Formalisms Every Computer Scientist Should Know

Contents

Class 1

0.1 Tools

- Lean or Coq
- CVC5 or Z3
- possibly a model checker
- professional version of ChatGPT

0.2 Syllabus

1. MATH ("Informalism")
   - proofs (natural deduction)
   - fixpoints (induction, coinduction)
2. DECLARATIVE LOGIC
   - syntax (rules) vs. semantics (models)
   - propositional, predicate, modal logic
   - decision procedures (SAT, SMT)
3. FUNCTIONS
   - λ calculus, typed λ
   - SOS (structured operational semantics), rewriting
   - "propositions-as-types" (connection to logic)
4. PROCESSES (CONCURRENT)*
   - CCS, Petri nets
   - (bi-) simulation
5. CIRCUITS*
   - boolean, sequential, dataflow (Kahn nets)
   - interfaces

6. STATE TRANSITION SYSTEMS*
   - (ω-) automata, games, timed, probabilistic, pushdown
   - programs, Turing machines
   - grammars (Chomsky hierarchy)
7. DECLARATIVE SPECIFICATION
   - Hoare logics, separation logic
   - temporal logics (LTL, CTL, ATL)
   - partial correctness vs. termination, safety vs. liveness

*: Operational.

0.3 Math

Definition 1. A real b is a <u>bound</u> of a function f from $\mathbb{R}$ to $\mathbb{R}$ if for all x in $\mathbb{R}$, we have $f(x) \leq b$.

Definition 2. Given two functions f and g from $\mathbb{R}$ to $\mathbb{R}$, their <u>sum</u> is the function $f+g$ such that for all x in $\mathbb{R}$, we have $(f+g)(x) = f(x)+g(x)$.

Theorem 1. For all functions f and g from $\mathbb{R}$ to $\mathbb{R}$, if f and g are bounded, then $f+g$ is bounded.

Proof.
1. Consider arbitrary functions $\hat{f}$ and $\hat{g}$ from $\mathbb{R}$ to $\mathbb{R}$.
2. Assume $\hat{f}$ and $\hat{g}$ are bounded.
3. Show that $\hat{f}+\hat{g}$ is bounded.
4. ($2 \rightarrow$) Let $\hat{a}$ be a bound for $\hat{f}$, and $\hat{b}$ be a bound for $\hat{g}$.
5. We show that $\hat{a}+\hat{b}$ is a bound for $\hat{f}+\hat{g}$.
6. Consider an arbitrary real $\hat{x}$.
7. Show $(\hat{f}+\hat{g})(x) \leq \hat{a}+\hat{b}$.
8. (Definition of sum) $(\hat{f}+\hat{g})(\hat{x}) = \hat{f}(\hat{x})+\hat{g}(\hat{x})$.
9. (Definition of bound) $\hat{f}(\hat{x}) \leq \hat{a}$ and $\hat{g}(\hat{x}) \leq \hat{b}$.
10. The rest follows from "arithmetic".

$\square$

Homework. Prove the Schröder-Bernstein theorem "in this style".

Definition 3. Two sets A and B are <u>equipollent</u> ("have the same size") if there is a bijection from A to B.

Definition 4. A function f from A to B is

1. <u>one-to-one</u> if for all x and y in A, if $x \neq y$, then $f(x) \neq f(y)$.
2. <u>onto</u> if for all z in B, there exists x in A such that $f(x) = z$.

3. bijective if f is one-to-one and onto.

Goals	Knowledge	Outermost symbol
Show for all x, $G(x)$. Consider arbitrary $\hat{x}$. Show $G(\hat{x})$.	We know for all x, $K(x)$. In particular, we know $K(\hat{t})$. $\hat{t}$: term containing only constants.	$\forall$
Show there exists x s.t. $G(x)$. We show that $G(\hat{t})$. $\hat{t}$: term containing only constants.	We know there exists x s.t. $K(x)$. Let $\hat{x}$ be s.t. $K(\hat{x})$.	$\exists$

Class 2

Goal	Knowledge	Outermost symbol
Show for all x, $G(x)$. Consider arbitrary $\hat{x}$. Show $G(\hat{x})$	We know for all x, $K(x)$ In particular we know $K(\hat{t})$ for constant $\hat{t}$	$\forall$
Show: exists x s.t. $G(x)$. We show $G(\hat{t})$	We know exists x s.t. $K(x)$ Let $\hat{x}$ be s.t. $K(x)$	$\exists$
Show G_1 iff G_2 1. Show if G_1 then G_2 2. Show if G_2 then G_1	We know K_1 iff K_2 In particular we know if K_1 then K_2 and if K_2 then K_1	$\iff$
Show if G_1 then G_2 Assume G_1 Show G_2	We know if K_1 then K_2 1. To show K_2 it suffices to show K_2 2. Know K_1, Also know K_2	$\Rightarrow$
Show G_1 and G_2 1. Show G_1 2. Show G_2	Know K_1 and K_2 1. Also Know K_1 2. Also Know K_2	$\wedge$
Show G_1 or G_2 1. Assume $\neg G_1$, show G_2 2. Assume $\neg G_2$, show G_1	We know K_1 or K_2. Show G. 1. Assume K_1, Show G 2. Assume K_2, Show G Case split $\uparrow$	$\vee$
Move Negation Inside, as far as possible		$\neg$

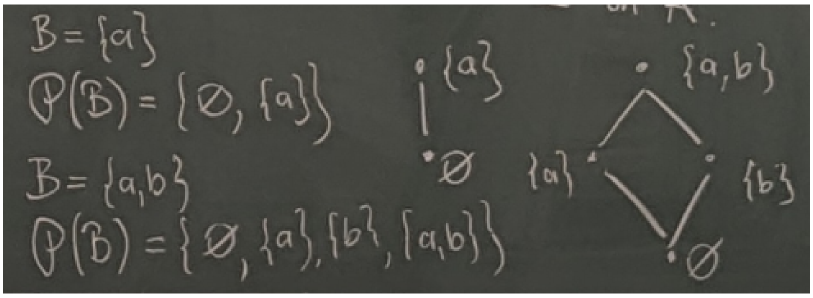
0.4 Lattices and Fixpoints

We begin by defining relations and their properties.

Definition 5. A binary relation R on a set A is a subset $R \subset A \times A$.
The relation R is reflexive if for all x in A, we have $R(x,x)$.
The relation R is Antisymmetric if for all x and y in A, if $R(x,y)$ and $R(y,x)$ then $x = y$.
The relation R is transitive if for all x,y and z in A, if $R(x,y)$ and $R(y,z)$ then $R(x,z)$.
The relation R is a partial order if R is reflexive, antisymmetric and transitive.
A Poset $(A, \sqsubseteq)$ is a set A and a partial order $\sqsubseteq$ on A.

Example 1. The pair $(\mathbb{N}, \leq)$ where $\mathbb{N}$ is the set of natural numbers, is a poset.
For every set B, we have $(\mathscr{P}(B), \subseteq)$ where $\mathscr{P}(B)$ is the powerset of B, is a poset.

Definition 6.
- Let $(A, \sqsubseteq)$ be a poset. A function F from A to A is monotone (order-preserving, homomorphism) if for all x and y in A, if $x \sqsubseteq y$, then $F(x) \sqsubseteq F(y)$.
- F has a fixpoint x in A if there exists x in A such that $F(x) = x$.



- x in A is a pre-fixpoint of F if $x \sqsubseteq F(x)$ and is a post-fixpoint of F, if $F(x) \sqsubseteq x$.

Definition 7. Let $(A, \sqsubseteq)$ be a poset.

- x in A is an upper bound(lower bound) on a subset B of A if for all y in B, it holds that $y \sqsubseteq x$ $(x \sqsubseteq y)$.
- x is the least upper bound of B if (i) x is an upper bound of B and (ii) for all upper bounds y of B, we have $x \sqsubseteq y$. We denote such x by $\bigsqcup B$.
- x is the greatest lower bound of B if (i) x is a lower bound of B and (ii) for all lower bounds y of B, we have $y \sqsubseteq x$. We denote such x by $\bigsqcap B$.

Example 2.
- Consider the poset $(\mathbb{N}, \leq)$. Then for any $B \subseteq \mathbb{N}$, if B is finite, $\bigsqcup B$ is well-defined and equal to $\max B$. If B is infinite, then $\bigsqcup B$ does not exist.
- Consider the poset $(\mathbb{N} \cup \{\infty\}, \leq)$ where for all x in $\mathbb{N}$, it holds that $x \leq \infty$. Then for all $B \subseteq \mathbb{N}$, the least upper bound $\bigsqcup B$ is well-define.
- Let A be any set and consider the poset $(\mathscr{P}(A), \subseteq)$. For any subset B of $\mathscr{P}(A)$, it holds that $\bigsqcup B = \bigcup B$ and $\bigsqcap B = \bigcap B$.

Definition 8. Poset $(A, \sqsubseteq)$ is a complete-lattice if for all $B \subseteq A$, both $\bigsqcap B$ and $\bigsqcup B$ exist.

Example 3. Let $(A, \sqsubseteq)$ be a complete-lattice.

- $\bigsqcup A = \top$
- $\bigsqcap A = \bot$
- $\bigsqcup \varnothing = \bot$
- $\bigsqcap \varnothing = \top$

Theorem 2 (Knaster-Tarski). For every complete lattice $(A, \sqsubseteq)$ and monotone function F on A, it holds that

1. $\bigsqcup \{x \in A | x \sqsubseteq F(x)\}$ is the unique greatest fixpoint of F.
2. $\bigsqcap \{x \in A | F(x) \sqsubseteq x\}$ is the unique least fixpoint of F.

Homework 1. Prove the Knaster Tarski Theorem.

Class 3

Definition 9 (Prefixpoint). Consider a lattice $(A, \sqsubseteq)$ and a function $f: A \to A$. The set of prefixes is

$$\{x \in A : x \sqsubseteq f(x)\}.$$

Definition 10 (Postfixpoint). Consider a lattice $(A, \sqsubseteq)$ and a function $f: A \to A$. The set of postfixes is

$$\{x \in A : f(x) \sqsubseteq x\}.$$

Definition 11 (gfp and lfp). Consider a complete lattice $(A, \sqsubseteq)$ and a function $f: A \to A$. Then,

$$\begin{aligned} \mathrm{gfp} f &:= \bigsqcup \{x \in A : x \sqsubseteq f(x)\} \\ \mathrm{lfp} f &:= \bigsqcap \{x \in A : f(x) \sqsubseteq x\}. \end{aligned}$$

Theorem 3 (Fixpoints). Consider a complete lattice $(A, \sqsubseteq)$ and a monotonic function $f: A \to A$. Then, $\mathrm{gfp} f$ and $\mathrm{lfp} f$ are fixpoints of f and, for all fixpoints x of f, we have $\mathrm{lfp} f \sqsubseteq x \sqsubseteq \mathrm{gfp} f$.

Definition 12 ($\bigsqcup$-continuous). Consider a complete lattice $(A, \sqsubseteq)$. A function $f: A \to A$ is $\bigsqcup$-continuous if, for all increasing sequences $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq x_2 \sqsubseteq \ldots$, we have

$$f\left(\bigsqcup \{x_n : n \in \mathbb{N}\}\right) = \bigsqcup \{f(x_n) : n \in \mathbb{N}\}.$$

Definition 13 ($\bigsqcap$-continuous). Consider a complete lattice $(A, \sqsubseteq)$. A function $f: A \to A$ is $\bigsqcap$-continuous if, for all increasing sequences $x_0 \sqsupseteq x_1 \sqsupseteq x_2 \sqsupseteq x_2 \sqsupseteq \ldots$, we have

$$f\left(\bigsqcap \{x_n : n \in \mathbb{N}\}\right) = \bigsqcap \{f(x_n) : n \in \mathbb{N}\}.$$

Lemma 1. $\bigsqcup$-continuous implies monotonicity and $\bigsqcap$-continuous implies monotonicity.

Theorem 4 (Constructive fixpoints). Consider a complete lattice $(A, \sqsubseteq)$ and a monotonic function $f: A \to A$. Then,

$$\begin{aligned} \mathrm{lfp} f &= \bigsqcup \{f^n(\bot) : n \in \mathbb{N}\} \\ \mathrm{gfp} f &= \bigsqcap \{f^n(\top) : n \in \mathbb{N}\}. \end{aligned}$$

Homework 2. Prove this theorem.

Definition 14 ($\mathbb{N}$). Define $\mathbb{N}$ as the smallest set X such that

1. $0 \in X$

2. if $n \in X$, then $Sn \in X$

In the definition of $\mathbb{N}$, we consider a universal set U sufficiently big, the complete lattice $(2^U, \subseteq)$ and the function on sets given by $f(Y) := \{0\} \cup \{Sn : n \in Y\}$. Then, $\mathsf{lfp} f = \mathbb{N}$.

Definition 15 (Set of words). Consider a finite alphabet Σ. Define Σ^* as the smallest set X such that

1. $\varepsilon \in X$
2. for all $a \in \Sigma$, we have $aX \subseteq X$.

A few remarks are in place.

- Inductively defined sets are countable and consists of finite elements.
- Inductively defined sets can be written as rules $x \Rightarrow f(x)$ meaning that, if $x \in X$, then $f(x) \in X$.
- Inductively defined sets allow proof by induction. Consider prove that for all $x \in X$ we have $G(x)$. This can be proven by showing
    1. $G(\bot)$
    2. For all $x \in X$, if $G(x)$, then $G(f(x))$

Definition 16 (Balanced binary sequences). Define the set S as the largest set X such that

1. $X \subseteq 01X \cup 10X$.

In the definition of balanced binary sequences, we consider the complete lattice $(\Sigma^\omega, \subseteq)$ and the function on sets given by $f(X) := 01X \cup 10X$. Then, balanced binary sequences corresponds to $\mathsf{gfp} f$.

Definition 17 (Interval $[0,1]$). Define the set S as the largest set X such that

1. $X \subseteq 0X \cup 1X \cup \ldots \cup 9X$.

A few remarks are in place.

- Coinductively defined sets are uncountable and consists of infinite elements.
- Coinductively defined sets can be written as rules $x \Leftarrow f(x)$ meaning that, for all $y \in X$, there exists x such that $y = f(x)$ and $x \in X$.
- Coinductively defined sets allow proof by induction. Consider prove that for all $x \in X$ we have $G(x)$. This can be proven by showing
    1. For all x and i, if $G(f_i(x))$, then $G(x)$

    where $\{f_1, \ldots, f_n\}$ is the set of rules that define the set X.

Homework 3 (Prove balanced binary sequences). Consider S generated by the rules $X \Leftarrow 01X$ and $X \Leftarrow 10X$. Prove that, for all binary words x, we have that $x\ inS$ if and only iff every finite prefix of even length of x has the same number of 0s and 1s.

Hints.

1. The direction $\Leftarrow$ can be proven by coinduction.
2. The direction $\Rightarrow$ can be proven by induction on the length of the prefix.