PACKET 1
(7.1) Integration by Parts, (7.2) Trigonometric Integrals

Integration by Parts (7.1)

Many integrals (specifically, antiderivatives) can be computed by u-substitution.

1) Find the antiderivative $\int xe^{x^2}dx$

   Solution: $\frac{1}{2}e^{x^2}+C$

However, many antiderivatives cannot be computed by a simple u-substitution, for example $\int xe^{x}dx$. Try a u-substitution – it doesn't accomplish anything, it just replaces *x* with *u*, but maintains the same form.

In such cases, integration by parts may be applied to compute the antiderivative. Recall that this technique is really just a more elaborate substitution method – a u-v substitution, if you will, where the integral is converted to the form $\int udv$, which results in

$$\int udv = uv - \int vdu.$$

The hope is that the new integral $\int vdu$ is "easier" to compute than the original. If not, either integration by parts must be performed again, or another technique must be applied.

The assignment of *u* to an appropriate function in the integrand is the first decision that must be made when performing integration by parts. One useful rule that works most of the time (but not all of the time) is the acronym "LIATE." "L" stands for "logarithmic function," "I" stands for "inverse trigonometric function," "A" stands for "algebraic function," "T" stands for "trigonometric function," and "E" stands for "exponential function." This is the order in which one chooses a function in the integrand to assign to *u*. For example, the integral $\int (x^2-1)\ln(2x)dx$ contains both an algebraic function (x^2-1) and a logarithmic function ($\ln(2x)$). Since "L" comes before "A" in the word "LIATE," then we let $u = \ln(2x)$. This relegates *dv* to $dv=(x^2-1)dx$. Please note that sometimes the "L" and "I" are reversed in the acronym: "ILATE." This rule and the "LIATE" rule are equally effective.

Furthermore, there exists a tabular method, which is a "shortcut" for integration by parts that is most helpful for integrals of products of polynomials and exponential functions or trigonometric functions. For example, for the integral, $\int (x^3+2x)\sin(3x)dx$, per the LIATE rule, one assigns $u=x^3+2x$ and

$dv = \sin(3x)dx$. Each of the functions used for u and dv are placed at the top of a column in a table, along with a "+" sign:

+	$x^3 + 2x$	$\sin(3x)$

One then writes alternating signs down the first column, successive derivatives down the second column, and successive antiderivatives down the third column – until the successive derivatives yield 0:

+	$x^3 + 2x$	$\sin(3x)$
-	$3x^2 + 2$	$-\frac{1}{3}\cos(3x)$
+	$6x$	$-\frac{1}{9}\sin(3x)$
-	6	$\frac{1}{27}\cos(3x)$
+	0	$\frac{1}{81}\sin(3x)$

Now, one can assemble the terms by following each of the four chains of arrows to form each product:

+	$x^3 + 2x$	$\sin(3x)$
-	$3x^2 + 2$	$-\frac{1}{3}\cos(3x)$
+	$6x$	$-\frac{1}{9}\sin(3x)$
-	6	$\frac{1}{27}\cos(3x)$
+	0	$\frac{1}{81}\sin(3x)$

The final antiderivative is:

$$\int (x^3+2x)\sin(3x)dx = +(x^3+2x)\left(-\frac{1}{3}\cos(3x)\right)$$

$$-(3x^2+2)\left(-\frac{1}{9}\sin(3x)\right)+(6x)\left(\frac{1}{27}\cos(3x)\right)-(6)\left(\frac{1}{81}\sin(3x)\right)$$

$$=-\frac{1}{3}(x^3+2x)\cos(3x)+\frac{1}{9}(3x^2+2)\sin(3x)+\frac{2}{9}x\cos(3x)-\frac{2}{27}\sin(3x)+C$$

Exercises:

2) Compute $\int xe^x dx$.

Solution: Let $u = x$ and $dv = e^x dx$. Then, $du = dx$ and $v = e^x$, so that

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

Alternately,

+	x	e^x
-	1	e^x
	0	e^x

Therefore, $\int xe^x dx = +xe^x - (1)e^x + C = xe^x - e^x + C$

3) Compute $\int_0^2 xe^x dx$.

Solution: $e^2 + 1$

Here are some basic antiderivative and derivative rules that might be helpful when performing integration by parts.

4) $\int \frac{1}{1+x^2} dx$

Solution: (basic arctangent integral) $\arctan x + C$

5) $\frac{d}{dx}\arctan x$

Solution: $\frac{1}{1+x^2}$

6) $\int \frac{1}{\sqrt{1-x^2}}\,dx$

Solution: (basic arcsine integral) $\arcsin x + C$

7) $\frac{d}{dx}\arcsin x$

Solution: $\frac{1}{\sqrt{1-x^2}}$

Also, polynomial division may be appropriate after performing integration by parts.

8) Divide: $\frac{2x^3 - x^2 + 3}{x^2 - 1}$

Solution:

$$\begin{array}{r@{}l} & \phantom{2x^3 - x^2 +{}} 2x - 1 \\ x^2 + 0x - 1 \big) & \overline{2x^3 - x^2 + 0x + 3} \\ & \underline{2x^3 + 0x^2 - 2x} \\ & - x^2 + 2x + 3 \\ & \underline{- x^2 + 0x + 1} \\ & \phantom{2x^3 - x^2 +{}} 2x + 2 \end{array}$$

Therefore, $\frac{2x^3 - x^2 + 3}{x^2 - 1} = 2x - 1 + \frac{2x + 2}{x^2 - 1}$

Here are some other integral forms that may occur while computing integration by parts. Compute the following, noting that these integrals do not require integration by parts.

9) $\int \frac{x^2}{2}\frac{1}{x}\,dx$

Solution: (integral of a power function) $\frac{x^2}{4} + C$

10) $\int x \frac{1}{\sqrt{1-x^2}}\,dx$

Solution: (u-substitution) $-\sqrt{1-x^2} + C$

11) $\int (x^2 - 1)\frac{2x}{x^2 + 1}\,dx$

Solution: (polynomial division and then integrate) $\int (x^2-1)\frac{2x}{x^2+1}dx = 2\int \frac{x^3-x}{x^2+1}dx$. Performing polynomial division, we have

$$\frac{x^3-x}{x^2+1} = x - \frac{2x}{x^2+1}$$

Therefore,

$$\int (x^2-1)\frac{2x}{x^2+1}dx$$

$$= 2\int \frac{x^3-x}{x^2+1}dx$$

$$= 2\int \left(x - \frac{2x}{x^2+1} \right)dx$$

$$= x^2 - 2\ln\left|x^2+1\right| + C$$

$$= x^2 - 2\ln\left(x^2+1\right) + C$$

Now, compute the following integrals, noting that some of these may require integration by parts, while others may not.

12) $\int e^{3x}dx$

Solution: (basic integration of an exponential function with $u = 3x$) $\frac{1}{3}e^{3x} + C$

(Note: In general, if $\int f(x)dx = F(x) + C$, then $\int f(Ax)dx = \frac{1}{A}F(Ax) + C$. This is a shortcut around using relative simple u-substitution for problems like $\int e^{3x}dx$.)

13) $\int xe^{3x}dx$

Solution: (integration by parts) $\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$

14) $\int x^2 e^{3x^3}dx$.

Solution: (u-substitution $u = 3x^3$) Let $u = 3x^3$, then $du = 9x^2dx$, so that

$$\int x^2 e^{3x^3}dx = \frac{1}{9}\int e^u du = \frac{1}{9}e^u + C = \frac{1}{9}e^{3x^3} + C$$

15) Answer with “T” for “true” and “F” for “false.”

$$\int (2-x-3x^2)\sin(2x)dx = \left(2x-\frac{x^2}{2}-x^3\right)\left(-\frac{1}{2}\cos(2x)\right)+C$$

Solution: F. $\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$

16) If you answered "F" for the last problem, then compute. $\int (2-x-3x^2)\sin(2x)dx$

Solution: (integration by parts twice)

+	$2-x-3x^2$	$\sin(2x)$
-	$-1-6x$	$-\frac{1}{2}\cos(2x)$
+	-6	$-\frac{1}{4}\sin(2x)$
-	0	$\frac{1}{8}\cos(2x)$

$$\int (2-x-3x^2)\sin(2x)dx$$

$$= +\left(2-x-3x^2\right)\left(-\frac{1}{2}\cos(2x)\right) - \left(-1-6x\right)\left(-\frac{1}{4}\sin(2x)\right) + \left(-6\right)\left(\frac{1}{8}\cos(2x)\right)+C$$

$$= -\frac{1}{2}\left(2-x-3x^2\right)\cos(2x) - \frac{1}{4}\left(1+6x\right)\sin(2x) - \frac{3}{4}\cos(2x)+C$$

17) $\int \arctan(2x)dx$

Solution: (integration by parts with $u = \arctan(2x)$ then a u-substitution)

Let $u = \arctan(2x)$ and $dv = dx$, so that $du = \dfrac{2}{1+4x^2}dx$ and $v = x$. Then,

$$\int \arctan(2x)dx = x\arctan(2x) - \int \frac{2x}{1+4x^2}dx$$

$$= x\arctan(2x) - \frac{1}{4}\ln\left|1+4x^2\right| + C$$

$$= x\arctan(2x) - \frac{1}{4}\ln\left(1+4x^2\right) + C$$

where u-substitution with $u = 1+4x^2$ was used for the last integral.

18) $\int \cos(4x)e^{5x}dx$

Solution: (integration by parts twice then solve for $\int \cos(4x)e^{5x}dx$)

Let $u=\cos(4x)$ and $dv=e^{5x}dx$, then $du=-4\sin(4x)dx$ and $v=\frac{1}{5}e^{5x}$, so that

$$\int \cos(4x)e^{5x}dx=\frac{1}{5}e^{5x}\cos(4x)+\frac{4}{5}\int e^{5x}\sin(4x)dx$$

Now, let $u=\sin(4x)$ and $dv=e^{5x}dx$, then $du=4\cos(4x)dx$ and $v=\frac{1}{5}e^{5x}$, so that

$$\begin{aligned}
&\int \cos(4x)e^{5x}dx\\
&=\frac{1}{5}e^{5x}\cos(4x)+\frac{4}{5}\int e^{5x}\sin(4x)dx\\
&=\frac{1}{5}e^{5x}\cos(4x)+\frac{4}{5}\left(\frac{1}{5}e^{5x}\sin(4x)-\frac{4}{5}\int e^{5x}\cos(4x)dx\right)\\
&=\frac{1}{5}e^{5x}\cos(4x)+\frac{4}{25}e^{5x}\sin(4x)-\frac{16}{25}\int e^{5x}\cos(4x)dx
\end{aligned}$$

In summary,

$$\int \cos(4x)e^{5x}dx=\frac{1}{5}e^{5x}\cos(4x)+\frac{4}{25}e^{5x}\sin(4x)-\frac{16}{25}\int e^{5x}\cos(4x)dx$$

Now, collecting the terms with $\int e^{5x}\cos(4x)dx$, we have

$$\int \cos(4x)e^{5x}dx+\frac{16}{25}\int e^{5x}\cos(4x)dx=\frac{1}{5}e^{5x}\cos(4x)+\frac{4}{25}e^{5x}\sin(4x)$$

$$\frac{41}{25}\int e^{5x}\cos(4x)dx=\frac{1}{5}e^{5x}\cos(4x)+\frac{4}{25}e^{5x}\sin(4x)$$

$$\int e^{5x}\cos(4x)dx=\frac{25}{41}\left(\frac{1}{5}e^{5x}\cos(4x)+\frac{4}{25}e^{5x}\sin(4x)\right)+C=\frac{5}{41}e^{5x}\cos(4x)+\frac{4}{41}e^{5x}\sin(4x)+C$$

19) $\int (x+1)\ln(x^2+1)dx$

Solution: (integration by parts then polynomial division) Let $u=\ln\left(x^2+1\right)$ so that $dv=(x+1)dx$,

$du=\frac{2x}{x^2+1}dx$, and $v=\frac{x^2}{2}+x$. Then,

$$\int (x+1)\ln(x^2+1)dx$$
$$=\left(\frac{x^2}{2}+x\right)\ln(x^2+1)-\int\left(\frac{x^2}{2}+x\right)\frac{2x}{x^2+1}dx$$
$$=\left(\frac{x^2}{2}+x\right)\ln(x^2+1)-\int\frac{x^3+2x^2}{x^2+1}dx$$
$$=\left(\frac{x^2}{2}+x\right)\ln(x^2+1)-\int\left(x+2-\frac{x+2}{x^2+1}\right)dx$$
$$=\left(\frac{x^2}{2}+x\right)\ln(x^2+1)-\int(x+2)dx+\int\frac{x}{x^2+1}dx+\int\frac{2}{x^2+1}dx$$
$$=\left(\frac{x^2}{2}+x\right)\ln(x^2+1)-\frac{x^2}{2}-2x+\frac{1}{2}\ln(x^2+1)+2\arctan x+C$$

Trigonometric Integrals (7.2)

First let's recall some basic integral rules for trigonometric functions.

Find the antiderivative:

1) $\int \sin x\, dx$
Ans: $-\cos x+C$
2) $\int \cos x\, dx$
Ans: $\sin x+C$
3) $\int \tan x\, dx$
Ans: $\ln|\sec x|+C$
4) $\int \sec x\, dx$
Ans: $\ln|\sec x+\tan x|+C$

Let's examine several cases.

I) $\int \sin^m(Ax)\cos^n(Ax)dx$ **, where neither power is equal to 1.**

First, compute this integral, where one of the powers is indeed unity:

1) $\int \sin^4(3x)\cos(3x)dx$

Ans: Let $u = \sin(3x)$. Then, $du = 3\cos(3x)dx$, and

$$\int \sin^4(3x)\cos(3x)dx$$
$$= \frac{1}{3}\int u^4 du$$
$$= \frac{1}{15}u^5 + C$$
$$= \frac{1}{15}\sin^5(3x) + C$$

Now, if neither power is 1, it may be helpful to apply trigonometric identities before u-substitution.

A) Either power (or both powers) of sine and cosine is odd. Then, one can factor out one of the functions and then apply the identity $\sin^2 x + \cos^2 x = 1$ **to aid with integration.**

Compute the integrals:

1) $\int \sin^4(x)\cos^5(x)dx$

Ans:

$$\int \sin^4(x)\cos^5(x)dx$$
$$= \int \sin^4(x)\cos^4(x)\cos(x)dx$$
$$= \int \sin^4(x)\left(\cos^2(x)\right)^2 \cos(x)dx$$
$$= \int \sin^4(x)\left(1-\sin^2(x)\right)^2 \cos(x)dx$$

Now, let $u = \sin(x)$. Then, $du = \cos(x)dx$, and

$$\int \sin^4(x)\left(1-\sin^2(x)\right)^2 \cos(x)dx$$
$$= \int u^4(1-u^2)^2 du$$
$$= \int \left(u^4 - 2u^6 + u^8\right)du$$
$$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C$$
$$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$$

Extra Problems

Some of these require integration by parts; others do not.

1) $\int (2-x-3x^2)e^{3x}dx$

Solution: (integration by parts twice)

$$\frac{1}{3}(2-x-3x^2)e^{3x}+\frac{1}{9}(1+6x)e^{3x}-\frac{2}{9}e^{3x}+C$$

2) $\int \sin(2x)e^{\cos(2x)}dx$

Solution: (u-substitution)

Let $u=\cos(2x)$, so that $du=-2\sin(2x)dx$, and $\int \sin(2x)e^{\cos(2x)}dx=-\frac{1}{2}e^{\cos(2x)}+C$

3) $\int \sqrt{x}e^{\sqrt{x}}dx$

Solution: (u-substitution $u=\sqrt{x}$ then integration by parts)

Let $u=\sqrt{x}$, so that

$$du=\frac{1}{2\sqrt{x}}dx$$

$$2\sqrt{x}du=dx$$

$$2udu=dx$$

Therefore, $\int \sqrt{x}e^{\sqrt{x}}dx=\int ue^u(2u)du=2\int u^2e^udu$. We perform integration by parts twice:

+	u^2	e^u
-	$2u$	e^u
+	2	e^u
-	0	e^u

Then,

$$\int \sqrt{x}e^{\sqrt{x}}dx$$

$$=\int ue^u(2u)du$$

$$=2\int u^2e^udu$$

$$=2\left(u^2e^u-2ue^u+2e^u\right)+C$$

$$=2xe^{\sqrt{x}}-4\sqrt{x}e^{\sqrt{x}}+4e^{\sqrt{x}}+C$$

4) $\int x^2 \arcsin x dx$

Solution: (integration by parts then u-substitution with $u = \sqrt{1-x^2}$ - tricky!!)

Let $u = \arcsin x$, then $dv = x^2 dx$, $du = \dfrac{1}{\sqrt{1-x^2}} dx$, and $v = \dfrac{x^3}{3}$. Then,

$$\begin{aligned}
&\int x^2 \arcsin x dx \\
&= \frac{x^3}{3} \arcsin x - \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx
\end{aligned}$$

Now, let $u = \sqrt{1-x^2}$, so that $du = -\dfrac{x}{\sqrt{1-x^2}} dx$ and $u^2 = 1-x^2$. Then,

$$\begin{aligned}
&\int x^2 \arcsin x dx \\
&= \frac{x^3}{3} \arcsin x - \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx \\
&= \frac{x^3}{3} \arcsin x - \frac{1}{3} \int x^2 \frac{x}{\sqrt{1-x^2}} dx \\
&= \frac{x^3}{3} \arcsin x - \frac{1}{3} \int (1-u^2)(-1) du \\
&= \frac{x^3}{3} \arcsin x + \frac{1}{3} \left(u - \frac{u^3}{3} \right) + C \\
&= \frac{x^3}{3} \arcsin x + \frac{1}{3} \left(\sqrt{1-x^2} - \frac{\left(\sqrt{1-x^2}\right)^3}{3} \right) + C
\end{aligned}$$

More trigonometric integrals:

5) $\int \sin^5(x) \cos^4(x) dx$

(factor out $\sin(x)$ and convert the cosines into sines)

Ans: $-\dfrac{\cos^5 x}{5} - \dfrac{2\cos^7 x}{7} - \dfrac{\cos^9 x}{9} + C$

$$\int \sec^3 x dx$$

$$= \int \sec x \sec^2 x dx$$

$$= \sec x \tan x - \int \tan x \sec x \tan x dx \quad (int\_by\_parts)$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x|$$

$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x|$$

$$\int \sec^3 x dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C$$

6) $\int_0^{\pi} \frac{\sin^3 x}{\cos^2 x} dx$

Ans:

$$\int \frac{\sin^3 x}{\cos^2 x} dx$$

$$= \int \frac{\sin^2 x \sin x}{\cos^2 x} dx$$

$$= \int \frac{(1 - \cos^2 x)\sin x}{\cos^2 x} dx$$

$$= \int \frac{\sin x - \cos^2 x \sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \sin x \right) dx$$

$$= \int \left(\cos^{-2} x \sin x - \sin x \right) dx$$

Let $u = \cos x$ in the first integral, etc… Final answer: -4.

Looking Ahead - Trigonometric Integrals (7.2) (cont.)

B) Note that, in problems like the example above, there is an odd power on the sine function, the cosine function, or both. However, what if there are only even powers of the sine and cosine function? One must apply power reduction formulas!!

First, answer each with True or False.

1) $\sin^2 x = \dfrac{1-\cos(2x)}{2}$ T

2) $\sin^2 x = \dfrac{1-\sin(2x)}{2}$ F

3) $\cos^2(3x) = \dfrac{1+\cos(3x)}{2}$ F

4) $\cos^2(3x) = \dfrac{1+\cos(6x)}{2}$ T

5) $\int \sin^2(4t)\cos^2(4t)dt = \dfrac{\sin^3(4t)}{12}\dfrac{\cos^3(4t)}{12} + C$ F

Compute the integral:

6) $\int \cos^2(3t)\sin^2(3t)dt$

Ans: (use power reduction formulas for each of $\cos^2(3t)$ and $\sin^2(3t)$, multiply out the result, then apply a power reduction one more time before integrating.)

$$\int \cos^2(3t)\sin^2(3t)dt$$
$$= \int \left(\frac{1+\cos(6t)}{2}\right)\left(\frac{1-\cos(6t)}{2}\right)dt$$
$$= \frac{1}{4}\int (1+\cos(6t))(1-\cos(6t))dt$$
$$= \frac{1}{4}\int (1-\cos^2(6t))dt$$
$$= \frac{1}{4}\int \left(1-\frac{1+\cos(12t)}{2}\right)dt$$
$$= \frac{1}{4}\int \left(\frac{1}{2}-\frac{1}{2}\cos(12t)\right)dt$$
$$= \frac{1}{4}\left(\frac{1}{2}t-\frac{1}{24}\sin(12t)\right)+C$$
$$= \frac{1}{8}t-\frac{1}{96}\sin(12t)+C$$

Note: in the 4th line, $1-\cos^2(6t)$ could be replace with $\sin^2(6t)$, so that

$$= \frac{1}{4}\int (1-\cos^2(6t))dt$$
$$= \frac{1}{4}\int (\sin^2(6t))dt$$
$$= \frac{1}{4}\int \left(\frac{1}{2}-\frac{1}{2}\cos(12t)\right)dt$$
$$= \frac{1}{4}\left(\frac{1}{2}t-\frac{1}{24}\sin(12t)\right)+C$$
$$= \frac{1}{8}t-\frac{1}{96}\sin(12t)+C$$

II) $\int \sin(Ax)\cos(Bx)dx$

In all the previous problems, the arguments of the sine and cosine functions are the same. However, if they are different, and the powers of both sine and cosine are unity, then product-to-sum formulas can be applied:

$$\sin u \cos v = \frac{1}{2}[\sin(u-v)+\sin(u+v)]$$

$$\sin u \sin v = \frac{1}{2}[\cos(u-v)-\cos(u+v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u-v)+\cos(u+v)]$$

Compute the integral:

1) $\int_0^{\pi} \cos(5x)\sin(7x)dx$

Ans:

$$\int \cos(5x)\sin(7x)dx$$

$$=\frac{1}{2}\int[\sin(7x-5x)+\sin(7x+5x)]dx$$

$$=\frac{1}{2}\int[\sin(2x)+\sin(12x)]dx$$

$$=\frac{1}{2}\left[-\frac{1}{2}\cos(2x)-\frac{1}{12}\cos(12x)\right]$$

$$-\frac{1}{4}\cos(2x)-\frac{1}{24}\cos(12x)$$

$$\int_0^{\pi}\cos(5x)\sin(7x)dx$$

$$=-\frac{1}{4}\cos(2x)-\frac{1}{24}\cos(12x)\Big|_0^{\pi}$$

$$=0$$

II) $\int \sec^m(Ax)\tan^n(Bx)dx$

First, compute these integrals:

1) $\int \tan x dx$ Answer: $\ln|\sec x|+C$

2) $\int \sec x dx$ Answer: $\ln|\sec x + \tan x| + C$

3) $\int \sec^2 x dx$ Answer: $\tan x + C$

4) $\int \sec x \tan x dx$ Answer: $\sec x + C$

5) $\int \tan(3x)\sec^2(3x)dx$ Answer: $\frac{1}{9}\tan^2(3x) + C$

A) Integrals with secant and tangent functions sometimes can be treated (roughly) similarly to integrals containing odd powers of sine or cosine. In such cases, after factoring an appropriate function , an identity for tangent and secant, related to $\sin^2 x + \cos^2 x = 1$, can be applied before u-substitution.

First, answer each with True or False.

1) $\tan^2 x + 1 = \sec^2 x$ T
2) $\sec^2 x + 1 = \tan^2 x$ F

Compute the following integrals:

3) $\int \sec^6(t)\tan^2(t)dt$

Ans:

$$\int \sec^6(t)\tan^2(t)dt$$

$$= \int \sec^4(t)\tan^2(t)\sec^2(t)dt$$

$$= \int \left(\sec^2(t)\right)^2 \tan^2(t)\sec^2(t)dt$$

$$= \int \left(1 + \tan^2(t)\right)^2 \tan^2(t)\sec^2(t)dt$$

Let $u = \tan(t)$. Then, $du = \sec^2(t)dt$, and

$$\int \left(1 + \tan^2(t)\right)^2 \tan^2(t)\sec^2(t)dt$$

$$\int \left(1 + u^2\right)^2 u^2 du$$

$$= \int \left(u^2 + 2u^4 + u^6\right)du$$

$$= \frac{u^3}{3} + \frac{2u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{\tan^3 x}{3} + \frac{2\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

4) $\int \sec^5(3\theta)\tan^5(3\theta)d\theta$

Ans:

$$\int \sec^4(3\theta)\tan^4(3\theta)\sec(3\theta)\tan(3\theta)d\theta$$

$$= \int \sec^4(3\theta)\left(\tan^2(3\theta)\right)^2 \sec(3\theta)\tan(3\theta)d\theta$$

$$= \int \sec^4(3\theta)\left(\sec^2(3\theta)-1\right)^2 \sec(3\theta)\tan(3\theta)d\theta$$

Let Let $u=\sec(3t)$. Then, $du = 3\sec(3t)\tan(3t)dt$, and

$$\int \sec^4(3\theta)\left(\sec^2(3\theta)-1\right)^2 \sec(3\theta)\tan(3\theta)d\theta$$

$$= \frac{1}{3}\int u^4\left(u^2-1\right)^2 du$$

$$= \frac{1}{3}\int \left(u^8-2u^6+u^4\right)du$$

$$= \frac{1}{3}\left(\frac{u^9}{9}-2\frac{u^7}{7}+\frac{u^5}{5}\right)+C$$

$$= \frac{\sec^9(3\theta)}{27}-\frac{2}{21}\sec^7(3\theta)+\frac{\sec^5(3\theta)}{15}+C$$

B) $\int \tan^n(Bx)dx$ **(no secant functions). In these cases, one may be able to first factor** $\tan^2(Bx)$**. At this stage, a trig identity, an integral "split" and u-substitution may be applied.**

Compute the integral:

1) $\int \tan^7(t)dt$

Ans:

$$
\begin{aligned}
&\int \tan^7(t)dt \\
&= \int \tan^5(t)\tan^2(t)dt \\
&= \int \tan^5(t)\left(\sec^2(t)-1\right)dt \\
&= \int \tan^5(t)\sec^2(t)dt - \int \tan^5(t)dt \\
&= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\tan^2(t)dt \\
&= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\left(\sec^2(t)-1\right)dt \\
&= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\sec^2(t)dt + \int \tan^3(t)dt \\
&= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\sec^2(t)dt + \int \tan(t)\tan^2(t)dt \\
&= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\sec^2(t)dt + \int \tan(t)\left(\sec^2(t)-1\right)dt \\
&= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\sec^2(t)dt + \int \tan(t)\sec^2(t)dt - \int \tan(t)dt \\
&= \frac{\tan^6 t}{6} - \frac{\tan^4 t}{4} + \frac{\tan^2 t}{2} - \ln\left|\sec t\right| + C
\end{aligned}
$$

where we let $u = \tan(t)$ in the first, second, and third integrals.