

# Elastic Pion Scattering Kernel Derivation

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This follows the conventions from the [BKM \(Bernard, Kaiser, Meissner\) review](#), including the use of  $F_\pi = 93.1$ , see table 2 on page 23 of the review, however it is not clear if  $F$  in the appendix, differs from  $F_\pi$ , by a factor of  $\pi$ . If this is the case a lot of things would make more sense. The BKM review also goes over this reaction on pg 115 for the two body case.

There are a few sources for pion scattering at zero energy:

[S-wave scattering length: Beane 2002](#)

[Weinberg 1992](#), includes isospin dependence. [ArXiV link](#) doesn't have diagrams. Note Weinberg uses  $F_\pi = 186\text{MeV}$ , so converting to the BKM convention requires a factor of 2.

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## 1 Identities, Definitions, and Useful Identities

$$\sigma_j \sigma_k = \delta_{jk} I + i \varepsilon_{jkl} \sigma_l \quad (1)$$

$$\mu = \frac{m_\pi}{m_{nucl}} \quad (2)$$

$$F_\pi = 93.1 \text{ MeV}, \quad g_A = 1.26 \quad g_{\pi N} = \frac{gm}{F} \quad (3)$$

For our purposes the incoming and outgoing pions have the same charges.

Notation with momentum four vectors can be confusing, so in this document, for 4-vectors  $p$  and  $q$ , use the notation  $p - q$  to represent:

$$p - q = (\sqrt{m_p + \vec{p}^2}, \vec{p}) + (\sqrt{m_q + \vec{q}^2}, -\vec{q}) \quad (4)$$

$$= \left( \sqrt{m_p + \vec{p}^2} + \sqrt{m_q + \vec{q}^2}, \vec{p} - \vec{q} \right) \quad (5)$$

$$= (p_0 + q_0, \vec{p} - \vec{q}) \quad (6)$$

I'm pretty sure this make sense to do, but I'm completely set on it. Additionally this means, that unless otherwise stated:

$$-p = \left( \sqrt{m_p^2 + \vec{p}^2}, -\vec{p} \right) \quad (7)$$

### 1.0.1 Prefactor

I'm pretty sure that each diagram comes with a prefactor of:

$$\frac{1}{2(1+\mu)} \quad \text{or} \quad \frac{1}{2(1+\mu)\pi^4} \quad (8)$$

### 1.0.2 Definition of the spin vector

There is still some confusion on the definition of the zeroth element of  $S$ .

[This source](#) implies its zero up to relativistic corrections. Consider the rest frame, and boosted (lab frame) spins:

$$\text{Rest frame: } S' = (0, s'_x, s'_y, s'_z) \quad \text{Lab frame: } S = (s_t, s_x, s_y, s_z) \quad (9)$$

This must be Lorentz invariant, so

$$s_t^2 - \vec{s} \cdot \vec{s} = -\vec{s}' \cdot \vec{s}' \quad (10)$$

Its easier to calculate the rest frame in terms of a boost on the lab frame, so:

$$S'^0 = \Lambda^0{}_\alpha S^\alpha = \Lambda^0{}_0 S^0 + \Lambda^0{}_i S^i = \gamma (S^0 - U_i S^i) \quad (11)$$

$$= \gamma (S^0 - u_i S^i) = U_0 S^0 - U_i S^i \quad (12)$$

$$= U_\alpha S^\alpha = 0 \quad (\text{invariant}) \quad (13)$$

$$S'^i = \Lambda^i{}_\alpha S^\alpha = \Lambda^i{}_0 S^0 + \Lambda^i{}_j S^j \quad (14)$$

$$= -\gamma U^i S^0 + \left[ \delta_{ij} + \frac{\gamma - 1}{U^2} U_i U_j \right] S^j \quad (15)$$

$$= S^i + \frac{\gamma^2}{\gamma + 1} U_i U_j S^j - \gamma U^i S^0 \quad (16)$$

Note that  $S'^0 = 0$ . Note  $U_\alpha S^\alpha = 0$

Where  $U$  is the 3-velocity that boosts the particle to the lab frame. Inverting the above gives the

spin in the lab frame from the particles rest frame:

$$s_t = \gamma \vec{U} \cdot \vec{s}' \quad (17)$$

$$\vec{s} = \vec{s}' + \frac{\gamma^2}{\gamma + 1} \vec{U} (\vec{U} \cdot \vec{s}') \quad (18)$$

And recall  $\gamma = (1 - \vec{U}^2)^{-1/2}$

### 1.1 Cross section and Scattering Length

For elastic, UN polarized scattering  $A + B \rightarrow A + B$ :

$$\left. \frac{d\sigma}{d\Omega} \right|_{\text{cm}} = \frac{1}{64\pi^2 s} |\bar{\mathcal{M}}|^2 \quad (19)$$

Here  $\mathcal{M}$  is the "transition amplitude, averaged over initial spins and summed over final ones". For our case we only have one ingoing and one outgoing state so I think this doesn't matter. In the center of mass frame, where  $p_1$  and  $p_2$  are the incident momenta, we have

$$s = (p_1 + p_2)^2 = [(m_1 + m_2) - \vec{0}^2] = m_1 + m_2 \quad (20)$$

And the scattering length is given by:

$$\lim_{k \rightarrow 0} \sigma = 4\pi a^2 \quad (21)$$

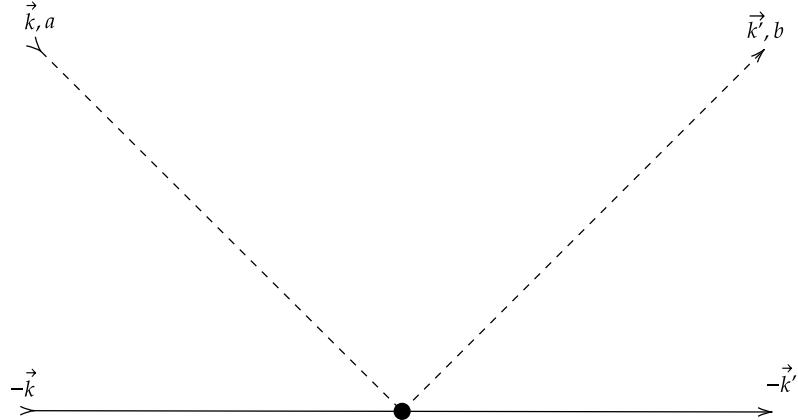
or equivalently

$$\lim_{k \rightarrow 0} k \cot \delta(k) = -\frac{1}{a} \quad (22)$$

## 2 1 Body Contributions

### 2.1 1 Body A

Diagram 1 A,  $\mathcal{O}(p^2)$



$$\mathcal{M}_{1,a} = \frac{1}{4F^2} v \cdot (\vec{k} + \vec{k}') \varepsilon^{abc} \tau_c \quad (23)$$

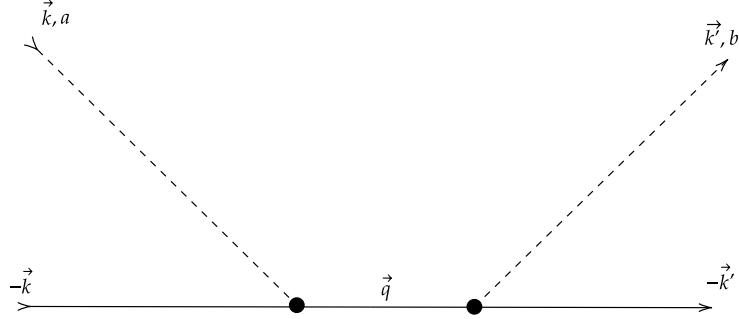
$$= \frac{1}{4F^2} (E_\pi + E_N) \varepsilon^{abc} \tau_c \quad (24)$$

We are in the CM frame, so  $\vec{k} + \vec{k}' = 0$ , but  $E_\pi \neq E_N$ , but:

$$\varepsilon^{abc} \tau_c = 0 \quad \text{for } a = b \quad (25)$$

So this diagram is zero.

## 2.2 1 Body B



$$\mathcal{M}_{1,b} = \left[ -\frac{g_A}{F} S \cdot k \tau^a \right] \left( \frac{i}{v \cdot q + i\varepsilon} \right) \left[ \frac{g_A}{F} S \cdot k' \tau^b \right] \quad (26)$$

$$= -i \frac{g_A^2}{F^2} \frac{(S \cdot k)(S \cdot k')}{q_0 + i\varepsilon} \tau^a \tau^b \quad (27)$$

$\vec{q} = 0 \implies q = (\sqrt{m_N^2 + \vec{q}'^2}, \vec{q}') = m_N$ , and letting  $S = (0, \frac{1}{2}\vec{\sigma})$  gives

$$\mathcal{M}_{1,b} = -i \frac{g_A^2}{F^2} \frac{(S \cdot k)(S \cdot k')}{m_N + i\varepsilon} \tau^a \tau^b \quad (28)$$

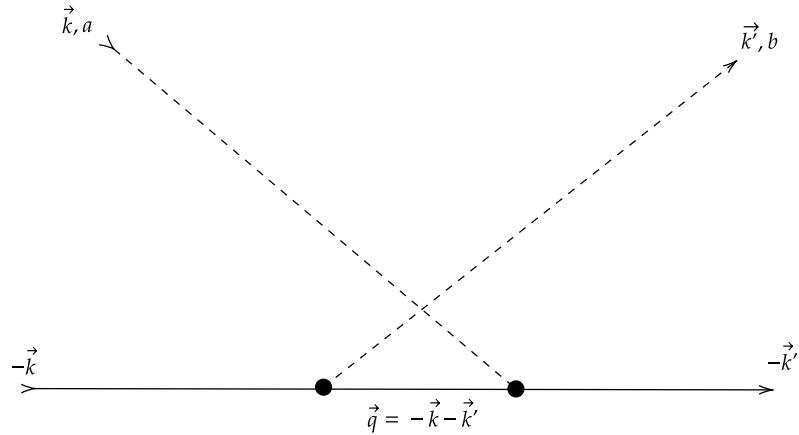
$$= -i \frac{g_A^2}{4F^2} \frac{\vec{\sigma} \cdot \vec{k} \vec{\sigma} \cdot \vec{k}'}{m_N + i\varepsilon} \tau^a \tau^b \quad (29)$$

Letting  $a = b$ , and  $\vec{k} = -\vec{k}'$  gives:

$$\mathcal{M}_{1,b} = i \frac{g_A^2}{4F^2} \frac{(\vec{\sigma} \cdot \vec{k})^2}{m_N + i\varepsilon} \tau^a \tau^a \quad (30)$$

Perhaps this propagator should be off shell, in which case  $m_N \rightarrow q_0$  and we have to integrate over it

### 2.3 1 Body C

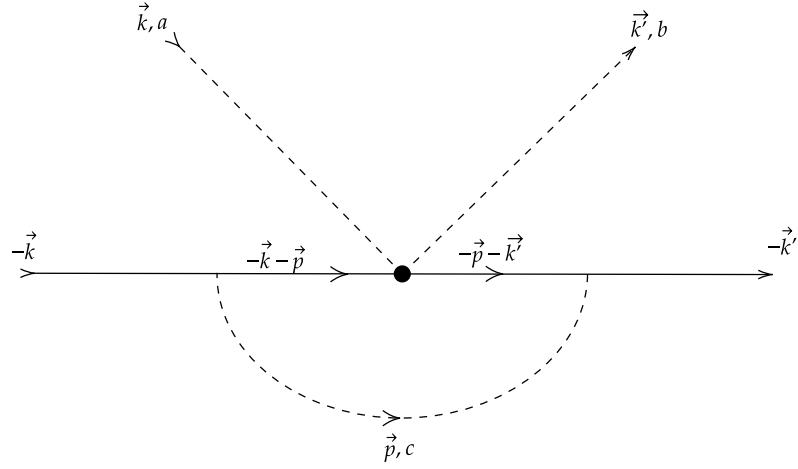


This is the same as diagram B but with  $\vec{q} = -\vec{k} - \vec{k}'$ , so  $q_0 = \sqrt{m_N^2 + \vec{q}^2}$

$$\mathcal{M}_{1,c} = -i \frac{g_A^2}{4F^2} \frac{\vec{\sigma} \cdot \vec{k} \vec{\sigma} \cdot \vec{k}'}{q_0 + i\varepsilon} \tau^b \tau^a \quad (31)$$

## 2.4 1 Body D

Diagram 1 B,  $\mathcal{O}(p^3)$



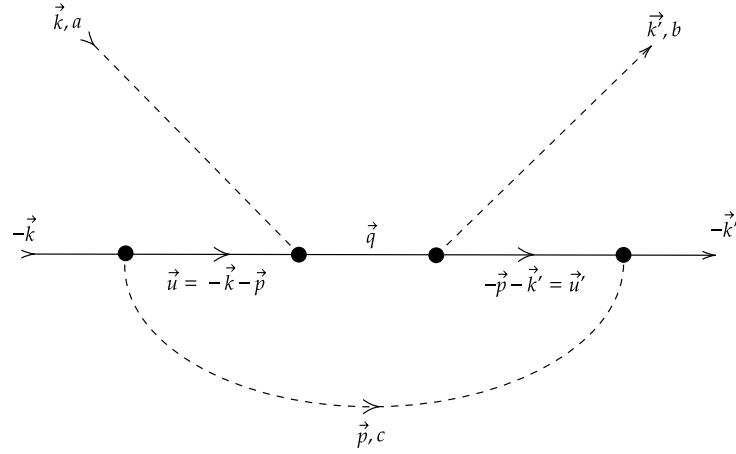
Let  $u = -k - p$  and  $u' = u + k - k' = -p - k'$

$$\begin{aligned} \mathcal{M}_{1,d} &= \left[ \frac{g_A}{F} S \cdot p \tau^c \right] i \left[ \vec{u} \cdot (-\vec{k} - \vec{p}) + i\varepsilon \right]^{-1} \left[ \frac{1}{4F^2} v \cdot (k + k') \varepsilon^{abd} \tau^d \right] \\ &\quad \times i \left[ \vec{u}' \cdot (-\vec{p} - \vec{k}') + i\varepsilon \right]^{-1} \left[ \frac{g_A}{F} S \cdot (-p) \tau^c \right] i [\vec{p}^2 + i\varepsilon]^{-1} \end{aligned} \quad (32)$$

$$= i \frac{g_A}{4F^4} \frac{(S \cdot p)^2}{(\vec{p}^2 + i\varepsilon) (\vec{u} \cdot (\vec{k} + \vec{p}) + i\varepsilon) (\vec{u}' \cdot (\vec{p} + \vec{k}') + i\varepsilon)} (E_\pi + E'_\pi) \varepsilon^{abd} \tau^d \tau^c \tau_c \quad (33)$$

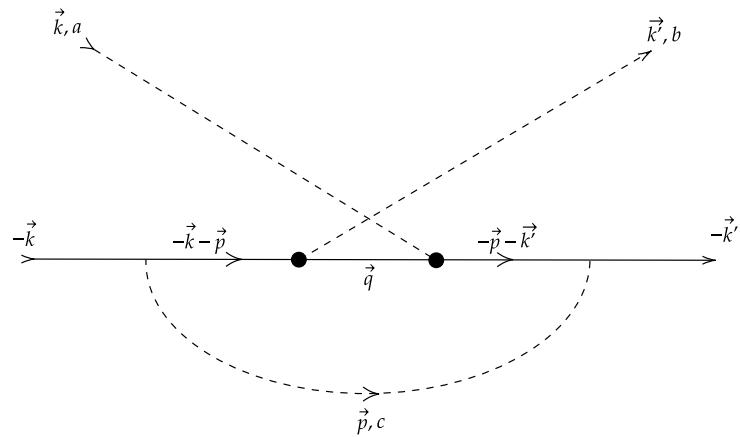
Check this, did a lot of mental calculations

## 2.5 1 Body E



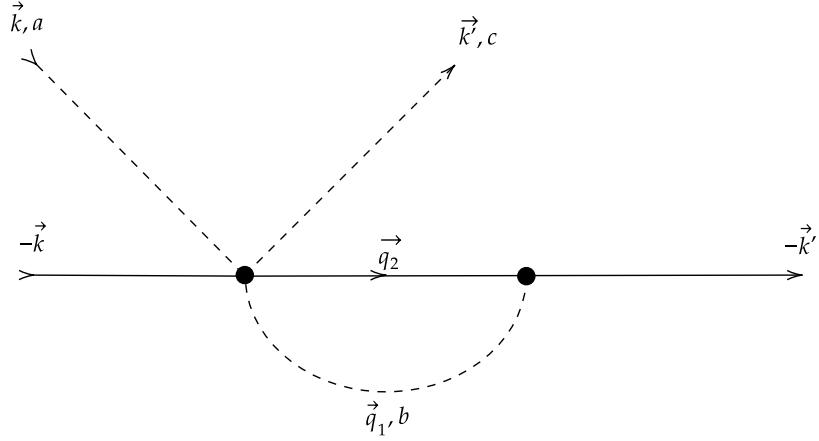
$$\begin{aligned}
 \mathcal{M}_{1,E} = & \left[ \frac{g_A}{F} S \cdot p \tau^c \right] \frac{i}{v \cdot (-k - p) + i\varepsilon} \left[ -\frac{g_A}{F} \S \cdot k \tau^a \right] \left( \frac{1}{v \cdot q + i\varepsilon} \right) \\
 & \times \left( \frac{1}{p^2 - m_\pi^2 + i\varepsilon} \right) \left[ \frac{g_A}{F} S \cdot k' \tau^b \right] \left( \frac{1}{v \cdot u' + i\varepsilon} \right) \left[ -\frac{g_A}{F} S \cdot p \tau^c \right]
 \end{aligned} \quad (34)$$

## 2.6 1 Body F



## 2.7 1 Body G

Diagram 1 D,  $\mathcal{O}(p^4)$

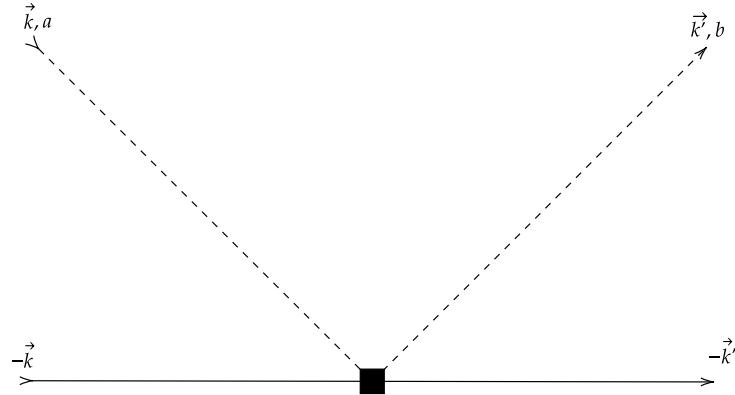


Using BKM A.16

$$\begin{aligned} \mathcal{M}_{1,d} = & \frac{g_A}{2F^3} \left[ \tau^a \delta^{bc} S \cdot (q_1 + k') + \tau^b \delta^{ac} S \cdot (-k + q_1) + \tau^c \delta^{ab} S \cdot (-k + q_1) \right] \\ & \times i [q_1^2 - m_\pi^2 + i\varepsilon]^{-1} i [v \cdot q_2 + i\varepsilon]^{-1} \left[ \frac{g_A}{F} S \cdot (-q_1) \tau^b \right] \end{aligned} \quad (35)$$

## 2.8 1 Body H

Diagram 1 C,  $\mathcal{O}(p^4)$



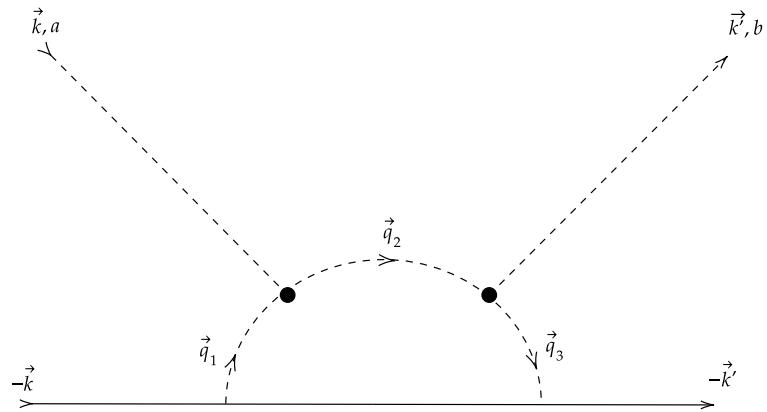
Only Feynman rule is BKM review, A.29, but I'm not going to write it down here since its rather long, but here is a screenshot of the rule.

2 pions ( $q_1$  in,  $q_2$  out):

$$\begin{aligned}
 & \frac{i\delta^{ab}}{F^2} \left[ -4c_1 M_\pi^2 + \left(2c_2 - \frac{g_A^2}{4m}\right) v \cdot q_1 v \cdot q_2 + 2c_3 q_1 \cdot q_2 \right] \\
 & + \frac{1}{8mF^2} \epsilon^{abc} \tau^c \left[ (p_1 + p_2) \cdot (q_1 + q_2) - v \cdot (p_1 + p_2) v \cdot (q_1 + q_2) \right] \\
 & - \frac{1}{F^2} \left( 2c_4 + \frac{1}{2m} \right) \epsilon^{abc} \tau^c [S \cdot q_1, S \cdot q_2]
 \end{aligned} \tag{A.29}$$

## 2.9 1 Body I

Diagram 1 C,  $\mathcal{O}(p^4)$



This diagram is 0, BKM A.3

### 3 2 Body Contributions

Note that for the scattering length at least, there is a prefactor:

$$\frac{1}{1+\mu} \equiv \alpha \quad (36)$$

which comes from considerations other than the diagrams. Additionally, see BKM review equation 5.29:

$$a_{ab} = \frac{1 + m_\pi/m_N}{1 + m_\pi/Am_N} \sum_r a_{ab}^{(r)} + a_{ab}^{\text{three-body}} \quad (37)$$

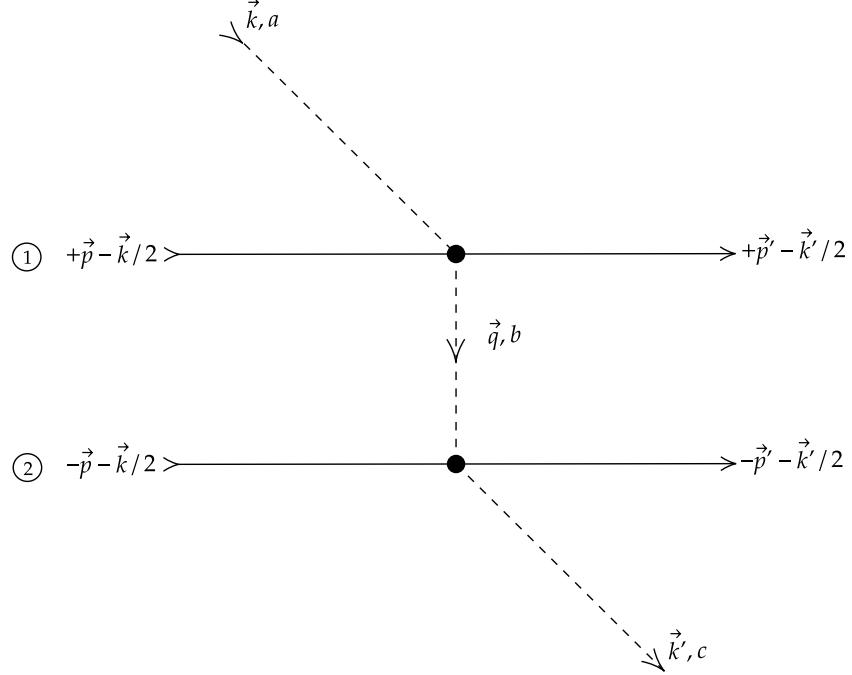
Note for the BKM review "three-body" means two nucleons and an external probe, which is what we call two body. In eq.(37)  $a, b$  are pion isospin indices, and  $r$  (and later  $s$  is used for this too) is for nucleon labeling.

The BKM review states:

$$(t_c^{(\pi)})_{ab} = -i\epsilon_{abc} \quad \text{is the pion isospin vector} \quad (38)$$

But this only appears in the last diagram, and is specifically the pion isospin operator, not the nucleon isospin operator.

#### 3.1 2 Body A



Note  $q = k/2 + p$

$$\mathcal{M}_{2,a} = \alpha \left[ \frac{1}{4F^2} v \cdot (k + q) \varepsilon^{abd} \tau_1^d \right] i [q^2 - m_\pi^2 + i\varepsilon]^{-1} \left[ \frac{1}{4F^2} v' \cdot (q + k') \varepsilon^{bce} \tau_2^e \right] \quad (39)$$

$$= \alpha \left( \frac{1}{2F} \right)^4 \frac{(E_\pi + q_0)(q_0 + E'_\pi)}{q^2 - m_\pi^2 + i\varepsilon} \varepsilon^{abd} \varepsilon^{bce} \tau_1^d \tau_2^e \quad (40)$$

Where:  $\vec{q} = \vec{p} - \vec{p}' + \frac{1}{2} (\vec{k} + \vec{k}')$ , and  $q_0 = \sqrt{m_{\pi_0}^2 + \vec{q}^2}$  We now restrict ourselves to just the inelastic process, where  $c = a$ , then computing the matrix dependence gives:

$$\varepsilon^{abd} \varepsilon^{bae} \tau_1^d \tau_2^e = -1 (\varepsilon^{bad} \varepsilon^{bae}) \tau_1^d \tau_2^e \quad (41)$$

$$= (\delta^{ae} \delta^{da} - \delta^{aa} \delta^{de}) \tau_1^d \tau_2^e \quad (42)$$

$$= (\delta^{ae} \delta^{da}) \tau_1^d \tau_2^e - \tau_1^e \tau_{2e} \quad (43)$$

$$= \tau_1^a \tau_2^a - \tau_1^e \tau_{2e} \quad (44)$$

Here, the index  $a$ , is not being summed over. For example in the case of neutral pion pion scattering  $a = 3$  and this reduces to

$$\tau_1^3 \tau_2^3 - \vec{\tau}_1 \cdot \vec{\tau}_2 \quad (45)$$

So the diagram contribution is then:

$$\mathcal{M}_{2,a} = \left( \frac{1}{2F} \right)^4 \frac{(E_\pi + q_0)(q_0 + E'_\pi)}{q^2 - m_\pi^2 + i\varepsilon} (\tau_1^a \tau_2^a - \vec{\tau}_1 \cdot \vec{\tau}_2) \quad (46)$$

Or in the threshold case:

$$\mathcal{M}_{2,a} = \left( \frac{1}{2F} \right)^4 \frac{m_\pi^2}{\vec{q}^2 + i\varepsilon} (\tau_1^a \tau_2^a - \vec{\tau}_1 \cdot \vec{\tau}_2) \quad (47)$$

For this diagram, at threshold, Beane gets the result:

$$\frac{M_\pi^2}{32\pi^4 F_\pi^4 (1 + \mu/2)} \frac{1}{\vec{q}^2} \quad (48)$$

And Weinberg for the threshold case writes the result as (eq 5):

$$\frac{M_\pi^2}{32\pi^4 F_\pi^4 (1 + \mu/2)} \sum_{r < s} \frac{1}{\vec{q}_{rs}^2} (2\vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} \delta_{ab} - t_a^{(r)} t_b^{(s)} - t_a^{(s)} t_b^{(r)}) \quad (49)$$

Taking  $a = b$  the above reduces to:

$$\frac{M_\pi^2}{16\pi^4 F_\pi^4 (1 + \mu/2)} \sum_{r < s} \frac{1}{\vec{q}_{rs}^2} (\vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} - t_a^{(r)} t_a^{(s)}) \quad (50)$$

**What do we do about this sum?**

### 3.2 2 Body B

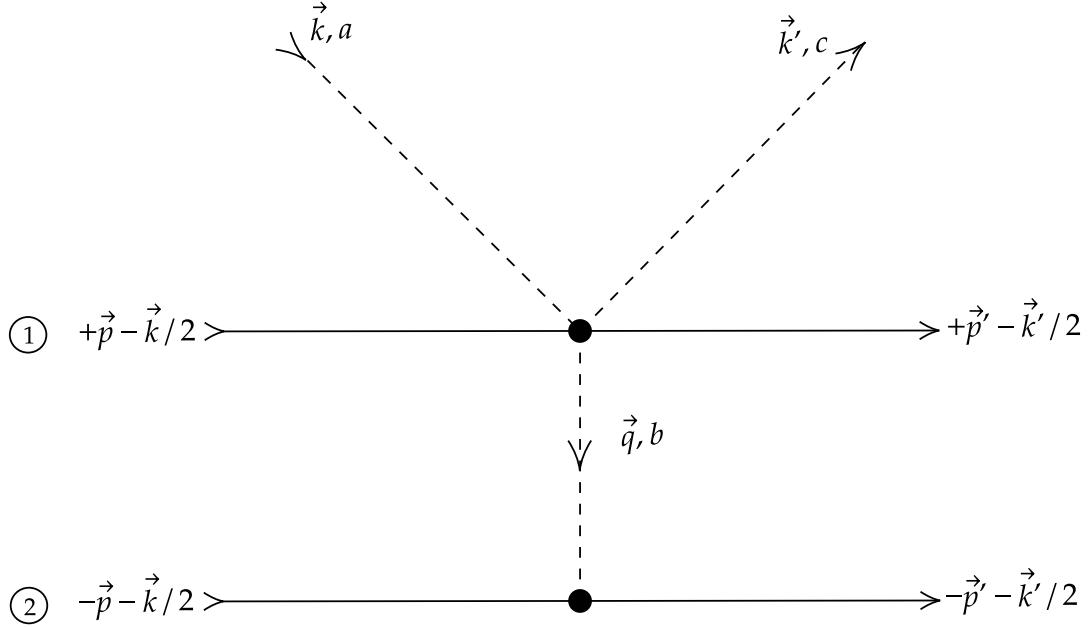
Diagram b (at threshold) according to Weinberg is:

$$-\frac{g_A^2 \delta_{ab}}{32\pi^4 F_\pi^4 (1+\mu)} \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{\vec{q}^2 + m_\pi^2} \quad (51)$$

and the Beane paper gives the result for diagram *b* and *c* together as:

$$-\frac{g_A^2 m_\pi^2}{128\pi^4 F_\pi^4 (1+\mu)} \frac{\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + m_\pi^2)^2} \quad (52)$$

Using BKM A.16 - check indicies.



$$\mathcal{M}_{2,b} = \frac{g_A}{2F^3} \left[ \tau^a \delta^{bc} S_1 \cdot (q + k') + \tau^b \delta^{ac} S_1 \cdot (k' - k) + \tau^c \delta^{ab} S_1 \cdot (q - k) \right] \times i [q^2 - m_\pi^2 + i\varepsilon]^{-1} \left[ \frac{g_A}{F} S_2 \cdot (-q) \tau^b \right] \quad (53)$$

$$(54)$$

The Feynman rule for 3 pions (all qs out) is just:

$$\frac{g_A}{2F^3} \left[ \tau^a \delta^{bc} S_1 \cdot (q_2 + q_3) + \tau^b \delta^{ac} S_1 \cdot (q_1 + q_3) + \tau^c \delta^{ab} S_1 \cdot (q_1 + q_2) \right] \quad (55)$$

And we have:

$$\vec{q}_1 = -\vec{k} \quad \vec{q}_2 = \vec{q} \quad \vec{q}_3 = \vec{k}' \quad (56)$$

The last  $\tau$  should be operating on the second nucleon, and the other  $\tau$  operators are supposed to be on nucleon 1. Additionally, the index *b*, must be summed over, whereas *a* and *c* are external

observables (pion isospin). The index  $b$  is the only one that is summed over.

$$\mathcal{M}_{2,b} = -i \frac{g_A^2}{8F^4} \frac{1}{q^2 - m_\pi^2 + i\varepsilon} \sum_{b=1}^3 S_1 \cdot \left[ \tau_1^a \delta^{bc} (q_2 + q_3) + \tau_1^b \delta^{ac} (q_1 + q_3) + \tau_1^c \delta^{ab} (q_1 + q_2) \right] S_2 \cdot q_2 \tau_2^b \quad (57)$$

$$= -i \frac{g_A^2}{8F^4} \frac{1}{q^2 - m_\pi^2 + i\varepsilon} \sum_{b=1}^3 S_1 \cdot \left[ \tau_1^a \tau_2^b \delta^{bc} (q_2 + q_3) + \tau_1^b \tau_2^b \delta^{ac} (q_1 + q_3) + \tau_2^b \tau_1^c \delta^{ab} (q_1 + q_2) \right] S_2 \cdot q_2 \quad (58)$$

$$= -i \frac{g_A^2}{8F^4} \frac{1}{q^2 - m_\pi^2 + i\varepsilon} \sum_{b=1}^3 S_1 \cdot \left[ \tau_1^a \tau_2^b \delta^{bc} (q_2 + q_3) + \vec{\tau}_1 \cdot \vec{\tau}_2 \delta^{ac} (q_1 + q_3) + \tau_2^b \tau_1^a \delta^{ab} (q_1 + q_2) \right] S_2 \cdot q_2 \quad (59)$$

$$= -i \frac{g_A^2}{8F^4} \frac{1}{q^2 - m_\pi^2 + i\varepsilon} S_1 \cdot [\tau_1^a \tau_2^c (q_2 + q_3) + 3\vec{\tau}_1 \cdot \vec{\tau}_2 (q_1 + q_3) \delta^{ac} + \tau_2^a \tau_1^a (q_1 + q_2)] S_2 \cdot q_2 \quad (60)$$

$$= -i \frac{g_A^2}{8F^4} \frac{1}{q^2 - m_\pi^2 + i\varepsilon} S_1 \cdot [\tau_1^a \tau_2^c (q_1 + 2q_2 + q_3) + 3\vec{\tau}_1 \cdot \vec{\tau}_2 \delta^{ac} (q_1 + q_3)] S_2 \cdot q_2 \quad (61)$$

Now taking  $a = c$ :

$$\mathcal{M}_{2,b} = -i \frac{g_A^2}{8F^4} \frac{1}{q^2 - m_\pi^2 + i\varepsilon} S_1 \cdot [\tau_1^a \tau_2^a (q_1 + 2q_2 + q_3) + 3\vec{\tau}_1 \cdot \vec{\tau}_2 (q_1 + q_3)] S_2 \cdot q_2 \quad (62)$$

Note that  $q_2$  is the propagator momentum and is therefore off shell. Now taking the limit of the threshold case,  $q_1 = q_3 = (m_\pi, \vec{0})$  and  $q_2 = (0, \vec{q})$ , and the non-relativistic limit of  $S = (0, \vec{\sigma})$ :

$$\mathcal{M}_{2,b} = i \frac{g_A^2}{16F^4} \frac{\tau_1^a \tau_2^a}{\vec{q}^2 + m_\pi^2 - i\varepsilon} (\vec{\sigma}_1 \cdot \vec{q}) (\vec{\sigma}_2 \cdot \vec{q}) \quad (63)$$

Which is really close to the Weinberg result, except for a factor of  $[2\pi^4 (1 + \mu)]^{-1}$  and the constant  $a$  should be a sum. Note that the factors of  $\pi$  are fixed if we use  $F \rightarrow \pi F_\pi$ . Also, I wonder if the sum over  $a$  comes from allowing any pion to propagate instead of just the neutral pion.

### 3.2.1 The Propagator

Weinberg writes the structure of the propagator as:  $(\vec{q}^2 + m_\pi^2)^{-1}$  Whereas Beane writes it as:  $(\vec{q}^2 + m_\pi^2)^{-2}$  But the "starting" propagator as defined in BKM A.1 is  $i\delta^{ab} (q^2 - m_\pi^2 + i\varepsilon)^{-1}$ , where  $q$  is the four momentum. Now we can write the propagator as :

$$[q^2 - m_\pi^2]^{-1} = [E^2 - \vec{q}^2 - m_\pi^2]^{-1} \quad (64)$$

$$= \frac{-1}{\vec{q}^2 + m_\pi^2} \left[ 1 - \left( \frac{E^2}{\vec{q}^2 + m_\pi^2} \right) \right]^{-1} \quad (65)$$

$$= \frac{-1}{\vec{q}^2 + m_\pi^2} \left[ 1 + \frac{E^2}{\vec{q}^2 + m_\pi^2} + \left( \frac{E^2}{\vec{q}^2 + m_\pi^2} \right)^2 + \dots \right] \quad (66)$$

Where  $E = \sqrt{m_\pi^2 + \vec{q}^2}$ . So now taking the threshold case  $m_\pi \gg \vec{q}^2$

$$[q^2 - m_\pi^2]^{-1} \approx \frac{-1}{\vec{q}^2 + m_\pi^2} \left[ 1 + \frac{m_\pi^2}{\vec{q}^2 + m_\pi^2} + \dots \right] \quad (67)$$

$$\approx \frac{-1}{\vec{q}^2 + m_\pi^2} \left[ 1 + \frac{m_\pi^2}{\vec{q}^2 + m_\pi^2} + \dots \right] \quad (68)$$

But I'm confused why Weinberg bothered with this, it's not that much more complicated to just program the initial propagator. Maybe its to avoid numerical zeros.

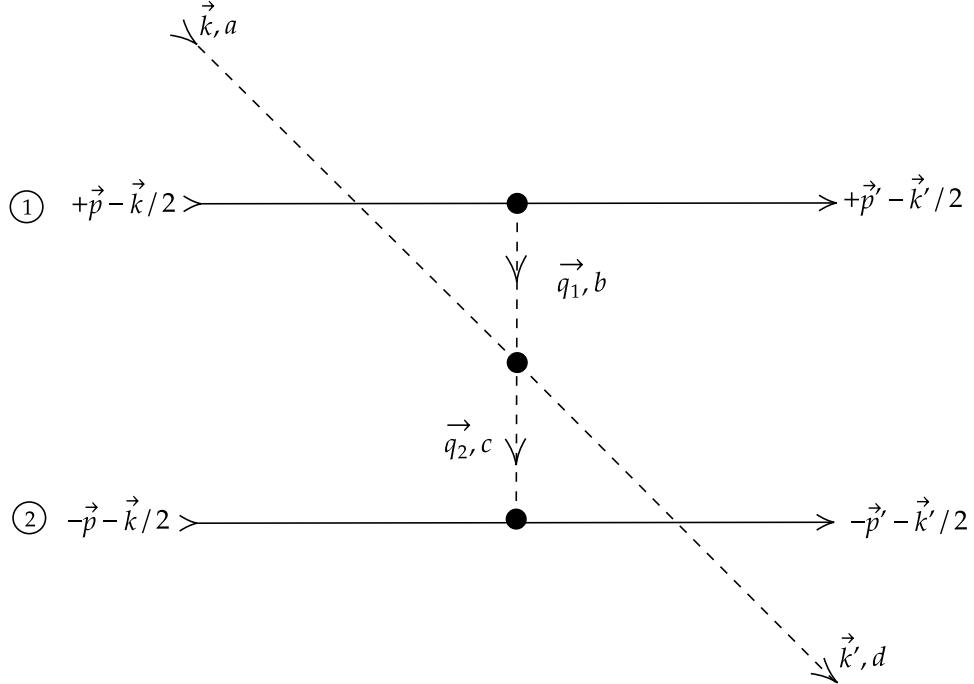
$$\mathcal{M}_{2,b} = -i \frac{g_A^2}{16F^4} \tau_1^a \tau_2^a \vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} \left[ \frac{-1}{\vec{q}^2 + m_\pi^2} + \mathcal{O}(q_0^2) \right] \quad (69)$$

So then:

$$\mathcal{M}_{2,b} = i \frac{g_A^2}{16F^4} \tau_1^a \tau_2^a \frac{\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q}}{\vec{q}^2 + m_\pi^2} \quad (70)$$

But this is still different than the Weinberg result by a factor of 2 and the isospin dependence.

### 3.3 2 Body C



With:

$$\vec{q}_1 = \vec{p} - \vec{p}' + \frac{1}{2}(\vec{k}' - \vec{k}) \quad (71)$$

$$\vec{q}_2 = \vec{p} - \vec{p}' + \frac{1}{2}(\vec{k}' + \vec{k}) \quad (72)$$

Using BKM A.10, with all  $q$ 's in

$$O^{abcd} = \frac{i}{F^2} \left\{ \delta^{ab} \delta^{cd} [(q_1 + q_2)^2 - m_\pi^2] + \delta^{ac} \delta^{bd} [(q_1 + q_3)^2 - m_\pi^2] + \delta^{ad} \delta^{bc} [(q_1 + q_4)^2 - m_\pi^2] \right\} \quad (73)$$

From BKM (left hand side), to our labels, (right hand side)

$$\text{Matrix indices } a, b, c, d \text{ remain the same} \quad (74)$$

$$\vec{q}_1 \rightarrow \vec{q}_1 \text{ index } b \quad (75)$$

$$\vec{q}_2 \rightarrow \vec{k} \text{ index } a \quad (76)$$

$$\vec{q}_3 \rightarrow -\vec{q}_2 \text{ index } c \quad (77)$$

$$\vec{q}_4 \rightarrow -\vec{k}' \text{ index } d \quad (78)$$

$$\mathcal{M}_{2,c} = \frac{g}{F} S_1 \cdot q_1 \tau_1^b i [q_1 - m_\pi^2 + i\varepsilon]^{-1} O^{abcd} \frac{g}{F} S_2 \cdot (-q_2) \tau_2^c i [q_2^2 - m_\pi^2 + i\varepsilon]^{-1} \quad (79)$$

$$= \frac{g}{F} S_1 \cdot q_1 \tau_1^b i [q_1 - m_\pi^2 + i\varepsilon]^{-1} \frac{i}{F^2} O^{abcd} \frac{g}{F} S_2 \cdot (-q_2) \tau_2^c i [q_2^2 - m_\pi^2 + i\varepsilon]^{-1} \quad (80)$$

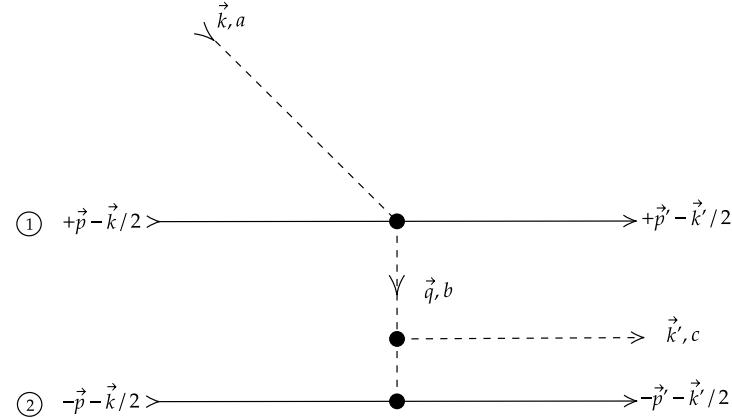
$$= -i \frac{g^2}{F^4} S_1 \cdot q_1 S_2 \cdot (-q_2) O^{abcd} \frac{\tau_1^b \tau_2^c}{(q_1^2 - m_\pi^2 + i\varepsilon)(q_2^2 - m_\pi^2 + i\varepsilon)} \quad (81)$$

I'm sure this is correct, but there is some weird stuff going on with the energy flow. In particular  $q_1^2 = -\vec{q}_1^2$ , but  $q_2^2 = E_2^2 - \vec{q}_2^2$ . Also note that  $O^{abcd}$  has no isospin dependence, so we can commute  $\tau$  with it. This gives:

$$\mathcal{M}_{2,c} = i \frac{g^2}{F^4} S_1 \cdot q_1 S_2 \cdot (-q_2) \frac{\tau_1^b \tau_2^c}{(\vec{q}_1^2 + m_\pi^2 + i\varepsilon)(E_2^2 - \vec{q}_2^2 - m_\pi^2 + i\varepsilon)} O^{abcd} \quad (82)$$

Where  $E_2$  is the

### 3.4 2 Body D



## 4 The Scattering Length

The total scattering length is given by BKM eq 5.29:

$$a_{ab} = \frac{1 + m_\pi/m_N}{1 + m_\pi/Am_N} \sum_r a_{ab}^{(r)} + \binom{A}{2} a_{ab}^{\text{Two Nucleon}} \quad (83)$$

Where  $a, b$  are the incoming and outgoing isospins of the pion, and  $m_N$  is the nucleon mass.

## 4.1 One Body contributions to the scattering length

The onebody case is typically parameterized in terms of  $a^\pm$ . Here  $a_{\pi^\pm n}$  stands for the scattering length of the reaction  $\pi^\pm n \rightarrow \pi^\pm n$

$$a_{\pi^- p} = a^+ + a^- \quad I = -1 \quad (84)$$

$$a_{\pi^0 p} = a^+ \quad I = 0 \quad (85)$$

$$a_{\pi^+ p} = a^+ - a^- \quad I = 1 \quad (86)$$

The isospin value on the right hand side refers to the pion isospin. We have something similar for the reaction with the neutron.

$$a_{\pi^- n} = a^+ - a^- \quad (87)$$

$$a_{\pi^0 n} = a^+ \quad (88)$$

$$a_{\pi^+ n} = a^+ + a^- \quad (89)$$

From "Isospin breaking in the pion–nucleon scattering lengths" Martin Hoferichter et. al..

$$a^+ = (1.5 \pm 2.2) \times 10^{-3} m_\pi^{-1} \quad (90)$$

$$a^- = (85.2 \pm 1.8) \times 10^{-3} m_\pi^{-1} \quad (91)$$

## 4.2 Two Body contributions to the scattering length

As stated in the BKM review pg 115-116 the two body contributions to the scattering length from the first six diagrams is:

$$a_{ab} = \frac{M_\pi^2}{32\pi^4 F_\pi^4 (1 + M_\pi/m_d)} \sum_{r < s} \left\langle \frac{1}{\vec{q}_{rs}^2} \left( 2\vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} \delta_{ab} - \tau_a^{(r)} \tau_b^{(s)} - \tau_a^{(s)} \tau_b^{(r)} \right) \right\rangle \quad (92)$$

$$- \frac{g_A^2 \delta_{ab}}{32\pi^4 F_\pi^4 (1 + M_\pi/m_d)} \sum_{r < s} \left\langle \vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} \frac{\vec{q}_{rs} \cdot \vec{\sigma}^{(r)} \vec{q}_{rs} \cdot \vec{\sigma}^{(s)}}{\vec{q}_{rs}^2 + M_\pi^2} \right\rangle \quad (93)$$

$$+ \frac{g_A^2}{32\pi^4 F_\pi^4 (1 + M_\pi/m_d)} \sum_{r < s} \left\langle \frac{\left[ \vec{q}_{rs}^2 \vec{\tau}^{(r)} \cdot \vec{\tau}^{(s)} \delta_{ab} + M_\pi^2 \left( \tau_a^{(r)} \tau_b^{(s)} + \tau_a^{(s)} \tau_b^{(r)} \right) \vec{q}_{rs} \cdot \vec{\sigma}^{(r)} \vec{q}_{rs} \cdot \vec{\sigma}^{(s)} \right]}{(\vec{q}_{rs}^2 + M_\pi^2)^2} \right\rangle \quad (94)$$

$$+ \frac{g_A^2 M_\pi}{132\pi^4 F_\pi^4 (1 + M_\pi/m_d)} \sum_{r < s} \left\langle \left( \vec{\tau}^{(r)} + \vec{\tau}^{(s)} \right) \cdot \left( \vec{\tau}^{(\pi)} \right)_{ab} \frac{\vec{q}_{rs} \cdot \vec{\sigma}^{(r)} \vec{q}_{rs} \cdot \vec{\sigma}^{(s)}}{(\vec{q}_{rs}^2 + M_\pi^2)^{3/2}} \right\rangle \quad (95)$$

Note that  $(\vec{\tau}^{(\pi)})_{ab} = -i\varepsilon_{abc}$ , where  $\varepsilon$  is the Levi-Civita Tensor, with  $a, b$  being the pion isospin indices. In this analysis we consider only elastic scattering so  $a = b \implies (\vec{\tau}^{(\pi)})_{aa} = -i\varepsilon_{aac} = \vec{0}$  so eq.(95) drops out.

In order to implement the above terms in our code I will write them in a manner more useful for us. The BKM review uses the notation  $(r), (s)$  to indicate nucleon number but we simply use 1, 2, and the integration over the other nucleons is accounted for elsewhere. Additionally we can simplify using  $a = b$ . Additionally we seek to write the isospin dependence in terms of the actual quantum numbers  $t_{12}, m_{12}$  etc. Now let:

$$\beta = \frac{1}{32\pi^4 F_\pi^4 (1 + M_\pi/M_{\text{nucl}})} \quad (96)$$

Simplifying we have:

$$\begin{aligned}
a &= 2\beta M_\pi^2 \left\langle \frac{1}{\vec{q}^2} (\vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^a \tau_2^a) \right\rangle \\
&\quad - \beta g_A^2 \left\langle \vec{\tau}_1 \cdot \vec{\tau}_2 \frac{\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{\vec{q}^2 + M_\pi^2} \right\rangle \\
&\quad + \beta g_A^2 \left\langle \frac{[\vec{q}^2 \vec{\tau}_1 \cdot \vec{\tau}_2 + 2M_\pi^2 (\tau_1^a \tau_2^a)] \vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + M_\pi^2)^2} \right\rangle
\end{aligned} \tag{97}$$

Recall:

$$\left\langle t'_{12} m'_{12} s'_{12} m'_{12} \middle| \vec{\tau}_1 \cdot \vec{\tau}_2 \middle| t_{12} m_{12}^t s_{12} m_{12} \right\rangle = \delta_{s'_{12} s_{12}} \delta_{m'_{12} m_{12}} \delta_{t'_{12} t_{12}} \delta_{m'_{12} m_{12}^t} \left[ 2t_{12}(t_{12} + 1) - 3 \right] \tag{98}$$

And from an internal communication with Harald Griesshammer

$$\left\langle t'_{12} m'_{12} \middle| \vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^a \tau_2^a \middle| t_{12} m_{12}^t \right\rangle = \begin{cases} -2(-1)^{t_{12}} & \text{if } t'_{12} = t_{12} \text{ and } m'_{12} = m_{12}^t = 0 \\ 0 & \text{otherwise} \end{cases} \tag{99}$$

Note the above holds for  $t = 0, 1$  and  $|m_{12}^t| \leq t$ . Equivalently we can write this as:

$$\left\langle t'_{12} m'_{12} \middle| \vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^a \tau_2^a \middle| t_{12} m_{12}^t \right\rangle = \delta_{t'_{12}, t_{12}} \delta_{m'_{12}, m_{12}^t} \delta_{m_{12}^t, 0} (-2)(-1)^{t_{12}} \tag{100}$$

So then, using  $M_J, M'_J$  as a shorthand for all the quantum numbers:

$$\langle M'_J | \tau_1^a \tau_2^a | M_J \rangle = \langle M'_J | \vec{\tau}_1 \cdot \vec{\tau}_2 - [\vec{\tau}_1 \cdot \vec{\tau}_2 - \tau_1^a \tau_2^a] | M_J \rangle \tag{101}$$

$$= \delta_{s_{12}, s'_{12}} \delta_{m_{12}, m'_{12}} \delta_{m'_{12} m_{12}^t} \delta_{t_{12}, t'_{12}} \left[ (2t_{12}(t_{12} + 1) - 3) - \delta_{m_{12}^t, 0} (-2)(-1)^{t_{12}} \right] \tag{102}$$

Now we only care about  $t_{12} = 0, 1$  and  $|m_{12}^t| \leq t_{12}$  so we can just evaluate the above for those cases.

$$(2t_{12}(t_{12} + 1) - 3) - \delta_{m_{12}^t, 0} (-2)(-1)^{t_{12}} = (-1)^{m_{12}^t + 1} \quad \text{for } t_{12} = 0, 1, \quad |m_{12}^t| \leq t_{12} \tag{103}$$

So:

$$\langle M'_J | \tau_1^a \tau_2^a | M_J \rangle = \delta_{s_{12}, s'_{12}} \delta_{m_{12}, m'_{12}} \delta_{m'_{12} m_{12}^t} \delta_{t_{12}, t'_{12}} (-1)^{m_{12}^t + 1} \tag{104}$$

The spin dependence  $\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2$  is taken care of in our code already so I will not include it here. So now making these substitutions for the isospin dependence we have:

$$\begin{aligned}
a &= \delta_{t'_{12}, t_{12}} \delta_{m'_{12}, m_{12}^t} \left[ -4\beta M_\pi^2 \frac{1}{\vec{q}^2} (-1)^{t_{12}} \delta_{s_{12}, s'_{12}} \delta_{m_{12}, m'_{12}} \delta_{m_{12}^t, 0} \right. \\
&\quad - \beta g_A^2 \left[ 2t_{12}(t_{12} + 1) - 3 \right] \frac{\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{\vec{q}^2 + M_\pi^2} \\
&\quad + \beta g_A^2 \frac{\vec{q}^2 \left[ 2t_{12}(t_{12} + 1) - 3 \right] \vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + M_\pi^2)^2} \\
&\quad \left. - 2\beta g_A^2 M_\pi^2 \frac{(-1)^{m_{12}^t} \vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + M_\pi^2)^2} \right]
\end{aligned} \tag{105}$$

We still need to account for the  $+(1 \leftrightarrow 2)$  to the extent that it is not already accounted for in the subroutines.

$$\begin{aligned}
a = & \delta_{t'_{12}, t_{12}} \delta_{m'_{12}, m^t_{12}} \left[ -8\beta M_\pi^2 \frac{1}{\vec{q}^2} (-1)^{t_{12}} \delta_{s_{12}, s'_{12}} \delta_{m_{12}, m'_{12}} \delta_{m^t_{12}, 0} \right. \\
& - \beta g_A^2 \left[ 2t_{12}(t_{12}+1) - 3 \right] \frac{\vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{\vec{q}^2 + M_\pi^2} + (1 \leftrightarrow 2) \\
& + \beta g_A^2 \frac{\vec{q}^2 \left[ 2t_{12}(t_{12}+1) - 3 \right] \vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + M_\pi^2)^2} + (1 \leftrightarrow 2) \\
& \left. - 2\beta g_A^2 M_\pi^2 \frac{(-1)^{m^t_{12}} \vec{q} \cdot \vec{\sigma}_1 \vec{q} \cdot \vec{\sigma}_2}{(\vec{q}^2 + M_\pi^2)^2} + (1 \leftrightarrow 2) \right] \quad (106)
\end{aligned}$$

In our implementation the term  $+(1 \leftrightarrow 2)$  is taken care of in subroutines that calculate  $\vec{A} \cdot \vec{\sigma} \vec{B} \cdot \vec{\sigma} \pm \vec{A} \cdot \vec{\sigma} \vec{B} \cdot \vec{\sigma}$ , with the  $\pm$  being determined by the symmetry of the nucleon. Note this  $+(1 \leftrightarrow 2)$  is taken care of in the first term simply by a factor of 2.