PACKET 3
(7.4) Partial Fraction Expansion, (7.5) Strategy for Integration, (7.8) Improper Integrals

(7.4) Partial Fraction Expansion

Partial fraction expansion is a technique often needed when an integrand consists of a ratio of two polynomials (i.e., a rational function).

1) First, write out the partial fraction expansion of the given function, using only general coefficients. You do NOT need to solve for these constants.

$$\frac{2x^2 - x + 7}{\left(x^2 - 9\right)^2 (3x^2 + 2)^3}$$

Ans:

$$\frac{2x^2 - x + 7}{\left(x^2 - 9\right)^2 (3x^2 + 2)^3} = \frac{A}{\left(x-3\right)^2} + \frac{B}{x-3} + \frac{C}{\left(x+3\right)^2} + \frac{D}{x+3} + \frac{Ex+F}{(3x^2+2)^3} + \frac{Gx+H}{(3x^2+2)^2} + \frac{Ix+J}{3x^2+2}$$

2) Apply the method of partial fraction expansion to compute the integral:

$$\int \frac{dx}{x^2 + 2x - 3}.$$

As you have probably already observed, many mathematical problems can be solved by more than one method. Note that one can also complete the square in the denominator of the integrand in problem #2 and apply trigonometric substitution to compute the integral. We will defer this exercise to the end of the packet.

Ans: Noting that $x^2 + 2x - 3 = (x+3)(x-1)$, we can write

$$\frac{1}{x^2+2x-3} = \frac{1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

for some constants A and B.

Multiplying both sides by $(x+3)(x-1)$, we have
$1 = A(x-1) + B(x+3)$, and collecting like terms, gives us

$$1 = (A+B)x + (3B - A).$$

It follows that

$A+B=0$ and $3B-A=1$ so that $A=-\frac{1}{4}$

and $B=\frac{1}{4}$.

Then,

$$\int \frac{dx}{x^2+2x-3}=\int\left(\frac{-\frac{1}{4}}{x+3}+\frac{\frac{1}{4}}{x-1}\right)dx=-\frac{1}{4}\int\frac{1}{x+3}dx+\frac{1}{4}\int\frac{1}{x-1}dx$$

$$=-\frac{1}{4}\ln|x+3|+\frac{1}{4}\ln|x-1|+C$$

An alternate problem:

$$\int \frac{dx}{x^2-3x-10}$$

Ans: $\frac{1}{7}\ln|x-5|-\frac{1}{7}\ln|x+2|+C$

Now, compute the following integrals.

3) $\int \frac{8x^2-9x+1}{4x^3-4x^2+x}dx$

Ans: This breaks down into $\frac{A}{x}+\frac{B}{(2x-1)^2}+\frac{C}{2x-1}$ to yield

$$\ln|x|+\frac{3}{2(2x-1)}+\ln|2x-1|+C$$

The first and last terms require a log rule for integration; the middle term requires a power rule.

An alternate problem:

$$\int \frac{9x^2+10x+2}{9x^3+6x^2+x}dx$$

Ans: $2\ln|x|-\frac{1}{3(3x+1)}-\ln|3x+1|+C$

4) $\int \frac{x^2 - x + 2}{2x^3 - 3x^2 + 12x + 7} dx$ Note: partial fraction expansion will be completely unnecessary here! Instead, you can apply a u-substitution that will greatly simplify this problem.

Ans: This does not need PFE; use u-substitution with $u = 2x^3 - 3x^2 + 12x + 7$ to yield $\frac{1}{6} \ln\left|2x^3 - 3x^2 + 12x + 7\right| + C$

An alternate problem:

$$\int \frac{x+2}{x^2 + 3x + 2} dx$$

Note: partial fraction expansion will be completely unnecessary here! Instead, you can first perform some algebra, which will greatly simplify this problem.

Ans: $\ln|x+1| + C$

5) $\int \frac{16 - 7x - x^2}{(x+2)(x^2+9)} dx$

Ans: This breaks down into $\frac{A}{x+2} + \frac{Bx + C}{x^2 + 9}$ to yield

$2\ln|x+2| - \frac{3}{2}\ln\left|x^2 + 9\right| - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$

The second fraction in the expansion itself splits into 2 terms, where the first applies a log rule and the second yields an arcctan function.

An alternate problem:

$$\int \frac{-13 + 8x - 5x^2}{(x-1)(x^2+4)} dx$$

Ans: $-2\ln|x-1| - \frac{3}{2}\ln\left|x^2 + 4\right| + \frac{5}{2}\arctan\left(\frac{x}{2}\right) + C$

If an integrand is a rational function whose numerator is of a greater degree than that of the denominator, long division must first be performed before partial fraction expansion can even be considered (this is needed in some of the practice problems.)

6) Perform long division on the following:

$$\frac{x^4 - 2x^3 + 4x - 1}{x^3 + x + 3}$$

Make sure to express your answer in the form: $quotient + \dfrac{remainder}{divisor}$

Ans: $x-2+\dfrac{-x^2+3x+5}{x^3+x+3}$

(7.5) Strategy for Integration

This section deals with one of the major challenges students face when encountering a "random" integral: what type of integral is it, and what is the first "move" one needs to make when computing it? Oftentimes, it boils down to deciding what the most appropriate substitution is for a given integral.

For example, the integral $\int \frac{x}{\sqrt{4x^2+9}}dx$ can be most easily solved by the u-substitution, $u = 4x^2+9$.

However, the integral $\int \frac{1}{\sqrt{4x^2+9}}dx$ is best computed with a trigonometric substitution, where $2x = 3\tan\theta$.

Sometimes, the best first move is not a substitution at all. For example, the integral $\int \frac{x^2-3x+1}{x^2+2}dx$ demands that we first perform polynomial division. Then, one can decide whether u-substitution or some other technique is appropriate.

Keep in mind the following techniques that you may need to apply when computing an integral:

1) Applying a basic integral rule, such as $\int \cos x dx = \sin x + C$
2) Applying algebraic or arithmetic operations before using another method
3) Applying u-substitution
4) Applying integration by parts
5) Applying techniques for trigonometric integrals
6) Applying partial fraction expansion
7) Applying trigonometric substitution
8) Applying some "trick" that may not be so obvious

Find the following integrals.

a) First, decide what you think is the best approach to solving each integral: a simple integral rule, u-subsitution, integration by parts, applying a trigonometric substitution, etc.
b) Then, if time, evaluate each integral.

1) $\int \frac{1}{1+x^2}dx$ Ans. $\arctan x + C$

2) $\int \frac{1}{x^2+7}dx$ Ans. $\frac{\sqrt{7}}{7}\arctan\left(\frac{\sqrt{7}\,x}{7}\right)+C$

3) $\int \frac{1}{\sqrt{1-x^2}}dx$ Ans. $\arcsin x+C$

4) $\int \frac{x}{\sqrt{1-x^2}}dx$ Ans. $-\sqrt{1-x^2}+C$ (u-substitution $u=1-x^2$)

5) $\int_{-1/2}^{0} \frac{x^2}{\sqrt{1-x^2}}dx$ Ans. $\left.\frac{1}{2}\arcsin x-\frac{1}{2}x\sqrt{1-x^2}\right|_{-1/2}^{0}=\frac{\pi}{12}-\frac{\sqrt{3}}{8}$ (trig. sub with sine function)

6) $\int \frac{x}{1-x^2}dx$ Ans. $-\frac{1}{2}\ln\left|1-x^2\right|+C$ (u-substitution $u=1-x^2$)

7) $\int \frac{x^2}{1-x^2}dx$ Ans. $-x+\frac{1}{2}\ln|1+x|+\frac{1}{2}\ln|1-x|+C$ (long division and partial fraction expan.)

8) $\int e^{3x}dx$ Ans. $\frac{1}{3}e^{3x}+C$

9) $\int \sin(7x)dx$ Ans. $-\frac{1}{7}\cos(7x)+C$

10) $\int_{0}^{\pi/2} \sin^3(2x)\cos^2(2x)dx$ Ans. $\left.-\frac{1}{6}\cos^3(2x)-\frac{1}{10}\cos^5(2x)\right|_{0}^{\pi/2}=\frac{8}{15}$ (factor sin(2x), apply trig. ident., and u-sub)

11) $\int (x-3)\sin(2x)dx$ Ans. $-\frac{1}{2}(x-3)\cos(2x)+\frac{1}{4}\sin(2x)+C$ (integration by parts)

12) $\int \frac{e^{1/x^2}}{x^3}dx$ Ans. $-\frac{1}{2}e^{1/x^2}+C$ (u-substitution $u=\frac{1}{x^2}$)

(7.8) Improper Integrals

Limits

It is necessary to compute limits when computing improper integrals.

First, find the following limits.

1) $\lim_{x\to\frac{3}{2}^-}\frac{1}{2x-3}$ Ans: $-\infty$

2) $\lim_{x \to \frac{3}{2}^+} \frac{1}{2x-3}$ Ans: ∞

3) $\lim_{x \to \infty} \frac{1}{\sqrt{2x-3}}$ Ans: 0

4) $\lim_{x \to 3^+} \sqrt{2x-6}$ Ans: 0

5) $\lim_{x \to \infty} xe^{-x}$ Ans: 0

6) $\lim_{x \to -\infty} xe^{-x}$ Ans: $-\infty$

7) $\lim_{x \to -\infty} \arctan(3x-1)$ Ans: $-\frac{\pi}{2}$

8) $\lim_{x \to 2^+} \ln|x-2|$ Ans: $-\infty$

9) $\lim_{x \to 2^+} -\ln|x-2|$ Ans: ∞

10) $\lim_{x \to -\infty} \ln|3x-2|$ Ans: ∞

11) $\lim_{x \to -\infty} \sin(3x-2)$ Ans: diverges

For all the definite integrals discussed up until now, including those in the last section, the integrands have been continuous and the limits of integration have been finite, i.e., for $\int_a^b f(x)dx$, a and b have been finite and $f(x)$ has been continuous on the interval $[a,b]$. However, in many applications, including statistics, a and/or b may be infinite or the function $f(x)$ may not be continuous on $[a,b]$. For example, in statistics, one may need to compute something like $\int_0^\infty xe^{-x^2} dx$, where the upper "limit" is infinite. (The function $f(x) = e^{-x^2}$ is related to the "bell curve" distribution.) In such cases, one first treats the integral as a definite integral with finite limits, then, one takes a limit, i.e.,

$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx = \lim_{b \to \infty} F(x)\Big|_a^b = \lim_{b \to \infty} [F(b) - F(a)]$, where $F(x)$ is an antiderivative of $f(x)$.

For example, $\int_1^\infty \frac{1}{x^2}dx = \lim_{b \to \infty} \int_1^b x^{-2}dx = \lim_{b \to \infty} \frac{-1}{x}\Big|_1^b = \lim_{b \to \infty}\left[-\frac{1}{b} + \frac{1}{1}\right] = 0 + 1 = 1$

12) Determine whether the integral

$$\int_1^{\infty} \frac{1}{x} dx$$

converges or diverges. If it converges, give its value. If it diverges, determine whether it diverges to ∞ or $-\infty$, if possible.

Ans.

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty}\big[\ln|t| - \ln|1|\big] = (\ln\infty) = \infty$$

Diverges to ∞

Determine whether each integral converges or diverges. If the interval converges, find its value; else show that it diverges. In the latter case determine if the integral converges to ∞ or $-\infty$, if possible.

13) $\displaystyle\int_0^{\infty} xe^{-x^2} dx$

Ans.

$$\int_0^{\infty} xe^{-x^2} dx = \lim_{t\to\infty} \int_0^{t} xe^{-x^2} dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-t^2} + \frac{1}{2}e^{0}\right] = 0 + \frac{1}{2} = \frac{1}{2}$$

Converges to $\frac{1}{2}$

14) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2} dx$

Ans.

$$\int_{-\infty}^{\infty} xe^{-x^2} dx$$

$$= \int_{-\infty}^{0} xe^{-x^2} dx + \int_{0}^{\infty} xe^{-x^2} dx +$$

$$= \lim_{t\to-\infty} \int_{t}^{0} xe^{-x^2} dx + \lim_{t\to\infty} \int_{0}^{t} xe^{-x^2} dx$$

$$= \lim_{t\to-\infty} = -\frac{1}{2}e^{-x^2}\Big|_t^0 + \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^t$$

$$= \lim_{t\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-t^2}\right) + \lim_{t\to\infty}\left(-\frac{1}{2}e^{-t^2} + \frac{1}{2}\right)$$

$$= -\frac{1}{2} + 0 - 0 + \frac{1}{2} = 0$$

Converges to 0

Note: you could also observe that the integrand is an odd function, so, from the results of the previous problem, the integral must be 0.

Extra Practice

Find these integrals:

1) $\int \frac{x^3 - 3x + 1}{x^2 - 1} dx$

Ans: Since the degree of the numerator is greater than, or equal to, that of the denominator, one must perform polynomial division before partial fraction expansion:

$$\int \frac{x^3 - 3x + 1}{x^2 - 1} dx$$
$$= \int \left(x + \frac{-2x + 1}{x^2 - 1} \right) dx \; (after \;\; polynomial \;\; division)$$
$$= \int \left(x - \frac{3}{2}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x-1} \right) dx \; (after \;\; PFE)$$
$$= \frac{1}{2}x^2 - \frac{3}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$$

2) $\int \frac{dx}{x^2 + 2x - 3}$ This is a problem you may have previously solved with partial fraction expansion. This is the most efficient way to evaluate this integral; however there is another way.

a) First, complete the square in the denominator, and then apply trigonometric substitution to compute the integral.

Ans: $\int \frac{dx}{x^2 + 2x - 3} = \int \frac{dx}{x^2 + 2x + 1 - 1 - 3} = \int \frac{dx}{(x+1)^2 - 4}$

Letting $x + 1 = 2\sec\theta$, then $dx = 2\sec\theta\tan\theta d\theta$, so that

$$\int \frac{dx}{(x+1)^2-4}$$

$$= \int \frac{2\sec\theta\tan\theta d\theta}{4\sec^2\theta-4}$$

$$= \frac{1}{2}\int \frac{\sec\theta\tan\theta}{\tan^2\theta}d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}d\theta$$

$$= \frac{1}{2}\int \csc\theta d\theta$$

$$= -\frac{1}{2}\ln\left|\csc\theta+\cot\theta\right|+C$$

Using a right triangle with acute angle θ,

$$-\frac{1}{2}\ln\left|\csc\theta+\cot\theta\right|+C$$

$$= -\frac{1}{2}\ln\left|\frac{x+1}{\sqrt{(x+1)^2-4}}+\frac{2}{\sqrt{(x+1)^2-4}}\right|+C$$

b) Believe it or not, the answer you found with trig sub is equivalent to one you found from partial fraction expansion earlier. Show that your two answers are equivalent. Note that partial fraction expansion is the preferred method here, needless to say!

3) $\int \frac{\sqrt{x}}{x-1}dx$ Hint: first let $u=\sqrt{x}$

Ans: Let $u=\sqrt{x}$. Then,

$$du = \frac{1}{2\sqrt{x}}dx,$$

$$2\sqrt{x}du = dx,$$

and $2udu = dx$, so that

$$\int \frac{u}{u^2-1} 2u\,du$$
$$= 2\int \frac{u^2}{u^2-1} du$$
$$= 2\int \frac{(u^2-1)+1}{u^2-1} du$$
$$= 2\left(\int 1 + \frac{1}{u^2-1} \right) du \text{ (} \textit{note: poly. div. could also be done here.} \text{)}$$
$$= 2\int \left(1 + \frac{1}{2}\frac{1}{u+1} - \frac{1}{2}\frac{1}{u-1} \right) du \text{ (} \textit{part frac } \exp \textit{ has been done here} \text{)}$$
$$= 2\left(u + \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u+1| \right) + C$$
$$= 2\left(\sqrt{x} + \frac{1}{2}\ln\left|\sqrt{x}+1\right| - \frac{1}{2}\ln\left|\sqrt{x}+1\right| \right) + C$$
$$= 2\sqrt{x} + \ln\left|\sqrt{x}+1\right| - \ln\left|\sqrt{x}+1\right| + C$$
$$= 2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$$

Evaluate each integral.

a) First decide what you think is the best approach to solving each integral: a simple integral rule, u-subsitution, integration by parts, applying a trigonometric substitution, etc.
b) Then, if time, evaluate each integral.

1) $\int \tan^2(3x)dx$ Ans. $\frac{1}{3}\tan(3x) - x + C$ (rewrite $\tan^2(3x)$ as $1-\sec^2(3x)$)

2) $\int \sec^2(3x)\tan^3(3x)dx$ Ans. $\frac{1}{12}\tan^4(3x) + C$ (u-substitution $u = \tan(3x)$)

3) $\int \frac{\sec^2(3x)}{\tan^3(3x)}dx$ Ans. $-\frac{1}{6\sin^2(3x)} + C$ (simplify to $\sin^{-2}(3x)\cos(3x)$, then use $u = \sin(3x)$)

4) $\int \frac{e^{3x}}{e^{3x}+2}dx$ Ans. $\frac{1}{3}\ln\left|e^{3x}+2\right| + C$ (u-substitution $u = e^{3x}+2$)

5) $\int \frac{e^{3x}}{e^{6x}+1}dx$ Ans $\frac{1}{3}\arctan(e^{3x}) + C$ (u-substitution $u = e^{3x}$)

6) $\int \sqrt{x}e^{\sqrt{x}}dx$ Ans. Let $u=\sqrt{x}$,and use int-by-parts to get $2e^{\sqrt{x}}(x-2\sqrt{x}+2)+C$

7) $\int \frac{\ln(\arcsin x)}{\sqrt{1-x^2}}dx$ Ans. $\arcsin x\ln(\arcsin x)-\arcsin x+C$ (u-substitution $u=\arcsin x$)

8) $\int \frac{x}{\sqrt{3-2x}}dx$ Ans. Let u = 3-2x to get $-\frac{3}{2}\sqrt{3-2x}+\frac{1}{6}(3-2x)^{3/2}+C$

9) $\int \frac{dx}{\sqrt{7+6x-x^2}}$ Ans: complete the square to get $\arcsin\left(\frac{x-3}{4}\right)$

10) $\int \frac{1}{\sin x+1}dx$ Ans. $\tan x-\sec x+C$ (multiply by the conjugate sin x – 1)

11) $\int \frac{3-4x}{1+x^2}dx$ Ans. $3\arctan x-2\ln\left|1+x^2\right|+C$ (split the integral first)

12) $\int \frac{x^4-x^2+3x-2}{x^3+x}dx$ Ans: $\frac{x^2}{2}-2\ln|x|+3\arctan x+C$ (poly. div., then part. frac. expan.)

13) $\int \frac{x^2+3x+3}{(x+1)^3}dx$ Ans: $\ln|x+1|-\frac{1}{x+1}-\frac{1}{2(x+1)^2}+C$ (part. frac. expan.)

14) $\int \frac{1}{x-\sqrt[3]{x}}dx$ Hint: let $u=\sqrt[3]{x}$

Ans:

$$du=\frac{1}{3\left(\sqrt[3]{x}\right)^2}dx$$

$$3\left(\sqrt[3]{x}\right)^2 du=dx$$

$$3u^2du=dx$$

Therefore,

$$\int \frac{1}{x-\sqrt[3]{x}}dx$$
$$= \int \frac{1}{\left(\sqrt[3]{x}\right)^3 - \sqrt[3]{x}}dx$$
$$= \int \frac{1}{u^3-u}3u^2du$$
$$= 3\int \frac{u^2}{u^3-u}du$$
$$= 3\int \frac{u}{u^2-1}du$$
$$= 3\int \left(\frac{\frac{1}{2}}{u+1} + \frac{\frac{1}{2}}{u-1} \right) du$$
$$= \frac{3}{2}\int \left(\frac{1}{u+1} + \frac{1}{u-1} \right) du$$
$$= \frac{3}{2}\left[\ln|u+1| + \ln|u-1|\right] + C$$
$$= \frac{3}{2}\left[\ln\left|\sqrt[3]{x}+1\right| + \ln\left|\sqrt[3]{x}-1\right|\right] + C$$

15) $\int \frac{\cos x}{\sin^3 x - \sin x}dx$ Hint: $u = \sin x$

Ans: Let $u = \sin x$ so that

$$\int \frac{\cos x}{\sin^3 x - \sin x}dx$$
$$= \int \frac{du}{u^3-u}$$
$$= \int \frac{du}{u(u+1)(u-1)}$$

and then apply PFE to yield $-\ln|\sin x| + \frac{1}{2}\ln|\sin x+1| + \frac{1}{2}\ln|\sin x-1| + C$

16) $\int \sqrt{4x^2-9}dx$

Ans: let $2x = 3\sec\theta$

$$\int \sqrt{4x^2 - 9}\,dx = ...$$
$$= 3\int \tan^2\theta \sec\theta d\theta$$
$$= 3\int \left(\sec^2\theta - 1\right)\sec\theta d\theta$$
$$= 3\int \left(\sec^3\theta - \sec\theta\right)d\theta$$

Note:

$$\int \sec^3\theta d\theta =$$
$$= \int \sec\theta \sec^2\theta d\theta$$
$$= \sec\theta\tan\theta - \int \tan^2\theta \sec\theta d\theta \ (\text{int}\, by \ \ parts)$$
$$= \sec\theta\tan\theta - \int (\sec^2 x - 1)\sec\theta d\theta$$
$$= \sec\theta\tan\theta - \int \tan^2\theta \sec\theta d\theta \ (\text{int}\, by \ \ parts)$$
$$= \sec\theta\tan\theta - \int \sec^3 x d\theta + \int \sec x d\theta$$

Then, solve for $\int \sec^3\theta d\theta = \dfrac{\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|}{2} + C$

Therefore,

$$\int \sqrt{4x^2 - 9}\,dx = \frac{3\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)}{2} - 3\ln\left|\sec\theta + \tan\theta\right|$$
$$= \frac{3}{2}\left[\sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|\right] + C$$
$$= \frac{3}{2}\left[\frac{4x}{3}\frac{\sqrt{4x^2 - 9}}{3} - \ln\left|\frac{4x}{3} + \frac{\sqrt{4x^2 - 9}}{3}\right|\right] + C$$

17) $\displaystyle\int_1^\infty \frac{\ln x}{x^4}dx$

Ans.

Let $u = \ln x$ and $dv = \dfrac{1}{x^4}dx$, so that $du = \dfrac{1}{x}dx$, $v = -\dfrac{1}{3x^3}$, and

$$\int \frac{\ln x}{x^4}dx = -\frac{\ln x}{3x^3} + \frac{1}{3}\int \frac{1}{x^3}\frac{1}{x}dx = -\frac{\ln x}{3x^3} - \frac{1}{9x^3} + C$$

Therefore,

$$\int_1^{\infty} \frac{\ln x}{x^4} dx$$

$$= \lim_{t\to\infty}\left[-\frac{\ln t}{3t^3} - \frac{1}{9t^3}\right] - \left[-\frac{\ln 1}{3} - \frac{1}{9}\right]$$

$$= [0-0] - \left[0 - \frac{1}{9}\right] = \frac{1}{9}$$

where $\lim_{t\to\infty}\left[-\frac{\ln t}{3t^3}\right] = \lim_{t\to\infty}\left[-\frac{1}{9t^3}\right] = 0$, per L'Hopital's Rule.

Ans. Converges to $\frac{1}{9}$ (integration by parts)

18) $\int_4^{\infty} \frac{5}{x^2 - x - 6} dx$

Ans:

Since the denominator factors into $(x-3)(x+2)$ the vertical asymptotes do not fall inside the limits of integration. Thus, we have only a Type I integral. Applying partial fraction expansion, we have

$$\int_4^{\infty} \frac{5}{x^2 - x - 6} dx$$

$$= \int_4^{\infty} \frac{5}{(x-3)(x+2)} dx$$

$$= \lim_{t\to\infty} \int_4^{t} \frac{5}{(x-3)(x+2)} dx$$

$$= \lim_{t\to\infty} \ln\left|\frac{x-3}{x+2}\right|_4^t$$

$$= \lim_{t\to\infty}\left(\ln\left|\frac{t-3}{t+2}\right| - \ln\frac{1}{6}\right)$$

We note that, since the function $\ln(x)$ is continuous, we can move the limit notation inside this function:

$$\lim_{t\to\infty} \ln\left|\frac{t-3}{t+2}\right| = \ln\left|\lim_{t\to\infty}\frac{t-3}{t+2}\right| = \ln\left|\frac{1}{1}\right| = 0$$

where we have applied L'Hôpital's Rule in the third line. Therefore,

$$\int_4^{\infty} \frac{5}{x^2 - x - 6} dx = \lim_{t\to\infty}\left(\ln\left|\frac{t-3}{t+2}\right| - \ln\frac{1}{6}\right) = -\ln\frac{1}{6} = \ln 6$$

so the given integral converges.