MATH 2418: Linear Algebra

Assignment# 3

Due : 09/14 Wednesday Term `Fall, 2016`

Solution.

[First Name] [Last Name] [Net ID]

Recommended Text Book Problems (do not turn in): [Sec 2.2: # 1, 2, 4, 6, 8, 12, 14, 32]; [Sec 2.3: # 1, 3, 4, 7, 9, 18, 30];

1. Given linear system $\begin{cases} 2x + 4y - 2z = 2 \\ 4x + 9y - 3z = 4 \\ -2x - 3y + 7z = 10 \end{cases}$.

   (a) Solve by reducing into upper triangular form and back substitution.

   (b) List all multipliers used and circle all the pivots.

(a) Aug. matrix

$$\begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix} \begin{matrix} \\ R_2-2R_1 \\ R_3+R_1 \end{matrix} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 5 & 12 \end{bmatrix}$$

$l_{21} = 2$

$l_{31} = -1$

$l_{32} = 1$

$$R_3 - R_2 \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 12 \end{bmatrix} = \text{U.T. form.}$$

$4z = 12$

$z = 3$

$y + z = 0$

$y = -3$

$2x + 4y - 2z = 2$

$2x + 4(-3) - 2(3) = 2$

$2x = 20$

$x = 10$

Sol:. $\boxed{x = 10, \ y = -3 \ z = 3}$

(b) Multipliers:

$l_{21} = 2$ $l_{31} = -1$, $l_{32} = 1$

$$\begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 12 \end{bmatrix} \rightarrow \text{pivots circled.}$$

2. Consider the linear system $\begin{cases} ax \quad + 2y = -2 \\ 3x + 6y = -6 \end{cases}$.

(a) For what values of a does the elimination break down (1) permanently (2) temporarily?

(b) Solve the system after fixing the temporary break down.

(c) Solve the system in case of permanent break down.

(a) (2) If $a=0$, the aug. matrix $\begin{bmatrix} 0 & 2 & -2 \\ 3 & 6 & -6 \end{bmatrix}$.

So elimination fails temporarily if $a=0$.

(1) Aug matrix $\begin{bmatrix} a & 2 & -2 \\ 3 & 6 & -6 \end{bmatrix}$

If $a=1$ $\quad \begin{bmatrix} 1 & 2 & -2 \\ 3 & 6 & 6 \end{bmatrix} \underset{\sim}{R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ elimination fails permanently if $a=1$.

(b) When $a=0$ $\quad \begin{bmatrix} 0 & 2 & -2 \\ 3 & 6 & -6 \end{bmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{bmatrix} 3 & 6 & -6 \\ 0 & 2 & -2 \end{bmatrix}$

$\Rightarrow \quad 2y = -2 \Rightarrow y = -1$

$3x + 6y = -6 \Rightarrow 3x + 6(-1) = -6 \Rightarrow x = 0$

Sol: $\boxed{x = 0, \quad y = -1}$

(c) In case of permanent breakdown.

$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$y = \text{free} = t$ (say)

$x + 2y = -2$

$x = -2t - 2$

Sol: $\boxed{x = -2t-2, \quad y = t}$

3. Consider a 3×3 matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

(a) Reduce matrix A into upper triangular form using elimination steps.

(b) Write down the matrices E_{21}, E_{31}, E_{32} that put A into triangular form.

(c) Write down the matrix M such that MA is upper triangular.

(a) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \underset{R_3 - 7R_1}{\overset{R_2 - 4R_1}{\sim}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \text{ is U.T.} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(b) $$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(c) $$M = E_{32} E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

4. Let $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 11 \\ 4 \\ 6 \end{bmatrix}$

(a) Write down the elementary matrices that reduce A into an upper triangular matrix.

(b) Write down the corresponding upper triangular system $U\mathbf{x} = \mathbf{c}$

(c) Solve the system

(a)

$$\begin{bmatrix} 0 & 2 & 3 & 11 \\ -2 & 3 & 1 & 4 \\ 1 & 4 & 0 & 6 \end{bmatrix}$$

$$R_1 \Leftrightarrow R_3 \quad \begin{bmatrix} 1 & 4 & 0 & 6 \\ -2 & 3 & 1 & 4 \\ 0 & 2 & 3 & 11 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_2 + 2R_1 \quad \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & 11 & 1 & 16 \\ 0 & 2 & 3 & 11 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 - \frac{2}{11}R_2 \quad \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & 11 & 1 & 16 \\ 0 & 0 & \frac{31}{11} & \frac{891}{11} \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{11} & 1 \end{bmatrix}$$

are elementary matrices.

(b) $Ux = c$

$$\left.\begin{aligned} x_1 + 4x_2 + \cdot &= 6 \\ 11x_2 + x_3 &= 16 \\ \frac{31}{11}x_3 &= \frac{89}{11} \end{aligned}\right\} \quad \text{OR} \quad \begin{bmatrix} 1 & 4 & 0 \\ 0 & 11 & 1 \\ 0 & 0 & \frac{31}{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ \frac{89}{11} \end{bmatrix}$$

(c) $\frac{31}{11}x_3 = \frac{89}{11} \Rightarrow x_3 = \frac{89}{31}$

$11x_2 + \frac{89}{31} = 16 \Rightarrow 11x_2 = 16 - \frac{89}{31} = \frac{407}{31} \Rightarrow x_2 = \frac{407}{11 \cdot 31} = \frac{37}{31}$

$x_1 + 4 \cdot \frac{37}{31} = 6 \Rightarrow x_1 = 6 - \frac{148}{31} = \frac{38}{31}$

Sol: $\boxed{x_1 = \frac{38}{31}, \quad x_2 = \frac{37}{31}, \quad x_3 = \frac{89}{31}}$

5. Let $E = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Find the products EF and FE. Is $EF = FE$?

(b) Write down the matrices E^{-1} and F^{-1}.

(c) Find the products $F^{-1}E^{-1}(EF)$ and $E^{-1}F^{-1}(EF)$.

(a) $EF = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 5 & 1 & -15 \\ 0 & 0 & 1 \end{bmatrix}$ ← perform $R_2 + 5R_1$

$FE = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← perform $R_1 = R_1 - 3R_3$.

$EF \neq FE$

(b) $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad F^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $F^{-1}E^{-1}(EF) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 5 & 1 & -15 \\ 0 & 0 & 1 \end{bmatrix}$ ← EF calculated in (a)

$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E^{-1}F^{-1}(EF) = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 5 & 1 & -15 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & -15 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -15 \\ 0 & 0 & 1 \end{bmatrix}$

6. True or False. Circle your answer.

(a) **T** **F** Consider the augmented matrix $[A \;\; \mathbf{b}] = \begin{bmatrix} 4 & 5 & b_1 \\ 0 & r & b_2 \end{bmatrix}$. The existence and uniqueness of the solution depends on all of r, b_1, b_2.

(b) **T** **F**: If $AB = I$ and $BC = I$ then $A = C$.

(c) **T** **F**: If P is a permutation matrix(a matrix that switches two rows) then $P^{100} = I$.

(d) **T** **F** If elimination fails permanently, the system has no solution.

(e) **T** **F** There is only one possible linear system which reduces to $\begin{cases} x + \;\; y = 1 \\ \quad\;\; 2y = 3 \end{cases}$ after one elimination step.

(a) Does not depend on b_1.

(b)
$$BC = I \Rightarrow A(BC) = AI$$
$$\Rightarrow (AB)C = A$$
$$\Rightarrow IC = A$$
$$\Rightarrow C = A$$

(c) Since $P^2 = I \Rightarrow P^{2k} = I$

(d) still can have infinitely many solutions.

(e) $\boxed{\begin{matrix} x + y = 1 \\ x + 3y = 4 \end{matrix}}$ & $\boxed{\begin{matrix} x + y = 1 \\ 2x + 4y = 5 \end{matrix}}$ are 2 different linear systems.