Chapter 14

Periodic Motion

PowerPoint® Lectures for
University Physics, Thirteenth Edition
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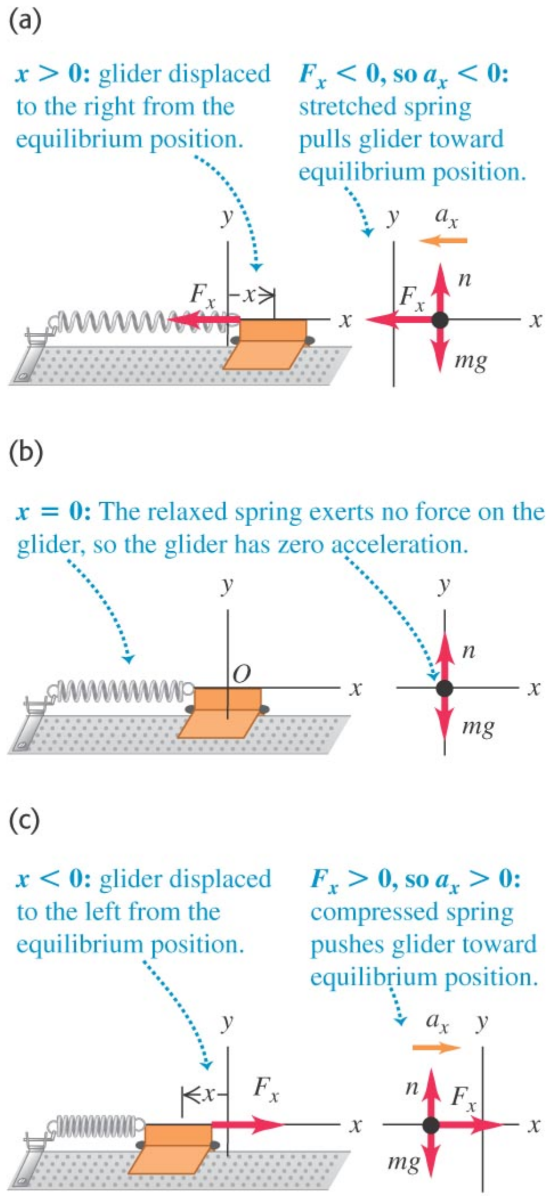
Introduction

- Our focus so far has been on *unbound* motion: objects that could translate or rotate over an infinite range of displacements and angles.
- In this chapter, we will consider instead *bound* motion: finite oscillations about an equilibrium position in which an object could remain at rest.
- Such oscillations (such as a pendulum, musical vibrations, and pistons in car engines) repeat themselves and are called *periodic motion*.

What causes periodic motion?

- The spring exerts a *restoring force* which tends to restore the object to its equilibrium position.
- This force causes *oscillation* of the system, or *periodic motion* if the system is undamped (no energy loss to friction).



Characteristics of periodic motion

- The *amplitude*, A, is the maximum magnitude of displacement from equilibrium.
- The *period*, T, is the time for one cycle.
- The *frequency*, f, is the number of cycles per unit time.
- The *angular frequency*, ω, is 2π times the frequency: $\omega = 2\pi f$.
- The frequency and period are reciprocals of each other: $f = 1/T$ and $T = 1/f$.

Example #1

If an object on a horizontal, frictionless surface is attached to a spring, displaced, and then released, it will oscillate. If it is displaced 0.120 m from its equilibrium position and released with zero initial speed, then after 0.800 s its displacement is found to be 0.120 m on the opposite side, and it has passed the equilibrium position once during this interval. Find

a. the amplitude

b. the period

c. the angular frequency

Solution #1

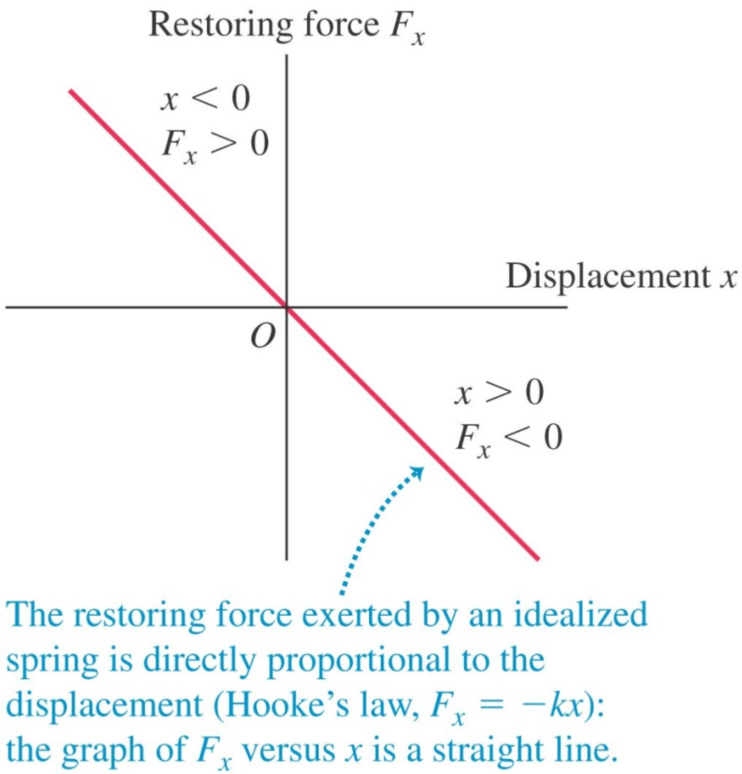
a. The amplitude is equal to the maximum displacement, A = 0.120 m.

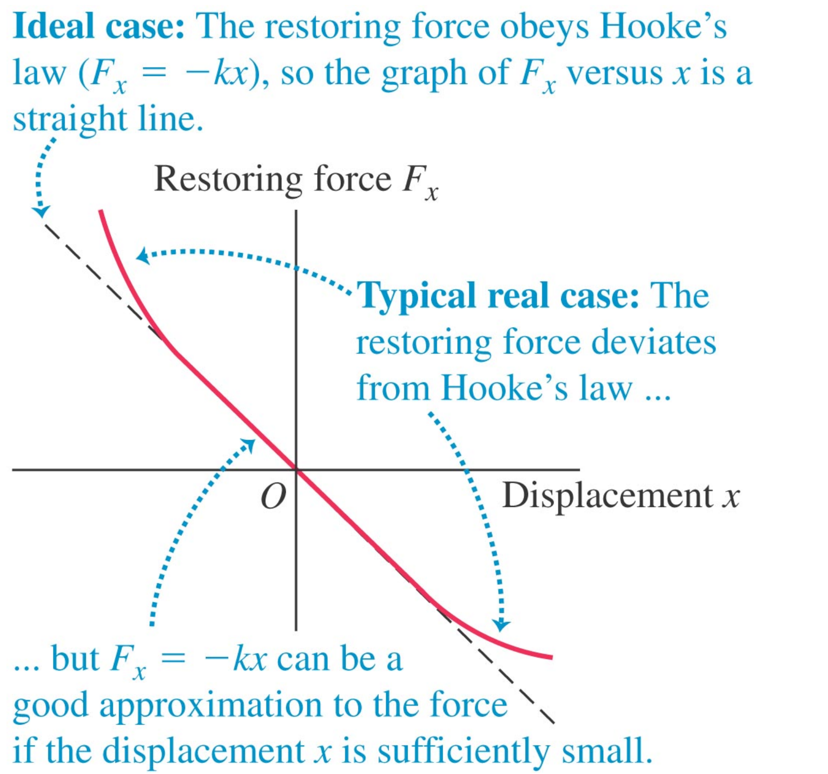
b. The period is the time to return to the initial displacement: T = 2(0.800 s) = 1.60 s.

c. The angular frequency is $\omega = 2\pi/T = 2\pi/(1.60\text{ s}) = 3.93$ rad/s.

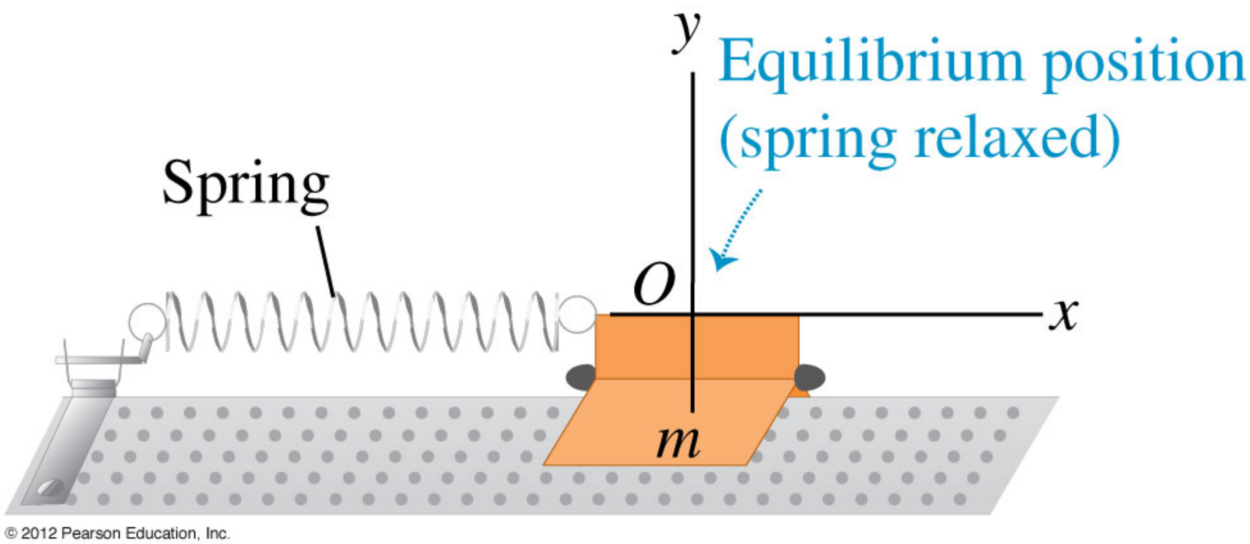
Simple harmonic motion (SHM)

- When the restoring force is *directly proportional* to the displacement from equilibrium, the resulting motion is called *simple harmonic motion* (SHM).
- An ideal spring obeys Hooke's law, so the restoring force is $F_x = -kx$, which results in simple harmonic motion.





Horizontal SHM



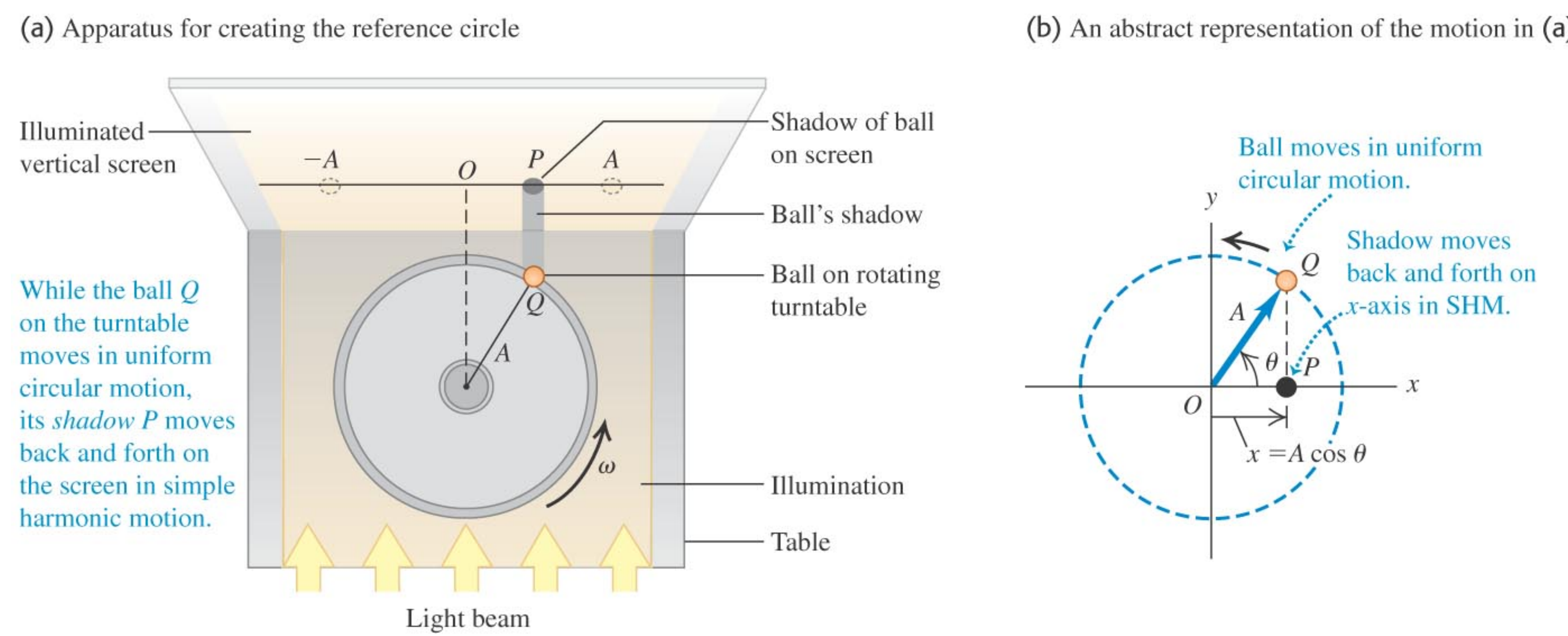
- A mass *m* attached to a spring with force constant *k* with displacement *x* from equilibrium experiences an acceleration:

$$a_x = d^2x/dt^2 = F_x/m = -(k/m)x$$

- This acceleration is proportional to the displacement. What is the motion *x(t)*, the solution to this 2^{nd}-order ordinary differential equation?

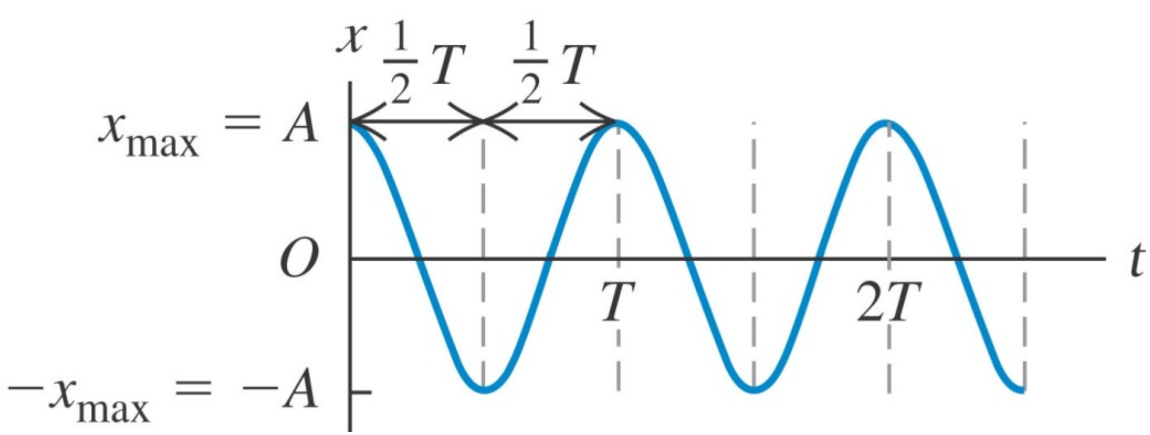
Simple harmonic motion viewed as a projection

- The ball Q moves in a circle in the xy plane with angle $\theta = \omega t + \phi$. The x component of the motion is $x = A \cos \theta$. Since the ball is in uniform circular motion, its acceleration is $\boldsymbol{a} = -\omega^2 \boldsymbol{r}$ with x component $a_x = -\omega^2 x = -\omega^2 A \cos(\omega t + \phi)$. This is identical to the equation for SHM with $\omega = \sqrt{(k/m)}$!
- The vector $\boldsymbol{A}$ is called a phasor and rotates with angular speed ω.



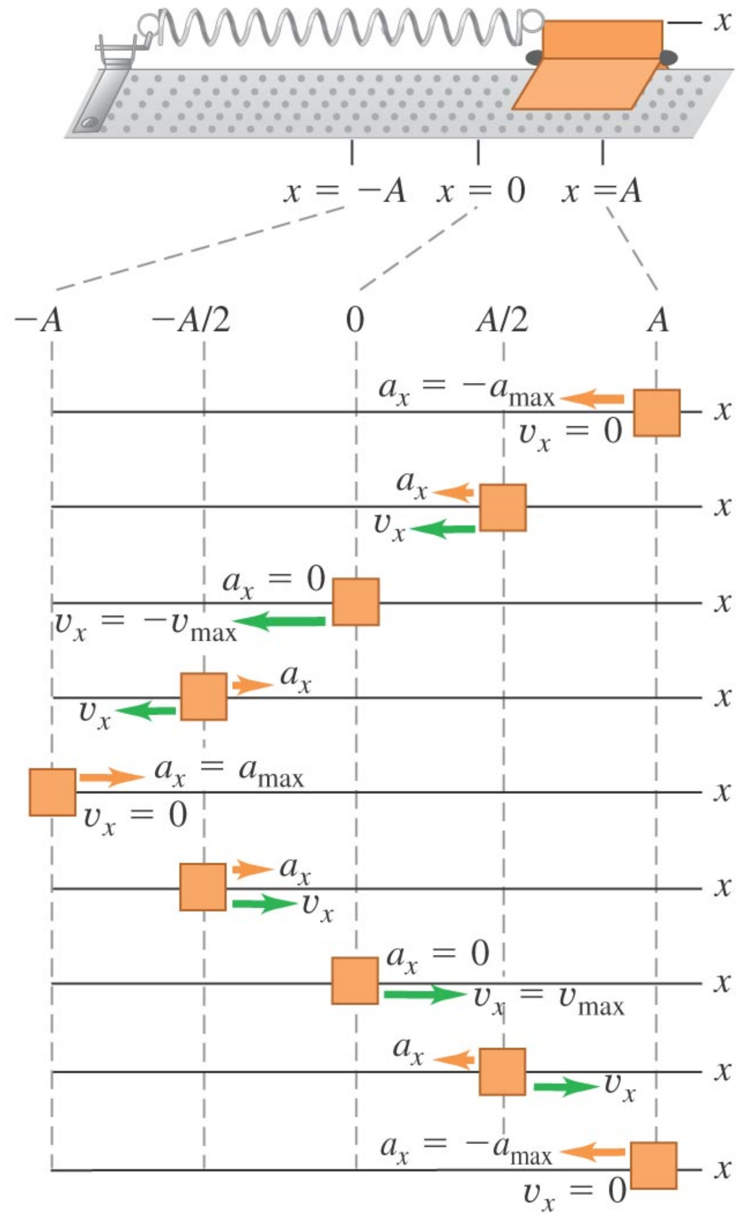
Displacement as a function of time in SHM

- SHM is described by the equation $x = A\cos(\omega t + \phi)$.
- It is periodic with amplitude A, angular frequency $\omega = \sqrt{(k/m)}$, frequency $f = \omega/(2\pi) = (2\pi)^{-1}\sqrt{(k/m)}$, and period $T = f^{-1} = (2\pi)\sqrt{(m/k)}$.
- The *phase* ϕ determines the time at which the oscillator reaches $x_{max} = +A$.



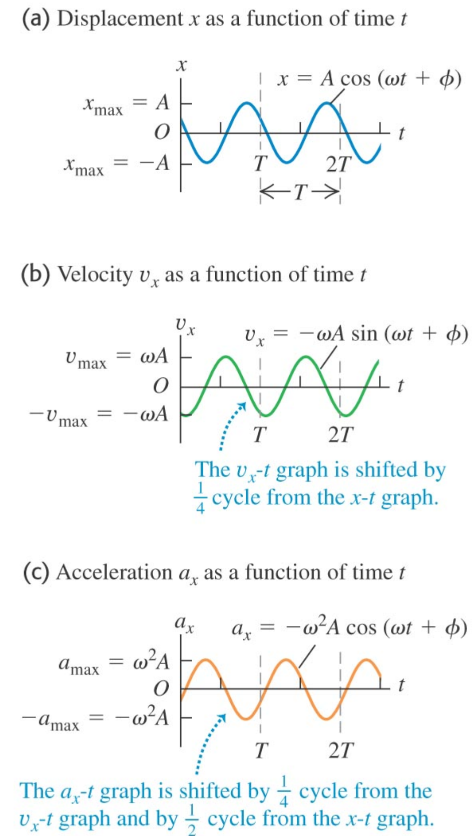
Behavior of v_x and a_x during one cycle

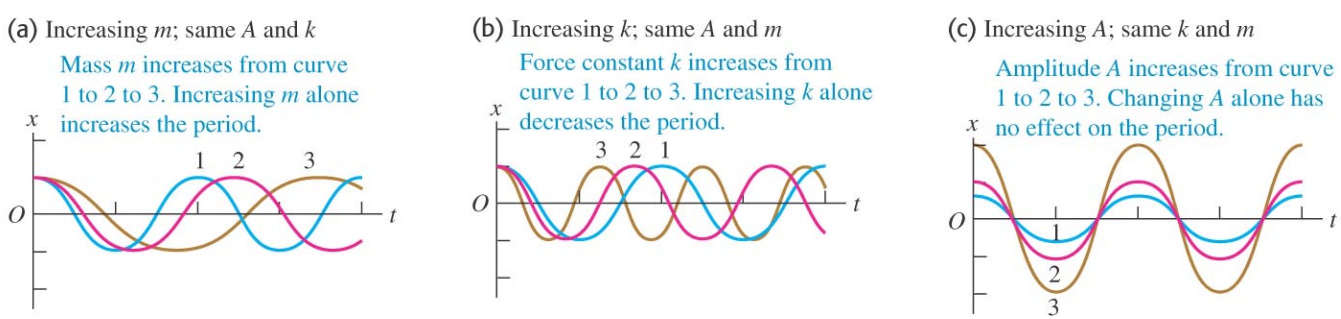
- The figure at right shows how v_x and a_x vary during one cycle.

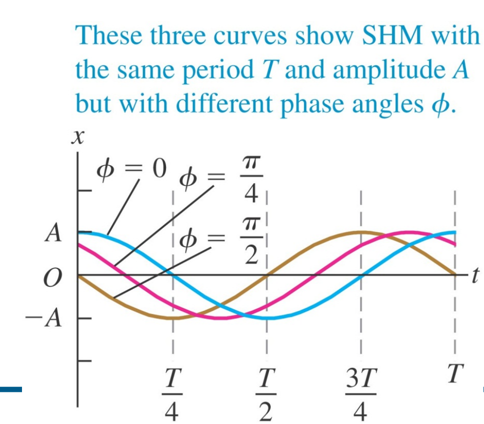


Graphs of displacement, velocity, and acceleration

- Changing *m* and *k* change the period $T = (2\pi)\sqrt{(m/k)}$.
- The period is independent of *A*, the defining characteristic of SHM.
- The initial conditions (ICs) determine the phase through $x = A\cos(\omega t + \phi)$.







Example #2

An object is undergoing SHM with period 1.200 s and amplitude 0.600 m. At t = 0 the object is at x = 0 and is moving in the negative x direction. How far is the object from the equilibrium position when t = 0.480 s?

Solution #2

The equation for SHM is $x = A\cos(\omega t + \phi)$. If the mass is at $x = 0$ at $t = 0$, we know $0 = A\cos\phi$ which implies that $\phi = \pm\pi/2$. The velocity is $dx/dt = -\omega A\sin(\omega t + \phi)$, implying that at $t = 0$ we have $dx/dt = -\omega A\sin\phi$. For this to be negative, $\phi = +\pi/2$:

$x = A\cos(\omega t + \pi/2) = -A\sin\omega t = -(0.6\text{ m})\sin[(2\pi)(t/1.2\text{ s})]$

after inserting the given amplitude and period. Evaluating:

$x(0.48\text{ s}) = = -(0.6\text{ m})\sin[(2\pi)(0.48\text{ s}/1.2\text{ s})] = -0.353\text{ m}$

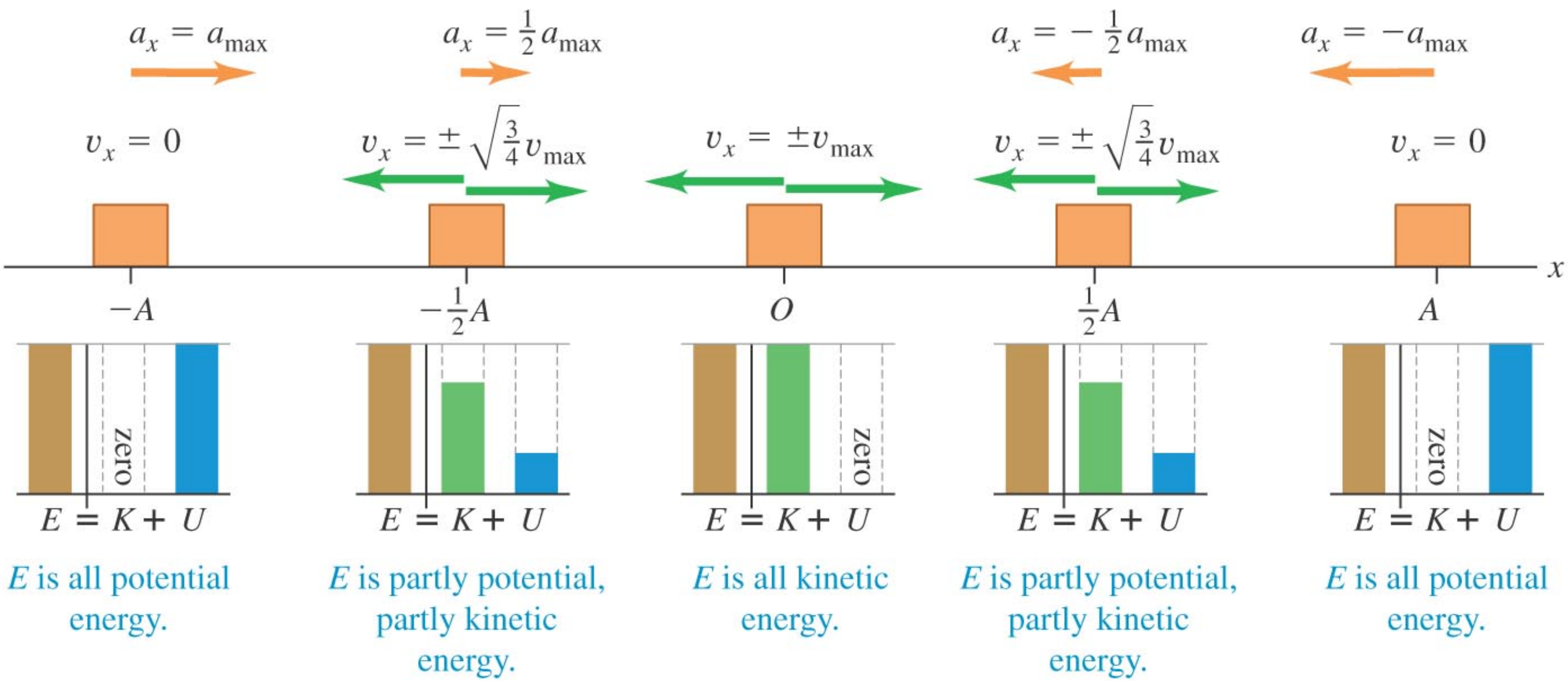
This is between $T/4$ and $T/2$ after $t = 0$, implying that the mass is returning towards the equilibrium position.

Energy in SHM

- Since Hooke's law is a conservative force, the total mechanical energy $E = K + U$ is conserved in SHM:

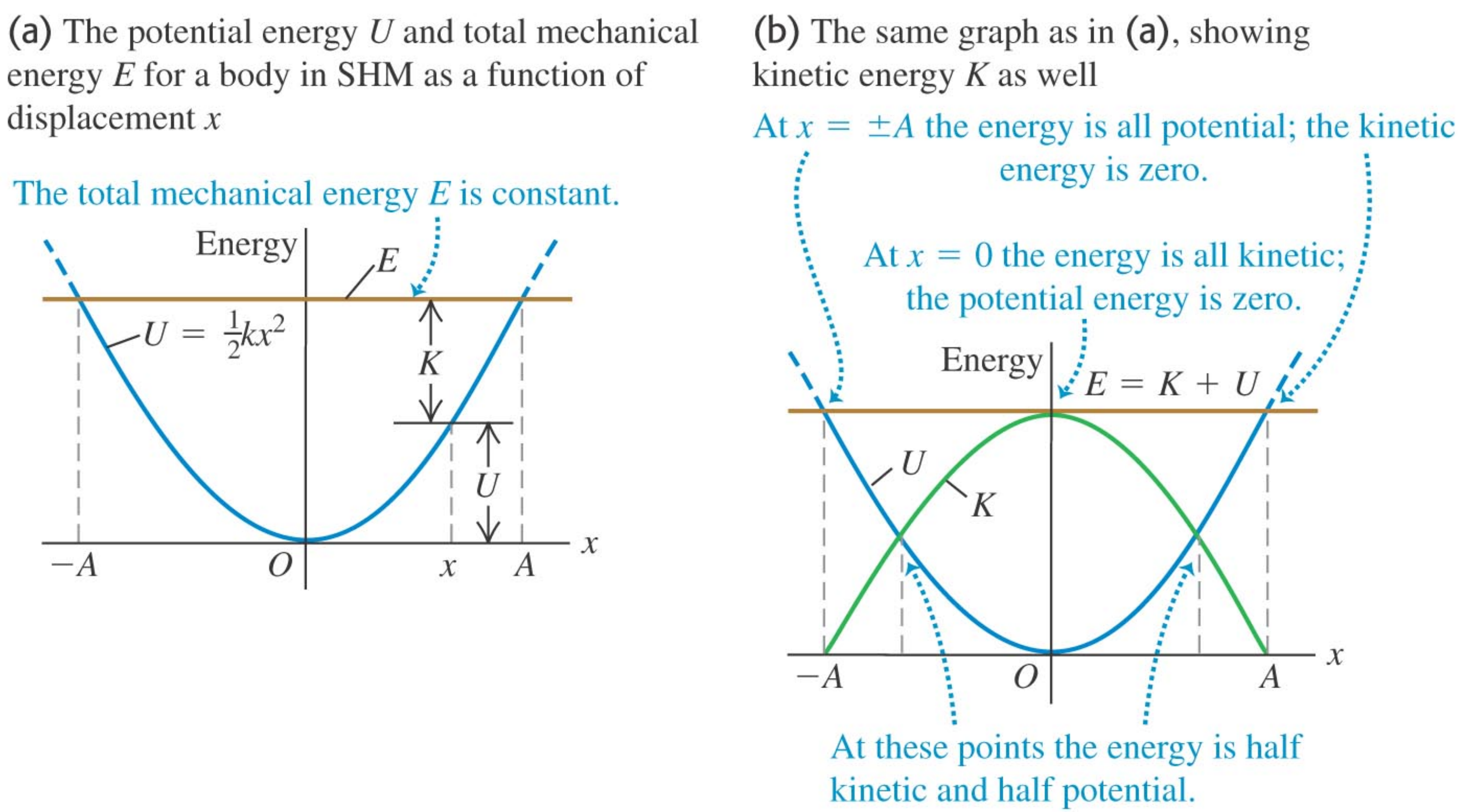
$$x = A\cos(\omega t + \phi),\ v_x = -\omega A\sin(\omega t + \phi)$$

$$E = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2 = \text{constant}$$



Energy diagrams for SHM

- The conserved energy E determines the turning points $\pm A$ at which $U(\pm A) = E$, $K = 0$.
- $U = \frac{1}{2}kx^2$, $K = \frac{1}{2}mv^2 = E - U = E - \frac{1}{2}kx^2$.



Example #3

A block with mass m = 0.300 kg is attached to one end of an ideal spring and moves on a horizontal frictionless surface. The other end of the spring is attached to a wall. When the block is at x = +0.240 m, its acceleration is a_x = -12.0 m/s^2 and its velocity is v_x = +4.00 m/s. What are

a. the spring's force constant k

b. the amplitude of the motion

c. the maximum speed of the block during its motion

d. the maximum magnitude of the block's acceleration

Solution #3

a. The force constant can be calculated as

$k = -F_x/x = -ma/x = -(0.3\text{ kg})(-12\text{ m/s}^2)/(0.24\text{ m}) = 15.0\text{ N/m}$

$\omega = (k/m)^{1/2} = [(15\text{ N/m})/(0.3\text{ kg})]^{1/2} = 7.07\text{ rad/s}$

b. $E = \frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

$A = [(v/\omega)^2 + x^2]^{1/2} = [(4\text{ m/s}/7.07\text{ rad/s})^2 + (0.24\text{ m})^2]^{1/2}$

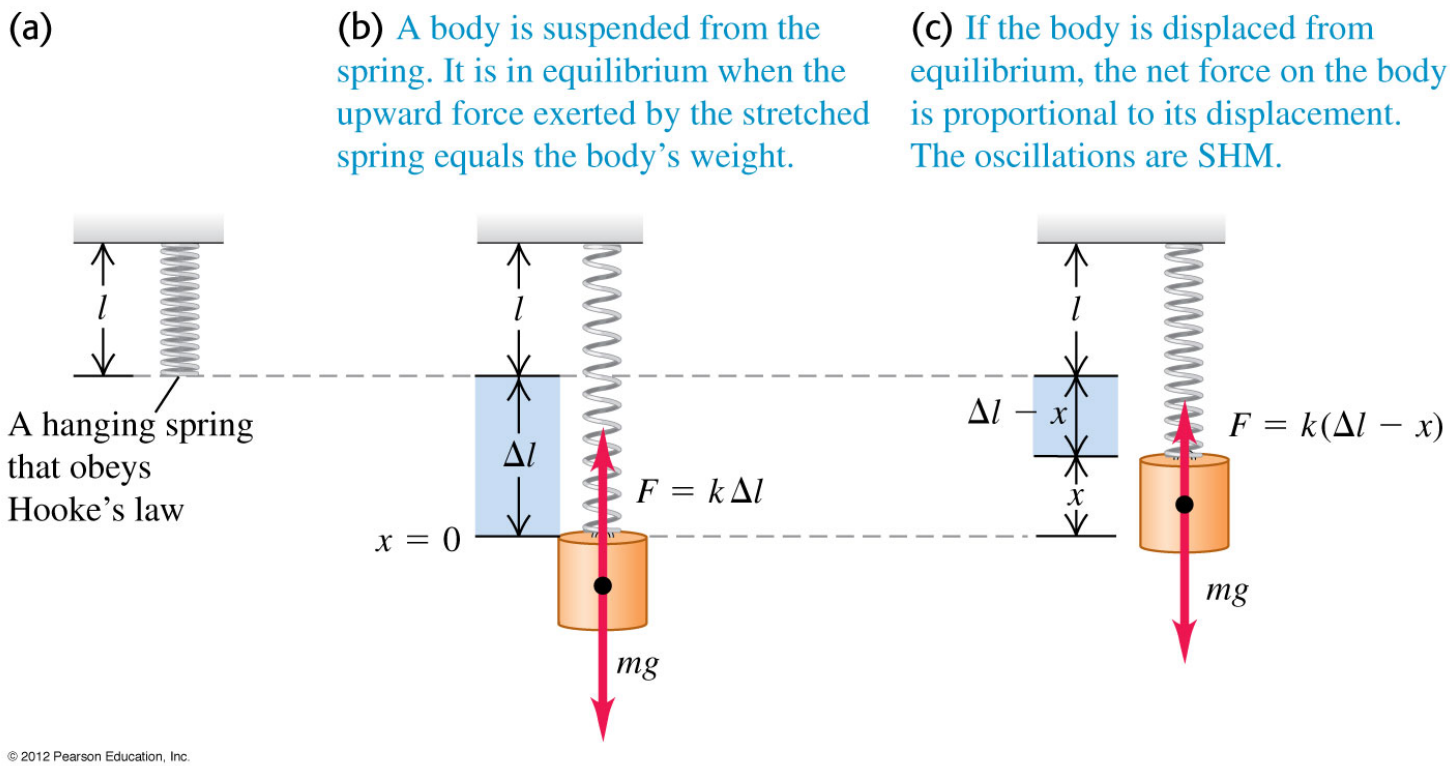
$= 0.614\text{ m}$

c. $\frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2 \Longrightarrow$

$v_{max} = \omega A = (7.07\text{ rad/s})(0.614\text{ m}) = 4.35\text{ m/s}$

d. $a_{max} = kx_{max}/m = \omega^2 A = (7.07\text{ rad/s})^2(0.614\text{ m}) = 30.7\text{ m/s}^2$

Vertical SHM



- If a body oscillates vertically from a spring, the restoring force has magnitude *kx*. Therefore the vertical motion is SHM.
- Let's try a PhET!

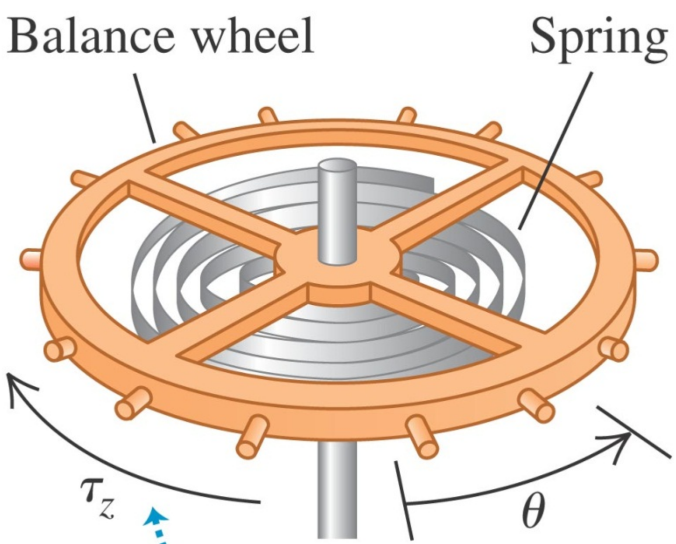
https://phet.colorado.edu/sims/mass-spring-lab/mass-spring-lab_en.html

Angular SHM

- A coil spring exerts a restoring torque $\tau_z = -\kappa\theta$, where κ is called the *torsion constant* of the spring.

- The result is *angular* simple harmonic motion:

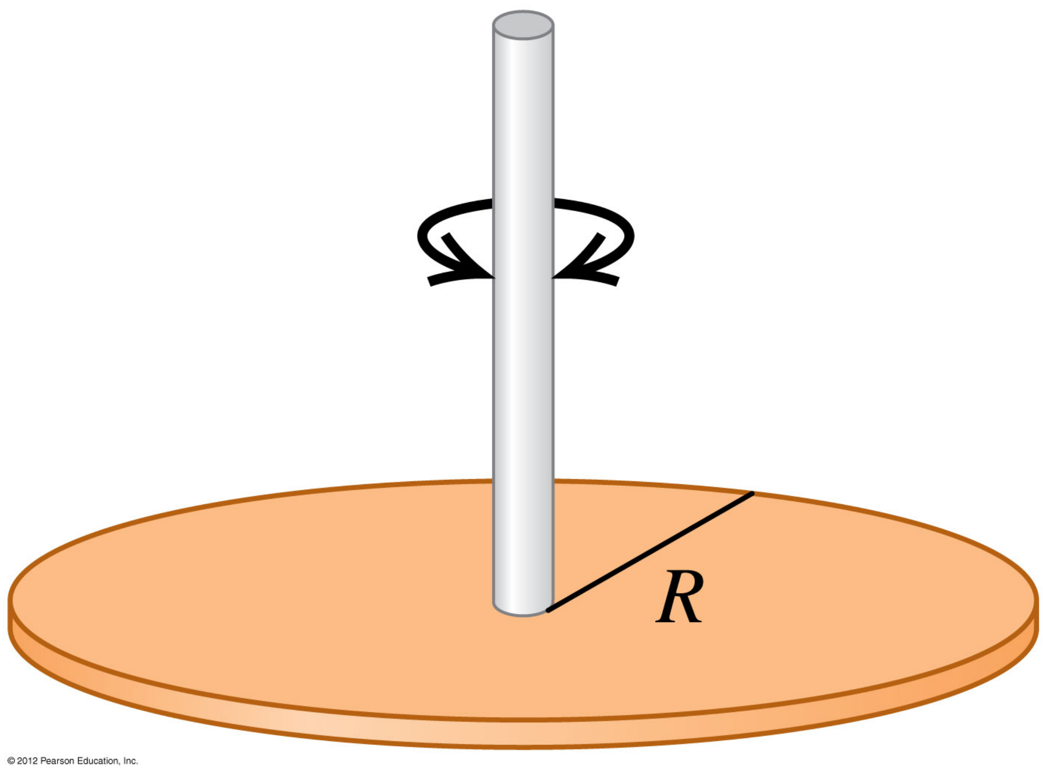
$$I\, d^2\theta/dt^2 = -\kappa\theta \Longrightarrow \theta(t) = A\cos(\omega t + \phi) \text{ with } \omega = \sqrt{(\kappa/I)}$$



The spring torque τ_z opposes the angular displacement θ.

Example #4

A thin metal disk with mass 2.00 g and radius 2.20 cm is attached at its center to a long fiber. The disk, when twisted and released, oscillates with a period of 1.00 s. Find the torsion constant of the fiber.



Solution #4

The disk has a moment inertia of

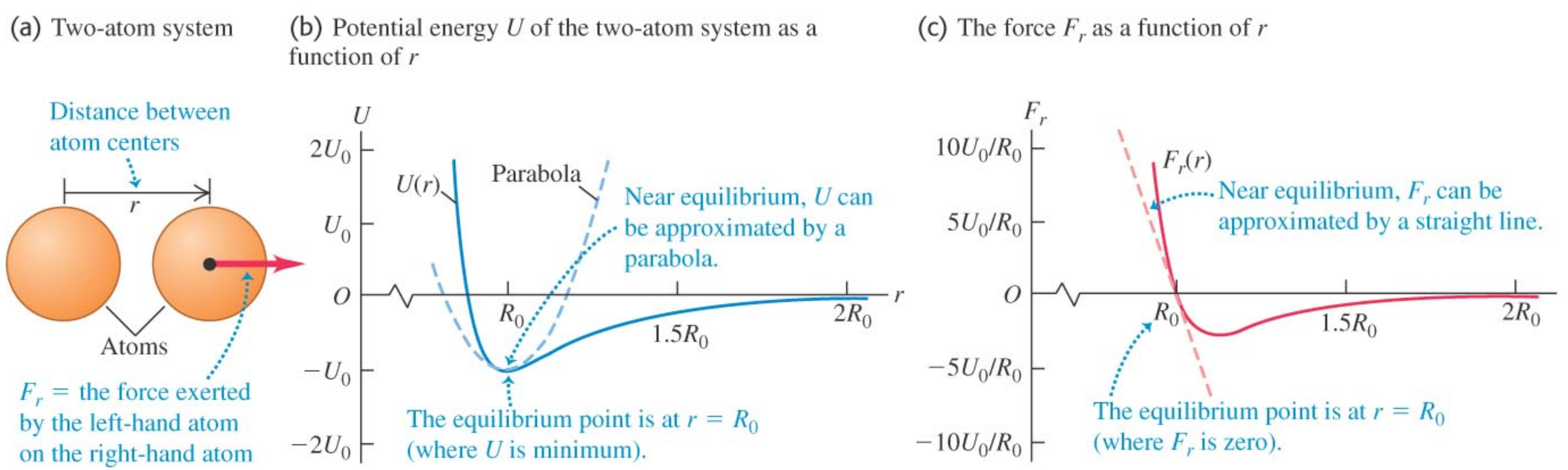
$I = ½MR^2 = ½(0.002\ \text{kg})(0.022\ \text{m})^2 = 4.84 \times 10^{-7}\ \text{kg}\bullet\text{m}^2$

From the angular version of Newton's 2nd law $I\, d^2\theta/dt^2 = -\kappa\theta$, implying SHM with $\omega = \sqrt{(\kappa/I)}$. Solving for κ:

$\kappa = I\omega^2 = I(2\pi/T)^2 = (4.84 \times 10^{-7}\ \text{kg}\bullet\text{m}^2)(2\pi/1.0\ \text{s})^2 = 1.91 \times 10^{-5}\ \text{N}\bullet\text{m/rad}$

Vibrations of molecules

- Two atoms a distance r apart exert van der Waals forces on each other with a stable equilibrium point at $r = R_0$.
- If they are displaced a small distance x from equilibrium, the restoring force is $F_r = -(72U_0/R_0^2)x$, so $k = 72U_0/R_0^2$ and the motion is SHM: $x(t) = A\cos(\omega t + \phi)$ with $\omega = \sqrt{(k/m)}$.



The simple pendulum

- A *simple pendulum* consists of a point mass suspended by a massless, string. From Newton's 2nd law:

$$m\, d^2x/dt^2 = mL\, d^2\theta/dt^2 = -mg \sin\theta$$

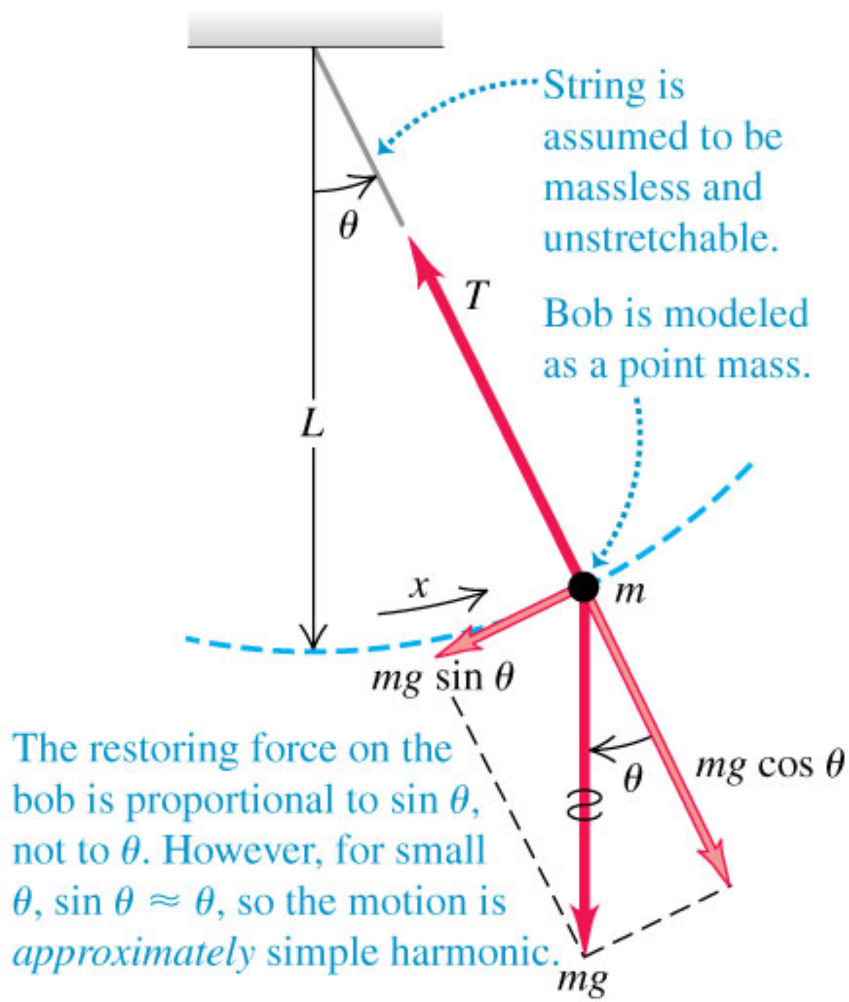
- If the pendulum swings with a small amplitude θ, $\sin\theta \approx \theta$:

$$L\, d^2\theta/dt^2 = -g\theta$$

- The restoring force is linear in θ implying SHM with frequency:

$$\omega = \sqrt{(g/L)}$$

(b) An idealized simple pendulum

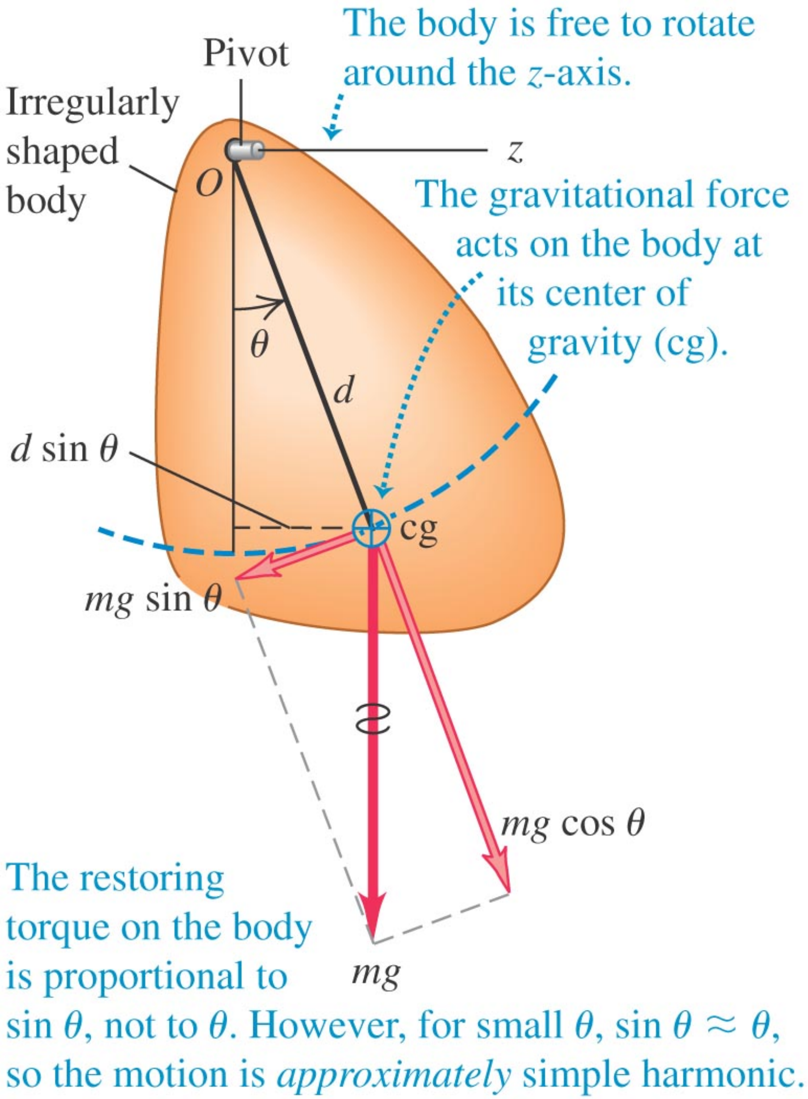


The physical pendulum

- A *physical pendulum* is an extended body instead of a point mass whose motion is rotational:

$$I\, d^2\theta/dt^2 = -mgd \sin\theta$$

- For small amplitudes, $\sin\theta \approx \theta$ and the restoring force is again linear in θ. The motion is simple harmonic with frequency $\omega = \sqrt{(mgd/I)}$.



Example #5

We want to hang a thin hoop on a horizontal nail and have the hoop make one complete small-angle oscillation each 2.0 s. What must the hoop's radius be?

Solution #5

A hoop's moment inertia about its center is $I_{cm} = MR^2$. By the parallel-axis theorem, its moment of inertia about its edge is:

$I = I_{cm} + MR^2 = 2MR^2$.

Its angular frequency will be

$\omega^2 = Mgd/I = MgR/(2MR^2) = \tfrac{1}{2}g/R$

$R = \tfrac{1}{2}g/\omega^2 = \tfrac{1}{2}g(T/2\pi)^2 = \tfrac{1}{2}(9.8\ \text{m/s}^2)(2\ \text{s}/2\pi)^2 = 0.496\ \text{m}$

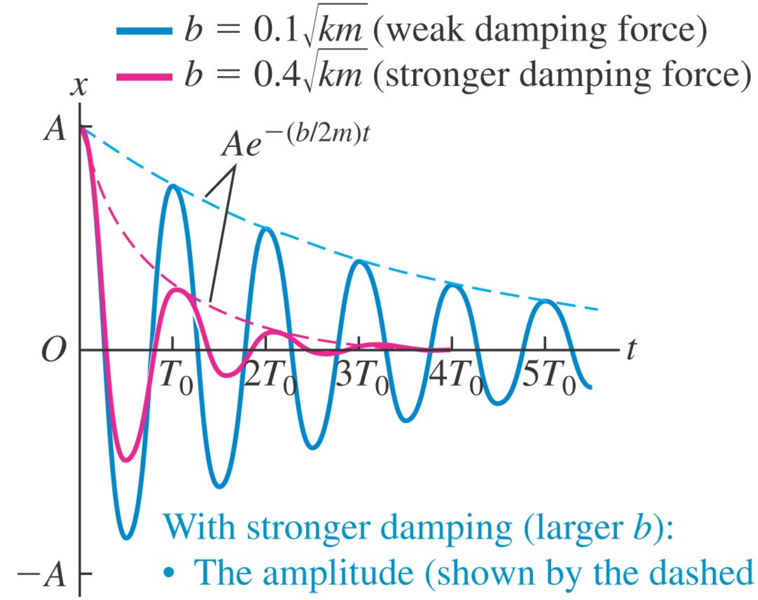
Damped oscillations

- Real systems have dissipation that act to restore the system to equilibrium.
- The damping force is often proportional to the velocity $F_x = -b(dx/dt)$.
- If $\gamma \equiv b/2m \geq \omega_0 = \sqrt{(k/m)}$, the system will no longer oscillate and instead will return directly to the equilibrium point.
- For an *underdamped* system, the solution is:

$$x(t) = A\, e^{-\gamma t} \cos(\omega' t + \phi)$$

where $\omega' = \sqrt{(\omega_0^2 - \gamma^2)}$.

- Damping causes the amplitude to exponentially decrease and reduces the frequency. The mechanical energy of a damped oscillator decreases continuously.



With stronger damping (larger b):
- The amplitude (shown by the dashed curves) decreases more rapidly.
- The period T increases (T_0 = period with zero damping).

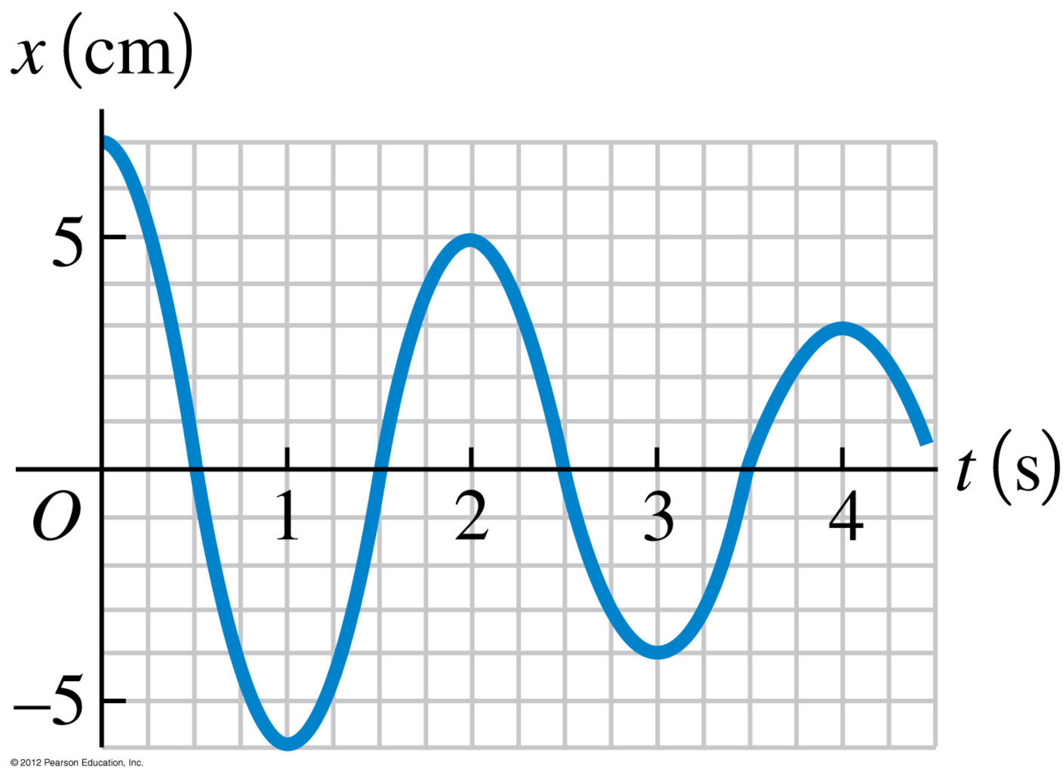
Example #6

A mass is vibrating at the end of a spring of force constant 225 N/m.

a. At what times is the mass not moving?

b. How much energy did this system originally contain?

c. How much energy did the system lose between t = 1.0 s and t = 4.0 s? Where did this energy go?



Solution #6

a. The mass is at rest when its velocity (the slope of the curve) is zero. This occurs at t = 0, 1s, 2s, 3s, 4s, …

b. At t = 0, the energy is

$E = ½kx^2$ = ½(225 N/m)(0.07 m)^2 = 0.551 J

c. The change in energy between t = 4.0 s and 1.0 s is

$\Delta E = E_2 - E_1 = ½k(x_2^2 - x_1^2)$ = ½(225 N/m)[(0.03 m)^2 – (0.06 cm)^2] = -0.304 J

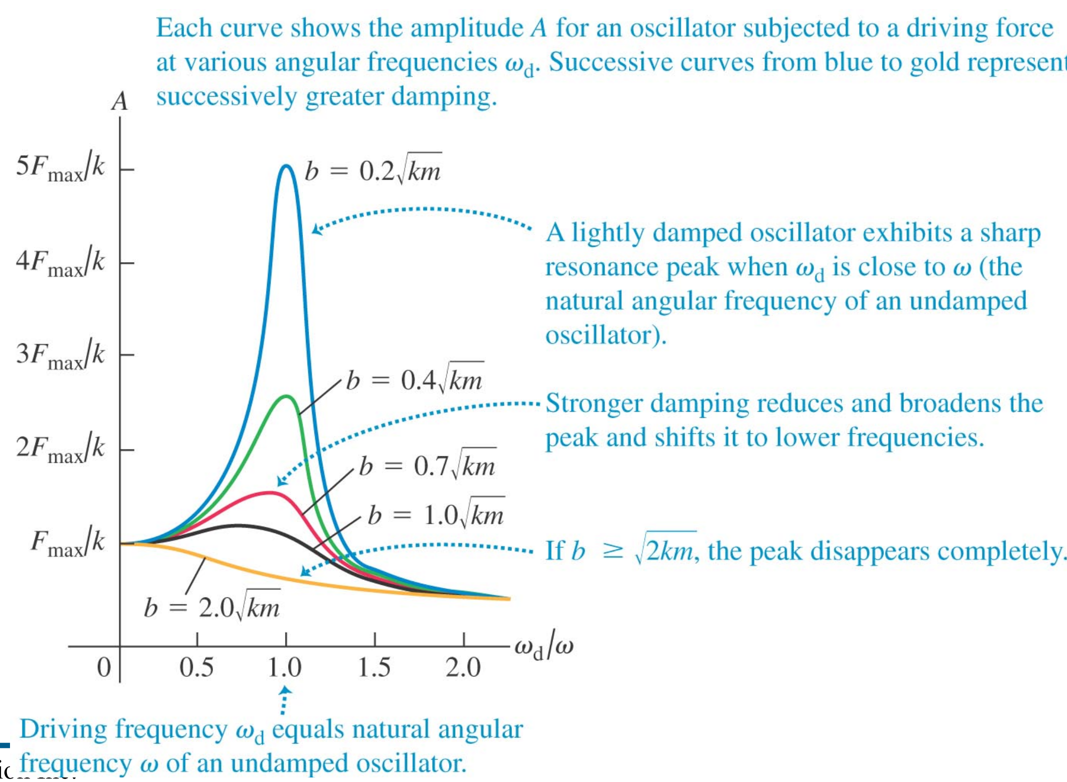
This energy is lost to dissipation.

Forced oscillations and resonance

- A *forced oscillation* occurs if a *driving force* acts on an oscillator. The amplitude of oscillations driven at frequency ω_d is:

$$A = (F_{max}/m)[(\omega_d^{\,2} - \omega_0^{\,2})^2 + (2\gamma\omega_d)^2]^{-1/2}$$

- *Resonance* occurs if the frequency of the driving force ω_d is near the *natural frequency* $\omega_0 = \sqrt{(k/m)}$ of the system.



Example #7

A sinusoidally varying driving force is applied to a damped harmonic oscillator of force constant k and mass m. If the damping constant has a value b_1, the amplitude is A_1 when the driving frequency $\omega_d = \omega_0$. In terms of A_1, what is the amplitude for the same driving frequency and the same force amplitude F_{max} if the damping constant is

a. $3b_1$

b. $\frac{1}{2}b_1$

Solution #7

We learned that the driven amplitude is:

$$A = (F_{max}/m)[(\omega_d^2 - \omega_0^2)^2 + (2\gamma\omega_d)^2]^{-1/2}$$

For $\omega_d = \omega_0$, $A = F_{max}/(2\gamma m\omega_0) = F_{max}/(b\omega_0)$

a. Tripling the damping constant *b* reduces the amplitude by ⅓.

b. Halving the damping constant *b* increases the amplitude by a factor of 2.