Another commonly used decomposition of the pion photoproduction amplitude is the so-called isospin parametrisation in terms of the three amplitudes $T^{(\frac{3}{2})}_{\gamma N}, T^{(\frac{1}{2})}_{\gamma p}, T^{(\frac{1}{2})}_{\gamma n}$ where the production amplitudes $T^{(I)}_{\gamma N}$ are related to the $T^{(0,\pm)}_{\gamma N}$ via

$$
\begin{aligned}
&T^{(\frac{3}{2})}_{\gamma N} = T^{(\frac{3}{2})}_{\gamma p} = T^{(\frac{3}{2})}_{\gamma n} = T^{(+)}_{\gamma N} - T^{(-)}_{\gamma N}\,,\\
&T^{(\frac{1}{2})}_{\gamma p} = T^{(0)}_{\gamma N} + \frac{1}{3}T^{(+)}_{\gamma N} + \frac{2}{3}T^{(-)}_{\gamma N}\,, \quad T^{(\frac{1}{2})}_{\gamma n} = T^{(0)}_{\gamma N} - \frac{1}{3}T^{(+)}_{\gamma N} - \frac{2}{3}T^{(-)}_{\gamma N}\,.
\end{aligned}
\tag{3.12}
$$

The isospin parametrisation is a natural choice when working in the isospin symmetric case of ChPT, thus we perform the fits of this work in this basis. For comparison with experimental results, the decomposition (3.11) in terms of the physical reaction channels is required.

3.3 Spin decomposition and multipole amplitudes

The matrix element of pion photoproduction is given by

$$
\mathcal{M} = \epsilon^\mu \mathcal{M}_\mu = -\,\mathrm{i}\,\mathrm{e}\;\epsilon^\mu \left\langle N(p')\pi^c(q) \,\middle|\, J_\mu(k^2) \,\middle|\, N(p)\right\rangle , \tag{3.13}
$$

with $J_\mu(k^2)$ the electromagnetic current operator, ϵ^μ the polarisation vector of the photon and $Q^2 = -k^2$ the virtuality of the photon. Different parametrisations of the matrix element are advantageous for different purposes. The first parametrisation is given in terms of the so-called Ball amplitudes [134][1]

$$
\mathcal{M}^\mu = \sum_{i=1}^{8} \bar{u}(p') B_i V_i^\mu u(p). \tag{3.14}
$$

We denote by $u(p)$ ($\bar{u}(p')$) the Dirac spinor of the incoming (outgoing) nucleon and suppress the spin index for the sake of brevity. The basis structures V_i^μ are all independent Lorentz-invariant matrices that can be formed using gamma matrices and the polarisation vector, and the coefficients B_i are scalar functions of the Mandelstam variables. Explicitly, the basis structures read

$$
\begin{aligned}
&V_1^\mu = \gamma^\mu\gamma_5, && V_2^\mu = \gamma_5 P^\mu, && V_3^\mu = \gamma_5 q^\mu, && V_4^\mu = \gamma_5 k^\mu,\\
&V_5^\mu = \gamma^\mu \not{k}\gamma_5, && V_6^\mu = \not{k}\gamma_5 P^\mu, && V_7^\mu = \not{k}\gamma_5 q^\mu, && V_8^\mu = \not{k}\gamma_5 k^\mu,
\end{aligned}
\tag{3.15}
$$

where $P = \frac{1}{2}(p+p')$ and $\not{a} = a_\mu\gamma^\mu$ is the Feynman slash notation for any four-vector contracted with the Dirac matrices. Note that the set of amplitudes (3.14)-(3.15) is not minimal, but convenient in practical calculations due to the simplicity of the basis structures. Imposing transversality of the amplitude $k_\mu\mathcal{M}^\mu = 0$ leads to the following conditions

$$
B_1 + B_6\, k\cdot P + B_7\, k\cdot q + B_8\, k^2 = 0, \quad B_2\, k\cdot P + B_3\, k\cdot q + B_4\, k^2 + B_5\, k^2 = 0. \tag{3.16}
$$

[1] We work with Hilt's convention, given in ref. [95], which is slightly different from Ball's original.

Thus, current conservation reduces the number of basis structures to six. In real pion photoproduction, due to the additional constraint $\epsilon \cdot k = 0$ and $k^2 = 0$, only four structures remain.

The second relevant parametrisation of the matrix element was worked out by Chew, Goldberger, Low and Nambu (CGLN) and constructed in such a way that the basis structures themselves fulfil transversality [134–136]:

$$\begin{aligned}
M_1^\mu &= -\tfrac{\mathrm{i}}{2}\gamma_5(\gamma^\mu \not{k} - \not{k}\gamma^\mu), \\
M_2^\mu &= 2\,\mathrm{i}\,\gamma_5 \left(P^\mu k\cdot(q - \tfrac{1}{2}k) - (q^\mu - \tfrac{1}{2}k^\mu)k\cdot P\right), \\
M_3^\mu &= -\,\mathrm{i}\,\gamma_5(\gamma^\mu k\cdot q - \not{k}q^\mu), \\
M_4^\mu &= -2\,\mathrm{i}\,\gamma_5(\gamma^\mu k\cdot P - \not{k}P^\mu) - 2m_N M_1^\mu.
\end{aligned} \tag{3.17}$$

with

$$\mathcal{M}^\mu = \sum_{i=1}^{4} \bar{u}(p')A_i M_i^\mu u(p). \tag{3.18}$$

Here, the number of basis structures is minimal. Furthermore, this decomposition is only possible for real photons $k^2 = 0$.

The third parametrisation for pion photoproduction is given by [135]

$$\epsilon_\mu \bar{u}(p')\left(\sum_{i=1}^{4} A_i M_i^\mu\right) u(p) = \frac{4\pi\sqrt{s}}{m_N}\chi_f^\dagger\,\mathcal{F}\,\chi_i. \tag{3.19}$$

Here, χ_i ($\chi_f^\dagger$) is the initial (final) Pauli spinor. $\mathcal{F}$ can be expressed in the so-called CGLN amplitudes:

$$\mathcal{F} = \mathrm{i}\,\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\,\mathcal{F}_1 + \frac{\boldsymbol{\sigma}\cdot\mathbf{q}\,\boldsymbol{\sigma}\cdot(\mathbf{k}\times\boldsymbol{\epsilon})}{|\mathbf{q}|\,|\mathbf{k}|}\,\mathcal{F}_2 + \mathrm{i}\,\frac{\boldsymbol{\sigma}\cdot\mathbf{k}\,\mathbf{q}\cdot\boldsymbol{\epsilon}}{|\mathbf{q}|\,|\mathbf{k}|}\,\mathcal{F}_3 + \mathrm{i}\,\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\,\mathbf{q}\cdot\boldsymbol{\epsilon}}{|\mathbf{q}|^2}\,\mathcal{F}_4. \tag{3.20}$$

Note that this parametrisation is only possible in the CM frame, in the case of real photons $k^2 = 0$ and when choosing a polarisation vector with a vanishing time-component $\epsilon^0 = 0$, which is possible in a gauge-invariant process. The $\mathcal{F}_i$'s can be expanded in a multipole series [134, 135]:

$$\begin{aligned}
\mathcal{F}_1 &= \sum_{l=0}^{\infty}\left\{[l\,M_{l+} + E_{l+}]P'_{l+1}(x) + [(l+1)M_{l-} + E_{l-}]P'_{l-1}(x)\right\}, \\
\mathcal{F}_2 &= \sum_{l=1}^{\infty}[(l+1)M_{l+} + lM_{l-}]P'_l(x), \\
\mathcal{F}_3 &= \sum_{l=1}^{\infty}\left\{[E_{l+} - M_{l+}]P''_{l+1}(x) + [E_{l-} + M_{l-}]P''_{l-1}(x)\right\},
\end{aligned}$$

$$\mathcal{F}_4 = \sum_{l=2}^{\infty} [M_{l+} - E_{l+} - M_{l-} - E_{l-}] P_l''(x), \tag{3.21}$$

where $x = \cos(\theta)$, $P_l(x)$ is a Legendre polynomial of degree l, $P_l'(x) = \frac{\mathrm{d}P_l}{\mathrm{d}x}$ is its first derivative and $P_l''(x)$ is the second derivative with respect to x. l is the orbital angular momentum of the outgoing pion-nucleon system. The subscript $\pm$ denotes the total angular momentum $j = l \pm 1/2$. To calculate multipole amplitudes, eq. (3.21) must be inverted. The angular dependence is integrated out, such that the multipoles

$$\begin{aligned}
E_{l+} &= \int_{-1}^{1} \frac{\mathrm{d}x}{2(l+1)} \left[P_l \mathcal{F}_1 - P_{l+1} \mathcal{F}_2 + \frac{l}{2l+1}(P_{l-1} - P_{l+1})\mathcal{F}_3 + \frac{l+1}{2l+3}(P_l - P_{l+2})\mathcal{F}_4 \right], \\
E_{l-} &= \int_{-1}^{1} \frac{\mathrm{d}x}{2l} \left[P_l \mathcal{F}_1 - P_{l-1} \mathcal{F}_2 - \frac{l+1}{2l+1}(P_{l-1} - P_{l+1})\mathcal{F}_3 + \frac{l}{2l-1}(P_l - P_{l-2})\mathcal{F}_4 \right], \\
M_{l+} &= \int_{-1}^{1} \frac{\mathrm{d}x}{2(l+1)} \left[P_l \mathcal{F}_1 - P_{l+1} \mathcal{F}_2 - \frac{1}{2l+1}(P_{l-1} - P_{l+1})\mathcal{F}_3 \right], \\
M_{l-} &= \int_{-1}^{1} \frac{\mathrm{d}x}{2l} \left[-P_l \mathcal{F}_1 + P_{l-1} \mathcal{F}_2 + \frac{1}{2l+1}(P_{l-1} - P_{l+1})\mathcal{F}_3 \right]
\end{aligned} \tag{3.22}$$

do not depend on the angle anymore. Here, we suppress the x-dependence of the Legendre polynomials P_l for the sake of brevity. The multipoles $E_{l\pm}$ and $M_{l\pm}$ depend on the total energy in the CM frame $\sqrt{s}$ and behave like $|\mathbf{q}|^l$ in the threshold region.

To calculate multipole amplitudes, we proceed as follows: First, we express the pion photoproduction amplitude in terms of the Ball amplitudes (eq. (3.14)). Then we rewrite the B_i's in terms of A_i's to obtain the representation of the amplitude in the minimal basis (eq. (3.18)). Finally, we use the coefficients A_i to calculate $\mathcal{F}_1 - \mathcal{F}_4$. The relation between these representations are given in the appendix A.

3.4 The cross section and polarisation asymmetry

In this subsection, we show how to calculate the unpolarised differential cross section and linear polarisation asymmetry of pion photoproduction. The general expression for the differential cross section of a scattering process with two incoming and n outgoing particles is given by

$$\mathrm{d}\sigma = \frac{(2\pi)^4}{2\sqrt{\lambda(s, m_1^2, m_2^2)}} \delta^{(4)} \left(\sum_{i=1}^{n} p_i' - p_1 - p_2 \right) |\mathcal{M}|^2 \prod_{i=1}^{n} \frac{\mathrm{d}^3 p_i'}{(2\pi)^3 2E_i'}. \tag{3.23}$$

In the above expression, p_i and m_i are momentum and mass of the incoming particle i, p_i' is the momentum of the outgoing particle i and $\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$ is the Källén function. The unpolarised squared matrix element $\left|\overline{\mathcal{M}}\right|^2$ of pion photoproduction can be obtained from

$$\left|\overline{\mathcal{M}}\right|^2 = \frac{1}{4}\sum_{\lambda=-1}^{1}\sum_{s,s'=-1}^{1}\left|\epsilon_\mu(k,\lambda)\mathcal{M}^\mu(k,p,s,p',s',q)\right|^2, \tag{3.24}$$

where λ is the helicity of the photon, s (s') is the spin of the incoming (outgoing) nucleon and the factor of $1/4$ arises from averaging over helicity and spin of the incoming particles. Using the simplifications of the CM frame and carrying out the integration, the unpolarised differential cross section reads

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{|\mathbf{q}|}{|\mathbf{k}|}\left|\overline{\mathcal{M}}\right|^2. \tag{3.25}$$

Furthermore, we calculate the linear polarised photon asymmetry Σ, which is given by

$$\Sigma = \frac{\mathrm{d}\sigma_\perp - \mathrm{d}\sigma_\parallel}{\mathrm{d}\sigma_\perp + \mathrm{d}\sigma_\parallel}. \tag{3.26}$$

In eq. (3.26), $\mathrm{d}\sigma_\perp$ and $\mathrm{d}\sigma_\parallel$ refer to angular cross section for photon polarisations perpendicular and parallel to the reaction plane, respectively. The difference to the unpolarised cross sections is that the squared matrix element is not averaged over both incoming polarisations, but calculated individually according to

$$\left|\mathcal{M}_{\perp/\parallel}\right|^2 = \frac{1}{2}\sum_{s,s'=-1}^{1}\left|\frac{1}{\sqrt{2}}\left(\epsilon_\mu(1) \pm \epsilon_\mu(-1)\right)\mathcal{M}^\mu\right| \tag{3.27}$$

with

$$\epsilon^\mu(1) = \frac{1}{\sqrt{2}}(0,-1,-\mathrm{i},0) \quad \text{and} \quad \epsilon^\mu(-1) = \frac{1}{\sqrt{2}}(0,1,-\mathrm{i},0). \tag{3.28}$$

3.5 Renormalisation

In this section, we discuss the subtleties of renormalisation, i.e. all steps necessary to remove unphysical infinities from the theory. Furthermore, we address the problem of the power-counting violating terms (PCVT) introduced in the section 2.3.1 and its effects for this work.

3.5.1 Renormalisation of subprocesses

To relate the bare constants of the effective Lagrangian to their physical counterparts, we must consider various subprocesses of pion photoproduction. We take care of the appearing integrals and their ultraviolet (UV) divergences by dimensional regularisation. Also, we choose