Chapter 9

Rotation of Rigid Bodies

PowerPoint® Lectures for
University Physics, Thirteenth Edition
– Hugh D. Young and Roger A. Freedman

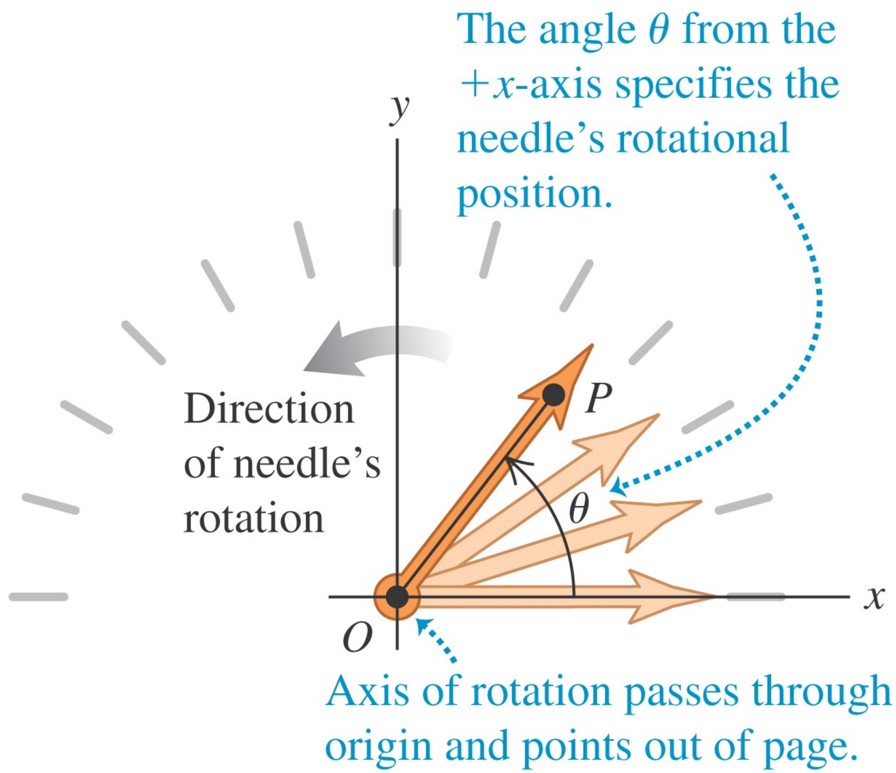
Lectures by Wayne Anderson

Introduction

- So far, we have only considered *translational* motion, changes in an object's displacement $\boldsymbol{x}$.
- If an object is *rigid* (the relative positions of its particles do not change), its motion is fully described by translation and rotation about an axis.
- A wind turbine is an object that rigidly rotates.

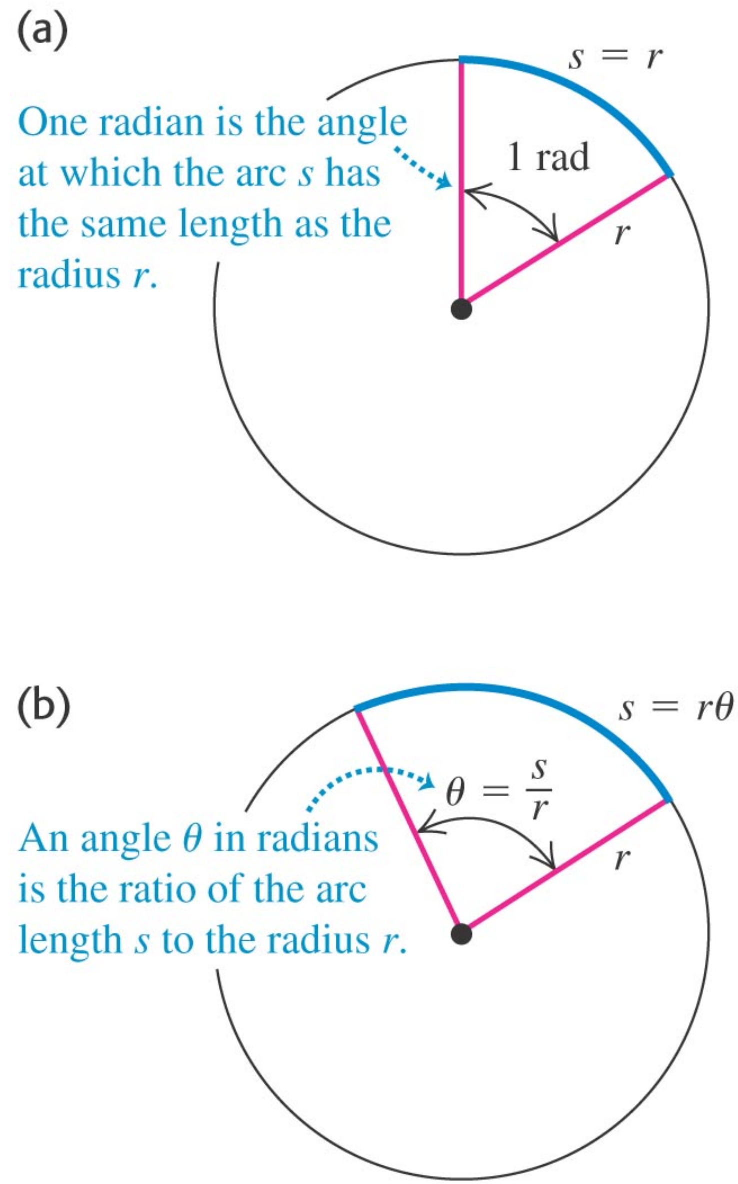
Angular coordinate

- A car's speedometer needle rotates about a *fixed axis*, as shown at the right.
- The angle θ that the needle makes with the $+x$-axis is a *coordinate* for rotation.
- This angular coordinate is dimensionless.



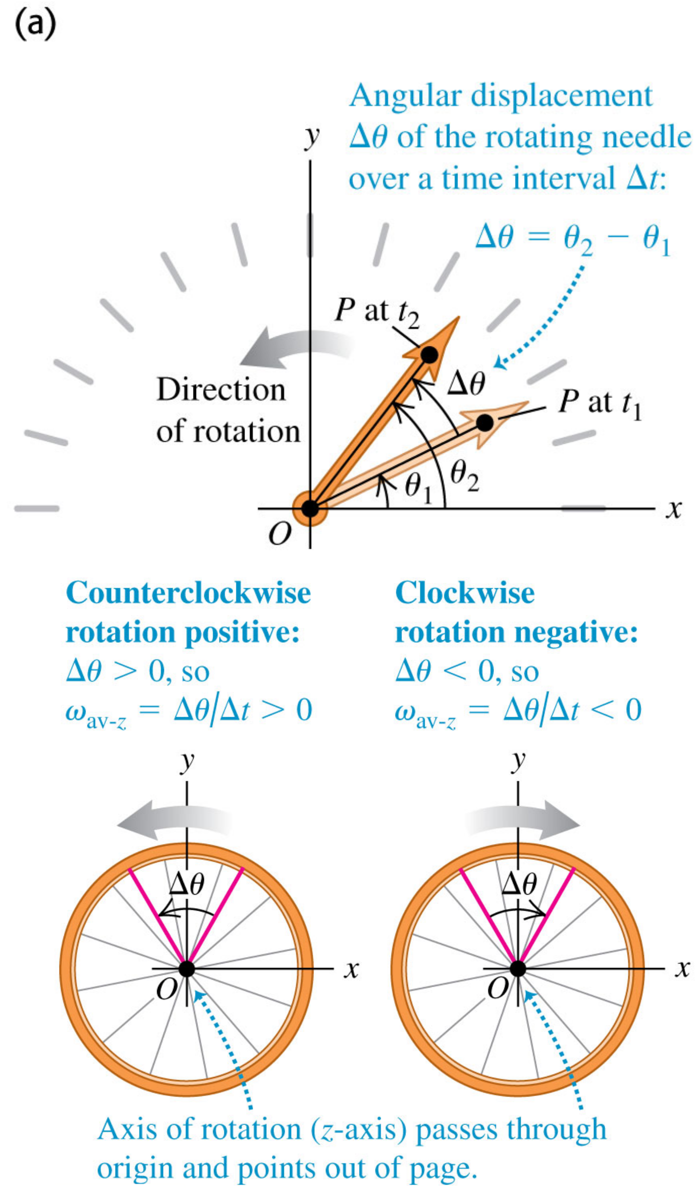
Units of angles

- One *radian* is the angle which subtends a length of arc s equal to the circle's radius r.
- If θ is measured in radians, the length of arc s subtended by θ is $s = \theta r$.
- For the full circle, $s = 2\pi r$ which implies the full circle has $\theta = s/r = 2\pi$ radians.
- One radian = 360°/2π ≃ 57.3°.



Angular velocity

- The *angular displacement* $\Delta\theta$ of a body is $\Delta\theta = \theta_2 - \theta_1$.
- The *average angular velocity* of a body is $\omega_{\text{av-}z} = \Delta\theta/\Delta t$.
- The subscript z means that the rotation is about the z-axis.
- The *instantaneous angular velocity is* $\omega_z = d\theta/dt$.
- This is analogous to earlier definitions of displacement x, average velocity $v_{av\text{-}x}$, and instantaneous velocity $v = dx/dt$.
- A counterclockwise rotation is positive; a clockwise rotation is negative.



Example #1

- An airplane propeller is rotating at 1900 rpm (rev/min).

a. Compute the propeller's angular velocity in rad/s.

b. How many seconds does it take the propeller to turn through 35°?

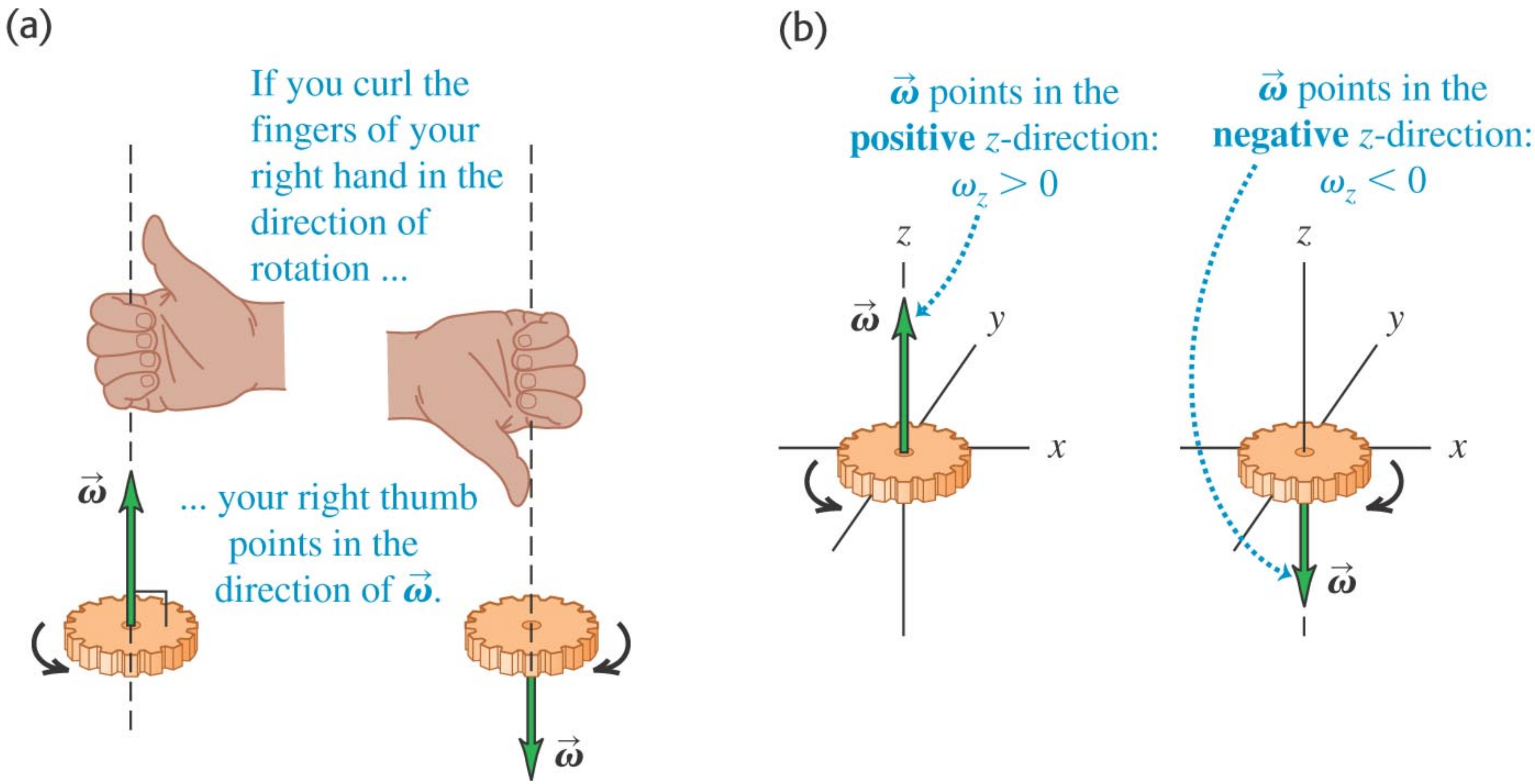
Solution #1

a. ω_z = 1900 rpm (1 min/60 s)(2π rad/1 rev) = 199 rad/s

b. $\Delta t = \Delta\theta/\omega_z$ = (35°)(π rad/180°)/(199 rad/s) = 3.07 ms

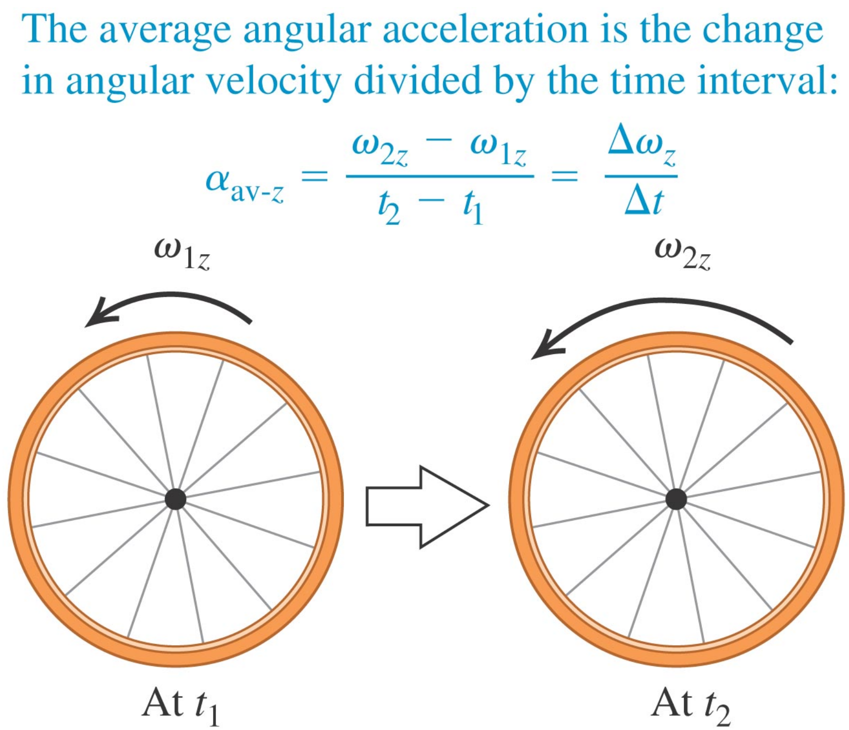
Angular velocity is a vector

- Angular velocity is defined as a vector whose direction is given by the right-hand rule:



Angular acceleration

- The *average angular acceleration* is $\alpha_{\text{av-}z} = \Delta\omega_z/\Delta t$.
- The *instantaneous angular acceleration is* $\alpha_z = d\omega_z/dt = d^2\theta/dt^2$.



Example #2

A fan blade rotates with angular velocity $\omega_z = \gamma - \beta t^2$ where $\gamma = 5.00$ rad/s and $\beta = 0.800$ rad/s^3.

a. Calculate the angular acceleration as a function of time.

b. Calculate the instantaneous angular acceleration α_z at $t = 3.00$ s and the average angular acceleration $\alpha_{av\text{-}z}$ for the time interval $t = 0$ to $t = 3.00$ s. How do these quantities compare?

Solution #2

a. $\alpha_z = d\omega_z/dt = -2\beta t$

b. At t = 3.00 s, $\alpha_z = -2(0.800 \text{ rad/s}^3)(3.00 \text{ s}) = -4.80 \text{ rad/s}^2$

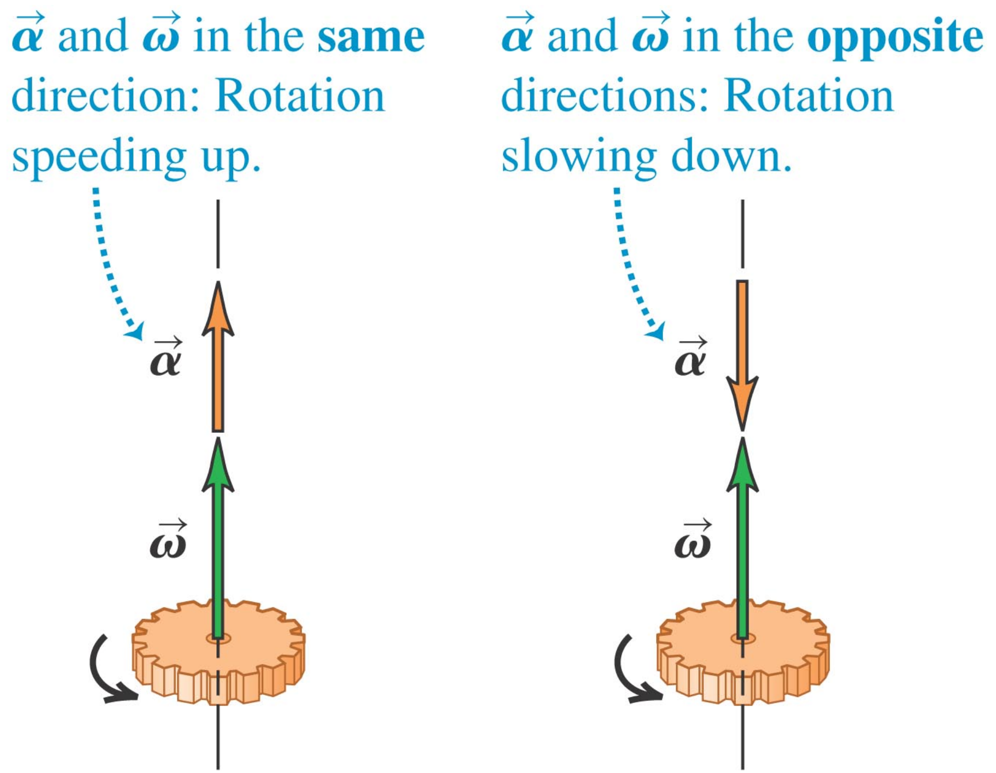
$\alpha_{av\text{-}z} = \Delta\omega_z/\Delta t = [\omega_z(3 \text{ s}) - \omega_z(0 \text{ s})]/(3 \text{ s})$

$= [-2.20 \text{ rad/s} - 5.00 \text{ rad/s}]/(3.00 \text{ s}) = -2.4 \text{ rad/s}^2$

The average angular acceleration has a smaller magnitude because the instantaneous angular acceleration is negative and decreasing over the interval.

Angular acceleration as a vector

- For a fixed rotation axis, the angular acceleration and angular velocity vectors both lie along that axis.
- Angular velocity increases in magnitude when aligned with angular acceleration, just like with linear velocities.



Rotation with constant angular acceleration

- The rotational formulas have the same form as the straight-line formulas, as shown in Table 9.1 below.

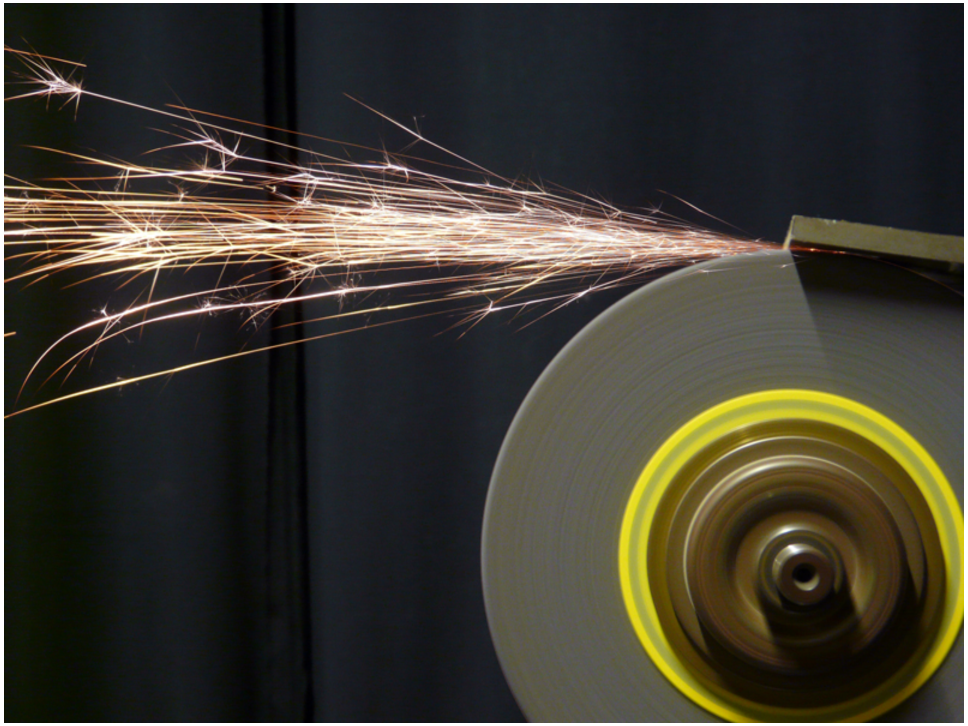
Table 9.1 Comparison of Linear and Angular Motion with Constant Acceleration

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_x + v_{0x})t$	$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t$

Example #3

- At $t = 0$, a grinding wheel has an angular velocity of 24.0 rad/s. It has a constant angular acceleration of 30.0 rad/s^2 until a circuit breaker trips at $t = 2.00$ s. From then on, it turns through 432 rad as it coasts to a stop at constant angular acceleration.

a. Through what total angle did the wheel turn between $t = 0$ and the time it stopped?



b. At what time did it stop?

c. What was its acceleration as it slowed down?

Solution #3

a. The wheel's motion during the first 2 s was:

$\theta(t) = \theta_0 + \omega_0 t + ½\alpha t^2 = 24.0t$ rad/s + $15.0t^2$ rad/s^2

$\theta(2.00\text{ s})$ = 24.0(2 s) rad/s + 15.0 (2 s)^2 rad/s^2 = 108 rad

108 rad + 432 rad = 540 rad

b. $\omega(t) = \omega_0 + \alpha t$ = 24.0 rad/s + $30.0t$ rad/s^2

$\omega(2.00\text{ s})$ = 84.0 rad/s

After t_0 = 2 s: $\theta(t) = \theta_0 + ½(\omega_0 + \omega)(t - t_0)$

$t = t_0 + 2(\theta - \theta_0)/(\omega_0 + \omega)$ = 2 s + 2(432 rad)/(84.0 rad/s + 0) = 12.3 s

c. $\alpha = \Delta\omega/\Delta t$ = (0 – 84.0 rad/s)/(10.3 s) = -8.16 s

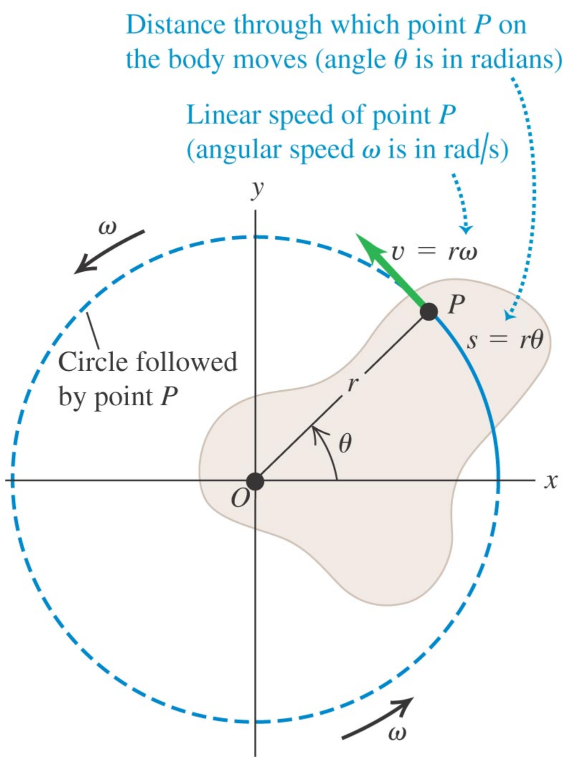
Relating linear and angular kinematics

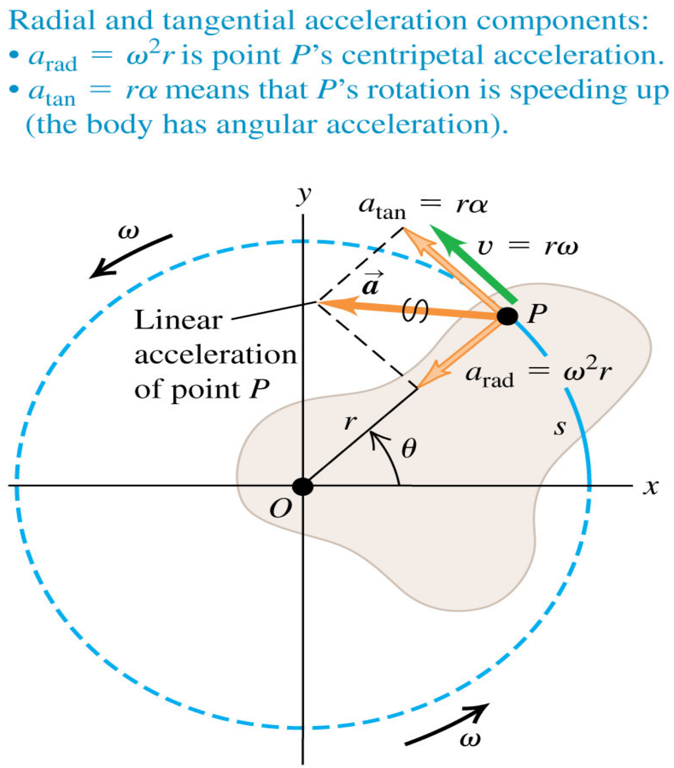
- For a point a distance r from the axis of rotation:

  its linear speed is $v = r\omega$

  its tangential acceleration is $a_{tan} = r\alpha$

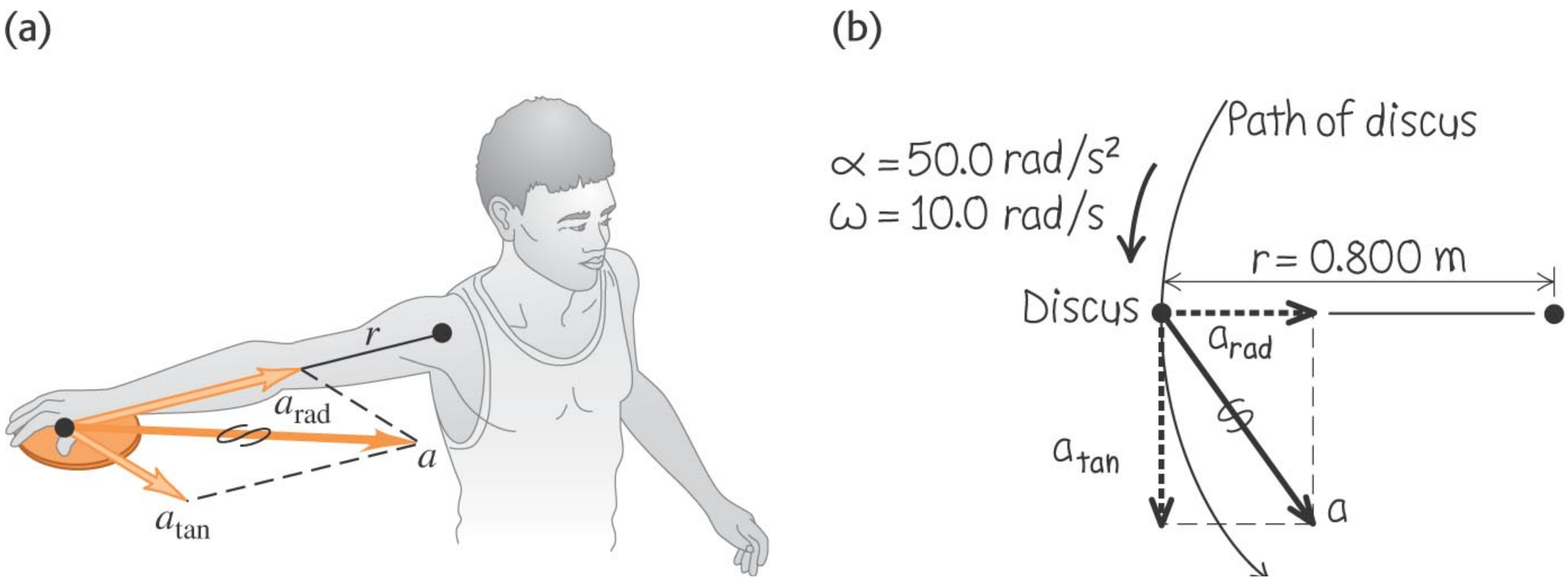
  its centripetal (radial) acceleration is $a_{rad} = v^2/r = r\omega^2$





An athlete throwing a discus

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. For this instant, find the tangential and centripetal components of the acceleration of the discus and its magnitude and direction.



Solution

The tangential acceleration is:

$a_{tan} = \alpha r = (50.0\ \text{rad/s}^2)(0.80\ \text{m}) = 40.0\ \text{m/s}^2$

The radial acceleration is:

$a_{rad} = \omega^2 r = (10.0\ \text{rad/s})^2(0.80\ \text{m}) = 80.0\ \text{m/s}^2$

The magnitude of the acceleration is:

$a = (a_{tan}{}^2 + a_{rad}{}^2)^{1/2} = [(40\ \text{m/s}^2)^2 + (80\ \text{m/s}^2)^2]^{1/2} = 89.4\ \text{m/s}^2$

Its direction is inwards from the tangent by an angle:

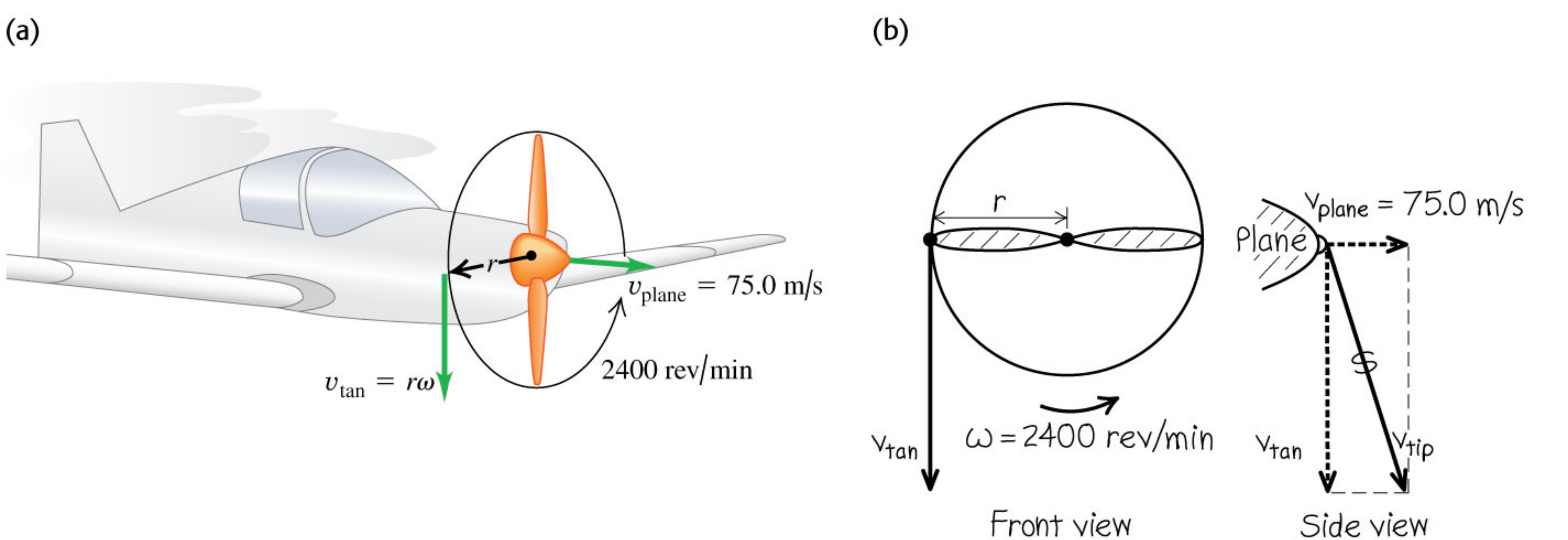
$\Theta = \tan^{-1}(a_{rad}/a_{tan}) = \tan^{-1}(80\ \text{m/s}^2/40\ \text{m/s}^2) = 63.4°$

Designing a propeller

You are designing an airplane that is to turn at 2400 rpm. The forward airspeed of the plane is to be 75.0 m/s and the speed of the propeller tips through the air must not exceed 270 m/s.

a. What is the maximum possible propeller radius?

b. With this radius, what is the acceleration of the propeller tip?



Solution

a. The propeller tip's speed is the magnitude of its velocity, which has components along the plane's flight and tangent to the propeller's motion:

$$v_{max} = (v_{plane}{}^2 + \omega^2 r^2)^{1/2} \Longrightarrow$$

$$r = (v_{max}{}^2 - v_{plane}{}^2)^{1/2}/\omega$$

$$= [(270\text{ m/s})^2 - (75\text{ m/s})^2]^{1/2}/[(2400\text{ rpm})(2\pi\text{ rad/rev})(1\text{ min}/60\text{ s})]$$

$$= 1.03\text{ m}$$

b. $a = a_{rad} = \omega^2 r = [(2400\text{ rpm})(2\pi\text{ rad/rev})(1\text{ min}/60\text{ s})]^2(1.03\text{ m})$

$$= 6.51 \times 10^4\text{ m/s}^2$$

Rotational kinetic energy

- If particles with masses m_i are located at distances r_i from an axis about which they are rotating with angular speed ω, their linear speeds are $v_i = \omega r_i$ and their kinetic energy is:

$$K = \tfrac{1}{2}\Sigma m_i v_i^2 = \tfrac{1}{2}\Sigma m_i \omega^2 r_i^2 = \tfrac{1}{2} I\omega^2$$

where the *moment of inertia* I is

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots = \Sigma m_i r_i^2$$

- The SI units of moment of inertia are kg • m^2.
- The moment of inertia depends on the arrangement of particles and the axis of rotation, but *not* the angular speed ω.

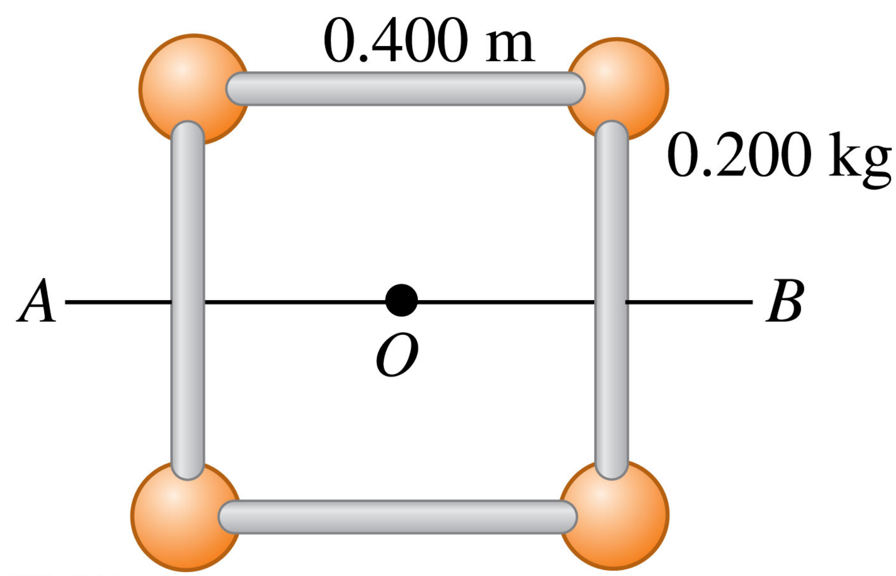
Example #6

Four small spheres, each of which you can regard as a point of mass 0.200 kg, are arranged in a square 0.400 m on a side and connected by extremely light rods. Find the moment of inertia about the following axes:

a. through the center of the square, perpendicular to the plane

b. bisecting 2 opposite sides of the square

c. that passes through the diagonal from upper left to lower right



Solution #6

a. Each of the 4 masses is a distance of $(\sqrt{2})(0.2\text{ m})$ from point O, so $I = \Sigma m_i r_i^2 = 4(0.2\text{ kg})2(0.2\text{ m})^2 = 0.064\text{ kg} \bullet \text{m}^2$.

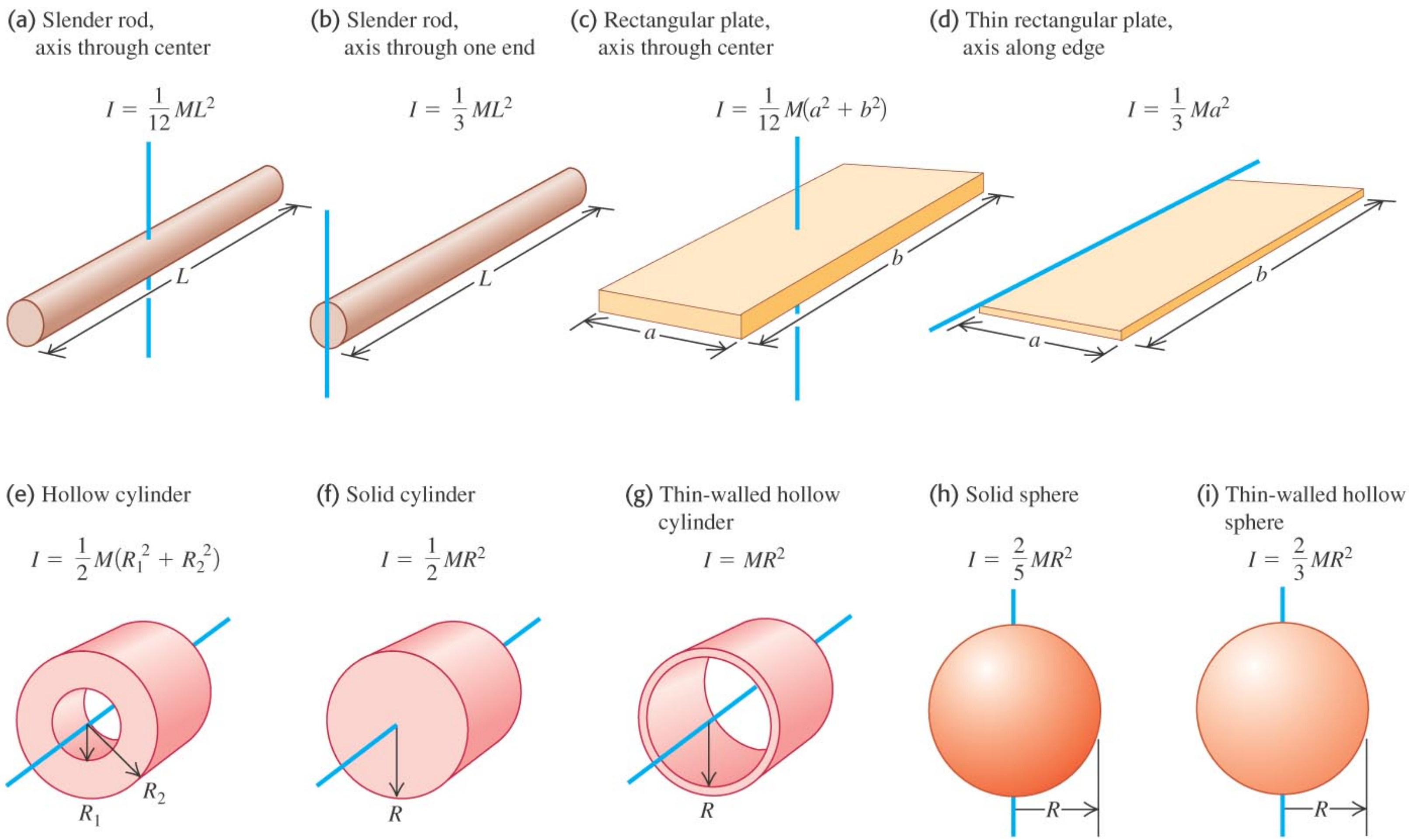
b. Each of the 4 masses is a distance of 0.2 m from this axis, so $I = \Sigma m_i r_i^2 = 4(0.2\text{ kg})(0.2\text{ m})^2 = 0.032\text{ kg} \bullet \text{m}^2$.

c. The 2 masses on this diagonal axis do not contribute to the moment of inertia, while the other 2 masses are a distance of $(\sqrt{2})(0.2\text{ m})$ from the axis, so:

$$I = \Sigma m_i r_i^2 = 2(0.2\text{ kg})2(0.2\text{ m})^2 = 0.064\text{ kg} \bullet \text{m}^2.$$

Moments of inertia of some common bodies

• Table 9.2 in Young and Friedman gives the moments of inertia of various bodies:

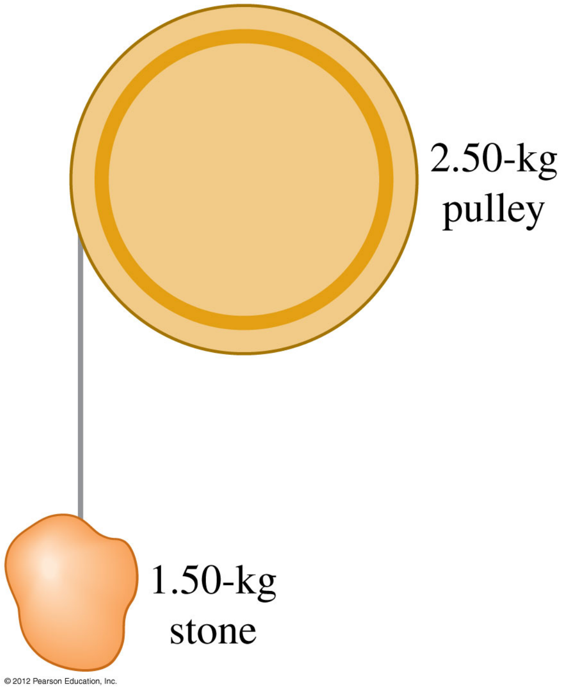


Example #7

A frictionless pulley has the shape of a uniform solid disk of mass 2.50 kg and radius 20.0 cm. A 1.50 kg stone is attached to a very light wire that is wrapped around the rim of the pulley, and the system is released from rest.

a. How far must the stone fall so that the pulley has 4.50 J of kinetic energy?

b. What percentage of the total kinetic energy does the pulley have?



Solution #7

Since there are no non-conservative external forces on the pulley-stone system, the mechanical energy is conserved. Gravitational potential energy U_g of the falling stone is converted into kinetic energy K_s of the stone and K_p pulley:

$$\Delta E = \Delta U_g + \Delta K_s + \Delta K_p = 0$$

$$-m_s g \Delta y = \tfrac{1}{2} m_s(\omega r)^2 + \tfrac{1}{2} I_p \omega^2 = \tfrac{1}{2}(\omega r)^2(m_s + \tfrac{1}{2} m_p)$$

$$= K_p(1 + 2m_s/m_p)$$

$$\Delta y = -K_p(m_s^{-1} + 2m_p^{-1})/g = -(4.5\ \text{J})[(1.5\ \text{kg})^{-1} + (2.5\ \text{kg})^{-1}]/(9.8\ \text{m/s}^2)$$

$$= -0.490\ \text{m}$$

$$K_p/(K_s + K_p) = (1 + 2m_s/m_p)^{-1} = 0.455$$

Gravitational potential energy of an extended body

- The gravitational potential energy of an extended body of particles m_i located at positions $\boldsymbol{r}_i$ is $U_g = \Sigma m_i g y_i = M g y_{cm}$.
- This is the same as that of a single particle of total mass M located at the extended body's center of mass.
- The pole vaulter below can leap over the bar without her center of mass ever going above it, reducing the required jump height.

© 2012 Pearson Education, Inc.

ConcepTest

A solid disk and a ring roll down an incline. The ring is slower than the disk if:

A. $m_{ring} = m_{disk}$

B. $r_{ring} = r_{disk}$

C. $m_{ring} = m_{disk}$ and $r_{ring} = r_{disk}$

D. The ring is always slower regardless of the relative values of m and r.

Solution

Gravitational potential energy is converted into kinetic energy. A rolling object has linear speed v and angular speed ω related by $v = \omega r$. Falling down a height h:

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}(1+\beta)mv^2 \Longrightarrow v = [2gh/(1+\beta)]^{1/2}$$

where $I = \beta mr^2$. We see that both m and r drop out of this expression for v. Since $\beta = 1$ for the ring and $\beta = ½$ for a solid disk, the ring is always slower.

The parallel-axis theorem

- If we know an object's moment of inertia I_{cm} about an axis passing through its center of mass, a simple formula gives its moment of inertia I_P about a parallel axis passing through any point P.
- If the origin is at the center of mass and P is located at (a, b):

$$I_P = \Sigma m_i[(x_i - a)^2 + (y_i - b)^2]$$

$$= \Sigma m_i[(x_i^2 + y_i^2) - 2(ax_i + by_i) + (a^2 + b^2)]$$

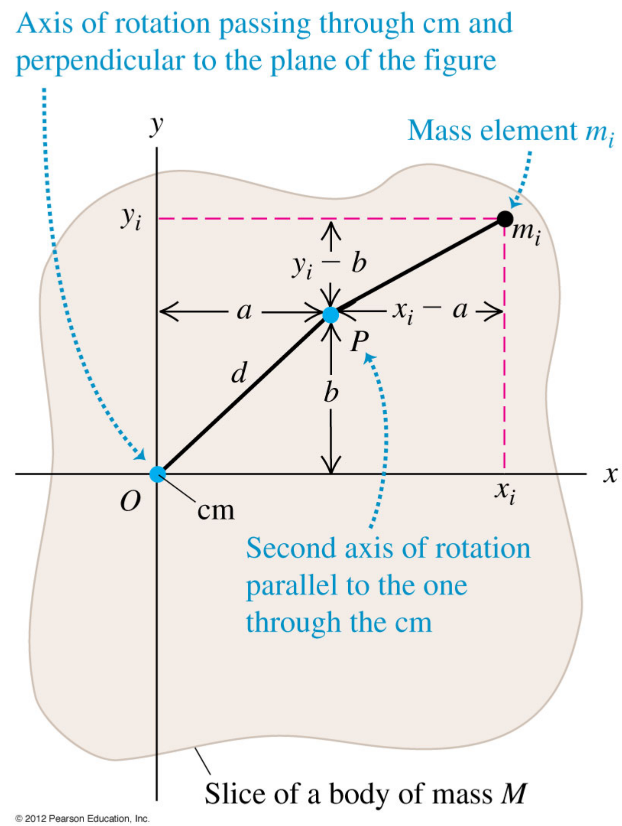
$$= I_{cm} + Md^2$$

M is the total mass and d is the

Distance between the origin and P.

The second term vanishes because

the center of mass is at the origin.



Example #8

A thin uniform rod of mass M and length L is bent at its center so that the two segments are now perpendicular to each other. Find its moment of inertia about an axis perpendicular to its plane and passing through:

a. the point where the two segments meet

b. the midpoint of the line connecting its two ends

Solution #8

a. Each segment has a mass ½M and length ½L and has a center of mass at its own center. About each center of mass, $I_{cm} = (½M)(½L)^2/12 = ML^2/96$. The point where the segments meet is a distance ¼L from each one, so by the parallel-axis theorem $I_P = ML^2/96 + (½M)(¼L)^2 = ML^2/24$. Since there are 2 segments, the total moment of inertia is:

$$I = ML^2/12$$

This agrees with our intuition that bending the rod doesn't change the distance of points from its center.

b. The midpoint of the line connecting the two ends is also a distance ½L from each center of mass, so again:

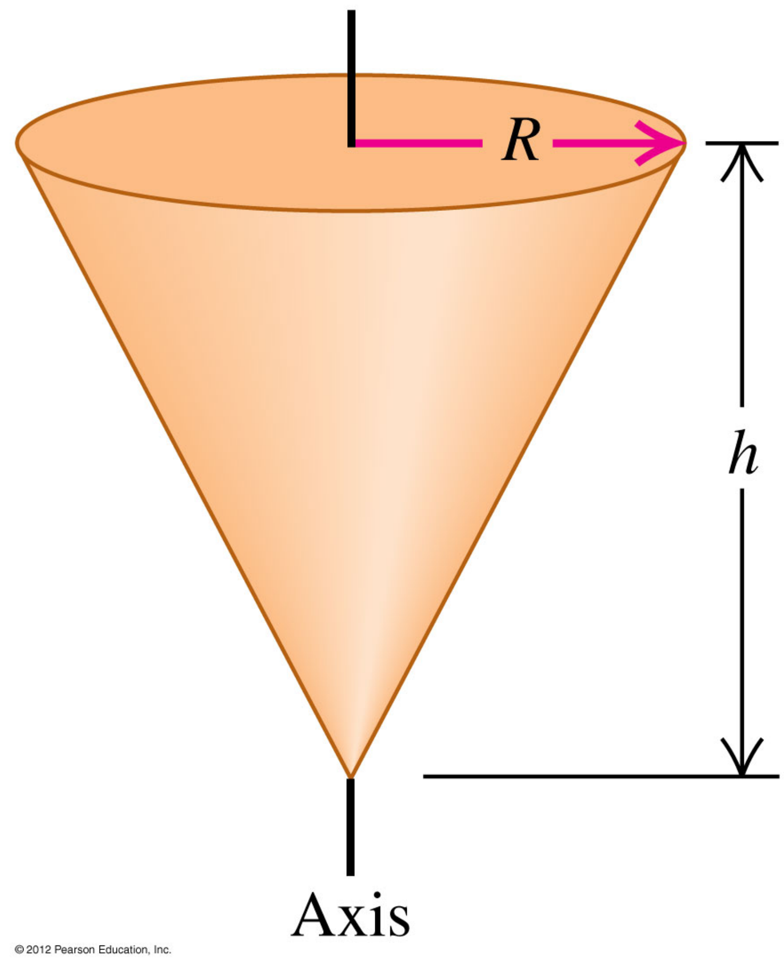
$$I = ML^2/12.$$

Moment of inertia of a solid

- For an object composed of discrete particles, the moment of inertia is $I = \Sigma m_i r_i^2$. If the object is a continuous solid, this expression becomes a volume integral $I = \int r^2 \rho(\mathbf{r}) d^3\mathbf{r}$.
- If the object is highly symmetrical, this volume integral can sometimes be expressed in terms of a single dimensional integral $I = \int dI$

Example #9

Calculate the moment of inertia of a uniform solid cone about an axis through its center. The cone has mass M and altitude h. The radius of its circular base is R.



Solution #9

A cone has volume $V = ⅓\pi R^2h$ implying a density $\rho = M/V = 3M/(\pi R^2h)$. If we slice the cone along the z axis with the origin at its tip, the radius of each section will be $r = zR/h$. Each slice will be a solid disk with moment of inertia $dI = ½r^2dM$ where $dM = \rho\, dV = \rho\pi r^2\, dz = (3Mz^2/h^3)\, dz$ implying that

$$dI = (3/2)(Mz^4R^2/h^5)\, dz$$

Integrating from $z = 0$ to $z = h$, we find:

$$I = (3/10)MR^2$$

This is independent of h as the moment of inertia only depends on the perpendicular distance from the rotation axis.