HW #6 – Key

(1) Augmented matrix:

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \longrightarrow \begin{matrix}\text{Elementary} \\ \text{row} \\ \text{operations}\end{matrix}$$

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \qquad\quad \uparrow \quad \uparrow \\ \qquad\quad \text{free} \end{array} \Rightarrow \begin{cases} x = -1 + w \\ y = -2z \\ z, w \in \mathbb{R} \end{cases}$$

(2) a) Solution: $(-3r + s,\ r - s,\ r,\ s)$

Special solutions:

$r = 1,\ s = 0 \longrightarrow (-3, 1, 1, 0) = \vec{s}_1$

$r = 0,\ s = 1 \longrightarrow (1, -1, 0, 1) = \vec{s}_2$

<u>Sol.:</u> $c_1 \vec{s}_1 + c_2 \vec{s}_2$; $c_1, c_2 \in \mathbb{R}$.

b) $\vec{x}_p = \begin{bmatrix} -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}$, $\underset{\substack{\uparrow \\ \text{Complete} \\ \text{solution}}}{\vec{x}} = \vec{x}_p + c_1 \vec{s}_1 + c_2 \vec{s}_2$; $c_1, c_2 \in \mathbb{R}$.

3 a. $R = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ → Reduced row echelon form.

$$\vec{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{Particular sol.}} + c \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\text{special sol.}}, \quad c \in \mathbb{R}.$$

b. $R = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} -2 \\ 7 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad c \in \mathbb{R}.$$

4 a) $\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

Test for linear independence:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$$\begin{cases} 1c_1 + 2c_2 + 1c_3 + 0c_4 = 0 \\ 0c_1 - 2c_2 - 1c_3 - 1c_4 = 0 \\ 1c_1 + 3c_2 + 1c_3 + 1c_4 = 0 \\ 1c_1 + 2c_2 + 0c_3 + 1c_4 = 0 \end{cases}$$

Elimination gives:
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

thus there are nontrivial solutions and the matrices are LD.

b) $\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\vec{u}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

test for independence:

$$c_1 \vec{u}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{u}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A\vec{x} = \vec{0}$$

where
$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$ Elimination gives

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow$$ trivial sol. $\Rightarrow$ linearly indep vectors.

Moreover, A is invertible $\Rightarrow \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ spans $M_{2\times2}$.

$\Rightarrow$ it is a basis.

c) $\vec{u}_1 = x^2+1$, $\vec{v}_2 = x^2-1$, $\vec{u}_3 = 2x-1$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span P_2 ? Every polynomial $a_0 + a_1 x + a_2 x^2$ should be written as: $C_1 \vec{u}_1 + C_2 \vec{u}_2 + C_3 \vec{v}_3$, or

$$c_1 (x^2+1) + c_2 (x^2-1) + C_3 (2x-1) = a_0 + a_1 x + a_2 x^2$$

this implies that

$$\begin{cases} c_1 - c_2 - c_3 = a_0 \\ c_1 + c_2 + 2c_3 = a_1 \\ c_1 + c_2 + 0c_3 = a_2 \end{cases} \rightarrow A\vec{x} = \vec{b}.$$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{Elimination}} U = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$$

A is invertible $\Rightarrow$ the system above has unique solution. $\Rightarrow \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ span P_2. Moreover $A\vec{x} = \vec{0}$ has only the trivial sol $\Rightarrow \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is also linearly independent.

thus: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for P_2.

5 $\underline{\underline{5}}$

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

the pivot columns in R (reduced row echelon form) give the columns of A that form a basis for the column space. In this case:

$$R = \begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Basis for the colomn space: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

⑥

⑥

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \\ \begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{array} \end{array}$$

Reduced row echelon form:

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns : 1, 2, 4 $\rightarrow$

Basis of spanned subspace : columns 1, 2, 4 of A.

<u>Basis:</u> $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$