0$, if $t = 0$ then $x = 0$.

Therefore, $\left.\frac{dx}{dt}\right|_{t=0} = \left.\frac{2t+1}{3x^2+e^x}\right|_{\substack{t=0\\x=0}} = 1$

Similarly, The equation $yt^2 + y^2 t - t + y = 0$ defines a function $y = y(t)$ implicitly. And we have

$2yt + t^2\frac{dy}{dt} + 2yt\frac{dy}{dt} + y^2 - 1 + \frac{dy}{dt} = 0$. This implies $\frac{dy}{dt} = \frac{1-y^2-2yt}{1+t^2+2yt}$.

For $yt^2 + y^2 t - t + y = 0$ if $t = 0$ then $y = 0$. Therefore, $\left.\frac{dy}{dt}\right|_{t=0} = \left.\frac{1-y^2-2yt}{1+t^2+2yt}\right|_{\substack{t=0\\y=0}} = 1$.

Also, $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = e^x \cos y\frac{dx}{dt} - e^x \sin y\frac{dy}{dt}$.

Finally, $\left.\frac{dz}{dt}\right|_{t=0} = \left. e^x \cos y\right|_{\substack{x=0\\y=0}} \left.\frac{dx}{dt}\right|_{t=0} - \left. e^x \sin y\right|_{\substack{x=0\\y=0}} \left.\frac{dy}{dt}\right|_{t=0} = 1$

□

Example

Suppose we solve $\begin{cases} x^2 + y^2 = \frac{1}{2}z^2 \\ x + y + z = 2 \end{cases}$ uniquely for x, y as functions of z near $x = 1, y = -1$. Find $\frac{dx}{dz}, \frac{dy}{dz}, \frac{d^2x}{dz^2}, \frac{d^2y}{dz^2}$ when $x = 1, y = -1, z = 2$.

Solution:

$$\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases} \Rightarrow \begin{cases} 2x\frac{dx}{dz}+2y\frac{dy}{dz}=z\cdots(1) \\ \frac{dx}{dz}+\frac{dy}{dz}+1=0\cdots(2) \end{cases} \Rightarrow \begin{cases} 2\left(\frac{dx}{dz}\right)^2+2x\frac{d^2x}{dz^2}+2\left(\frac{dy}{dz}\right)^2+2y\frac{d^2y}{dz^2}=1\cdots(3) \\ \frac{d^2x}{dz^2}+\frac{d^2y}{dz^2}=0\cdots(4) \end{cases}$$

Put $x=1, y=-1, z=2$ in (1)&(2), we have

$$\begin{cases} 2x\frac{dx}{dz}+2y\frac{dy}{dz}=z\cdots(1) \\ \frac{dx}{dz}+\frac{dy}{dz}+1=0\cdots(2) \end{cases} \Rightarrow 4\frac{dx}{dz}=0, 4\frac{dy}{dz}=-4 \Rightarrow \frac{dx}{dz}=0, \frac{dy}{dz}=-1$$

Put $x=1, y=-1, z=2$ and $\dfrac{dx}{dz}=0, \dfrac{dy}{dz}=-1$ in (3)&(4), we have $\dfrac{d^2x}{dz^2}=-\dfrac{1}{4}, \dfrac{d^2y}{dz^2}=\dfrac{1}{4}$.

□

<u>Example</u>

Suppose that near the point $(x,y,u,v)=(1,1,1,1)$ we can solve $\begin{cases} xu+yvu^2=2 \\ xu^3+y^2v^4=2 \end{cases}$ uniquely for u and v as functions $u(x,y), v(x,y)$ of x and y. Compute $\left(\frac{\partial u}{\partial x}\right)(1,1)$.

Solution:

Differentiate all equations with respect to x of $\begin{cases} xu+yvu^2=2 \\ xu^3+y^2v^4=2 \end{cases}$. We have

$$\begin{cases} u+x\frac{\partial u}{\partial x}+yv2u\frac{\partial u}{\partial x}+yu^2\frac{\partial v}{\partial x}=0 \\ u^3+x3u^2\frac{\partial u}{\partial x}+y^24v^3\frac{\partial v}{\partial x}=0 \end{cases}$$

Since $x=1, y=1, u=1, v=1$, we have

$$\begin{cases} 1+3\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0 \\ 1+3\frac{\partial u}{\partial x}+4\frac{\partial v}{\partial x}=0 \end{cases} \Rightarrow \begin{cases} 4+12\frac{\partial u}{\partial x}+4\frac{\partial v}{\partial x}=0 \\ 1+3\frac{\partial u}{\partial x}+4\frac{\partial v}{\partial x}=0 \end{cases} \Rightarrow 9\frac{\partial u}{\partial x}=-3 \Rightarrow \frac{\partial u}{\partial x}(1,1)=-\frac{1}{3}$$

□

Example (for reference only)

Functions $x = x(\rho,\phi,\theta), y = y(\rho,\phi,\theta), z = z(\rho,\phi,\theta)$ can be regarded as equations involving $x, y, z, \rho, \phi, \theta$. Assume $x = x(\rho,\phi,\theta), y = y(\rho,\phi,\theta), z = z(\rho,\phi,\theta)$ with appropriate differentiability may define ρ, ϕ, θ each as implicit functions of x, y, z.

The determinant $J \equiv \frac{\partial(y_1,\cdots,y_n)}{\partial(x_1,\cdots,x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$, is called the Jacobian of $y_1, \cdots, y_n$ with respect to $x_1, \cdots, x_n$

(a) Using Cramer's Rule, deduce that $\frac{\partial \rho}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(\phi,\theta)}, \frac{\partial \phi}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(\theta,\rho)}, \frac{\partial \theta}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(\rho,\phi)}$, where

$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$.

(b) Guess $\frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z}$, then prove your guess at $\frac{\partial \rho}{\partial v}$

(c) Let $\nabla \rho \equiv \frac{\partial \rho}{\partial x}\vec{i} + \frac{\partial \rho}{\partial y}\vec{j} + \frac{\partial \rho}{\partial z}\vec{k}$ and $\frac{\partial \vec{r}}{\partial \phi} \equiv \frac{\partial x}{\partial \phi}\vec{i} + \frac{\partial y}{\partial \phi}\vec{j} + \frac{\partial z}{\partial \phi}\vec{k}, \frac{\partial \vec{r}}{\partial \theta} \equiv \frac{\partial x}{\partial \theta}\vec{i} + \frac{\partial y}{\partial \theta}\vec{j} + \frac{\partial z}{\partial \theta}\vec{k}$, show that $\nabla \rho = \frac{\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}}{J}$

(d) Let $x = x(\rho,\phi,\theta) = \rho \sin\phi \cos\theta, y = y(\rho,\phi,\theta) = \rho \sin\phi \sin\theta, z = z(\rho,\phi,\theta) = \rho \cos\phi$. Evaluate $J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}, \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z}$.

Solution:

(a)

$x = x(\rho,\phi,\theta), y = y(\rho,\phi,\theta), z = z(\rho,\phi,\theta)$

$$\Rightarrow \begin{cases} 1 = \frac{\partial x}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial x}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial x} \\ 0 = \frac{\partial y}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial x} \\ 0 = \frac{\partial z}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} \end{cases} \underset{\text{Cramer's Rule}}{\Rightarrow} \frac{\partial \rho}{\partial x} = \frac{\det\begin{pmatrix} 1 & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ 0 & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ 0 & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}} = \frac{\frac{\partial(y,z)}{\partial(\phi,\theta)}}{\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}} = \frac{1}{J}\frac{\partial(y,z)}{\partial(\phi,\theta)}$$

$$, \frac{\partial \phi}{\partial x} = \frac{\det\begin{pmatrix} \frac{\partial x}{\partial \rho} & 1 & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & 0 & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & 0 & \frac{\partial z}{\partial \theta} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}} = \frac{-\frac{\partial(y,z)}{\partial(\rho,\theta)}}{\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}} = \frac{1}{J}\frac{\partial(y,z)}{\partial(\theta,\rho)}, \ \frac{\partial \theta}{\partial x} = \frac{\det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & 1 \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & 0 \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & 0 \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}} = \frac{1}{J}\frac{\partial(y,z)}{\partial(\rho,\phi)}$$

(b)

$\frac{\partial \rho}{\partial y} = \frac{\frac{\partial(z,x)}{\partial(\phi,\theta)}}{\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}} = \frac{1}{J}\frac{\partial(z,x)}{\partial(\phi,\theta)}, \frac{\partial \rho}{\partial z} = \frac{\frac{\partial(x,y)}{\partial(\phi,\theta)}}{\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}} = \frac{1}{J}\frac{\partial(x,y)}{\partial(\phi,\theta)}.$

$x = x(\rho,\phi,\theta), y = y(\rho,\phi,\theta), z = z(\rho,\phi,\theta)$

$$\Rightarrow \begin{cases} 0 = \frac{\partial x}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial x}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial x} \\ 1 = \frac{\partial y}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial x} \\ 0 = \frac{\partial z}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} \end{cases} \underset{\text{Cramer's Rule}}{\Rightarrow} \frac{\partial \rho}{\partial x} = \frac{\det\begin{pmatrix} 0 & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ 1 & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ 0 & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}} = \frac{-\frac{\partial(x,z)}{\partial(\phi,\theta)}}{\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}} = \frac{\frac{\partial(z,x)}{\partial(\phi,\theta)}}{\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}} = \frac{1}{J}\frac{\partial(z,x)}{\partial(\phi,\theta)}$$

(c)

$$\nabla\rho = \frac{\partial\rho}{\partial x}\vec{i} + \frac{\partial\rho}{\partial y}\vec{j} + \frac{\partial\rho}{\partial z}\vec{k} = \frac{1}{J}\frac{\partial(y,z)}{\partial(\phi,\theta)}\vec{i} + \frac{1}{J}\frac{\partial(z,x)}{\partial(\phi,\theta)}\vec{j} + \frac{1}{J}\frac{\partial(x,y)}{\partial(\phi,\theta)}\vec{k}$$

$$= \frac{\frac{\partial(y,z)}{\partial(\phi,\theta)}\vec{i}+\frac{\partial(z,x)}{\partial(\phi,\theta)}\vec{j}+\frac{\partial(x,y)}{\partial(\phi,\theta)}\vec{k}}{J} = \frac{\begin{vmatrix} \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta} \end{vmatrix}\vec{i}+\begin{vmatrix} \frac{\partial z}{\partial\phi} & \frac{\partial x}{\partial\phi} \\ \frac{\partial z}{\partial\theta} & \frac{\partial x}{\partial\theta} \end{vmatrix}\vec{j}+\begin{vmatrix} \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} \end{vmatrix}\vec{k}}{J}$$

$$= \frac{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \end{vmatrix}}{J} = \frac{\left(\frac{\partial x}{\partial\phi}\vec{i}+\frac{\partial y}{\partial\phi}\vec{j}+\frac{\partial z}{\partial\phi}\vec{k}\right)\times\left(\frac{\partial x}{\partial\theta}\vec{i}+\frac{\partial y}{\partial\theta}\vec{j}+\frac{\partial z}{\partial\theta}\vec{k}\right)}{J} = \frac{\frac{\partial\vec{r}}{\partial\phi}\times\frac{\partial\vec{r}}{\partial\theta}}{J}$$

(d)

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{pmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$\underset{\text{Expand along }3rd\text{ row}}{=} \cos\phi\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} - (-\rho\sin\phi)\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$= \cos\phi\left(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta\right) + \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta\right)$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi$$

$$\frac{\partial\rho}{\partial x} = \frac{1}{J}\frac{\partial(y,z)}{\partial(\phi,\theta)} = \frac{\begin{vmatrix} \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ -\rho\sin\phi & 0 \end{vmatrix}}{\rho^2\sin\phi} = \frac{\rho^2\sin^2\phi\cos\theta}{\rho^2\sin\phi} = \sin\phi\cos\theta$$

$$\frac{\partial\rho}{\partial y} = \frac{1}{J}\frac{\partial(z,x)}{\partial(\phi,\theta)} = \frac{\begin{vmatrix} -\rho\sin\phi & 0 \\ \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \end{vmatrix}}{\rho^2\sin\phi} = \frac{\rho^2\sin^2\phi\sin\theta}{\rho^2\sin\phi} = \sin\phi\sin\theta$$

$$\frac{\partial\rho}{\partial z} = \frac{1}{J}\frac{\partial(x,y)}{\partial(\phi,\theta)} = \frac{\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}}{\rho^2\sin\phi} = \frac{\rho^2\sin\phi\cos\phi\cos^2\theta+\rho^2\sin\phi\cos\phi\sin^2\theta}{\rho^2\sin\phi}$$

$$= \frac{\rho^2\sin\phi\cos\phi}{\rho^2\sin\phi} = \cos\phi$$

$\square$

6 Taylor's Theorem

If a function $f(x)$ of one variable is differentiable through order N in $a-l<x<a+l$ where $l>0$, then for any real number h such that $a-l<a+h<a+l$, we have

$f(a+h)=f(a)+\frac{h}{1!}f'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{N-1}}{(N-1)!}f^{(N-1)}(a)+R_N(h)$ where

$R_N(h)=\frac{h^N}{N!}f^{(N)}(a+\theta h)$ for some $0<\theta<1$.

OR

If a function $f(x)$ of one variable is differentiable through order $n+1$ in $a-l<x<a+l$ where $l>0$, then for any real number x such that $a-l<x<a+l$, we have

$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(N-1)}(a)}{(N-1)!}(x-a)^{N-1}+R_N(x)$ where

$R_N(x)=\frac{f^{(N)}[a+\theta(x-a)]}{N!}(x-a)^N$ for some $0<\theta<1$.

If $a=0$ we have Maclaurin's Theorem:

$f(h)=f(0)+\frac{h}{1!}f'(0)+\frac{h^2}{2!}f''(0)+\cdots+\frac{h^{N-1}}{(N-1)!}f^{(N-1)}(0)+R_N(h)$ with $R_N(h)=\frac{h^N}{N!}f^{(N)}(\theta h)$ for some $0<\theta<1$.

OR

$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(N-1)}(0)}{(N-1)!}x^{N-1}+R_N(x)$ with $R_N(x)=\frac{f^{(N)}(\theta x)}{N!}x^N$ for some $0<\theta<1$.

Example

Express the Maclaurin series of $\cos x$ (up to x^4) and $\frac{1}{1+y}$, hence find the Maclaurin series of $\frac{1}{\cos x}$ (up to x^4) .

Solution:

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$, $\frac{1}{1+y}=1-y+y^2-y^3+y^4-\cdots$.

$\frac{1}{\cos x} = \frac{1}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots} = \frac{1}{1+\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}$

$= 1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^2 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^3 + \cdots$

$= 1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!}\right) + \left(\frac{x^4}{4} + \cdots\right) + \cdots = 1 + \frac{x^2}{2!} + \frac{5x^4}{24} + \cdots$

(up to x^4).

□

Taylor's Theorem generalizes to a function of two variables $f(x, y)$ which has continuous partial derivatives of all orders up to N inside the rectangle with vertices $(a-p, b-q), (a-p, b+q)$, $(a+p, b-q), (a+p, b+q)$, where $p, q > 0$.

At the point $(a+h, b+k)$, where $a-p < a+h < a+p, b-q < b+k < b+q$, we have

$f(a+h, b+k) = f(a, b) + \frac{1}{1!}\left[hf_x(a, b) + kf_y(a, b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)\right]$

$+\frac{1}{3!}\left[h^3 f_{xxx}(a, b) + 3h^2 k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)\right] + \cdots + R_N(h, k)$

$= \sum_{i=0}^{N-1} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(a, b) + R_N(h, k)$

where

$R_N(h, k) = \frac{1}{N!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^N f(a+\theta h, b+\theta k) = \frac{1}{N!}\left[\sum_{i=0}^{N} \begin{pmatrix} N \\ i \end{pmatrix} h^i k^{N-i} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a+\theta h, b+\theta k)\right]$

$= \frac{1}{N!}\left[\sum_{i=0}^{N} \frac{N(N-1)\cdots(N-i+1)h^i k^{N-i}}{i!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a+\theta h, b+\theta k)\right]$

$= \sum_{i=0}^{N} \frac{h^i k^{N-i}}{i!(N-i)!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a+\theta h, b+\theta k)$

for some $0 < \theta < 1$.

If $R_N(h, k) \to 0$ as $N \to \infty$ we have the <u>Taylor Series</u> representation of $f(a+h, b+k)$.

<u>OR</u>

For the case of two variables:

$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] +$

$\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots + R_N(x,y)$

$= \sum_{i=0}^{N-1} \frac{1}{i!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^i f(a,b) + R_N(x,y)$

Where

$R_N(x,y) = \frac{1}{N!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^N f[a+\theta(x-a), b+\theta(y-b)]$

$= \frac{1}{N!}\left[\sum_{i=0}^{N} \left(\begin{array}{c} N \\ \\ i \end{array}\right) (x-a)^i (y-b)^{N-i} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a+\theta(x-a), b+\theta(y-b))\right]$

$= \frac{1}{N!}\left[\sum_{i=0}^{N} \frac{N(N-1)\cdots(N-i+1)(x-a)^i(y-b)^{N-i}}{i!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a+\theta(x-a), b+\theta(y-b))\right]$

$= \sum_{i=0}^{N} \frac{(x-a)^i(y-b)^{N-i}}{i!(N-i)!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a+\theta(x-a), b+\theta(y-b))$

If $R_N(x,y) \to 0$ as $N \to \infty$ we have the <u>Taylor Series</u> representation of $f(x,y)$.

Successive terms given by Taylor's Theorem give the tangent plane approximation to the surface $z = f(x,y)$ (Taking the first two terms of the series), a quadratic surface approximation (the first three terms), a cubic surface (the first four terms) etc, all meeting $z = f(x,y)$ at the point (a,b).

<u>Example</u>

Find the tangent plane and quadratic surface approximations given by Taylor's Theorem at the point $(1,0)$ for the cone $f(x,y) = (x^2+y^2)^{\frac{1}{2}}$. Hence estimate $f(0.9, 0.1)$.

Solution:

$f_x(x,y) = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}, f_y(x,y) = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$

$f_{xx}(x,y) = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}, f_{xy}(x,y) = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}, f_{yy}(x,y) = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$ which exist everywhere except at the origin $(0,0)$. The tangent plane approximation is:

$p_1(x,y) = f(1,0) + \frac{1}{1!}\left[(x-1)f_x(1,0) + (y-0)f_y(1,0)\right] \Rightarrow p_1(x,y) = x.$

The quadratic surface approximation is:

$p_2(x,y) = f(1,0) + \frac{1}{1!}\left[(x-1)f_x(1,0) + (y-0)f_y(1,0)\right]$

$+\frac{1}{2!}\left[(x-1)^2 f_{xx}(1,0) + 2(x-1)(y-0)f_{xy}(1,0) + (y-0)^2 f_{yy}(1,0)\right]$

$\Rightarrow p_2(x,y) = x + \frac{1}{2}y^2$

Hence, using the tangent plane $f(0.9, 0.1) \approx p_1(0.9, 0.1) = 0.9$. And, using the quadratic surface $f(0.9, 0.1) \approx p_2(0.9, 0.1) = 0.905$. The correct value is $f(0.9, 0.1) = 0.9055$ (to 4 s.f)

$\square$

Example

$z^3 - 2xz + y = 0$ determines an implicit function $z = z(x,y)$ of x and y about the point $(1,1)$. Use the Taylor theorem to expand $z = z(x,y)$ about the point $(1,1)$ up to the linear terms.

Solution:

Note that from $z^3 - 2xz + y = 0$ when $x = 1, y = 1$ we have $z = 1$.

$z^3 - 2xz + y = 0 \Rightarrow 3z^2\frac{\partial z}{\partial x} - 2z - 2x\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{2z}{3z^2-2x} \& \frac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=1}} = 2$

$z^3 - 2xz + y = 0 \Rightarrow 3z^2\frac{\partial z}{\partial y} - 2x\frac{\partial z}{\partial y} + 1 = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{-1}{3z^2-2x} \& \frac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=1}}$

$z(x,y) = z(1,1) + \left[\frac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=1}}(x-1) + \frac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=1}}(y-1)\right] + \cdots$

So $z(x,y) = 1 + 2(x-1) - 1(y-1) + \cdots$

$\square$

Example

Let $f(x)$ be a smooth function, that is, $f(x)$ has continuous derivatives up to any order.

Then the Taylor series of $f(x)$ about the point a up to order 2 is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Hence or otherwise, find the Taylor series of $g(x,y,z) = \cos(x+y+z) - \cos x \cos y \cos z$ about the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ up to order 2 .

Solution:

$\cos(x+y+z) = \cos x \cos(y+z) - \sin x \sin(y+z)$

$= \cos x(\cos y \cos z - \sin y \sin z) - \sin x(\sin y \cos z + \cos y \sin z)$

$= \cos x \cos y \cos z - \cos x \sin y \sin z - \sin x \sin y \cos z - \sin x \cos y \sin z$

Observe that

$\cos x = 1 - \frac{x^2}{2} + \cdots$, $\cos x = -\sin\left(x - \frac{\pi}{2}\right) + \cdots$, $\sin x = x - \cdots$, $\sin x = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \cdots$

$g(x,y,z) = \cos(x+y+z) - \cos x \cos y \cos z = -\cos x \sin y \sin z - \sin x \sin y \cos z - \sin x \cos y \sin z$

$= -\left(1 - \frac{x^2}{2} + \cdots\right)\left[1 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \cdots\cdots\right] 1 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + \cdots\cdots\bigg]$

$-(x - \cdots)\left[1 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \cdots\cdots\right] - \left(z - \frac{\pi}{2}\right) - \cdots\bigg] - (x -$

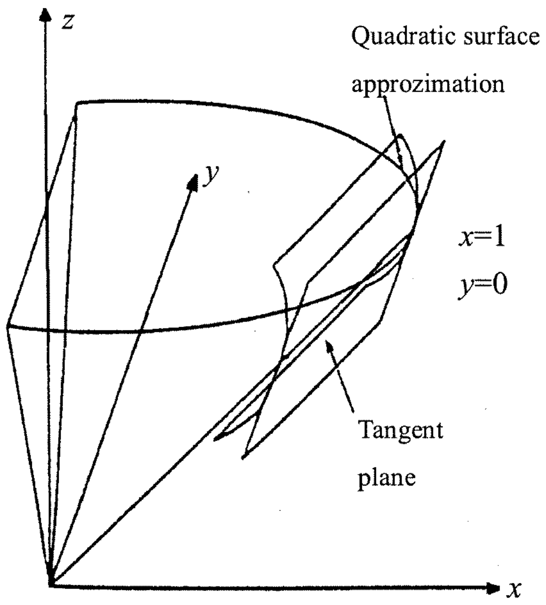
$\cdots)\left[-\left(y - \frac{\pi}{2}\right) - \cdots\right]\left[1 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + \cdots\cdots\right]$

$= -\left[1 - \frac{x^2}{2} - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2\right] - \left[-x\left(z - \frac{\pi}{2}\right)\right] - \left[-x\left(y - \frac{\pi}{2}\right)\right]$

$= -1 + \frac{x^2}{2} + \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + x\left(z - \frac{\pi}{2}\right) + x\left(y - \frac{\pi}{2}\right) + \cdots\cdots.$

$\square$

A bound for the error involved in approximating a function near a point (a,b) may be obtained by summing over estimates of the moduli of the terms in $R_N(x,y)$.



Linear and quadratic approximations to the cone $z = \sqrt{x^2 + y^2}, \quad (x, y) = (0, 0)$

Example (Estimation of Small Changes)

For (x, y) near (a, b) we may expect

$f(x, y) \approx f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b)f(x, y)$

Writing $f(x, y) - f(a, b) = \delta f, x - a = \delta x, y - b = \delta y$ So from

$f(x, y) - f(a, b) \approx (x - a)f_x(a, b) + (y - b)f_y(a, b)$ we have

$\delta f \approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$, which gives the approximate change in f due to small changes δx and δy in x and y respectively.

$\square$

Example

(a) Use Taylor's formula to find a linear approximation of $f(x,y) = \cos x \cos y$ near the origin.

(b) Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

Solution:

(a)

By Taylor's Theorem, we have

$f(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2}\left[x^2 f_{xx}(\theta x, \theta y) + 2xyf_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)\right]$, where

$0 < \theta < 1$.

Then the linear approximation of $f(x,y) = \cos x \cos y$ near the origin is

$p_1(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)]$.

And $f(0,0) = \cos x \cos y|_{\substack{x=0\\y=0}} = 1$, $f_x(0,0) = -\sin x \cos y|_{\substack{x=0\\y=0}} = 0$, $f_y(0,0) = -\cos x \sin y|_{\substack{x=0\\y=0}} = 0$.

We have $p_1(x,y) = 1$. If (x,y) is close to $(0,0)$, $f(x,y) = \cos x \cos y \approx p_1(x,y) = 1$.

(b)

The error in the approximation is

$E(x,y) = f(x,y) - f(0,0) - [xf_x(0,0) + yf_y(0,0)]$

$= \frac{1}{2}\left[x^2 f_{xx}(\theta x, \theta y) + 2xyf_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)\right]$

Observe that $f_{xx}(x,y) = -\cos x \cos y$, $f_{yy}(x,y) = -\cos x \cos y$, $f_{xy}(x,y) = \sin x \sin y$

$|E(x,y)| = |f(x,y) - f(0,0) - [xf_x(0,0) + yf_y(0,0)]|$

$= \left|\frac{1}{2}\left[x^2 f_{xx}(\theta x, \theta y) + 2xyf_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)\right]\right|$

$= \frac{1}{2}\left|\left[x^2 f_{xx}(\theta x, \theta y) + 2xyf_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)\right]\right|$

$|E(x,y)| \le \frac{1}{2}\left(|x^2 f_{xx}(\theta x,\theta y)| + |2xy f_{xy}(\theta x,\theta y)| + |y^2 f_{yy}(\theta x,\theta y)|\right)$

$= \frac{1}{2}\left(|x|^2\,|f_{xx}(\theta x,\theta y)| + 2|x||y|\,|f_{xy}(\theta x,\theta y)| + |y|^2\,|f_{yy}(\theta x,\theta y)|\right)$

$= \frac{1}{2}\left(|x|^2|-\cos\theta x\cos\theta y| + 2|x||y||\cos\theta x\cos\theta y| + |y|^2|-\cos\theta x\cos\theta y|\right)$

$= \frac{1}{2}\left(|x|^2|\cos\theta x||\cos\theta y| + 2|x||y||\cos\theta x||\cos\theta y| + |y|^2|\cos\theta x||\cos\theta y|\right)$

$\underset{\substack{|\cos\theta x|\le 1,\\ |\cos\theta y|\le 1}}{\le} \frac{1}{2}\left(|x|^2 + 2|x||y| + |y|^2\right) \underset{\substack{|x\le 0.1|,\\ |y\le 0.1|}}{\le} 2(0.1)^2 = 0.02$

□

<u>Example</u>

(a) Use Taylor's formula to find a quadratic approximation of $f(x,y) = e^x \sin y$ at the origin.

(b) Show that in the approximation the error $|E(x,y)| < 1.474 \times 10^{-3}$ if $|x| \le 0.1, |y| \le 0.1$ and $e^{0.1} \approx 1.1052$.

(c) Find an approximation of $e^{-0.09}\sin 0.06$ with an error less than 1.474×10^{-3}.

Solution:

(a)

By Taylor's Theorem, we have

$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

$+\frac{1}{6}\left[x^3 f_{xxx}(\theta x,\theta y) + 3x^2 y f_{xxy}(\theta x,\theta y) + 3xy^2 f_{xyy}(\theta x,\theta y) + y^3 f_{yyy}(\theta x,\theta y)\right]$

,where $0 < \theta < 1$.

Then the quadratic approximation of $f(x,y) = e^x \sin y$ at the origin is:

$p_2(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$

And

$f(0,0) = e^x \sin y|_{\substack{x=0\\ y=0}} = 0, f_x(0,0) = e^x \sin y|_{\substack{x=0\\ y=0}} = 0, f_y(0,0) = e^x \cos y|_{\substack{x=0\\ y=0}} = 1$

$f_{xx}(0,0) = e^x \sin y|_{\substack{x=0\\ y=0}} = 0, f_{xy}(0,0) = e^x \cos y|_{\substack{x=0\\ y=0}} = 1, f_{yy}(0,0) = -\,e^x \sin y|_{\substack{x=0\\ y=0}} = 0$

So $p_2(x,y) = f(0,0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right] = y + xy.$

If (x,y) is close to $(0,0)$, $f(x,y) = e^x \sin y \approx p_2(x,y) = y + xy$.

(b)

The error in the approximation is

$E(x,y) = f(x,y) - p_2(x,y) =$

$\frac{1}{6}\left[x^3 f_{xxx}(\theta x,\theta y) + 3x^2 y f_{xxy}(\theta x,\theta y) + 3xy^2 f_{xyy}(\theta x,\theta y) + y^3 f_{yyy}(\theta x,\theta y)\right]$

Observe that

$f_{xx}(x,y) = e^x \sin y, f_{xy}(x,y) = e^x \cos y, f_{yy}(x,y) = -e^x \sin y$

$\Rightarrow f_{xxx}(x,y) = e^x \sin y, f_{xxy}(x,y) = e^x \cos y, f_{xyy}(x,y) = -e^x \sin y, f_{yy}(x,y) = -e^x \cos y$

$|E(x,y)| = |f(x,y) - p_2(x,y)| =$

$\frac{1}{6}\left|\left[x^3 f_{xxx}(\theta x,\theta y) + 3x^2 y f_{xxy}(\theta x,\theta y) + 3xy^2 f_{xyy}(\theta x,\theta y) + y^3 f_{yy}(\theta x,\theta y)\right]\right|$

$\leq \frac{1}{6}\left[|x|^3 \left|f_{xxx}(\theta x,\theta y)\right| + 3|x|^2|y| \left|f_{xxy}(\theta x,\theta y)\right| + 3|x||y|^2 \left|f_{xyy}(\theta x,\theta y)\right| + |y|^3 \left|f_{yyy}(\theta x,\theta y)\right|\right]$

$= \frac{1}{6}\left[|x|^3 \left|e^{\theta x}\sin\theta y\right| + 3|x|^2|y| \left|e^{\theta x}\cos\theta y\right| + 3|x||y|^2 \left|-e^{\theta x}\sin\theta y\right| + |y|^3 \left|-e^{\theta x}\cos\theta y\right|\right]$

$= \frac{1}{6}\left[|x|^3 \left|e^{\theta x}\right| |\sin\theta y| + 3|x|^2|y| \left|e^{\theta x}\right| |\cos\theta y| + 3|x||y|^2 \left|e^{\theta x}\right| |\sin\theta y| + |y|^3 \left|e^{\theta x}\right| |\cos\theta y|\right]$

$0 < \theta < 1, -0.1 \leq x \leq 0.1 \Rightarrow -0.1 < \theta x < 0.1 \Rightarrow |\theta x| < 0.1$

Observe that $|\sin\theta y|, |\cos\theta y| \leq 1, \left|e^{\theta x}\right| = e^{\theta x} \leq e^{|\theta x|} < e^{0.1}$.

Thus,

$|E(x,y)| \leq \frac{1}{6}\left[|x|^3 \left|e^{\theta x}\right| |\sin\theta y| + 3|x|^2|y| \left|e^{\theta x}\right| |\cos\theta y| + 3|x||y|^2 \left|e^{\theta x}\right| |\sin\theta y| + |y|^3 \left|e^{\theta x}\right| |\cos\theta y|\right]$

$\underset{\substack{|\sin\theta y||\cos\theta y|\leq 1 \\ |x|,|y|\leq 0.1, \\ |e^{\theta x}|<e^{0.1}}}{<} \frac{1}{6}\left[(0.1)^3 e^{0.1} + 3(0.1)^2(0.1)e^{0.1} + 3(0.1)(0.1)^2 e^{0.1} + (0.1)^3 e^{0.1}\right] = \frac{0.008e^{0.1}}{6} \approx 1.474 \times 10^{-3}$

(c)

$f(-0.09, 0.06) = e^{-0.09}\sin 0.06 \approx p_2(-0.09, 0.06) = (y + xy)\big|_{\substack{x=-0.09 \\ y=0.06}} = 0.06 - 0.09 \times 0.06.$

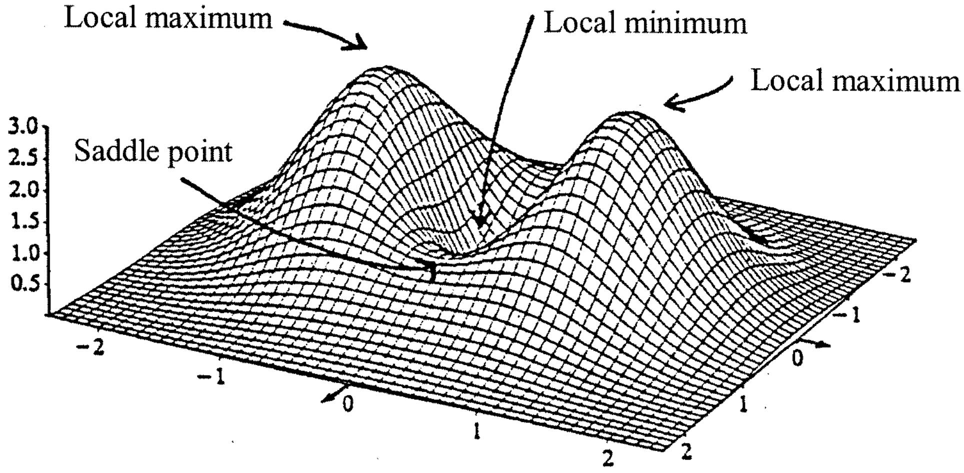
$= 0.06 \times 0.91 = 0.0546$

7 Maxima and Minima

A function $f(x, y)$ has a local maximum at a point (a, b) if for all points (x, y) close to (a, b), we $f(x, y) \leq f(a, b)$ and the point (a, b) is called a local maximum point.

A function $f(x, y)$ has a local minimum at a point (a, b) if for all points (x, y) close to (a, b), we have $f(x, y) \geq f(a, b)$ and the point (a, b) is called a local minimum point.

In a region the global (absolute) maximum (minimum) of a function is either the greatest (least) of the local maxima (minima) or occurs on the boundary of the region.



$$z = f(x, y) = (x^2 + 3y^2)\, e^{1-x^2-y^2}$$

For a local maximum or minimum at (a, b) the curves lying in the two planes $x = a$ and $y = b$ must also have a maximum or minimum at (a, b). Hence we must have $\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$

If this condition holds we say (a, b) is a stationary point.

It is possible to have a stationary point which is neither a local maximum nor a local minimum. If along one path through (a, b) the point is a local minimum, whilst along a different path it is a local maximum, we say that the point (a, b) is a saddle point. There is no equivalent point for functions of one variable.

Theorem

Assume $f(x,y)$ has the partial derivative up to the order we need and all partial derivatives are continuous, also $f_x(a,b)=f_y(a,b)=0$. If $f_{xx}(a,b)>0, f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b)>0$ for $f(x,y)$ at (a,b) then $f(x,y)$ has a local minimum there. If $f_{xx}(a,b)<0, f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b)>0$ for $f(x,y)$ at (a,b) then $f(x,y)$ has a local maximum there.

If $f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b)<0$, then $f(x,y)$ has a saddle point at (a,b).

Examples

(a) Determine the nature of the stationary points of $f(x,y)=x^2+2y^2$.

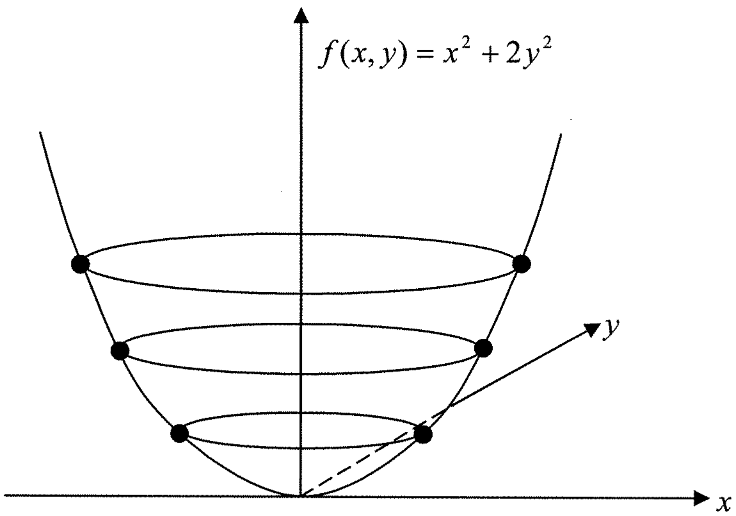
Solution:

$$f(x,y)=x^2+2y^2 \Rightarrow \begin{cases} f_x(x,y)=2x \\ f_y(x,y)=4y \end{cases}$$

So $f_x(x,y)=2x=0 \& f_y(x,y)=4y=0 \Rightarrow (x,y)=(0,0)$ is the only stationary point.

$f_{xx}(x,y)=2, f_{yy}(x,y)=4, f_{xy}(x,y)=0, f_{xx}(0,0)=2>0$ and $f_{xx}(0,0)f_{yy}(0,0)-f_{xy}^2(0,0)=8>0$.

$f(0,0)=0$ is a local minimum and $(0,0)$ is a local minimum point and clearly $f(x,y)\geq 0$ hence



$f(0,0)=0$ is a global minimum.

(b) Determine the nature of the stationary points of $f(x,y)=2x-x^2-2y^2$.

Solution:

$f_x(x,y)=2-2x, f_y(x,y)=-4y$

$f_x(x,y)=2-2x=0 \& f_y(x,y)=-4y=0 \Rightarrow (x,y)=(1,0)$, which is the only stationary point.

$f_{xx}(x,y) = -2, f_{yy}(x,y) = -4, f_{xy}(x,y) = 0, \quad f_{xx}(1,0) = -2 < 0$ and

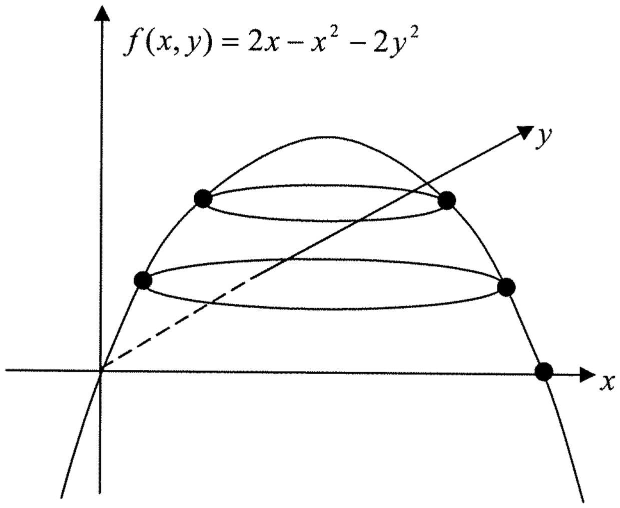
$f_{xx}(1,0)f_{yy}(1,0) - f_{xy}^2(1,0) = 8 > 0.$

$f(1,0) = 1$ is a local maximum. In addition,

$f(x,y) - f(1,0) = 2x - x^2 - 2y^2 - 1 = -(x-1)^2 - 2y^2 \leq 0$

Thus $f(x,y) \leq f(1,0)$ and $(1,0)$ is a global maximum point.

$\square$



(c) Determine the nature of the stationary points of $f(x, y) = x^2 - 2y^2$.

Solution:

$f_x(x, y) = 2x, f_y(x, y) = -4y$.

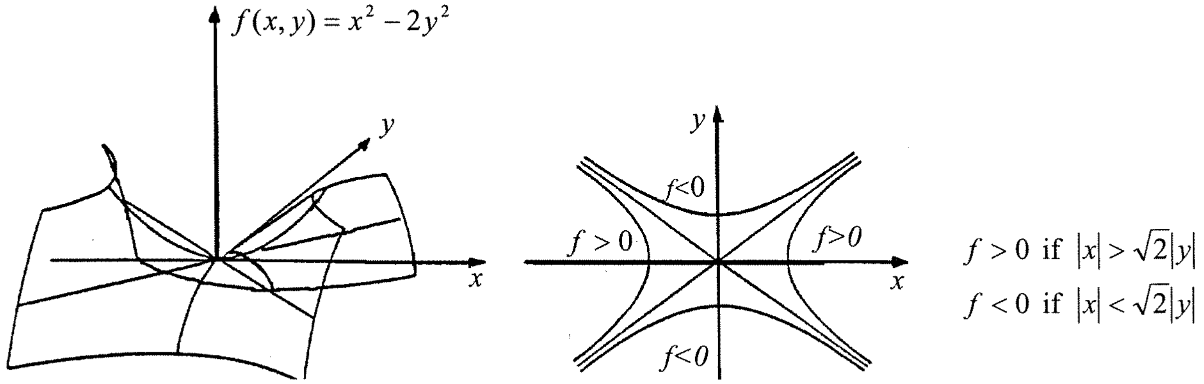
$f_x(x, y) = 2x = 0, f_y(x, y) = -4y = 0 \Rightarrow (x, y) = (0, 0)$, which is the only stationary point.

$f_{xx}(x, y) = 2, f_{yy}(x, y) = -4, f_{xy}(x, y) = 0$ Since $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = -8, (0, 0)$ is a saddle point.

$(0, 0)$ being a saddle point can also be observed by noting that $f(x, y)$ may be either positive or negative depending on the relative magnitudes of $|x|$ and $|y|$

If $y = 0, x \neq 0, \quad f(x, 0) - f(0, 0) = x^2 > 0$. If $x = 0, y \neq 0, \quad f(0, y) - f(0, 0) = -2y^2 \leq 0$.

Thus $(0, 0)$ is a saddle point



Example

Determine the nature of the stationary points of $f(x,y)=x^3+y^3-3(x+y)$.

Solution:

$f(x,y)=x^3+y^3-3(x+y) \Rightarrow \frac{\partial f}{\partial x}=3x^2-3, \frac{\partial f}{\partial y}=3y^2-3$

$$\begin{cases} \frac{\partial f}{\partial x}=3x^2-3=0 \\ \\ \frac{\partial f}{\partial y}=3y^2-3-0 \end{cases} \Rightarrow \begin{cases} x=\pm 1 \\ \\ y=\pm 1 \end{cases}$$

Hence there are 4 stationary points $(1,1),(1,-1),(-1,1),(-1,-1)$. In addition,

$\frac{\partial^2 f}{\partial x^2}=6x, \frac{\partial^2 f}{\partial y^2}=6y, \quad \frac{\partial^2 f}{\partial x \partial y}=0$

Stationary Point	$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial x\partial y}$	$\frac{\partial^2 f}{\partial y^2}$	$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$	Conclusion
$(1,1)$	6	0	6	36	Local Minimum Point
$(-1,-1)$	-6	0	-6	36	Local Maximum Point
$(1,-1)$	6	0	-6	-36	Saddle Point
$(-1,1)$	-6	0	6	-36	Saddle Point

□

Example

Determine the nature of the stationary points of $f(x,y) = x^4 + y^4$.

Solution:

$f(x,y) = x^4 + y^4 \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 4x^3 \\ \frac{\partial f}{\partial y} = 4y^3 \end{cases}$. We observe $(0,0)$ is the only stationary point.

And $\Delta(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = 0$ the test inconclusive, but clearly $(0,0)$ is a local minimum Point.

□

Example

Consider a point $P\left(x_0, y_0\right)$ and a straight line $y = ax + b$. The value $\left[y_0 - \left(ax_0 + b\right)\right]^2$ is called the square of the vertical displacement of the data point $P\left(x_0, y_0\right)$ from the line $y = ax + b$.

Determine the straight line $y = ax + b$ such that the sum S of the squares of the vertical displacements of the data points $P_1(1, 3.1), P_2(2, 6.1), P_3(3, 9.2)$ from the line $y = ax + b$ is the smallest.

In other words, determine the values of a, b such that S is the smallest where

$S = (3.1 - a - b)^2 + (6.1 - 2a - b)^2 + (9.2 - 3a - b)^2$

Solution:

$\mathrm{S(a,b)} = (3.1 - \mathrm{a} - \mathrm{b})^2 + (6.1 - 2\mathrm{a} - \mathrm{b})^2 + (9.2 - 3\mathrm{a} - \mathrm{b})^2$

$$\Rightarrow \begin{cases} 0 = \frac{\partial S}{\partial a} = -2[(3.1 - a - b) + 2(6.1 - 2a - b) + 3(9.2 - 3a - b)] \\ 0 = \frac{\partial S}{\partial b} = -2[(3.1 - a - b) + (6.1 - 2a - b) + (9.2 - 3a - b)] \end{cases}$$

$$\Leftrightarrow \begin{cases} \left(1 + 2^2 + 3^2\right) a + (1 + 2 + 3)b = 1 \times 1.3 + 2 \times 6.1 + 3 \times 9.2 \\ (1 + 2 + 3)a + 3b = 3.1 + 6.1 + 9.2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 14a + 6b = 42.9 \\ 6a + 3b = 18.4 \end{cases} \Leftrightarrow \begin{cases} 14a + 6b = 42.9 \\ 12a + 6b = 36.8 \end{cases} \Leftrightarrow \begin{cases} a = \frac{6.1}{2} = 3.05 \\ b = \frac{18.4 - 18.3}{3} = \frac{0.1}{3} = \frac{1}{30} \end{cases}$$

Then $S(a,b)=(3.1-a-b)^2+(6.1-2a-b)^2+(9.2-3a-b)^2 \Rightarrow \frac{\partial^2 S}{\partial a^2}=28, \frac{\partial^2 S}{\partial b^2}=6, \frac{\partial^2 S}{\partial b \partial a}=12$

Observe that $\frac{\partial^2 S}{\partial a^2}=28>0$ and

$\Delta\left(3.05, \frac{1}{30}\right)=S_{aa}\left(3.05, \frac{1}{30}\right) S_{bb}\left(3.05, \frac{1}{30}\right)-S_{ab}^2\left(3.05, \frac{1}{30}\right)=28 \times 6-144=168-144=24>0.$

It follows that $S(a,b)$ has a local minimum value at $a=3.05, b=\frac{1}{30}$.

Observe that $S(a,b)=(3.1-a-b)^2+(6.1-2a-b)^2+(9.2-3a-b)^2$ is a degree 2 polynomial in a,b and $\left(3.05, \frac{1}{30}\right)$ is the only local minimum point, $S(a,b)$ actually has the global minimum value $S\left(3.05, \frac{1}{30}\right)=\frac{1}{600}$ at $\left(3.05, \frac{1}{30}\right)$ and this fact can be shown as follows:

Due to Taylor's expansion theorem and noticing that $S(a,b)$ is a degree 2 polynomial in a,b, we have $S(a,b)-S\left(3.05, \frac{1}{30}\right)$

$$=\frac{1}{2}\left[S_{aa}\left(3.05, \tfrac{1}{30}\right)(a-3.05)^2+2S_{ab}\left(3.05, \tfrac{1}{30}\right)(a-3.05)\left(b-\tfrac{1}{30}\right)+S_{bb}\left(3.05, \tfrac{1}{30}\right)\left(b-\tfrac{1}{30}\right)^2\right]$$

$$\underset{\left.\frac{\partial^2 S}{\partial a^2}\right|_{\substack{x=3.05\\y=\frac{1}{30}}}=28,\ \left.\frac{\partial^2 S}{\partial b^2}\right|_{\substack{x=3.05\\y=\frac{1}{30}}}=6,\ \left.\frac{\partial^2 S}{\partial b\partial a}\right|_{\substack{x=3.05\\y=\frac{1}{30}}}=28}{=} \frac{1}{2}\left[28(a-3.05)^2+24(a-3.05)\left(b-\tfrac{1}{30}\right)+6\left(b-\tfrac{1}{30}\right)^2\right]$$

$$=\frac{1}{2}\left[\begin{pmatrix} a-3.05 & b-\frac{1}{30} \end{pmatrix}\begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix}\begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix}\right]$$

See that $\begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix}$ is a 2×2 real symmetric matrix which eigenvalues are $17 \pm \sqrt{265}(>0)$. Then there exist a 2×2 orthogonal matrix P and the diagonal matrix $D=\begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix}$ such that

$$\begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix}=P\begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix}P^{-1} \underset{p^{-1}=P^T}{=} P\begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix}P^{\mathrm{T}}$$

Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=P^{\mathrm{T}}\begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix}$. Then

$$\begin{pmatrix} a-3.05 & b-\frac{1}{30} \end{pmatrix} \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix}$$

$$= \begin{pmatrix} a-3.05 & b-\frac{1}{30} \end{pmatrix} P \begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix} P^{\mathrm{T}} \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix}$$

$$= \left[P^{\mathrm{T}} \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix} \right]^{\mathrm{T}} \begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix} P^{\mathrm{T}} \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix}$$

$$= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (17+\sqrt{265})y_1^2 + (17-\sqrt{265})y_2^2 \geq 0$$

It follows that $S(a,b) - S\left(3.05, \frac{1}{30}\right) = \frac{1}{2} \left[\begin{pmatrix} a-3.05 & b-\frac{1}{30} \end{pmatrix} \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix} \right] \geq 0.$

Therefore, $S(a,b)$ has the global minimum value $S\left(3.05, \frac{1}{30}\right) = \frac{1}{600}$ at $\left(3.05, \frac{1}{30}\right)$.

$\square$

Example

Find the global (absolute) maximum and global minimum values of the function

$f(x,y) = x^2 + y^2 - x - y + 1$ in the disk defined by $x^2 + y^2 \leq 1$.

Solution:

Remark: Since $f(x,y) = x^2 + y^2 - x - y + 1$ is a continuous function defined on a bounded closed disk $x^2 + y^2 \leq 1$, according to some theorems not mentioned $f(x,y)$ can attain global maximum and global minimum at stationary points or at those points where $f(x,y)$ do not have partial derivatives or at the points on the boundary.

Consider $f(x,y) = x^2 + y^2 - x - y + 1$ on $x^2 + y^2 = 1$, where $x^2 + y^2 = 1$ is the boundary of $x^2 + y^2 \leq 1$.

Let $\quad x = \cos t, y = \sin t, 0 \leq t \leq 2\pi, \quad$ consider

$g(t) \equiv f(\cos t, \sin t) = \cos^2 t + \sin^2 t - \cos t - \sin t + 1 = 2 - \cos t - \sin t$, where $0 \leq t \leq 2\pi$

Then $g'(t) = \sin t - \cos t = 0 \Rightarrow \sin t = \cos t \Rightarrow t = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

$g''(t) = \cos t + \sin t$ and $g''\left(\frac{\pi}{4}\right) > 0, g''\left(\frac{5\pi}{4}\right) < 0.$ $\quad g(0) = g(2\pi) = 1, g\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}, g\left(\frac{5\pi}{4}\right) = 2 + \sqrt{2}.$

It is obvious that $g(t)$ attains the greatest value $g\left(\frac{5\pi}{4}\right) = 2 + \sqrt{2}$ at $t = \frac{5\pi}{4}$ and the smallest value $g\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$ at $t = \frac{\pi}{4}$

When $t = \frac{5\pi}{4}, x = -\frac{\sqrt{2}}{2}, y = -\frac{\sqrt{2}}{2}.$

When $t = \frac{\pi}{4}, x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}$

Consider $f(x,y) = x^2 + y^2 - x - y + 1$ inside $x^2 + y^2 = 1.$

$$\begin{cases} \frac{\partial f}{\partial x} = 2x - 1 = 0 \\ \\ \frac{\partial f}{\partial y} = 2y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1/2 \\ \\ y = 1/2 \end{cases} \quad \text{and} \quad f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$$

$\Delta\left(\frac{1}{2}, \frac{1}{2}\right) = f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right) f_{yy}\left(\frac{1}{2}, \frac{1}{2}\right) - f_{xy}^2\left(\frac{1}{2}, \frac{1}{2}\right) = 4$ and $f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right) = 2$

$f(x,y) = x^2 + y^2 - x - y + 1$ has a local minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$

Compare $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$ and $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}.$

We conclude that $f(x,y) = x^2 + y^2 - x - y + 1$ has a absolute minimum value at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}.$

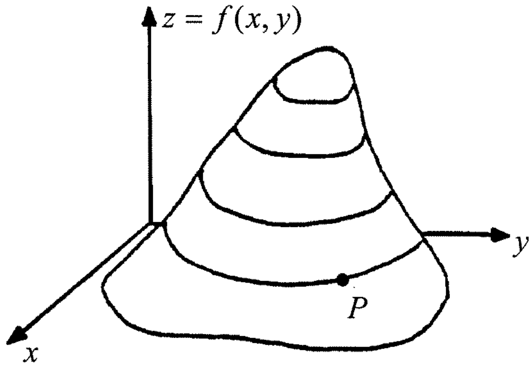
Also $f(x,y) = x^2 + y^2 - x - y + 1$ has a absolute maximum value at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2}$

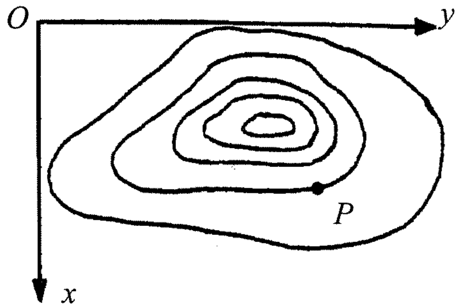
□

□

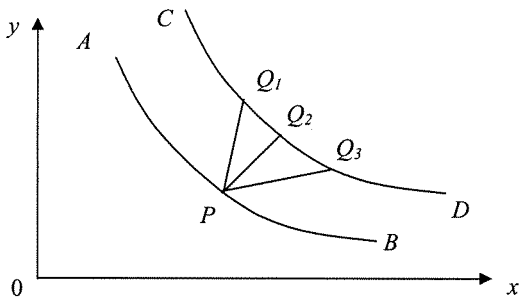
8 Directional Derivative and Gradient

For the scalar function $z = f(x, y)$ the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ measure the rates of change of $z = f(x, y)$ in the x and y directions respectively. We would like to know the rate of change in any direction.





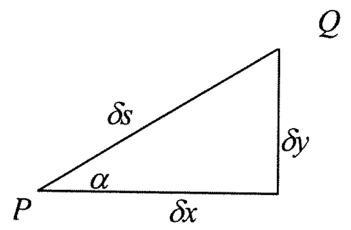
Consider a point P which lies on some level curve AB and a (variable) point Q on a nearby level curve CD.



The rate of change of f in the direction PQ is known as the <u>directional derivative</u> of $f(x, y)$ in the direction PQ and is $\frac{df}{ds} = \lim_{\delta s \to 0} \frac{\delta f}{\delta s}$ where $\frac{\delta f}{\delta s} = \frac{f(P) - f(Q)}{PQ}$, $PQ = \delta s$

Geometrically it is clear that $\frac{df}{ds}$ will be a maximum in the direction normal to the level curve at P (since $f(P) - f(Q)$ is constant and PQ is least in that direction). $\frac{df}{ds}$ is also denoted as $D_{\vec{v}} f$ where $\vec{v}$ is the unit vector in the direction PQ.

If the direction PQ is inclined an angle α to the x -axis we have



$$\delta x = \delta s \cos\alpha, \delta y = \delta s \sin\alpha$$

From Taylor's Theorem $\delta f \approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$, thus $\frac{\delta f}{\delta s} \approx \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha$

And as $\delta s \to 0$, $\frac{df}{ds} = D_{\vec{v}} f = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha \cdots (*)$, where $\vec{v} = \vec{i}\cos\alpha + \vec{j}\sin\alpha$.

By considering $\frac{df}{ds}$ as a function of α it is easy to show that $\frac{df}{ds}$ has a maximum value of $\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]^{\frac{1}{2}}$ when $\alpha = \tan^{-1}\left(\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}\right)$. If we define a vector $\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$ known as the gradient of f, its magnitude is that of the maximum value of $\frac{df}{ds}$ and its direction is that in which $\frac{df}{ds}$ its maximum value.

Hence ∇f gives us the magnitude and direction (normal to the level curve at P) of the greatest rate of change of f.

Also, from (*), if $\vec{v}$ is a unit vector in any other direction PQ, then $\vec{v} = (\cos\alpha)\vec{i} + (\sin\alpha)\vec{j}$ and $\frac{df}{ds} = D_{\vec{v}} f = \operatorname{grad} f \cdot \vec{v} = \nabla f \cdot \vec{v}$

The above discussion can be generalized to n independent variables. For example, if $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

then $D_{\vec{v}} f(a_1, a_2, a_3) = \begin{bmatrix} f_x(a_1, a_2, a_3) \\ f_y(a_1, a_2, a_3) \\ f_z(a_1, a_2, a_3) \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f_x(a_1, a_2, a_3)$

Now, we would to show that the gradient $\nabla f(x_1, \ldots, x_n) = \begin{bmatrix} f_{x_1}(x_1, \ldots, x_n) \\ \vdots \\ f_{x_n}(x_1, \ldots, x_n) \end{bmatrix} \neq \overrightarrow{0}$ of f at

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ points in the directions in which f increases most rapidly.

Let $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ be a unit vector i.e. $|\vec{v}| = 1$. Then by Cauchy-Schawz inequality, we have

$|D_{\vec{v}} f\left(a_1, a_2, a_3\right)| = |\nabla f\left(a_1, a_2, a_3\right) \cdot \vec{v}| \leq |\nabla f\left(a_1, a_2, a_3\right)| \, |\vec{v}| = |\nabla f\left(a_1, a_2, a_3\right)| \;|$

In addition, let $\vec{w} = \frac{\nabla f(a_1,\ldots,a_n)}{|\nabla f(a_1,\ldots,a_n)|}$.

Then,

$D_{-w} f\left(a_1, \ldots, a_n\right) = \nabla f\left(a_1, \ldots, a_n\right) \cdot \vec{w} = \nabla f\left(a_1, \ldots, a_n\right) \cdot \frac{\nabla f(a_1,\ldots,a_n)}{|\nabla f(a_1,\ldots,a_n)|}$

$= \frac{|\nabla f(a_1,\ldots,a_n)|^2}{|\nabla f(a_1,\ldots,a_n)|} = |\nabla f\left(a_1, \ldots, a_n\right)|$

Example

Let $f(x,y,z) = 10 + xy + xz + yz$ and $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T$. Find $D_{\vec{v}}f(1,1,1)$.

Solution:

Method 1 :

$D_{\vec{v}}f(1,1,1) = \lim\limits_{t\to 0} \frac{f\left(1+t\frac{1}{\sqrt{2}}, 1+t\frac{1}{\sqrt{2}}, 1\right) - f(1,1,1)}{t}$

$= \lim\limits_{t\to 0} \frac{10+\left(1+\frac{t}{\sqrt{2}}\right)\left(1+\frac{t}{\sqrt{2}}\right)+\left(1+\frac{t}{\sqrt{2}}\right)+\left(1+\frac{t}{\sqrt{2}}\right)-(10+1+1+1)}{t}$

$= \lim\limits_{t\to 0} \frac{\left(1+\frac{t}{\sqrt{2}}\right)^2+2\left(1+\frac{t}{\sqrt{2}}\right)-3}{t-0}$

Let $g(t) = \left(1+\frac{t}{\sqrt{2}}\right)^2 + 2\left(1+\frac{t}{\sqrt{2}}\right)$. Then

$D_{\vec{v}}f(1,1,1) = \lim\limits_{t\to 0} \frac{\left(1+\frac{t}{\sqrt{2}}\right)^2+2\left(1+\frac{t}{\sqrt{2}}\right)-3}{t-0} = \lim\limits_{t\to 0} \frac{g(t)-g(0)}{t-0}$

$= \frac{dg}{dt}\Big|_{t=0} = \left.\frac{d\left[\left(1+\frac{t}{\sqrt{2}}\right)^2+2\left(1+\frac{t}{\sqrt{2}}\right)\right]}{dt}\right|_{t=0} = \left[2\left(1+\frac{t}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right]\Big|_{t=0} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

Method 2 :

$g(t) = f\left(1+t\frac{1}{\sqrt{2}}, 1+t\frac{1}{\sqrt{2}}, 1\right)$. Then using chain rule, we have

$\frac{dg}{dt}\Big|_{t=0} = \frac{\partial f}{\partial x}(1,1,1)\frac{dx}{dt}(0) + \frac{\partial f}{\partial y}(1,1,1)\frac{dy}{dt}(0) + \frac{\partial f}{\partial z}(1,1,1)\frac{dz}{dt}(0)$

$$= \begin{bmatrix} \frac{\partial f}{\partial x}(1,1,1) \\ \frac{\partial f}{\partial y}(1,1,1) \\ \frac{\partial f}{\partial z}(1,1,1) \end{bmatrix} \cdot \begin{bmatrix} \frac{dx}{dt}(0) \\ \frac{dy}{dt}(0) \\ \frac{dz}{dt}(0) \end{bmatrix} = \left.\begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}\right|_{x=1,y=1,z=1} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$\square$

Example

Suppose that the temperature (in degree Celsius) in the point (x, y, z) is

$w = f(x, y, z) = \frac{1}{180}(7400 - 4x - 9y - 0.03xy) - 2z$. Suppose a hawk hovering at the point

$P(200, 200, 5)$ above the airport suddenly dives at a speed of 3 km/min in the direction specified by

the vector $(3, 4, -12)^T$. What instantaneous rate of change of temperature does the bird experience?

Solution:

The unit vector in the direction of the given vector $3\vec{i} + 4\vec{j} - 12\vec{k}$ is

$\vec{u} = \frac{3\vec{i}+4\vec{j}-12\vec{k}}{\sqrt{3^2+4^2+(-12)^2}} = \frac{3}{13}\vec{i} + \frac{4}{13}\vec{j} - \frac{12}{13}\vec{k}$

The temperature gradient vector $\nabla f(x, y, z) = -\frac{4+0.03y}{180}\vec{i} - \frac{9+0.03x}{180}\vec{j} - 2\vec{k}$ has the value

$\nabla f(200, 200, 5) = -\frac{4+0.03\times 200}{180}\vec{i} - \frac{9+0.03\times 200}{180}\vec{j} - 2\vec{k} = -\frac{10}{180}\vec{i} - \frac{15}{180}\vec{j} - 2\vec{k}$ at the initial position

$P(200, 200, 5)$ of the hawk. Therefore the hawk's initial rate of change of temperature

with respect to distance is:

$D_{\vec{u}}f(200, 200, 5) = \nabla f(200, 200, 5) \cdot \vec{u} = \left(-\frac{10}{180}\vec{i} - \frac{15}{180}\vec{j} - 2\vec{k}\right) \cdot \left(\frac{3}{13}\vec{i} + \frac{4}{13}\vec{j} - \frac{12}{13}\vec{k}\right) = 1.808\frac{{}^0C}{\text{km}}$

Its speed is $\frac{ds}{dt} = 3\frac{\text{km}}{\text{min}}$, so the time rate of change of temperature experienced by the hawk is

$\frac{dw}{dt} = \frac{dw}{ds}\frac{ds}{dt} = D_{-u}w\frac{ds}{dt} = \left(1.808\frac{{}^0C}{\text{km}}\right)\left(3\frac{\text{km}}{\text{min}}\right) = 5.424\frac{{}^0\text{C}}{\text{mim}}$

Thus the hawk initially gets warmer by almost 5.5 degrees per minute as it dives toward the ground.

$\square$

<u>Example (for reference only)</u>

Consider $f(x,y,z) = axy^2 + byz + cz^2x^3$.

(a) Determine the value of a, b, c such that the value of the partial derivative $\frac{\partial f}{\partial z}$ at $P(1,2,-1)$ is the greatest among all directional derivatives of f at $P(1,2,-1)$.

(b) Let a, b, c be the values found in (i), find the directional derivative $D_{\vec{v}}f(1,2,-1)$ with $\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k}$

Solution:

(a)

$f(x,y,z) = axy^2 + byz + cz^2x^3 \Rightarrow \nabla f(x,y,z) = (ay^2 + 3cz^2x^2)\,\vec{i} + (2axy + bz)\vec{j} + (by + 2czx^3)\,\vec{k}$

And $\nabla f(1,2,-1) = (4a+3c)\vec{i} + (4a-b)\vec{j} + (2b-2c)\vec{k}$.

In order the value of the partial derivative $\frac{\partial f}{\partial z}$ at $P(1,2,-1)$ is the greatest among all directional derivatives of f at $P(1,2,-1)$ we have

$$\begin{cases} 4a+3c=0 \\ 4a-b=0 \\ 2b-2c>0 \end{cases} \Rightarrow \begin{cases} c = -\frac{4a}{3} \\ b = 4a \\ b-c>0 \end{cases} \Rightarrow 4a + \frac{4a}{3} > 0 \Rightarrow \frac{16a}{3} > 0 \Rightarrow a > 0$$

Then $\nabla f(1,2,-1) = \left(8a + \frac{8a}{3}\right)\vec{k} = \frac{32a}{3}\vec{k}, a > 0$.

(b)

$D_{\vec{i}+2\vec{j}+3\vec{k}}f(1,2,-1) = \nabla f(1,2,-1) \cdot \frac{\vec{i}+2\vec{j}+3\vec{k}}{|\vec{i}+2\vec{j}+3\vec{k}|} = \frac{32a}{3}\vec{k} \cdot \frac{\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{14}} = \frac{32a}{\sqrt{14}}, \quad a > 0$

□

Example (for reference only)

Consider the function $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

Recall that $f(x,y)$ is not differentiable at $(0,0)$.

(a) Given $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ where $a^2+b^2=1$ and $a \neq 0, b \neq 0$, the straight line passing through the origin with direction $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is $\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} a \\ b \end{pmatrix}$, where t is any real number. Find the directional derivative $f_v(0,0)$.

(b) Show that the chain rule is not applicable to find $f_v(0,0)$.

Solution:

(a) $f(at,bt) = \begin{cases} \frac{ab^2}{a^2+b^2}t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \Rightarrow f(at,bt) = \frac{ab^2}{a^2+b^2}t$

Then $f_v(0,0) = \lim_{t \to 0} \frac{\frac{ab^2}{a^2+b^2}t}{t} = \frac{ab^2}{a^2+b^2}$.

$f(x,y)$ has directional derivatives in all directions.

(b)

By (a), $f_v(0,0)$ exists and $f_v(0,0) = \frac{ab^2}{a^2+b^2} \neq 0$ where $a^2+b^2=1$ and $a \neq 0, b \neq 0$.

However, $\nabla f(0,0) \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_x(0,0) \\ f_y(0,0) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0 \neq f_v(0,0) = \frac{ab^2}{a^2+b^2}$.

Remark: This example shows the fact that the chain rule is not applicable if f is not differentiable.

(For reference only)

Let $f : R^2 \to R$. We say $f(x,y)$ is differentiable at (x_0,y_0) if and only if there exists a 1×2 matrix T such that $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-f(x_0,y_0)-T\begin{pmatrix} x-x_0 \\ \\ y-y_0 \end{pmatrix}}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$

Observe that if $T=(a,b)$ then

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-f(x_0,y_0)-T\begin{pmatrix} x-x_0 \\ \\ y-y_0 \end{pmatrix}}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0 = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-f(x_0,y_0)-a(x-x_0)-b(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

<u>Theorem</u>

Suppose $f : R^2 \to R$ is differentiable at (x_0,y_0), then the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (x_0,y_0) and the 1×2 matrix T is given by $T=\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

<u>Theorem</u>

Suppose $f : R^2 \to R$ is differentiable at (x_0,y_0), then $f : \mathrm{R}^2 \to \mathrm{R}$ is continuous at (x_0,y_0).

<u>Example</u>

Consider the function $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

Show that

(a) $\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)$ exist.

Proof:

$\frac{\partial f}{\partial x}(0,0) = \lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim\limits_{x\to 0}\frac{0-0}{x-0} = 0;\quad \frac{\partial f}{\partial y}(0,0) = \lim\limits_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0}\frac{0-0}{y-0} = 0$

(b)

Show that $f(x,y)$ is not differentiable at $(0,0)$.

Proof:

We observe that if (x, y) is on the line $y = t, x = t$, then if $t \neq 0$

$$\frac{f(x,y)-f(0,0)-\left[\frac{\partial f}{\partial x}(0,0)\right](x-0)-\left[\frac{\partial f}{\partial y}(0,0)\right](y-0)}{\sqrt{(x-0)^2+(y-0)^2}} = \frac{f(t,t)}{\sqrt{t^2+t^2}} = \frac{t^3}{\frac{t^2+t^2}{\sqrt{2t^2}}} = \frac{1}{2\sqrt{2}}\frac{t}{|t|}$$

Suppose now (x, y) approaches $(0, 0)$ along the straight line $y = t, x = t$, $\lim_{t\to 0}\frac{1}{2\sqrt{2}}\frac{t}{|t|}$ does not exist.

It follows that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-f(x_0,y_0)-\left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0)-\left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \text{ does not exist.}$$

So $f(x, y)$ is not differentiable at $(0, 0)$.

$\square$
