PACKET 5

(8.2) Area of a Surface of Revolution, (9.1) DE Models

(8.2) Area of a Surface of Revolution

1) Recall the general formula for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the x-axis:

   $S = \int_a^b 2\pi r\, ds$. This is related to the surface area of a cylinder with height *l* and radius *r* (without top and bottom): $2\pi rl$, where *l* is replaced with *ds*, the differential arc length.

   a) Rewrite the integral $S = \int_a^b 2\pi r\, ds$ as an integral with respect to *x* for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the x-axis.

      Solution: $S = \int_a^b 2\pi r\, ds = \int_a^b 2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

   b) Rewrite the integral $S = \int_a^b 2\pi r\, ds$ as an integral with respect to *y* for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the *x*-axis. (Assume $f(a) \le f(b)$ and $x = g(y)$)

      Solution: $S = \int_{f(a)}^{f(b)} 2\pi r\, ds = \int_{f(a)}^{f(b)} 2\pi y\sqrt{1+(g'(y))^2}\, dy = \int_{f(a)}^{f(b)} 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

2) Recall the general formula for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the y-axis:

   $S = \int_a^b 2\pi r\, ds$

   a) Rewrite the integral $S = \int_a^b 2\pi r\, ds$ as an integral with respect to *x* for the area of a surface of revolution of $y = f(x)$ on $a \le x \le b$ about the y-axis.

Solution: $S=\int_a^b 2\pi r\,ds=\int_a^b 2\pi x\sqrt{1+\left(f'(x)\right)^2}\,dx=\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

b) Rewrite the integral $S=\int_a^b 2\pi r\,ds$ as an integral with respect to y for the area of a surface of revolution of $y=f(x)$ on $a\le x\le b$ about the y-axis. (Assume $f(a)\le f(b)$ and $x=g(y)$)

Solution: $S=\int_{f(a)}^{f(b)} 2\pi r\,ds=\int_{f(a)}^{f(b)} 2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\,dx=\int_{f(a)}^{f(b)} 2\pi g(y)\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dx$

3) Given the curve $y=e^{2x}$, $1\le y\le e^2$ set up, but do not evaluate, an integral for the area of the surface obtained by rotating the curve about:

a) the y-axis with the independent variable x.

b) the y-axis with the independent variable y

c) the x-axis with the independent variable x

d) the x-axis with the independent variable y

Solution: (a) x the independent variable. $x=\frac{1}{2}\ln y$, so if $1\le y\le e^2$, then $0\le x\le 1$

$ds=\sqrt{1+(2e^{2x})^2}\,dx$ $\qquad SA=\int_0^1 2\pi x\,ds=\int_0^1 2\pi x\sqrt{1+(2e^{2x})^2}\,dx$

(b) y the independent variable. $x=\frac{1}{2}\ln y$, $1\le y\le e^2$

$ds=\sqrt{1+\left(\frac{1}{2y}\right)^2}\,dy$ $\qquad SA=\int_1^{e^2} 2\pi x\,ds=\int_1^{e^2} 2\pi\,\frac{1}{2}\ln y\sqrt{1+\left(\frac{1}{2y}\right)^2}\,dy$

(c) x the independent variable. $\int_0^1 2\pi e^{2x}\sqrt{1+(2e^{2x})^2}\,dx$

(d) y the independent variable $\int_1^{e^2} 2\pi\,y\sqrt{1+\left(\frac{1}{2y}\right)^2}\,dy$

4) Consider the curve $y=2\sqrt{1+2x}$, $2\le x\le 10$.

a) If this segment of the curve is rotated about the x-axis, find the area of the resulting surface. Use the version of the integral with respect to x.

Solution;

Since $ds = \sqrt{1+\left(\dfrac{2}{(1+2x)^{1/2}}\right)^2}\,dx$, the surface area is $\displaystyle\int_2^{10} 2\pi\, r\, ds = \int_2^{10} 2\pi(2)\sqrt{1+2x}\sqrt{\frac{5+2x}{1+2x}}\,dx$

$\displaystyle = \int_2^{10} 4\pi\sqrt{5+2x}\,dx = 4\pi\frac{2}{3}(5+2x)^{3/2}\Big|_2^{10} = \frac{4}{3}\pi\left[25^{3/2}-27\right]$ sq units.

b) Compute the integral with independent variable *y*, representing the same area of revolution about the x-axis.

Solution:

$x=2 \Rightarrow y = 2\sqrt{1+2(2)} = 2\sqrt{5}$ and $x=10 \Rightarrow y = 2\sqrt{1+2(10)} = 2\sqrt{21}$

$y^2 = 4(1+2x)$

$\dfrac{y^2}{4} = 1+2x$

$x = \dfrac{y^2}{8} - \dfrac{1}{2}$

$\dfrac{dx}{dy} = \dfrac{y}{4}$ so that $SA = \displaystyle\int_{2\sqrt{5}}^{2\sqrt{21}} 2\pi y\sqrt{1+\frac{y^2}{4}}\,dy$. Let $u = 1+\dfrac{y^2}{4}$ … Ans. $\dfrac{8}{3}\pi\left[25^{3/2}-27\right]$

c) If this segment of the curve is rotated about the *y*-axis, set up, but to not compute, the integral with respect to *x* that represents the area of revolution. Note that you will have to change only one item in your integral from part (a)!

Solution: $\displaystyle\int_2^{10} 2\pi x\sqrt{\frac{5+2x}{1+2x}}\,dx$

d) If this segment of the curve is rotated about the *y*-axis, set up, but to not compute, the integral with respect to *y* that represents the area of revolution. Note that you will have to change only one item in your integral from part (b)!

Solution: $\displaystyle\int_{2\sqrt{5}}^{2\sqrt{21}} 2\pi\left(\frac{y^2}{8}-\frac{1}{2}\right)\sqrt{1+\frac{y^2}{4}}\,dy$

5) Consider the area of the surface formed by revolving the curve $y = \arcsin x$ on $0 \le x \le \dfrac{1}{2}$ about the *y*-axis.

a) First, set up the integral with respect to x that represents this area. Do not evaluate the integral.

Solution: $\int_0^{1/2} 2\pi x\sqrt{1+\left(\frac{1}{\sqrt{1-x^2}}\right)^2}\,dx = \int_0^{1/2} 2\pi x\sqrt{1+\frac{1}{1-x^2}}\,dx$

b) It turns out that using the integral with respect to y will make the problem more tractable, but not trivial! Set up the integral with respect to y that represents the same area as your integral in part (a). Then, if time permits, evaluate this integral. (This can be reserved for the Extra Problems.)

Solution: Since $y = \arcsin x$, then $\sin y = x$ on $0 \le x \le \frac{1}{2}$. Also, when $x = 0$, $y = \arcsin 0 = 0$, and when $x = \frac{1}{2}$, $y = \arcsin \frac{1}{2} = \frac{\pi}{6}$

Therefore,

$$\int_0^{1/2} 2\pi x\sqrt{1+\left(\frac{1}{\sqrt{1-x^2}}\right)^2}\,dx$$

$$= \int_0^{\pi/6} 2\pi \sin y\sqrt{1+\cos^2 y}\,dy$$

Now, let $u = \cos y$. Then, $du = -\sin y\,dy$, and

$$\int 2\pi \sin y\sqrt{1+\cos^2 y}\,dy$$

$$= -2\pi\int \sqrt{1+u^2}\,du$$

Now, let $u = \tan\theta$ so that $du = \sec^2\theta\,d\theta$ and

$$-2\pi\int \sqrt{1+u^2}\,du$$

$$= -2\pi\int \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta$$

$$= -2\pi\int \sec\theta\sec^2\theta\,d\theta$$

We then perform integration by parts (with a few other steps!!) to obtain

$$-2\pi\int \sec\theta \sec^2\theta\, d\theta$$

$$= -2\pi\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right]$$

$$= -\pi\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right]$$

$$= -\pi\left[u\sqrt{u^2+1} + \ln\left|\sqrt{u^2+1} + u\right|\right]$$

$$= -\pi\left[\cos y\sqrt{\cos^2 y+1} + \ln\left|\sqrt{\cos^2 y+1} + \cos y\right|\right]$$

Finally,

$$\int_0^{\pi/6} 2\pi \sin y\sqrt{1+\cos^2 y}\,dy$$

$$= -\pi\left[\cos y\sqrt{\cos^2 y+1} + \ln\left|\sqrt{\cos^2 y+1} + \cos y\right|\right]\Bigg|_0^{\pi/6}$$

$$= \pi\left[\sqrt{2} + \ln\left(\sqrt{2}+1\right) - \frac{\sqrt{21}}{4} - \ln\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)\right]$$

(9.1) DE Models

1. For the differential equation

$$\frac{dy}{dt} = y^3 - y^2 - 6y$$

a) Find the equilibrium solutions for the DE.

Solution:
These solutions occur when dy/dt=0, or when the right side is 0:

$$y^3 - y^2 - 6y = 0$$

$$y(y^2 - y - 6) = 0$$

$$y(y-3)(y+2) = 0$$

$$y = 0, y = 3, y = -2$$

so y = 0 , y = 3, and y = -2 are the three solution curves (horizontal lines) where y will never change.

b) For what values of *y* is the solution increasing? Give your answer in interval notation.

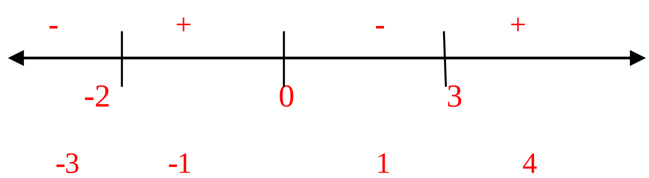
Solution:
This occurs when dy/dt>0, or when

$$y^3 - y^2 - 6y > 0$$

or

$y(y-3)(y+2) > 0$.

We perform a sign analysis on the function $f(y) = y(y-3)(y+2)$, which is the right side of the differential equation. We use test points in the four intervals:



and we find that

$f(-3) < 0,\ f(-1) > 0,\ f(1) < 0,$

and

$f(4) > 0$

so that

$y(y-3)(y+2)$ is positive on $(-2,0) \cup (3,\infty)$ and negative on $(-\infty,-2) \cup (0,3)$.

Therefore

dy/dt>0

on

$(-2,0) \cup (3,\infty)$.

c) For what values of y is the solution decreasing? Give your answer in interval notation.

Solution:
These are values of y where dy/dt<0 , so, from the sign analysis of part (b), y is decreasing on $(-\infty,-2) \cup (0,3)$

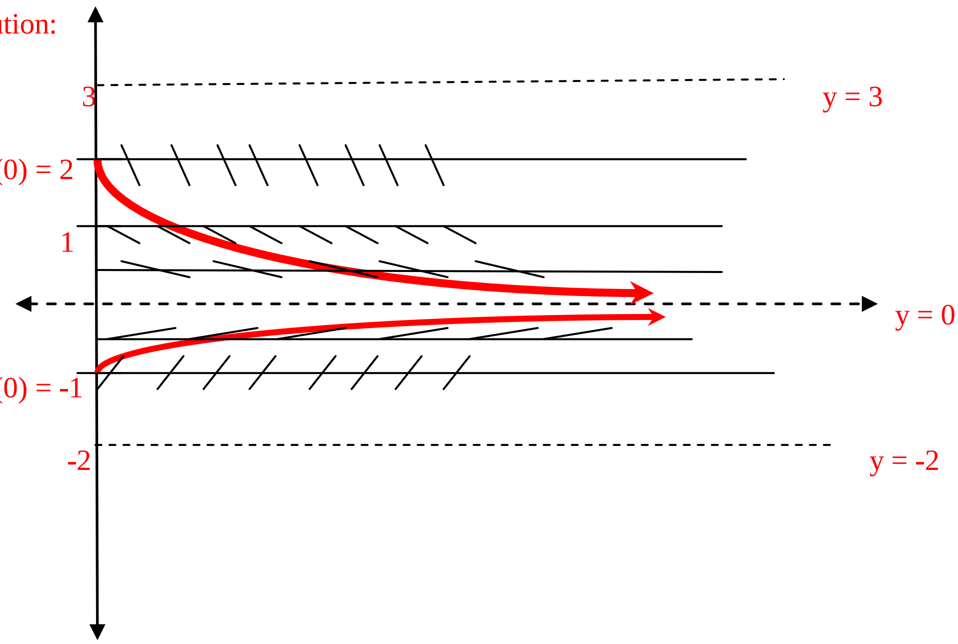
d) Make a rough sketch of the solution curves for each of the following initial values for $t \geq 0$.

$y(0) = -1$

$y(0) = 2$

To aid in your sketching, add a slope field (direction field) by finding the slope of solutions on the lines $y = -1, y = -\frac{1}{2}, y = \frac{1}{2}, y = 1, y = 2$.

Solution:



The equilibrium solution y = 0 serves as a horizontal asymptote. A direction field for different values of y can be used to verify these solution curves, e.g., if y = 1, then for the right side of the DE, f(y),

$f(1) = 1(1-3)(1+2) = -6$

so the tangent slopes for all solution curves at the point where they cross the line y = 1 is -6, so we place line segments with slope -6 (roughly) on the horizontal line y = 1. We can do this for other values of y as well.

Note that that the DE $\dfrac{dy}{dt} = y^3 - y^2 - 6y$

can be solved !! If time, we will solve this equation in the "Extra Problems" section.

e) Use your answer to part (d) to estimate $\lim_{t\to\infty} y$ for any initial condition where $y(0) > -2$ and $y(0) < 3$

Solution:

$\lim_{t\to\infty} y = 0$

2. Show that the function

$y = \sin(2t)$

is a solution of the IVP (initial value problem),

$$y''' - y'' + 4y' = 4y$$
$$y\left(\frac{\pi}{4}\right) = 1$$

Solution:

If $y = \sin(2t)$

Then
$$y' = 2\cos(2t)$$
$$y'' = -4\sin(2t)$$
$$y''' = -8\cos(2t)$$

Checking the left side of the equation, we have

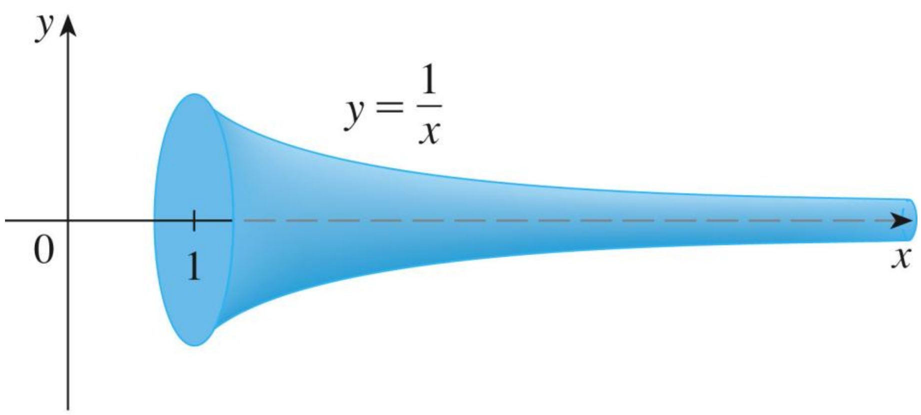
$$y''' - y'' + 4y' = -8\cos(2t) + 4\sin(2t) + 8\cos(2t) = 4\sin(2t)$$

Checking the right side, we have
$$4y = 4\sin(2t),$$
which is equal to the left side.

Finally, checking the initial condition, we have $y = \sin\left(2\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1$, which matches the given initial condition.

Extra Problems

1) Consider the function $y = \dfrac{1}{x}$ on $1 \le x \le \infty$.
   a) Set up the integral that represents the surface area generated by revolving this curve about the x-axis. A depiction of this surface is below. (This is sometimes referred to as Gabriel's Horn.)



Ans. Since $y' = -\frac{1}{x^2}$, then the surface area is

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx$$

$$= \int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

$$= \int_1^\infty 2\pi \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx$$

$$= \int_1^\infty 2\pi \frac{1}{x}\frac{\sqrt{x^4+1}}{x^2}\,dx$$

$$= \int_1^\infty 2\pi \frac{\sqrt{x^4+1}}{x^3}\,dx$$

b) Apply the Comparison Test for Improper Integrals to show whether your integral in part (a) diverges or converges.

Ans. Since $\frac{\sqrt{x^4+1}}{x^3} > \frac{\sqrt{x^4}}{x^3} = \frac{1}{x}$ and $\int_0^\infty \frac{1}{x}dx$ diverges to ∞, then $\int_1^\infty 2\pi \frac{\sqrt{x^4+1}}{x^3}\,dx$ must also diverge to ∞.

c) The following integral represents the volume inside Gabriel's Horn:

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

Compute this integral. Are the results of this computation surprising, given your results in part (b)?

Ans.

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \pi\int_1^\infty x^{-2}dx$$

$$= \lim_{t\to\infty} \pi\int_1^t x^{-2}dx$$

$$= \lim_{t\to\infty} \left.\frac{-\pi}{x}\right|_1^t$$

$$= \lim_{t\to\infty}\left(\frac{-\pi}{t}+\pi\right) = \pi$$

Even though the surface area of Gabriel's Horn is infinite, its volume is finite!!

2) Verify that the equation

$$ye^y - e^y = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

is a solution to the DE

$$yy' = \frac{1}{9e^y + x^2e^y}$$

Hint: you will need implicit differentiation.

Solution:

Using implicit differentiation,

$$\frac{d}{dx}\left(ye^y - e^y\right) = \frac{d}{dx}\left(\frac{1}{3}\arctan\left(\frac{x}{3}\right) + C\right)$$

$$y'e^y + ye^y y' - e^y y' = \frac{1}{3}\frac{1}{1+\left(\frac{x}{3}\right)^2}\frac{1}{3}$$

$$y'e^y + ye^y y' - e^y y' = \frac{1}{9}\frac{1}{1+\frac{x^2}{9}}$$

$$y'e^y + ye^y y' - e^y y' = \frac{1}{9+x^2}$$

where we have applied the chain rule on the right hand side to compute the derivative of $\arctan\left(\frac{x}{3}\right)$.

Solving for y' ,we have

$$y'e^y + ye^y y' - e^y y' = \frac{1}{9+x^2}$$

$$y'\left(e^y + ye^y - e^y\right) = \frac{1}{9+x^2}$$

$$y'\left(ye^y\right) = \frac{1}{9+x^2}$$

$$y' = \frac{1}{ye^y\left(9+x^2\right)}$$

Substituting this expression for y' into the the left hand side of the DE, we have

$$yy' = y\frac{1}{ye^y\left(9+x^2\right)} = \frac{1}{e^y\left(9+x^2\right)} = \frac{1}{9e^y + x^2e^y}$$

which yields the right hand side of the DE. The solution is verified.

3) Find all values of r such that $y = x^r$ satisfies the DE $2x^2y'' + 3xy' - y = 0$. Use your result to find two different functions that are solutions of this DE.

Solution: Note that

$$y' = rx^{r-1}$$

$$y'' = r(r-1)x^{r-2}.$$

Substituting these functions into the DE leads to

$$2x^2r(r-1)x^{r-2} + 3xrx^{r-1} - x^r = 0$$

$$2r(r-1)x^2x^{r-2} + 3rxx^{r-1} - x^r = 0$$

$$2r(r-1)x^r + 3rx^r - x^r = 0$$

$$[2r(r-1) + 3r - 1]x^r = 0$$

$$2r(r-1) + 3r - 1 = 0$$

$$2r^2 + r - 1 = 0$$

$$(2r-1)(r+1) = 0$$

$$r = \frac{1}{2}, -1$$

The two solutions are then $y = \sqrt{x}, \quad y = \frac{1}{x}$

Looking Ahead

Methods do exist for solving certain types of differential equations. If, for example, a differential equation can be rewritten in the form,

$$f(x)dx = g(y)dy$$

then the equation is said to be separable (one variable is only on the left side, and the other is only on the right side). Separable DEs can then be solved by integrating each side:

$$\int f(x)dx = \int g(y)dy$$

For example, the DE

$$\frac{dy}{dx} = xy + x$$

is separable since it can be rewritten as

$$\frac{dy}{dx} = x(y+1)$$
$$dy = x(y+1)dx$$
$$\frac{1}{y+1}dy = xdx$$

We can then integrate both sides:

$$\int \frac{1}{y+1}dy = \int xdx$$
$$\ln|y+1| + C_1 = \frac{x^2}{2} + C_2$$
$$\ln|y+1| = \frac{x^2}{2} + C_2 - C_1$$
$$\ln|y+1| = \frac{x^2}{2} + C$$

Note that, since we have not solved for *y*, this type of solution is called an <u>explicit solution.</u> Also note that the two arbitrary constants arising from the antiderivatives merge into one, called "*C*." This can always be done, in general, so that one only needs to add a "*C*" to the right side of the equation after all the antiderivatives are computed: one does not need to formally show both constants after each integration.

1) Solve the following separable equation.

$$\frac{dy}{dx} = \frac{xe^{2y} - e^{2y}}{x^2 + 1}$$

Ans.
The DE

$$\frac{dy}{dx} = \frac{xe^{2y} - e^{2y}}{x^2 + 1}$$

is separable since it can be rewritten as

$$\frac{dy}{dx} = \frac{e^{2y}(x-1)}{x^2+1}$$
$$dy = \frac{e^{2y}(x-1)}{x^2+1}dx$$
$$e^{-2y}dy = \frac{x-1}{x^2+1}dx$$

We can then integrate both sides:

$$\int e^{-2y}dy = \int \frac{x-1}{x^2+1}dx$$
$$\int e^{-2y}dy = \int \frac{x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$$
$$-\frac{1}{2}e^{-2y} = \frac{1}{2}\ln\left(x^2+1\right) - \arctan x + C$$