Continuous Time to Discrete Time

$$\{u[n]\} \longrightarrow \boxed{H} \xrightarrow{\{u(t)\}} \boxed{G_c} \xrightarrow{\{y(t)\}} \boxed{S} \xrightarrow{\{y[n]\}}$$

zero-order hold
$u(t)=u[n]\ \forall t\in[nT_s,(n+1)T_s[$

uniform sampling
$y[n]=y(nT_s)$

State-Space Representation

$\dot q(t)=A_c q(t)+B_c u(t)$ | $q[n+1]=A_d q[n]+B_d u[n]$

$y(t)=C_c q(t)+D_c u(t)$ | $y[n]=C_d q[n]+D_d u[n]$

Exact Discretization

Choose $C_d:=C_c$ and $D_d:=D_d$. Construct $M=\begin{bmatrix}A_c & B_c\\ 0 & 0\end{bmatrix}$ and compute $e^{MT_s}=I+MT_s+\frac{(MT_s)^2}{2!}+\frac{(MT_s)^3}{3!}+\dots=\begin{bmatrix}A_d & B_d\\ 0 & I\end{bmatrix}$

Euler Forward Discretization

Choose $C_d:=C_c$, $D_d:=D_d$, $A_d:=I+A_cT_s$, $B_d:=B_cT_s$.

Linear Constant-Coefficient Difference Equations

LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k] \qquad a_k,b_k\in\mathbb{R},\ N,M\in\mathbb{N}$$

Convert to state-space by choosing $A_d:=\begin{bmatrix}0 & I\\ -a_N & \cdots\ -a_1\end{bmatrix}$, $B_d:=\begin{bmatrix}0\\ \vdots\\ b_0\end{bmatrix}$, $C_d:=[-a_N\dots -a_1]$, $D_d:=[b_0]$ if $b_k=0$ for all $k>0$ and $a_0=1$. Otherwise, solve manually.

$$y[n]=\sum_{k=0}^{M} b_k u[n-k] \Rightarrow \text{the system has a FIR}$$

Convert to impulse response assuming $q[n]=0$ for $n\le 0$ by computing $h=\{D, CB, CAB, CA^2B, \dots, CA^{n-1}B\}$

System Properties

memoryless $\Leftrightarrow y[n]=f_n(\{u[n]\})$

Causal $\Leftrightarrow y[n]=f_n(\{u[n-k]\mid k\in\mathbb{N}_0\})$
$\Leftrightarrow h[n]=0$ for $n<0$ if G is LTI
(or via transfer function)

linear $\Leftrightarrow \forall$ input seqs $u_1,u_2.\ \forall \alpha_1,\alpha_2\in\mathbb{R}.$
$G(\alpha_1u_1+\alpha_2u_2)=\alpha_1Gu_1+\alpha_2Gu_2$

time-invariant $\Leftrightarrow G\{u[n-k]\}=\{(Gu)[n-k]\}$

stable $\Leftrightarrow \exists M\in\mathbb{R}.\ \forall n\in\mathbb{N}.\ |u[n]|<1 \quad |(Gu)[n]|<M$
$\Leftrightarrow \sum_{k=-\infty}^{\infty}|h[k]|<\infty$ if G is LTI
(or via transfer function)

LTI-system response

Unit Impulse & Step sequence

$\{\delta[n]\}=\{\dots,\underset{\uparrow}{1},0,0,\dots\}$ | $\{s[n]\}=\{\dots,1,1,1,\dots\}$

It holds that $s[n]=\sum_{k=-\infty}^{n}\delta[k]$ and $\{s[n]\}-\{s[n-1]\}=\{\delta[n]\}$.

Impulse Response

An LTI-system G is fully characterized by $G\delta=:h$. Now, $Gu=G\left(\sum_{k=-\infty}^{\infty}u[k]\{\delta[n-k]\}\right)=\sum_{k=-\infty}^{\infty}u[k]\{h[n-k]\}=u*h$

If $h[n]=0$ for $n\ge N$, then G has a FIR. If no such $N\in\mathbb{N}$ exists, then G has an IIR.

Convolution Properties

commutative $u*h=h*u$, associative $(x*y)*z=x*(y*z)$,
distributive $x*(h_1+h_2)=x*h_1+x*h_2$

Exponential Series, Eigenfunctions & z-Transform

Complex Exponential Series

For $z\in\mathbb{C}$ with the DT-frequency $\Omega\in[0,2\pi[$ in $z=|z|e^{j\Omega}$, define $x[n]:=z^n=|z|^n(\cos(\Omega n)+j\sin(\Omega n))$.

Periodic Signals

x is M-periodic $\Leftrightarrow \{x[n]\}=\{x[n+M]\}$

N is the fundamental period $\Leftrightarrow N=\min_{x\text{ is }M\text{-periodic}} M$

Periodicity Constraint

Sampling CT $t\mapsto\cos(\omega t)$ to $\{\cos(\Omega n)\}$ with $\Omega=\omega T_s$ and $\omega=2\pi f$ where f is the frequency in Hz.

is N periodic $\Leftrightarrow \exists M,N\in\mathbb{N}_0.\ \frac{\Omega}{2\pi}=\frac{M}{N}$

N is the fundamental period $\Leftrightarrow \frac{M}{N}$ is irreducible.

Eigenfunctions & z-Transform

$$G\{z^n\}=\sum_{k=-\infty}^{\infty}h[k]\{z^{n-k}\}=\underbrace{\left(\sum_{k=-\infty}^{\infty}h[k]z^{-k}\right)}_{\text{Eigenvalue computed by z-transform } H(z) \text{ of } h}\underbrace{\{z^n\}}_{\text{Eigenfunction}}$$

z-Transform Properties

Linearity $\alpha_1x_1+\alpha_2x_2 \leftrightarrow \alpha_1X_1(z)+\alpha_2X_2(z)$

Time-Shifting $\{x[n-k]\}\leftrightarrow X(z)z^{-k}$

Convolution $x_1*x_2\leftrightarrow X_1(z)\cdot X_2(z)$

Accumulation $\{\sum_{k=-\infty}^{n}x[k]\}\leftrightarrow X(z)\frac{z}{z-1}$

Differentiation $\{nx[n]\}\leftrightarrow -z\frac{d}{dz}X(z)$

Transfer Function

An LTI system has the transfer function $H(z)=\frac{Y(z)}{U(z)}$ where $y=u*h \leftrightarrow Y(z)=U(z)H(z)$.

Determine from LCCDE using linearity and time-shifting.

Determine from state-space by computing $H(z)=C(zI-A)^{-1}B+D$.

$H(z)$ has stable and causal interpretation $\Leftrightarrow |H(p)|=\infty\to|p|<1$

$H(z)$ has stable interpretation $\Leftrightarrow |H(p)|=\infty\to|p|\neq 1$

system causal $\Rightarrow (|H(p)|=\infty\to|p|<1)\leftrightarrow$ system stable

system anti-causal $\Rightarrow (|H(p)|=\infty\to|p|\to|p|\geq 1)\leftrightarrow$ system stable

Fourier Transform

Frequencies & DT Fourier-Transform

$$G\{e^{j\Omega n}\}=\sum_{k=-\infty}^{\infty}h[k]\{e^{j\Omega(n-k)}\}=\underbrace{\sum_{k=-\infty}^{\infty}h[k]e^{-j\Omega k}}_{\text{Frequency response computed by DT-FT } H(\Omega) \text{ of } h}\underbrace{\{e^{j\Omega n}\}}_{\text{Frequency}}$$

DT-FT Properties

z-transform evaluated on unit circle $X(\Omega)=X(z=e^{j\Omega})$

inverse $\{x[n]\}=\{\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\Omega)e^{j\Omega n}d\Omega\}$

linearity $\alpha_1x_1+\alpha_2x_2\leftrightarrow\alpha_1X_1(\Omega)+\alpha_2X_2(\Omega)$

convolution $x_1*x_2\leftrightarrow X_1(\Omega)\cdot X_2(\Omega)$

Parseval's Theorem $\sum_{n=-\infty}^{\infty}|x[n]|^2\leftrightarrow\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(\Omega)|^2d\Omega$

Magnitude Symmetry $x\in\mathbb{R}^\infty \Rightarrow X(\Omega)=X(\Omega-2\pi)^*$

Exists if $\sum_{k=-\infty}^{\infty}|x[k]|<\infty$ or $x=\{e^{j\Omega_0 n}\}\leftrightarrow X(\Omega)=2\pi\delta(\Omega-\Omega_0)$

Fourier Series & Discrete Fourier Transform

DFS & DT-DFT

An N-periodic signal $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}n}$

for N-periodic DFS coefficients $X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}$

Any finite signal can be extended to a periodic one and transformed via DFS. That is the DT-DFT.

DFS & DT-DFT Properties

Linearity $\alpha_1 x_1 + \alpha_2 x_2 \longleftrightarrow \alpha_1 \{X_1[k]\} + \alpha_2 \{X_2[k]\}$

Parseval's Theorem $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$

Magnitude symmetry $x \in \mathbb{R}^\infty \Rightarrow X[N-k] = X^*[k]$ (so $X[0] \in \mathbb{R}$)

Response to Complex Exponential Series

$Y[k] = H(e^{jk\frac{2\pi}{N}})\, U[k]$ holds for n-periodic $\{u[n]\}$

$G\{\mathbb{1}_{n\geq 0}\, e^{j\Omega}\} \xrightarrow{n\to\infty} H(e^{j\Omega})\{e^{j\Omega n}\}$ for causal exp. series

Aliasing

The effect of multiple CT frequency mapping to the same DT frequency. Solution: Analog-filter s.t. $|w| < \frac{\pi}{T_s}$.

System Identification

White Noise

A sequence x of random variables with $\mathbb{E}[x[n]] = 0$ and $\mathbb{E}[x[n]x[\ell]] = \mathbb{1}_{n=\ell}$, usually with $Var[x[n]] = 1$, has in expectation a flat frequency spectrum. Simulate via `sqrt(3) * (2* np.random.uniform() - 1)` or `np.random.normal()`

Unit Impulse Method

① Measure $G\{\delta[n]\} =: y$ in N samples.

② Estimate $\hat{H}(k\frac{2\pi}{N}) := Y[k] = H(k\frac{2\pi}{N}) - \sum_{n=N}^{\infty} h[n] e^{-jk\frac{2\pi}{N}n}$.

Problem: MSE grows with N if measurement noise is involved.

Frequency Response Method

① Fix a target frequency $\Omega_\ell = \ell\frac{2\pi}{N}$, transient N_T, N, and $A \in \mathbb{R}$.

② Measure $G\{A\cos(\Omega_\ell n)\} = y$, and compute $Y[\ell] = \sum_{n=N_T}^{N_T+N-1} y_m[n] e^{-j\Omega_\ell n}$

③ Estimate $\hat{H}(\Omega_\ell) := \frac{2\,Y[\ell]}{NA} = \frac{Y[\ell]}{U[\ell]}$

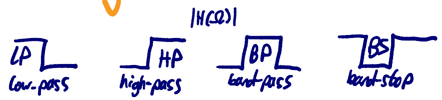
④ Repeat for more ℓ until satisfied.

Finding H(z) from $\hat{H}(\Omega_\ell)$

$$H(z) = \frac{\sum_{k=0}^{B-1} b_k z^{-k}}{1 + \sum_{k=1}^{A-1} a_k z^{-k}} \overset{!}{=} \hat{H}(z) \text{ when } z = e^{j\Omega_\ell},\quad \hat{H}(\Omega_\ell) = R_\ell e^{j\phi_\ell}$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} \cdots R_1\cos(\phi_1 - (A-1)\Omega_1) & \cdots & -\cos((B-1)\Omega_1) \\ \cdots R_1\sin(\phi_1 - (A-1)\Omega_1) & \cdots & \sin((B-1)\Omega_1) \\ & \vdots & \end{bmatrix}}_{2L \times A+B-1} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ b_0 \\ \vdots \end{bmatrix}}_{A+B-1 \times 1} \overset{LS}{=} \underbrace{\begin{bmatrix} -R_1\cos(\phi_1) \\ -R_1\sin(\phi_1) \\ \vdots \end{bmatrix}}_{2L \times 1}$$

Filtering



filters are commonly used.

Non-causal Filtering

Chain a real, causal filter G with $H(z)$ and its anticausal counterpart $\tilde{G}$ with $H(z^{-1})$ to create $y = \tilde{G}Gu$ with $Y(e^{j\Omega}) = |H(e^{j\Omega})|^2 U(e^{j\Omega})$. Pro: No phase delay! (*)

Anti-Aliasing CT-Filter

$\frac{Y(s)}{U(s)} = \frac{1}{RCs+1} = \frac{w_c}{s+w_c}$ for cut-off $w_c = \frac{1}{RC}$

FIR-Filters

An FIR filter is characterized by $h = \{b_0, b_1, \dots, b_{M-1}\}$ with $H(z) = \sum_{k=0}^{M-1} b_k z^{-1}$, where $M-1$ is the filter order.

Moving-Average Filter

$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} u[n-k] \Leftrightarrow H(\Omega) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-j\Omega k} = \frac{1-e^{-j\Omega M}}{1-e^{-j\Omega}}$

Trick: Multiply both sides with $e^{-j\Omega}$ and subtract modified from original.

Alternative: $y[n] = y[n-1] + \frac{u[n]-u[n-M]}{M}$ identical if no numerical errors happen. (*) Conversion yields non-causal WMA with $2M-1$ coefficients that has 0 phase and good mag.

Weighted MA-Filter

$y[n] = \frac{1}{S}\sum_{k=0}^{M-1} w_k u[n-k]$ with $S = \sum w_k$. Usually $w_k = M-k$, $S = \frac{M(M+1)}{2}$.

IIR-Filters

An IIR filter is characterized by $b_0, \dots, b_{M-1}$ and $a_1, \dots, a_{N-1}$ with $y[n] = \sum_{k=0}^{M-1} b_k u[n-k] - \sum_{k=1}^{N-1} a_k y[n-k]$ and $H(z)$ above, where $\max(M-1, N-1)$ is the order.

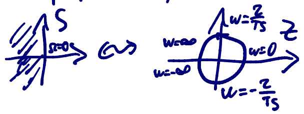
Butterworth Filter

In CT, $R(w) = \frac{1}{\sqrt{1+w^{2K}}}$ is a K-th order filter with corner frequency $1\frac{\text{rad}}{s}$. It has no ripples and is maximally flat. The stable TF with that freq. resp. is $H(s) = (\prod_{k=1}^{K}(s - s_k))^{-1}$ where $s_k = \exp\left(\frac{j(2k+K-1)\pi}{2K}\right)$. Replacing $s \mapsto \frac{s}{w_c}$ changes the cut-off frequency to w_c.

Bilinear Transform

Convert from CT to DT by $s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right) \Leftrightarrow z = \frac{1+s\frac{T_s}{2}}{1-s\frac{T_s}{2}}$

With frequencies, $\Omega = 2\arctan(w\frac{T_s}{2}) \overset{\text{small}}{\approx} wT_s$



Designing High-Pass in CT / DT

Design Low-Pass for corner frequency $\frac{1}{w_c}$ / $\pi - \Omega_c$ with $H_{LP}(s/z)$. Then, $H_{HP}(s/z) = H_{LP}(s^{-1}/-z)$.

Designing Band-Pass in CT

Design low-pass for corner frequency $w_1 - w_0$ with $H_{LP}(s)$. Then $H_{BP}(s) = H_{LP}\left(\frac{s^2 + \sqrt{w_0 w_1}^2}{s}\right)$

Prewarping

We want $\frac{\Omega_c}{T_s} = w_c$ but with large w_c, this doesn't hold. So, instead design for $\tilde{w}_c = \frac{2}{T_s}\tan\left(\frac{w_c T_s}{2}\right)$

$e^{j\theta} = \cos\theta + j\sin\theta \qquad e^{-j\theta} = \cos\theta - j\sin\theta$

$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2}$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots = e$

$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{9} + \dots = \frac{\pi^2}{6}$

$\sum_{k=0}^{\infty} \left(\frac{1}{n}\right)^k = \frac{n}{n-1} \qquad \sum_{k=0}^{\infty} \left(-\frac{1}{n}\right)^k = \frac{n}{n+1}$ for $n > 1$