Neutral Pion Photoproduction

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0.1 Lenkewitz convention vs ours

Lenkewtiz calls the incoming momenta of the particles $\vec{k_1}$ and $\vec{k_2}$, so to transfer that convention to our convention we use:

$$\vec{k}_1 = \vec{p} - \vec{k}/2 \tag{1}$$
$$\vec{k}_2 = -\vec{p} - \vec{k}/2 \tag{2}$$
$$\vec{k}_1' = \vec{p}' - \vec{k}'/2 \tag{3}$$
$$\vec{k}_2' = -\vec{p}' - \vec{k}'/2 \tag{4}$$

Throughout this document I will stick to the conventions present in the BKM review, link here: https://arxiv.org/pdf/hep-ph/9501384. In particular I'm pretty sure the BKM review uses $F = 2f_\pi$ Here are a list of the Feynman rules used, as they are listed in the BKM review, along with their equation number, using:

l	Momentum of a pion or nucleon propagator
k	Momentum of an external vector or axial source
q	Momentum of an external pion
ϵ	Photon polarization vector
ϵ_A	Polarization vector of an axial source
p	Momentum of a nucleon in a heavy mass formulation
v_μ	nucleon 4-velocity
S_μ	covariant spin-vector of the nucleon

And the pion isospin indices are a, b, c, and

Pion propagator (A.1)

$$\frac{i\delta^{ab}}{l^2 - m_\pi^2 + i0} \tag{5}$$

1 pion (q out) A.12

$$\frac{g_A}{F} S \cdot q \tau^a \tag{6}$$

2 pions (q_1 in q_2 out) A.14

$$\frac{1}{4F^2} v \cdot (q_1 + q_2) \epsilon^{abc} \tau^c \tag{7}$$

1 pion 1 photon where a is the isospin of the outgoing pion A.15

$$\frac{ieg_A}{F} \epsilon \cdot S \epsilon^{a3b} \tau^b \tag{8}$$

1 pion (q out) A.12

$$\frac{g_A}{F} S \cdot q \tau^a \tag{9}$$

2 pions, photon A.6

$$e \epsilon^{a3b} \epsilon \cdot (q_1 + q_2) \tag{10}$$

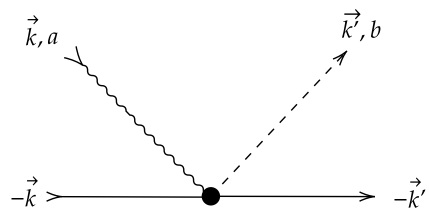
Note $\epsilon = (0, \vec{\epsilon}\,)$, where the zeroth element has been set to zero with a choice of gauge. Neglecting relativistic effects $S = (0, \vec{\sigma}/2)$. So then $\epsilon \cdot S = -\frac{\vec{\sigma}}{2}$

1 Two Body Scattering

The kinematic pre-factor is:

$$K_{1N} = \frac{M_N + m_\pi}{m_{3N} + m_\pi} \frac{m_{3N}}{m_N} \tag{11}$$

1.1 Diagram A



This is just BKM A.15, with $a = 15$

$$\begin{aligned} \mathcal{M} &= \frac{ieg_A}{F} \epsilon \cdot S \varepsilon^{a3c} \tau^c \qquad (12)\\ &= -\frac{ieg_A}{2F} \epsilon \cdot \vec{\sigma} \varepsilon^{33c} \tau^c \qquad (13)\end{aligned}$$

Note that $\varepsilon^{33c}\tau^c = 0$ since $\varepsilon^{33c} = 0 \,\forall\, c$, but this doesn't make sense to me, since it should mean this diagram is zero. A.28 is the other diagrams that is 1 pion and photon, note that I am not sure what mass it is referring to by m

$$\begin{aligned} \mathcal{M} &= -\frac{eg_A}{2mF} S \cdot \epsilon v \cdot k' (\tau^3 + 1) \qquad (14)\\ &= \frac{eg_A}{4mF} \vec{\epsilon} \cdot \vec{\sigma} \left(\sqrt{m_\pi^2 + \vec{k}'^2} - \vec{v} \cdot \vec{k} \right) (\tau_3 + 1) \qquad (15)\\ &\approx \frac{eg_A}{4F} \vec{\epsilon} \cdot \vec{\sigma} \sqrt{m_\pi^2 + \vec{k}'^2}\, (\tau_3 + 1) \qquad (16)\end{aligned}$$

Now lets assume $m = m_\pi$ and take the threshold limit $\vec{k}' \to \vec{0}$, then we have

$$\mathcal{M} = \frac{eg_A}{4F} \vec{\epsilon} \cdot \vec{\sigma}\, (\tau_3 + 1) \tag{17}$$

Which is the same as Lenkewitz diagrams a and b added together modulo some pre-factors.

2 Three Body Scattering

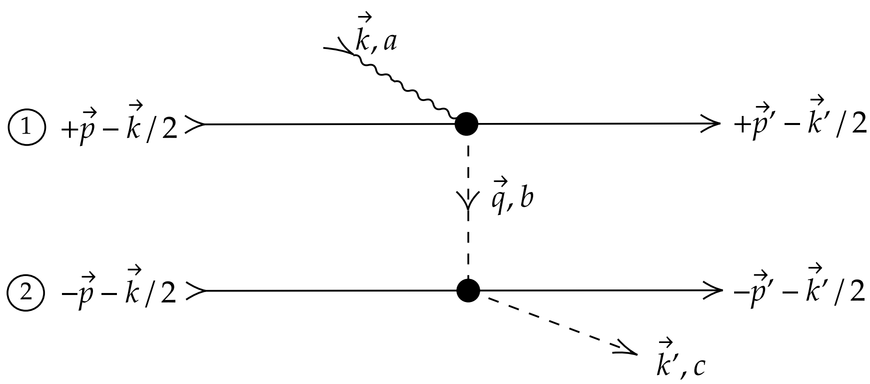
Lenkewitz uses the pre-factor

$$K_{2N} = \frac{1}{(2\pi)^3}\frac{m_\pi m_{3N}}{m_\pi + m_{3N}}\frac{e g_A}{4 f_\pi^3}\frac{1}{(4\pi)} \tag{18}$$

Where m_{3N} refers to the mass of the target m_t, so with that motivation, let

$$\mu = \frac{1}{(2\pi)^3}\frac{m_\pi m_t}{m_\pi + m_t} \tag{19}$$

2.1 Diagram A



From top to bottom we have

$$\mathcal{M}_{1\to 2} = \mu \left[ie\frac{g_A}{F}\epsilon \cdot S_1 \epsilon^{b3d}\tau_1^d\right]\left[\frac{i}{q^2 - m_\pi^2 + i0}\right]\left[\frac{1}{4F^2} v \cdot \left(q + k'\right)\epsilon^{bce}\tau_2^e\right] \tag{20}$$

The momenta into $\vec{q}$ is

$$\vec{q} = \left(\vec{p} - \vec{k}/2\right) + \vec{k} - \left(\vec{p}' - \vec{k}'/2\right) = \vec{p} - \vec{p}' + \frac{1}{2}(\vec{k} - \vec{k}') \tag{21}$$

The energy associated with the propagator is $q_0 = E_1 + k_0 - E_1'$ and we have:

$$E_1 = \frac{\left(\vec{p} - \vec{k}/2\right)^2}{2M_\mathrm{N}} + M_\mathrm{N} \qquad E_1' = \frac{\left(\vec{p}' - \vec{k}'/2\right)^2}{2M_\mathrm{N}} + M_\mathrm{N} \qquad k_0 = \omega \tag{22}$$

So then

$$q_0 = E_1 - E_1' + \omega \tag{23}$$

$$= \frac{1}{2M_\mathrm{N}}\left[\left(\vec{p} - \vec{k}/2\right)^2 - \left(\vec{p}' - \vec{k}'/2\right)^2\right] + \omega \tag{24}$$

Now we let

$$q^2 = q_0^2 - \vec{q}^2 \tag{25}$$

$$v \cdot \left(q + k'\right) = q_0 + k_0' = q_0 + \sqrt{m_\pi + \vec{k}'^2} \tag{26}$$

We may need to include the relativistic contributions of v at higher order.

Now to evaluate the isospin dependence recall $\pi_0 \implies c = 3$, and that the implicit sum is over $b, d, e = 1, 2, 3$. This can be evaluated with the following Mathematica code:

$$\epsilon = \text{LeviCivitaTensor}[3] \tag{27}$$

$$c = 3; \tag{28}$$

$$\sum_{b=1}^{3}\sum_{e=1}^{3}\sum_{d=1}^{3} \epsilon[[b,3,d]], \epsilon[[b,c,e]]\tau_{1,d}\tau_{2,e} \tag{29}$$

$$= \tau_{1,1}\tau_{2,1} + \tau_{1,2}\tau_{2,2} \tag{30}$$

$$= \vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_{1,3}\tau_{2,3} \tag{31}$$

So placing the spin indices up, and the nucleon labeling down we have

$$\epsilon^{b3d}\epsilon^{bce}\tau_1^d\tau_2^e = \vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^3\tau_2^3 \tag{32}$$

We will leave everything in terms of q_0 since that expression is so long.

$$\mathcal{M}_{1\to2} = \mu\left[-ie\frac{g_A}{F}\frac{1}{2}\epsilon\cdot\sigma_1\right]\left[\frac{i}{q^2 - m_\pi^2 + i0}\right]\left[\frac{1}{4F^2}v\cdot\left(q+k'\right)\right]\left(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^3\tau_2^3\right) \tag{33}$$

$$= \mu\frac{eg_A}{8F^3}\frac{q_0 + \sqrt{m_\pi^2 + \vec{k}'^2}}{q_0^2 - \vec{q}^2 - m_\pi^2 + i0}\left(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^3\tau_2^3\right)\vec{\epsilon}\cdot\vec{\sigma}_1 \tag{34}$$

With q_0 given by eq.24 so the full result for scattering off an A body nucleus is:

$$\mathcal{M} = \binom{A}{2}\left(\mathcal{M}_{1\to2} + \mathcal{M}_{2\to1}\right) \tag{35}$$

This gives the final result

$$\mathcal{M} = \mu\frac{eg_A}{8F^3}\binom{A}{2}\frac{q_0 + \sqrt{m_\pi^2 + \vec{k}'^2}}{q_0^2 - \vec{q}^2 - m_\pi^2 + i0}\left(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^3\tau_2^3\right)\vec{\epsilon}\cdot\left(\vec{\sigma}_1 + \vec{\sigma}_2\right) \tag{36}$$

And to program this we use:

$$\vec{q} = \vec{p} - \vec{p}' + \frac{1}{2}(\vec{k} - \vec{k}') \tag{37}$$

$$q_0 = \frac{1}{2M_\text{N}}\left[\left(\vec{p} - \vec{k}/2\right)^2 - \left(\vec{p}' - \vec{k}'/2\right)^2\right] + \omega \tag{38}$$

2.2 Reduction to the threshold case

In the threshold case we have

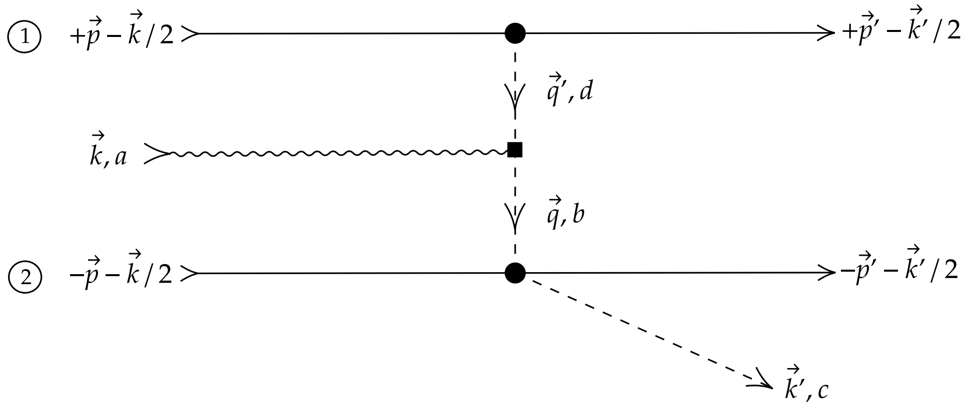
$$\vec{k}' = (m_\pi, \vec{0}) \tag{39}$$

$$q_0 = \omega + \mathcal{O}\left(\frac{1}{M_N}\right) = m_\pi + \mathcal{O}\left(\frac{1}{M_N}\right) \tag{40}$$

Making these substitutions gives:

$$\mathcal{M} = -\mu\frac{em_\pi g_A}{4F^3}\frac{\epsilon\cdot\left(\vec{\sigma}_1 + \vec{\sigma}_2\right)}{\left(\vec{p} - \vec{p}' + \vec{k}/2\right)^2}\left(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^3\tau_2^3\right) \tag{41}$$

2.3 Diagram B



If I recall correctly there is some weirdness on q', such that $q_0' = 0$. This is probably because:

$$q_0' = \frac{1}{2M_{\rm N}}\left[\left(\vec{p}-\vec{k}/2\right) - \left(\vec{p}'-\vec{k}'/2\right)^2\right] = \mathcal{O}\left(1/M_N\right) \tag{42}$$

Again we will go from top to bottom, starting with A.12, then the propagator, then A.6 etc. I will use ϵ for photon polarization and ε for the Levi-Civita tensor to avoid ambiguity.

$$\begin{aligned}\mathcal{M}_{1\to 2} = \mu&\left[\frac{g_A}{F}S\cdot q'\tau_1^d\right]\left[\frac{i}{q'^2-m_\pi^2+i0}\right]\left[e\varepsilon^{d3b}\epsilon\cdot\left(q'+q\right)\right]\left[\frac{i}{q^2-m_\pi^2+i0}\right]\\ &\times\left[\frac{1}{4F^2}v\cdot\left(q+k'\right)\varepsilon^{bce}\tau_2^e\right]\end{aligned} \tag{43}$$

Note $c = 3$ and:

$$\begin{aligned}\varepsilon^{b3e}\varepsilon^{d3b}\tau_1^d\tau_2^e &= -\tau_1^1\tau_2^1 - \tau_1^2\tau_2^2 & (44)\\ &= -(\vec{\tau}_1\cdot\vec{\tau}_2 - \tau_1^3\tau_2^3) & (45)\end{aligned}$$

See the Mathematica doc if you want a more general implementation. Letting $\epsilon = (0, \vec{\epsilon}\,)$ and $v = (1, \vec{0})$ gives.

$$\begin{aligned}\mathcal{M}_{1\to 2} &= \mu\left[\frac{-g_A}{2F}\vec{q}'\cdot\vec{\sigma}_1\right]\left[\frac{-ie\vec{\epsilon}\cdot\left(\vec{q}'+\vec{q}\right)}{q'^2-m_\pi^2}\right]\left[\frac{i}{q^2-m_\pi^2}\right]\left[\frac{1}{4F^2}\left(q_0+\sqrt{m_\pi^2+\vec{k}'^2}\right)\right] & (46)\\ &\quad\times\left[-1(\vec{\tau}_1\cdot\vec{\tau}_2-\tau_1^3\tau_2^3)\right] & (47)\\ &= \mu\frac{eg_A}{8F^3}\frac{\vec{\epsilon}\cdot\left(\vec{q}+\vec{q}'\right)\ \vec{q}'\cdot\vec{\sigma}_1}{(q'^2-m_\pi^2)(q^2-m_\pi^2)}\left(q_0+\sqrt{m_\pi^2+\vec{k}'^2}\right)(\vec{\tau}_1\cdot\vec{\tau}_2-\tau_1^3\tau_2^3) & (48)\end{aligned}$$

So in an A body nucleus the total contribution to the matrix element is

$$\begin{aligned}\mathcal{M} &= \binom{A}{2}\left(\mathcal{M}_{1\to 2}+\mathcal{M}_{2\to 1}\right) & (49)\\ &= \mu\frac{eg_A}{8F^3}\binom{A}{2}\frac{\vec{\epsilon}\cdot\left(\vec{q}+\vec{q}'\right)\ \vec{q}'\cdot\left(\vec{\sigma}_1+\vec{\sigma}_2\right)}{(q'^2-m_\pi^2)(q^2-m_\pi^2)}\left(q_0+\sqrt{m_\pi^2+\vec{k}'^2}\right)(\vec{\tau}_1\cdot\vec{\tau}_2-\tau_1^3\tau_2^3) & (50)\end{aligned}$$

Recall:

$$\vec{q}' = \vec{p} - \vec{p}' + \frac{1}{2}\left(\vec{k}' - \vec{k}\right) \tag{51}$$

$$q_0' = \frac{1}{2M_{\mathrm{N}}}\left[\left(\vec{p} - \vec{k}/2\right) - \left(\vec{p}' - \vec{k}'/2\right)^2\right] \tag{52}$$

$$\vec{q} = \vec{q}' + \vec{k} = \vec{p} - \vec{p}' + \frac{1}{2}\left(\vec{k}' + \vec{k}\right) \tag{53}$$

$$q_0 = \frac{1}{2M_{\mathrm{N}}}\left[\left(\vec{p} - \vec{k}/2\right) - \left(\vec{p}' - \vec{k}'/2\right)^2\right] + \omega \tag{54}$$

2.4 Reduction to the threshold case

In the threshold case we have:

$$k' = (m_\pi, \vec{0}), \tag{55}$$

$$q_0 = m_\pi' \approx \omega \tag{56}$$

$$\vec{q} + \vec{q}' = 2(\vec{p} - \vec{p}') \tag{57}$$

And for the time being we let $q_0' = 0$ with the motivation being comparison to Lenkewitz. This gives:

$$\mathcal{M} = \mu \frac{e m_\pi g_A}{4F^3}\binom{A}{2}\frac{\vec{q}' \cdot (\vec{\sigma}_1 + \vec{\sigma}_2)\vec{\epsilon} \cdot \left(\vec{q} + \vec{q}'\right)}{\vec{q}^2\left(\vec{q}'^2 + m_\pi^2\right)}(\vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^3\tau_2^3) \tag{58}$$

$$= \mu \frac{e m_\pi g_A}{2F^3}\binom{A}{2}\frac{\left(\vec{p} - \vec{p}' - \vec{k}/2\right) \cdot (\vec{\sigma}_2 + \vec{\sigma}_2)\vec{\epsilon} \cdot (\vec{p} - \vec{p}')}{\left(\vec{p} - \vec{p}' + \vec{k}/2\right)^2\left[\left(\vec{p} - \vec{p}' - \vec{k}/2\right)^2 + m_\pi^2\right]}(\vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^3\tau_2^3) \tag{59}$$

3 Comparison to Lenkewtiz

The literature for some reason always writes the result as diagram A minus diagram B which accounts for the difference in the negative sign.

We have already shown my results for 2 body diagrams A and B are the same as Lenkewtiz up to a pre-factor. Lenkewtiz assigns

$$K_{2N} = \frac{1}{(2\pi)^3}\frac{m_\pi m_{3N}}{m_\pi + m_{3N}}\frac{e g_A}{4f_\pi^3}\frac{1}{(4\pi)} \tag{60}$$

Where Lenkewitz assigns $f_\pi = 93\,\mathrm{MeV}$, and $g_A = 1.26$. The BKM review uses the same values.

The factor 3 in the Lenkewitz paper comes from $\binom{A}{2}$.
The factor $1/(2\pi)^3$ comes from the integration measure.
The factor $m_\pi m_{3N}/(m_\pi + m_{3N})$ comes from kinematic considerations.
The factor $1/4\pi$ is a mystery, and is the only term that I don't understand.

I am off by a factor of 2 in both terms, perhaps this comes from an additional symmetry factor. Also, the sign on diagram A doesn't match with Lenkewitz, but it is implemented with the negative sign, and reproduces the Lenkewitz results.

4 Variable Substitution

In the threshold case a variable substitution is used to remove the singularity. The kernel is proportional to:

$$\int \mathrm{d}^3p' \frac{1}{\left(\vec{p}-\vec{p}'-\frac{1}{2}\vec{k}\right)^2} f(\vec{p},\vec{p}') = \int \mathrm{d}^3q \frac{1}{\vec{q}^2} f(\vec{p},\vec{p}-\vec{\ell}-\vec{k}/2) \tag{61}$$

$$= \int \mathrm{d}\Omega\, \mathrm{d}\ell\, f(\vec{p},\vec{p}-\vec{\ell}-\vec{k}/2) \tag{62}$$

With $\vec{q}=\vec{p}-\vec{p}'-\vec{k}/2 \implies \vec{p}'=\vec{p}-\vec{q}-\vec{k}/2$. This allows for the cancelation of the singularity in the threshold case.

4.1 Finite Energy

At finite energy the situation is more complicated, we now have an integral of the form

$$\int \frac{f(p,p')}{q_0^2-\left(\vec{p}-\vec{p}'-\vec{k}/2\right)^2-m_\pi^2} \tag{63}$$

We seek an additional transformation that eliminates the zero in the denominator.

$$\int \frac{f(p,p')}{q_0^2-\left(\vec{p}-\vec{p}'-\vec{k}/2\right)^2-m_\pi^2} = \int \mathrm{d}^3p' \frac{f(p,p')}{q_0^2-\vec{q}^2-m_\pi^2} \tag{64}$$

$$= -\int \mathrm{d}^3p' \frac{f(p,p')}{\vec{q}^2-\ell^2} \tag{65}$$

$$= -\int \mathrm{d}^3u \frac{f(p,p')}{u^2} \tag{66}$$

$$= -\int \mathrm{d}\Omega \mathrm{d}u f(p,p') \tag{67}$$

$$q_0 = \frac{1}{2M_\mathrm{N}}\left[\left(\vec{p}-\vec{k}/2\right)^2-\left(\vec{p}'-\vec{k}'/2\right)^2\right]+\omega \tag{68}$$

$$= \sqrt{\vec{k}^2}+\mathcal{O}\left(1/M_N\right) \tag{69}$$

$$\vec{q}=\vec{q}'+\vec{k}=\vec{p}-\vec{p}'+\frac{1}{2}\left(\vec{k}'+\vec{k}\right) \tag{70}$$

Where $\sqrt{\vec{k}^2}=\omega$. So then $\ell^2=q_0^2-m_\pi^2$ and we seek a transformation of the form $p'\to u$ such that

$$\vec{u}=\left(p'_x+a\right)\hat{x}+\left(p'_y+b\right)\hat{y}+\left(p'_z+c\right)\hat{z} \tag{71}$$

So then:

$$\vec{u}^2=\vec{p}'^2+2\left(ap'_x+bp'_y+cp'_z\right)+a^2+b^2+c^2 \tag{72}$$

So we want:

$$ap'_x + bp'_y + cp'_z = 0 \tag{73}$$
$$a^2 + b^2 + c^2 = \ell^2 \tag{74}$$

a, b and c are free, so WLOG let $a = b$ then:

$$a(p'_x + p'_y) + cp'{}_z = 0 \qquad \Longrightarrow \qquad c = -a\left(\frac{p'_x + p'_y}{p'_z}\right) \tag{75}$$

This gives:

$$a^2 + b^2 + c^2 = \ell^2 = a^2\left[2 + \left(\frac{p'_x + p'_y}{p'_z}\right)^2\right] \tag{76}$$
$$\Longrightarrow a = \frac{\ell p'_z}{\sqrt{2{p'_z}^2 + (p'_x + p'_z)^2}} \tag{77}$$

This satisfies $\vec{u}^2 = \ell^2 + \vec{p}'^2$. The only wrinkle is that ℓ is dependent on $\vec{p}'$ so the Jacobian gets rather complicated. Recall that if you have a transformation $x = g(a, b, c),\ y = f(a, b, c),\ z = h(a, b, c)$ the Jacobian of the integral transform taking it from coordinates x, y, z to a, b, c is given by:

$$J = \mathrm{Det}\begin{bmatrix} \partial x/\partial a & \partial x/\partial b & \partial x/\partial c \\ \partial y/\partial a & \partial y/\partial b & \partial y/\partial c \\ \partial z/\partial a & \partial z/\partial b & \partial z/\partial c \end{bmatrix} \tag{78}$$

But in our situation we have $\vec{p}' \to \vec{u}$ and have access to the function $\vec{u} = F(\vec{p}')$ so in our situation we need $\frac{1}{J}$