MATH 2418: Linear Algebra

Assignment 11

Due `November 30, 2016` Term `Fall, 2016`

Instructions: Recommended exercises from the textbook: Section 6.2: 1, 3, 6, 7, 9, 11, 12, 14, 26, 31; Section 6.3: 1, 4, 8, 13; Section 6.4: 1, 2, 3, 5, 7, 13, 18, 20; Section 6.5: 1, 2, 3, 4,9, 14, 15, 16, 18, 20, 21, 22, 25, 26.

1. Find (i) the characteristic equation, (ii) the distinct eigenvalues, (iii) basis for the eigenspaces, and (iv) diagonalizability of the following matrices

   (i) [4 points] $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   (ii) [6 points] $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

2. Let A be the matrix representation of the counterclockwise rotation in $\mathbb{R}^2$ through the origin by angle $0 < \phi < \pi/2$.

   (i) [5 points] Find A;

   (ii) [5 points] Is there a diagonal matrix D with real valued entries that is similar to A?

3. [10 points] Find a 3×3 matrix A that has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, and for which $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ are their respective eigenvectors.

4. Let $u = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$ be an $1 \times n$ matrix and I the $n \times n$ identity matrix. Let A be the $n \times n$ matrix such that

$$A = u^T u + I.$$

(i) [6 points] Find all eigenvalues of A;

(ii) [2 points] Find trace(A);

(iii) [2 points] Find $\det(A)$.

5. [10 points] Consider the system of differential equations $\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t)$ with initial value $x(0) = (-1, 1, 0)$, where $A = \begin{bmatrix} 5 & -28 & -18 \\ -1 & 5 & 3 \\ 3 & -16 & -10 \end{bmatrix}$.

(i) Find the eigenvalues of A;

(ii) Find a matrix P such that $P^{-1}AP$ is diagonal.

(iii) Find the solution $x(t)$ of the equations with $x(0) = (-1, 1, 0)$.

6. Let Q be an $n \times n$ real orthogonal matrix.[1]

   (i) [2 points] Justify that for every $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

   (ii) [2 points] Justify that for every $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, $(Q(\mathbf{x} - \mathbf{y})) \cdot (Q(\mathbf{x} - \mathbf{y})) = (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$.

   (iii) [2 points] Suppose that Q has real eigenvalues. Find all possible real eigenvalues of Q.

   (iv) [2 points] Suppose that Q has real eigenvalues. Is it true Q must have all possible eigenvalues obtained in iii)? Justify your answer.

   (v) [2 points] Is it possible that an orthogonal matrix does not have any real eigenvalues? Justify your answer.[2]

[1] May use matrix multiplication to represent dot product.

[2] Use the rotation matrix, e.g., the matrix obtained in Question 2.

7. [10 points] Let $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$. Find a spectral decomposition of A.

8. [10 points] Let f be a polynomial of $\mathbf{x} = (x_1,\, x_2,\, x_3,\, x_4) \in \mathbb{R}^4$ defined by

$$f(\mathbf{x}) = 4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 8x_1x_2 - 6x_1x_4 + 6x_2x_3 + 8x_3x_4.$$

(i) [2 points] Find the symmetric matrix A, such that $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

(ii) [6 points] Find a change of variables $\mathbf{x} = Q\mathbf{y}$, $\mathbf{y} = (y_1,\, y_2,\, y_3,\, y_4)$ such that $f(Q\mathbf{y})$ does not contain cross-product terms y_iy_j with $i,\, j = 1,\, 2,\, 3,\, 4$ and $i \neq j$.

(iii) [2 points] Determine whether A is positive definite or not.

9. [15 points] True or False. For all subquestions below, assume that A is an $n \times n$ real matrix.

(i) **T** **F**: λ is an eigenvalue of A if and only if $-\lambda$ is an eigenvalue of $-A$.

(ii) **T** **F**: If A is an $n \times n$ matrix and λ is one of its eigenvalues, then $\operatorname{rank}(A - \lambda I_n) < n$.

(iii) **T** **F**: The eigenvalues of a 2×2 real matrix A are both negative if $\operatorname{tr}(A)^2 > 4\det(A) > 0$ and $\operatorname{tr}(A) < 0$.

(iv) **T** **F**: If $\lambda = 0$ is not an eigenvalue of A, then the column space of A is $\mathbb{R}^n$.

(v) **T** **F**: If the column vectors of a square matrix A are linearly independent, then $\mathbf{0}$ is not an eigenvalue of A.

(vi) **T** **F**: Let E be an elementary matrix and A an $n \times n$ matrix. Then EA and A have the same eigenvalues.

(vii) **T** **F**: Two eigenvectors of a symmetric matrix A corresponding to two distinct eigenvalues are orthogonal to each other.

(viii) **T** **F**: If a square matrix A is diagonalizable with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then there is a unique matrix P such that $P^{-1}AP = \operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, where $\operatorname{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is the diagonal matrix with the eigenvalues on the main diagonal.

(ix) **T** **F**: If a square matrix A is diagonalizable, then so is A^T.

(x) **T** **F**: If λ is an eigenvalue of a square matrix A, then λ^k must be an eigenvalue of A^k for any positive integer k.

(xi) **T** **F**: Let A be an $n \times n$ real matrix. Then A and A^T have the same characteristic polynomial.

(xii) **T** **F**: Let $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ be a polynomial. Then every eigenvector of A is also an eigenvector of $f(A) = a_0 I + a_1 A + \cdots + a_n A^n$.

(xiii) **T** **F**: Let A be an $n \times n$ positive definite real symmetric matrix. Then every eigenvalue of A is positive.

(xiv) **T** **F**: Every real symmetric matrix is diagonalizable.

(xv) **T** **F**: Every eigenvalue of a real symmetric matrix is real.