PACKET 6

(8.2) Area of a Surface of Revolution, (9.1) DE Models, (9.2) Direction Fields, (9.3) Separable Equations, (9.4) Models of Population Growth

(8.2) Area of a Surface of Revolution

1) Recall the general formula for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the x-axis:

   $S = \int_a^b 2\pi r\,ds$. This is related to the surface area of a cylinder with height *l* and radius *r* (without top and bottom): $2\pi rl$, where *l* is replaced with *ds*, the differential arc length.

   a) Rewrite the integral $S = \int_a^b 2\pi r\,ds$ as an integral with respect to *x* for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the x-axis.

      Solution: $S = \int_a^b 2\pi r\,ds = \int_a^b 2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

   b) Rewrite the integral $S = \int_a^b 2\pi r\,ds$ as an integral with respect to *y* for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the *x*-axis. (Assume $f(a) \le f(b)$ and $x = g(y)$)

      Solution: $S = \int_{f(a)}^{f(b)} 2\pi r\,ds = \int_{f(a)}^{f(b)} 2\pi y\sqrt{1+(g'(y))^2}\,dy = \int_{f(a)}^{f(b)} 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

2) Recall the general formula for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the y-axis:

   $S = \int_a^b 2\pi r\,ds$

a) Rewrite the integral $S = \int_a^b 2\pi r\, ds$ as an integral with respect to x for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the y-axis.

Solution: $S = \int_a^b 2\pi r\, ds = \int_a^b 2\pi x \sqrt{1 + \left(f'(x)\right)^2}\, dx = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

b) Rewrite the integral $S = \int_a^b 2\pi r\, ds$ as an integral with respect to y for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the y-axis. (Assume $f(a) \le f(b)$ and $x = g(y)$)

Solution: $S = \int_{f(a)}^{f(b)} 2\pi r\, ds = \int_{f(a)}^{f(b)} 2\pi g(y) \sqrt{1 + \left(g'(y)\right)^2}\, dx = \int_{f(a)}^{f(b)} 2\pi g(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx$

3) Given the curve $y = e^{2x}$, $1 \le y \le e^2$ set up, but do not evaluate, an integral for the area of the surface obtained by rotating the curve about:

a) the y-axis with the independent variable x.

b) the y-axis with the independent variable y

c) the x-axis with the independent variable x

d) the x-axis with the independent variable y

Solution: (a) x the independent variable. $x = \frac{1}{2}\ln y$, so if $1 \le y \le e^2$, then $0 \le x \le 1$

$ds = \sqrt{1 + (2e^{2x})^2}\ dx \qquad SA = \int_0^1 2\pi x\, ds = \int_0^1 2\pi x \sqrt{1 + (2e^{2x})^2}\ dx$

(b) y the independent variable. $x = \frac{1}{2}\ln y$, $1 \le y \le e^2$

$ds = \sqrt{1 + \left(\frac{1}{2y}\right)^2}\ dy \qquad SA = \int_1^{e^2} 2\pi x\, ds = \int_1^{e^2} 2\pi\ \frac{1}{2}\ln y \sqrt{1 + \left(\frac{1}{2y}\right)^2}\ dy$

(c) x the independent variable. $\int_0^1 2\pi e^{2x} \sqrt{1 + (2e^{2x})^2}\ dx$

(d) y the independent variable $\int_1^{e^2} 2\pi\, y \sqrt{1 + \left(\frac{1}{2y}\right)^2}\ dy$

4) Consider the curve $y = 2\sqrt{1+2x}$, $2 \le x \le 10$.

a) If this segment of the curve is rotated about the *x*-axis, find the area of the resulting surface. Use the version of the integral with respect to *x*.

Solution;

Since $ds = \sqrt{1+\left(\frac{2}{(1+2x)^{1/2}}\right)^2}\,dx$, the surface area is $\int_2^{10} 2\pi r\,ds = \int_2^{10} 2\pi(2)\sqrt{1+2x}\sqrt{\frac{5+2x}{1+2x}}\,dx$

$$= \int_2^{10} 4\pi\sqrt{5+2x}\,dx = 4\pi\frac{2}{3}(5+2x)^{3/2}\Big|_2^{10} = \frac{4}{3}\pi\left[25^{3/2} - 27\right] \text{ sq units.}$$

b) Compute the integral with independent variable *y*, representing the same area of revolution about the x-axis.

Solution:

$x = 2 \Rightarrow y = 2\sqrt{1+2(2)} = 2\sqrt{5}$ and $x = 10 \Rightarrow y = 2\sqrt{1+2(10)} = 2\sqrt{21}$

$$y^2 = 4(1+2x)$$

$$\frac{y^2}{4} = 1+2x$$

$$x = \frac{y^2}{8} - \frac{1}{2}$$

$\frac{dx}{dy} = \frac{y}{4}$ so that $SA = \int_{2\sqrt{5}}^{2\sqrt{21}} 2\pi y\sqrt{1+\frac{y^2}{4}}\,dy$. Let $u = 1+\frac{y^2}{4}$ … Ans. $\frac{8}{3}\pi\left[25^{3/2} - 27\right]$

c) If this segment of the curve is rotated about the *y*-axis, set up, but to not compute, the integral with respect to *x* that represents the area of revolution. Note that you will have to change only one item in your integral from part (a)!

Solution: $\int_2^{10} 2\pi x\sqrt{\frac{5+2x}{1+2x}}\,dx$

d) If this segment of the curve is rotated about the *y*-axis, set up, but to not compute, the integral with respect to *y* that represents the area of revolution. Note that you will have to change only one item in your integral from part (b)!

Solution: $\displaystyle\int_{2\sqrt{5}}^{2\sqrt{2}} 2\pi\left(\frac{y^2}{8}-\frac{1}{2}\right)\sqrt{1+\frac{y^2}{4}}\,dy$

5) Consider the area of the surface formed by revolving the curve $y=\arcsin x$ on $0\le x\le \frac{1}{2}$ about the y-axis.

a) First, set up the integral with respect to x that represents this area. Do not evaluate the integral.

Solution: $\displaystyle\int_0^{1/2} 2\pi x\sqrt{1+\left(\frac{1}{\sqrt{1-x^2}}\right)^2}\,dx=\int_0^{1/2} 2\pi x\sqrt{1+\frac{1}{1-x^2}}\,dx$

b) It turns out that using the integral with respect to y will make the problem more tractable, but not trivial! Set up the integral with respect to y that represents the same area as your integral in part (a). Then, if time permits, evaluate this integral. (This can be reserved for the Extra Problems.)

Solution: Since $y=\arcsin x$, then $\sin y = x$ on $0\le x\le\frac{1}{2}$. Also, when $x=0$, $y=\arcsin 0=0$, and when $x=\frac{1}{2}$, $y=\arcsin\frac{1}{2}=\frac{\pi}{6}$

Therefore,

$$\int_0^{1/2} 2\pi x\sqrt{1+\left(\frac{1}{\sqrt{1-x^2}}\right)^2}\,dx$$

$$=\int_0^{\pi/6} 2\pi\sin y\sqrt{1+\cos^2 y}\,dy$$

Now, let $u=\cos y$. Then, $du=-\sin y\,dy$, and

$$\int 2\pi\sin y\sqrt{1+\cos^2 y}\,dy$$

$$=-2\pi\int\sqrt{1+u^2}\,du$$

Now, let $u=\tan\theta$ so that $du=\sec^2\theta\,d\theta$ and

$$-2\pi\int\sqrt{1+u^2}\,du$$

$$=-2\pi\int\sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta$$

$$=-2\pi\int\sec\theta\sec^2\theta\,d\theta$$

We then perform integration by parts (with a few other steps!!) to obtain

$$-2\pi\int\sec\theta\sec^2\theta\,d\theta$$

$$=-2\pi\left[\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln|\sec\theta+\tan\theta|\right]$$
$$=-\pi\left[\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right]$$
$$=-\pi\left[u\sqrt{u^2+1}+\ln\left|\sqrt{u^2+1}+u\right|\right]$$
$$=-\pi\left[\cos y\sqrt{\cos^2 y+1}+\ln\left|\sqrt{\cos^2 y+1}+\cos y\right|\right]$$

Finally,

$$\int_0^{\pi/6}2\pi\sin y\sqrt{1+\cos^2 y}\,dy$$
$$=-\pi\left[\cos y\sqrt{\cos^2 y+1}+\ln\left|\sqrt{\cos^2 y+1}+\cos y\right|\right]\Big|_0^{\pi/6}$$
$$=\pi\left[\sqrt{2}+\ln\left(\sqrt{2}+1\right)-\frac{\sqrt{21}}{4}-\ln\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)\right]$$

(9.1) DE Models

1. For the differential equation

$$\frac{dy}{dt}=y^3-y^2-6y$$

a) Find the equilibrium solutions for the DE.

Solution:
These solutions occur when dy/dt=0, or when the right side is 0:

$$y^3 - y^2 - 6y = 0$$
$$y(y^2 - y - 6) = 0$$
$$y(y-3)(y+2) = 0$$
$$y = 0, y = 3, y = -2$$

so y = 0 , y = 3, and y = -2 are the three solution curves (horizontal lines) where y will never change.

b) For what values of *y* is the solution increasing? Give your answer in interval notation.

Solution:
This occurs when dy/dt>0, or when

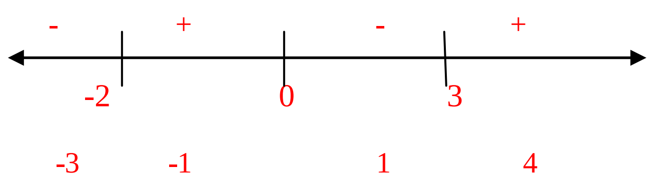
$$y^3 - y^2 - 6y > 0$$

or

$$y(y-3)(y+2) > 0.$$

We perform a sign analysis on the function $f(y) = y(y-3)(y+2)$,

which is the right side of the differential equation. We use test points in the four intervals:



and we find that

$f(-3) < 0,\ f(-1) > 0,\ f(1) < 0,$

and

$f(4) > 0$

so that

$y(y-3)(y+2)$ is positive on $(-2,0) \cup (3,\infty)$ and negative on

$(-\infty,-2) \cup (0,3)$.

Therefore
dy/dt>0
on

$(-2,0) \cup (3,\infty)$.

c) For what values of *y* is the solution decreasing? Give your answer in interval notation.

Solution:
These are values of y where dy/dt<0 , so, from the sign analysis of part (b), y is decreasing on
$(-\infty,-2) \cup (0,3)$

d) Make a rough sketch of the solution curves for each of the following initial values for

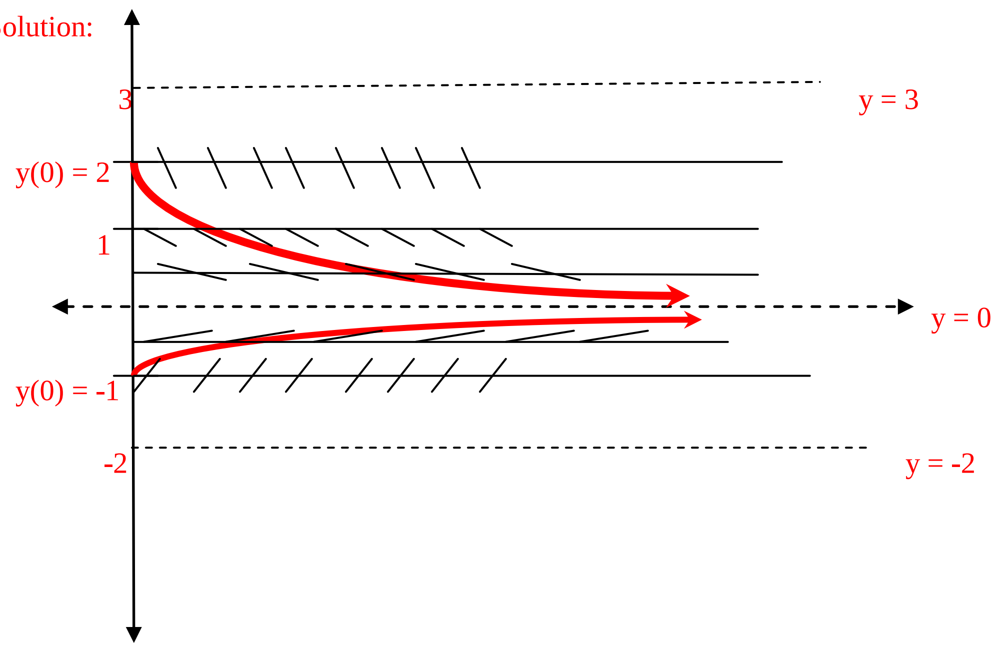
$t \geq 0.$

$y(0) = -1$

$y(0) = 2$

To aid in your sketching, add a slope field (direction field) by finding the slope of solutions on the lines $y = -1, y = -\frac{1}{2}, y = \frac{1}{2}, y = 1, y = 2$.

Solution:



The equilibrium solution y = 0 serves as a horizontal asymptote. A direction field for different values of y can be used to verify these solution curves, e.g., if y = 1, then for the right side of the DE, f(y),

$f(1) = 1(1-3)(1+2) = -6$

so the tangent slopes for all solution curves at the point where they cross the line y = 1 is -6, so we place line segments with slope -6 (roughly) on the horizontal line y = 1. We can do this for other values of y as well.

Note that that the DE $\frac{dy}{dt} = y^3 - y^2 - 6y$

can be solved !! If time, we will solve this equation in the "Extra Problems" section.

e) Use your answer to part (d) to estimate $\lim_{t \to \infty} y$ for any initial condition where $y(0) > -2$ and $y(0) < 3$

Solution:

$$\lim_{t\to\infty} y = 0$$

2. Show that the function

$$y = \sin(2t)$$

is a solution of the IVP (initial value problem),

$$y''' - y'' + 4y' = 4y$$
$$y\left(\frac{\pi}{4}\right) = 1$$

Solution:

If $y = \sin(2t)$

Then
$$y' = 2\cos(2t)$$
$$y'' = -4\sin(2t)$$
$$y''' = -8\cos(2t)$$

Checking the left side of the equation, we have

$$y''' - y'' + 4y' = -8\cos(2t) + 4\sin(2t) + 8\cos(2t) = 4\sin(2t)$$

Checking the right side, we have
$4y = 4\sin(2t)$,
which is equal to the left side.

Finally, checking the initial condition, we have $y = \sin\left(2\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1$, which matches the given initial condition.

(9.2) Direction Fields

Recall that any first order ODE,

$$\frac{dy}{dx} = f(x, y),$$

is essentially a formula for computing the tangent slope of a solution curve passing through a point (x,y).

1.Given the DE, $\frac{dy}{dx} = 3x^2 - 2y,$

find the slope of the solution curve that passes through each point.

a) (-2,1) Ans. $\frac{dy}{dx} = 3(-2)^2 - 2(1) = 10$

b) (0,5) Ans. $\frac{dy}{dx} = 3(0)^2 - 2(5) = -10$

2.Decide whether solutions of the DE, $\frac{dy}{dx} = \frac{1-e^x}{y}$, in each of the four quadrants will be increasing or decreasing.

a) 1st quadrant Ans. Since $x > 0$ and $y > 0,$ $\frac{dy}{dx} = \frac{1-e^x}{y} < 0$, so solutions will be decreasing.

b) 2nd quadrant Ans. Since $x < 0$ and $y > 0,$ $\frac{dy}{dx} = \frac{1-e^x}{y} > 0$, so solutions will be increasing.

c) 3rd quadrant Ans. Since $x < 0$ and $y < 0,$ $\frac{dy}{dx} = \frac{1-e^x}{y} < 0$, so solutions will be decreasing.

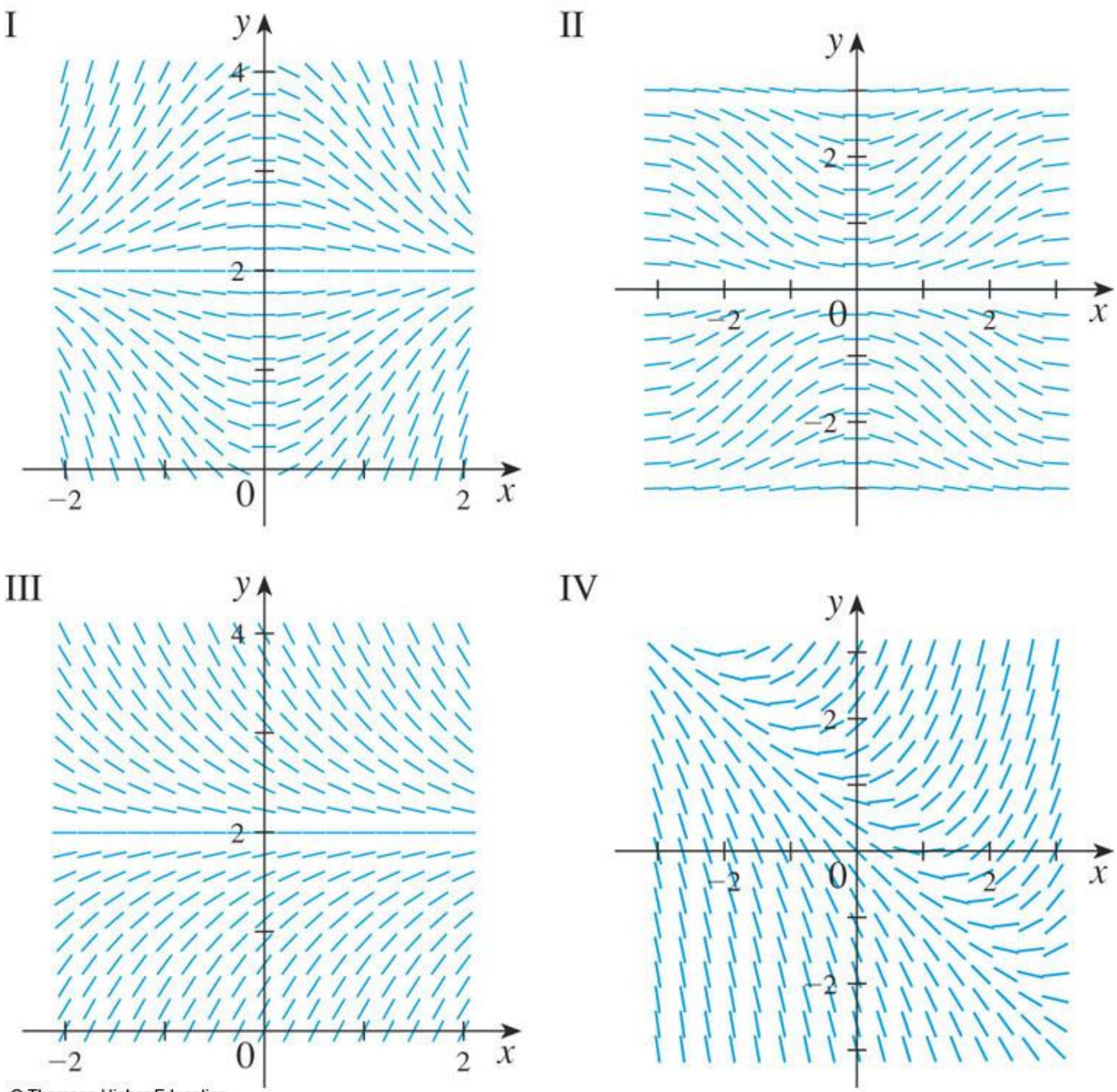
d) 4rd quadrant Ans. Since $x > 0$ and $y < 0,$ $\frac{dy}{dx} = \frac{1-e^x}{y} > 0$, so solutions will be increasing.

3.Select the correct direction field for each DE below.

a. $\frac{dy}{dx} = x(2-y)$ Ans. I

b. $\frac{dy}{dx} = \sin x \sin y$ Ans. II

Careful – the center of the two graphs on the left is not at the origin.



4.For which of the direction fields above does $\lim_{x\to\infty} y(x) = 2$ or all solutions y(x) (that are represented) ?

Ans. I and III

(9.3) Separable Equations

Methods do exist for solving certain types of differential equations. If, for example, a differential equation can be rewritten in the form,

$$f(x)dx = g(y)dy$$

then the equation is said to be <u>separable</u> (one variable is only on the left side, and the other is only on the right side). Separable DEs can then be solved by integrating each side:

$$\int f(x)dx = \int g(y)dy$$

This method can be used to solve the population models discussed in section 9.1

1. First, suppose $y = \sqrt{3x^2 - e^x + Cx}$

is a solution of a first-order DE. Which functions below are also possible solutions of the same DE? Circle the letter next to each function that is also a solution.

a) $y = \sqrt{3x^2 - e^x + (C-2)x}$

b) $y = \sqrt{3x^2 - e^x + (C+3)x}$

c) $y = \sqrt{3x^2 - e^x + 3Cx}$

d) $y = \sqrt{3x^2 - e^x + Kx}$

e) $y = \sqrt{3x^2 - e^x + \frac{C}{3}x}$

f) $y = \sqrt{3x^2 - e^x + C}$

g) $y = \sqrt{3x^2 - e^x + x}$

h) $y = \sqrt{x^2 - e^x + x}$

Answers: a,b,c,d,e,g

2. Consider the DE

$$\frac{dy}{dx} = \frac{x^2 - x}{y}$$
$$y(3) = -2$$

a) Find the general solution for the DE. Leave your answer in implicit form, i.e. you do not need to solve your solution for *y*.

Solution:

$$\frac{dy}{dx} = \frac{x^2 - x}{y}$$
$$\int y\,dy = \int \left(x^2 - x\right) dx$$
$$\frac{y^2}{2} = \frac{x^3}{3} - \frac{x^2}{2} + C$$
$$y^2 = \frac{2x^3}{3} - \frac{2x^2}{2} + 2C$$
$$y^2 = \frac{2x^3}{3} - x^2 + K$$

b) Now, solve the IVP (initial value problem). Give your answer in explicit form, i.e., solve your solution for *y*.

Solution:

$$\frac{dy}{dx} = \frac{x^2 - x}{y}$$
$$y(3) = -2$$

$$y = \pm\sqrt{\frac{2x^3}{3} - x^2 + K}$$

However, since $y(3) = -2 < 0$, we take the negative solution,

$$y = -\sqrt{\frac{2x^3}{3} - x^2 + K}$$

Applying the IC (initial condition), we have

$$-2=-\sqrt{\frac{2(27)}{3}-9+K}$$
$$2=\sqrt{18-9+K}$$
$$2=\sqrt{9+K}$$
$$4=9+K$$
$$K=-5$$

So that the solution is

$$y=-\sqrt{\frac{2x^3}{3}-x^2-5}$$

3. Solve the initial value problem.

$$\frac{dy}{dx}=\frac{xy^2-x}{2}$$
$$y(0)=2$$

a) Find the explicit solution, i.e., solve your solution for y.

b) Compute $\lim_{x\to\infty} y$. Explain why you were able to apply the method you used to compute the limit.

Solution:

a) This is actually a separable equation: (next page…)

$$\frac{dy}{dx}=\frac{xy^2-x}{2}$$

$$\frac{dy}{dx}=\frac{x(y^2-1)}{2}$$

$$\frac{2}{y^2-1}\frac{dy}{dx}=x$$

$$\frac{2}{y^2-1}dy=xdx$$

$$\int\frac{2}{(y-1)(y+1)}dy=\int xdx$$

$$\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy=\int xdx$$

$$\ln|y-1|-\ln|y+1|=\frac{x^2}{2}+C$$

$$\ln\left|\frac{y-1}{y+1}\right|=\frac{x^2}{2}+C$$

$$\left|\frac{y-1}{y+1}\right|=e^{\frac{x^2}{2}+C}$$

$$\left|\frac{y-1}{y+1}\right|=e^{\frac{x^2}{2}}e^C$$

$$\left|\frac{y-1}{y+1}\right|=e^Ce^{\frac{x^2}{2}}$$

$$\frac{y-1}{y+1}=\pm e^Ce^{\frac{x^2}{2}}$$

$$\frac{y-1}{y+1}=Ke^{\frac{x^2}{2}}$$

$$y-1=Ke^{\frac{x^2}{2}}(y+1)$$

$$y-1=Ke^{\frac{x^2}{2}}y+Ke^{\frac{x^2}{2}}$$

$$y\left(1-Ke^{\frac{x^2}{2}}\right)=Ke^{\frac{x^2}{2}}+1$$

$$y=\frac{1+Ke^{\frac{x^2}{2}}}{1-Ke^{\frac{x^2}{2}}}$$

This is the general solution.

To find *K*, we apply the initial condition:

$$\frac{2-1}{2+1} = Ke^{0}$$

$$K = \frac{1}{3}$$

Therefore,

$$y = \frac{1+\frac{1}{3}e^{\frac{x^2}{2}}}{1-\frac{1}{3}e^{\frac{x^2}{2}}}$$

b) $\lim_{x\to\infty} y = \lim_{x\to\infty} \dfrac{1+\frac{1}{3}e^{\frac{x^2}{2}}}{1-\frac{1}{3}e^{\frac{x^2}{2}}} = \lim_{x\to\infty} \dfrac{\frac{1}{3}xe^{\frac{x^2}{2}}}{-\frac{1}{3}xe^{\frac{x^2}{2}}} = -1$, applying L'Hopital's Rule.

4. Consider the DE.

$$yy' = \frac{1}{9\cos y + x^2 \cos y}$$

Find the general solution for the differential equation. Leave your answer in implicit form.

Solution:

Again, this is a separable equation.

$$yy' = \frac{1}{(9+x^2)\cos y}$$

$$y\frac{dy}{dx} = \frac{1}{(9+x^2)\cos y}$$

$$y\cos y\frac{dy}{dx} = \frac{1}{(9+x^2)}$$

$$y\cos y dy = \frac{1}{(9+x^2)}dx$$

$$\int y\cos y dy = \int \frac{1}{(9+x^2)}dx$$

$$y\sin y + \cos y = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

where we have applied integration-by-parts to compute the integral on the left side. Note that we must leave the solution in implicit form since we cannot solve for y.

(9.4) Models of Population Growth

1. Suppose that a population develops according to the logistic differential equation (limited growth model.)

$\frac{dP}{dt} = 2P - 0.001P^2$,where t represents years.

a) What is the carrying capacity? (You can get this directly from the DE, but it will also be the positive equilibrium solution.)
b) For what values of P is the population increasing? Give your answer in interval notation.
c) At what value of P is the population increasing the *fastest*, i.e., where the slope is the greatest? Note: this occurs at inflection points!
d) For what values of P will P remain the same, i.e., what are the equilibrium solutions?
e) Draw a coordinate system with P vs. t, and sketch the curves in part (d).
f) On this same coordinate system, make a rough sketch of curves for each of the following initial populations (initial conditions)

   $P(0) = 100$

   $P(0) = 3000$

   In the long term, what level does the population seem to approach?
g) Recall that the logistic population above includes a term due to birth rate, $2P$, and a term due to competition, $-0.001P^2$. Suppose that, in addition to the terms on the right side of the DE above, there is a term due to immigration. Specifically, suppose the population increases by an additional 5250 per year, due to immigration. Write the new DE, which includes this term.
h) Find the new nonnegative equilibrium solution for your equation in part (g).
i) Make a rough sketch of the solution curves for each of the initial populations:

   $P(0) = 0$

   $P(0) = 7000$

   In the long term, what level does the population seem to approach?

j) If $P(10) = 100$, find $P'(10)$. Hint: you don't need to solve for $P(t)$!

k) Extra Problem: If time, solve your equation from part (g) with $P(0)=7000$ (and $P > 3500$), and compute $\lim_{t\to\infty} P(t)$.

l) The solution to the general logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{1}{M}P\right)$ with initial condition $P(0) = P_0$ is

$$P(t) = \frac{M}{1 + Ae^{-kt}} \quad \text{where} \quad A = \frac{M - P_0}{P_0}.$$

If $\frac{dP}{dt} = 2P - 0.001P^2$ and $P(0) = 100$, after how many years will the population reach 80% of its carrying capacity? Round your answer to two decimal digits.

Solution:

a) Rewriting the DE in standard form,

$$\frac{dP}{dt} = 2P(1 - \frac{0.001}{2}P)$$

$$\frac{dP}{dt} = 2P(1 - \frac{1}{2000}P)$$

so, the carrying capacity must be $K = 2000$.

b) We know from the logistic equation, that the population P will increase to the carrying capacity when $P < K$ and will decrease to the carrying capacity when $P > K$. Therefore, when $0 < P < 2000$, P will increase.

c)

This occurs when $\frac{dP}{dt}$

is maximum, or, equivalently, when the right side

$f(P) = 2P - 0.001P^2$

is maximum, so locating the critical point to maximize the function,

$$f'(P) = 2 - 0.002P = 0$$

$$P = 1000$$

Therefore, when P = 1000,

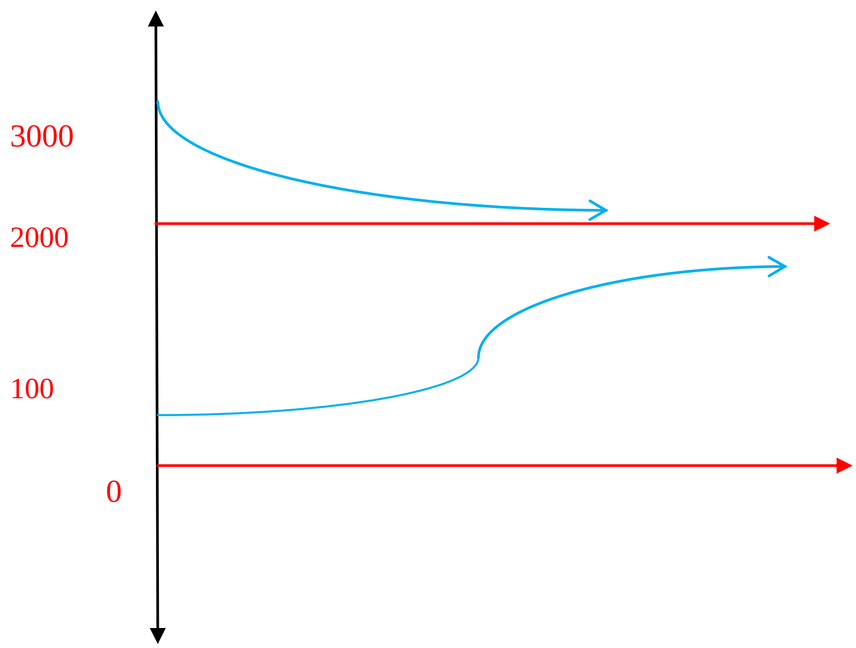
$\frac{dP}{dt}$

will be maximum.

(Of course, we know from the logistics equation that, in general, P will increase the fastest when it is half the carrying capacity: 2000/2=1000.

d) $P = 0$ and $P = 2000$.

e) and f)



g) $\dfrac{dP}{dt} = 2P - 0.001P^2 + 5250$

h) Let the right hand side equal $f(P)$:

$$f(P) = 2P - 0.001P^2 + 5250$$
$$f(P) = -0.001P^2 + 2P + 5250$$

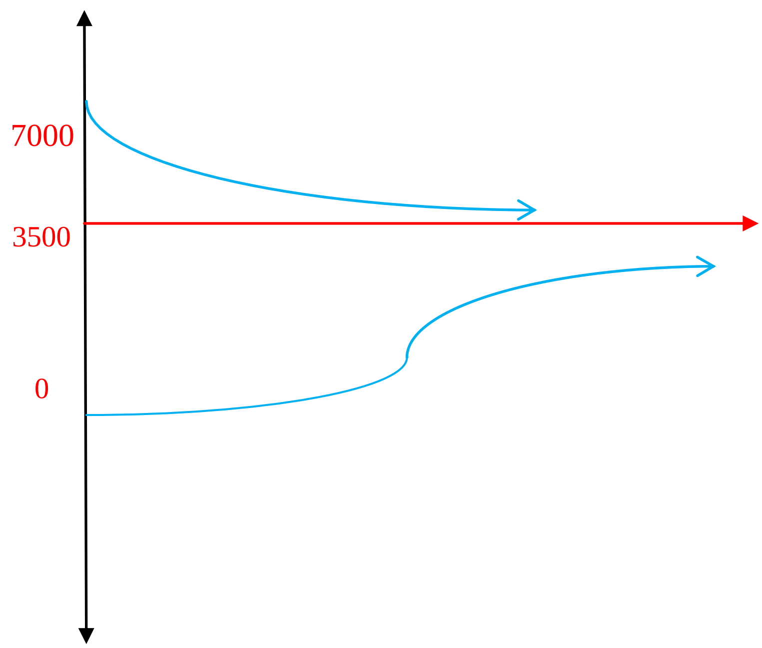
Applying the quadratic formula for the roots, we have

$$\frac{-2 \pm \sqrt{4 - 4(-0.001)(5250)}}{2(-0.001)} = 3500, -1500$$

Therefore, $\dfrac{dP}{dt} = 2P - 0.001P^2 + 5250 = 0$

when P= 3500 and P = -1500. The only relevant solution is P= 3500. The population will now asymptotically approach this solution instead of 2000, as in the logistics model.

i)



j) If $P(10) = 100$, find $P'(10)$. Hint: you don't need to solve for $P(t)$!

Note that $\frac{dP}{dt} = 2P - 0.001P^2$ means that $P'(t) = 2P(t) - 0.001(P(t))^2$. In other words, the derivative of P at any point t (in the domain) is equal to twice P at that same point t less 0.001 times the square of P at that same point. Therefore,

$$P'(10) = 2P(10) - 0.001(P(10))^2 = 2(100) - 0.001(100)^2 = 190.$$

k) (next page)

$$\frac{dP}{dt} = 2P - 0.001P^2 + 5250$$

$$\int \frac{dP}{-0.001P^2 + 2P + 5250} = \int dt$$

$$-\int \frac{dP}{(P+1500)(P-3500)} = \int dt$$

$$\int \left(\frac{\frac{1}{5000}}{P+1500} - \frac{\frac{1}{5000}}{P-3500} \right) dP = \int dt$$

$$\frac{1}{5000} \ln|P+1500| - \frac{1}{5000} \ln|P-3500| = t + C$$

$$\frac{1}{5000} \ln\left|\frac{P+1500}{P-3500}\right| = t + C$$

$$\frac{1}{5000} \ln\left(\frac{P+1500}{P-3500}\right) = t + C, \quad P > 3500$$

$$\ln\left(\frac{P+1500}{P-3500}\right) = 5000t + C$$

$$\frac{P+1500}{P-3500} = e^C e^{5000t}$$

$$\frac{P+1500}{P-3500} = Ke^{5000t}$$

$$P + 1500 = PKe^{5000t} - 3500Ke^{5000t}$$

$$P - PKe^{5000t} = -3500Ke^{5000t} - 1500$$

$$P = \frac{-3500Ke^{5000t} - 1500}{1 - Ke^{5000t}}$$

$$P = \frac{3500Ke^{5000t} - 1500}{Ke^{5000t} - 1}$$

where partial fraction expansion had been performed in the fourth line. Applying the initial condition, we have

$$7000 = \frac{3500K - 1500}{K - 1}$$

$$7000K - 7000 = 3500K - 1500$$

$$3500K = 5500$$

$$K = \frac{11}{7}$$

So that

$$P(t)=\frac{3500\left(\frac{11}{7}\right)e^{5000t}-1500}{\left(\frac{11}{7}\right)e^{5000t}-1}$$

Indeed, $\lim_{t\to\infty} P(t)=3500$, which is what we expected from our knowledge of the equilibrium solutions.

l) The solution to the general logistic equation $\frac{dP}{dt}=kP\left(1-\frac{1}{M}P\right)$ with initial condition

$P(0)=P_0$ is

$P(t)=\frac{M}{1+Ae^{-kt}}$ where $A=\frac{M-P_0}{P_0}$.

If $\frac{dP}{dt}=2P-0.001P^2$ and $P(0)=100$, after how many years will the population reach 80% of its carrying capacity? Round your answer to two decimal digits.

We are given that $P_0=100$, $M=2000$, and $k=2$, so we want to know how long it will take before $P=0.8M=0.8(2000)=1600$.

Also, $A=\frac{2000-100}{100}=19$, so that $P(t)=\frac{M}{1+Ae^{-kt}}$ means $1600=\frac{2000}{1+19e^{-2t}}$.

We then solve for *t:*

$$\left(1+19e^{-2t}\right)(1600)=2000$$

$$1600+30400e^{-2t}=2000$$

$$30400e^{-2t}=400$$

$$e^{-2t}=\frac{400}{30400}$$

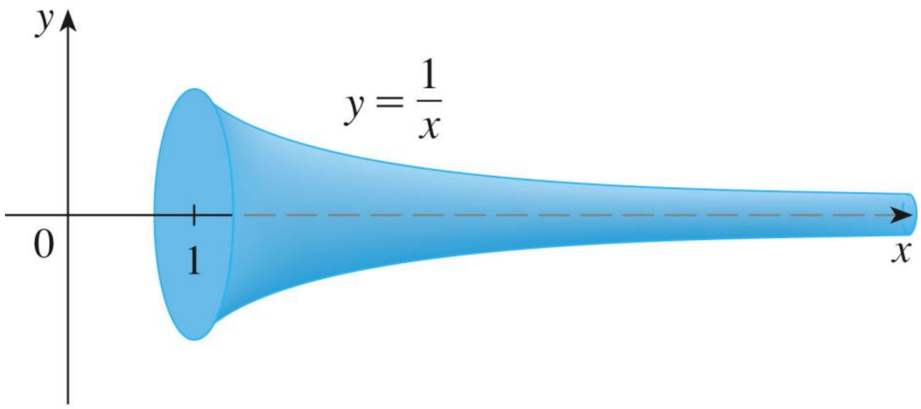
$$-2t=\ln\left(\frac{400}{30400}\right)$$

$$t=-\frac{1}{2}\ln\left(\frac{400}{30400}\right)\approx 2.16\ yrs$$

Extra Problems

1) Consider the function $y = \frac{1}{x}$ on $1 \le x \le \infty$.

   a) Set up the integral that represents the surface area generated by revolving this curve about the *x*-axis. A depiction of this surface is below. (This is sometimes referred to as Gabriel's Horn.)



Ans. Since $y' = -\frac{1}{x^2}$, then the surface area is

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx$$

$$= \int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

$$= \int_1^\infty 2\pi \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx$$

$$= \int_1^\infty 2\pi \frac{1}{x}\frac{\sqrt{x^4+1}}{x^2}\,dx$$

$$= \int_1^\infty 2\pi \frac{\sqrt{x^4+1}}{x^3}\,dx$$

   b) Apply the Comparison Test for Improper Integrals to show whether your integral in part (a) diverges or converges.

Ans. Since $\frac{\sqrt{x^4+1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x}$ and $\int_0^\infty \frac{1}{x}dx$ diverges to ∞, then $\int_1^\infty 2\pi \frac{\sqrt{x^4+1}}{x^3}dx$ must also diverge to ∞.

   c) The following integral represents the volume inside Gabriel's Horn:

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

Compute this integral. Are the results of this computation surprising, given your results in part (b)?

Ans.

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \pi\int_1^\infty x^{-2}dx$$

$$= \lim_{t\to\infty} \pi\int_1^t x^{-2}dx$$

$$= \lim_{t\to\infty} \left.\frac{-\pi}{x}\right|_1^t$$

$$= \lim_{t\to\infty}\left(\frac{-\pi}{t}+\pi\right) = \pi$$

Even though the surface area of Gabriel's Horn is infinite, its volume is finite!!

2) Verify that the equation

$$ye^y - e^y = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

is a solution to the DE

$$yy' = \frac{1}{9e^y + x^2e^y}$$

Hint: you will need implicit differentiation.

Solution:

Using implicit differentiation,

$$\frac{d}{dx}\left(ye^y - e^y\right) = \frac{d}{dx}\left(\frac{1}{3}\arctan\left(\frac{x}{3}\right) + C\right)$$

$$y'e^y + ye^y y' - e^y y' = \frac{1}{3}\frac{1}{1+\left(\frac{x}{3}\right)^2}\frac{1}{3}$$

$$y'e^y + ye^y y' - e^y y' = \frac{1}{9}\frac{1}{1+\frac{x^2}{9}}$$

$$y'e^y + ye^y y' - e^y y' = \frac{1}{9+x^2}$$

where we have applied the chain rule on the right hand side to compute the derivative of $\arctan\left(\frac{x}{3}\right)$.

Solving for y' ,we have

$$y'e^y + ye^y y' - e^y y' = \frac{1}{9+x^2}$$

$$y'\left(e^y + ye^y - e^y\right) = \frac{1}{9+x^2}$$

$$y'\left(ye^y\right) = \frac{1}{9+x^2}$$

$$y' = \frac{1}{ye^y\left(9+x^2\right)}$$

Substituting this expression for y' into the the left hand side of the DE, we have

$$yy' = y\frac{1}{ye^y\left(9+x^2\right)} = \frac{1}{e^y\left(9+x^2\right)} = \frac{1}{9e^y + x^2e^y}$$

which yields the right hand side of the DE. The solution is verified.

3) Find all values of r such that $y = x^r$ satisfies the DE $2x^2y'' + 3xy' - y = 0$. Use your result to find two different functions that are solutions of this DE.

Solution: Note that

$$y' = rx^{r-1}$$

$$y'' = r(r-1)x^{r-2}.$$

Substituting these functions into the DE leads to

$$2x^2 r(r-1)x^{r-2} + 3xrx^{r-1} - x^r = 0$$

$$2r(r-1)x^2 x^{r-2} + 3rxx^{r-1} - x^r = 0$$

$$2r(r-1)x^r + 3rx^r - x^r = 0$$

$$[2r(r-1) + 3r - 1]x^r = 0$$

$$2r(r-1) + 3r - 1 = 0$$

$$2r^2 + r - 1 = 0$$

$$(2r-1)(r+1) = 0$$

$$r = \frac{1}{2}, -1$$

The two solutions are then $y = \sqrt{x}, \quad y = \frac{1}{x}$

4) Solve the following separable equation explicitly.

$$\frac{dy}{dx} = \frac{xy^2 - y^2}{x^2 + 1}$$

Ans.

The DE

$$\frac{dy}{dx} = \frac{xy^2 - y^2}{x^2 + 1}$$

is separable since it can be rewritten as

$$\frac{dy}{dx} = \frac{y^2(x-1)}{x^2 + 1}$$

$$dy = \frac{y^2(x-1)}{x^2 + 1}dx$$

$$\frac{1}{y^2}dy = \frac{x-1}{x^2+1}dx$$

We can then integrate both sides:

$$\int \frac{1}{y^2}\,dy = \int \frac{x-1}{x^2+1}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$-\frac{1}{y} = \frac{1}{2}\ln\left(x^2+1\right) - \arctan x + C$$

Finally, we solve for y:

$$\frac{1}{y} = -\frac{1}{2}\ln\left(x^2+1\right) + \arctan x + C$$

$$y = \frac{1}{-\frac{1}{2}\ln\left(x^2+1\right) + \arctan x + C}$$

5) The equation with arbitrary constant,

$$\ln y + y = \sin x + C$$

represents a family of curves.

Find another equation (with arbitrary constant) that represents another family of curves, each of which is orthogonal (perpendicular) to one of the given curves at any point (x,y) through which is passes. Hint: you will need to apply implicit differentiation to the given equation to find a differential equation. You will then need to modify this DE in an appropriate way before solving it to find the desired curves.

Solution:

$$\frac{d}{dx}\left(\ln y + y\right) = \frac{d}{dx}\left(\sin x + C\right)$$

$$\frac{1}{y}\frac{dy}{dx} + \frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx}\left(\frac{1}{y}+1\right) = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{\frac{1}{y}+1}$$

$$\frac{dy}{dx} = \frac{y\cos x}{1+y}$$

which gives the slope of any solution curve at point (x,y), which passes through this point. The slope of any orthogonal curve that passes through this point is:

$$\frac{dy}{dx} = -\frac{1+y}{y\cos x}$$

Solving this equation will yield the set of curves orthogonal to the given curves:

$$\frac{dy}{dx} = -\frac{1+y}{y\cos x}$$

$$y\cos x\frac{dy}{dx} = -(1+y)$$

$$\int \frac{y}{1+y}dy = -\int \frac{1}{\cos x}dx$$

$$\int \left(1-\frac{1}{1+y}\right)dy = -\int \sec x dx$$

$$y - \ln|1+y| = -\ln|\sec x + \tan x| + C$$

6) Find the equation for the curve that passes through the point (0,1) whose slope at any point (x,y) is $\frac{x^2 y + y}{(\ln y)(x^2-1)}$.

Solution:

$$\frac{dy}{dx} = \frac{x^2 y + y}{(\ln y)(x^2-1)}$$

$$\frac{dy}{dx} = \frac{y(x^2+1)}{(\ln y)(x^2-1)}$$

$$\frac{\ln y}{y}dy = \frac{x^2+1}{x^2-1}dx$$

$$\int \frac{\ln y}{y}dy = \int \frac{x^2+1}{x^2-1}dx$$

For the left integral, let $u = \ln y$ and, for the right integral, perform polynomial division to yield:

$$\int \frac{\ln y}{y}dy = \int \frac{x^2+1}{x^2-1}dx$$

$$\frac{(\ln y)^2}{2} = \int \left(1+\frac{2}{x^2-1}\right)dx$$

Next, apply partial fraction expansion to the right integral to obtain

$$\frac{(\ln y)^2}{2} = \int \left(1 + \frac{2}{x^2 - 1}\right) dx$$

$$\frac{(\ln y)^2}{2} = \int \left(1 - \frac{1}{x+1} + \frac{1}{x-1}\right) dx$$

$$\frac{(\ln y)^2}{2} = x - \ln|x+1| + \ln|x-1| + C$$

Applying the initial condition (0,1), we have

$$\frac{(\ln 1)^2}{2} = 0 - \ln|0+1| + \ln|0-1| + C$$

$$0 = 0 + C$$

$$C = 0$$

Therefore, the curve passing through (0,1) is

$$\frac{(\ln y)^2}{2} = x - \ln|x+1| + \ln|x-1|$$

7) Find the general solution for the DE

$$\frac{dy}{dt} = y^3 - y^2 - 6y$$

(In the previous packet, you determined when solutions to this DE were increasing, decreasing, and constant. Now, you can solve it.)

Solution:

This is a separable equation:

$$\frac{dy}{y^3 - y^2 - 6y} = dt$$

$$\frac{dy}{y(y^2 - y - 6)} = dt$$

$$\frac{dy}{y(y-3)(y+2)} = dt$$

$$\int \frac{dy}{y(y-3)(y+2)} = \int dt$$

Applying partial fraction expansion, we have

$$\frac{1}{y(y-3)(y+2)} = \frac{A}{y} + \frac{B}{y-3} + \frac{C}{y+2}$$
$$1 = A(y-3)(y+2) + By(y+2) + Cy(y-3)$$
$$1 = (A+B+C)y^2 + (-A+2B-3C)y + (-6A)$$
$$A+B+C=0$$
$$-A+2B-3C=0$$
$$-6A=1$$
$$A=-\frac{1}{6}, \quad B=\frac{1}{15}, \quad C=\frac{1}{10}$$

Therefore,

$$\int \frac{dy}{y(y-3)(y+2)} = \int dt$$
$$\int\left(-\frac{1}{6}\frac{1}{y} + \frac{1}{15}\frac{1}{y-3} + \frac{1}{10}\frac{1}{y+2}\right)dy = \int dt$$
$$-\frac{1}{6}\ln|y| + \frac{1}{15}\ln|y-3| + \frac{1}{10}\ln|y+2| = t + C$$