Some Monte Carlo Simulations in Finance with GBM Stock Price

Alex Lu

Mean and Variance of Stock Price under GBM

A stock price $S(t)$ at time $t \geq 0$ can be modelled by the Geometric Brownian Motion (GBM):

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}$$

where $W(t)$ is a Standard Brownian Motion, μ is the drift, $\sigma \geq 0$ is the volatility, and $S(0) \geq 0$ is the initial stock price.

The point at time t of a Brownian Motion realization can be simulated using a normal distribution with mean 0 and variance t:

$$W(t) \sim \mathcal{N}(0, t)$$

or with a standard normal distribution

$$W(t) \sim \sqrt{t}Z \quad \text{where} \quad Z \sim \mathcal{N}(0, 1)$$

The point at time t of a GBM realization can therefore be simulated using

$$S(t) \sim S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z}$$

We can perform Monte Carlo Simulations to estimate the mean of $S(t)$ by simulating the points

$$S^1(t), S^2(t), ..., S^M(t)$$

and taking the average

$$\mathrm{E}[\widehat{S(t)}] = \frac{1}{M}\sum_{m=1}^{M} S^m(t)$$

This equivalently means we are performing Monte Carlo integration to estimate the integral

$$\begin{aligned}
\mathrm{E}[S(t)] &= \mathrm{E}\left[S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}\right] \\
&= \int_{-\infty}^{\infty} S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz
\end{aligned}$$

which we can also compute directly

$$
\begin{aligned}
\mathrm{E}[S(t)] &= \int_{-\infty}^{\infty} S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= S(0)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\mu t}e^{-\frac{z^2}{2}+\sigma\sqrt{t}z-\frac{\sigma^2 t}{2}}dz \\
&= S(0)e^{\mu t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2\sigma\sqrt{t}z+\sigma^2 t)}dz \\
&= S(0)e^{\mu t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2}dz \\
&= S(0)e^{\mu t}
\end{aligned}
$$

In the last step we note that

$$
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2}dz = 1
$$

is the integral over the sample space of the pdf of the normal distribution $\mathcal{N}(\sigma\sqrt{t},1)$. We could also let $y = z-\sigma\sqrt{t}$ so $dy = dz$ hence

$$
\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2}dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}dy = 1
$$

We can also use these simulations to estimate the variance

$$
\mathrm{Var}\widehat{(S}(t)) = \frac{1}{M}\sum_{m=1}^{M}(S^m(t) - E[\widehat{S(t)}])^2
$$

which approximates the integral for $\mathrm{Var}(S(t))$ that can be evaluated directly

$$
\begin{aligned}
\mathrm{E}\left[(S(t))^2\right] &= \mathrm{E}\left[\left(S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W(t)}\right)^2\right] \\
&= \int_{-\infty}^{\infty}\left(S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}z}\right)^2\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(S(0))^2 e^{2\mu t-\sigma^2 t+2\sigma\sqrt{t}z}e^{-\frac{z^2}{2}}dz \\
&= (S(0))^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2\mu t}e^{-\frac{z^2}{2}+2\sigma\sqrt{t}z-\sigma^2 t}dz \\
&= (S(0))^2 e^{2\mu t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\sigma^2 t}e^{-\frac{z^2}{2}+2\sigma\sqrt{t}z-2\sigma^2 t}dz \\
&= (S(0))^2 e^{2\mu t+\sigma^2 t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z^2-4\sigma\sqrt{t}z+4\sigma^2 t\right)}dz \\
&= (S(0))^2 e^{2\mu t+\sigma^2 t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z-2\sigma\sqrt{t}\right)^2}dz \\
&= (S(0))^2 e^{2\mu t+\sigma^2 t}
\end{aligned}
$$

We note that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(z-2\sigma\sqrt{t}\right)^2} dz = 1$$

is the integral over the sample space of the pdf of the normal distribution $\mathcal{N}(2\sigma\sqrt{t}, 1)$. We could also let $y = z - 2\sigma\sqrt{t}$ so $dy = dz$ hence

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-2\sigma\sqrt{t})^2} dz = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy = 1$$

Thus, the variance is

$$\begin{aligned}
\operatorname{Var}(S(t)) &= E\left[(S(t))^2\right] - (E[S(t)])^2 \\
&= (S(0))^2 e^{2\mu t + \sigma^2 t} - \left(S(0)e^{\mu t}\right)^2 \\
&= (S(0))^2 e^{2\mu t + \sigma^2 t} - (S(0))^2 e^{2\mu t} \\
&= (S(0))^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)
\end{aligned}$$

European Call and Put Option

An European Call Option $C(t)$ is a contract that allows the right but not the obligation to buy the underlying stock at the expiry time $T \geq t$ and strike price $K \geq 0$. At time T, if $S(T) \geq K$ the option holder will exercise the contract and profit $S(T) - K$ (Buy stock at K and sell at $S(T)$) and if $S(T) < K$ the option holder will let the contract expire worthless. Thus, the payoff function of the option is

$$C(T) = \max(S(T) - K, 0) \quad \text{or} \quad (S(T) - K)\mathbb{1}_{(S(t) \geq K)}$$

Options are priced according to a risk-neutral measure (*) where additional risk is not compensated by additional expected return. We can simulate an endpoint for a realization of the underlying stock price using

$$S^*(T) \sim S(0)e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z}$$

where we have replaced μ with the risk free interest rate r. Once we simulate the points $S^{1*}(T), S^{2*}(T), ..., S^{M*}(T)$ we can compute the discounted payoff of that realization with

$$C^m(0) = e^{-rT}\max(S^{m*}(T) - K, 0)$$

Finally, we can estimate the call option price with

$$\widehat{C(0)} = \frac{1}{M}\sum_{m=1}^{M} C^m(0)$$

which estimates the integral

$$
\begin{aligned}
C(0) &= e^{-rT}\mathrm{E}[\max(0, S^*(T) - K)] \\
&= e^{-rT}\mathrm{E}\left[\max\left(S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W(T)} - K, 0\right)\right] \\
&= e^{-rT}\int_{-\infty}^{\infty} \max\left(S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} - K, 0\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz
\end{aligned}
$$

We can also compute the integral directly. First we note that

$$
f(z) = S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} - K
$$

is an increasing function of z and is equal to zero at a single point z_0. Setting $f(z) \geq 0$ we have

$$
\begin{aligned}
S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} - K &\geq 0 \\
S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} &\geq K \\
e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} &\geq \frac{K}{S(0)} \\
\left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}z &\geq \ln(K) - \ln(S(0)) \\
\sigma\sqrt{T}z &\geq \ln(K) - \ln(S(0)) - rT + \frac{\sigma^2 T}{2} \\
z \geq z_0 &= \frac{\ln(K) - \ln(S(0))}{\sigma\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2}
\end{aligned}
$$

We can now compute the price of a call option

$$
\begin{aligned}
e^{-rT}\mathrm{E}[S^*(T)\mathbb{1}_{(S^*(T)\geq K)}] &= e^{-rT}\int_{z_0}^{\infty} S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz \\
&= e^{-rT}S(0)\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{rT}e^{-\frac{z^2}{2}+\sigma\sqrt{T}z-\frac{\sigma^2 T}{2}}\,dz \\
&= e^{-rT}S(0)\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{rT}e^{-\frac{1}{2}\left(z^2-2\sigma\sqrt{T}z+\sigma^2 T\right)}dz \\
&= S(0)\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}dz
\end{aligned}
$$

Letting $y = z - \sigma\sqrt{T}$ we have

$$
dy = dz, \quad y_0 = z_0 - \sigma\sqrt{T} = \frac{\ln(K) - \ln(S)}{\sigma\sqrt{T}} - \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}
$$

and

$$
\begin{aligned}
e^{-rT}\mathrm{E}[S^*(T)\mathbb{1}_{(S^*(T)\geq K)}] &= S(0)\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}\,dz \\
&= S(0)\frac{1}{\sqrt{2\pi}}\int_{y_0}^{\infty} e^{-\frac{1}{2}y^2}\,dy \\
&= S(0)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-y_0} e^{-\frac{1}{2}y^2}\,dy \\
&= S(0)N(-y_0)
\end{aligned}
$$

Next

$$
\begin{aligned}
e^{-rT}\mathrm{E}[K\mathbb{1}_{(S^*(T)\geq K)}] &= e^{-rT}\int_{z_0}^{\infty} K\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz \\
&= e^{-rT}K\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{z^2}{2}}\,dz \\
&= e^{-rT}K\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-z_0} e^{-\frac{z^2}{2}}\,dz \\
&= e^{-rT}KN(-z_0)
\end{aligned}
$$

Finally, the price of a call option is

$$
\begin{aligned}
C(0) &= e^{-rT}\mathrm{E}[(S^*(T)-K)\mathbb{1}_{(S^*(T)\geq K)}] \\
&= e^{-rT}\left(\mathrm{E}[S^*(T)\mathbb{1}_{(S^*(T)\geq K)}] - \mathrm{E}[K\mathbb{1}_{(S^*(T)\geq K)}]\right) \\
&= S(0)N(d_+) - Ke^{-rT}N(d_-)
\end{aligned}
$$

where

$$
\begin{aligned}
d_+ &= \frac{\ln(S(0))-\ln(K)}{\sigma\sqrt{T}} + \frac{r\sqrt{T}}{\sigma} + \frac{\sigma\sqrt{T}}{2} \\
d_- &= \frac{\ln(S(0))-\ln(K)}{\sigma\sqrt{T}} + \frac{r\sqrt{T}}{\sigma} - \frac{\sigma\sqrt{T}}{2}
\end{aligned}
$$

A European Put Option $P(t)$ is a contract that allows the right but not the obligation to sell the underlying stock at the expiry time $T \geq t$ and strike price $K \geq 0$. At time T, if $S(T) < K$ the option holder will exercise the contract and profit $K - S(T)$ (Buy stock at $S(T)$ and sell at K) and if $S(T) \geq K$ the option holder will let the contract expire worthless. Thus, the payoff function of the option is

$$
P(T) = \max(K - S(T), 0) \quad \text{or} \quad (K - S(T))\mathbb{1}_{(S(T)<K))}
$$

Like the call option we can estimate the price of a put option by simulating realizations of the risk neutral stock price and then computing the discounted payoff

$$
P^m(0) = e^{-rT}\max(K - S^{m*}(T), 0)
$$

Our estimate for the put option price is

$$\widehat{P(0)} = \frac{1}{M}\sum_{m=1}^{M} P^m(0)$$

which estimates the integral

$$\begin{aligned} P(0) &= e^{-rT}\mathrm{E}[\max(0, K - S^*(T))] \\ &= e^{-rT}\mathrm{E}\left[\max\left(K - S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma W(T)}, 0\right)\right] \\ &= e^{-rT}\int_{-\infty}^{\infty} \max\left(K - S(0)e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z}, 0\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz \end{aligned}$$

We can evaluate this integral directly or we can use the Put-Call Parity formula which relates the price of a stock with it's call, put, and strike price. We start with the payoff of buying a call and selling a put

$$\begin{aligned} C(T) - P(T) &= \max(S(T) - K, 0) - \max(K - S(T), 0) \\ &= \max(S(T) - K, 0) + \min(S(T) - K, 0) \\ &= S(T) - K \end{aligned}$$

Using the No Arbitrage Principal (NAP), we can bring the time back to 0 and get the Put-Call Parity

$$C(0) - P(0) = S(0) - Ke^{-rT}$$

The price of a Put option is

$$\begin{aligned} P(0) &= C(0) - S(0) + Ke^{-rT} \\ &= S(0)N(d_+) - Ke^{-rT}N(d_-) - S(0) + Ke^{-rT} \\ &= Ke^{-rT}(1 - N(d_-)) - S(0)(1 - N(d_+)) \\ &= Ke^{-rT}(N(-d_-)) - S(0)(N(-d_+)) \end{aligned}$$

In the last step we note that

$$\begin{aligned} 1 - N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\,dz \\ &= \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-\frac{z^2}{2}}\,dz \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{-\frac{z^2}{2}}\,dz \\ &= N(-x) \end{aligned}$$

Procedure

For the procedure we will fix μ, σ, $S(0)$, r, and K. We will simulate M sample paths from time 0 to T by discretizing the time into N points per unit of time

$$t_n = \frac{n-1}{N} \quad \text{for} \quad n = 1, 2, ..., NT+1$$

The stochastic portion will be generated with the standard normal distributions

$$Z_n^m \sim \mathcal{N}(0,1) \quad \text{for} \quad m = 1, 2, ..., M \quad n = 1, 2, ..., NT$$

The initial points of the Brownian motion and Stock price will be set to

$$\begin{aligned} W^m(0) &= W^m(t_1) = W_1^m = 0 \\ S^m(0) &= S^m(t_1) = S_1^m = S(0) \end{aligned}$$

and the rest of the points will be computed with

$$\begin{aligned} &W^m(t_n) = W_n^m = W_{n-1}^m + \frac{1}{\sqrt{N}} Z_{n-1}^m \\ &S^m(t_n) = S_n^m = S(0) e^{\left(\mu - \frac{\sigma^2}{2}\right) t_n + \sigma W_n^m} \\ &\text{for} \quad m = 1, 2, ..., M \quad n = 2, 3, ..., NT+1 \end{aligned}$$

The means and variances will be computed

$$\begin{aligned} &\mathrm{E}[\widehat{S(t_n)}] = \frac{1}{M} \sum_{m=1}^{M} S_n^m \\ &\mathrm{Var}(\widehat{S(t_n)}) = \frac{1}{M} \sum_{m=1}^{M} (S_n^m - E[\widehat{S(t_n)}])^2 \\ &\text{for} \quad n = 1, 2, ..., NT+1 \end{aligned}$$

and compared to the true values

$$\begin{aligned} &\mathrm{E}[S(t_n)] = S(0) e^{\mu t_n} \\ &\mathrm{Var}(S(t_n)) = (S(0))^2 e^{2\mu t_n} \left(e^{\sigma^2 t_n} - 1 \right) \\ &\text{for} \quad n = 1, 2, ..., NT+1 \end{aligned}$$

The risk neutral stock price will be computed with

$$S^{m*}(0) = S^{m*}(t_1) = S_1^{m*} = S(0)$$

and

$$\begin{aligned} &S^{m*}(t_n) = S_n^{m*} = S(0) e^{\left(r - \frac{\sigma^2}{2}\right) t_n + \sigma W_n^m} \\ &\text{for} \quad m = 1, 2, ..., M \quad n = 2, 3, ..., NT+1 \end{aligned}$$

We will denote a the price of a call option at time t_1 that expires at time t_2 as $C(t_1, t_2)$, $t_1 \leq t_2$ and the price of a put option as $P(t_1, t_2)$. We will be computing the value of a call option at time $t_1 = 0$ while varying the expiry time $t_2 = t$. The discounted realized Call and Put payoffs will be computed with

$$\begin{aligned}
&C^m(0, t_n) = C_n^m = e^{-rt_n} \max(S_n^{m*} - K, 0) \\
&P^m(0, t_n) = P_n^m = e^{-rt_n} \max(K - S_n^{m*}, 0) \\
&\text{for} \quad m = 1, 2, ..., M \quad n = 1, 2, ..., NT + 1
\end{aligned}$$

The call and put prices will be computed

$$\begin{aligned}
&\widehat{C(0, t_n)} = \frac{1}{M} \sum_{m=1}^{M} C_n^m \\
&\widehat{P(0, t_n)} = \frac{1}{M} \sum_{m=1}^{M} P_n^m \\
&\text{for} \quad n = 1, 2, ..., NT + 1
\end{aligned}$$

and compared to the true values

$$\begin{aligned}
C(0, 0) = C(0, t_1) = \max(S(0) - K, 0) \\
P(0, 0) = C(0, t_1) = \max(K - S(0), 0)
\end{aligned}$$

and

$$\begin{aligned}
&C(0, t_n) = S(0)N(d_+^n) - Ke^{-rt_n}N(d_-^n) \\
&P(0, t_n) = Ke^{-rt_n}(N(-d_-^n)) - S(0)(N(-d_+^n)) \\
&\text{for} \quad n = 2, 3, ..., NT + 1
\end{aligned}$$

where

$$\begin{aligned}
&d_+^n = \frac{\ln(S(0)) - \ln(K)}{\sigma\sqrt{t_n}} + \frac{r\sqrt{t_n}}{\sigma} + \frac{\sigma\sqrt{t_n}}{2} \\
&d_-^n = \frac{\ln(S(0)) - \ln(K)}{\sigma\sqrt{t_n}} + \frac{r\sqrt{t_n}}{\sigma} - \frac{\sigma\sqrt{t_n}}{2} \\
&\text{for} \quad n = 2, 3, ..., NT + 1
\end{aligned}$$

Results

Using $S(0) = 1$, $\mu = 0.25$, $\sigma = 0.5$, $r = 0.15$, $K = 1$, $T = 1$, $M = 10000$, and $N = 10000$ we obtain the following results from the Matlab code.

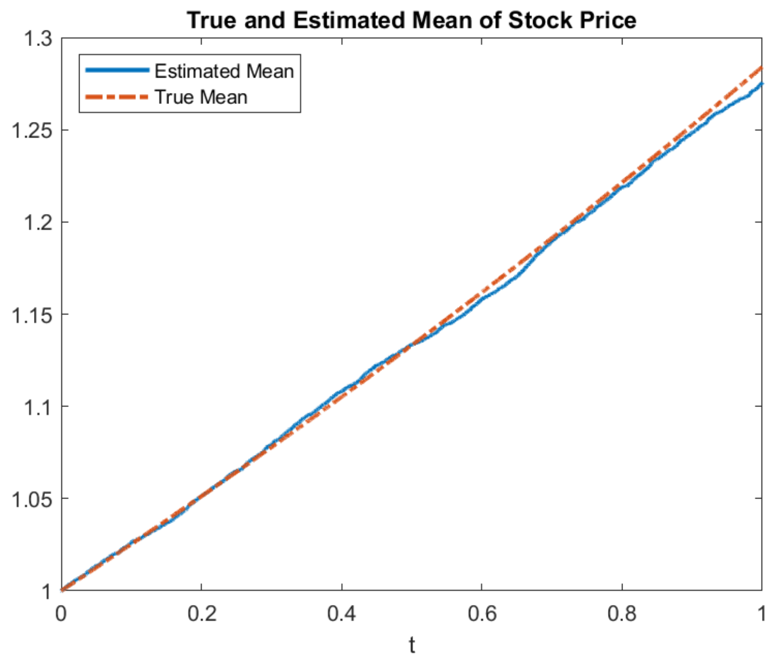

Figure 1: Plot of $\mathrm{E}[\widehat{S(t)}]$ and $\mathrm{E}[S(t)]$

Figure 2: Plot of $\mathrm{Var}[\widehat{S(t)}]$ and $\mathrm{Var}[S(t)]$

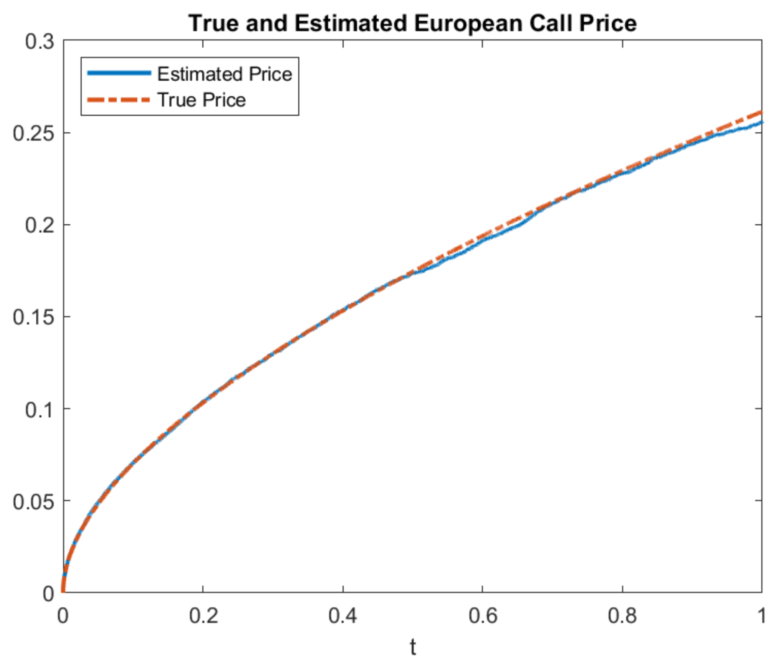

Figure 3: Plot of $\widehat{C(0,t)}$ and $C(0,t)$

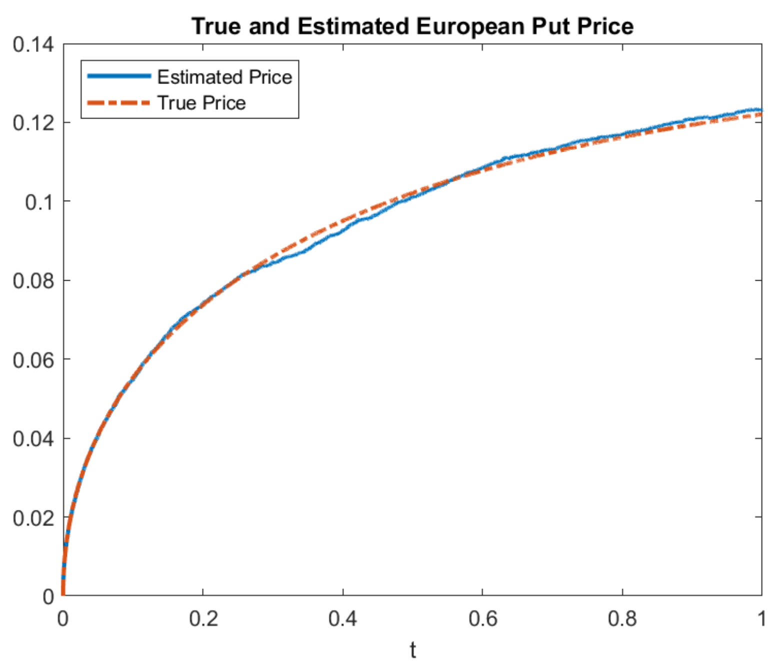

Figure 4: Plot of $\widehat{P(0,t)}$ and $P(0,t)$