Lab 3: Basic Motion and Graphing

1 Introduction

The purpose of this lab is to learn about how equations of motion describe measurable physical quantities and to learn how to test a theory by performing experiments. We will develop graphing skills and use the method of regression analysis to analyze our data, both of which facilitate comparing our results to those expected from theory.

2 Key Concepts

- Position, velocity, and acceleration
- Data acquisition
- Regression analysis
- Graphing

3 Theory

3.1 Equation of Motion

Our goal is to use the data we acquire during the lab to determine the form of our **equations of motion**. Motion can be described in terms of an object's location at an instant in time (position) and how fast that position is changing (velocity). Some of the quantities of interest are vectors (like displacement, velocity, and acceleration), and some are scalars (like distance, speed, and time).

A few definitions will be helpful.

- **Displacement** is the vector change in an object's position between an initial and final point in its motion. Its sign is determined by the direction of motion and the relevant coordinate system for the problem.
$$\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1$$
- **Distance** is the actual (scalar) length traveled by the object and is always positive.
- The **average velocity** between two points along the object's path is the displacement divided by time elapsed between the initial and final points. It is also a vector, and its sign depends on the direction of motion and coordinate system used. The bar above $\vec{\mathbf{v}}$ tells you that it is an average quantity.
$$\bar{\vec{\mathbf{v}}} = \frac{\Delta\vec{\mathbf{x}}}{\Delta t} = \frac{\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1}{t_2 - t_1}$$

- The **instantaneous velocity** is the derivative of displacement with respect to time at a point on the object's trajectory. It can also be thought of as the slope of the tangent to the graph of position vs. time at a position of interest.

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{x}}}{dt}$$

- **Speed** is the distance traveled divided by time and is a scalar that is always positive. It is also the magnitude of the instantaneous velocity.

$$v = |\vec{\mathbf{v}}|$$

- The **average acceleration** between two points is the change in velocity divided by time elapsed. It also is a vector, and its sign depends on the direction of motion and coordinate system used.

$$\bar{\vec{\mathbf{a}}} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{\vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1}{t_2 - t_1}$$

- The **instantaneous acceleration** is the derivative of the instantaneous velocity with respect to time. It can also be thought of as the slope of the tangent to the graph of velocity vs. time at a position of interest.

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{x}}}{dt^2}$$

These ideas can be put together to describe the motion of an object. In the special case of constant acceleration, the three-dimensional equation of motion is given by

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_0 + \vec{\mathbf{v}}_0 t + \frac{1}{2}\vec{\mathbf{a}}t^2.$$

Keep in mind that in our 3-D Cartesian coordinate system, $\vec{\mathbf{x}}(t) = x(t)\hat{\mathbf{\imath}} + y(t)\hat{\mathbf{\jmath}} + z(t)\hat{\mathbf{k}}$, so the above equation is really just compact notation for three equations—one for each component of $\vec{\mathbf{x}}(t)$. Since we will be looking at motion along one dimension (call it x) in this lab, our 1-D equation of motion takes the simpler form

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2.$$

For further explanation, see the sections on motion in your textbook and the appendix discussing the derivation of the equation of motion.

3.2 Data Analysis

As mentioned above, our goal is to use the data we acquire during the lab to determine the form of our equations of motion. In this lab, the data that we will collect will be pairs of position and time measurements: $\{(x_0, t_0), (x_1, t_1), (x_2, t_2), \ldots, (x_n, t_n)\}$. In the first part of the lab, we will measure times associated with different distances traveled by the air track glider using a stopwatch. In the second part of the lab, we will track the motion of the glider using a computer program that calculates its position, velocity, and acceleration.

We want to determine the form of the equation describing one variable as a function of the other. To do this, we will use a technique known as **regression analysis** to determine the form of $x(t)$. We will then be able to compare our expression with the theoretical equation of motion solved for our variable of interest. (To solve an equation for a variable means to rewrite the function with the variable of interest by itself on the left hand side of the equation and everything else on the right hand side.)

Relationship	Experimental Equation	Linearization Equation
Linear	$y = mx + b$	already a line
Power	$y = bx^m$	$\log y = m \log x + \log b$
Exponential	$y = be^{mx}$	$\ln y = mx + \ln b$

Table 1: Table of functional relationships, experimental equations, and linearization equations.

3.2.1 Regression Analysis

Regression analysis takes a collection of data points like the ones above and finds the function $x(t)$ that comes closest to matching up with it. As an example, consider the data set

$$\{(x, t)\} = \{(4, 1), (7, 2), (10, 3)\}.$$

The function that best describes these data points is the function $x(t) = 3t + 1$. In other words, if you graphed the three points above, this $x(t)$ is the line that comes closest to hitting all the points. We call this function our **experimental equation**. In this case, it is the equation of a line with the slope

$$m = \frac{\text{rise}}{\text{run}} = \frac{x_i - x_{i-1}}{t_i - t_{i-1}} = 3.$$

However, not all functions of our variables will be linear, and so we will need to consider two other possible types of experimental equation. The first, **power law functions**, describe situations in which $x(t) = bt^m$, for some numbers b and m. An example is the equation of motion in this lab with the initial position and velocity set to zero. The second, **exponential functions**, describe situations in which $x(t) = be^{mt}$. An example might be the formula for the exponential decay of a substance with a certain half-life. These equations are given in Table 1.

3.2.2 Graphing

To determine the form of $x(t)$ for a data set, we first make a plots of our data. Since a straight line is the easiest type of function to graph and analyze, we try to fit linear functions to the data. By plotting the data in different ways, we can see which form is best described by a straight line, and therefore determine the true relationship between x and t by the process below.

There are three basic types of functions that can describe our data: linear functions, power functions, and exponential functions. Their different forms are detailed in Table 1. There are then three different ways we will graph our data, which can be deduced by looking at the Linearization Equation column in the table.

This technique is called **linearization**. If x is the independent variable, and y is the dependent variable of a data set, we plot y vs. x, $\log y$ vs. $\log x$, and $\ln y$ vs. x. We fit (straight) trendlines to each graph and determine which form of the data is best approximated by a line. The equation of the trendline in this best-fit case is called the **linearization equation**. We can recover the **experimental equation**—i.e., the real relationship between our variables—from the linearization equation after some algebra based on the relationships between the two types of equations as seen in the table. These steps involve graphing our data correctly and understanding what the slope and y intercept of our linearization equation correspond to in our experimental equation. See the file called Graphing Skills on eLearning for a detailed example.

1. For a linear experimental equation: We don't have to do anything to linearize, because our function is already that of a line. The slope of the linearization equation corresponds to the slope of the experimental equation, and the y-intercept of the linearization equation corresponds to the y-intercept of the experimental equation.

2. For a power experimental equation: Take the logarithm of both variables and plot them. The slope of the linearization equation corresponds to the power of the variable in the experimental equation, and the y-intercept of the linearization equation corresponds to the base-10 log of the constant that multiplies the independent variable of the experimental equation.

3. For an exponential experimental equation: Take the natural logarithm of the variable on the vertical axis, and plot it against the variable on the horizontal axis. The slope of the linearization equation here corresponds to the constant multiplying the independent variable (in the exponent) of the experimental equation, and the y-intercept of the linearization equation corresponds to the natural log of the coefficient of e in the experimental equation.

In order to determine which of the best fit lines most closely matches our plots, we associate what's called an R^2 value to each of the best fit lines. The R^2 value always lies between 0 and 1, and the closer it is to 1, the better it describes our data. Therefore, when we make the three plots for a data set and add best fit lines, we pick the one with the R^2 value closet to 1 to proceed with in determining the experimental equation.

Be aware that due to experimental error, it is possible that the R^2 value for the expected linearization equation is not the closest to 1. This can happen if you are not careful enough in taking data. Some of these labs will be graded on accuracy, and it is therefore extremely important that you take data as accurately as possible.

3.3 Reaction Times

We will be taking times by human measurement in the first part of this experiment. We will be using a stopwatch to measure the time required for the air track glider to move a set distance. There will be a lot of error in these measurements due to the imperfect **reaction time** of the person with the stopwatch.

To reduce such errors, consider the following way to judge what a person's reaction time is. First determine the theoretical value of the time expected for the glider to move a particular distance from the equation $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$. Then look at the difference between the time measured and the theoretical time. Performing many trials this way, the average of the differences in many trials should correspond to the person's reaction time. You could also look online for the average human reaction time.

4 Experiment

4.1 Equipment

- Triple beam balance
- Set of masses
- Mass hangers (2)
- Air track with air supply and glider
- Red cables (2)

- Rotary motion sensor
- Computer with Logger Pro software
- Marble tiles (6) and aluminum squares (2)
- Stopwatch
- USB flash drive

4.2 Procedure

4.2.1 Setup

Turn on the air supply, and make sure your setup looks like Figure 1 below.

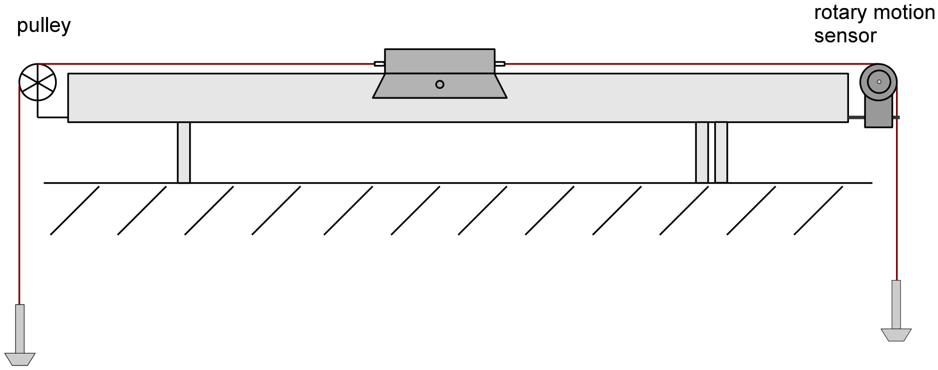

Figure 1: Diagram of air track setup.

4.2.2 Time of Transit Measurements

1. Place one marble tile beneath the single leg of the air track so that the end with the pulley is elevated.
2. Put 20 g masses on each hanger, and turn on the air supply.
3. Consider the glider's leading edge—the one that will reach the bottom of the air track first when released—as your reference point. Move the glider so that this edge lines up at the 70 cm mark as measured from the high end of the track. This might be the 130 cm mark as measured from the low end if you're looking at the other side.
4. Using the stopwatch, measure the time (in seconds) it takes for the glider to travel 30 cm down the air track. Record this as t_1 in Table 2 for one tile.
5. To minimize the human error component, have the person with the stopwatch place his finger on the air track at the final location of the glider and close his eyes. He will start the time when the person holding the glider says they have released it, and stop it when he feels the glider touch his finger. This should help to reduce the effect of anticipation in starting and stopping the stopwatch.

6. Repeat two more times for this distance and elevation, and calculate the average of the three times measured.

7. Repeat these steps for all of the distances and track elevations listed in Table 2.

	One Tile				Two Tiles			
Δx	t_1	t_2	t_3	$t_{\text{avg 1}}$	t_1	t_2	t_3	$t_{\text{avg 2}}$
30 cm								
40 cm								
50 cm								
60 cm								
70 cm								
80 cm								

Table 2: Time of transit measurements with stopwatch. All time values are in seconds.

4.2.3 Position vs. Time Measurements

1. Open Microsoft Excel and the Lab 3 Air Track file in Logger Pro. Both have shortcut icons on the desktop. The Logger Pro window should have a data table on the left and three graphs on the right: position (m), velocity (m/s), and acceleration (m/s^2) vs. time (s).

2. Place 10 grams on the mass hanger at the end of the track with the rotary sensor and remove all masses from the other hanger. Put 200 g on the glider itself (100 g on each side).

3. Turn on the air supply, and with the air track level, have one partner hold the glider again at the 70 cm mark down from the end of the track with the pulley.

4. Click the green Collect button in Logger Pro to start collecting data, wait for it to turn red, and then release the glider. Click Stop after the hanger hits the floor and the glider has traveled as far as it can.

5. If you can't see the curves in the graphs very well, you can autoscale them by hitting Ctrl+J.

6. Do a few practice runs, and proceed when the curves look smooth enough and you are satisfied with the results.

7. Type "time (s)" in cell A1 and "position (m)" in cell B1 of the Excel spreadsheet.

8. Copy the time and position columns of data in Logger Pro, and paste them into columns A and B in Excel. Your first time and position data points should be in cells A2 and B2.

9. In columns C and D, take the base-10 logarithms of the time and position values, respectively, for each row. This is most easily accomplished by the procedure below.

(a) Click to select cell C2. Type "=LOG10(A2)" and press enter. Don't worry if #NUM! appears in the cell. Your first time value is probably 0 s, and $\log_{10}(0)$ is undefined, but continue anyway, and the remaining values should work.

(b) Select cell C2 again, and click and drag the lower right corner of the cell down to the bottom of your data. For example, if your last data values are in row 20, drag down to cell C20. This automatically applies the equation you entered to the corresponding cells, and all the logarithms of the times should be calculated.

(c) Now select cell D2, and repeat for the position measurements.

You will now make graphs of these two sets of data.

1. First, highlight the time and position data in columns A and B.

2. Click Insert in the menu bar, and then click Chart.

3. Choose XY (Scatter) as the chart type, and proceed through the chart wizard, entering a title and labels for the axes as you go.

4. When you finish, the graph should be added to the spreadsheet, and it should look like the position vs. time graph you saw in Logger Pro.

5. Right click on the data points in the graph, and click to add a trendline.

6. The Linear type should already be selected. Click over to the Options tab, and check the two boxes marked "Display equation on chart" and "Display R-squared value on chart." Click OK.

7. Repeat this procedure to make a graph of log(position) vs. log(time). Note that you should always exclude undefined log values when graphing.

8. Save your work in Excel on a flash drive to be used in answering the analysis questions.

5 Analysis

1. Using the graphing methods described above, plot the position (in meters) vs. average time data you obtained in Table 2, and determine the experimental equation that best describes it. In your lab write up, be sure to include printouts of **all six graphs (three for each data set)** with appropriate titles, properly labeled axes with units, trendlines and linearization equations for each, and R^2 values. For each elevation of the track, you are plotting position vs. time, log(position) vs. log(time), and ln(position) vs. time.

2. In the second part of the experiment with the computer, we are interested only in the glider's motion when it has constant acceleration, since then $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$ applies. We will learn later that this is a consequence of Newton's second law and that constant acceleration arises from a constant force. The data you took, then, most likely has parts that don't satisfy the constant force assumption. Make two new plots—position vs. time and log(position) vs. log(time)—that exclude the part of the data you think isn't useful in testing the validity of the above equation of motion. Include **all four plots** in your write-up.

   (a) Which part(s) of the data are you leaving out and why?

   (b) Compare the R^2 values of your two log-log plots. Which one has the better trendline fit?

x_1	y_1
3	22.5
5	35.5
7	48.5
9	61.5
11	74.5

x_2	y_2
7	2675.4
12	13478.4
14	21403.2
17	38321.4
21	72235.8

x_3	y_3
0	2.4
1	3.95693
2	6.52388
3	10.7561
4	29.238

Table 3: Tables for data sets (x_1, y_1), (x_2, y_2), and (x_3, y_3).

(c) What are the slopes of your trendlines in the log-log plots, and what do you expect them to be from theory?

(d) Based on your results, do you think you have confirmed or contradicted theory?

(e) What sources of error are present in this part of the experiment?

3. Make graphs for the data sets in Table 3 and determine the experimental equations that would produce the data in each case. Here, be sure to include print outs of **all nine of your graphs** with all the information asked for in question 1. Also, pay close attention to which variables should be plotted on which axes.

6 Appendix A: Derivation of Equations of Motion

Mathematics is a powerful tool for describing the behavior of physical systems. An example of this is the use of equations to describe the motion of objects through space. Here we will construct a simple set of examples of equations of motion.

6.1 Equations of Motion

Consider holding a ball in your hand a distance x_0 from the ground and then letting go. The ball falls until it hits the ground and, for our purposes here, let's assume it stops right where it lands. At each moment in time after the ball has been released, up until it hits the ground, the ball is a little bit closer to the ground than it was before. If we were to move to any other time whatsoever in this interval between release and contact, the ball would be in a different place. Thus, the ball is at a unique position for every moment we could choose. At each moment before a given moment it's a little bit further from the ground, and each moment later it's a little bit closer.

The location of the ball along its path from hand to ground is called its position, and can be described using a function we will call $x(t)$. $x(t)$ tells us where between the hand and the ground the ball is at time t.

The first problem we have is how to choose the values for the time t. Since t runs between when the ball is released and when it hits the ground, we are free to choose any set of values that is long enough to cover the interval of time where the ball is in motion. Let's say that the ball takes an *amount* of time t_{ground} to fall to the ground. Then, whatever interval of time we choose should be at least t_{ground} in length with any extra time covering some of when the ball is not moving (either in your hand or on the ground). Since the ball is at a unique position for each moment in time, we only need to choose one value of t and assign it to a known position. Since we know that the ball is at a position x_0 at the instant we let go, we only need to choose a time (value of t) at which this occurred. Let's call this value t_0. Then the time at which the ball hits the ground is $t_0 + t_{ground}$. We call t_0 our *initial time* and x_0 our *initial position.* The equation $x(t_0) = x_0$ defining the value of our function $x(t)$ at time t_0 to be x_0 is called an *initial condition.*

6.1.1 Displacement

Let's pick two moments in time during which our ball is falling, call them t_1 and t_2, and say that moment t_1 comes before moment t_2. The *displacement* of the ball (or any object) is the change in its position between two moments in time. It is written as

$$\Delta x = x_2 - x_1,$$

where $x(t_1) = x_1$ and $x(t_2) = x_2$. At any moment in time t, the displacement of the ball from its initial position is then given by $\Delta x = x(t) - x_0$. However, we still have no idea what the function $x(t)$ looks like. In this lab, it will be our goal to use the data we collect to determine this equation.

6.1.2 Velocity

There are several other physical quantities that will be involved in determining our equations of motion. We already know that as the time changes, the position changes as well. The rate at which position changes with respect to time is called the *velocity.* Let's choose our two points in time t_1 and t_2 for which we know the positions x_1 and x_2 of the ball. We can calculate the ratio

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1},$$

which is called the *average velocity* of the ball between times t_1 and t_2.

As the interval of time between t_1 and t_2 becomes smaller and smaller, our average velocity gets closer and closer to the velocity that the ball has at times t_1 and t_2. In order to obtain the velocity at any given instant in time, we need to let the difference between times t_1 and t_2 go to zero. This gives us the function

$$v(t) = \frac{dx}{dt}$$

for the *instantaneous velocity* of the ball at any given time. Another way to look at instantaneous velocity is the slope of the tangent at a point of interest of a graph of displacement (vertical axis) vs. time (horizontal axis). When the instantaneous velocity is zero, the ball is not moving. As with position, we define the initial velocity, $v(t_0) = v_0$. This is another initial condition similar to the one defined above.

6.1.3 Acceleration

There is no guarantee that the velocity of an object will stay the same during any given motion. Our ball is a great example of this, because when we let it go the ball isn't moving, yet it falls to the ground. This change in velocity is called *acceleration* and has a completely analogous relationship to velocity that velocity does to position.

As with displacement, we can define the difference in velocity between two times as

$$\Delta v = v_2 - v_1,$$

where $v(t_1) = v_1$ and $v(t_2) = v_2$. Using this, we can define the ratio

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1},$$

which gives the *average acceleration* of the ball between times t_1 and t_2. Again, as we let the difference between times t_1 and t_2 go to zero, we obtain the derivative

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2},$$

which gives us the function for the *instantaneous velocity* of the ball (or any other object) at time t. Again we can think of instantaneous acceleration as the slope of the tangent of a graph of velocity vs. time at our instant of interest. As an exercise, we could continue this process, however it turns out that for this lab (and most physical situations), this level of depth is all we need.

6.1.4 Motion with constant acceleration

As we've seen above, our ball is obviously accelerating—it has zero velocity to begin with, but still ends up hitting the ground. If we had some idea about how our ball was accelerating, we could use the quantities we've just defined to make some theoretical prediction for what our equation of motion should look like. The simplest possibility for how our ball (or any object) could be accelerating is to consider that the ball has *constant acceleration*, i.e.

$$\frac{da}{dt} = 0.$$

In this case then, the average acceleration is the instantaneous acceleration, and so we can write

$$\bar{a} = a = \frac{v_2 - v_1}{t_2 - t_1}.$$

Since the acceleration is constant, it doesn't matter what we choose for t_1 or t_2, so let's let $t_1 = t_0$ and $t_2 = t$. But wait! We still haven't fixed t_0 to any particular number! However, as was pointed out at the beginning, we are free to choose whatever number we want for t_0. Looking at this, it would seem to make things easier if we simply let $t_0 = 0$, at which point our equation becomes

$$a = \frac{v(t) - v_0}{t},$$

which we can then solve to get the equation

$$v(t) = at + v_0$$

for $v(t)$, the equation of a line.

For the choices we've made, the average velocity, as we've previously defined it, is given by

$$\bar{v} = \frac{x(t) - x_0}{t},$$

but since the equation for $v(t)$ is the equation of a line, the average velocity at time t is also just

$$\bar{v} = \frac{v(t) + v_0}{2},$$

which we can then set equal to one another. Doing so and substituting in appropriately gives us the equation

$$x(t) = \frac{1}{2}at^2 + v_0t + x_0,$$

which is the equation of motion for an object (in this case a ball) under constant acceleration.