HW 5 - Part I

(1)
$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$$

a) $R_3 \to R_3 - 2R_1$, $l_{31} = 2$

$R_3 \to R_3 + 4R_2$, $l_{32} = -4$

$$U = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$$

$c_1 = 1$, $c_2 = 0$,

$c_3 = -2c_1 + 4c_2 + 1$

$c_3 = -1$

c)
$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow$$

$y_3 = -1/14$

$y_2 = -3y_3 = 3/14$

$y_1 = 1 + 3y_2$

$y_1 = 10/7$

d) $A\vec{x} = \vec{b} \to LU\vec{x} = \vec{b} \to U\vec{x} = L^{-1}\vec{b} = \vec{c}$

$\underbrace{L\vec{c} = \vec{b}}_{\text{(b) above}}$, $\underbrace{U\vec{x} = \vec{c}}_{\text{(c) above}}$ $\Rightarrow \vec{y}$ is the solution to $A\vec{x} = \vec{b}$.

2 $A = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

$R_2 \to R_2 + R_1 \,, \quad l_{21} = -1$

$R_3 \to R_3 - R_1 \,, \quad l_{31} = 1$

$A = LU \longrightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Now: write $U = D\hat{U}$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally: $\boxed{A = L D \hat{U}}$

③ $A = LU$, $L = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ $U = \begin{bmatrix} c & e \\ 0 & d \end{bmatrix}$ ③ S-1

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & e \\ ac & ae+d \end{bmatrix} \Rightarrow \left.\begin{matrix} c = 0 \\ ac = 1 \end{matrix}\right\} \text{impossible.}$$

④ $S = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 8 \\ 2 & 8 & 23 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $l_{21} = 1$

$R_3 \to R_3 - 2R_1$, $l_{31} = 2$

$R_3 \to R_3 - 3R_2$, $l_{32} = 3$

$U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

$\hat{U} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\boxed{A = L D \underbrace{L^T}_{\hat{U}}}$$

5 $A = \begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ 4

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$R_2 \Leftrightarrow R_3$

$R_3 \to R_3 - R_1$, $l_{31} = 1$

$U = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

- $L\vec{c} = P\vec{b} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$

$c_1 = -2$, $c_2 = 4$, $c_3 = 1 - c_1 = 3$

- $U\vec{x} = \vec{c} \to \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$

$\boxed{x_3 = 3}$, $2x_2 + x_3 = 4 \Rightarrow \boxed{x_2 = \frac{1}{2}}$,

$3x_1 = -2 + x_2 = -\frac{3}{2} \Rightarrow \boxed{x_1 = -\frac{1}{2}}$

HW 5 - Part 2

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1 (a) • $(0,0,0)$ is in the set.

• Closed under $\oplus$: (addition)

$(a, a+c, c) + (a', a'+c', c') = (a+a', (a+a')+(c+c'), c+c')$ ✓

• Closed under $\odot$: (scalar mult.)

$k(a, a+ac, c) = (ka, ka+kc, kc)$ ✓

→ the set <u>is</u> a subspace of $\mathbb{R}^3$.

(b) The set is <u>not</u> <u>closed</u> under $\oplus$:

$(a, a+c+1, c) + (a', a'+c'+1+c') =$

$= (a+a', \underbrace{(a+a') + (c+c') + 2}_{\neq (a+a') + (c+c') + 1}, c+c')$

→ the set is <u>NOT</u> a subspace.

(c) The set is <u>not</u> closed under multiplication by scalars: take $\underline{c=0}$, cA is the zero matrix and $0\vec{x} = \vec{0}$ has nontrivial solutions.

→ The set is <u>NOT</u> a subspace.

(d) • the zero $n \times n$ matrix belongs to this set.

• If A and B belong to the set:

$$A^T = -A \quad , \quad B^T = -B$$

$$(A+B)^T = A^T + B^T = -A - B = -(A+B) \Rightarrow$$

the set is closed under addition.

• $(kA)^T = kA^T = k(-A) = -kA \Rightarrow$

the set is closed under scaler mult.

→ The set is a subspace.

2 $\vec{b} = (b_1, b_2, b_3)$. Can we write any $\vec{b}$ as

$$\vec{b} = k_1 \vec{v}_1 + k_2 \vec{v}_2 + k_3 \vec{v}_3 \ ?$$

This eq. implies:

$$\begin{cases} k_1 + k_2 + 2k_3 = b_1 \\ k_1 + k_3 = b_2 \\ 2k_1 + k_2 + 3k_3 = b_3 \end{cases}$$

this system is solvable only if $b_3 - b_2 - b_1 = 0$.

thus, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ DOES NOT span $\mathbb{R}^3$.

(3) Solving the system $A\vec{x} = \vec{b}$ by elimination gives: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

this implies that:

$$\vec{b} = \underbrace{2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}}_{\text{columns of } A} \Rightarrow$$

$\Rightarrow \vec{b}$ is in the column space of A.

(4) (a) Reduced row echelon form:

$$R = \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $A\vec{x} = \vec{0}$, $\vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$

x_2: Free, x_4: Free, x_5: Free, x_7: Free

Null space: $(-3x_2 - 4x_4 - 2x_5, x_2, -2x_4, x_4, x_5, 0, x_7)$

where $x_2, x_4, x_5, x_7 \in \mathbb{R}$.

(c). $x_2 = 1,\ x_4 = x_5 = x_7 = 0 \Rightarrow$

$(-3, 1, 0, 0, 0, 0, 0) = \vec{s}_1$

- $x_2 = 0,\ x_4 = 1,\ x_5 = x_7 = 0 \Rightarrow$

$(-4, 0, -2, 1, 0, 0, 0) = \vec{s}_2$

- $x_2 = 0,\ x_4 = 0,\ x_5 = 1,\ x_7 = 0 \Rightarrow$

$(-2, 0, 0, 0, 1, 0, 0) = \vec{s}_3$

$\rightarrow$ Answer: $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$

(d) Rank = 3 (3 pivots)

5 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & r & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & r & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

where r and s are real numbers.

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⑥ • Only the trivial solution: either the 3 lines intersect at the origin, or two of them completely overlap and the other one intersects them at the origin.

• Nontrivial solutions: all 3 lines completely overlap one another.

⑦ (a) We have to consider 2 cases:

(i) $a = 0 \Rightarrow b \neq 0$ and $c \neq 0 \Rightarrow$

$ad - bc \neq 0$

reduced row echelon form is obtained by:

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \quad \begin{matrix} R_1 \to R_1/c \\ R_2 \to R_2/b \end{matrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_1 \to R_1 + \left(-\frac{d}{c}\right) R_2$$

✓

(ii) $a \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \quad R_1 \to R_1/a$$

$$\begin{bmatrix} 1 & b/a \\ 0 & \frac{ad-bc}{a} \end{bmatrix} \quad R_2 \to R_2 - cR_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_2 \to R_2 \times \left(\frac{a}{ad-bc}\right)$$

(b) Applying the same elimination steps as above the augmented matrix $\begin{bmatrix} a & b & k \\ c & d & e \end{bmatrix}$ is reduced to row echelon form:

$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \end{bmatrix}, \quad p, q \text{ real numbers}$$

We have one solution: $\begin{cases} x = p \\ y = q \end{cases}$