Formalisms Every Computer Scientist Should Know

Contents

Class 1

Class 2

Goal	Knowledge	Outermost symbol
Show for all x, $G(x)$. Consider arbitrary $\hat{x}$. Show $G(\hat{x})$	We know for all x, $K(x)$ In particular we know $K(\hat{t})$ for constant $\hat{t}$	$\forall$
Show: exists x s.t. $G(x)$. We show $G(\hat{t})$	We know exists x s.t. $K(x)$ Let $\hat{x}$ be s.t. $K(x)$	$\exists$
Show G_1 iff G_2 1. Show if G_1 then G_2 2. Show if G_2 then G_1	We know K_1 iff K_2 In particular we know if K_1 then K_2 and if K_2 then K_1	$\iff$
Show if G_1 then G_2 Assume G_1 Show G_2	We know if K_1 then K_2 1. To show K_2 it suffices to show K_2 2. Know K_1, Also know K_2	$\Rightarrow$
Show G_1 and G_2 1. Show G_1 2. Show G_2	Know K_1 and K_2 1. Also Know K_1 2. Also Know K_2	$\wedge$
Show G_1 or G_2 1. Assume $\neg G_1$, show G_2 2. Assume $\neg G_2$, show G_1	We know K_1 or K_2. Show G. 1. Assume K_1, Show G 2. Assume K_2, Show G Case split $\uparrow$	$\vee$
Move Negation Inside, as far as possible		$\neg$

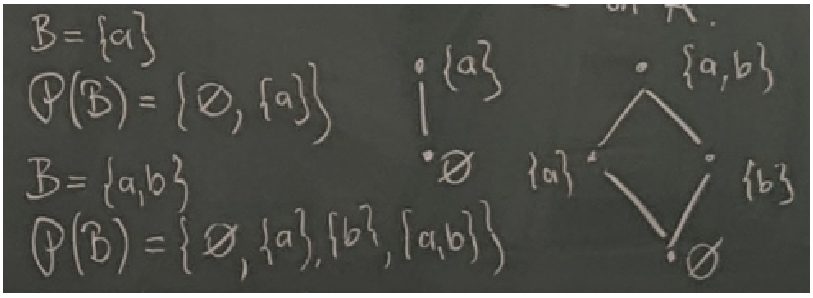
0.1 Lattices and Fixpoints

We begin by defining relations and their properties.

Definition 1. A binary relation R on a set A is a subset $R \subset A \times A$.
The relation R is reflexive if for all x in A, we have $R(x,x)$.
The relation R is Antisymmetric if for all x and y in A, if $R(x,y)$ and $R(y,x)$ then $x = y$.
The relation R is transitive if for all x, y and z in A, if $R(x,y)$ and $R(y,z)$ then $R(x,z)$.
The relation R is a partial order if R is reflexive, antisymmetric and transitive.
A Poset $(A, \sqsubseteq)$ is a set A and a partial order $\sqsubseteq$ on A.

Example 1. The pair $(\mathbb{N}, \leq)$ where $\mathbb{N}$ is the set of natural numbers, is a poset.
For every set B, we have $(\mathscr{P}(B), \subseteq)$ where $\mathscr{P}(B)$ is the powerset of B, is a poset.

Definition 2.
- Let $(A, \sqsubseteq)$ be a poset. A function F from A to A is monotone (order-preserving, homomorphism) if for all x and y in A, if $x \sqsubseteq y$, then $F(x) \sqsubseteq F(y)$.
- F has a fixpoint x in A if there exists x in A such that $F(x) = x$.



- x in A is a pre-fixpoint of F if $x \sqsubseteq F(x)$ and is a post-fixpoint of F, if $F(x) \sqsubseteq x$.

Definition 3. Let $(A, \sqsubseteq)$ be a poset.

- x in A is an upper bound(lower bound) on a subset B of A if for all y in B, it holds that $y \sqsubseteq x$ ($x \sqsubseteq y$).
- x is the least upper bound of B if (i) x is an upper bound of B and (ii) for all upper bounds y of B, we have $x \sqsubseteq y$. We denote such x by $\bigsqcup B$.
- x is the greatest lower bound of B if (i) x is a lower bound of B and (ii) for all lower bounds y of B, we have $y \sqsubseteq x$. We denote such x by $\bigsqcap B$.

Example 2.
- Consider the poset $(\mathbb{N}, \leq)$. Then for any $B \subseteq \mathbb{N}$, if B is finite, $\bigsqcup B$ is well-defined and equal to $\max B$. If B is infinite, then $\bigsqcup B$ does not exist.
- Consider the poset $(\mathbb{N} \cup \{\infty\}, \leq)$ where for all x in $\mathbb{N}$, it holds that $x \leq \infty$. Then for all $B \subseteq \mathbb{N}$, the least upper bound $\bigsqcup B$ is well-define.
- Let A be any set and consider the poset $(\mathscr{P}(A), \subseteq)$. For any subset B of $\mathscr{P}(A)$, it holds that $\bigsqcup B = \bigcup B$ and $\bigsqcap B = \bigcap B$.

Definition 4. Poset $(A, \sqsubseteq)$ is a complete-lattice if for all $B \subseteq A$, both $\bigsqcap B$ and $\bigsqcup B$ exist.

Example 3. Let $(A, \sqsubseteq)$ be a complete-lattice.

- $\bigsqcup A = \top$
- $\bigsqcap A = \bot$
- $\bigsqcup \varnothing = \bot$
- $\bigsqcap \varnothing = \top$

Theorem 1 (Knaster-Tarski). For every complete lattice $(A, \sqsubseteq)$ and monotone function F on A, it holds that

1. $\bigsqcup \{x \in A | x \sqsubseteq F(x)\}$ is the unique greatest fixpoint of F.
2. $\bigsqcap \{x \in A | F(x) \sqsubseteq x\}$ is the unique least fixpoint of F.

Homework 1. Prove the Knaster Tarski Theorem.

Class 3

Definition 5 (Prefixpoint). Consider a lattice $(A, \sqsubseteq)$ and a function $f \colon A \to A$. The set of prefixes is

$$\{x \in A : x \sqsubseteq f(x)\}.$$

Definition 6 (Postfixpoint). Consider a lattice $(A, \sqsubseteq)$ and a function $f \colon A \to A$. The set of postfixes is

$$\{x \in A : f(x) \sqsubseteq x\}.$$

Definition 7 (gfp and lfp). Consider a complete lattice $(A, \sqsubseteq)$ and a function $f \colon A \to A$. Then,

$$\begin{aligned} \mathrm{gfp} f &:= \bigsqcup \{x \in A : x \sqsubseteq f(x)\} \\ \mathrm{lfp} f &:= \bigsqcap \{x \in A : f(x) \sqsubseteq x\}. \end{aligned}$$

Theorem 2 (Fixpoints). Consider a complete lattice $(A, \sqsubseteq)$ and a monotonic function $f \colon A \to A$. Then, $\mathrm{gfp} f$ and $\mathrm{lfp} f$ are fixpoints of f and, for all fixpoints x of f, we have $\mathrm{lfp} f \sqsubseteq x \sqsubseteq \mathrm{gfp} f$.

Definition 8 ($\bigsqcup$-continuous). Consider a complete lattice $(A, \sqsubseteq)$. A function $f \colon A \to A$ is $\bigsqcup$-continuous if, for all increasing sequences $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq x_2 \sqsubseteq \ldots$, we have

$$f\left(\bigsqcup \{x_n : n \in \mathbb{N}\}\right) = \bigsqcup \{f(x_n) : n \in \mathbb{N}\}.$$

Definition 9 ($\bigsqcap$-continuous). Consider a complete lattice $(A, \sqsubseteq)$. A function $f \colon A \to A$ is $\bigsqcap$-continuous if, for all increasing sequences $x_0 \sqsupseteq x_1 \sqsupseteq x_2 \sqsupseteq x_2 \sqsupseteq \ldots$, we have

$$f\left(\bigsqcap \{x_n : n \in \mathbb{N}\}\right) = \bigsqcap \{f(x_n) : n \in \mathbb{N}\}.$$

Lemma 1. $\bigsqcup$-continuous implies monotonicity and $\bigsqcap$-continuous implies monotonicity.

Theorem 3 (Constructive fixpoints). Consider a complete lattice $(A, \sqsubseteq)$ and a monotonic function $f \colon A \to A$. Then,

$$\begin{aligned} \mathrm{lfp} f &= \bigsqcup \{f^n(\bot) : n \in \mathbb{N}\} \\ \mathrm{gfp} f &= \bigsqcap \{f^n(\top) : n \in \mathbb{N}\}. \end{aligned}$$

Homework 2. Prove this theorem.

Definition 10 ($\mathbb{N}$). Define $\mathbb{N}$ as the smallest set X such that

1. $0 \in X$

2. if $n \in X$, then $Sn \in X$

In the definition of $\mathbb{N}$, we consider a universal set U sufficiently big, the complete lattice $(2^U, \subseteq)$ and the function on sets given by $f(Y) := \{0\} \cup \{Sn : n \in Y\}$. Then, $\mathsf{lfp} f = \mathbb{N}$.

Definition 11 (Set of words). Consider a finite alphabet Σ. Define Σ^* as the smallest set X such that

1. $\varepsilon \in X$
2. for all $a \in \Sigma$, we have $aX \subseteq X$.

A few remarks are in place.

- Inductively defined sets are countable and consists of finite elements.
- Inductively defined sets can be written as rules $x \Rightarrow f(x)$ meaning that, if $x \in X$, then $f(x) \in X$.
- Inductively defined sets allow proof by induction. Consider prove that for all $x \in X$ we have $G(x)$. This can be proven by showing
  1. $G(\perp)$
  2. For all $x \in X$, if $G(x)$, then $G(f(x))$

Definition 12 (Balanced binary sequences). Define the set S as the largest set X such that

1. $X \subseteq 01X \cup 10X$.

In the definition of balanced binary sequences, we consider the complete lattice $(\Sigma^\omega, \subseteq)$ and the function on sets given by $f(X) := 01X \cup 10X$. Then, balanced binary sequences corresponds to $\mathsf{gfp} f$.

Definition 13 (Interval $[0,1]$). Define the set S as the largest set X such that

1. $X \subseteq 0X \cup 1X \cup \ldots \cup 9X$.

A few remarks are in place.

- Coinductively defined sets are uncountable and consists of infinite elements.
- Coinductively defined sets can be written as rules $x \Leftarrow f(x)$ meaning that, for all $y \in X$, there exists x such that $y = f(x)$ and $x \in X$.
- Coinductively defined sets allow proof by induction. Consider prove that for all $x \in X$ we have $G(x)$. This can be proven by showing
  1. For all x and i, if $G(f_i(x))$, then $G(x)$

  where $\{f_1, \ldots, f_n\}$ is the set of rules that define the set X.

Homework 3 (Prove balanced binary sequences). Consider S generated by the rules $X \Leftarrow 01X$ and $X \Leftarrow 10X$. Prove that, for all binary words x, we have that $x\ inS$ if and only iff every finite prefix of even length of x has the same number of 0s and 1s.

Hints.

1. The direction $\Leftarrow$ can be proven by coinduction.
2. The direction $\Rightarrow$ can be proven by induction on the length of the prefix.