

Variable transform

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Lets analyze

$$\alpha = \int d^3 p' \int d^3 p \frac{f(\vec{p}, \vec{p}')}{(\vec{p} - \vec{p}' + \vec{k}/2)^2} \quad (1)$$

Make the substitution:

$$\vec{q} = \vec{p} - \vec{p}' + \vec{k}/2 \quad (2)$$

$$\implies \vec{p} = \vec{q} + \vec{p}' - \vec{k}/2 \quad (3)$$

So with respect to the first integral $d^3 p$ we have $d^3 p = d^3 q$. Do we have to account for the fact that we are substituting a variable that is later integrated over? Certainly if we consider only the first integral:

$$\beta = \int d^3 p \frac{f(\vec{p}, \vec{p}')}{(\vec{p} - \vec{p}' + \vec{k}/2)^2} \quad (4)$$

$$= \int d^3 q \frac{f(\vec{q} + \vec{p}' - \vec{k}/2, \vec{p}')}{\vec{q}^2} \quad (5)$$

$$= \int dq \int d\hat{q} f(\vec{q} + \vec{p}' - \vec{k}/2, \vec{p}') \quad (6)$$

Where in the last step we write this in spherical coordinates, and $d\hat{q} = d\phi_q d\theta_q \sin \theta_q$ represents the radial integration. If we don't have to worry about doing a substitution with a variable we are later integrating with then we can write α as:

$$\alpha = \int dp' d\hat{p}' p'^2 \int dq d\hat{q} f(\vec{q} + \vec{p}' - \vec{k}/2, \vec{p}') \quad (7)$$

But the mathematician in me is uneasy since it appears the dependence just vanishes. To explain this first consider a trivial example:

$$1 = \int_0^\infty \int_0^\infty dx dy e^{-(x+y)} = \int_0^\infty \int_y^{\infty+y} e^{-u} du dy \quad (8)$$

$$= \int_0^\infty \int_y^\infty e^{-u} du dy = 1 \quad (9)$$

So the dependence goes into the bounds of the integration. In our example we have:

$$\alpha = \int d^3 p' \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \frac{f(\vec{p}, \vec{p}')}{(\vec{p} - \vec{p}' + \vec{k}/2)^2} \quad (10)$$

$$\begin{aligned} &= \int d^3 p' \int_{-\infty + (-\vec{p}' + \vec{k}/2)_x = q_x}^{\infty + (-\vec{p}' + \vec{k}/2)_x = q_x} dq_x \int_{-\infty + (-\vec{p}' + \vec{k}/2)_y = q_y}^{\infty + (-\vec{p}' + \vec{k}/2)_y = q_y} dq_y \\ &\quad \times \int_{-\infty + (-\vec{p}' + \vec{k}/2)_z = q_z}^{\infty + (-\vec{p}' + \vec{k}/2)_z = q_z} dq_z \frac{f(\vec{q} + \vec{p}' - \vec{k}/2, \vec{p}')}{\vec{q}^2} \end{aligned} \quad (11)$$

$$= \int d^3 p' \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \int_{-\infty}^{\infty} dq_z \frac{f(\vec{q} + \vec{p}' - \vec{k}/2, \vec{p}')}{\vec{q}^2} \quad (12)$$

$$= \int d^3 p' \int d^3 q \frac{f(\vec{q} + \vec{p}' - \vec{k}/2, \vec{p}')}{\vec{q}^2} \quad (13)$$

$$= \int dp' d\hat{p}' p'^2 \int dq d\hat{q} f(\vec{q} + \vec{p}' - \vec{k}/2, \vec{p}') \quad (14)$$