Elastic Pion Scattering Kernel Derivation

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This follows the conventions from the BKM (Bernard, Kaiser, Meissner) review, including the use of $F_\pi = 93.1$, see table 2 on page 23 of the review, however it is not clear if F in the appendix, differs from F_π, by a factor of π. If this is the case a lot of things would make more sense. The BKM review also goes over this reaction on pg 115 for the two body case.

There are a few sources for pion scattering at zero energy:
S-wave scattering length: Beane 2002

Weinberg 1992, includes isospin dependence. ArXiV link doesn't have diagrams. Note Weinberg uses $F_\pi = 186\text{MeV}$, so converting to the BKM convention requires a factor of 2.

Contents

1 Identities, Definitions, and Useful Identities

$$\sigma_j \sigma_k = \delta_{jk} I + i\varepsilon_{jk\ell}\sigma_\ell \tag{1}$$

$$\mu = \frac{m_\pi}{m_{nucl}} \tag{2}$$

$$F_\pi = 93.1\text{MeV}, \quad g_A = 1.26 \quad g_{\pi N} = \frac{gm}{F} \tag{3}$$

For our purposes the incoming and outgoing pions have the same charges.

Notation with momentum four vectors can be confusing, so in this document, for 4-vectors p and q, use the notation $p - q$ to represent:

$$\begin{aligned} p - q &= (\sqrt{m_p + \vec{p}^2}, \vec{p}) + (\sqrt{m_q + \vec{q}^2}, -\vec{q}) & (4) \\ &= \left(\sqrt{m_p + \vec{p}^2} + \sqrt{m_q + \vec{q}^2}, \vec{p} - \vec{q}\right) & (5) \\ &= (p_0 + q_0, \vec{p} - \vec{q}) & (6) \end{aligned}$$

I'm pretty sure this make sense to do, but I'm completely set on it. Additionally this means, that unless otherwise stated:

$$-p = \left(\sqrt{m_p^2 + \vec{p}^2}, -\vec{p}\right) \tag{7}$$

1.0.1 Prefactor

I'm pretty sure that each diagram comes with a prefactor of:

$$\frac{1}{2(1+\mu)} \quad \text{or} \quad \frac{1}{2(1+\mu)\pi^4} \tag{8}$$

1.0.2 Definition of the spin vector

There is still some confusion on the definition of the zeroth element of S.

This source implies its zero up to relativistic corrections. Consider the rest frame, and boosted (lab frame) spins:

$$\text{Rest frame:} \quad S' = (0, s'_x, s'_y, s'_z) \qquad \text{Lab frame: } S = (s_t, s_x, s_y, s_z) \tag{9}$$

This must be Lorentz invariant, so

$$s_t^2 - \vec{s} \cdot \vec{s} = -\vec{s}\,' \cdot \vec{s}\,' \tag{10}$$

Its easier to calculate the rest frame in terms of a boost on the lab frame, so:

$$\begin{aligned} S'^0 &= \Lambda^0{}_\alpha S^\alpha = \Lambda^0{}_0 S^0 + \Lambda^0{}_i S^i = \gamma\left(S^0 - U_i S^i\right) & (11) \\ &= \gamma\left(S^0 - u_i S^i\right) = U_0 S^0 - U_i S^i & (12) \\ &= U_\alpha S^\alpha = 0 \quad \text{(invariant)} & (13) \end{aligned}$$

$$\begin{aligned} S'^i &= \Lambda^i{}_\alpha S^\alpha = \Lambda^i{}_0 S^0 + \Lambda^i{}_j S^j & (14) \\ &= -\gamma U^i S^0 + \left[\delta_{ij} + \frac{\gamma - 1}{\vec{U}^2} U_i U_j\right] S^j & (15) \\ &= S^i + \frac{\gamma^2}{\gamma + 1} U_i U_j S^j - \gamma U^i S^0 & (16) \end{aligned}$$

Note that $S'^0 = 0$ Note $U_\alpha S^\alpha = 0$

Where U is the 3-velocity that boosts the particle to the lab frame. Inverting the above gives the spin in the lab frame from the particles rest frame:

$$s_t = \gamma \vec{U} \cdot s' \tag{17}$$

$$\vec{s} = \vec{s}\,' + \frac{\gamma^2}{\gamma + 1} \vec{U} \left(\vec{U} \cdot \vec{s}\,' \right) \tag{18}$$

And recall $\gamma = \left(1 - \vec{U}^{\,2}\right)^{-1/2}$

1.1 Cross section and Scattering Length

For elastic, UN polarized scattering $A + B \to A + B$:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{cm}} = \frac{1}{64\pi^2 s} \left|\overline{\mathcal{M}}\right|^2 \tag{19}$$

Here $\mathcal{M}$ is the "transition amplitude, averaged over initial spins and summed over final ones". For our case we only have one ingoing and one outgoing state so I think this doesn't matter. In the center of mass frame, where p_1 and p_2 are the incident momenta, we have

$$s = (p_1 + p_2)^2 = \left[(m_1 + m_2) - \vec{0}^{\,2}\right] = m_1 + m_2 \tag{20}$$

And the scattering length is given by:

$$\lim_{k \to 0} \sigma = 4\pi a^2 \tag{21}$$

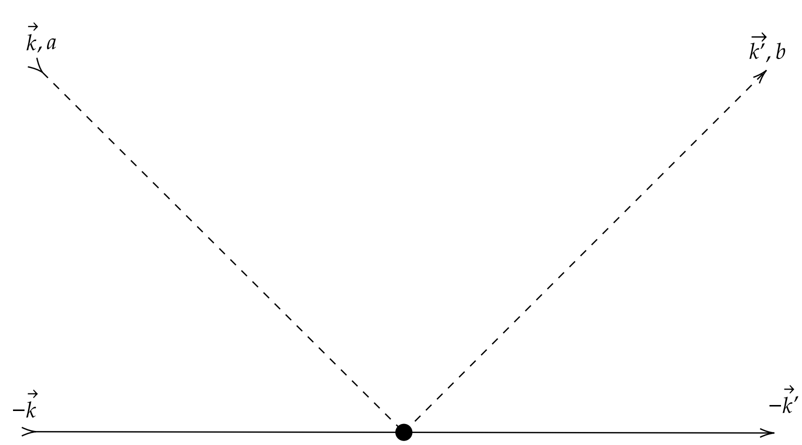
or equivalently

$$\lim_{k \to 0} k \cot \delta(k) = -\frac{1}{a} \tag{22}$$

2 1 Body Contributions

2.1 1 Body A

Diagram 1 A, $\mathcal{O}(p^2)$



$$\mathcal{M}_{1,a} = \frac{1}{4F^2} v \cdot \left(\vec{k} + \vec{k}'\right) \varepsilon^{abc} \tau_c \tag{23}$$

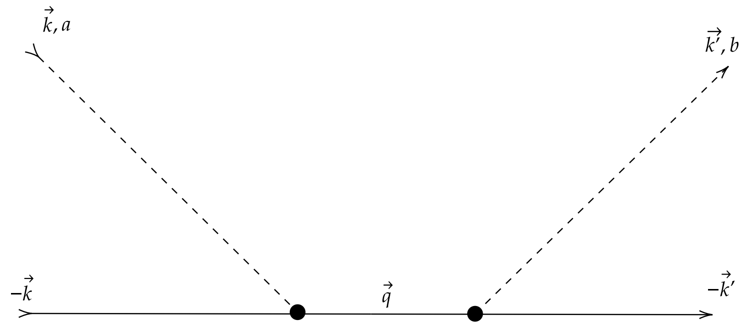
$$= \frac{1}{4F^2} (E_\pi + E_N) \varepsilon^{abc} \tau_c \tag{24}$$

We are in the CM frame, so $\vec{k} + \vec{k}' = 0$, but $E_\pi \neq E_N$, but:

$$\varepsilon^{abc} \tau_c = 0 \quad \text{for} \quad a = b \tag{25}$$

So this diagram is zero.

2.2 1 Body B



$$\mathcal{M}_{1,b} = \left[-\frac{g_A}{F} S \cdot k \tau^a\right] \left(\frac{i}{v \cdot q + i\varepsilon}\right) \left[\frac{g_A}{F} S \cdot k' \tau^b\right] \tag{26}$$

$$= -i \frac{g_A^2}{F^2} \frac{(S \cdot k)(S \cdot k')}{q_0 + i\varepsilon} \tau^a \tau^b \tag{27}$$

$\vec{q} = 0 \implies q = (\sqrt{m_N^2 + \vec{q}^2}, \vec{q}) = m_N$, and letting $S = (0, \frac{1}{2}\vec{\sigma})$ gives

$$\mathcal{M}_{1,b} = -i \frac{g_A^2}{F^2} \frac{(S \cdot k)(S \cdot k')}{m_N + i\varepsilon} \tau^a \tau^b \tag{28}$$

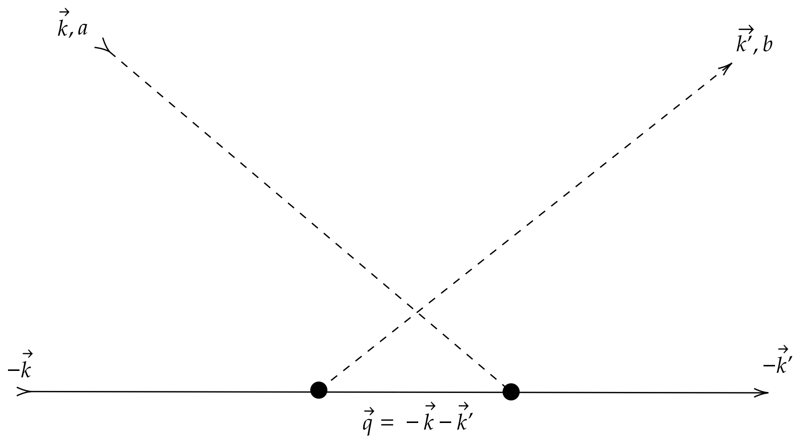
$$= -i \frac{g_A^2}{4F^2} \frac{\vec{\sigma} \cdot \vec{k}\, \vec{\sigma} \cdot \vec{k}\,'}{m_N + i\varepsilon} \tau^a \tau^b \tag{29}$$

Letting $a = b$, and $\vec{k} = -\vec{k}'$ gives:

$$\mathcal{M}_{1,b} = i \frac{g_A^2}{4F^2} \frac{\left(\vec{\sigma} \cdot \vec{k}\right)^2}{m_N + i\varepsilon} \tau^a \tau^a \tag{30}$$

Perhaps this propagator should be off shell, in which case $m_N \to q_0$ and we have to integrate over it

2.3 1 Body C

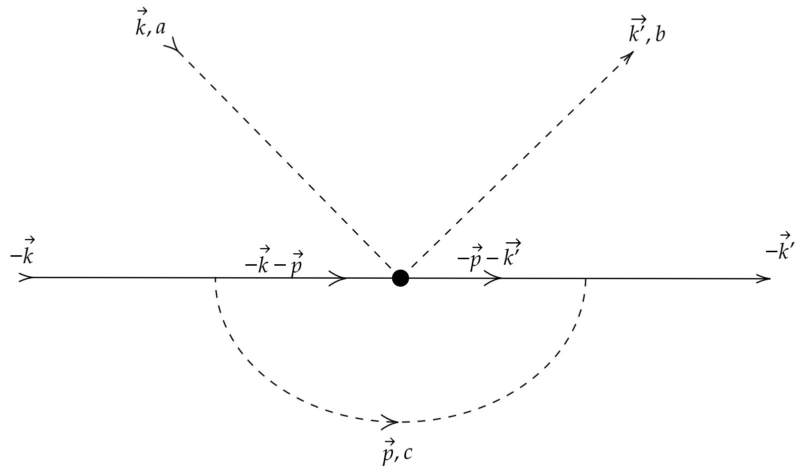


This is the same as diagram B but with $\vec{q} = -\vec{k} - \vec{k}'$, so $q_0 = \sqrt{m_{_N}^2 + \vec{q}^{\,2}}$

$$\mathcal{M}1, c = -i\frac{g_A^2}{4F^2}\,\frac{\vec{\sigma}\cdot\vec{k}\;\vec{\sigma}\cdot\vec{k}\,'}{q_0 + i\varepsilon}\,\tau^b\tau^a \tag{31}$$

2.4 1 Body D

Diagram 1 B, $\mathcal{O}(p^3)$

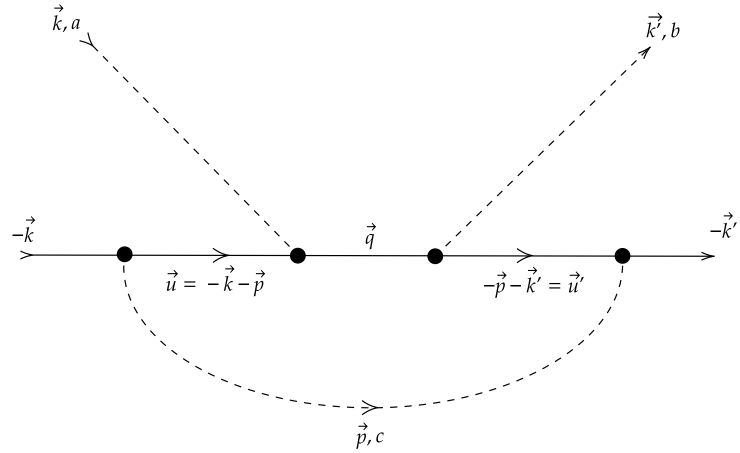


Let $u = -k - p$ and $u' = u + k - k' = -p - k'$

$$\begin{aligned}\mathcal{M}_{1,d} = &\left[\frac{g_A}{F} S \cdot p\, \tau^c\right] i \left[\vec{u} \cdot \left(-\vec{k} - \vec{p}\right) + i\varepsilon\right]^{-1} \left[\frac{1}{4F^2} v \cdot \left(k + k'\right) \varepsilon^{abd} \tau^d\right] \\ &\times i \left[\vec{u}' \cdot \left(-\vec{p} - \vec{k}'\right) + i\varepsilon\right]^{-1} \left[\frac{g_A}{F} S \cdot (-p\,)\, \tau^c\right] i[\vec{p}^2 + i\varepsilon]^{-1} \qquad (32)\\ &= i \frac{g_A}{4F^4} \frac{(S \cdot p\,)^2}{\left(\vec{p}^2 + i\varepsilon\right) \left(\vec{u} \cdot \left(\vec{k} + \vec{p}\right) + i\varepsilon\right) \left(\vec{u}' \cdot \left(\vec{p} + \vec{k}'\right) + i\varepsilon\right)} (E_\pi + E'_\pi) \varepsilon^{abd} \tau^d \tau^c \tau_c \qquad (33)\end{aligned}$$

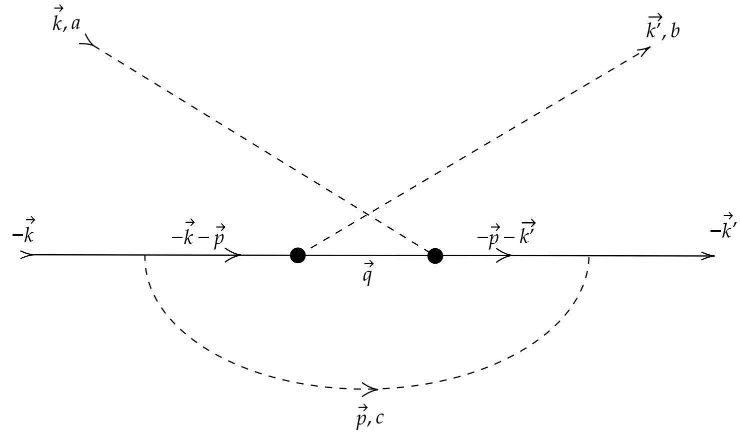
Check this, did a lot of mental calculations

2.5 1 Body E



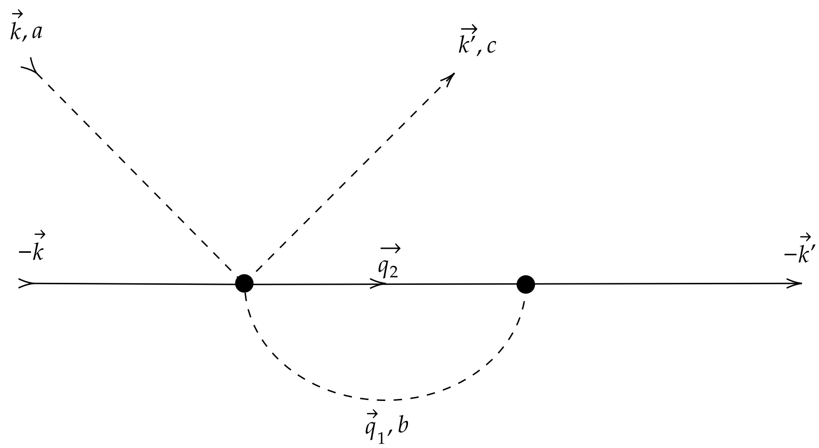
$$\begin{aligned}\mathcal{M}_{1,E} = &\left[\frac{g_A}{F} S \cdot p \tau^c\right] \frac{i}{v \cdot (-k - p) + i\varepsilon} \left[-\frac{g_A}{F} \S \cdot k \tau^a\right] \left(\frac{1}{v \cdot q + i\varepsilon}\right) \\ &\times \left(\frac{1}{p^2 - m_\pi^2 + i\varepsilon}\right) \left[\frac{g_A}{F} S \cdot k' \tau^b\right] \left(\frac{1}{v \cdot u' + i\varepsilon}\right) \left[-\frac{g_A}{F} S \cdot p \tau^c\right] \end{aligned} \tag{34}$$

2.6 1 Body F



2.7 1 Body G

Diagram 1 D, $\mathcal{O}(p^4)$

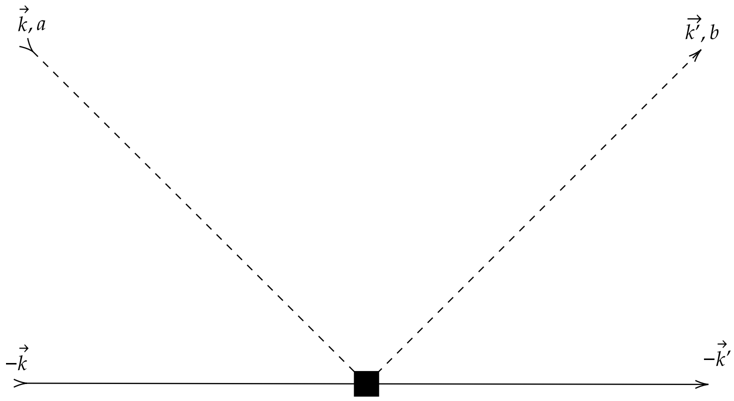


Using BKM A.16

$$\mathcal{M}_{1,d} = \frac{g_A}{2F^3}\left[\tau^a\delta^{bc}S\cdot\left(q_1+k'\right)+\tau^b\delta^{ac}S\cdot(-k+q_1)+\tau^c\delta^{ab}S\cdot(-k+q_1)\right] \times i\left[q_1^2-m_\pi^2+i\varepsilon\right]^{-1} i\left[v\cdot q_2+i\varepsilon\right]^{-1}\left[\frac{g_A}{F}S\cdot(-q_1)\tau^b\right] \tag{35}$$

2.8 1 Body H

Diagram 1 C, $\mathcal{O}(p^4)$



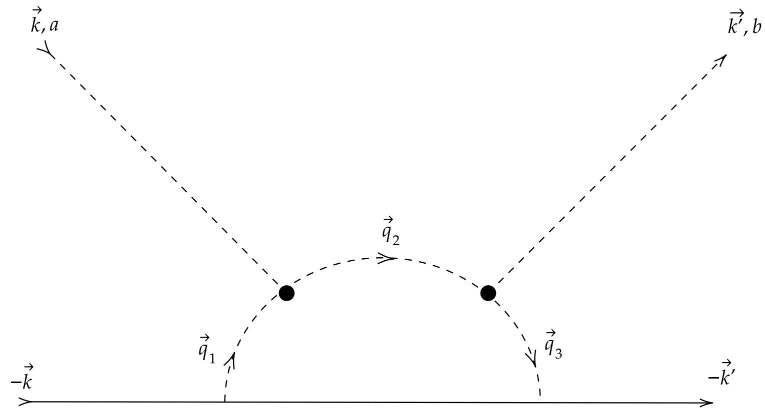
Only Feynman rule is BKM review, A.29, but I'm not going to write it down here since its rather long, but here is a screenshot of the rule.

2 pions (q_1 in, q_2 out):

$$\begin{aligned}
&\frac{i\delta^{ab}}{F^2}\left[-4c_1 1M_\pi^2 + (2c_2 - \frac{g_A^2}{4m})v\cdot q_1 v\cdot q_2 + 2c_3 q_1\cdot q_2\right] \\
&+ \frac{1}{8mF^2}\epsilon^{abc}\tau^c\left[(p_1+p_2)\cdot(q_1+q_2) - v\cdot(p_1+p_2)v\cdot(q_1+q_2)\right] \\
&- \frac{1}{F^2}\left(2c_4 + \frac{1}{2m}\right)\epsilon^{abc}\tau^c[S\cdot q_1, S\cdot q_2]
\end{aligned} \tag{A.29}$$

2.9 1 Body I

Diagram 1 C, $\mathcal{O}(p^4)$



This diagram is 0, BKM A.3

3 2 Body Contributions

Note that for the scattering length at least, there is a prefactor:

$$\frac{1}{1+\mu} \equiv \alpha \tag{36}$$

which comes from considerations other than the diagrams. Additionally, see BKM review equation 5.29:

$$a_{ab} = \frac{1 + m_\pi/m_N}{1 + m_\pi/Am_N} \sum_r a_{ab}^{(r)} + a_{ab}^{\text{three-body}} \tag{37}$$

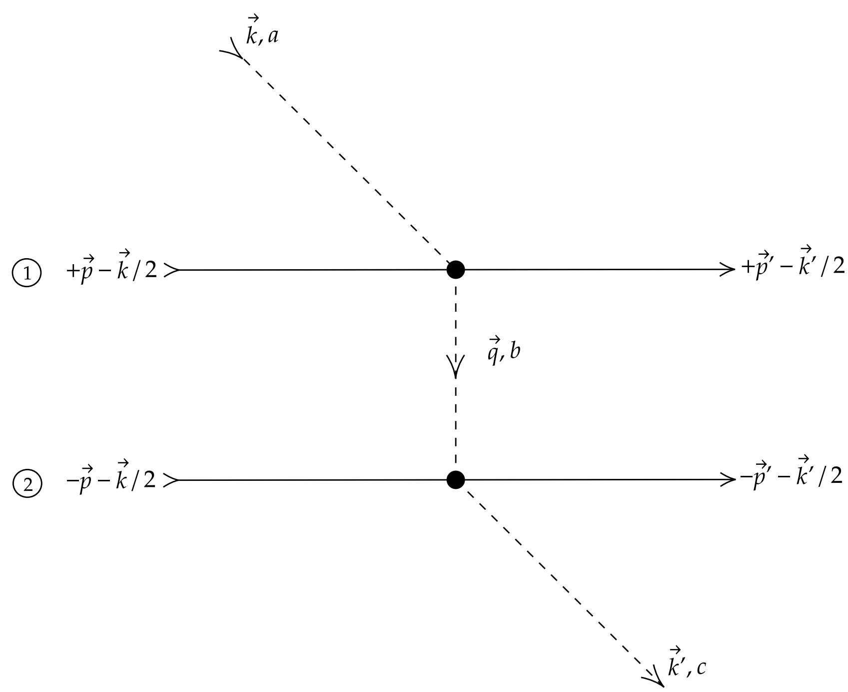
Note for the BKM review "three-body" means two nucleons and an external probe, which is what we call two body. In eq.(37) a, b are pion isospin indices, and r (and later s is used for this too) is for nucleon labeling.

The BKM review states:

$$(t_c^{(\pi)})_{ab} = -i\epsilon_{abc} \quad \text{is the pion isospin vector} \tag{38}$$

But this only appears in the last diagram, and is specifically the pion isospin operator, not the nucleon isospin operator.

3.1 2 Body A



Note $q = k/2 + p$

$$\mathcal{M}_{2,a} = \alpha \left[\frac{1}{4F^2} v \cdot (k+q)\, \varepsilon^{abd} \tau_1^d \right] i \left[q^2 - m_\pi^2 + i\varepsilon \right]^{-1} \left[\frac{1}{4F^2} v' \cdot \left(q + k'\right) \varepsilon^{bce} \tau_2^e \right] \tag{39}$$

$$= \alpha \left(\frac{1}{2F} \right)^4 \frac{(E_\pi + q_0)(q_0 + E'_\pi)}{q^2 - m_\pi^2 + i\varepsilon} \varepsilon^{abd} \varepsilon^{bce} \tau_1^d \tau_2^e \tag{40}$$

Where: $\vec{q} = \vec{p} - \vec{p}\,' + \frac{1}{2}\left(\vec{k} + \vec{k}\,'\right)$, and $q_0 = \sqrt{m_{\pi 0}^2 + \vec{q}^{\,2}}$ We now restrict ourselves to just the inelastic process, where $c = a$, then computing the matrix dependence gives:

$$\varepsilon^{abd} \varepsilon^{bae} \tau_1^d \tau_2^e = -1 \left(\varepsilon^{bad} \varepsilon^{bae} \right) \tau_1^d \tau_2^e \tag{41}$$

$$= \left(\delta^{ae} \delta^{da} - \delta^{aa} \delta^{de} \right) \tau_1^d \tau_2^e \tag{42}$$

$$= (\delta^{ae} \delta^{da}) \tau_1^d \tau_2^e - \tau_1^e \tau_{2e} \tag{43}$$

$$= \tau_1^a \tau_2^a - \tau_1^e \tau_{2e} \tag{44}$$

Here, the index a, is not being summed over. For example in the case of neutral pion pion scattering $a = 3$ and this reduces to

$$\tau_1^3 \tau_2^3 - \vec{\tau}_1 \cdot \vec{\tau}_2 \tag{45}$$

So the diagram contribution is then:

$$\mathcal{M}_{2,a} = \left(\frac{1}{2F} \right)^4 \frac{(E_\pi + q_0)(q_0 + E'_\pi)}{q^2 - m_\pi^2 + i\varepsilon} \left(\tau_1^a \tau_2^a - \vec{\tau}_1 \cdot \vec{\tau}_2 \right) \tag{46}$$

Or in the threshold case:

$$\mathcal{M}_{2,a} = \left(\frac{1}{2F} \right)^4 \frac{m_\pi^2}{\vec{q}^{\,2} + i\varepsilon} \left(\tau_1^a \tau_2^a - \vec{\tau}_1 \cdot \vec{\tau}_2 \right) \tag{47}$$

For this diagram, at threshold, Beane gets the result:

$$\frac{M_\pi^2}{32\pi^4 F_\pi^4 (1 + \mu/2)} \frac{1}{\vec{q}^{\,2}} \tag{48}$$

And Weinberg for the threshold case writes the result as (eq 5):

$$\frac{M_\pi^2}{32\pi^4 F_\pi^4 (1 + \mu/2)} \sum_{r<s} \frac{1}{\vec{q}_{rs}^{\;2}} \left(2\vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} \delta_{ab} - t_a^{(r)} t_b^{(s)} - t_a^{(s)} t_b^{(r)} \right) \tag{49}$$

Taking $a = b$ the above reduces to:

$$\frac{M_\pi^2}{16\pi^4 F_\pi^4 (1 + \mu/2)} \sum_{r<s} \frac{1}{\vec{q}_{rs}^{\;2}} \left(\vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} - t_a^{(r)} t_a^{(s)} \right) \tag{50}$$

What do we do about this sum?

3.2 2 Body B

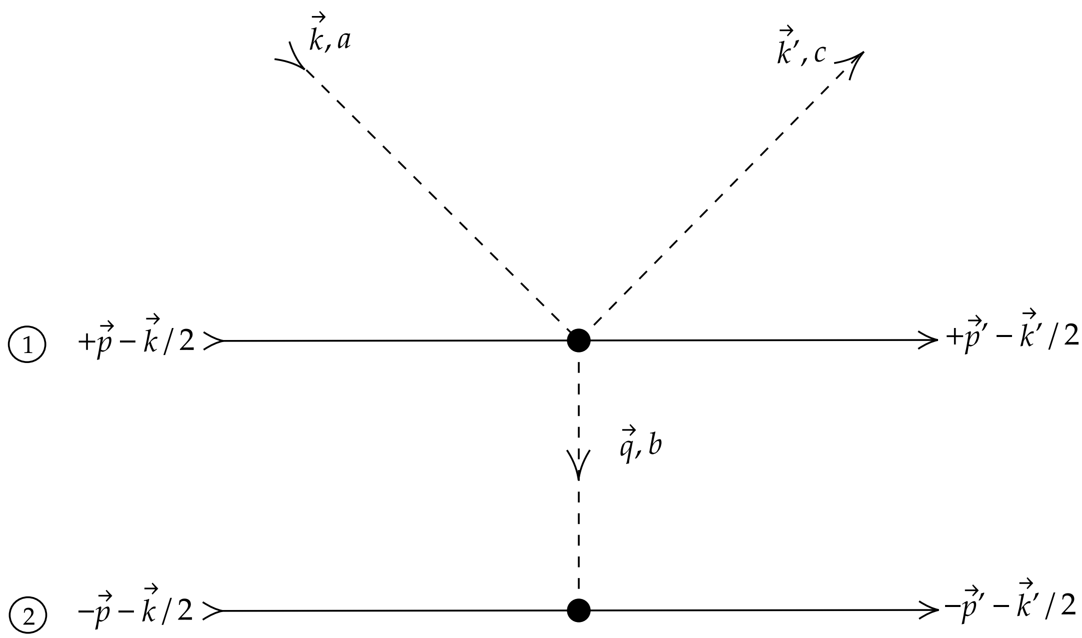
Diagram b (at threshold) according to Weinberg is:

$$-\frac{g_A^2\delta_{ab}}{32\pi^4F_\pi^4(1+\mu)}\vec{\tau}_1\cdot\vec{\tau}_2\frac{\vec{q}\cdot\vec{\sigma}_1\vec{q}\cdot\vec{\sigma}_2}{\vec{q}^2+m_\pi^2} \tag{51}$$

and the Beane paper gives the result for diagram b and c together as:

$$-\frac{g_A^2m_\pi^2}{128\pi^4F_\pi^4(1+\mu)}\frac{\vec{q}\cdot\vec{\sigma}_1\vec{q}\cdot\vec{\sigma}_2}{(\vec{q}^2+m_\pi^2)^2} \tag{52}$$

Using BKM A.16 - check indicies.



$$\begin{aligned}\mathcal{M}_{2,b} = \frac{g_A}{2F^3}&\left[\tau^a\delta^{bc}S_1\cdot(q+k')+\tau^b\delta^{ac}S_1\cdot(k'-k)+\tau^c\delta^{ab}S_1\cdot(q-k)\right]\\ &\times i\left[q^2-m_\pi^2+i\varepsilon\right]^{-1}\left[\frac{g_A}{F}S_2\cdot(-q)\tau^b\right] \end{aligned}\tag{53}$$

$$\tag{54}$$

The Feynman rule for 3 pions (all qs out) is just:

$$\frac{g_A}{2F^3}\left[\tau^a\delta^{bc}S_1\cdot(q_2+q_3)+\tau^b\delta^{ac}S_1\cdot(q_1+q_3)+\tau^c\delta^{ab}S_1\cdot(q_1+q_2)\right] \tag{55}$$

And we have:

$$\vec{q}_1=-\vec{k}\quad \vec{q}_2=\vec{q}\quad \vec{q}_3=\vec{k}' \tag{56}$$

The last τ should be operating on the second nucleon, and the other τ operators are supposed to be on nucleon 1. Additionally, the index b, must be summed over, whereas a and c are external observables (pion isospin). The index b is the only one that is summed over.

$$\mathcal{M}_{2,b} = -i\frac{g_A^2}{8F^4}\frac{1}{q^{\,2}-m_\pi^2+i\varepsilon}\sum_{b=1}^{3} S_1\cdot\left[\tau_1^a\delta^{bc}(q_2+q_3)+\tau_1^b\delta^{ac}(q_1+q_3)+\tau_1^c\delta^{ab}\left(q_1+q_2\right)\right]S_2\cdot q_2\tau_2^b \tag{57}$$

$$= -i\frac{g_A^2}{8F^4}\frac{1}{q^{\,2}-m_\pi^2+i\varepsilon}\sum_{b=1}^{3} S_1\cdot\left[\tau_1^a\tau_2^b\delta^{bc}(q_2+q_3)+\tau_1^b\tau_2^b\delta^{ac}(q_1+q_3)+\tau_2^b\tau_1^c\delta^{ab}\left(q_1+q_2\right)\right]S_2\cdot q_2 \tag{58}$$

$$= -i\frac{g_A^2}{8F^4}\frac{1}{q^{\,2}-m_\pi^2+i\varepsilon}\sum_{b=1}^{3} S_1\cdot\left[\tau_1^a\tau_2^b\delta^{bc}(q_2+q_3)+\vec{\tau}_1\cdot\vec{\tau}_2\delta^{ac}(q_1+q_3)+\tau_2^b\tau_1^a\delta^{ab}\left(q_1+q_2\right)\right]S_2\cdot q_2 \tag{59}$$

$$= -i\frac{g_A^2}{8F^4}\frac{1}{q^{\,2}-m_\pi^2+i\varepsilon}S_1\cdot\left[\tau_1^a\tau_2^c(q_2+q_3)+3\vec{\tau}_1\cdot\vec{\tau}_2(q_1+q_3)\delta^{ac}+\tau_2^a\tau_1^a\left(q_1+q_2\right)\right]S_2\cdot q_2 \tag{60}$$

$$= -i\frac{g_A^2}{8F^4}\frac{1}{q^{\,2}-m_\pi^2+i\varepsilon}S_1\cdot\left[\tau_1^a\tau_2^c(q_1+2q_2+q_3)+3\vec{\tau}_1\cdot\vec{\tau}_2\delta^{ac}(q_1+q_3)\right]S_2\cdot q_2 \tag{61}$$

Now taking $a = c$:

$$\mathcal{M}_{2,b} = -i\frac{g_A^2}{8F^4}\frac{1}{q^{\,2}-m_\pi^2+i\varepsilon}S_1\cdot\left[\tau_1^a\tau_2^a(q_1+2q_2+q_3)+3\vec{\tau}_1\cdot\vec{\tau}_2(q_1+q_3)\right]S_2\cdot q_2 \tag{62}$$

Note that q_2 is the propagator momentum and is therefore off shell. Now taking the limit of the threshold case, $q_1 = q_3 = (m_\pi, \vec{0})$ and $q_2 = (0, \vec{q})$, and the non-relativistic limit of $S = (0, \vec{\sigma})$:

$$\mathcal{M}_{2,b} = i\frac{g_A^2}{16F^4}\frac{\tau_1^a\tau_2^a}{\vec{q}^{\,2}+m_\pi^2-i\varepsilon}\left(\vec{\sigma}_1\cdot\vec{q}\right)\left(\vec{\sigma}_2\cdot\vec{q}\right) \tag{63}$$

Which is really close to the Weinberg result, except for a factor of $\left[2\pi^4\left(1+\mu\right)\right]^{-1}]$ and the constant a should be a sum. Note that the factors of π are fixed if we use $F \to \pi F_\pi$. Also, I wonder if the sum over a comes from allowing any pion to propagate instead of just the neutral pion.

3.2.1 The Propagator

Weinberg writes the structure of the propagator as: $(\vec{q}^{\,2}+m_\pi^2)^{-1}$ Whereas Beane writes it as: $(\vec{q}^{\,2}+m_\pi^2)^{-2}$ But the "starting" propagator as defined in BKM A.1 is $i\delta^{ab}\left(q^2-m_\pi^2+i\varepsilon\right)^{-1}$, where q is the four momentum. Now we can write the propagator as :

$$\left[q^2-m_\pi^2\right]^{-1} = \left[E^2-\vec{q}^{\,2}-m_\pi^2\right]^{-1} \tag{64}$$

$$= \frac{-1}{\vec{q}^{\,2}+m_\pi^2}\left[1-\left(\frac{E^2}{\vec{q}^{\,2}+m_\pi^2}\right)\right]^{-1} \tag{65}$$

$$= \frac{-1}{\vec{q}^{\,2}+m_\pi^2}\left[1+\frac{E^2}{\vec{q}^{\,2}+m_\pi^2}+\left(\frac{E^2}{\vec{q}^{\,2}+m_\pi^2}\right)^2+...\right] \tag{66}$$

Where $E = \sqrt{m_\pi^2+\vec{q}^{\,2}}$. So now taking the threshold case $m_\pi \gg \vec{q}^{\,2}$

$$\left[q^2 - m_\pi^2\right]^{-1} \approx \frac{-1}{\vec{q}^2 + m_\pi^2}\left[1 + \frac{m_\pi^2}{\vec{q}^2 + m_\pi^2} + ...\right] \tag{67}$$

$$\approx \frac{-1}{\vec{q}^2 + m_\pi^2}\left[1 + \frac{m_\pi^2}{\vec{q}^2 + m_\pi^2} + ...\right] \tag{68}$$

But I'm confused why Weinberg bothered with this, it's not that much more complicated to just program the initial propagator. Maybe its to avoid numerical zeros.

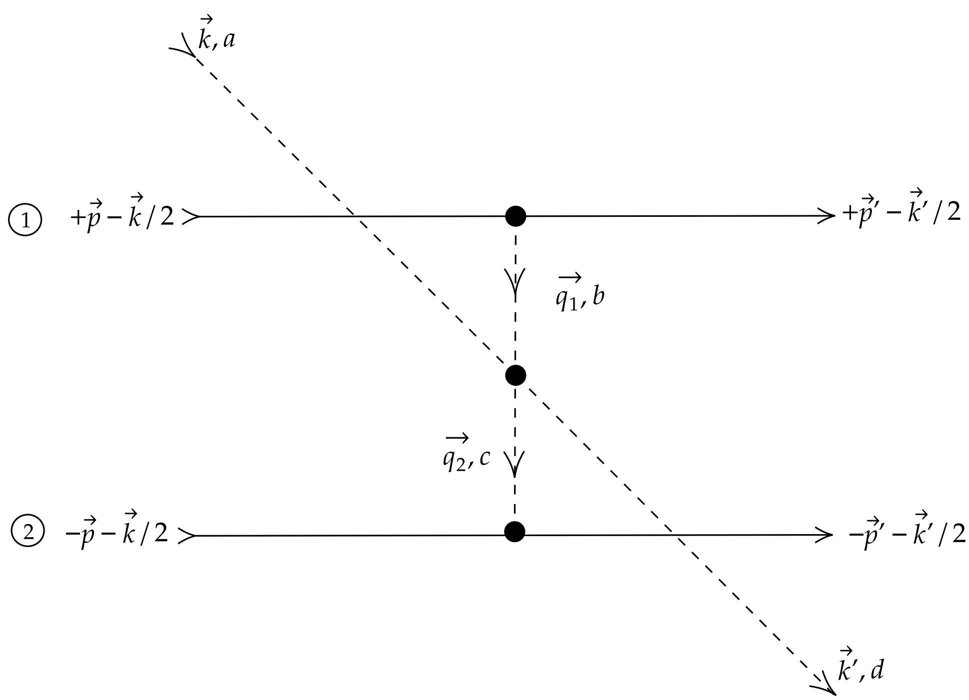
$$\mathcal{M}_{2,b} = -i\frac{g_A^2}{16F^4}\tau_1^a\tau_2^a\vec{\sigma}_1 \cdot \vec{q}\,\vec{\sigma}_2 \cdot \vec{q}\left[\frac{-1}{\vec{q}^2 + m_\pi^2} + \mathcal{O}(q_0^2)\right] \tag{69}$$

So then:

$$\mathcal{M}_{2,b} = i\frac{g_A^2}{16F^4}\tau_1^a\tau_2^a\,\frac{\vec{\sigma}_1 \cdot \vec{q}\,\vec{\sigma}_2 \cdot \vec{q}}{\vec{q}^2 + m_\pi^2} \tag{70}$$

But this is still different than the Weinberg result by a factor of 2 and the isospin dependence.

3.3 2 Body C



With:

$$\vec{q}_1 = \vec{p} - \vec{p}' + \frac{1}{2}(\vec{k}' - \vec{k}) \tag{71}$$

$$\vec{q}_2 = \vec{p} - \vec{p}' + \frac{1}{2}(\vec{k}' + \vec{k}) \tag{72}$$

Using BKM A.10, with all q's in

$$O^{abcd} = \frac{i}{F^2}\left\{\delta^{ab}\delta^{cd}\left[(q_1+q_2)^2 - m_\pi^2\right] + \delta^{ac}\delta^{bd}\left[(q_1+q_3)^2 - m_\pi^2\right] + \delta^{ad}\delta^{bc}\left[(q_1+q_4)^2 - m_\pi^2\right]\right\} \tag{73}$$

From BKM (left hand side), to our labels, (right hand side)

$$\begin{aligned}
&\text{Matrix indices } a, b, c, d \;\text{ remain the same} && (74)\\
&\vec{q}_1 \to \vec{q}_1 \quad \text{index } b && (75)\\
&\vec{q}_2 \to \vec{k} \quad \text{index } a && (76)\\
&\vec{q}_3 \to -\vec{q}_2 \quad \text{index } c && (77)\\
&\vec{q}_4 \to -\vec{k}' \quad \text{index } d && (78)
\end{aligned}$$

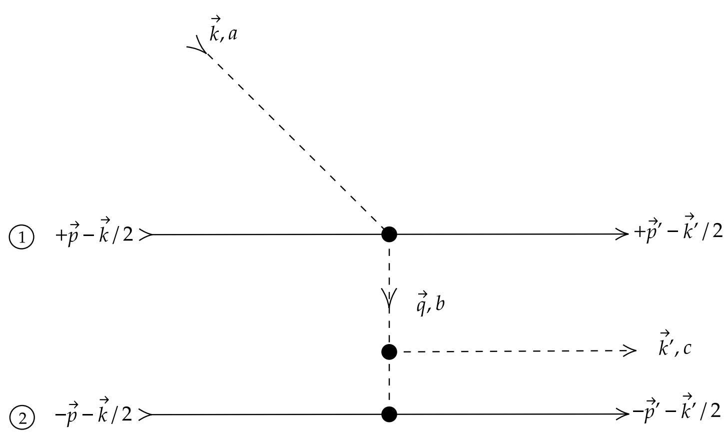
$$\begin{aligned}
\mathcal{M}_{2,c} &= \frac{g}{F} S_1 \cdot q_1 \tau_1^b i \left[q_1 - m_\pi^2 + i\varepsilon\right]^{-1} O^{abcd} \frac{g}{F} S_2 \cdot (-q_2) \tau_2^c i \left[q_2^2 - m_\pi^2 + i\varepsilon\right]^{-1} && (79)\\
&= \frac{g}{F} S_1 \cdot q_1 \tau_1^b i \left[q_1 - m_\pi^2 + i\varepsilon\right]^{-1} \frac{i}{F^2} O^{abcd} \frac{g}{F} S_2 \cdot (-q_2) \tau_2^c i \left[q_2^2 - m_\pi^2 + i\varepsilon\right]^{-1} && (80)\\
&= -i \frac{g^2}{F^4} S_1 \cdot q_1 \; S_2 \cdot (-q_2) O^{abcd} \frac{\tau_1^b \tau_2^c}{(q_1^2 - m_\pi^2 + i\varepsilon)(q_2^2 - m_\pi^2 + i\varepsilon)} && (81)
\end{aligned}$$

I'm sure this is correct, but there is some weird stuff going on with the energy flow. In particular $q_1^2 = -\vec{q}_1^{\,2}$, but $q_2^2 = E_2^2 - \vec{q}_2^{\,2}$. Also note that O^{abcd} has no isospin dependence, so we can commute τ with it. This gives:

$$\mathcal{M}_{2,c} = i \frac{g^2}{F^4} S_1 \cdot q_1 \; S_2 \cdot (-q_2) \frac{\tau_1^b \tau_2^c}{(\vec{q}_1^{\,2} + m_\pi^2 + i\varepsilon)(E_2^2 - \vec{q}_2^{\,2} - m_\pi^2 + i\varepsilon)} O^{abcd} \tag{82}$$

Where E_2 is the

3.4 2 Body D



4 The Scattering Length

The total scattering length is given by BKM eq 5.29:

$$a_{ab} = \frac{1 + m_\pi/m_N}{1 + m_\pi/Am_N} \sum_r a_{ab}^{(r)} + \binom{A}{2} a_{ab}^{\text{Two Nucleon}} \tag{83}$$

Where a, b are the incoming and outgoing isospins of the pion, and m_N is the nucleon mass.

4.1 One Body contributions to the scattering length

The onebody case is typically parameterized in terms of $a^{\pm}$. Here $a_{\pi^{\pm}n}$ stands for the scattering length of the reaction $\pi^{\pm}n \to \pi^{\pm}n$

$$a_{\pi^-p} = a^+ + a^- \quad I = -1 \tag{84}$$
$$a_{\pi^0p} = a^+ \quad I = 0 \tag{85}$$
$$a_{\pi^+p} = a^+ - a^- \quad I = 1 \tag{86}$$

The isospin value on the right hand side refers to the pion isospin. We have something similar for the reaction with the neutron.

$$a_{\pi^-n} = a^+ - a^- \tag{87}$$
$$a_{\pi^0n} = a^+ \tag{88}$$
$$a_{\pi^+n} = a^+ + a^- \tag{89}$$

From "Isospin breaking in the pion–nucleon scattering lengths" Martin Hoferichter et. al..

$$a^+ = (1.5 \pm 2.2) \times 10^{-3} m_\pi^{-1} \tag{90}$$
$$a^- = (85.2 \pm 1.8) \times 10^{-3} m_\pi^{-1} \tag{91}$$

4.2 Two Body contributions to the scattering length

As stated in the BKM review pg 115-116 the two body contributions to the scattering length from the first six diagrams is:

$$a_{ab} = \frac{M_\pi^2}{32\pi^4 F_\pi^4 \left(1 + M_\pi/m_d\right)} \sum_{r<s} \left\langle \frac{1}{\vec{q}_{rs}^{\,2}} \left(2\vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} \delta_{ab} - \tau_a^{(r)}\tau_b^{(s)} - \tau_a^{(s)}\tau_b^{(r)}\right) \right\rangle \tag{92}$$

$$- \frac{g_A^2 \delta_{ab}}{32\pi^4 F_\pi^4 \left(1 + M_\pi/m_d\right)} \sum_{r<s} \left\langle \vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} \frac{\vec{q}_{rs} \cdot \vec{\sigma}^{(r)} \vec{q}_{rs} \cdot \vec{\sigma}^{(s)}}{\vec{q}_{rs}^{\,2} + M_\pi^2} \right\rangle \tag{93}$$

$$+ \frac{g_A^2}{32\pi^4 F_\pi^4 \left(1 + M_\pi/m_d\right)} \sum_{r<s} \left\langle \frac{\left[\vec{q}_{rs}^{\,2} \vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} \delta_{ab} + M_\pi^2 \left(\tau_a^{(r)}\tau_b^{(s)} + \tau_a^{(s)}\tau_b^{(r)}\right) \vec{q}_{rs} \cdot \vec{\sigma}^{(r)} \vec{q}_{rs} \cdot \vec{\sigma}^{(s)}\right.}{\left(\vec{q}_{rs}^{\,2} + M_\pi^2\right)^2} \right\rangle \tag{94}$$

$$+ \frac{g_A^2 M_\pi}{132\pi^4 F_\pi^4 \left(1 + M_\pi/m_d\right)} \sum_{r<s} \left\langle \left(\vec{\tau}^{(r)} + \vec{\tau}^{(s)}\right) \cdot \left(\vec{\tau}^{(\pi)}\right)_{ab} \frac{\vec{q}_{rs} \cdot \vec{\sigma}^{(r)} \vec{q}_{rs} \cdot \vec{\sigma}^{(s)}}{\left(\vec{q}_{rs}^{\,2} + M_\pi^2\right)^{3/2}} \right\rangle \tag{95}$$

Note that $\left(\vec{\tau}^{(\pi)}\right)_{ab} = -i\varepsilon_{abc}$, where ε is the Levi-Civita Tensor, with a, b being the pion isospin indices. In this analysis we consider only elastic scattering so $a = b \implies \left(\vec{\tau}^{(\pi)}\right)_{aa} = -i\varepsilon_{aac} = \vec{0}$ so eq.(95) drops out.

In order to implement the above terms in our code I will write them in a manner more useful for us. The BKM review uses the notation $(r), (s)$ to indicate nucleon number but we simply use $1, 2$, and the integration over the other nucleons is accounted for elsewhere. Additionally we can simplify using $a = b$. Additionally we seek to write the isospin dependence in terms of the actual quantum numbers t_{12}, m_{12} etc. Now let:

$$\beta = \frac{1}{32\pi^4 F_\pi^4 (1 + M_\pi/M_{\text{nucl}})} \tag{96}$$

Simplifying we have:

$$\begin{aligned} a = {} & 2\beta M_\pi^2 \left\langle \frac{1}{\vec{q}^2} \left(\vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^a \tau_2^a \right) \right\rangle \\ & - \beta g_A^2 \left\langle \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{\vec{q}^2 + M_\pi^2} \right\rangle \\ & + \beta g_A^2 \left\langle \frac{\left[\vec{q}^2 \vec{\tau}_1 \cdot \vec{\tau}_2 + 2M_\pi^2 \ (\tau_1^a \tau_2^a) \ \right] \vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + M_\pi^2)^2} \right\rangle \end{aligned} \tag{97}$$

Recall:

$$\left\langle t'_{12} m_{12}^{t'} \ s'_{12} m'_{12} \right| \vec{\tau}_1 \cdot \vec{\tau}_2 \left| t_{12} m_{12}^t \ s_{12} m_{12} \right\rangle = \delta_{s'_{12} s_{12}} \delta_{m'_{12} m_{12}} \delta_{t'_{12} t_{12}} \delta_{m_{12}^{t'} m_{12}^t} \Big[2t_{12}(t_{12}+1) - 3 \Big] \tag{98}$$

And from an internal communication with Harald Griesshammer

$$\left\langle t'_{12} m_{12}^{t'} \right| \vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^a \tau_2^a \left| t_{12} m_{12}^t \right\rangle = \begin{cases} -2(-1)^{t_{12}} & \text{if} \quad t'_{12} = t_{12} \quad \text{and} \quad m_{12}^{t'} = m_{12}^t = 0 \\ 0 & \text{otherwise} \end{cases} \tag{99}$$

Note the above holds for $t = 0, 1$ and $|m_{12}^t| \leq t$. Equivalently we can write this as:

$$\left\langle t'_{12} m_{12}^{t'} \right| \vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^a \tau_2^a \left| t_{12} m_{12}^t \right\rangle = \delta_{t'_{12}, t_{12}} \delta_{m_{12}^{t'}, m_{12}^t} \delta_{m_{12}^t, 0} (-2)(-1)^{t_{12}} \tag{100}$$

So then, using M_J, M'_J as a shorthand for all the quantum numbers:

$$\left\langle M'_J \right| \tau_1^a \tau_2^a \left| M_J \right\rangle = \left\langle M'_J \right| \vec{\tau}_1 \cdot \vec{\tau}_2 - \left[\vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^a \tau_2^a \right] \left| M_J \right\rangle \tag{101}$$

$$= \delta_{s_{12}, s'_{12}} \delta_{m_{12}, m'_{12}} \delta_{m_{12}^{t'} m_{12}^t} \delta_{t_{12}, t'_{12}} \left[\Big(2t_{12} \left(t_{12} + 1 \right) - 3 \Big) - \delta_{m_{12}^t, 0} \left(-2 \right) \left(-1 \right)^{t_{12}} \right] \tag{102}$$

Now we only care about $t_{12} = 0, 1$ and $|m_{12}^t| \leq t_{12}$ so we can just evaluate the above for those cases.

$$\Big(2t_{12} \left(t_{12} + 1 \right) - 3 \Big) - \delta_{m_{12}^t, 0} \left(-2 \right) \left(-1 \right)^{t_{12}} = (-1)^{m_{12}^t + 1} \quad \text{for} \quad t_{12} = 0, 1, \ \ |m_{12}^t| \leq t_{12} \tag{103}$$

So:

$$\left\langle M'_J \right| \tau_1^a \tau_2^a \left| M_J \right\rangle = \delta_{s_{12}, s'_{12}} \delta_{m_{12}, m'_{12}} \delta_{m_{12}^{t'} m_{12}^t} \delta_{t_{12}, t'_{12}} (-1)^{m_{12}^t + 1} \tag{104}$$

The spin dependence $\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2$ is taken care of in our code already so I will not include it here. So now making these substitutions for the isospin dependence we have:

$$\begin{aligned} a = \delta_{t'_{12}, t_{12}} \delta_{m_{12}^{t'}, m_{12}^t} \Bigg[& - 4\beta M_\pi^2 \frac{1}{\vec{q}^2} (-1)^{t_{12}} \delta_{s_{12}, s'_{12}} \delta_{m_{12}, m'_{12}} \delta_{m_{12}^t, 0} \\ & - \beta g_A^2 \Big[2t_{12}(t_{12}+1) - 3 \Big] \frac{\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{\vec{q}^2 + M_\pi^2} \\ & + \beta g_A^2 \frac{\vec{q}^2 \Big[2t_{12} \left(t_{12} + 1 \right) - 3) \Big] \vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + M_\pi^2)^2} \\ & - 2\beta g_A^2 M_\pi^2 \frac{(-1)^{m_{12}^t} \ \vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + M_\pi^2)^2} \Bigg] \end{aligned} \tag{105}$$

We still need to account for the $+(1 \leftrightarrow 2)$ to the extent that it is not already accounted for in the subroutines.

$$
\begin{aligned}
a = \delta_{t'_{12},t_{12}} \delta_{m^{t'}_{12},m^t_{12}} \Bigg[& -8\beta M_\pi^2 \frac{1}{\vec{q}^2} (-1)^{t_{12}} \delta_{s_{12},s'_{12}} \delta_{m_{12},m'_{12}} \delta_{m^t_{12},0} \\
& - \beta g_A^2 \Big[2t_{12}(t_{12}+1) - 3 \Big] \frac{\vec{q}\cdot\vec{\sigma}_1 \vec{q}\cdot\vec{\sigma}_2}{\vec{q}^2 + M_\pi^2} + (1 \leftrightarrow 2) \\
& + \beta g_A^2 \frac{\vec{q}^2 \Big[2t_{12}\left(t_{12}+1\right) - 3) \Big] \vec{q}\cdot\vec{\sigma}_1 \vec{q}\cdot\vec{\sigma}_2}{\left(\vec{q}^2 + M_\pi^2\right)^2} + (1 \leftrightarrow 2) \\
& - 2\beta g_A^2 M_\pi^2 \frac{(-1)^{m^t_{12}}\ \vec{q}\cdot\vec{\sigma}_1 \vec{q}\cdot\vec{\sigma}_2}{\left(\vec{q}^2 + M_\pi^2\right)^2} + (1 \leftrightarrow 2) \Bigg]
\end{aligned}
\tag{106}
$$

In our implementation the term $+(1 \leftrightarrow 2)$ is taken care of in subroutines that calculate $\vec{A}\cdot\vec{\sigma}\vec{B}\cdot\vec{\sigma} \pm \vec{A}\cdot\vec{\sigma}\vec{B}\cdot\vec{\sigma}$, with the $\pm$ being determined by the symmetry of the nucleon. Note this $+(1 \leftrightarrow 2)$ is taken care of in the first term simply by a factor of 2.