Homework 1 Answers

Problem 1

Write a recursive method in pseudocode that returns the count of ones digits in the binary representation of N. Use the fact that this is equal to the count of ones digits in the binary representation of $N/2$, plus one if N is odd.

Procedure *count*

Input: N $\{N \in \mathbb{N}\}$

Output: count of ones digits in binary representation of N

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if N < 2 then
    return N
else
    return (N mod 2) + count(⌊N/2⌋)
end if
```

Problem 2a

Evaluate the following sum: $\sum_{i=0}^{\infty} \frac{1}{4^i}$

$$
\begin{aligned}
\text{Let } S &= \sum_{i=0}^{\infty} \frac{1}{4^i} && \text{(given)} \\
&= \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i \\
&= \sum_{i=0}^{\infty} 4^{-i}
\end{aligned}
$$

$$
\begin{aligned}
4S &= 4 \cdot \sum_{i=0}^{\infty} 4^{-i} && \text{(multiply } S \text{ by 4)} \\
&= \sum_{i=0}^{\infty} \left(4 \cdot 4^{-i}\right) \\
&= \sum_{i=0}^{\infty} 4^{1-i} \\
&= 4 + \sum_{i=1}^{\infty} 4^{1-i} \\
&= 4 + \sum_{i=0}^{\infty} 4^{-i}
\end{aligned}
$$

$$
\begin{aligned}
4S - S &= 4 + \sum_{i=0}^{\infty} 4^{-i} - \sum_{i=0}^{\infty} 4^{-i} && \text{(subtract } S \text{ from } 4S\text{)} \\
3S &= 4
\end{aligned}
$$

$$
\therefore\ S = \frac{4}{3} \qquad \text{(divide } 3S \text{ by 3)}
$$

Problem 2b

Evaluate the following sum: $\sum_{i=0}^{\infty} \frac{i}{4^i}$

$$
\begin{aligned}
\text{Let } S &= \sum_{i=0}^{\infty} \frac{i}{4^i} && \text{(given)} \\
&= 0 + \sum_{i=1}^{\infty} \frac{i}{4^i} \\
&= \sum_{i=1}^{\infty} \frac{i}{4^i} \\
&= \sum_{i=1}^{\infty} \left(i \cdot \frac{1}{4^i} \right) \\
&= \sum_{i=1}^{\infty} \left(i \cdot \left(\frac{1}{4} \right)^i \right) \\
&= \sum_{i=1}^{\infty} \left(i \cdot 4^{-i} \right)
\end{aligned}
$$

$$
\begin{aligned}
4S &= 4 \cdot \sum_{i=1}^{\infty} \left(i \cdot 4^{-i} \right) && \text{(multiply } S \text{ by 4)} \\
&= \sum_{i=1}^{\infty} \left(4 \cdot i \cdot 4^{-i} \right) \\
&= \sum_{i=1}^{\infty} \left(i \cdot 4 \cdot 4^{-i} \right) \\
&= \sum_{i=1}^{\infty} \left(i \cdot 4^{1-i} \right) \\
&= 1 + \sum_{i=2}^{\infty} \left(i \cdot 4^{1-i} \right) \\
&= 1 + \sum_{i=1}^{\infty} \left((i+1) \cdot 4^{-i} \right)
\end{aligned}
$$

$$
\begin{aligned}
4S - S &= 1 + \sum_{i=1}^{\infty} \left((i+1) \cdot 4^{-i}\right) - \sum_{i=1}^{\infty} \left(i \cdot 4^{-i}\right) && \text{(subtract } S \text{ from } 4S\text{)} \\
3S &= 1 + \sum_{i=1}^{\infty} \left(((i+1) \cdot 4^{-i}) - (i \cdot 4^{-i}) \right) \\
&= 1 + \sum_{i=1}^{\infty} \left(((i+1) - i) \cdot 4^{-i} \right) \\
&= 1 + \sum_{i=1}^{\infty} \left((1) \cdot 4^{-i} \right) \\
&= 1 + \sum_{i=1}^{\infty} 4^{-i} \\
&= 1 + \sum_{i=0}^{\infty} (4^{-i}) - 1 \\
&= \sum_{i=0}^{\infty} 4^{-i} \\
&= \frac{4}{3} && \text{(from Problem 2a)} \\
\therefore\ S &= \frac{4}{9} && \text{(divide } 3S \text{ by 3)}
\end{aligned}
$$

Problem 3

Let F_i denote the i-th Fibonacci number where $F_0 = F_1 = 1$.

Prove the following: $\sum_{i=1}^{N-2} F_i = F_N - 2$

Proof. The proof of the claim $\sum_{i=1}^{N-2} F_i = F_N - 2$ is by induction over N.

<u>Base Case</u>
Consider the case when $N = 3$.

$$\begin{aligned}
\sum_{i=1}^{3-2} F_i &= F_3 - 2 \\
\sum_{i=1}^{1} F_i &= F_3 - 2 \\
F_1 &= F_1 \\
1 &= 1
\end{aligned}$$

This is a tautology, ergo the claim is true when $N = 3$.

<u>Inductive Case</u>
Assume $\sum_{i=1}^{k-2} F_i = F_k - 2$ is true for some arbitrary value $k \in \mathbb{N}$ where $k \geq 3$.
Consider the case when $N = k + 1$.

$$\begin{aligned}
\sum_{i=1}^{(k+1)-2} F_i &= \sum_{i=1}^{k-1} F_i \\
&= F_{k-1} + \sum_{i=1}^{k-2} F_i \\
&= F_{k-1} + F_k - 2 \quad &&\text{(by our inductive hypothesis)} \\
&= F_{k+1} - 2 &&\text{(by the definition of the Fibonacci numbers, and that } k + 1 > 1\text{)}
\end{aligned}$$

Ergo, if the claim is true or some arbitrary value $k \in \mathbb{N}$ where $k \geq 3$, then the claim is also true for $k + 1$.

$\therefore \sum_{i=1}^{N-2} F_i = F_N - 2$ is true for $\forall N \in \mathbb{N} \mid N \geq 3$ by the principle of mathematical induction. $\square$

Problem 4

Prove the following by induction: $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

Proof. The proof of the claim $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$ is by induction over n.

<u>Base Case</u>
Consider the case when $n = 1$.

$$
\begin{aligned}
1^3 &= \frac{1^2(1+1)^2}{4} \\
1^3 &= \frac{1^2(2)^2}{4} \\
1 &= \frac{1(4)}{4} \\
1 &= 1
\end{aligned}
$$

This is a tautology, ergo the claim is true when $n = 1$.

<u>Inductive Case</u>
Assume $1^3 + 2^3 + 3^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$ is true for some arbitrary value $k \in \mathbb{N}$ where $k \geq 1$.
Consider the case when $n = k + 1$.

$$
\begin{aligned}
1^3 + 2^3 + 3^3 + ... + (k+1)^3 &= 1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 && \text{(by our inductive hypothesis)} \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} && \text{(factor out a common } (k+1)^2\text{)} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} && \text{(factorize } (k^2 + 4k + 4)\text{)} \\
&= \frac{(k+1)^2((k+1)+1)^2}{4}
\end{aligned}
$$

Ergo, if the claim is true for some arbitrary value $k \in \mathbb{N}$ where $k \geq 1$, then the claim is also true for $k + 1$.

$\therefore 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$ is true for $\forall n \in \mathbb{N} \mid n \geq 1$ by the principle of mathematical induction. □