Neutral Pion Photoproduction

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1 Diagram 1

I will not be inserting the Feynman diagrams into latex right now, but "Diagram 1" refers to the photon coming in, hitting nucleon one, a virtual pion propagating to nucleon 2, and a real pion coming out.

From an in person discussion, the matrix element is:

$$\mathcal{M}_{1\to 2} = -\frac{eg_A}{16f_\pi^3}\frac{1}{8}\frac{\left(\omega + \sqrt{m_\pi^2 + \vec{k}'^2}\right)}{\omega^2 - q^2 - m_\pi^2 + i\varepsilon}\,\vec{\epsilon}\cdot\vec{\sigma}_1\,\varepsilon^{a3c}\tau_1^c\,\varepsilon^{abd}\tau_2^d \tag{1}$$

But we also have the contribution from the photon hitting nucleon two and spitting out a pion on nucleon one, so we need to include this symmetry for the full contribution from this diagram

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_{1\to 2} + \mathcal{M}_{2\to 1} && (2)\\
&= -\frac{eg_A}{16f_\pi^3}\,\frac{1}{8}\frac{\left(\omega + \sqrt{m_\pi^2 + \vec{k}'^2}\right)}{\omega^2 - q^2 - m_\pi^2 + i\varepsilon}\,\left(\vec{\epsilon}\cdot\vec{\sigma}_1\,\varepsilon^{a3c}\tau_1^c\,\varepsilon^{abd}\tau_2^d + \vec{\epsilon}\cdot\vec{\sigma}_2\,\varepsilon^{a3c}\tau_2^c\,\varepsilon^{abd}\tau_1^d\right) && (3)\\
&= -\frac{eg_A}{16f_\pi^3}\,\frac{1}{8}\frac{\left(\omega + \sqrt{m_\pi^2 + \vec{k}'^2}\right)}{\omega^2 - q^2 - m_\pi^2 + i\varepsilon}\,\vec{\epsilon}\cdot(\vec{\sigma}_1 + \vec{\sigma}_2)\left[\varepsilon^{a3c}\tau_1^c\,\varepsilon^{abd}\tau_2^d + \varepsilon^{a3c}\tau_2^c\,\varepsilon^{abd}\tau_1^d\right] && (4)
\end{aligned}$$

Note τ_1 and τ_2 operate in different spaces so $\left[\tau_1^a, \tau_2^b\right] = 0\ \forall\, a, b$. Now focus on just the τ matrix structure and recall $\pi_0 \implies b = 3$

$$\begin{aligned}
\varepsilon^{a3c}\tau_1^c\,\varepsilon^{abd}\tau_2^d &= (\delta^{3b}\delta^{dc} - \delta^{3d}\delta^{cb})\tau_1^b\tau_2^c && (5)\\
&= \delta^{3b}\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^3\tau_2^b && (6)\\
&= \vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^3\tau_2^3 && (7)
\end{aligned}$$

So now just plug this into (4)

$$\mathcal{M} = -\frac{eg_A}{64f_\pi^3}\,\frac{\left(\omega + \sqrt{m_\pi^2 + \vec{k}'^2}\right)}{\omega^2 - q^2 - m_\pi^2 + i\varepsilon}\,\vec{\epsilon}\cdot(\vec{\sigma}_1 + \vec{\sigma}_2)\ \left(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_2^3\tau_1^3\right) \tag{8}$$

At threshold we have:

$$\vec{k}' \to 0 \tag{9}$$

$$\omega \to m_\pi \tag{10}$$

Including $\binom{A}{2}$ this now gives

$$\mathcal{M}_t = -\frac{eg_A}{64f_\pi^3}\binom{A}{2}\frac{\left(\omega + \sqrt{m_\pi^2 + \vec{k}'^2}\right)}{\omega^2 - q^2 - m_\pi^2 + i\varepsilon}\,\vec{\epsilon}\cdot(\vec{\sigma}_1 + \vec{\sigma}_2)\left(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_2^3\tau_1^3\right) \tag{11}$$

$$= \frac{e\,m_\pi\,g_A}{32f_\pi^3}\binom{A}{2}\frac{\vec{\epsilon}\cdot(\vec{\sigma}_1 + \vec{\sigma}_2)\left(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_2^3\tau_1^3\right)}{q^2 + i\varepsilon} \tag{12}$$

Now lets make the assumption that q can be written in terms of just $k/2$ this gives an additional factor of $1/4$ prefactor becomes

$$\frac{e\,m_\pi\,g_A}{32f_\pi^3}\binom{A}{2}\frac{1}{4} = \frac{e\,m_\pi\,g_A}{128f_\pi^3}\binom{A}{2} \tag{13}$$

This 128 now matches the Lenkewitz prefactor. Including the factor K_{2N} the Lenkewitz result for this diagram is:

$$\mathcal{M}_{1,\text{Lenkewitz}} = \frac{m_{3N}}{(m_{3N} + m_\pi)}\frac{m_\pi e g_A}{128\pi\,(2\pi)^3\,f_\pi^3}\binom{A}{2}\frac{\vec{\epsilon}\cdot(\vec{\sigma}_1 + \vec{\sigma}_2)\,(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^z\tau_2^z)}{\left[\vec{k}_1 - \vec{k}_2 - \vec{k}_1' + \vec{k}_2' + \vec{k}_\gamma\right]^2} \tag{14}$$

Recall $m_{3N}/(m_{3N} + m_\pi) \approx 1$ and is a prefactor from other considerations.

I think in the Lenkewitz paper they use a different convention. Their denominator works out to:

$$p_{12} - p_{12}' + k_\gamma/2 = \frac{1}{2}\left[(k_1 - k_2) - \left(k_1' - k_2'\right) + k_\gamma\right] \tag{15}$$

Whereas with our convention q works out to

$$q = \left(p - p'\right) + \frac{1}{2}\left(k' - k\right) + k_\gamma \qquad \text{Conservation at nucleon 1} \tag{16}$$

$$= \left(p - p'\right) + \frac{3}{2}k' - \frac{1}{2}k \qquad \text{Conservation at nucleon 2} \tag{17}$$

Additionally there is the factor of $1/\pi(2\pi)^3$, which possibly comes from the integration but it is hard to be sure.

2 Diagram 2

In this diagram the incoming photon strikes a virtual pion propagating between nucleons. q_1 is the initial momentum of the propagating pion, and q_2 is the momentum after.

$$\begin{aligned}\mathcal{M}_{1\to2} = \frac{1}{8}\left[\frac{g_A}{2f_\pi}\vec{\sigma}_1\cdot\vec{q}_1\tau_1^a\right] i\left[-\vec{q}_1^{\,2} - m_\pi^2 + i\varepsilon\right]^{-1}\left[e\varepsilon^{a3b}\vec{\epsilon}\cdot(\vec{q}_1+\vec{q}_2)\right]\\ i\left[E_\pi^2 - \vec{q}_2^{\,2} - m_\pi^2 + i\varepsilon\right]^{-1}\left[\frac{1}{8f_\pi^2}v\cdot\left(q_2+k'\right)\varepsilon^{bcd}\tau_2^d\right]\end{aligned} \tag{18}$$

$$= -\frac{eg_A}{16f_\pi^3}\frac{1}{8}\frac{\left[\vec{\sigma}_1\cdot\vec{q}_1\tau_1^a\right]\left[\varepsilon^{a3b}\vec{\epsilon}\cdot(\vec{q}_1+\vec{q}_2)\right]\left[v\cdot\left(q_2+k'\right)\varepsilon^{bcd}\tau_2^d\right]}{\left[-\vec{q}_1^{\,2} - m_\pi^2 + i\varepsilon\right]\left[E_\pi^2 - \vec{q}_2^{\,2} - m_\pi^2 + i\varepsilon\right]} \tag{19}$$

Now $v\cdot(q_2+k') = (1,0,0,0)\cdot(q_2+k')$.
The matrix structure is

$$\tau_1^a\varepsilon^{a3b}\varepsilon^{bcd}\tau_2^d = \varepsilon^{a3b}\varepsilon^{bcd}\tau_1^a\tau_2^d \tag{20}$$
$$= -\varepsilon^{b3a}\varepsilon^{bcd}\tau_1^a\tau_2^d \tag{21}$$
$$= -\vec{\tau}_1\cdot\vec{\tau}_2 + \tau_1^3\tau_2^3 \tag{22}$$

$$\mathcal{M}_{1\to2} = -\frac{eg_A}{16f_\pi^3}\frac{\left[\vec{\sigma}_1\cdot\vec{q}_1\right]\left[\vec{\epsilon}\cdot(\vec{q}_1+\vec{q}_2)\right]\left[v\cdot\left(q_2+k'\right)\right]}{\left[-\vec{q}_1^{\,2} - m_\pi^2 + i\varepsilon\right]\left[E_\pi^2 - \vec{q}_2^{\,2} - m_\pi^2 + i\varepsilon\right]}\left[-\vec{\tau}_1\cdot\vec{\tau}_2 + \tau_1^3\tau_2^3\right] \tag{23}$$

This in the pion going from $1\to2$, we also need $2\to1$

$$\mathcal{M} = \mathcal{M}_{2\to1} + \mathcal{M}_{1\to2} \tag{24}$$
$$= -\frac{eg_A}{16f_\pi^3}\frac{1}{8}\frac{(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\vec{q}_1\left[\vec{\epsilon}\cdot(\vec{q}_1+\vec{q}_2)\right]\left[E_{q_2}+E_{k'}\right]}{\left[-\vec{q}_1^{\,2} - m_\pi^2 + i\varepsilon\right]\left[E_\pi^2 - \vec{q}_2^{\,2} - m_\pi^2 + i\varepsilon\right]}\left[-\vec{\tau}_1\cdot\vec{\tau}_2 + \tau_1^3\tau_2^3\right] \tag{25}$$

In the threshold limit we have

$$E_\pi \to m_\pi \tag{26}$$
$$E_{q_2} \to m_\pi \tag{27}$$
$$E_{k'} \to m_\pi \tag{28}$$

Which gives

$$\mathcal{M} = \frac{e\,m_\pi g_A}{128f_\pi^3}\;\frac{\vec{q}_1\cdot(\vec{\sigma}_1+\vec{\sigma}_2)\left[\vec{\epsilon}\cdot(\vec{q}_1+\vec{q}_2)\right]}{\left(\vec{q}_1^{\,2}+m_\pi^2-i\varepsilon\right)\left(\vec{q}_2^{\,2}+i\varepsilon\right)}\left[\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^3\tau_2^3\right] \tag{29}$$

This is extremely similar to Lenkewitz, since this term is more complicated I think it is better to wait until we have Diagram 1 figured out until we compare this result directly to the Lenkewitz result.

2.1 The factors of π

Lenkewitz has an additional factor of $1/\pi(2\pi)^3$ in his definition of K_{2N}. Is there any good way to figure out if this is from the integration ahead of time? I haven't found additional mention of this term in the thesis.

It seems to me likely that it comes from the integration since there are no Feynman rules which contain factors like this (to my knowledge)