Tutorial 11

1. Let L be the number of flip(s) of a coin in an experiment until the first head occurs. Given a hypothesis $\mathcal{H}$ that the coin is fair, it is proposed to reject $\mathcal{H}$ if $L > r$. Determine the value of r when the significance level is $\alpha \le 0.05$.

   What is the shortcoming of this significance test?

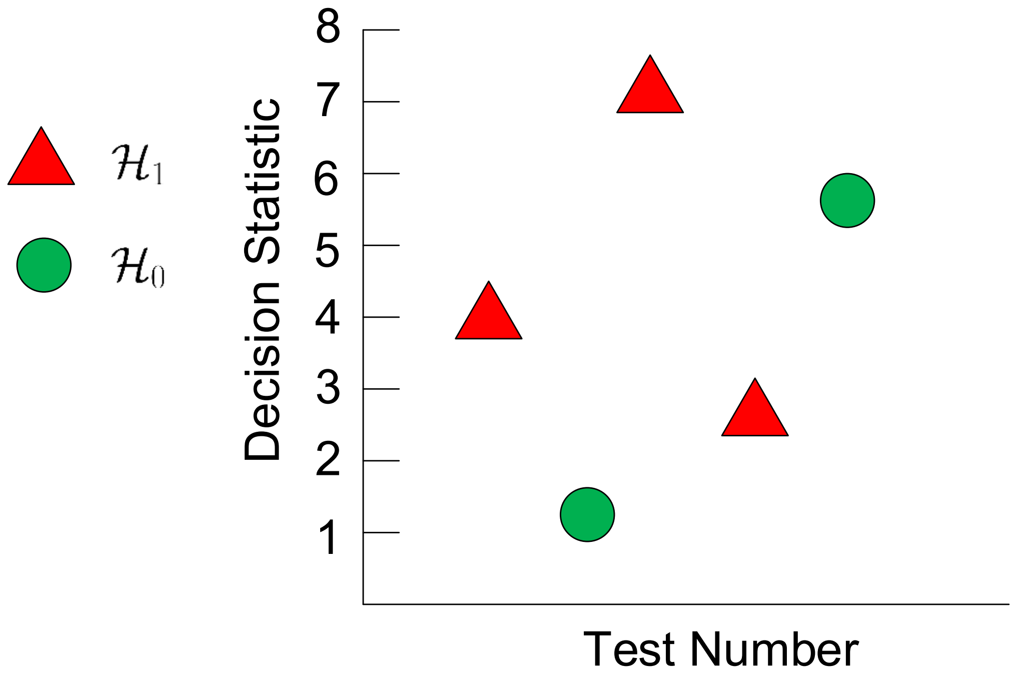
2. Suppose the time duration of a voice call is an exponential random variable t in min. and its probability density function (PDF) is:

$$p(t) = \begin{cases} \frac{1}{3}e^{-t/3}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Data calls tend to be longer than voice calls on average. A significance test is designed as follows. When a call is received, the hypothesis that the call is a voice call is rejected if the call duration is greater than t_0 min.

(a) Determine the mean value of voice call duration $\mathbb{E}\{T\}$.
(b) Express the significance level α in terms of t_0.
(c) Find the value of t_0 for $\alpha = 0.05$.

3. Suppose an experiment of binary detection is performed. There are 5 test cases where 3 and 2 correspond to signal presence $\mathcal{H}_1$ and signal absence $\mathcal{H}_0$, respectively. When the detection statistic is greater than a certain threshold, $\mathcal{H}_1$ is chosen, and otherwise $\mathcal{H}_0$ is chosen. Draw the receiver operating characteristic (ROC) curve.



4. Consider the binary hypothesis testing problem using a single observation x:

$$\mathcal{H}_0 : x \text{ corresponds to } p_0(x) = \lambda_0 e^{-\lambda_0 x},$$

$$\mathcal{H}_1 : x \text{ corresponds to } p_1(x) = \lambda_1 e^{-\lambda_1 x}, \lambda_1 > \lambda_0 > 0,\ x \geq 0$$

That is, we need to choose between two exponential distributions with parameters λ_0 and λ_1.

Based on the Neyman-Pearson theorem, suggest a decision statistic for this binary hypothesis test. It is assumed that λ_0 and λ_1 are unknown.

Solution

1.

Denote H and T as Head and Tail, respectively. The hypothesis of a fair coin means that $p(H) = p(T) = 0.5$.

Rejecting $\mathcal{H}$ if $L > r$ means that there is no head up to the rth trial, i.e., all tosses give T. Given $\mathcal{H}$, this probability is:

$$p(\text{all } r \text{ tosses give } T) = (0.5)^r = \alpha \leq 0.05$$

$$\Rightarrow \log((0.5)^r) \leq \log(0.05) \Rightarrow r \geq 4.32 = 5$$

That is, $\mathcal{H}$ is rejected if the number of tosses is at least 6. Or $\mathcal{H}$ is accepted if $L = 1, 2, 3, 4, 5$.

Note that for discrete random variables, we may not be able to set α exactly equal to an arbitrary value.

The experiment corresponds to geometric distribution. Recall (2.6) and Example 2.7:

$$p(r) = P(X = r) = (1-p)^{r-1}p, \quad 1 \leq r < \infty$$

$$F(r) = P(X \leq r) = \sum_{i=1}^{r} q^{i-1}p = \frac{p(1-q^r)}{1-q} = 1-(1-p)^r$$

With $p = 0.5$, the probability that $\mathcal{H}$ is accepted is:

$$F(5) = 1-(1-0.5)^5 = 1-(0.5)^5 = 0.9688$$

This aligns with

$$\alpha = (0.5)^5 = 0.0313$$

This significance test accepts $\mathcal{H}$ if $L = 1, 2, 3, 4, 5$. When the coin is biased to H with $p > 0.5$, $\mathcal{H}$ has a higher chance to be accepted, i.e., higher probability of accepting H when it is false.

2.(a)
Recall (2.17) and Question 3 of Tutorial 5:

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\mathbb{E}\{X\} = \frac{1}{\lambda}$$

Clearly, now $\lambda = 1/3$, hence we obtain:

$$\mathbb{E}\{T\} = 3$$

2.(b)

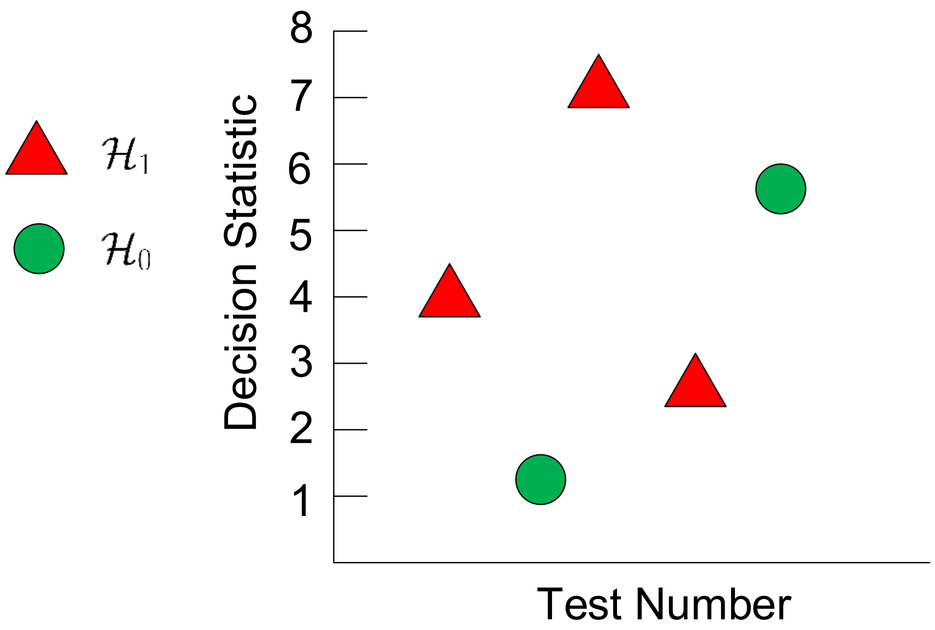
The rejection region can be expressed as $R = \{T > t_o\}$. As significance level α is the probability of rejecting $\mathcal{H}$ when it is true, we have:

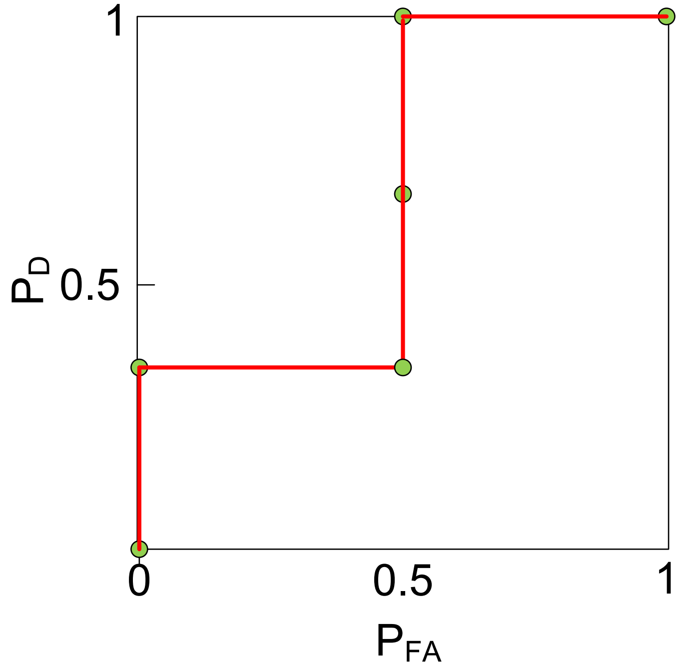
$$\alpha = P(T > t_0) = \int_{t_0}^{\infty} p(t)dt = \int_{t_0}^{\infty} \frac{1}{3} e^{-t/3} dt = e^{-t_0/3}$$

2.(c)

$$\alpha = 0.05 = e^{-t_0/3} \Rightarrow t_0 = -3\ln(0.05) = 8.9872$$

3.





We start with a threshold less than all decision statistics, producing the first ROC point $(P_{\text{FA}}, P_{\text{D}}) = (1, 1)$.

Increasing the threshold gradually, we then have the points: $(0.5, 1)$, $(0.5, 2/3)$, $(0.5, 1/3)$, $(0, 1/3)$, $(0, 0)$.

Connecting these 6 points yields the ROC curve.

4.

We apply (5.6) to obtain:

$$L(x) = \frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = \frac{\lambda_1 e^{-\lambda_1 x}}{\lambda_0 e^{-\lambda_0 x}} = \frac{\lambda_1}{\lambda_0} e^{-(\lambda_1-\lambda_0)x} > \gamma_{\rm NP}$$

$$\begin{aligned}
&\Rightarrow e^{-(\lambda_1-\lambda_0)x} > \frac{\lambda_0}{\lambda_1}\gamma_{\rm NP} \\
&\Rightarrow (\lambda_0 - \lambda_1)x > \ln(\lambda_0\gamma_{\rm NP}/\lambda_1) \\
&\Rightarrow x < \frac{\ln(\lambda_0\gamma_{\rm NP}/\lambda_1)}{\lambda_1 - \lambda_0} = \gamma, \quad \gamma \geq 0
\end{aligned}$$

That is, we can directly use the measurement x as the decision statistic. If $x < \gamma$, then we choose $\mathcal{H}_1$. Otherwise, $\mathcal{H}_0$ is chosen.

Since $\lambda_1 > \lambda_0$, this means that it is more probable a random variable drawn from $p_0(x)$ is larger than that from $p_1(x)$. Note also that the mean values of the random variables are $1/\lambda_1$ and $1/\lambda_0$, and apparently, $1/\lambda_0 > 1/\lambda_1$.

For example, the PDFs for $\lambda_1 = 5$ and $\lambda_0 = 0.5$ are illustrated as: