Complex Numbers

imaginary unit i with $i^2 := -1$

complex numbers $\mathbb{C} := \{a+bi \mid a,b \in \mathbb{R}\}$

$z \in \mathbb{C}$ has $\mathrm{Re}(z) := a$, $\mathrm{Im}(z) := b$,

$$r = |z| = \sqrt{x^2+y^2} = \sqrt{z \cdot \bar{z}}$$

$$\theta = \arg(z) = \arctan\left(\frac{y}{x}\right) \begin{cases} +0 & \text{if } Q1 \\ +\pi & \text{if } Q2/3 \\ +2\pi & \text{if } Q4 \end{cases}$$

cartesian form $z = a+bi$

trigonometric form $z = r(\cos\theta + i\sin\theta)$

exponential form $z = re^{i\theta}$

complex conjugate $\bar{z} = x - yi$

$|z| = |\bar{z}|$

$\overline{z_1+z_2} = \bar{z_1} + \bar{z_2}$

$\overline{z_1 \cdot z_2} = \bar{z_1} \cdot \bar{z_2}$

$\overline{z_1/z_2} = \bar{z_1}/\bar{z_2}$

$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

$|z_1/z_2| = |z_1|/|z_2|$

triangle inequality $|z_1+z_2| \le |z_1| + |z_2|$

multiplication and division

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ $\qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

exponentiation and taking roots

$z^n = r^n e^{in\theta}$

$\sqrt[n]{z} = \sqrt[n]{|r|}\, e^{i\frac{\theta+2k\pi}{n}}$ with $k \in \{0, \dots, n-1\}$

Gauss-Algorithm on LSEs $\quad Ax = b$

$m \times n$ matrix, i.e. m equations, n unknowns

- subtract multiples of the first equation from the other equations
- exchange equations to restore a pivot

$\Big\}$ => matrix in row echelon („Zeilenstufen") form of rank r (D1.30) (# pivots)

number of ops. $\approx \frac{1}{3} n^3$

Solutions (exist only if $r < m$ consistency conditions („Verträglichkeitsbedingungen") are met or if $r = m$):

- $r = n \rightarrow$ exactly one solution
- $r < n \rightarrow$ infinitely many solutions using $n - r$ parameters

(1.7) A nonsingular („regulär") $\Leftrightarrow m = n = r$

$\Leftrightarrow Ax = 0$ only has trivial solution

A singular $\Leftrightarrow m = n > r$

$\Leftrightarrow Ax = 0$ has nontrivial solutions

Triangular Factorization $\quad A$ invertible

Use Gauss to perform LU-decomposition:

$$A = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \end{pmatrix} \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} = L \cdot U$$

L stores multiply operations

U stores output (with pivots)

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} -\tfrac{1}{2}\cdot = \begin{pmatrix} 1 & 0 & 0 \\ \tfrac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & \tfrac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} -\tfrac{2}{3}\cdot$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \tfrac{1}{2} & 1 & 0 \\ 0 & \tfrac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & \tfrac{3}{2} & 1 \\ 0 & 0 & \tfrac{4}{3} \end{pmatrix}$$

Row exchanges must be captured before Gauss and articulated using a permutation matrix:

$$PA = LU \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ „swap the first and second rows"

Solve $Ax = b$ in two steps:

- $Lc = Pb$
- $Ux = c$

$\Big\}$ number of ops. $\approx n^2$

Direct formula (without row exchanges):

$l_{ii} = 1$

$$l_{ik} := \left(a_{ik} - \sum_{j=1}^{k-1} l_{ij} r_{jk}\right) \frac{1}{r_{kk}} \quad (k = 1 \dots i-1)$$

$$r_{ik} := \left(a_{ik} - \sum_{j=1}^{i-1} l_{ij} r_{jk}\right) \frac{1}{l_{ii}} \quad (k = i \dots n)$$

Matrix Structure & Arithmetics

Matrix $A \in \mathbb{K}^{m \times n}$ with entries a_{ij}

row $0 \le i \le m$, column $0 \le j \le n$

diagonal matrix $D = \begin{pmatrix} d_{11} & * \\ * & d_{nn} \end{pmatrix}$

identity matrix $I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

triangular matrices (see ←)

matrix-vector multiply ($A \in \mathbb{K}^{m \times n}$, $x \in \mathbb{K}^{n \times 1}$):

Ax

- by rows $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 2 \\ 3\cdot 1 + 4\cdot 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$
- by columns $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$

matrix-matrix-multiply ($A \in \mathbb{K}^{m\times n}$, $B \in \mathbb{K}^{n\times p}$):

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \qquad \begin{pmatrix} 1&2&3\\4&5&6\\7&8&9 \end{pmatrix} \begin{pmatrix} 1&2\\3&4\\5&6 \end{pmatrix} = \begin{pmatrix} 22&28\\49&64\\76&100 \end{pmatrix}$$

$$A \cdot B_j = (AB)_j \qquad -A_i- \cdot B = -(AB)_i-$$

column perspective — row perspective

S2.1) arithmetic rules:

$$(\alpha\beta)A = \alpha(\beta A)$$
$$(\alpha A)B = \alpha(AB) = A(\alpha B)$$
$$(\alpha+\beta)A = \alpha A + \beta A$$
$$\alpha(A+B) = \alpha A + \alpha B$$

$A+B = B+A$ commutativity of +

$(A+B)+C = A+(B+C)$ associativity of +

$(AB)C = A(BC)$ associativity of •

$$\left.\begin{aligned}(A+B)C &= AC+BC\\ A(B+C) &= AB+AC\end{aligned}\right\}\text{ distributive laws}$$

no commutativity of • in general!

linear combination of vectors $a_1, \dots, a_n$ using scalars $\alpha_1, \dots, \alpha_n$:

$$\alpha_1 a_1 + \dots + \alpha_n a_n$$

zero divisor A

$\Leftrightarrow$ $AB = 0$ or $BA = 0$ for $A, B \neq 0$

Inverses

If $A \in \mathbb{K}^{n\times n}$ and $Ax=b$ has a solution, A is called invertible:

$$Ax = b \iff x = A^{-1}b$$

Definition:

$$AA^{-1} = A^{-1}A = I$$

S2.18) Rules:

$$(A^{-1})^{-1} = A$$
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A^H)^{-1} = (A^{-1})^H$$

The Gauss-Jordan method can be used to determine A^{-1}!

$$(A\ I) \xrightarrow{\text{Gauss}} (U\ L^{-1}) \xrightarrow[\text{divide by pivots}]{\text{zeros above pivots}} (I\ A^{-1})$$

Symmetry & Transposition

Every $A \in \mathbb{R}^{m\times n}$ or $C \in \mathbb{C}^{m\times n}$ can be transposed to A^T or C^H:

$$(A^T)_{ij} = A_{ji} \qquad (C^H)_{ij} = \overline{C_{ij}}$$

S2.6) Rules:

$$(A^T)^T = A \qquad (C^H)^H = C$$
$$(\alpha A)^T = \alpha A^T \qquad (\alpha C)^H = \bar{\alpha} C^H$$
$$(A+B)^T = A^T + B^T \qquad (C+D)^H = C^H + D^H$$
$$(AB)^T = B^T A^T \qquad (CD)^H = D^H C^H$$

$A \in \mathbb{R}^{n\times n}$ is symmetrical if $A^T = A$
skew-symmetrical if $A^T = -A$
(„schiefsymmetrisch")

$C \in \mathbb{C}^{n\times n}$ is hermitical if $C^H = C$
skew-hermitical if $C^H = -C$
(„schiefhermitesch")

S2.7) A, B are symmetrical / hermitical and $AB = BA$
$\iff$ AB is symmetrical / hermitical

A^TA and AA^T are symmetrical for any A
A^HA and AA^H are hermitical for any A

Norm & Inner Product

A norm $\|\cdot\| : V \to \mathbb{R}$ must be

(N1) positive definite: $\|x\| \geq 0 \ \forall x \in V$
$\|x\| = 0 \iff x = 0$

(N2) homogenous: $\|\alpha x\| = |\alpha| \cdot \|x\|$
$\forall x \in V \ \forall \alpha \in \mathbb{E}$

(N3) able to satisfy the rectangle inequality:
$\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

An inner product (scalar product)
$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{E}$ must be

(S1) linear in the 2nd factor: $\forall x,y,z \in V \ \forall \alpha \in \mathbb{E}$
$\langle x, y+z \rangle = \langle x,y \rangle + \langle x,z \rangle$
$\langle x, \alpha y \rangle = \alpha \langle x,y \rangle$

(S2) symmetrical / hermitical:
$\langle x,y \rangle = \langle y,x \rangle$ / $\langle x,y \rangle = \overline{\langle y,x \rangle}$

(S3) positive definite: $\langle x,x \rangle \geq 0 \ \forall x,y \in V$
$\langle x,x \rangle = 0 \iff x = 0$

Note: $\langle x,x \rangle \in \mathbb{R}$, even if $x \in \mathbb{C}$.

Euclidian inner product and 2-Norm:

$\langle x,y\rangle_2 = x^H y = \sum_{k=1}^{n} \bar{x}_k y_k$ for $x,y\in\mathbb{C}^n$

$\langle x,y\rangle_2 = x^T y = \sum_{k=1}^{n} x_k y_k$ for $x,y\in\mathbb{R}^n$

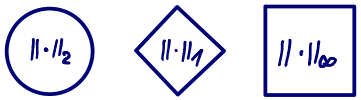
$\|x\|_2 = \sqrt{\langle x,x\rangle} = \sqrt{\sum_{k=1}^{n} x_k^2}$

P-Norm: $\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$

1-Norm: „Manhattan distance"

Maximum-Norm: $\|a\|_\infty = \max_{0\le x\le 1} |a(x)|$ for $a\in P$

unit spheres:



hermitical and positive definite (>0) matrices define inner products: $(u,v)\mapsto u^H A v$

induced norm $\|x\| = \sqrt{\langle x,x\rangle}$

In a vector space over $\mathbb{E}$, the norm represents length. The minimal angle is

$$\sphericalangle(x,y) = \arccos\frac{Re(\langle x,y\rangle)}{\|x\|\cdot\|y\|} \quad \text{for } x,y\in V$$

orthogonality $x\perp y \iff \langle x,y\rangle = 0$

S6.2) Pythagoras $\Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$

S6.1) Cauchy-Schwarz inequality $|\langle x,y\rangle| \le \|x\|\cdot\|y\|$

Orthogonal Projection

The outer product is a matrix defined by xy^T / xy^H.

A matrix $P_y\in\mathbb{E}^{n\times n}$ that orthogonally projects $x\in\mathbb{E}^n$ onto the line through $y\in\mathbb{E}^n$ by $P_y x$ is defined using

S2.15) $$P_y := \frac{1}{\|y\|^2} yy^H = \frac{yy^H}{\langle y,y\rangle} = uu^H \quad \left(u := \frac{y}{\|y\|}\right)$$

It satisfies $P_y x = \alpha y$ and $x - P_y x \perp y$.

Orthogonal Matrices

$A\in\mathbb{R}^{n\times n}$ is orthogonal if $A^TA = I \iff A^{-1} = A^T$

$C\in\mathbb{C}^{n\times n}$ is unitary if $C^HC = I \iff C^{-1} = C^H$

S2.20) A, B is unitary $\Rightarrow$ AB unitary

A unitary $\iff$ A^{-1} unitary

S2.21) Its projections are

- length preserving $\|Ax\| = \|x\|$
- angle preserving $\langle Ax,Ay\rangle = \langle x,y\rangle$

Vector Spaces

Definition: Space V over $\mathbb{K}$ with addition and scalar multiplication that satisfies

(V1) $x+y = y+x$ (V2) $(x+y)+z = x+(y+z)$

(V3) existance of 0 with $x+0 = x$

(V4) inverses for every x, i.e. $x+(-x) = 0$

(V5) $\alpha(x+y) = \alpha x + \alpha y$

(V6) $(\alpha+\beta)x = \alpha x + \beta x$

(V7) $(\alpha\beta)x = \alpha(\beta x)$

(V8) $1x = x$

S4.1) $0x = 0$
$\alpha 0 = 0$
$\alpha x = 0 \Rightarrow \alpha = 0$ or $x = 0$
$(-\alpha)x = \alpha(-x) = -(\alpha x)$

S4.2) subtraction, i.e. $x+z = y$ for all $x,y\in V$ for a unique z.

Definition: A subspace $U\subseteq V$ is

- closed under addition $\forall x,y\in U\,(x+y\in U)$
- closed under scalar multiplication $\forall x\in U\ \forall\alpha\in\mathbb{K}\,(\alpha x\in U)$
- in possession of the zero vector

S4.3) and also a vector space

A spanning set („Erzeugendensystem") $S\subseteq V$ generates a subspace (its linear hull) using

$$\operatorname{span} S := \left\{ \sum_{k=1}^{m} \gamma_k s_k \;\middle|\; m\in\mathbb{N},\ \gamma_i\in\mathbb{K},\ s_i\in S \right\}$$

Span generates a subspace that contains linear combinations of vectors in S.

Linear Dependence, Basis & Dimension

$v_i \in \mathbb{K}^n$ are **linearly independent** if for any $\gamma_i \in \mathbb{K}$: $\gamma_1 v_1 + \dots + \gamma_m v_m = 0$
$\Rightarrow \gamma_i = 0$ for all $i \le m$

$S \subseteq V$ is linearly independent if $\forall X \subseteq S$ all $v_i \in X$ are linearly independent

A vector space V has a **basis B** if $V = \operatorname{span} B$ and all vectors in B are linearly independent

S4.12) every $v \in V$ is a linear combination of vectors in B with unique coefficients that form the **coordinate vector ξ** of v regarding basis B with
$$v = \sum_{i=1}^{|B|} \xi_i b_i$$

L4.7) V has a finite spanning set $\Leftrightarrow$ every B has the same size, **dimension** $\dim V$

L4.8) $\operatorname{span}\{v_1, \dots, v_m\} = V$
$\Rightarrow w_1, \dots, w_n \in V$ with $n \le m$ are linearly independent

$U, U' \subseteq V$ are **complementary** and form V
$\Leftrightarrow \forall v \in V \ \exists$ unique $u \in U, u' \in U'$ with $v = u + u'$
$\Leftrightarrow V = U \oplus U'$ **direct sum** and $U \cap U' = \{0\}$

Basis transformation from B to B':

- Determine every $b'_k \in B'$'s coordinates regarding B, i.e. $b'_k = \sum_{i=1}^{|B|} \tau_{ik} b_i$
- Define the transformation matrix
$$T = \begin{pmatrix} | & | & & | \\ \tau_{i1} & \tau_{i2} & \cdots & \tau_{in} \\ | & | & & | \end{pmatrix} \in \mathbb{K}^{n \times n}$$

alternatively (for $\mathbb{K}^n$):
$$T = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} | & & | \\ b'_1 & \cdots & b'_n \\ | & & | \end{pmatrix}$$

Coordinate transformation from ξ regarding B to ξ' regarding B':
$$\xi = T\xi' \quad \text{and} \quad \xi' = T^{-1}\xi$$

Linear Transformations

$F: X \to Y$ is called linear if
- $F(x+y) = F(x) + F(y)$ for all $x, y \in X$
- $F(\gamma x) = \gamma F(x)$ $\quad \gamma \in \mathbb{K}$

By choosing appropriate bases, every linear transformation can be expressed using a matrix:

$$\begin{tikzcd} X \arrow[r, "F(x)"] \arrow[d, shift right, "k_X"'] & Y \arrow[d, shift right, "k_Y"'] \\ \mathbb{E}^n \arrow[r, "A\xi"] \arrow[u, shift right, "k_X^{-1}"'] & \mathbb{E}^m \arrow[u, shift right, "k_Y^{-1}"'] \end{tikzcd}$$

$F(x) = k_Y^{-1}(A \cdot k_X(x))$
$A\xi = (k_Y \circ F \circ k_X^{-1})(\xi)$

F **injective** $\Leftrightarrow f(x) = f(y) \Rightarrow x = y$ (S5.6)
$\Leftrightarrow \ker F = \{0\}$
$\Leftrightarrow \operatorname{rank} F = \dim X$

F **surjective** $\Leftrightarrow f(X) = Y$
$\Leftrightarrow \operatorname{im} F = Y$
$\Leftrightarrow \operatorname{rank} F = \dim Y$

F **bijective** $\Leftrightarrow F$ is an **isomorphism**
and $X = Y$ $\Leftrightarrow F$ is an **automorphism**

S5.9) $V_1 \cong V_2 \Leftrightarrow \dim V_1 = \dim V_2$

L5.3) The composition of two linear transformations is itself linear.

L5.5) $U \subseteq X$ is a subspace $\Rightarrow F(U) \subseteq Y$ too
$W \subseteq Y$ is a subspace $\Rightarrow F^{-1}(W) \subseteq X$ too

The **kernel** is the preimage of 0:
$\ker F := \{x \in X \mid F(x) = 0\}$
$\ker A := \{x \in \mathbb{E}^n \mid Ax = 0\}$
$= \mathcal{N}(A)$ **nullspace**

The **image** is the set of all reached values:
$\operatorname{im} F := f(X) = \{f(x) \in Y \mid x \in X\}$
$\operatorname{im} A := \{b \in \mathbb{E}^m \mid Ax = b, x \in \mathbb{E}^n\}$
$= \mathcal{C}(A)$ **column space**

The **rank** is the dimension of the image:
$\operatorname{rank} F/A := \dim(\operatorname{im} F/A)$

S5.7) **rank nullity theorem** („Dimensionsformel")
$$\dim X = \dim(\ker F) + \dim(\operatorname{im} F)$$

Affine Transformations

affine space $v_0 + U := \{v_0 + u \mid u \in U\}$
for $v_0 \in V$ subspace $U \subseteq V$

affine transformation $H: X \to Y$
$x \mapsto y_0 + F(x)$
for linear $F: X \to Y$

S5.19) Choose some $y_0 \in Y \setminus \{0\}$ (i.e. one particular solution $F(x)$). The solution set to F then is $x_0 + \ker F$.

The 4 Matrix Subspaces

$A \in \mathbb{E}^{m \times n}$

$C(A) = \operatorname{im} A \qquad N(A) = \ker A$
rowspace $A = C(A^H)$
left nullspace $A = N(A^H)$

$\dim(\operatorname{im} A) = \dim(\operatorname{im} A^H) = \operatorname{rank} A$
$\dim(\ker A) = n - \operatorname{rank} A$
$\dim(\ker A^H) = m - \operatorname{rank} A$

Fundamental Theorem of Algebra II:

$N(A)$ is orthogonal complement to $C(A^H)$
$\Leftrightarrow N(A) = C(A^H)^\perp \subset \mathbb{E}^n$
$\Leftrightarrow N(A) \oplus C(A^H) = \mathbb{E}^n$

$N(A^H)$ is orthogonal complement to $C(A)$
$\Leftrightarrow N(A^H) = C(A)^\perp \subset \mathbb{E}^m$
$\Leftrightarrow N(A^H) \oplus C(A) = \mathbb{E}^m$

Orthonormal Bases

Some basis $\mathcal{B} = (b_1, \dots, b_n)$ is **orthogonal** if all its vectors are pairwise orthogonal, i.e. $b_k \perp b_l \Leftrightarrow \langle b_k, b_l \rangle = 0 \ \forall k, l,\ k \neq l$.

It is **orthonormal**, if additionally all vectors have length 1, i.e. $\|b_k\| = \sqrt{\langle b_k, b_k \rangle} = 1$
$\Rightarrow \langle b_k, b_k \rangle = 1$

Then, $\langle b_k, b_l \rangle = \delta_{kl}$ (**Kronecker-Delta**)
$= \begin{cases} 0 \text{ if } k \neq l \\ 1 \text{ if } k = l \end{cases}$

S6.4) An orthonormal basis makes expressing vectors using coordinates easy:

$$x = \sum_{i=1}^{n} \langle b_k, x \rangle b_k$$
$$\Rightarrow \xi_k = \langle b_k, x \rangle \quad \text{in} \quad \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

S6.5) **Parseval's formula**
If an orthonormal basis over some vector space V with inner product $\langle \cdot, \cdot \rangle_V$ exists,

$$\langle x, y \rangle_V = \sum_{i=1}^{n} \overline{\xi_k} \eta_k = \xi^H \eta = \langle \xi, \eta \rangle_2$$

(The abstract inner product of two vectors always equals the Euclidean Scalar Product in the coordinate space relative to the basis)

$\Rightarrow \|x\|_V = \|\xi\|_2$
$\measuredangle(x, y) = \measuredangle(\xi, \eta)$
$x \perp y \Leftrightarrow \xi \perp \eta$

Gram-Schmidt Orthonormalization Process

given: linearly independent $\{a_1, \dots, a_n\} \in V$
returns: pairwise orthonormal $\{b_1, \dots, b_n\} \in V$

$b_1 := \frac{1}{\|a_1\|_V} a_1$

for $k = 2 \dots m$
$\tilde{b}_k := a_k - \sum_{j=1}^{k-1} \langle b_j, a_k \rangle_V b_j$

$b_k := \frac{1}{\|\tilde{b}_k\|_V} \cdot \tilde{b}_k$

S6.6) After k iterations, $\operatorname{span}\{b_1, \dots, b_k\} = \operatorname{span}\{a_1, \dots, a_k\}$

$\{a_1, \dots, a_n\}$ is a basis
$\Leftrightarrow \{b_1, \dots, b_n\}$ is an orthonormal basis

C6.7) Every vector space of finite dimension has an orthonormal basis.

S6.10) The **basis transformation matrix between** two orthonormal bases is unitary/orthogonal, i.e. $T^{-1} = T^H / T^T$

Orthogonal Transformations

A linear transformation $F: X \to Y$ is orthogonal / unitary (rigid) if
$$\langle F(v), F(w) \rangle_Y = \langle v, w \rangle_X$$

S6.13)
- **length preserving** $\|Ax\| = \|x\|$
- **angle preserving** $\langle Ax, Ay \rangle = \langle x, y \rangle$
- **injective**, i.e. $\ker F = \{0\}$

Additionally, if $\dim X = \dim Y$:

- F is an isomorphism
- F^{-1} is also orthogonal / unitary
- $\{b_1, \dots, b_n\}$ is an orthonormal base for X $\Leftrightarrow$ $\{F(b_1), \dots, F(b_n)\}$ is an orthonormal base for Y
- The projection matrix in the coordinate space regarding the orthonormal bases is orthogonal / unitary

Orthogonal Complements

A subspace $U \subset V$ has exactly one orthogonal complement

$$U^\perp = \{ y \in V \mid y \perp U \}$$

$\Rightarrow$ $U^\perp \perp U$ and $V = U \oplus U^\perp$

$\Rightarrow$ $\dim U + \dim U^\perp = \dim V$

QR-Decomposition

given a matrix $A \in \mathbb{E}^{m \times n}$ of rank $A = n$, i.e. with linearly independent columns, run Gram-Schmidt on those columns to produce Q. Then, calculate $R = Q^T A$:

$$\underbrace{\begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} q_1^T a_1 & \cdots & q_1^T a_n \\ & \ddots & \vdots \\ & & q_n^T a_n \end{pmatrix}}_{R \quad r}$$

Linear Least Squares Method

goal: given an overdetermined LSE, i.e. $Ax = b$ with $A \in \mathbb{E}^{m \times n}$ of rank $A = n$ and $b \in \mathbb{E}^m$, determine

$$x^* = \operatorname*{argmin}_{x \in \mathbb{E}^n} \|Ax - b\|_2^2$$

(Find $x^* \in C(A)$ with minimal distance to b, i.e. with $(Ax^* - b) \perp C(A)$)

$\Leftrightarrow$ $A^H A x^* = A^H b$ normal equation

$\Leftrightarrow$ $x^* = A^+ b$ where $A^+ := (A^H A)^{-1} A^H$ denotes the Moore-Penrose pseudoinverse (for $A \in \mathbb{E}^{m \times n}$ with rank $A = n$, i.e. linearly independent columns).

S7.9) given $A = QR$, $Rx^* = Q^H b$

Determinant

The determinant of $A \in \mathbb{E}^{n \times n}$ (square!) is defined as

$$\det A = |A| = \sum_{p \in S_n} \operatorname{sign}(p) \prod_{i=1}^{n} a_{i,p(i)}$$

where S_n holds all $n!$ permutations (i.e. unique ways to order the columns) of A using v swap operations.

$$\operatorname{sign}(p) := \begin{cases} +1 & \text{if } v \equiv_2 0 \text{ (even)} \\ -1 & \text{if } v \equiv_2 1 \text{ (odd)} \end{cases}$$

Using cofactors, this big calculation can be split up into multiple little ones:

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{(develop by row } i\text{)}$$
$$= \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{(develop by column } j\text{)}$$

Cofactor $C_{ij} := (-1)^{i+j} \det M_{ij}$ with M_{ij} being matrix A without row i and column j.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

S8.3) Properties:

(i) det is linear regarding rows/cols.

$$\begin{vmatrix} \vdots \\ — \alpha v_e + \beta w_e — \\ \vdots \end{vmatrix} = \alpha \begin{vmatrix} \vdots \\ — v_e — \\ \vdots \end{vmatrix} + \beta \begin{vmatrix} \vdots \\ — w_e — \\ \vdots \end{vmatrix}$$

(BUT $\det(\alpha A + \beta B) \neq \alpha \det(A) + \beta \det(B)$)

(ii) swapping two rows/cols. reverses the sign of det

$$\begin{vmatrix} \vdots \\ = a_e = \\ = a_k = \\ \vdots \end{vmatrix} = - \begin{vmatrix} \vdots \\ = a_k = \\ = a_e = \\ \vdots \end{vmatrix}$$

(iii) $\det I = 1$

S8.4) Further,

(iv) A has a zero-row/cols. $\Leftrightarrow \det A = 0$

(v) $\det(\gamma A) = \gamma^n \det(A)$

(vi) A has two equal rows/cols. $\Leftrightarrow \det A = 0$

(vii) adding/subtracting a multiple of one row/col. from another leaves $\det A$ unchanged

(viii) A is a diagonal matrix or (ix) triangular $\Leftrightarrow \det A$ is the sum of all diagonal entries $\Leftrightarrow \det A = \sum_{i=1}^{n} a_{ii}$

S8.5) Performing Gauss using v swaps and result R:

$$\det A = (-1)^v \underbrace{\prod_{i=1}^{n} r_{ii}}_{\text{(product of pivots)}}$$

$\det A \neq 0 \Leftrightarrow \operatorname{rank} A = n \Leftrightarrow A$ invertible

S8.7) $\det(AB) = \det(A) \cdot \det(B)$

C8.8) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

S8.9) $\det(A^T) = \det A$
$\det(A^H) = \overline{\det A}$

Eigenvalues & Eigenvectors

An eigenvalue $\lambda \in \mathbb{E}$ of a linear endomorphism $F: V \to V$ scales at least one eigenvector $v \in V, v \neq 0$ to its function value:

$$F(v) = \lambda v$$

All eigenvectors for λ and 0 form the subspace

$$E_\lambda = \{ v \in V \mid F(v) = \lambda v \} \quad \text{eigenspace}$$

L9.2) $= \ker(F - \lambda I)$

The spectrum of F is the set of all its eigenvalues.

L9.1) The vector space and its coordinate spaces share eigenvalues:

$$\left.\begin{matrix} \lambda \text{ of } F \\ x \text{ of } F \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \lambda \text{ of } A \\ \xi \text{ of } A \end{matrix}\right.$$

C9.3) λ is an eigenvalue of $A \in \mathbb{E}^{n \times n}$
$\Leftrightarrow A - \lambda I$ is singular
$\Leftrightarrow E_\lambda = \ker(A - \lambda I) \neq \{0\}$

The geometric multiplicity of λ^* is

$$\dim E_\lambda = \dim \ker(A - \lambda^* I) = n - \operatorname{rank}(A - \lambda^* I)$$

The characteristic polynomial of A is

$$\chi_A(\lambda) := \det(A - \lambda I)$$

L9.4) $= (-1)^n \lambda^n + (-1)^{n-1} \operatorname{trace}(A) \lambda^{n-1} + \dots + \det(A) \cdot \lambda^0$

where $\operatorname{trace}(A) = \sum_{i=1}^{n} a_{ii}$ denotes the sum of all diagonal elements,

and the characteristic equation is

$\chi_A(\lambda) = 0 \Leftrightarrow \lambda$ is an eigenvalue.

The algebraic multiplicity of λ^* is the root multiplicity of λ^* in $\chi_A(\lambda)$.

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \quad \chi_A(\lambda) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$$

$\Rightarrow \lambda^* = 3$ is an eigenvalue of A with alg. mult. of 2

$(A - 3I)x = 0$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = 0$

$\dim E_\lambda = \dim \ker(A - 3I) = \dim \operatorname{span}\{\binom{1}{0}\} = 1$

$\Rightarrow \lambda^* = 3$ has a geo. mult. of 1

S9.13) geometric mult. $\leq$ algebraic mult.

L9.6) A singular $\Leftrightarrow A$ has eigenvalue 0

S9.7) Similar matrices A and B ($B = T^{-1}AT$) share $\chi_\cdot(\lambda)$, det, trace, and eigenvalues

S9.11) eigenvectors for different eigenvalues are linearly independent.

Find all eigenvalues of $A \in \mathbb{E}^{n\times n}$ and one eigenvector each:

- Compute $\chi_A(\lambda)$
- Solve $\chi_A(\lambda) = 0$ for $\lambda \Rightarrow \lambda_1, \dots, \lambda_k$
- Solve $(A - \lambda_i I)x = 0$ for all $i \le k$ $\Rightarrow x_1, \dots, x_k$

Check: $\prod_{i=1}^{k} \lambda_i = \det A$, $\sum_{i=1}^{k} \lambda_i = \operatorname{trace} A$

(factor $\chi_A(\lambda)$ using all determined λ_i and choose $\lambda = 0$) (by Vieta's formula the sum of $\chi_A(\lambda)$ $= (-1)^{n-1}$? the factor before λ^{n-1} = trace A according to L9.4)

Eigenvalue Decomposition

S9.9) $A \in \mathbb{E}^{n\times n}$ is diagonalizable

$\Leftrightarrow$ A has n linearly independent eigenvectors that form an eigenbasis $\{v_1, \dots, v_n\}$ in $\mathbb{E}^n$

$\Leftrightarrow$ A factorization

$$A = V \Lambda V^{-1} \quad (AV = V\Lambda)$$

exists with $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$

and invertible $V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ (where λ_i denotes the eigenvalue of v_i)

S9.14) $A \in \mathbb{C}^{n\times n}$ is diagonalizable over $\mathbb{C}$

$\Leftrightarrow$ geo. mult. = alg. mult. for all eigenvalues

More generally, if $\{v_1, \dots, v_n\}$ is an eigenbasis regarding the linear transformation $F: V \to V$,

$$x = \sum_{i=1}^{n} \xi_k v_k \mapsto F(x) = \sum_{i=1}^{n} \lambda_k \xi_k v_k$$

(C9.15/16) **Spectral Theorem**

If $A \in \mathbb{C}^{n\times n} / \mathbb{R}^{n\times n}$ is hermitical/symmetrical,

(i) all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

(ii) the complex/real eigenvectors for different eigenvalues are pairwise orthogonal

(iii) an orthonormal eigenbasis of A over $\mathbb{C}^n / \mathbb{R}^n$ exists

(iv) Q is unitary/orthogonal in

$$Q^{H/T} A Q = \Lambda \Leftrightarrow A = Q \Lambda Q^{H/T}$$

$A \in \mathbb{E}^{n\times n}$ is normal $\overset{\text{def}}{\Leftrightarrow} A^H A = A A^H$

$\Leftrightarrow$ A is diagonalizable using a unitary matrix (and may have complex eigenvectors)

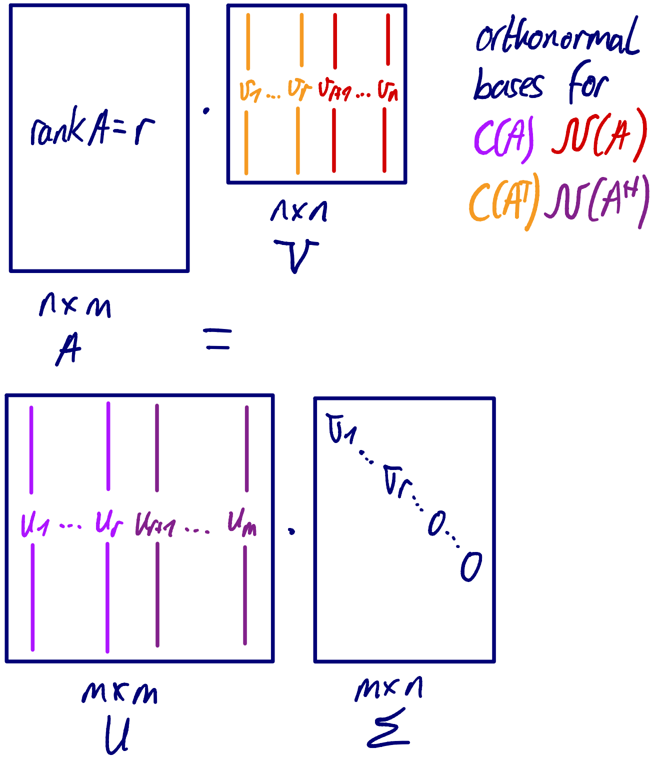
$A \in \mathbb{E}^{n\times n}$ is hermitical and positively (semi-) definite $\Rightarrow$ all eigenvalues are ≥ 0

Singular Value Decomposition

Every matrix $A \in \mathbb{E}^{m\times n}$ can be factored to

$$A = U \Sigma V^H \qquad (AV = U\Sigma)$$

with $\Sigma = \operatorname{diag}(\sigma_1, \dots \sigma_r, 0, \dots, 0) \in \mathbb{E}^{m\times n}$, and orthogonal $U \in \mathbb{E}^{m\times m}$ and $V \in \mathbb{E}^{n\times n}$.



How? Since $A^H A$ is always hermitical and positively definite, an eigenvalue decomposition exists.

Let $\sigma_k^2 := \lambda_k$ for $k \in \{1, \dots, n\}$ and choose the indices such that $\sigma_1 \geq \dots \geq \sigma_r > 0 = \sigma_{r+1} = \dots$ where $r := \operatorname{rank} A^H A$.

$$A^H A \begin{pmatrix} V_r & V_\perp \end{pmatrix} = \begin{pmatrix} V_r & V_\perp \end{pmatrix} \begin{pmatrix} \Sigma_r^2 & 0 \end{pmatrix}$$

Ignore $V_\perp$ since its columns are multiplied by zero. Instead, inspect

(note that V_r and Σ_r are nonsingular)

$$A^H A V_r = V_r \Sigma_r^2$$

$$\Leftrightarrow V_r^H A^H A V_r = \Sigma_r^2$$

$$\Leftrightarrow \underbrace{(\Sigma_r^{-1} V_r^H A^H)}_{:= U_r^{-1} = U_r^H} \underbrace{(A V_r \Sigma_r^{-1})}_{:= U_r} = I$$

An SVD's reduced form is

$$A V_r = U_r \Sigma_r$$

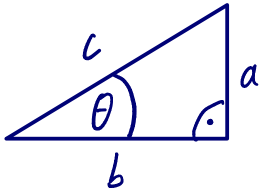
with $\Sigma_r \in \mathbb{E}^{r\times r}$ and orthogonal $U_r \in \mathbb{E}^{m\times r}$ and $V_r \in \mathbb{E}^{n\times r}$. Expand it by defining $U := (U_r \; U^\perp)$ and $V = (V_r \; V^\perp)$.

The general Moore-Penrose pseudoinverse is defined as

$$A^+ = V \Sigma^+ U^H$$

and returns the least-squares solution for $Ax = b$ in $x^* = A^+ b$.

Trigonometry



$\sin\theta = \frac{a}{c}$

$\cos\theta = \frac{b}{c}$

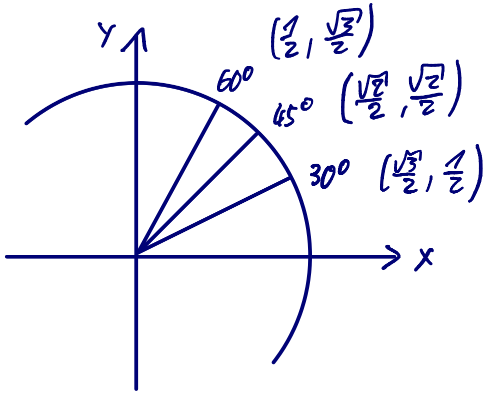
$\tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta}$

$\sin^2 x + \cos^2 x = 1$

$\sin(x\pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x\pm y) = \cos x \cos y \mp \sin x \sin y$

θ in °	0	30	45	60	90
θ in RAD	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	—



Symmetry helps derive missing values

Miscallaneous Matrices

rotation in $\mathbb{R}^2$: (orthogonal) $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

rotation in $\mathbb{R}^3$: (orthogonal) $\begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$

permutation: (orthogonal) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Vandermonde: (used for polynomial interpolation in linear least squares) $\begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \end{pmatrix}$

Miscallaneous Vector Spaces

all matrices over $\mathbb{K}$	$\mathbb{K}^{m\times n}$
sequences	$\mathbb{K}^\infty$
infinite matrices	$\mathbb{K}^{\infty\times\infty}$
polynomials with max. degree	$\mathcal{P}_d(\mathbb{K})$
polynomials	$\mathcal{P}(\mathbb{K})$
real functions	$\mathcal{F}(\mathbb{R};\mathbb{R})$