Chapter 10

Dynamics of Rotational Motion

PowerPoint® Lectures for
University Physics, Thirteenth Edition
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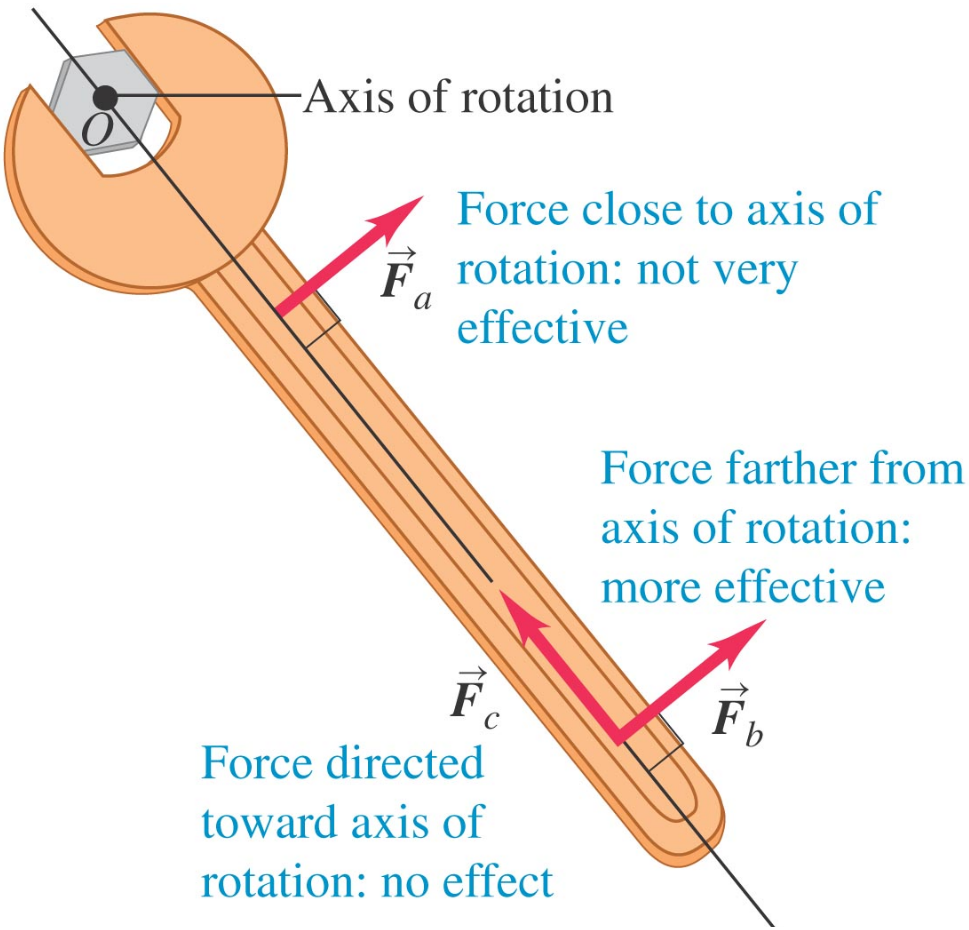
Introduction

- In the previous lecture, we learned about the rotation of rigid bodies which is generally determined by their time-varying angular velocities $\boldsymbol{\omega}$.
- In this lecture, we will introduce the concept of *torque* which gives rise to angular accelerations.
- We will introduce the *angular momentum* $\boldsymbol{L}$ and discover under which conditions it is conserved.
- These concepts will allow us to solve general problems of rotational dynamics.

Loosen a bolt

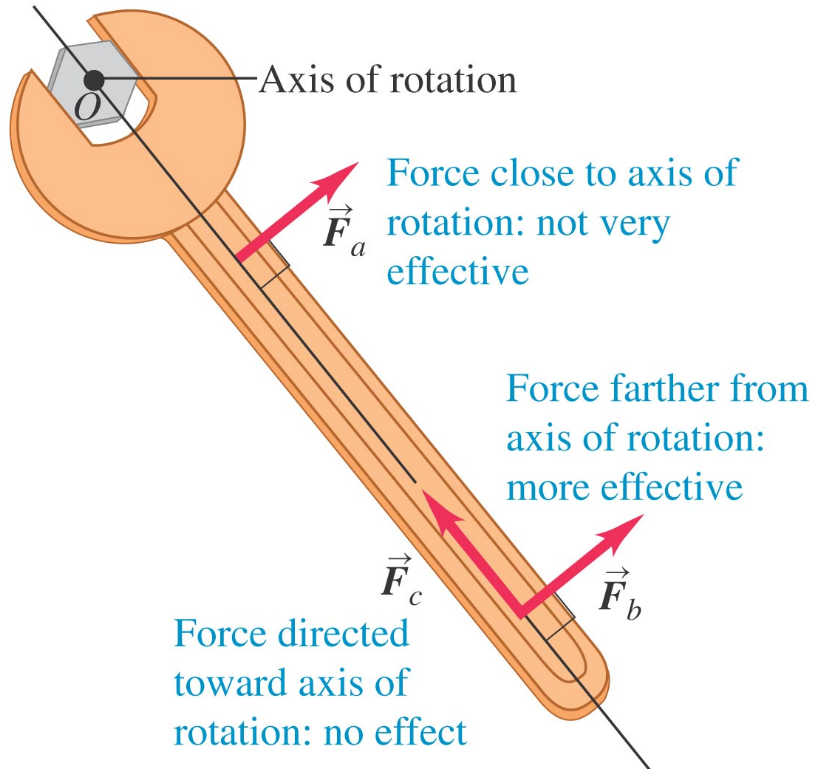
- Which of the three equal-magnitude forces in the figure is most likely to loosen the bolt?
- “Give me a lever long enough and a fulcrum on which to place it, and I shall move the world.”

- *Archimedes, 3rd century BC*



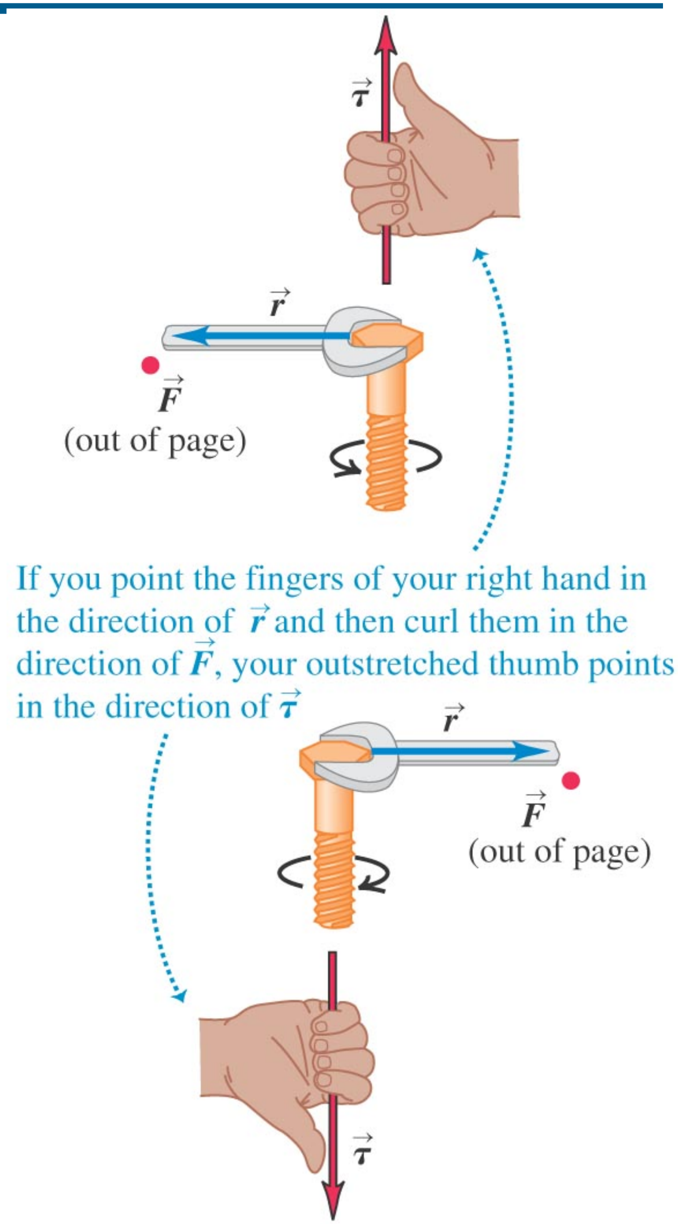
Torque

- A force $\boldsymbol{F}$ exerted on a rigid body at a displacement $\boldsymbol{r}$ from the origin O will induce a *torque* $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$ about the axis parallel to $\boldsymbol{\tau}$ passing through the origin O.
- The SI unit of torque is the N•m. Although this is equivalent to a Joule, we use N•m to avoid confusion with work.
- $\boldsymbol{F}_b$ exerts a greater torque than $\boldsymbol{F}_a$ about O because it is exerted at a greater distance r from O.
- $\boldsymbol{F}_c$ exerts no torque because $\theta_{rF} = 180°$ and the vector product is proportional to $\sin\theta_{rF} = \sin 180° = 0$.



Direction of the torque

- The direction of the torque $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$ is given by the right-hand rule for vector products.
- A circled dot ⊙ is used to denote vectors pointing "out of the page" while a circled X ⊗ denotes vectors pointing "into the page". This is supposed to look like a arrow coming towards and away from you.

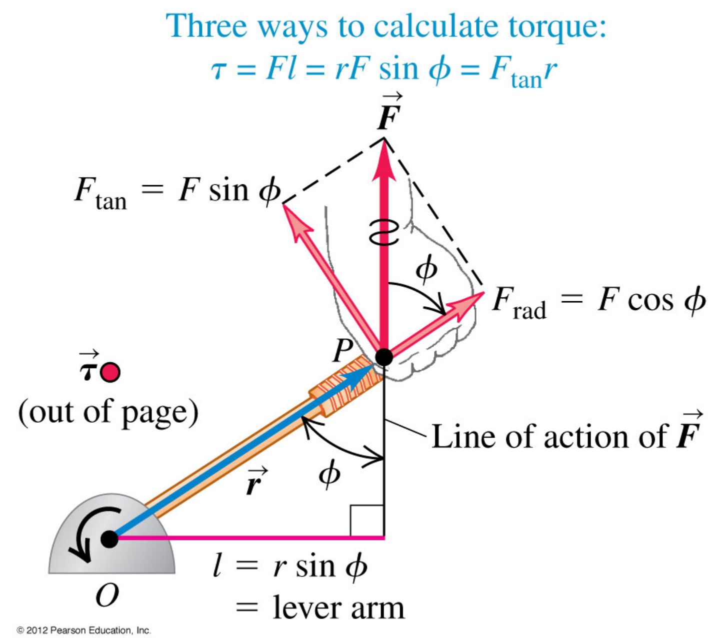


Magnitude of the torque

- The magnitude τ can be calculated in 3 ways:

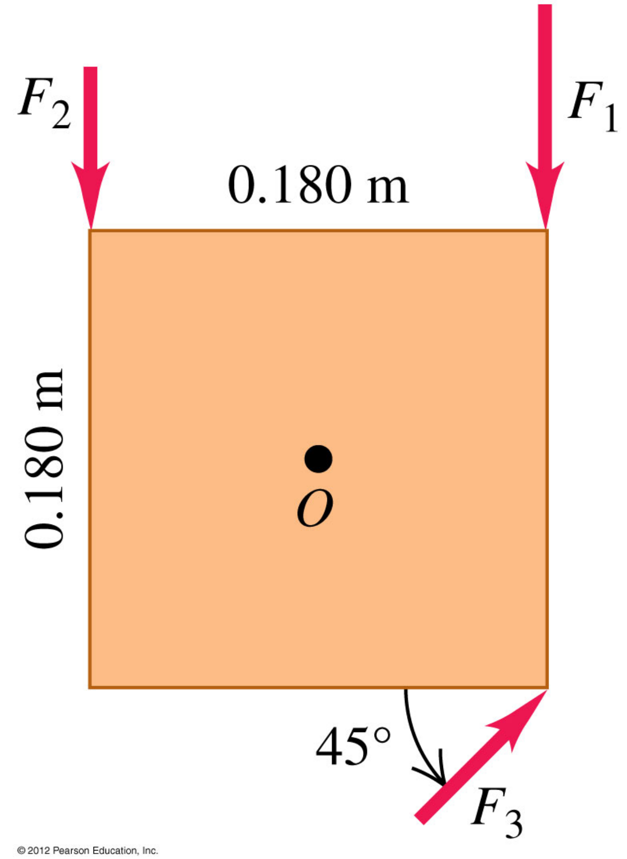
$$\tau = rF \sin \phi = lF = rF_{\tan}$$

- The *line of action* of a force is the line along which the force vector lies.
- The *lever arm* l for a force is the perpendicular distance from O to the line of action of the force (see figure).
- The torque of a force with respect to O is the product of the force and its lever arm.
- The torque magnitude τ is also the product of the distance r and the tangential component of the force $F_{\tan}$.



Example #1

- A square metal plate 0.180 m on each side is pivoted about an axis through point *O* at its center and perpendicular to the plate. Calculate the net torque about this axis due to the three forces shown in the figure if the magnitudes of the forces are $F_1 = 18.0$ N, $F_2 = 26.0$ N, and $F_3 = 14.0$ N.



Solution #1

$\boldsymbol{F}_1$ exerts a torque τ_1 into the page by the right-hand rule. Since the lever arm for this force is 0.09 m, the torque is

$$\tau_1 = (18.0 \text{ N})(0.09 \text{ m}) = 1.62 \text{ N}\bullet\text{m}$$

$\boldsymbol{F}_2$ exerts a torque τ_2 out the page by the right-hand rule. Since the lever arm for this force is also 0.09 m, the torque is

$$\tau_2 = -(26.0 \text{ N})(0.09 \text{ m}) = -2.43 \text{ N}\bullet\text{m}$$

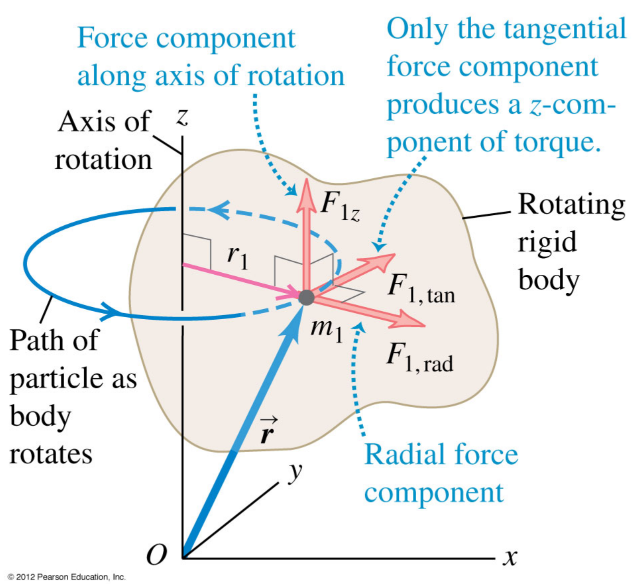
if we define into the page to be positive.

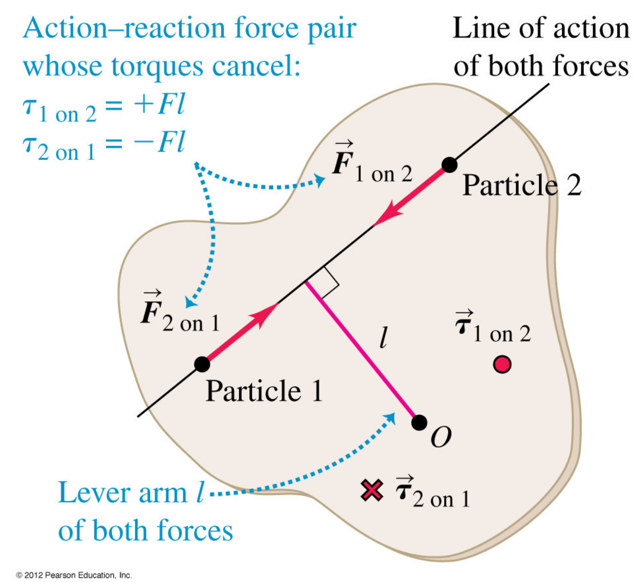
$\boldsymbol{F}_3$ exerts a torque τ_3 out the page by the right-hand rule, however its lever arm is $(0.09 \text{ m})\sqrt{2}$. The torque is

$$\tau_2 = -\sqrt{2}(14.0 \text{ N})(0.09 \text{ m}) = -1.78 \text{ N}\bullet\text{m}$$

Torque and angular acceleration for a rigid body

- We learned last lecture that a particle of mass m a distance r from the rotation axis experiencing a tangential acceleration a_{tan} undergoes an angular acceleration $\alpha_z = a_{tan}/r$.
- By Newton's 2nd law, $a_{tan} = F_{tan}/m = rF_{tan}/(mr) = \tau/(mr)$ where we have used our definition of the torque $\tau = rF_{tan}$.
- This implies $\alpha_z = a_{tan}/r = \tau_z/(mr^2) = \tau_z/I$ where $I = mr^2$ is the moment of inertia of our particle. Summing over all the particles in our body: $\Sigma\tau_z = I\alpha_z$, the rotational analogue of Newton's 2nd law. Only external torques enter this sum.





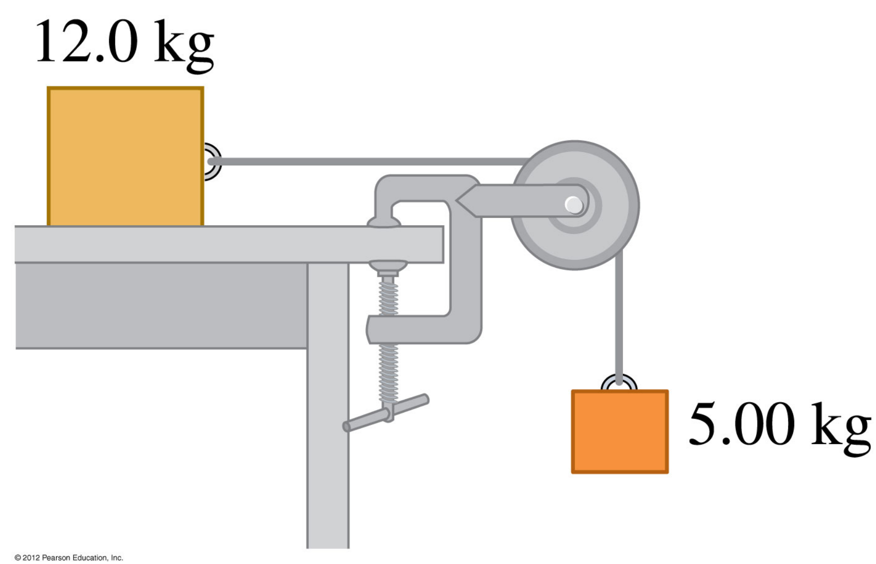
Example #2

A 12.0 kg box resting on a horizontal, frictionless surface is attached to a 5.00 kg weight by a thin, light wire that passes over a frictionless pulley. The pulley has the shape of a uniform solid disk of mass 2.00 kg and diameter 0.500 m. After the system is released, find

a. the tension in the wire on both sides of the pulley

b. the acceleration of the box

c. the horizontal and vertical components of the force that the axle exerts on the pulley



Solution #2

Since the length of the wire is fixed, $v_b = v_w = \omega_z r_p$. Differentiating with respect to time gives $a_b = a_w = \alpha_z r_p$. Now we apply Newton's 2nd law to all 3 objects:

Box: $T_1 = m_b a$ Weight: $m_w g - T_2 = m_w a$

Pulley: $r_p(T_2 - T_1) = I_p \alpha_z = ½ m_p r_p{}^2 \alpha_z = ½ m_p r_p a$

Inserting the 1st and 2nd equations into the 3rd:

$m_w(g - a) - m_b a = ½ m_p a \Longrightarrow a = m_w g / (½ m_p + m_b + m_w)$

a = (5 kg)(9.8 m/s^2)/[½(2 kg) + (12 kg) + (5 kg)] = 2.72 m/s^2

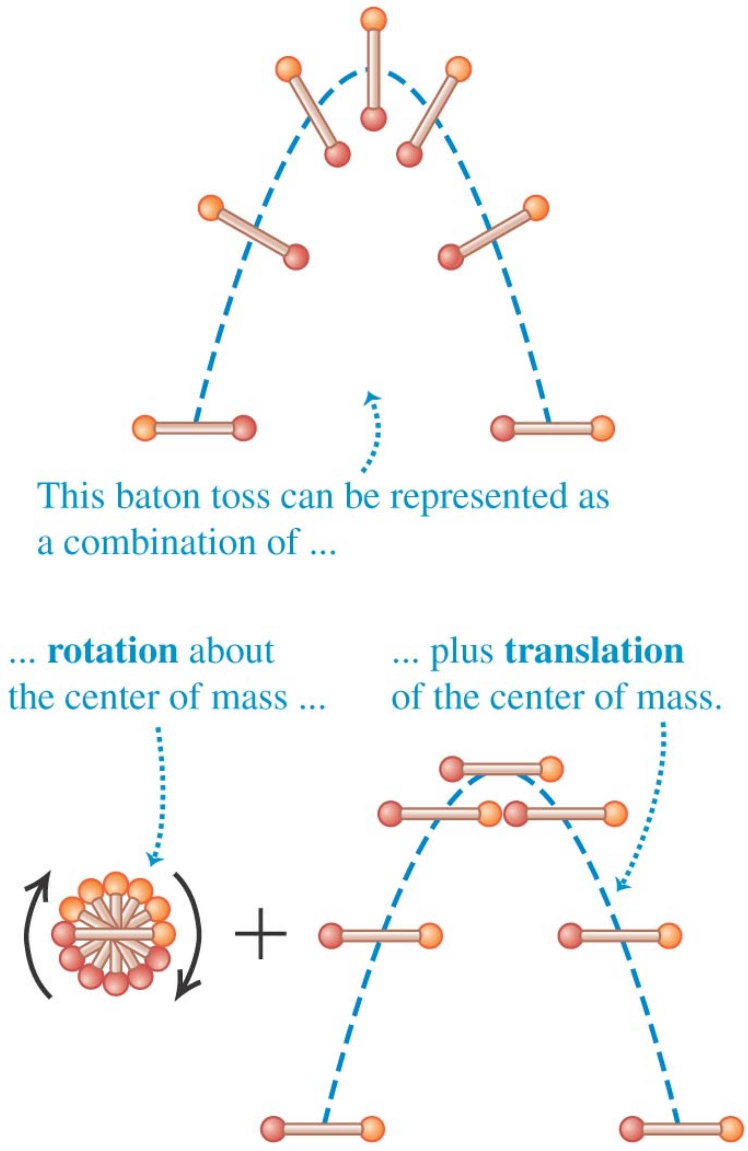
$T_1 = m_b a$ = (12 kg)(2.72 m/s^2) = 32.7 N

$T_2 = m_w(g - a)$ = (5 kg)(9.8 m/s^2 – 2.72 m/s^2) = 35.4 N

Since the center of mass of the pulley is at rest, the axle must exert a force to the right of A_x = 32.7 N to cancel T_1. It must also exert a force $A_y = T_2 + m_p g$ = 35.4 N + (2 kg)(9.8 m/s^2) = 55 N upwards to cancel T_2 and the pulley's weight. These forces exert no torques since their lever arms are zero.

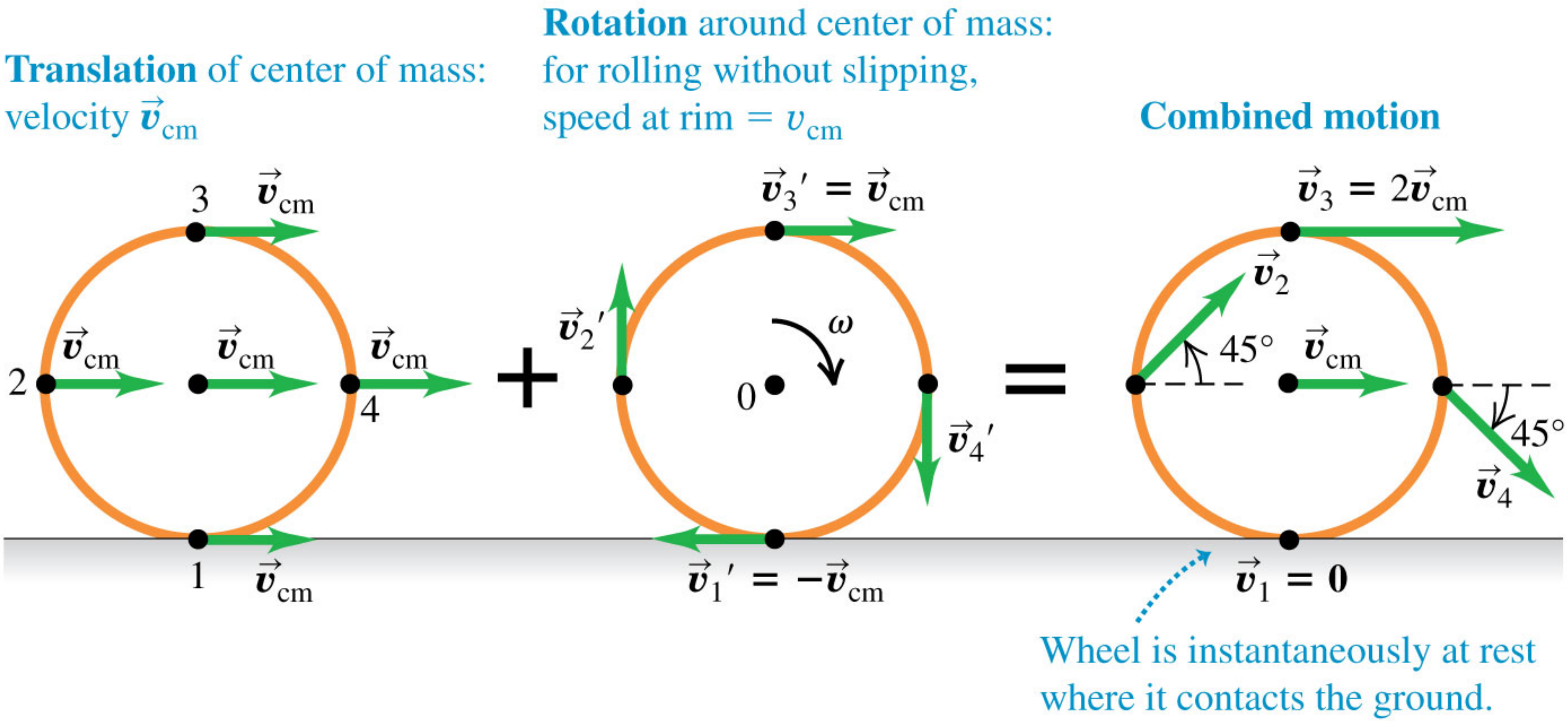
Rigid body rotation about a moving axis

- The motion of a rigid body is a combination of translational motion of the center of mass and rotation about the center of mass.
- The kinetic energy of a rotating and translating rigid body is $K = \frac{1}{2}Mv_{cm}{}^2 + \frac{1}{2}I_{cm}\omega^2$.



Rolling without slipping

- An example of combined translation and rotation is a rolling wheel.
- For the bottom of the wheel (point 1) to be at rest with respect to the ground (no slipping), the velocity of the center of mass must be $v_{cm} = R\omega$.



Example #3

A solid ball is released from rest and slides down a hillside that slopes downwards at 65.0° from the horizontal.

a. What minimum value must the coefficient of static friction between the hill and ball surfaces have for no slipping to occur?

b. Would the coefficient of friction calculated in part (a) be sufficient to prevent a hollow ball (such as a soccer ball) from slipping?

c. In part (a), why did we use the coefficient of static friction and not coefficient of kinetic friction?

Solution #3

If the ball rolls without slipping, $v = \omega R \Longrightarrow a = R\alpha$. We apply Newton's 2nd law to the translational and rotational motion:

x translation: $mg \sin \theta - f = ma$

y translation: $n - mg \cos \theta = 0 \Longrightarrow f = \mu_s n = \mu_s mg \cos \theta$

rotation: $Rf = I\alpha = \frac{2}{5}mR^2\alpha = \frac{2}{5}mRa = \frac{2}{5}R(mg \sin \theta - f\,)$

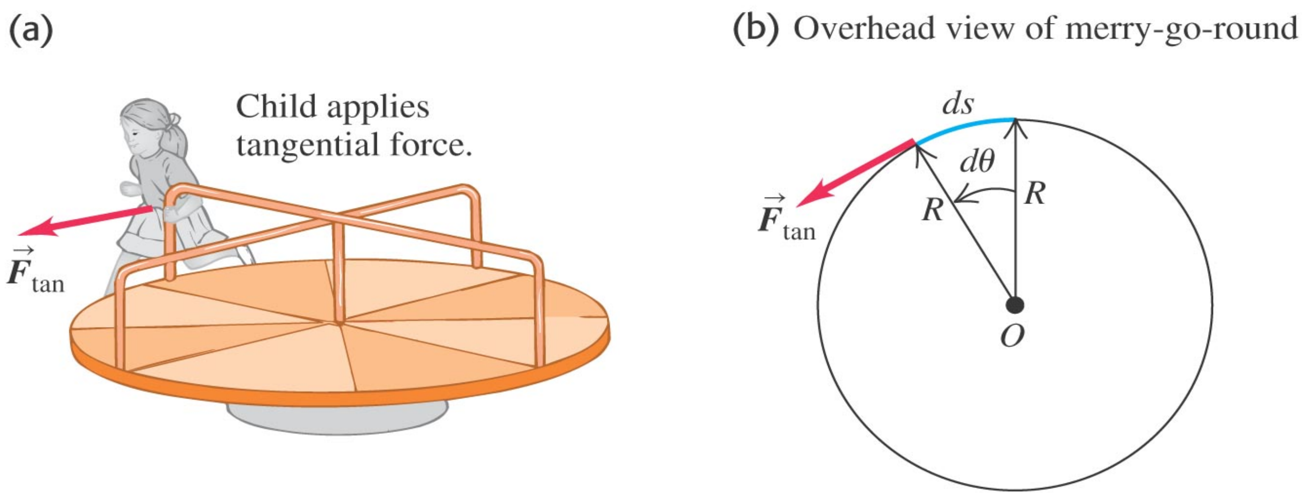
$7f/5 = 7(\mu_s mg \cos \theta)/5 = \frac{2}{5}mg \sin \theta \Longrightarrow \mu_s = (2/7) \tan \theta$

If the sphere had been hollow $(I = \frac{2}{3}mR^2)$, a larger coefficient $\mu_s = (2/5) \tan \theta$ would have been required.

The bottom of the sphere does not slide against the hillside, so static friction applies.

Work and power in rotational motion

- The forces that give rise to torques can also perform work on an object, changing its rotational kinetic energy.
- The child below exerts a force $\boldsymbol{F}_{\text{tan}}$ on the merry-go-round, exerting a torque $\tau = RF_{\text{tan}}$ in the vertical direction.
- In the time dt over which the wheel turns by $d\theta$, she performs work $dW = F_{\text{tan}} ds = F_{\text{tan}}\, R\, d\theta = \tau\, d\theta$ exerting power $P = dW/dt = \tau\omega$.



Example #4

a. Compute the torque developed by an industrial motor whose output is 150 kW at an angular speed of 400 rpm.

b. A drum with negligible mass, 0.400 m in diameter, is attached to the motor shaft, and the power output of the motor is used to raise a weight hanging from a rope wrapped around the drum. How heavy a weight can the motor lift at constant speed?

c. At what constant speed will the weight rise?

Solution #4

a. ω = 400 rpm (2π rad/rev)(1 min/60 s) = 41.9 rad/s

$\tau = P/\omega$ = (150 kW)/(41.9 rad/s) = 3,580 N•m

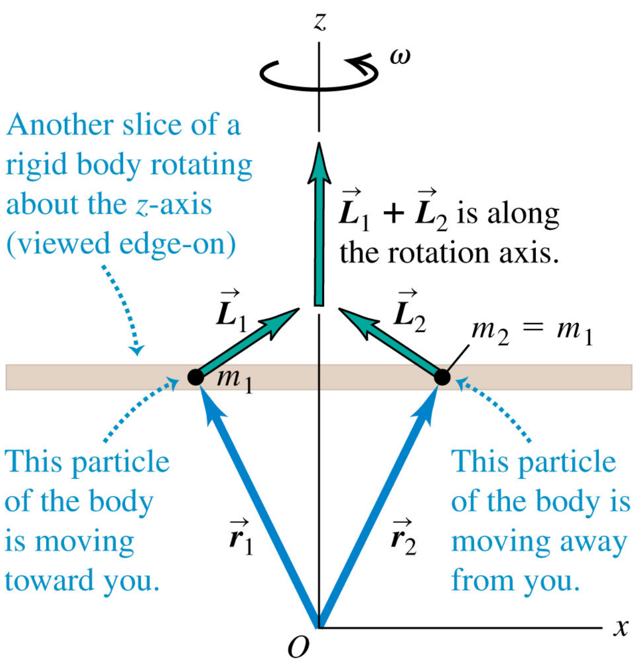
b. $w = \tau/R$ = (3,580 N•m)/(0.2 m) = 17,900 N

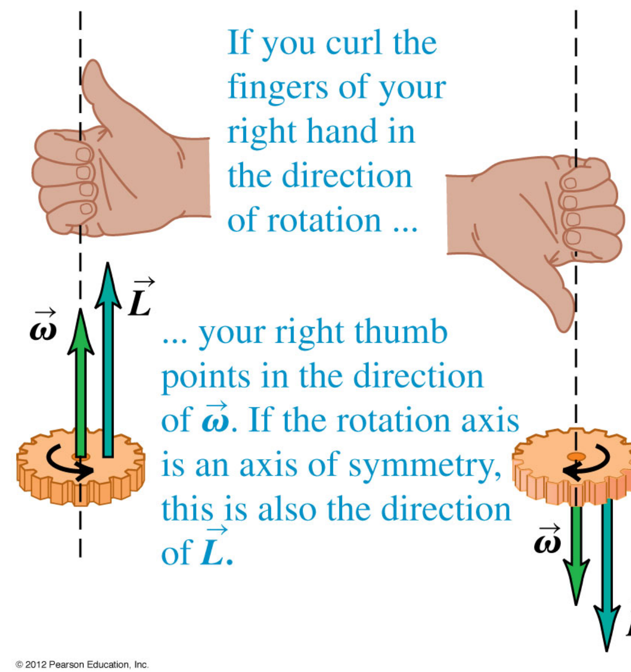
c. $v = P/w$ = (150 kW)/(17,900 N) = 8.38 m/s

equivalently, $v = \omega R$ = (41.9 rad/s)(0.2 m) = 8.38 m/s

Angular momentum

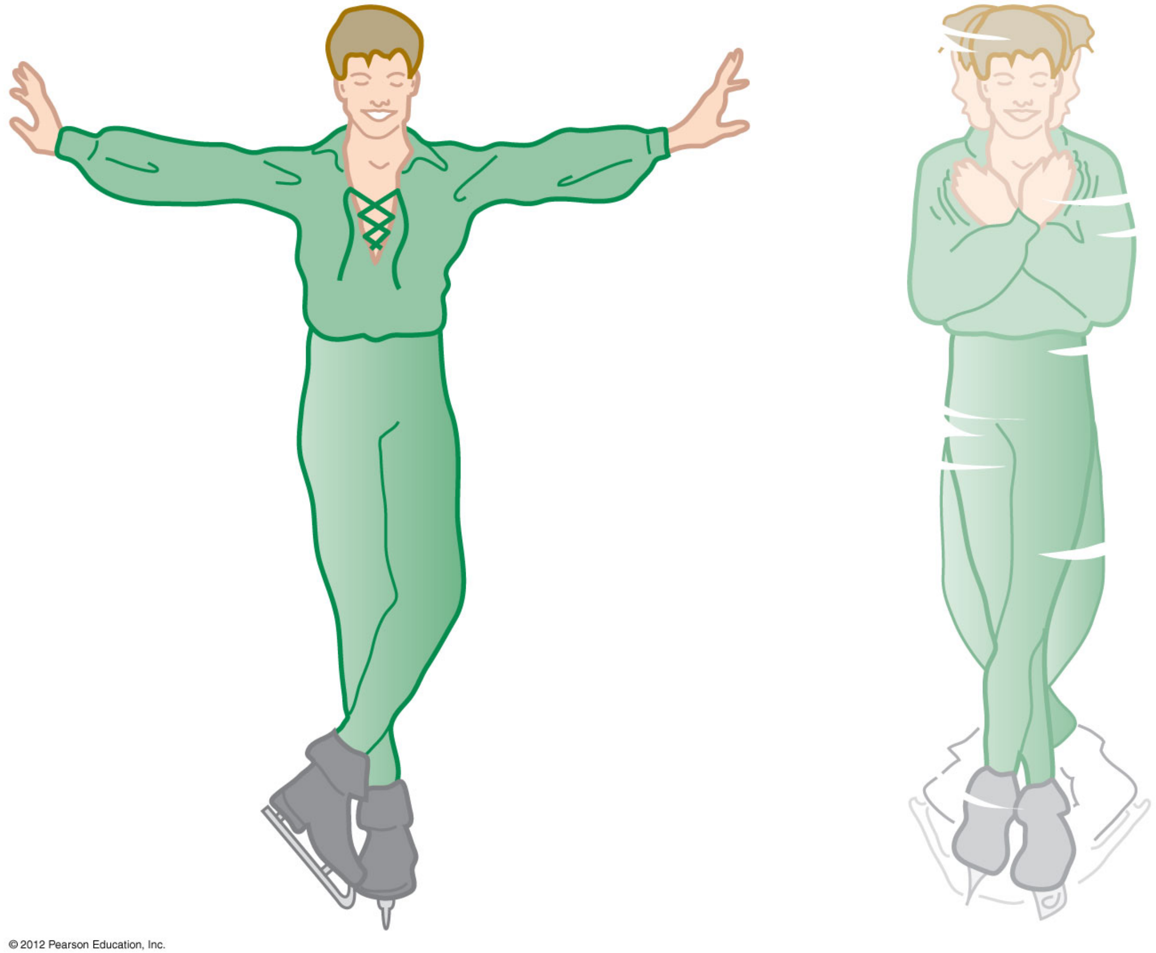
- A particle with mass m, displacement $\boldsymbol{r}$ from the origin O, and velocity $\boldsymbol{v}$ will have *angular momentum* $\boldsymbol{L} = \boldsymbol{r} \times m\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{p}$.
- If an object is rotating with angular velocity $\boldsymbol{\omega}$ about a symmetry axis (the z axis), each particle with mass dm and velocity $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$ will contribute $dL_z = \omega(r \sin\theta)^2\, dm = \omega\, dI$ to the total angular momentum implying that $\boldsymbol{L} = I_z\, \boldsymbol{\omega}$ for the entire object.
- $d\boldsymbol{L}/dt = d(\boldsymbol{r} \times m\boldsymbol{v})/dt = \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times m\boldsymbol{a} = \boldsymbol{r} \times (\Sigma\boldsymbol{F}) = \Sigma\boldsymbol{\tau}$





Conservation of angular momentum

- When the net external torque is zero, the total angular momentum of the system is constant (conserved): $d\boldsymbol{L}/dt = \Sigma\boldsymbol{\tau} = 0$.



Example #5

The outstretched hands and arms of a figure skater preparing for a spin can be considered a slender rod pivoting about an axis through its center. When the skater's hands and arms are brought in and wrapped about his body to execute the spin, the hands and arms can be considered a thin-walled, hollow cylinder. His hands and arms have a combined mass of 8.0 kg, When outstretched, they span 1.8 m; when wrapped, they form a cylinder of radius 25 cm. The moment of inertia about the rotation axis of the remainder of his body is constant constant and equal to 0.40 kg•m^2. If his original angular speed is 0.40 rev/s, what is his final angular speed?

Solution #5

Since there are no external torques, angular momentum is conserved:

$I_1 = I_b + ml^2/12 = 0.40\ \text{kg•m}^2 + (8\ \text{kg})(1.8\ \text{m})^2/12 = 2.56\ \text{kg•m}^2$

$I_2 = I_b + mR^2 = 0.40\ \text{kg•m}^2 + (8\ \text{kg})(0.25\ \text{m})^2 = 0.9\ \text{kg•m}^2$

$L_1 = I_1\omega_1 = L_2 = I_2\omega_2 \Longrightarrow$

$\omega_2 = (I_1/I_2)\omega_1 = (2.56/0.9)(0.4\ \text{rev/s}) = 1.14\ \text{rev/s}$

Example #6

A thin, uniform bar, 2.00 m long and weighing 90.0 N, is hanging vertically from the ceiling by a frictionless pivot. Suddenly it is struck 1.50 m below the ceiling by small 3.00 kg ball , initially travelling at 10.0 m/s. The ball rebounds in the opposite direction with a speed of 6.00 m/s.

a. Find the angular speed of the bar just after the collision.

b. During the collision, why is the angular momentum conserved but not the linear momentum?

Solution #6

a. Angular momentum is conserved in the collision:

$L_1 = m_b v_1 d = L_2 = m_b v_2 d + I\omega$

$I = ml^2/12 = (90\text{ N})(2\text{ m})^2/[12(9.8\text{ m/s}^2)] = 3.06\text{ kg}\bullet\text{m}^2$

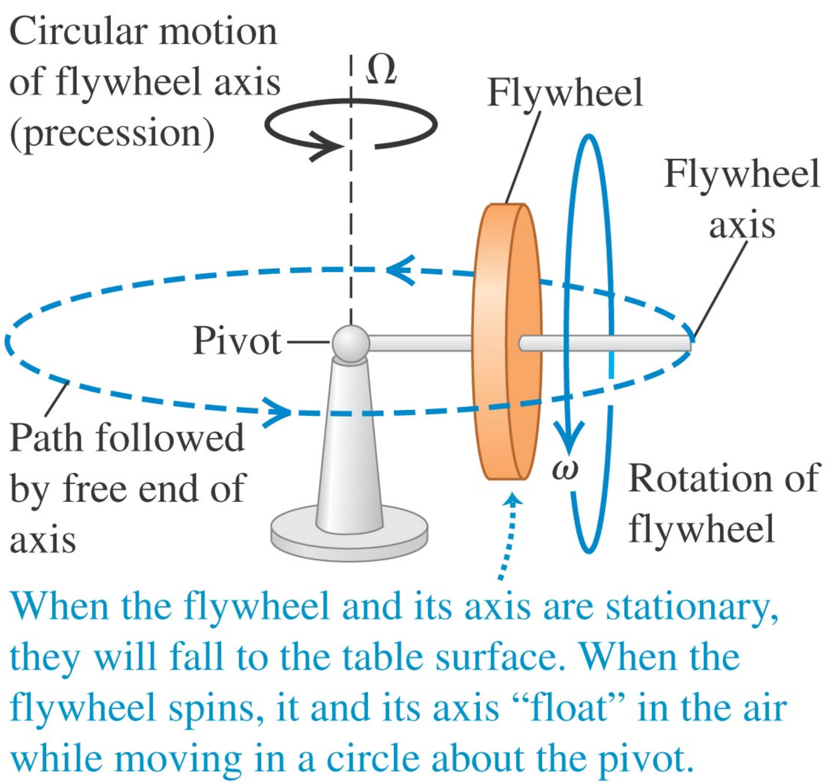
$\omega = m_b d(v_1 - v_2)/I$

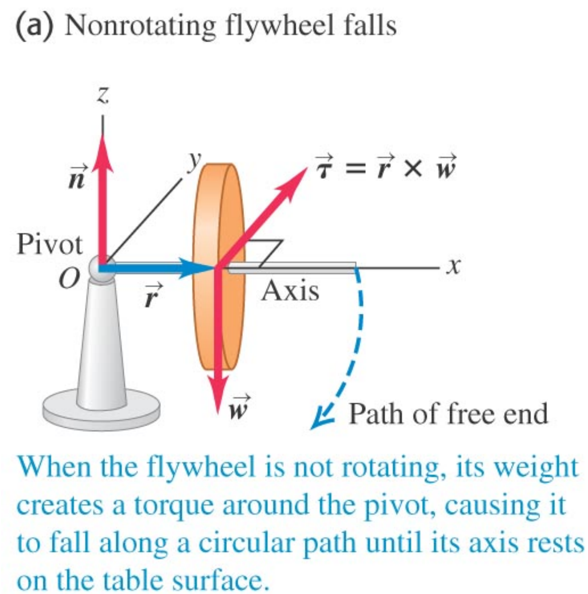
$= (3\text{ kg})(1.5\text{ m})[(10\text{ m/s}) - (-6\text{ m/s})]/(3.06\text{ kg}\bullet\text{m}^2) = 23.5\text{ rad/s}$

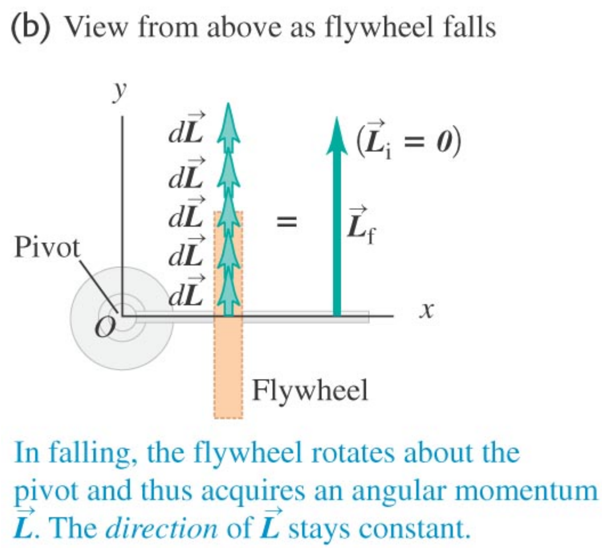
b. The pivot can exert a force that changes the linear momentum, but since the lever arm is zero this force cannot exert a torque, conserving angular momentum.

Gyroscopes and precession

- For a gyroscope, the axis of rotation changes direction. The motion of this axis is called *precession*.

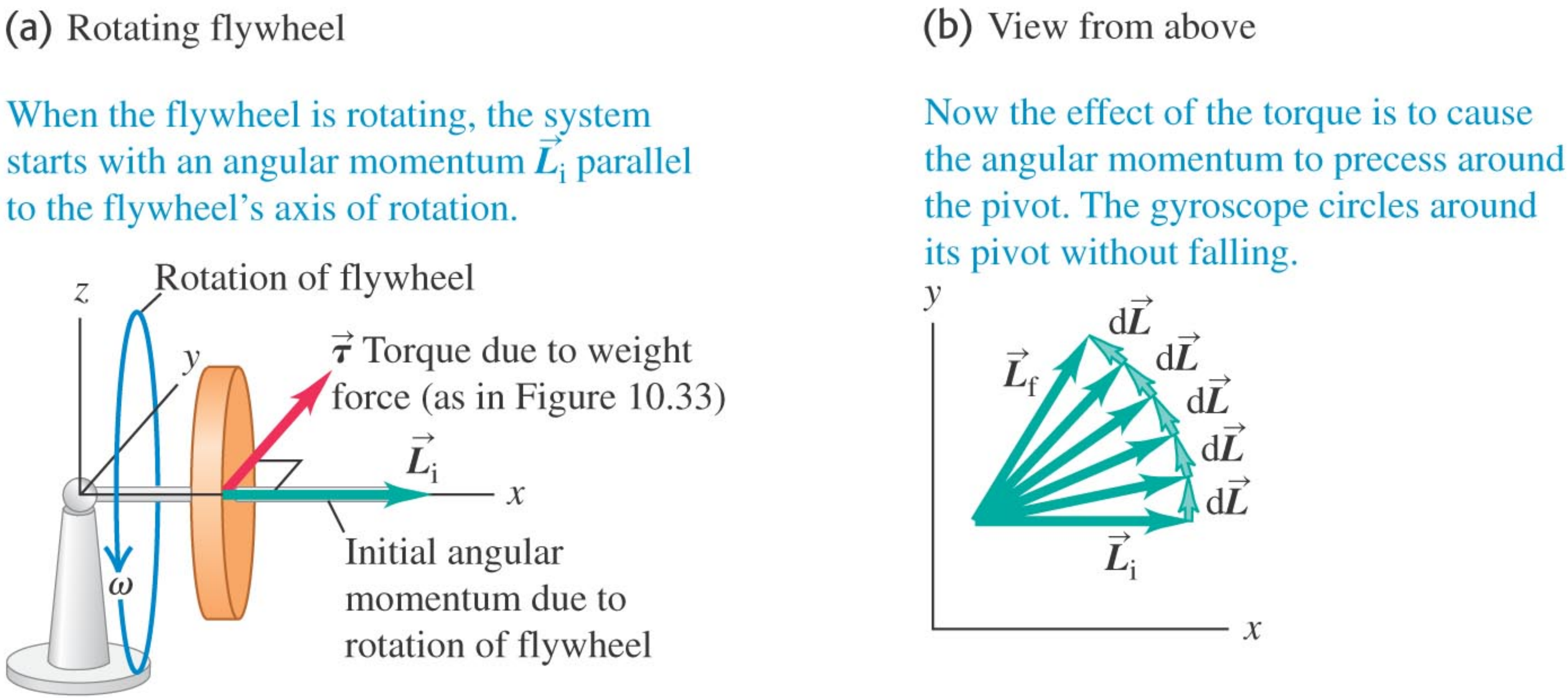






A rotating flywheel

- For a spinning flywheel, the magnitude of the angular momentum stays the same, but its direction changes continuously.

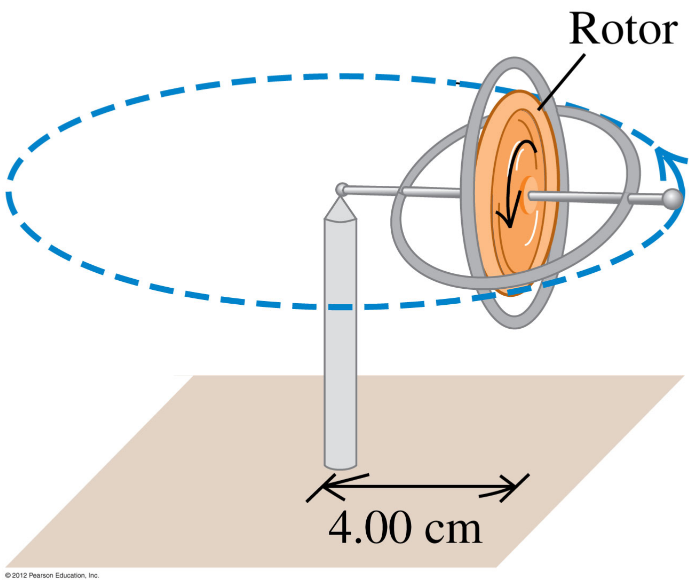


Example #7

The rotor (flywheel) of a toy gyroscope has a mass 0.140 kg. Its moment of inertia about its axis is 1.20×10^{-4} kg•m^2. The mass of the frame is 0.0250 kg. The gyroscope is supported on a single pivot with its center of mass a horizontal distance of 4.00 cm from the pivot. The gyroscope is precessing in a horizontal plane at the rate of one revolution in 2.20 s.

a. Find the upwards force exerted by the pivot.

b. Find the speed with which the rotor is spinning about its axis.



Solution #7

a. The upwards force exerted by the pivot balances the weight of the gyroscope and frame:

$F = (m_g + m_f)g = (0.140 \text{ kg} + 0.025 \text{ kg})(9.8 \text{ m/s}^2) = 1.62 \text{ N}$

b. $\tau = Fd = dL/dt = \Omega L = \Omega I\omega \Longrightarrow$

$\omega = Fd/(\Omega I) = PFd/(2\pi I)$

$= (2.2 \text{ s})(1.62 \text{ N})(0.04 \text{ m})/[2\pi(1.2 \times 10^{-4} \text{ kg}\bullet\text{m}^2)]$

$= 189 \text{ rad/s}$