14.3.23

Derivation of partial wave decomposition of π^0 production operator

operator : $\dfrac{\vec{\varepsilon}\cdot(\vec{\sigma}_1+\vec{\sigma}_2)\,(\vec{\tau}_1\cdot\vec{\tau}_2-\tau_1^z\tau_2^z)}{(\vec{p}_{12}-\vec{p}_{12}\,'+\vec{k}_\gamma/2)^2}$

Since we would like to stick with density, I will not try to shift $\vec{p}_{12}\,'$ momentum.

direct calculation of matrix element

$$\langle p_{12}'\,(l_{12}'s_{12}')j_{12}'\,m_{12}'\,t_{12}'\,m_{12}^{t'}\,|\,\frac{\vec{\varepsilon}\cdot(\vec{\sigma}_1+\vec{\sigma}_2)\,(\vec{\tau}_1\cdot\vec{\tau}_2-\tau_1^z\tau_2^z)}{(\vec{p}_{12}-\vec{p}_{12}\,'+\vec{k}_\gamma/2)^2}\,|\,p_{12}\,(l_{12}s_{12})j_{12}\,m_{12}\,t_{12}\,m_{12}^t\rangle$$

1) isospin part

$\langle t_{12}'m_{12}^{t}\,|\,\vec{\tau}_1\cdot\vec{\tau}_2-\tau_1^z\tau_2^z\,|\,t\,m_{12}^t\rangle$ both parts are symmetrical $\rightarrow$ $\delta_{t_{12}'t_{12}}$!

$= 2\,t_{12}(t_{12}+1)-3+(-1)^{m_{12}^t}$

2) spin matrix element

$\langle s_{12}'\,m_{12}^{s'}\,|\,\vec{\varepsilon}\cdot(\vec{\sigma}_1+\vec{\sigma}_2)\,|\,s\,m_{12}^s\rangle$

use: $\vec{a}\cdot\vec{b}=-\sqrt{3}\,\{a,b\}^{00}$

$\{$ coupling of tensor operators of rank 1.

$= -\sqrt{3}\sum_\lambda (1\,1\,0,\ \lambda\,-\lambda\,0)\ \varepsilon_\lambda\ \langle s_{12}'\,m_{12}^{s'}\,|(\vec{\sigma}_1+\vec{\sigma}_2)^{-\lambda}\,|s_{12}\,m_{12}^s\rangle$

$a_{+1}=-\frac{1}{\sqrt{2}}(a_x+i\,a_y)$

$a_0=a_z$ or $a_\lambda=a\sqrt{\frac{4\pi}{3}}\,Y_{1\lambda}(\hat{a})$

$a_{-1}=\frac{1}{\sqrt{2}}(a_x-i\,a_y)$

I use the „Rose" definition of a reduced matrix element

$$\langle s_{12}'\,m_{12}^{s'}\,|(\vec{\sigma}_1+\vec{\sigma}_2)^{-\lambda}\,|s_{12}\,m_{12}^s\rangle=\langle s_{12}'\,||(\vec{\sigma}_1+\vec{\sigma}_2)||\,s_{12}\rangle\,(s_{12}\,1\,s_{12}',\ m_{12}^s\,-\lambda\ m_{12}^{s'})$$

With this definition, $\langle\frac{1}{2}\,||\,\sigma\,||\,\frac{1}{2}\rangle=\dfrac{\langle\frac{1}{2}\frac{1}{2}\,|\,\sigma_z\,|\,\frac{1}{2}\frac{1}{2}\rangle}{(\frac{1}{2}\,1\,\frac{1}{2},\ \frac{1}{2}\,0\,\frac{1}{2})}=\sqrt{3}.$

Use „master formula"

$$\langle(j_1'j_2')J'\,||\,\{T_1(k_1),T_2(k_2)\}^K\,||\,(j_1j_2)J\rangle$$

$$=\sqrt{\hat{j}_1'\,\hat{j}_2'\,\hat{J}\,\hat{K}}\begin{Bmatrix} j_1' & j_1 & k_1\\ j_2' & j_2 & k_2\\ J' & J & K\end{Bmatrix}\langle j_1'\,||\,T_1(k_1)\,||\,j_1\rangle\ \langle j_2'\,||\,T_2(k_2)\,||\,j_2\rangle$$

$$\langle s_{12}'\,||\,\sigma_1\,||\,s_{12}\rangle=\sqrt{\hat{\tfrac{1}{2}}\,\hat{\tfrac{1}{2}}\,\hat{s}_{12}\,\hat{1}}\begin{Bmatrix}1/2 & 1/2 & 1\\ 1/2 & 1/2 & 0\\ s_{12}' & s_{12} & 1\end{Bmatrix}\langle\tfrac{1}{2}\,||\,\sigma_1\,||\,\tfrac{1}{2}\rangle\ \langle\tfrac{1}{2}\,||\,\mathbb{1}\,||\,1/2\rangle$$

$$=2\sqrt{\hat{s}_{12}}\,\sqrt{3}\,(-)^{s_{12}}\,\sqrt{6}\begin{Bmatrix}1/2 & 1/2 & 1\\ s_{12} & s_{12}' & 1/2\end{Bmatrix}\sqrt{3}\cdot 1$$

$$=6\sqrt{6}\,\sqrt{\hat{s}_{12}}\,(-)^{s_{12}}\begin{Bmatrix}1/2 & 1/2 & 1\\ s_{12} & s_{12}' & 1/2\end{Bmatrix}$$

second σ by permutation of particles 1 and 2 as additional factor $(1+(-)^{s_{12}+s_{12}'})$

$$\Rightarrow \langle s_{12}' m_{12}^{s\,\prime} | \vec{\varepsilon}\cdot(\vec{\sigma}_1+\vec{\sigma}_2) | s m_{12}^s \rangle$$

$$= -\sqrt{3} \sum_\lambda (1\,1\,0, \lambda\, -\lambda\, 0)(s_{12}\, 1\, s_{12}', m_{12}^s\, -\lambda\, m_{12}^{s\prime})\, \varepsilon_\lambda$$

$$6\sqrt{6}\sqrt{\hat{s}_{12}}\,(-)^{s_{12}} \begin{Bmatrix} 1/2 & 1/2 & 1 \\ s_{12} & s_{12}' & 1/2 \end{Bmatrix} (1+(-)^{s_{12}+s_{12}'})$$

3) orbital part

$$\langle p_{12}' l_{12}' m_{12}^{l\prime} | \frac{1}{(\vec{p}_{12}-\vec{p}_{12}'+\vec{k}_\gamma/2)^2} | p_{12}\, l_{12}\, m_{12}^l \rangle$$

I assume that $\vec{k}_\gamma = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$ (positive z-direction)

problem is the singularity that could also happen in angular dependence!

$$\int d^3\tilde{p}_{12}\, d^3\tilde{p}_{12}' \langle p_{12}' l_{12}' m_{12}^{l\prime} | \vec{\tilde{p}}_{12}{}' \rangle \frac{1}{(\underbrace{\vec{\tilde{p}}_{12}-\vec{\tilde{p}}_{12}{}'+\vec{k}_\gamma/2}_{})^2} \langle \vec{\tilde{p}}_{12} | p_{12}\, l_{12}\, m_{12}^l \rangle$$

$\vec{\tilde{p}}_{12} \equiv \vec{\bar{p}}_{12} + \vec{k}_\gamma/4$

$\vec{\tilde{p}}_{12}{}' \equiv \vec{\bar{p}}_{12}{}' - \vec{k}_\gamma/4$

$$= \int d^3\bar{p}_{12}\, d^3\bar{p}_{12}' \langle p_{12}' l_{12}' m_{12}^{l\prime} | \vec{\bar{p}}_{12}{}' + \vec{k}_\gamma/4 \rangle \frac{1}{(\vec{\bar{p}}_{12}-\vec{\bar{p}}_{12}{}')^2} \langle \vec{\bar{p}}_{12} - \vec{k}_\gamma/4 | p_{12}\, l_{12}\, m_{12}^l \rangle$$

now I use the transition to $\vec{q} = \vec{\bar{p}}_{12} - \vec{\bar{p}}_{12}{}'$ and $\vec{P} = \frac{1}{2}(\vec{\bar{p}}_{12} + \vec{\bar{p}}_{12}{}')$

For these coordinates the Jacobian should be 1. $\det \begin{array}{c|cc} & \vec{q} & \vec{P} \\ \bar{p}_{12} & 1 & 1/2 \\ \bar{p}_{12}' & -1 & 1/2 \end{array}$

$$\vec{\bar{p}}_{12} = \vec{P} + \tfrac{1}{2}\vec{q} \qquad \vec{\bar{p}}_{12}{}' = \vec{P} - \tfrac{1}{2}\vec{q}$$

$$= \int d^3q\, d^3P \langle p_{12}' l_{12}' m_{12}^{l\prime} | \vec{P} - \tfrac{1}{2}\vec{q} + \vec{k}_\gamma/4 \rangle \frac{1}{q^2} \langle \vec{P} + \tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4 | p_{12}\, l_{12}\, m_{12}^l \rangle$$

q^2 cancels!

$$= \int d\hat{q} \int d\hat{P} \int dP\, P^2 \int dq\; Y^*_{l_{12}' m_{12}^{l\prime}}(\vec{P} - \tfrac{1}{2}\vec{q} + \vec{k}_\gamma/4)\, Y_{l_{12} m_{12}^l}(\vec{P} + \tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4)$$

$$\frac{\delta(p_{12} - |\vec{P} + \tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4|)}{p_{12}^2} \frac{\delta(p_{12}' - |\vec{P} - \tfrac{1}{2}\vec{q} + \vec{k}_\gamma/4|)}{p_{12}'^2}$$

$$\delta(\rho_{12} - |\vec{p} + \tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4|)\, \delta(\rho_{12}' - |\vec{p} - (\tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4)|)\, \frac{1}{|\vec{p}+\vec{\tilde{p}}|^{l_{12}} |\vec{p}-\vec{\tilde{p}}|^{l_{12}'}} \frac{1}{|\frac{\vec{q}}{2} - \frac{\vec{k}_\gamma}{4}|^{k}}$$

depends on angle between $\vec{p}$, $\frac{1}{2}\vec{q} - \vec{k}_\gamma/4$, $\vec{p}^2$ and $|\frac{1}{2}\vec{q} - \vec{k}_\gamma/4|^2$

Therefore, I can expand this function in terms of Legendre polynomials

$$\delta(\rho_{12} - |\vec{p} + \tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4|)\, \delta(\rho_{12}' - |\vec{p} - (\tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4)|)\, \frac{1}{|\vec{p}+\vec{\tilde{p}}|^{l_{12}} |\vec{p}-\vec{\tilde{p}}|^{l_{12}'}}$$

$$= \sum_k P_k(x) \frac{\hat{k}}{2} \underbrace{\int_{-1}^{1} dx' P_k(x') \frac{\delta(\rho_{12} - \sqrt{\vec{p}^2 + \vec{\tilde{p}}^2 + 2\vec{p}\cdot\vec{\tilde{p}}})}{\rho_{12}^2} \frac{\delta(\rho_{12}' - \sqrt{\vec{p}^2 + \vec{\tilde{p}}^2 - 2\vec{p}\cdot\vec{\tilde{p}}})}{\rho_{12}'^2} \frac{1}{|\vec{p}+\vec{\tilde{p}}|^{l_{12}} |\vec{p}-\vec{\tilde{p}}|^{l_{12}'}}}_{\equiv g_k(p, \tilde{p})}$$

where $x = \hat{p}\cdot\hat{\tilde{p}}$ and $\vec{\tilde{p}} = \frac{1}{2}\vec{q} - \frac{1}{4}\vec{k}_\gamma$.

using $P_k(x) = \frac{4\pi}{\sqrt{\hat{k}}} (-)^k\, y^{00}_{kk}(\hat{p}, \hat{\tilde{p}})$

$$= \sum_k 2\pi (-)^k \sqrt{\hat{k}}\; y^{00}_{kk}(\hat{p}, \hat{\tilde{p}}) \cdot g_k(p, |\tilde{p}|)$$

Doing a second expansion in $y = \hat{q}\cdot\hat{k}_\gamma$

$$\frac{1}{|\frac{\vec{q}}{2} - \frac{\vec{k}_\gamma}{4}|^{k}} g_k(p, |\tilde{p}|) = \sum_{k'} 2\pi (-)^{k'} \sqrt{\hat{k}'}\; y^{00}_{k'k'}(\hat{q}, \hat{k}_\gamma)\; \tilde{g}_{kk'}(p, q, \omega)$$

with $\tilde{g}_{kk'}(p, q, \omega) \equiv \int_{-1}^{1} dy' P_{k'}(y)\, g_k(p, |q^2/4 + \frac{\omega^2}{16} - \frac{1}{4} q\omega y|) \frac{1}{|\frac{\vec{q}}{2} - \frac{\vec{k}_\gamma}{4}|^{k}}$

Altogether, the delta functions are expressed as

$$\delta(p_{12} - |\vec{p} + \tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4|)\ \delta(p_{12}' - |\vec{p} - \tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4|)$$

$$= \sum_{k} 2\pi (-)^{k} \sqrt{\hat{k}}\ \mathcal{Y}^{00}_{kk}\left(\hat{p}, \widehat{\frac{\vec{q}}{2} - \frac{\vec{k}_\gamma}{4}}\right),$$

$$\sum_{k'} 2\pi (-)^{k'} \sqrt{\hat{k}'}\ \mathcal{Y}^{00}_{k'k'}(\hat{q}, \hat{k}_\gamma)\ \tilde{g}_{kk'}(p, q, \omega)$$

The final angular integral is still quite complicated $\Rightarrow$

$$= \int d\hat{q} \int d\hat{p}\ Y^{*}_{l_{12}' m_{12}^{l'}}(\vec{p} - \tfrac{1}{2}\vec{q} + \vec{k}_\gamma/4)\, Y_{l_{12} m_{12}^{l}}(\vec{p} + \tfrac{1}{2}\vec{q} - \vec{k}_\gamma/4)$$

$$\mathcal{Y}^{00}_{kk}\left(\hat{p}, \widehat{\frac{\vec{q}}{2} - \frac{\vec{k}_\gamma}{4}}\right) \mathcal{Y}^{00}_{k'k'}(\hat{q}, \hat{k}_\gamma)$$

Addition theorem for spherical harmonics helps

$$Y_{l_{12} m_{12}^{l}}(\widehat{\vec{p} + \vec{\tilde{p}}}) = \sum_{\lambda_1 + \lambda_2 = l_{12}} \sqrt{\frac{4\pi\, \hat{l}_{12}!}{\hat{\lambda}_1!\ \hat{\lambda}_2!}}\ \frac{p^{\lambda_1} \tilde{p}^{\lambda_2}}{|\vec{p} + \vec{\tilde{p}}|^{l_{12}}}\ \mathcal{Y}^{l_{12} m_{12}^{l}}_{\lambda_1 \lambda_2}(\hat{p}\, \hat{\tilde{p}})$$

$$= \sum_{\lambda_1 + \lambda_2 = l_{12}} \sqrt{\frac{4\pi\, \hat{l}_{12}!}{\hat{\lambda}_1!\ \cancel{\hat{\lambda}_2!}}}\ \frac{p^{\lambda_1} \cancel{\tilde{p}^{\lambda_2}}}{|\vec{p} + \vec{\tilde{p}}|^{l_{12}}} \sum_{\Lambda_1 + \Lambda_2 = \lambda_2} \sqrt{\frac{4\pi\, \cancel{\hat{\lambda}_2!}}{\hat{\Lambda}_1!\ \hat{\Lambda}_2!}}\ \frac{\left(\frac{q}{2}\right)^{\Lambda_1} \left(-\frac{\omega}{4}\right)^{\Lambda_2}}{\cancel{|\tfrac{1}{2}\vec{q} - \tfrac{\vec{k}_\gamma}{4}|^{\lambda_2}}}$$

$$\left\{ Y_{\lambda_1}(\hat{p}),\ \mathcal{Y}^{\lambda_2}_{\Lambda_1 \Lambda_2}(\hat{q}, \hat{k}_\gamma) \right\}^{l_{12} m_{12}^{l}}$$

$$Y^{*}_{l'_{12} m'_{l2}}\left(\widehat{\vec p + \tilde{\vec p}}\right) = \sum_{\lambda'_1+\lambda'_2 = l'_{12}} \sqrt{\frac{4\pi\, \hat{l}'_{12}!}{\hat\lambda'_1!\,\hat\lambda'_2!}}\; \frac{p^{\lambda'_1}(-\tilde p)^{\lambda'_2}}{|\vec p - \tilde{\vec p}|^{l'_{12}}}\; \mathcal{Y}^{*\, l'_{12} m'_{l2}}_{\lambda'_1\lambda'_2}(\hat p, \hat{\tilde p})$$

$$(-)^{\lambda'_2}$$

$$= \sum_{\lambda'_1+\lambda'_2 = l'_{12}} \sqrt{\frac{4\pi\, \hat{l}'_{12}!}{\hat\lambda'_1!\,\cancel{\hat\lambda'_2!}}}\; \frac{p^{\lambda'_1}(\cancel{-}\tilde p)^{\lambda'_2}}{|\vec p - \tilde{\vec p}|^{l'_{12}}}\; \mathcal{Y}^{*\, l'_{12} m'_{l2}}_{\lambda'_1\lambda'_2}(\hat p, \hat{\tilde p}) \sum_{\Lambda'_1+\Lambda'_2=\lambda'_2} \sqrt{\frac{4\pi\, \cancel{\hat\lambda'_2!}}{\hat\Lambda'_1!\,\hat\Lambda'_2!}}\; \frac{\left(\frac{q}{2}\right)^{\Lambda'_1}\left(-\frac{\omega}{4}\right)^{\Lambda'_2}}{\cancel{\left|\frac{1}{2}\vec q - \frac{\vec k_\gamma}{4}\right|^{\lambda'_2}}}$$

$$\left\{ Y^{*}_{\lambda'_1}(\hat p),\ \mathcal{Y}^{\lambda'_2 *}_{\Lambda'_1\Lambda'_2}(\hat q, \hat k_\gamma)\right\}^{l'_{12} m'_{l2}}$$

$$\mathcal{Y}^{00}_{kk}\left(\hat p,\ \widehat{\frac{\vec q}{2} - \frac{\vec k_\gamma}{4}}\right) = \sum_{\mu_1+\mu_2 = k} \sqrt{\frac{4\pi\, \hat k!}{\hat\mu_1!\,\hat\mu_2!}}\; \frac{\left(\frac{1}{2}q\right)^{\mu_1}\left(-\frac{1}{4}\omega\right)^{\mu_2}}{\left|\frac{\vec q}{2} - \frac{\vec k_\gamma}{4}\right|^{k}}$$

$$\left\{ Y_k(\hat p),\ \mathcal{Y}^{k}_{\mu_1\mu_2}(\hat q, \hat k_\gamma)\right\}^{00}$$

Note that all dependence of $\vec q$, k_γ, $\vec p$ etc is now added to $\tilde g_{kk'}$ coefficient

The angular integral then reads

$$\int d\hat{q} \int d\hat{p} \; \left\{ Y^*_{\lambda_1'}(\hat{p}), \mathcal{Y}^{\lambda_2' *}_{\Lambda_1' \Lambda_2'}(\hat{q}, \hat{q}_\gamma) \right\}^{l_{12}' q_{12}^{\ell'}} \left\{ Y_{\lambda_1}(\hat{p}), \mathcal{Y}^{\lambda_2}_{\Lambda_1 \Lambda_2}(\hat{q}, \hat{q}_\gamma) \right\}^{l_{12} q_{12}^{\ell}}$$

$$\left\{ Y_k(\hat{p}) \, \mathcal{Y}^{k}_{\mu_1 \mu_2}(\hat{q}, \hat{q}_\gamma) \right\}^{00} \quad \mathcal{Y}^{00}_{q' q'}(\hat{q}, \hat{q}_\gamma)$$

Now do recoupling to combine spherical harmonics with the same angular dependence.

$$\left\{ \left\{ Y_{\lambda_1}(\hat{p}), \mathcal{Y}^{\lambda_2}_{\Lambda_1 \Lambda_2}(\hat{q}, \hat{q}_\gamma) \right\}^{l_{12}}, \left\{ Y_k(\hat{p}) \, \mathcal{Y}^{k}_{\mu_1 \mu_2}(\hat{q}, \hat{q}_\gamma) \right\}^{0} \right\}^{l_{12} q_{12}^{\ell}}$$

$$= \sum_{g_1 g_2} \sqrt{\hat{l}_{12} \hat{0} \hat{g}_1 \hat{g}_2} \begin{Bmatrix} \lambda_1 & \lambda_2 & l_{12} \\ k & k & 0 \\ g_1 & g_2 & l_{12} \end{Bmatrix} \left\{ \left\{ Y_{\lambda_1}(\hat{p}), Y_k(\hat{p}) \right\}^{g_1}, \left\{ \mathcal{Y}^{\lambda_2}_{\Lambda_1 \Lambda_2}(\hat{q}, \hat{q}_\gamma) \, \mathcal{Y}^{k}_{\mu_1 \mu_2}(\hat{q}, \hat{q}_\gamma) \right\}^{g_2} \right\}^{l_{12} q_{12}^{\ell}}$$

Now simplification using $\mathcal{Y}^{g_1}_{\lambda_1 k}(\hat{p}\hat{p}) = \sqrt{\frac{\hat{\lambda}_1 \hat{k}}{4\pi \hat{g}_1}} \, (\lambda_1 k \, g_1, 000) \, Y_{g_1}(\hat{p})$

$$= \sum_{g_1 g_2} \sqrt{\hat{l}_{12} \hat{0} \hat{g}_1 \hat{g}_2} \begin{Bmatrix} \lambda_1 & \lambda_2 & l_{12} \\ k & k & 0 \\ g_1 & g_2 & l_{12} \end{Bmatrix} \sqrt{\frac{\hat{\lambda}_1 \hat{k}}{4\pi \hat{g}_1}} \, (\lambda_1 k \, g_1, 000)$$

$$\left\{ Y_{g_1}(\hat{p}), \left\{ \mathcal{Y}^{\lambda_2}_{\Lambda_1 \Lambda_2}(\hat{q}, \hat{q}_\gamma) \, \mathcal{Y}^{k}_{\mu_1 \mu_2}(\hat{q}, \hat{q}_\gamma) \right\}^{g_2} \right\}^{l_{12} q_{12}^{\ell}}$$

$$= \sum_{g_1 g_2} \sqrt{\hat{l}_{12} \hat{0} \hat{g}_1 \hat{g}_2} \begin{Bmatrix} \lambda_1 & \lambda_2 & l_{12} \\ k & k & 0 \\ g_1 & g_2 & l_{12} \end{Bmatrix} \sqrt{\frac{\hat{\lambda}_1 \hat{k}}{4\pi \hat{g}_1}} \, (\lambda_1 k \, g_1, 000) \sum_{f_1 f_2} \sqrt{\hat{f}_1 \hat{f}_2 \hat{\lambda}_2 \hat{k}} \begin{Bmatrix} \Lambda_1 & \Lambda_2 & \lambda_2 \\ \mu_1 & \mu_2 & k \\ f_1 & f_2 & g_2 \end{Bmatrix}$$

$$\left\{ Y_{g_1}(\hat{p}), \left\{ \mathcal{Y}^{f_1}_{\Lambda_1 \mu_1}(\hat{q}, \hat{q}) \, \mathcal{Y}^{f_2}_{\Lambda_2 \mu_2}(\hat{q}_\gamma, \hat{q}_\gamma) \right\}^{g_2} \right\}^{l_{12} q_{12}^{\ell}}$$

$$= \sum_{g_1 g_2} \sqrt{\hat{l}_{12}\, \hat{0}\, \hat{g}_1 \hat{g}_2} \begin{Bmatrix} \lambda_1 & \lambda_2 & l_{12} \\ k & k & 0 \\ g_1 & g_2 & l_{12} \end{Bmatrix} \sqrt{\frac{\hat{\lambda}_1 \hat{k}}{4\pi \hat{g}_1}} (\lambda_1 k\, g_1, 000) \sum_{f_1 f_2} \sqrt{\hat{f}_1 \hat{f}_2\, \hat{\lambda}_2\, \hat{k}} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_2 \\ \mu_1 & \mu_2 & k \\ f_1 & f_2 & g_2 \end{Bmatrix}$$

$$\cdot \sqrt{\frac{\hat{\lambda}_1 \hat{\mu}_1\, \hat{\lambda}_2 \hat{\mu}_2}{(4\pi)^2\, \hat{f}_1 \hat{f}_2}} (\lambda_1 \mu_1 f_1, 000)(\lambda_2 \mu_2 f_2, 000)$$

$$\left\{ Y_{g_1}(\hat{p}),\ \mathcal{Y}^{g_2}_{f_1 f_2}(\hat{q}\, \hat{k}_\gamma) \right\}^{l_{12} q^{l}_{12}}$$

Next step: couple additionally $\mathcal{Y}^{00}_{q' q'}(\hat{q}, \hat{k}_\gamma)$

$$\left\{ \left\{ Y_{g_1}(\hat{p}),\ \mathcal{Y}^{g_2}_{f_1 f_2}(\hat{q}\, \hat{k}_\gamma) \right\}^{l_{12}} \mathcal{Y}^{0}_{q' q'}(\hat{q}, \hat{k}_\gamma) \right\}^{l_{12} q^{l}_{12}}$$

$$= \sum_{f_3} (-)^{g_1 + g_2 + 0 + l_{12}} \sqrt{\hat{f}_3\, \hat{l}_{12}} \begin{Bmatrix} g_1 & g_2 & l_{12} \\ 0 & l_{12} & f_3 \end{Bmatrix}$$

$$\left\{ Y_{g_1}(\hat{p}),\ \left\{ \mathcal{Y}^{g_2}_{f_1 f_2}(\hat{q}\, \hat{k}_\gamma),\ \mathcal{Y}^{0}_{q' q'}(\hat{q}, \hat{k}_\gamma) \right\}^{f_3} \right\}^{l_{12} q^{l}_{12}}$$

$$= \sum_{f_3} (-)^{g_1 + g_2 + 0 + l_{12}} \sqrt{\hat{f}_3\, \hat{l}_{12}} \begin{Bmatrix} g_1 & g_2 & l_{12} \\ 0 & l_{12} & f_3 \end{Bmatrix}$$

$$\sum_{h_1 h_2} \sqrt{\hat{h}_1 \hat{h}_2\, \hat{g}_2\, \hat{0}} \begin{Bmatrix} f_1 & f_2 & g_2 \\ q' & q' & 0 \\ h_1 & h_2 & f_3 \end{Bmatrix}$$

$$\left\{ Y_{g_1}(\hat{p}),\ \left\{ \mathcal{Y}^{h_1}_{f_1 q'}(\hat{q}\, \hat{q}),\ \mathcal{Y}^{h_2}_{f_2 q'}(\hat{k}_\gamma, \hat{k}_\gamma) \right\}^{f_3} \right\}^{l_{12} q^{l}_{12}}$$

$$= \sum_{f_3} (-)^{g_1+g_2+0+l_{12}} \sqrt{\hat{f}_3 \hat{l}_{12}} \begin{Bmatrix} g_1 & g_2 & l_{12} \\ 0 & l_{12} & f_3 \end{Bmatrix}$$

$$\sum_{h_1 h_2} \sqrt{\hat{h}_1 \hat{h}_2 \hat{g}_2 \hat{0}} \begin{Bmatrix} f_1 & f_2 & g_2 \\ q' & q' & 0 \\ h_1 & h_2 & f_3 \end{Bmatrix}$$

$$\sqrt{\frac{\hat{f}_1 \hat{q}' \hat{f}_2 \hat{q}'}{(4\pi)^2 \hat{h}_1 \hat{h}_2}} (f_1 q' h_1, 000)(f_2 q' h_2, 000)$$

$$\left\{ Y_{g_1}(\hat{p}), \mathcal{Y}^{f_3}_{h_1 h_2}(\hat{q}\, \hat{q}_\gamma) \right\}^{l_{12} m^l_{12}}$$

The angular integral then reads

$$\int d\hat{q} \int d\hat{p} \left\{ Y^*_{\lambda_1'}(\hat{p}), \mathcal{Y}^{\lambda_2' *}_{\Lambda_1' \Lambda_2'}(\hat{q}, \hat{q}_\gamma) \right\}^{l'_{12} m^{l'}_{12}} \left\{ Y_{g_1}(\hat{p}), \mathcal{Y}^{f_3}_{h_1 h_2}(\hat{q}\, \hat{q}_\gamma) \right\}^{l_{12} m^l_{12}}$$

$$= \sum_{\substack{n_1 n_2 \\ n_1' n_2'}} (\lambda_1' \lambda_2' l'_{12}, n_1' m^{l'}_{12} - n_1')(g_1 f_3 l_{12}, n_1 m^l_{12} - n_1)$$
$$(\Lambda_1' \Lambda_2' \lambda_2', n_2' m^{l'}_{12} - n_1' - n_2')(h_1 h_2 f_3, n_2 m^l_{12} - n_1 - n_2)$$

$$\int d\hat{q} \int d\hat{p}\; Y^*_{\lambda_1' n_1'}(\hat{p})\, Y_{g_1 n_1}(\hat{p})\, Y^*_{\Lambda_1' n_2'}(\hat{q})\, Y_{h_1 n_2}(\hat{q})$$
$$Y^*_{\Lambda_2'\, m^{l'}_{12} - n_1' - n_2'}(\hat{q}_\gamma)\, Y_{h_2\, m^l_{12} - n_1 - n_2}(\hat{q}_\gamma)$$

$$= \sum_{n_1 n_2} (\lambda_1' \lambda_2' l'_{12}, n_1 m^{l'}_{12} - n_1)(\lambda_1' f_3 l_{12}, n_1 m^l_{12} - n_1)$$
$$(\Lambda_1' \Lambda_2' \lambda_2', n_2 m^{l'}_{12} - n_1 - n_2)(\Lambda_1' h_2 f_3, n_2 m^l_{12} - n_1 - n_2)\, \delta_{\lambda_1' g_1} \delta_{\Lambda_1' h_1}$$

$$Y^*_{\Lambda_2'\, m^{l'}_{12} - n_1 - n_2}(\hat{q}_\gamma)\, Y_{h_2\, m^l_{12} - n_1 - n_2}(\hat{q}_\gamma)$$

aim: 1CG coefficient involving $l_2\, l_{12}'$ to match tensor operator

1CG coefficient involving $\lambda_2'\, l_2$ to make coupling of $Y(\hat{k}_\gamma)$.

use $(a\, b\, e,\ \alpha\, \beta\, \alpha+\beta)(e\, d\, c\ \ \alpha+\beta\ \ \gamma\ \ \alpha+\beta+\gamma) = \sum_f \sqrt{\hat{e}\,\hat{f}}\ (-)^{a+b+d+c} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}$

$(b\, d\, f,\ \beta\ \gamma\ \beta+\gamma)\ (a\, f\, c,\ \alpha\ \beta+\gamma)$

$$(\lambda_1'\ \lambda_2'\ l_{12}',\ n_1\ m_{12}^{l'}-n_1)\ (\lambda_1'\ f_3\ l_{12},\ n_1\ m_{12}^{l}-n_1)$$

$$= (l_{12}\ f_3\ \lambda_1',\ -m_{12}^{l}\ \ m_{12}^{l}-n_1\ \ -n_1)\ (\lambda_1'\ \lambda_2'\ l_{12}',\ -n_1\ -m_{12}^{l'}+n_1\ -m_{12}^{l'})\ (-)^{f_3+m_{12}^{l}-n_1} \sqrt{\frac{\hat{l}_{12}}{\hat{f}_3}}$$

$$(-)^{\lambda_1'+\lambda_2'-l_{12}'}$$

$$= \sum_{g_3} \sqrt{\hat{g}_3\ \hat{\lambda}_1'} \begin{Bmatrix} l_{12} & f_3 & \lambda_1' \\ \lambda_2' & l_{12}' & g_3 \end{Bmatrix} (f_3\ \lambda_2'\ g_3,\ m_{12}^{l}-n_1\ \ -m_{12}^{l'}+n_1\ \ m_{12}^{l}-m_{12}^{l'})$$

$$(l_{12}\ g_3\ l_{12}',\ -m_{12}^{l}\ \ m_{12}^{l}-m_{12}^{l'}\ \ -m_{12}^{l'})$$

$$(-)^{f_3+m_{12}^{l}-n_1} \sqrt{\frac{\hat{l}_{12}}{\hat{f}_3}}\ (-)^{\lambda_1'+\lambda_2'-l_{12}'}\ (-)^{l_{12}+f_3+\lambda_2'+l_{12}'}$$

$$(\Lambda_1'\ \Lambda_2'\ \lambda_2',\ n_2\ m_{12}^{l'}-n_1-n_2)\ (\Lambda_1'\ l_2\ f_3,\ n_2\ m_{12}^{l}-n_1-n_2)$$

$$= (\lambda_2'\ \Lambda_2'\ \Lambda_1',\ -m_{12}^{l'}+n_1\ \ m_{12}^{l'}-n_1-n_2\ \ -n_2)\ (\Lambda_1'\ l_2\ f_3,\ -n_2\ -m_{12}^{l}+n_1+n_2\ \ -m_{12}^{l}+n_1)$$

$$\sqrt{\frac{\hat{\lambda}_2'}{\hat{\Lambda}_1'}}\ (-)^{\Lambda_2'+m_{12}^{l'}-n_1-n_2}\ \ (-)^{\Lambda_1'+l_2-f_3}$$

$$= \sum_{g_4} \sqrt{\hat{g}_4\ \hat{\Lambda}_1'} \begin{Bmatrix} \lambda_2' & \Lambda_2' & \Lambda_1' \\ l_2 & f_3 & g_4 \end{Bmatrix} \begin{matrix} (\lambda_2'\ l_2\ g_4,\ m_{12}^{l'}-n_1-n_2\ \ -m_{12}^{l}+n_1+n_2\ \ m_{12}^{l'}-m_{12}^{l}) \\ (\lambda_2'\ g_4\ f_3,\ -m_{12}^{l'}+n_1\ \ m_{12}^{l'}-m_{12}^{l}\ \ -m_{12}^{l}+n_1) \end{matrix}$$

$$\sqrt{\frac{\hat{\lambda}_2'}{\hat{\Lambda}_1'}}\ (-)^{\Lambda_2'+m_{12}^{l'}-n_1-n_2}\ \ (-)^{\Lambda_1'+l_2-f_3}\ (-)^{\lambda_2'+\Lambda_2'+l_2+f_3}$$

$$= \sum_{g_3} \sqrt{\hat g_3 \hat\lambda_1'} \begin{Bmatrix} l_{12} & f_3 & \lambda_1' \\ \lambda_2' & l_{12}' & g_3 \end{Bmatrix} \sum_{g_4} \sqrt{\hat g_4 \hat\Lambda_1'} \begin{Bmatrix} \lambda_2' & \Lambda_2' & \Lambda_1' \\ h_2 & f_3 & g_4 \end{Bmatrix}$$

$$(-)^{f_3+m_{12}^l-n_1} \sqrt{\frac{\hat l_{12}'}{\hat f_3}} (-)^{\lambda_1'+\lambda_2'-l_{12}'} \quad (-)^{l_{12}+f_3+\lambda_2'+l_{12}'}$$

$$\sqrt{\frac{\hat\lambda_2'}{\hat\Lambda_1'}} (-)^{\Lambda_2'+m_{12}^{l'}-n_1-n_2} \quad (-)^{\Lambda_1'+h_2-f_3} (-)^{\lambda_2'+\Lambda_2'+h_2+f_3}$$

$$\sum_{n_1 n_2} \left(f_3\ \lambda_2'\ g_3,\ m_{12}^l-n_1\ \ -m_{12}^{l'}+n_1\ \ m_{12}^l-m_{12}^{l'}\right) \left(\lambda_2'\ g_4\ f_3,\ -m_{12}^{l'}+n_1\ \ m_{12}^{l'}-m_{12}^l\ \ -m_{12}^l+n_1\right)$$

$$\left(\Lambda_2'\ h_2\ g_4,\ m_{12}^{l'}-n_1-n_2\ \ -m_{12}^l+n_1+n_2\ \ m_{12}^{l'}-m_{12}^l\right)$$

$$\varphi^*_{\Lambda_2' m_{12}^{l'}-n_1-n_2}(\hat r_\gamma)\, \varphi_{h_2 m_{12}^l-n_1-n_2}(\hat r_\gamma)\, \delta_{\lambda_1' g_1}\, \delta_{\Lambda_1' h_1}$$

$$\left(l_{12}\ g_3\ l_{12}',\ -m_{12}^l\ \ m_{12}^l-m_{12}^{l'}\ \ -m_{12}^{l'}\right)$$

$$= \sum_{n_1 \tilde n_2} \sum_{g_3} \sqrt{\hat g_3 \hat\lambda_1'} \begin{Bmatrix} l_{12} & f_3 & \lambda_1' \\ \lambda_2' & l_{12}' & g_3 \end{Bmatrix} \sum_{g_4} \sqrt{\hat g_4 \hat\Lambda_1'} \begin{Bmatrix} \lambda_2' & \Lambda_2' & \Lambda_1' \\ h_2 & f_3 & g_4 \end{Bmatrix}$$

$$(-)^{f_3+m_{12}^l-n_1} \sqrt{\frac{\hat l_{12}'}{\hat f_3}} (-)^{\lambda_1'+\lambda_2'-l_{12}'} \quad (-)^{l_{12}+f_3+\lambda_2'+l_{12}'}$$

$$\sqrt{\frac{\hat\lambda_2'}{\hat\Lambda_1'}} (-)^{\Lambda_2'\cancel{+m_{12}^{l'}}-\tilde n_2} \quad (-)^{\Lambda_1'+h_2-f_3} (-)^{\lambda_2'+\Lambda_2'+h_2+f_3}$$

$$\left(f_3\ \lambda_2'\ g_3,\ m_{12}^l-n_1\ \ -m_{12}^{l'}+n_1\ \ m_{12}^l-m_{12}^{l'}\right) \left(f_3\ \lambda_2'\ g_4,\ +m_{12}^l-n_1\ \ -m_{12}^{l'}+n_1\ \ -m_{12}^{l'}+m_{12}^l\right)$$

$$\left(\Lambda_2'\ h_2\ g_4,\ -m_{12}^{l'}+\tilde n_2\ \ +m_{12}^l-\tilde n_2\ \ -m_{12}^{l'}+m_{12}^l\right) (-)^{\Lambda_2'+h_2-g_4} (-)^{\lambda_2'-m_{12}^{l'}+n_1} \sqrt{\frac{\hat f_3}{\hat g_4}}$$

$$\cancel{(-)^{\tilde n_2-m_{12}^{l'}}}\, \varphi_{\Lambda_2' -m_{12}^{l'}+\tilde n_2}(\hat r_\gamma)\, \varphi_{h_2 m_{12}^l-\tilde n_2}(\hat r_\gamma)\, \delta_{\lambda_1' g_1}\, \delta_{\Lambda_1' h_1} \quad (-)^{f_3+\lambda_2'-g_4}$$

$$\left(l_{12}\ g_3\ l_{12}',\ -m_{12}^l\ \ m_{12}^l-m_{12}^{l'}\ \ -m_{12}^{l'}\right)$$

$$= \sum_{g_3} \sqrt{\hat{g}_3 \hat{\lambda}'_1} \begin{Bmatrix} l_{12} & f_3 & \lambda'_1 \\ \lambda'_2 & l'_{12} & g_3 \end{Bmatrix} \sqrt{\hat{g}_3 \hat{\Lambda}'_1} \begin{Bmatrix} \lambda'_2 & \Lambda'_2 & \Lambda'_1 \\ h_2 & f_3 & g_3 \end{Bmatrix}$$

$$(-)^{f_3} \sqrt{\frac{\hat{l}_{12}}{\hat{f}_3}} (-)^{\lambda'_1+\lambda'_2-l'_{12}} (-)^{l_{12}+f_3+\lambda'_2+l'_{12}}$$

$$\sqrt{\frac{\hat{\lambda}'_2}{\hat{\Lambda}'_1}} (-)^{\Lambda'_2} (-)^{\Lambda'_1+h_2-f_3} (-)^{\lambda'_2+\Lambda'_2+h_2+f_3} (-)^{\Lambda'_2+h_2-g_3} (-)^{\lambda'_2}$$

$$\sqrt{\frac{\hat{f}_3}{\hat{g}_3}} (-)^{f_3+\lambda'_2-g_3} \delta_{\lambda'_1 g_1} \delta_{\Lambda'_1 h_1} (l_{12}\, g_3\, l'_{12} | -m^{l}_{12}\;\; m^{l}_{12}-m^{l'}_{12}\;\; -m^{l'}_{12})$$

$$(-1)^{+m^{l}_{12}-m^{l'}_{12}} Y^{g_3\,-m^{l'}_{12}+m^{l}_{12}}_{\lambda'_2 h_2} (\hat{k}_\gamma\, \hat{k}_\gamma)$$

$$= \sum_{g_3} \sqrt{\hat{g}_3 \hat{\lambda}'_1} \begin{Bmatrix} l_{12} & f_3 & \lambda'_1 \\ \lambda'_2 & l'_{12} & g_3 \end{Bmatrix} \sqrt{\hat{g}_3 \hat{\Lambda}'_1} \begin{Bmatrix} \lambda'_2 & \Lambda'_2 & \Lambda'_1 \\ h_2 & f_3 & g_3 \end{Bmatrix}$$

$$(-)^{f_3} \sqrt{\frac{\hat{l}_{12}}{\hat{f}_3}} (-)^{\lambda'_1+\lambda'_2-l'_{12}} (-)^{l_{12}+f_3+\lambda'_2+l'_{12}}$$

$$\sqrt{\frac{\hat{\lambda}'_2}{\hat{\Lambda}'_1}} (-)^{\Lambda'_2} (-)^{\Lambda'_1+h_2-f_3} (-)^{\lambda'_2+\Lambda'_2+h_2+f_3} (-)^{\Lambda'_2+h_2-g_3} (-)^{\lambda'_2}$$

$$\sqrt{\frac{\hat{f}_3}{\hat{g}_3}} (-)^{f_3+\lambda'_2-g_3} \delta_{\lambda'_1 g_1} \delta_{\Lambda'_1 h_1} (l_{12}\, g_3\, l'_{12} | -m^{l}_{12}\;\; m^{l}_{12}-m^{l'}_{12}\;\; -m^{l'}_{12})$$

$$\sqrt{\frac{\hat{\Lambda}'_2\, \hat{h}_2}{4\pi\, \hat{g}_3}} (\Lambda'_2\, h_2\, g_3, 0\, 0)\; Y^{*}_{g_3+m^{l'}_{12}-m^{l}_{12}} (\hat{k}_\gamma)$$

$$= \sum_{g_1 g_2} \sqrt{\hat{\ell}_{12}\, \hat{0}\, \hat{g}_1 \hat{g}_2} \begin{Bmatrix} \lambda_1 & \lambda_2 & \ell_{12} \\ k & k & 0 \\ g_1 & g_2 & \ell_{12} \end{Bmatrix} \sqrt{\frac{\hat{\lambda}_1 \hat{k}}{4\pi \hat{g}_1}} (\lambda_1 k\, g_1, 000) \sum_{f_1 f_2} \sqrt{\hat{f}_1 \hat{f}_2\, \hat{\lambda}_2\, \hat{g}_2} \begin{Bmatrix} \Lambda_1 & \Lambda_2 & \lambda_2 \\ \mu_1 & \mu_2 & k \\ f_1 & f_2 & g_2 \end{Bmatrix}$$

$$\cdot \sqrt{\frac{\hat{\Lambda}_1 \hat{\mu}_1\, \hat{\Lambda}_2 \hat{\mu}_2}{(4\pi)^2\, \hat{f}_1\, \hat{f}_2}} (\Lambda_1 \mu_1 f_1, 000)(\Lambda_2 \mu_2 f_2, 000)$$

$$\sum_{\mu_1 + \mu_2 = k} \sqrt{\frac{4\pi\, \hat{k}!}{\hat{\mu}_1!\, \hat{\mu}_2!}} \frac{\left(\frac{1}{2} q\right)^{\mu_1} \left(-\frac{1}{4}\omega\right)^{\mu_2}}{\left|\frac{\vec{q}}{2} - \frac{\vec{\omega}_\gamma}{4}\right|^{k}}$$

$$\sum_{\lambda_1 + \lambda_2 = \ell_{12}} \sum_{\Lambda_1 + \Lambda_2 = \lambda_2} \sqrt{\frac{(4\pi)^2 \hat{\ell}_{12}!}{\hat{\lambda}_1!\, \hat{\Lambda}_1!\, \hat{\Lambda}_2!}} \frac{p^{\lambda_1} \left(\frac{q}{2}\right)^{\Lambda_1} \left(-\frac{\omega}{4}\right)^{\Lambda_2}}{|\vec{p} + \vec{p}\,|^{\ell_{12}}}$$

$$\sum_{\lambda_1' + \lambda_2' = \ell_{12}'} \sum_{\Lambda_1' + \Lambda_2' = \lambda_2'} \sqrt{\frac{(4\pi)^2 \hat{\ell}_{12}'!}{\hat{\lambda}_1'!\, \hat{\Lambda}_1'!\, \hat{\Lambda}_2'!}} \frac{p^{\lambda_1'} (-)^{\lambda_2'} \left(\frac{q}{2}\right)^{\Lambda_1'} \left(-\frac{\omega}{4}\right)^{\Lambda_2'}}{|\vec{p} - \vec{p}\,|^{\ell_{12}'}}$$

$$\int d\hat{q} \int d\hat{p} \left\{ Y_{g_1}(\hat{p})\, \mathcal{Y}^{g_2}_{f_1 f_2}(\hat{q}\, \hat{\omega}_\gamma) \right\}^{\ell_{12} q^\ell_{2}}$$

$$\left\{ Y^*_{\lambda_1'}(\hat{p}),\, \mathcal{Y}^{\lambda_2' *}_{\Lambda_1' \Lambda_2'}(\hat{q}, \hat{\omega}_\gamma) \right\}^{\ell_{12}' q^{\ell'}_2} \mathcal{Y}^{00}_{\ell \ell'}(\hat{q}, \hat{\omega}_\gamma)$$

In summary:

$$\int d\hat q \int d\hat p \left\{ Y^{*}_{\lambda_1' n_1'}(\hat p), \, \mathcal{Y}^{\lambda_2'*}_{\lambda_1'\lambda_2'}(\hat q, \hat x_\gamma) \right\}^{l_{12}' m_{12}^{l'}} \left\{ Y_{g_1 n_1}(\hat p), \, \mathcal{Y}^{f_3}_{h_1 h_2}(\hat q \, \hat x_\gamma) \right\}^{l_{12} m_{12}^{l}}$$

$$= \sum_{n_1 n_2} (\lambda_1' \lambda_2' l_{12}', n_1' m_{12}^{l'} - n_1') (g_1 f_3 l_{12}, n_1 m_{12}^{l} - n_1)$$
$$n_1' n_2' (\lambda_1' \Lambda_2' \lambda_2', n_2' m_{12}^{l'} - n_1' - n_2') (h_1 h_2 f_3, n_2 m_{12}^{l} - n_1 - n_2)$$
$$\int d\hat q \int d\hat p \; Y^{*}_{\lambda_1' n_1'}(\hat p) \, Y_{g_1 n_1}(\hat p) \, Y^{*}_{\lambda_1' n_2'}(\hat q) \, Y_{h_1 n_2}(\hat q) \; Y^{*}_{\Lambda_2' m_{12}^{l'} - n_1' - n_2'}(\hat x_\gamma) \, Y_{h_2 m_{12}^{l} - n_1 - n_2}(\hat x_\gamma)$$

$$= \sum_{g_3} \sqrt{\hat g_3 \hat\lambda_1'} \begin{Bmatrix} l_{12} & f_3 & \lambda_1' \\ \lambda_2' & l_{12}' & g_3 \end{Bmatrix} \sum_{g_4} \sqrt{\hat g_4 \hat\Lambda_1'} \begin{Bmatrix} \lambda_2' & \Lambda_2' & \Lambda_1' \\ h_2 & f_3 & g_4 \end{Bmatrix}$$

$$(-)^{f_3 + m_{12}^{l} - n_1} \sqrt{\frac{\hat l_{12}}{\hat f_3}} (-)^{\lambda_1' + \lambda_2' - l_{12}'} (-)^{l_{12} + f_3 + \lambda_2' + l_{12}'}$$

$$\sqrt{\frac{\hat\lambda_2'}{\hat\Lambda_1'}} (-)^{\Lambda_2' + m_{12}^{l'} - n_1 - n_2} (-)^{\Lambda_1' + h_2 - f_3} (-)^{\lambda_2' + \Lambda_2' + h_2 + f_3}$$

$$\sum_{n_1 n_2} (f_3 \lambda_2' g_3, m_{12}^{l} - n_1 \; -m_{12}^{l'} + n_1 \; m_{12}^{l} - m_{12}^{l'}) (\lambda_2' g_4 f_3, -m_{12}^{l'} + n_1 \; m_{12}^{l'} - m_{12}^{l} \; -m_{12}^{l} + n_1)$$
$$(\Lambda_2' h_2 g_4, m_{12}^{l'} - n_1 - n_2 \; -m_{12}^{l} + n_1 + n_2 \; m_{12}^{l'} - m_{12}^{l}) \; Y^{*}_{\Lambda_2' m_{12}^{l'} - n_1 - n_2}(\hat x_\gamma) \, Y_{h_2 m_{12}^{l} - n_1 - n_2}(\hat x_\gamma)$$
$$\delta_{\lambda_1' g_1} \delta_{\Lambda_1' h_1} (l_{12} g_3 l_{12}', -m_{12}^{l} \; m_{12}^{l} - m_{12}^{l'} \; -m_{12}^{l'})$$

$$= \sum_{g_3} \sqrt{\hat g_3 \hat\lambda_1'} \begin{Bmatrix} l_{12} & f_3 & \lambda_1' \\ \lambda_2' & l_{12}' & g_3 \end{Bmatrix} \sqrt{\hat g_3 \hat\Lambda_1'} \begin{Bmatrix} \lambda_2' & \Lambda_2' & \Lambda_1' \\ h_2 & f_3 & g_3 \end{Bmatrix}$$

$$(-)^{f_3} \sqrt{\frac{\hat l_{12}}{\hat f_3}} (-)^{\lambda_1' + \lambda_2' - l_{12}'} (-)^{l_{12} + f_3 + \lambda_2' + l_{12}'}$$
$$\sqrt{\frac{\hat\lambda_2'}{\hat\Lambda_1'}} (-)^{\Lambda_2'} (-)^{\Lambda_1' + h_2 - f_3} (-)^{\lambda_2' + \Lambda_2' + h_2 + f_3} (-)^{\Lambda_2' + h_2 - g_3} (-)^{\lambda_2'}$$
$$\sqrt{\frac{\hat f_3}{\hat g_3}} (-)^{f_3 + \lambda_2' - g_3} \delta_{\lambda_1' g_1} \delta_{\Lambda_1' h_1} (l_{12} g_3 l_{12}', -m_{12}^{l} \; m_{12}^{l} - m_{12}^{l'} \; -m_{12}^{l'})$$

$$\sqrt{\frac{\hat\Lambda_2' \hat h_2}{4\pi \hat g_3}} (\Lambda_2' h_2 g_3, 0 0) \, Y^{*}_{g_3 \, m_{12}^{l'} - m_{12}^{l}}(\hat x_\gamma)$$

The complete result is then

$$\langle p_{12}' \, l_{12}' \, m_{l_{12}}' \, | \, \frac{1}{(\vec{p}_{12} - \vec{p}_{12}' + \vec{k}_8/2)^2} \, | \, p_{12} \, l_{12} \, m_{l_{12}} \rangle = \sum_{\alpha\alpha'} \int dP \, P^2 \, dq \; g_{\alpha\alpha'}(P q \omega) \; G^{\alpha\alpha'}_{\alpha\alpha'}(P q \omega)$$

$$(l_2 \, g_3 \, l_{12}' \, | \, m_{l_{12}} - m_{l_{12}}' \, m_{l_{12}}')$$

with $g_{\alpha\alpha'}(P q \omega) = \int_{-1}^{1} dy' \, P_{k'}(y') \, \frac{g_k(P \, |\frac{q}{2} - \frac{k_8}{4}| \, \omega)}{|\frac{q}{2} - \frac{k_8}{4}|^k}$

and $g_k(P \tilde{P} \omega) = \int_{-1}^{1} dx' \, P_k(x') \, \frac{\delta(p_{12} - |\vec{P} + \vec{\tilde{P}}|)}{p_{12}^{l_{12}+2}} \, \frac{\delta(p_{12}' - |\vec{P} - \vec{\tilde{P}}|)}{p_{12}'^{\,l_{12}'+2}}$

$\tilde{P} = |\frac{\vec{q}}{2} - \frac{\vec{k}_8}{4}|$

$x = \hat{P} \cdot \hat{\tilde{P}} \qquad y = \hat{k}_8 \, \hat{q}$

where $G^{\alpha\alpha'}_{\alpha'\alpha}(P q \omega) = \sum_{\lambda_1+\lambda_2 = l_{12}} \sqrt{\frac{4\pi \, \hat{l}_{12}!}{\hat{\lambda}_1!}} \sum_{\Lambda_1+\Lambda_2 = \lambda_2} \sqrt{\frac{4\pi}{\hat{\Lambda}_1! \, \hat{\Lambda}_2!}} \sum_{\lambda_1'+\lambda_2' = l_{12}'} \sqrt{\frac{4\pi \, \hat{l}_{12}'!}{\hat{\lambda}_1'!}} \; (-)^{\lambda_2'} \sum_{\Lambda_1'+\Lambda_2' = \lambda_2'} \sqrt{\frac{4\pi}{\hat{\Lambda}_1'! \, \hat{\Lambda}_2'!}}$

$$\sum_{\mu_1+\mu_2 = k} \sqrt{\frac{4\pi \, \hat{k}!}{\hat{\mu}_1! \, \hat{\mu}_2!}} \; \left(\frac{q}{2}\right)^{\Lambda_1+\Lambda_1'+\mu_1} \left(\frac{\omega}{4}\right)^{\Lambda_2+\Lambda_2'+\mu_2} P^{\lambda_1+\lambda_1'}$$

$$\sum_{g_2} \sqrt{\hat{l}_{12} \, \hat{0} \, \hat{\lambda}_1' \, \hat{g}_2} \begin{Bmatrix} \lambda_1 & \lambda_2 & l_{12} \\ k & k & 0 \\ \lambda_1' & g_2 & l_{12} \end{Bmatrix} \sqrt{\frac{\hat{\lambda}_1 \hat{k}}{4\pi \, \hat{\lambda}_1'}} \, (\lambda_1 k \lambda_1', 000) \sum_{f_1 f_2} \sqrt{\hat{f}_1 \hat{f}_2 \hat{\lambda}_2 \hat{k}} \begin{Bmatrix} \Lambda_1 & \Lambda_2 & \lambda_2 \\ \mu_1 & \mu_2 & k \\ f_1 & f_2 & g_2 \end{Bmatrix}$$

$$\sqrt{\frac{\hat{\Lambda}_1 \hat{\mu}_1 \hat{\Lambda}_2 \hat{\mu}_2}{(4\pi)^2 \hat{f}_1 \hat{f}_2}} \, (\Lambda_1 \mu_1 f_1, 000)(\Lambda_2 \mu_2 f_2, 000) \sum_{f_3} (-)^{\lambda_1'+g_2+0+l_{12}} \sqrt{\hat{f}_3 \hat{l}_{12}} \begin{Bmatrix} \lambda_1' & g_2 & l_{12} \\ 0 & l_{12} & f_3 \end{Bmatrix}$$

$$\sum_{h_2} \sqrt{\hat{\Lambda}_1' \hat{h}_2 \hat{g}_2 \hat{0}} \begin{Bmatrix} f_1 & f_2 & g_2 \\ k' & k' & 0 \\ \Lambda_1' & h_2 & f_3 \end{Bmatrix} \sqrt{\frac{\hat{f}_1 \hat{k}' \hat{f}_2 \hat{k}'}{(4\pi)^2 \hat{\Lambda}_1' \hat{h}_2}} \, (f_1 k' \Lambda_1', 000)(f_2 k' h_2, 000)$$

$$\sum_{g_3} \sqrt{\hat{g}_3 \hat{\lambda}_1'} \begin{Bmatrix} l_{12} & f_3 & \lambda_1' \\ \lambda_2' & l_{12}' & g_3 \end{Bmatrix} \sqrt{\hat{g}_3 \hat{\Lambda}_1'} \begin{Bmatrix} \lambda_2' & \Lambda_2' & \Lambda_1' \\ h_2 & f_3 & g_3 \end{Bmatrix}$$

$$(-)^{f_3} \sqrt{\frac{\hat{l}_{12}}{\hat{f}_3}} \, (-)^{\lambda_1'+\lambda_2'-l_{12}'} \, (-)^{l_{12}+f_3+\lambda_2'+l_{12}'}$$

$$\sqrt{\frac{\hat{\lambda}_2'}{\hat{\Lambda}_1'}} \, (-)^{\Lambda_2'} \, (-)^{\Lambda_1'+h_2-f_3} \, (-)^{\lambda_2'+\Lambda_2'+h_2+f_3} \, (-)^{\Lambda_2'+h_2-g_3} \, (-)^{\lambda_2'}$$

$$\sqrt{\frac{\hat{f}_3}{\hat{g}_3}} \, (-)^{f_3+\lambda_2'-g_3} \sqrt{\frac{\hat{\Lambda}_2' \hat{h}_2}{4\pi \, \hat{g}_3}} \, (\Lambda_2' h_2 g_3, 00)$$

$$Y^*_{g_3\, m_{l_{12}}' - m_{l_{12}}}(\hat{k}_8) \, (-)^{l_{12}+g_3+l_{12}'}$$

The part without the CG coefficient is the reduced matrix element. Therefore, I can use the general relation for reduced matrix element to combine spin & orbital dependence. The relevant reduced spin matrix element is

$$\langle s_{12}' \| (\vec{\sigma}_1 + \vec{\sigma}_2)^1 \| s_{12} \rangle = 6 \sqrt{6} \sqrt{\hat{s}_{12}} \, (-)^{s_{12}} \begin{Bmatrix} 1/2 & 1/2 & 1 \\ s_{12} & s_{12}' & 1/2 \end{Bmatrix} (1 + (-)^{s_{12}+s_{12}'})$$

$$\langle p_{12}' (l_{12}' s_{12}') j_{12}' m_{12}' \,|\, \frac{1}{(\vec{p}_{12} - \vec{p}_{12}' + \vec{k}_\gamma/2)^2} (\sigma_1 + \sigma_2)^{1-\gamma} \,|\, p_{12} (l_{12} s_{12}) j_{12} m_{12} \rangle$$

$$= \sum_{\tilde{\mu}\tilde{\mu}'} (l_{12}' s_{12}' j_{12}', \tilde{\mu}' \; m_{12}' - \tilde{\mu}' \; m_{12}') (l_{12} s_{12} j_{12}, \tilde{\mu} \; m_{12} - \tilde{\mu} \; m_{12})$$

$$\langle p_{12}' \; l_{12}' \; \tilde{\mu}' \,|\, \frac{1}{(\vec{p}_{12} - \vec{p}_{12}' + \vec{k}_\gamma/2)^2} \,|\, p_{12} \; l_{12}' \; \tilde{\mu} \rangle$$

$$\langle s_{12}' \| (\sigma_1 + \sigma_2)^1 \| s_{12} \rangle (s_{12} \; 1 \; s_{12}', m_{12} - \tilde{\mu} \; -\gamma \; m_{12}' - \tilde{\mu}')$$