

Scattering Observables from Few-Body Densities and Application in Light Nuclei

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The dynamics of scattering on light nuclei is well understood, but calculation is numerically difficult using standard methods. Fortunately using recent developments, the relevant quantities can be factored into a product of the n -body transition densities and the interaction kernel of a chosen probe. These transition densities depend only on the target, and not the probe; they are calculated once and stored. The kernels depend on only the probe and not the target. The calculation of the transition densities becomes numerically difficult for $n \geq 4$, but we discuss a solution through use of a renormalization group transformation. This technique allows for extending the density transition method to ${}^4\text{He}$ and ${}^6\text{Li}$. Throughout this work Compton scattering is used as test bed but extension to pion-photoproduction is discussed as well.

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1. Introduction

This can be 10 pages. Anything is red is a question or something that needs to be fixed.

I am a little unclear on how much discussion of effective field theories, and general intro information is needed here.

The transition density method was pioneered by Griesshammer *et al.*[1]. An incoming probe striking an A body nucleus may interact with 1, 2... A nucleons. The n nucleons it interacts with we call *active* and $A - n$ it does not we call *spectators*. The mathematical description of these two parts are completely separate; the active nucleons contribute to the kernel and spectator nucleons contribute to the density. The complete separation of these two parts means if one has access to a different kernels and b different nucleus descriptions, then ab different results can be produced. Figure 1 provides an example for the case $A = 3$.

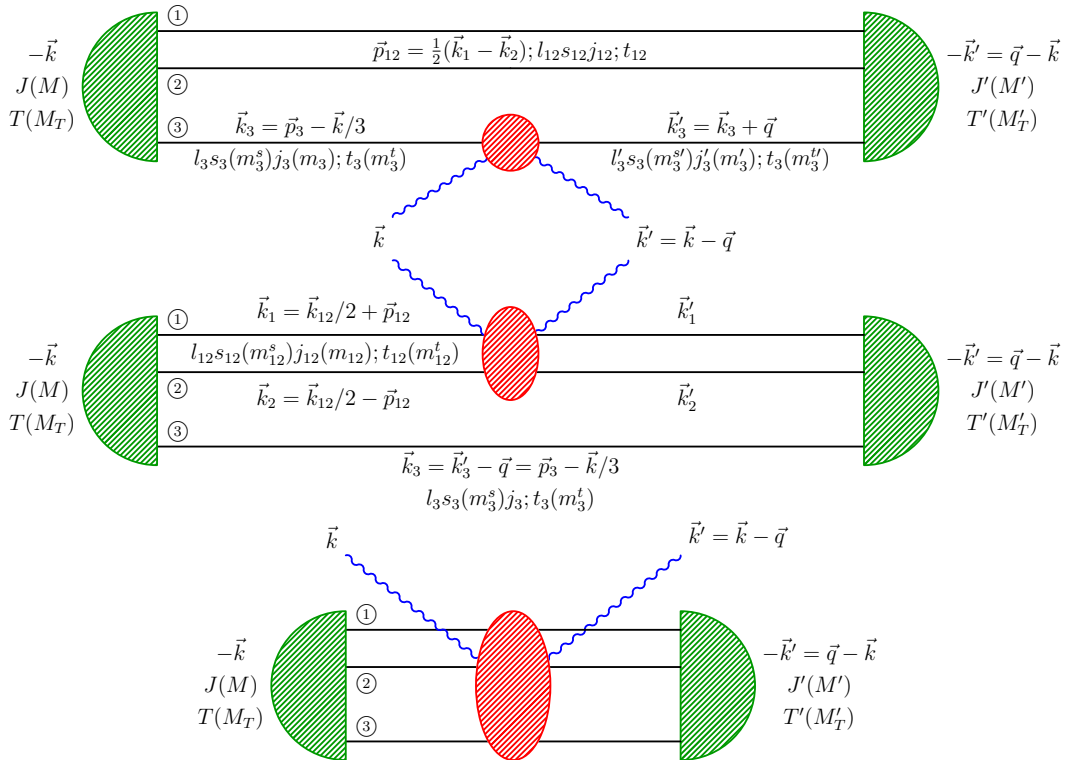


Figure 1: Kinematics in the center of mass frame and quantum numbers for an $A = 3$ system in the case of Compton scattering. Generalization to other reactions only changes the kind of ingoing/outgoing probe. Top: one-body processes, center: two-body processes, bottom: three-body processes. Green represents the densities, and red represents the kernels. Griesshammer *et al.*[1]

For scattering off an A body nucleus, the total scattering amplitude is given by

$$\begin{aligned} A_M^{M'}(\vec{k}, \vec{q}) = & \binom{A}{1} \langle M' | \hat{O}_3(\vec{k}, \vec{q}) | M \rangle + \binom{A}{2} \langle M' | \hat{O}_{12}(\vec{k}, \vec{q}) | M \rangle \\ & + \binom{A}{3} \langle M' | \hat{O}_{123}(\vec{k}, \vec{q}) | M \rangle + \binom{A}{4} \langle M' | \hat{O}_{1234}(\vec{k}, \vec{q}) | M \rangle \\ & + \dots + \binom{A}{A} \langle M' | \hat{O}_{1\dots A}(\vec{k}, \vec{q}) | M \rangle, \end{aligned} \quad (1)$$

where M, M' is the spin of the target nucleus, and there are $\binom{A}{i}$ ways for a probe to hit i nucleons. Fortunately χ EFT provides a hierarchy of scales which predicts decreasing contributions for higher order terms for energies greater than ~ 40 MeV. To this end we use only the first two terms

$$A_M^{M'}(\vec{k}, \vec{q}) = \binom{A}{1} \langle M' | \hat{O}_3(\vec{k}, \vec{q}) | M \rangle + \binom{A}{2} \langle M' | \hat{O}_{12}(\vec{k}, \vec{q}) | M \rangle$$

In practice this is enough for accuracy on the 5% level.

2. Kernels and Densities

The one-body and two-body kernel need to be considered separately. Their form is completely different, and they require a one and two body density respectively. We now write the wave function in the three body system as the partial-wave decomposition of Jacobi momenta p_i and the relevant quantum numbers α . The wave function in momentum space is given by

$$\psi_\alpha(p_{12}, p_3) = \langle p_{12} p_3 \alpha | M \rangle. \quad (2)$$

The nucleus being described has total angular momentum J and spin-projection M . The momenta are $\vec{p}_{12} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2)$ where $\vec{p}_3 = \vec{k} + \frac{1}{3}\vec{k}_3$, and $p_{12} = |\vec{p}_{12}|$ and $p_1 = |\vec{p}_1|$. Recall \vec{k}_i are the individual nucleon momenta, and \vec{k} is the probe momentum in the CM frame. The quantity α represents all the quantum numbers of the nucleons inside the nucleus [1]:

$$|\alpha\rangle = |[(l_{12}s_{12}) j_{12} (l_3 s_3) j_3] JM, (t_{12} t_3) T M_T\rangle, \quad (3)$$

where s, l and j are the spin, orbital and total angular momentum respectively. The quantum number s_3 simply represent the spin of nucleon 3, whereas s_{12} represents the total spin of the 1-2 subsystem; the quantities l_{12} and j_{12} combine the 1-2 subsystem similarly. Furthermore, s_{12} and l_{12} combine to j_{12} and likewise for l_3, s_3 and j_3 . Finally j_{12} and j_3 combine to the total nucleus spin J . The same combinations are done for t_{12}, t_3 and T , with isospin projection M_T . Here t_3 and t_{12} are the isospin of the nucleon labeled 3 and the 1-2 subsystem respectively, T is the isospin of the entire nucleus and M_T is the number of protons minus neutrons over 2. We now seek to describe the scattering amplitudes.

We restrict ourselves to elastic processes, which simplifies the following discussion by requiring that the probe does not change the charge of any of the nucleons. For the one body density, the

one-body kernel of the probe interaction with nucleon 3 is O_3 . Leaving nucleons 1 and 2 as spectators [1],

$$\begin{aligned} \langle \vec{k}'_3 | \langle s_3 m_3^{s'} | \langle t_3 m_3^{t'} | \hat{O}_3(\vec{k}, \vec{q}) | t_3 m_3^t | s_3 m_3^s | \vec{k}_3 \rangle \\ \equiv \delta_{m_3^{t'} m_3^t} \delta^{(3)}(\vec{k}'_3 - \vec{k}_3 - \vec{q}) O_3(m_3^{s'} m_3^s m_3^t; \vec{k}_3; \vec{k}, \vec{q}), \end{aligned} \quad (4)$$

where m_t and m_t' are the isospin of the active nucleon before and after the interaction (recall $m_t = \pm \frac{1}{2}$ is the proton/neutron). Symbolically, the matrix element \hat{O}_3 is:

$$\begin{aligned} \langle M' | \hat{O}_3(\vec{k}, \vec{q}) | M \rangle = \sum_{\alpha\alpha'} \int dp_{12} p_{12}^2 dp_3 p_3^2 dp'_{12} p_{12}'^2 dp'_3 p_3'^2 \psi_{\alpha'}^\dagger(p'_{12} p'_3) \psi_{\alpha}(p_{12} p_3) \\ \times \langle p'_{12} p'_3 [(l'_{12} s'_{12}) j'_{12} (l'_3 s_3) j'_3] J' M' (t'_{12} t_3) T' M_T | \hat{O}_3(\vec{k}, \vec{q}) \\ | p_{12} p_3 [(l_{12} s_{12}) j_{12} (l_3 s_3) j_3] J M (t_{12} t_3) T M_T \rangle. \end{aligned} \quad (5)$$

The central result is that up to relativistic corrections, this can be written as:

$$\langle M' | \hat{O}_3(\vec{k}, \vec{q}) | M \rangle = \sum_{\substack{m_3^{s'} m_3^s \\ m_3^t}} \hat{O}_3(m_3^{s'} m_3^s, m_3^t; \vec{k}, \vec{q}) \rho_{m_3^{s'} m_3^s}^{m_3^t M_T, M' M}(\vec{k}, \vec{q}). \quad (6)$$

Here ρ , is the *one-body transition density amplitude* for the nucleus which was discussed previously and can truly be interpreted as the probability amplitude that nucleon m_3^t absorbs momentum \vec{q} , changes its spin projection from m_3^s to $m_3^{s'}$ and changes the spin-projection of the nucleus from M to M' . Its operator form is

$$\rho_{m_3^{s'} m_3^s}^{m_3^t M_T, M' M}(\vec{k}, \vec{q}) = \langle M' | s_3 m_3^{s'}, t_3 m_3^t \rangle e^{i \frac{1}{2} \vec{q} \cdot \vec{r}_3} \langle s_3 m_3^s, t_3 m_3^t | M \rangle. \quad (7)$$

The two body case works similarly, and results in

$$\langle M' | \hat{O}_{12} | M \rangle = \sum_{\alpha'_{11}, \alpha_{12}} \int dp_{12} p_{12}^2 dp'_{12} p_{12}'^2 O_{12}^{\alpha'_{11} \alpha_{12}}(p'_{12}, p_{12}) \rho_{\alpha'_{11} \alpha_{12}}^{M_T, M' M}(p'_{12}, p_{12}; \vec{q}). \quad (8)$$

This two-body density $\rho_{\alpha'_{11} \alpha_{12}}^{M_T, M' M}$ is of course completely distinct from the one-body density and has a more involved expression equivalent to (7). This two-body density can also be interpreted as a probability density. It is dependent on the incoming and outgoing quantum numbers α_{12} and α'_{12} of the 1-2 system, and also on the relative momenta of the two nucleons which are integrated over. As a result, the total file size for the two nucleon densities are on the order of a few GB per energy and angle, whereas those of the one nucleon densities are on the order of a few KB. Importantly, ρ can be computed generically from a nuclear potential, such as the chiral SMS potential without reference to the kernel \hat{O}_3 or \hat{O}_{12} [2]. *Maybe this is too much detail.*

3. SRG Transform

Previous work using the transition density method has looked at ^3He and ^4He but to extend this ^6Li involves many body iterations which are much more complicated, and numerically expensive

to compute [1, 3]. To this end a similarity renormalization group (SRG) transformation is used [4]. In general a nuclear potential, such as the chiral SMS potential does not fall off quickly for high momentum meaning we would have to extend the cutoff much further than we would like which in turn increases computation cost. The SRG transform is a unitary transformation that allows us to shovel all the dependence into the low momentum region making calculation for $A = 6$ possible. The SRG transformation can be thought of as a local averaging or smoothing out of the potential, resulting in decreased "resolution" has the SRG is applied. In the unevolved, high resolution figure

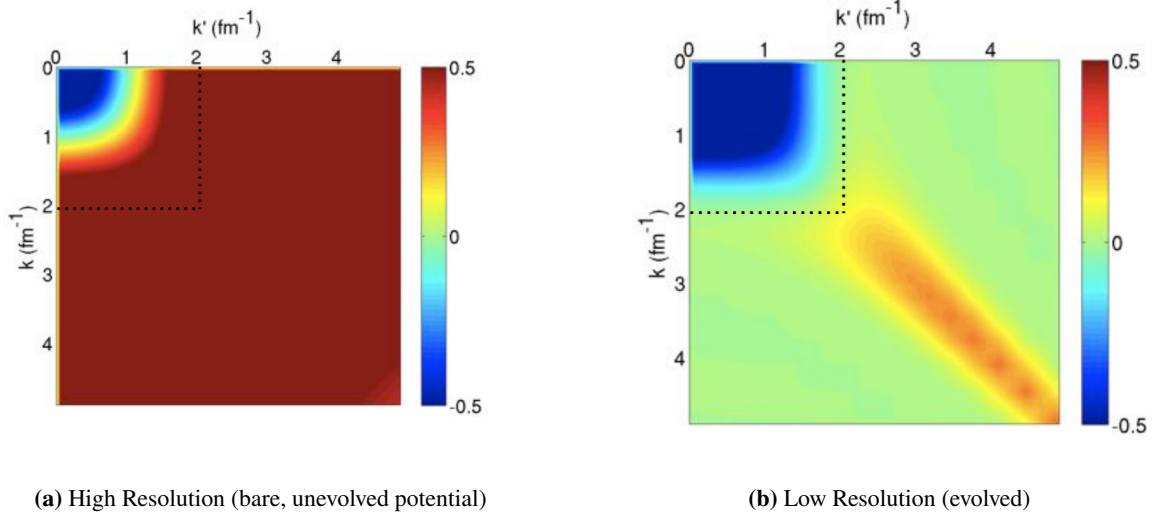


Figure 2: Nuclear potentials $V(k, k')$. Figures from Kai Hebeler: “Chiral Effective Field Theory and Nuclear Forces: overview and applications” presentation at TALENT school at MITP 2022

2a one can see the potential does not go to zero quickly whereas once the transformation is applied in figure 2b it does. As a result a cutoff can be made at $k, k' = 2\text{fm}^{-1}$ without losing much accuracy. In the example above, if the unevolved potential had to be consider up to $k, k' = 5$, this means an efficiency gain of $(5/2)^2 = 6.25$.

However, this creates another problem; the SRG transform creates a change in physical meaning of the momenta free variables. To understand this, let us first consider an example the reader is certainly more familiar with - a Fourier transformation which changes a potential from position to momentum space.

$$V(\vec{r}, \vec{r}') = \langle r' | V | r \rangle = \int d^3p d^3p' \langle r' | p' \rangle \langle p' | V | p \rangle \langle p | r \rangle = V(\vec{p}, \vec{p}') \quad (9)$$

After the transform our free variables have different physical meaning. In fact, any unitary transform, also transforms the coordinates.

$$\begin{aligned} \langle p' | V | p \rangle &= \langle p' | \mathbb{1} V \mathbb{1} | p \rangle \\ &= \langle p' | U^\dagger U V U^\dagger U | p \rangle \\ &= \left(\langle p' | U^\dagger \right) \left(U V U^\dagger \right) (U | p \rangle) \\ &= \langle \tilde{p}' | V_{eff} | \tilde{p} \rangle = V_{eff}(\tilde{p}, \tilde{p}') \end{aligned} \quad (10)$$

So calling the free variables in an SRG transformed potential “momenta” is abuse of notation. They are not physical states in the sense that $\tilde{p} = 50\text{MeV}$ does not correspond to a state of 50MeV that an experimentalist can prepare. The Lagrangians that generate the Feynman diagrams in the kernel however have dependence on physical momenta. To solve this, previous work with SRG transformations has transformed the Lagrangians into the SRG evolved space, however in the case of the density formalism this would mean adding SRG information into the kernel, thereby breaking kernel-density independence. Additionally, the SRG transformation can take many different forms, changed over the years, and we wish to allow for these developments without having to re-write the kernel code. Therefore Xiang-Xiang Sun *et al.* have chosen to apply an inverse transformation to the densities. This work was begun with the highest nucleon number system we can calculate densities for without an SRG transformation, which is ^4He . In this way we have gained confidence in ^4He before moving to the more complex ^6Li . *Are they not publishing a paper on this? How do I credit Xiang-Xiang effectively*

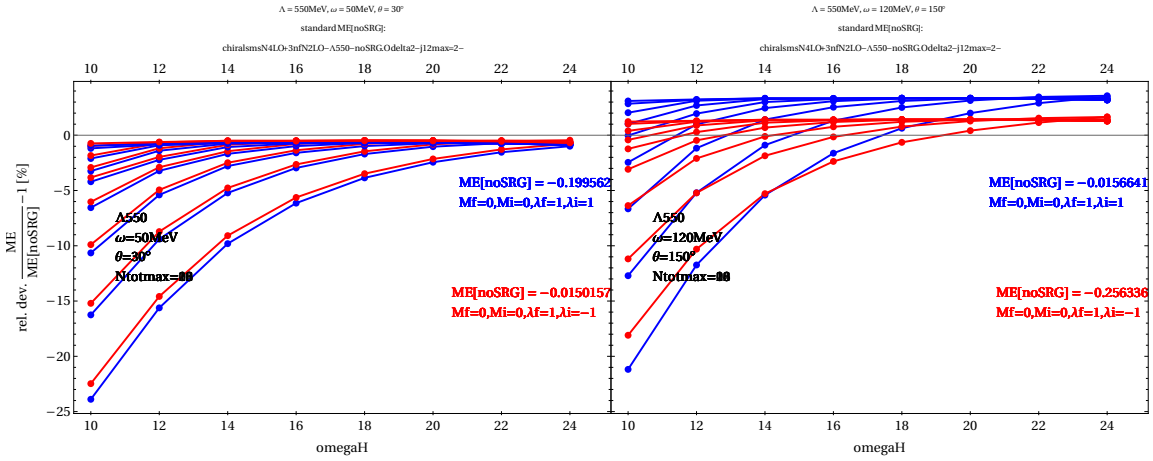


Figure 3: A convergence plot for ^4He Compton scattering showing the relative deviation of the matrix elements (ME) of the reaction.

In figure 3 we see the effectiveness of the results. The SRG transformation uses the harmonic oscillator basis, and the value ‘omegaH’ on the x-axis is the width of the harmonic oscillator potential, and ‘Ntotmax’ is the total number of states used. *^6Li results go here maybe? And talk about extraction of polarizabilities*

How much more do we talk about? We have many more results since the actual chiral dynamics presentation. Also do we really want to put that "preliminary" plot with the fuzzy values on here?

4. Extension to Other Interactions

Beyond Compton scattering we are interested in extending the density formalism to other processes. In particular pion-photoproduction, and pion-pion scattering are of interest. Fortunately their kernels share remarkable similarity since there are only so many ways for one particle to go in and one to go out. *I don’t want to include that diagram since I stole the figures from somewhere, but I don’t remember where*

4.1 Pion-Photoproduction

For the one-body kernel we use the results of pion-photoproduction on a single nucleon, $\gamma N \rightarrow \pi N$ which has been studied extensively. Its differential cross section can be decomposed in terms of the electric and magnetic multipoles $E_{l\pm}, M_{l\pm}$ which are angle independent [5]. There have been many different experimental results over the years which measure these multipoles to high order and with good precision such as ‘Unified Chew-Mandelstam SAID analysis...’ Workman *et al.*[6]. The resulting scattering matrices \mathcal{M} are exactly what enters as $\hat{\mathcal{O}}_3$ equation (6). This solves a significant problem since calculation of the one-body pion-photoproduction kernel to high accuracy directly through Feynman diagrams requires including many terms in the chiral expansion due to the proximity of the $\Delta(1232)$ resonance at $\sim 200\text{MeV}$ [7].

The two-body contributions do not decompose into multipole interactions, and therefore must be calculated directly through Feynman diagrams. At threshold energy the kernel has been analyzed before by Lenkewitz *et al.*[8, 9]. We must extend this to finite energy, and to higher order if necessary.

We have a stable result now, do we include this information? We didn't have this at the time of the conference.

4.2 Pion-Pion scattering and other reactions

Talk about we want to look at pion-pion, and we could look at Dark matter.

5. Conclusion

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