Variable transform

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Lets analyze

$$\alpha = \int d^3p' \int d^3p \frac{f(\vec{p}, \vec{p}')}{(\vec{p} - \vec{p}' + \vec{k}/2)^2} \tag{1}$$

Make the substitution:

$$\vec{q} = \vec{p} - \vec{p}' + \vec{k}/2 \tag{2}$$
$$\implies \vec{p} = \vec{q} + \vec{p}' - \vec{k}/2 \tag{3}$$

So with respect to the first integral d^3p we have $d^3p = d^3q$. Do we have to account for the fact that we are substituting a variable that is later integrated over? Certainly if we consider only the first integral:

$$\beta = \int d^3p \frac{f(\vec{p}, \vec{p}')}{(\vec{p} - \vec{p}' + \vec{k}/2)^2} \tag{4}$$
$$= \int d^3q \frac{f(\vec{q} + \vec{p}' - \vec{k}/2,\ \vec{p}')}{\vec{q}^2} \tag{5}$$
$$= \int dq \int d\widehat{q}\ f(\vec{q} + \vec{p}' - \vec{k}/2,\ \vec{p}') \tag{6}$$

Where in the last step we write this in spherical coordinates, and $d\widehat{q} = d\phi_q d\theta_q \sin\theta_q$ represents the radial integration. If we don't have to worry about doing a substitution with a variable we are later integrating with then we can write α as:

$$\alpha = \int dp' d\widehat{p}' p'^2 \int dq\ d\widehat{q}\ f(\vec{q} + \vec{p}' - \vec{k}/2,\ \vec{p}') \tag{7}$$

But the mathematician in me is uneasy since it appears the dependence just vanishes. To explain this first consider a trivial example:

$$1 = \int_0^\infty \int_0^\infty dxdy\ \mathrm{e}^{-(x+y)} = \int_0^\infty \int_y^{\infty+y} e^{-u} dudy \tag{8}$$
$$= \int_0^\infty \int_y^\infty e^{-u} dudy = 1 \tag{9}$$

So the dependence goes into the bounds of the integration. In our example we have:

$$\alpha = \int d^3p' \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \frac{f(\vec{p},\vec{p}')}{(\vec{p}-\vec{p}'+\vec{k}/2)^2} \tag{10}$$

$$\begin{aligned} &= \int d^3p' \int_{-\infty+(-\vec{p}'+\vec{k}/2)_x=q_x}^{\infty+(-\vec{p}'+\vec{k}/2)_x=q_x} dq_x \int_{-\infty+(-\vec{p}'+\vec{k}/2)_y=q_y}^{\infty+(-\vec{p}'+\vec{k}/2)_y=q_y} dq_y \\ &\qquad \times \int_{-\infty+(-\vec{p}'+\vec{k}/2)_z=q_z}^{\infty+(-\vec{p}'+\vec{k}/2)_z=q_z} dq_z \frac{f(\vec{q}+\vec{p}'-\vec{k}/2,\ \vec{p}')}{\vec{q}^2} \end{aligned} \tag{11}$$

$$= \int d^3p' \int_{-\infty}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \int_{-\infty}^{\infty} dq_z \frac{f(\vec{q}+\vec{p}'-\vec{k}/2,\ \vec{p}')}{\vec{q}^2} \tag{12}$$

$$= \int d^3p' \int d^3q \frac{f(\vec{q}+\vec{p}'-\vec{k}/2,\ \vec{p}')}{\vec{q}^2} \tag{13}$$

$$= \int dp' d\widehat{p}' p'^2 \int dq\ d\widehat{q}\ f(\vec{q}+\vec{p}'-\vec{k}/2,\ \vec{p}') \tag{14}$$