PACKET 2
(7.2) Trigonometric Integrals, (7.3) Trigonometric Substitution

Trigonometric Integrals (7.2)

First let's recall some basic integral rules for trigonometric functions.

Find the antiderivative:

1) $\int \sin x\, dx$

Ans: $-\cos x + C$

2) $\int \cos x\, dx$

Ans: $\sin x + C$

3) $\int \tan x\, dx$

Ans: $\ln|\sec x| + C$

4) $\int \sec x\, dx$

Ans: $\ln|\sec x + \tan x| + C$

I) $\int \sin^m(Ax)\cos^n(Ax)dx$ **, where neither power is equal to 1.**

First, compute this integral, where one of the powers is indeed unity:

1) $\int \sin^4(3x)\cos(3x)dx$

Ans: Let $u = \sin(3x)$. Then, $du = 3\cos(3x)dx$, and

$$\int \sin^4(3x)\cos(3x)dx$$

$$= \frac{1}{3}\int u^4 du$$

$$= \frac{1}{15}u^5 + C$$

$$= \frac{1}{15}\sin^5(3x) + C$$

Now, if neither power is 1, it may be helpful to apply trigonometric identities before u-substitution.

A) Either power (or both powers) of sine and cosine is odd. Then, one can factor out one of the functions and then apply the identity $\sin^2 x + \cos^2 x = 1$ to aid with integration.

Compute the integrals:

1) $\int \sin^4(x)\cos^5(x)dx$

Ans:

$$\int \sin^4(x)\cos^5(x)dx$$
$$= \int \sin^4(x)\cos^4(x)\cos(x)dx$$
$$= \int \sin^4(x)\left(\cos^2(x)\right)^2 \cos(x)dx$$
$$= \int \sin^4(x)\left(1-\sin^2(x)\right)^2 \cos(x)dx$$

Now, let $u = \sin(x)$. Then, $du = \cos(x)dx$, and

$$\int \sin^4(x)\left(1-\sin^2(x)\right)^2 \cos(x)dx$$
$$= \int u^4(1-u^2)^2 du$$
$$= \int \left(u^4 - 2u^6 + u^8\right)du$$
$$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C$$
$$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$$

B) Note that, in problems like the example above, there is an odd power on the sine function, the cosine function, or both. However, what if there are only even powers of the sine and cosine function? One must apply power reduction formulas!!

First, answer each with True or False.

1) $\sin^2 x = \dfrac{1-\cos(2x)}{2}$ T

2) $\sin^2 x = \dfrac{1-\sin(2x)}{2}$ F

3) $\cos^2(3x) = \dfrac{1+\cos(3x)}{2}$ F

4) $\cos^2(3x) = \dfrac{1+\cos(6x)}{2}$ T

5) $\int \sin^2(4t)\cos^2(4t)dt = \frac{\sin^3(4t)}{12}\frac{\cos^3(4t)}{12} + C$ F

Compute the integral:

6) $\int \cos^2(3t)\sin^2(3t)dt$

Ans: (use power reduction formulas for each of $\cos^2(3t)$ and $\sin^2(3t)$, multiply out the result, then apply a power reduction one more time before integrating.)

$$\begin{aligned}
&\int \cos^2(3t)\sin^2(3t)dt \\
&= \int \left(\frac{1+\cos(6t)}{2}\right)\left(\frac{1-\cos(6t)}{2}\right)dt \\
&= \frac{1}{4}\int (1+\cos(6t))(1-\cos(6t))dt \\
&= \frac{1}{4}\int (1-\cos^2(6t))dt \\
&= \frac{1}{4}\int \left(1-\frac{1+\cos(12t)}{2}\right)dt \\
&= \frac{1}{4}\int \left(\frac{1}{2}-\frac{1}{2}\cos(12t)\right)dt \\
&= \frac{1}{4}\left(\frac{1}{2}t-\frac{1}{24}\sin(12t)\right)+C \\
&= \frac{1}{8}t-\frac{1}{96}\sin(12t)+C
\end{aligned}$$

Note: in the 4^{th} line, $1-\cos^2(6t)$ could be replace with $\sin^2(6t)$, so that

$$\begin{aligned}
&= \frac{1}{4}\int (1-\cos^2(6t))dt \\
&= \frac{1}{4}\int (\sin^2(6t))dt \\
&= \frac{1}{4}\int \left(\frac{1}{2}-\frac{1}{2}\cos(12t)\right)dt \\
&= \frac{1}{4}\left(\frac{1}{2}t-\frac{1}{24}\sin(12t)\right)+C \\
&= \frac{1}{8}t-\frac{1}{96}\sin(12t)+C
\end{aligned}$$

II) $\int \sin(Ax)\cos(Bx)dx$

In all the previous problems, the arguments of the sine and cosine functions are the same. However, if they are different, and the powers of both sine and cosine are unity, then product-to-sum formulas can be applied:

$$\sin u \cos v = \frac{1}{2}[\sin(u-v)+\sin(u+v)]$$

$$\sin u \sin v = \frac{1}{2}[\cos(u-v)-\cos(u+v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u-v)+\cos(u+v)]$$

Compute the integral:

1) $\int_0^{\pi} \cos(5x)\sin(7x)dx$

Ans:

$$\int \cos(5x)\sin(7x)dx$$

$$=\frac{1}{2}\int [\sin(7x-5x)+\sin(7x+5x)]dx$$

$$=\frac{1}{2}\int [\sin(2x)+\sin(12x)]dx$$

$$=\frac{1}{2}\left[-\frac{1}{2}\cos(2x)-\frac{1}{12}\cos(12x)\right]$$

$$-\frac{1}{4}\cos(2x)-\frac{1}{24}\cos(12x)$$

$$\int_0^{\pi} \cos(5x)\sin(7x)dx$$

$$=-\frac{1}{4}\cos(2x)-\frac{1}{24}\cos(12x)\Big|_0^{\pi}$$

$$=0$$

II) $\int \sec^m(Ax)\tan^n(Bx)dx$

First, compute these integrals:

1) $\int \tan x dx$ Answer: $\ln|\sec x| + C$

2) $\int \sec x dx$ Answer: $\ln|\sec x + \tan x| + C$

3) $\int \sec^2 x dx$ Answer: $\tan x + C$

4) $\int \sec x \tan x dx$ Answer: $\sec x + C$

5) $\int \tan(3x)\sec^2(3x)dx$ Answer: $\frac{1}{9}\tan^2(3x) + C$

A) Integrals with secant and tangent functions sometimes can be treated (roughly) similarly to integrals containing odd powers of sine or cosine. In such cases, after factoring an appropriate function , an identity for tangent and secant, related to $\sin^2 x + \cos^2 x = 1$**, can be applied before u-substitution.**

First, answer each with True or False.

$\tan^2 x + 1 = \sec^2 x$ Ans:T
$\sec^2 x + 1 = \tan^2 x$ Ans: F

Compute the following integrals:

1) $\int \sec^6(t)\tan^2(t)dt$

Ans:

$\int \sec^6(t)\tan^2(t)dt$

$= \int \sec^4(t)\tan^2(t)\sec^2(t)dt$

$= \int \left(\sec^2(t)\right)^2 \tan^2(t)\sec^2(t)dt$

$= \int \left(1+\tan^2(t)\right)^2 \tan^2(t)\sec^2(t)dt$

Let $u = \tan(t)$. Then, $du = \sec^2(t)dt$, and

$$\int \left(1+\tan^2(t)\right)^2 \tan^2(t)\sec^2(t)dt$$
$$\int \left(1+u^2\right)^2 u^2 du$$
$$= \int \left(u^2 + 2u^4 + u^6\right)du$$
$$= \frac{u^3}{3} + \frac{2u^5}{5} + \frac{u^7}{7} + C$$
$$= \frac{\tan^3 x}{3} + \frac{2\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

2) $\int \sec^5(3\theta)\tan^5(3\theta)d\theta$

Ans:

$$\int \sec^4(3\theta)\tan^4(3\theta)\sec(3\theta)\tan(3\theta)d\theta$$
$$= \int \sec^4(3\theta)\left(\tan^2(3\theta)\right)^2 \sec(3\theta)\tan(3\theta)d\theta$$
$$= \int \sec^4(3\theta)\left(\sec^2(3\theta)-1\right)^2 \sec(3\theta)\tan(3\theta)d\theta$$

Let Let $u = \sec(3t)$. Then, $du = 3\sec(3t)\tan(3t)dt$, and

$$\int \sec^4(3\theta)\left(\sec^2(3\theta)-1\right)^2 \sec(3\theta)\tan(3\theta)d\theta$$
$$= \frac{1}{3}\int u^4\left(u^2-1\right)^2 du$$
$$= \frac{1}{3}\int \left(u^8 - 2u^6 + u^4\right)du$$
$$= \frac{1}{3}\left(\frac{u^9}{9} - 2\frac{u^7}{7} + \frac{u^5}{5}\right) + C$$
$$= \frac{\sec^9(3\theta)}{27} - \frac{2}{21}\sec^7(3\theta) + \frac{\sec^5(3\theta)}{15} + C$$

B) $\int \tan^n(Bx)dx$ **(no secant functions). In these cases, one may be able to first factor** $\tan^2(Bx)$**. At this stage, a trig identity, an integral "split" and u-substitution may be applied.**

Compute the integral:

1) $\int \tan^7(t)dt$

Ans:

$$\int \tan^7(t)dt$$
$$= \int \tan^5(t)\tan^2(t)dt$$
$$= \int \tan^5(t)\left(\sec^2(t) - 1\right)dt$$
$$= \int \tan^5(t)\sec^2(t)dt - \int \tan^5(t)dt$$
$$= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\tan^2(t)dt$$
$$= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\left(\sec^2(t) - 1\right)dt$$
$$= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\sec^2(t)dt + \int \tan^3(t)dt$$
$$= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\sec^2(t)dt + \int \tan(t)\tan^2(t)dt$$
$$= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\sec^2(t)dt + \int \tan(t)\left(\sec^2(t) - 1\right)dt$$
$$= \int \tan^5(t)\sec^2(t)dt - \int \tan^3(t)\sec^2(t)dt + \int \tan(t)\sec^2(t)dt - \int \tan(t)dt$$
$$= \frac{\tan^6 t}{6} - \frac{\tan^4 t}{4} + \frac{\tan^2 t}{2} - \ln\left|\sec t\right| + C$$

where we let $u = \tan(t)$ in the first, second, and third integrals.

(7.3) Trigonometric Substitution

Given a real number $a > 0$ and a function $u = g(x)$, there are many integrals containing the forms $\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, or $\sqrt{u^2 - a^2}$, which do not require trigonometric substitution, or some other more complicated method.

Explain what method you would use to evaluate each integral below. (Some of these integrals may be rules, so you can evaluate them immediately.) You do not need to solve the integral. Keep in mind that some can be evaluated by more than one method, but you should choose the most efficient one!

Please note that, in section 7.4, you will learn that $\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$, for any positive constant a.

1) $\int \frac{1}{\sqrt{1 - x^2}}dx$

   Ans: $\arcsin(x) + C$

2) $\int \frac{x}{\sqrt{1 - x^2}}dx$

Ans: Let $u=1-x^2$ to yield $\int \frac{x}{\sqrt{1-x^2}}dx = \frac{-1}{2}\int u^{-1/2}du = \ldots = -\sqrt{1-x^2}+C$

3) $\int \frac{x}{1+x^2}dx$

Ans: Let $u=1+x^2$ to yield $\int \frac{x}{1+x^2}dx = \frac{1}{2}\int \frac{du}{u} = \ldots = \frac{1}{2}\ln\left|1+x^2\right|+C$

4) $\int \frac{1}{1+x^2}dx$

Ans: $\arctan(x)+C$

5) $\int \frac{1}{5+x^2}dx$

Ans: $\frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)+C$

6) $\int \frac{5+x}{5+x^2}dx$

Ans: Split the integral into two:

$$\int \frac{5+x}{5+x^2}dx = \int \frac{5}{5+x^2}dx + \int \frac{x}{5+x^2}dx = \ldots = \sqrt{5}\arctan\left(\frac{x}{\sqrt{5}}\right)+\frac{1}{2}\ln\left|5+x^2\right|+C$$

However, for many other integrals, trigonometric substitutions may be appropriate if the integral contains one of these three forms: $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, or $\sqrt{u^2-a^2}$. Often, after one of these substitutions is made, the integrand can be reduced into a form that may require one of the techniques from the last section (trigonometric integrals).

Try $u=a\sin\theta$ for integrands containing $\sqrt{a^2-u^2}$

Try $u=a\tan\theta$ for integrands containing $\sqrt{a^2+u^2}$

Try $u=a\sec\theta$ for integrands containing $\sqrt{u^2-a^2}$

Answer each with True or False. Explain!

7) If $x=4\sec\theta$ then $dx=4\sec\theta\tan\theta$
8) If $x=4\sec\theta$ then $dx=4\sec\theta d\theta$
9) For the integral $\int\sqrt{x^2-4}\,dx$, the substitution $x=2\sec\theta$ leads to

$$\int\sqrt{x^2-4}\,dx = \int\sqrt{4\sec^2\theta-4}\,d\theta = 2\int\sqrt{\sec^2\theta-1}\,d\theta = 2\int\sqrt{\tan^2\theta}\,d\theta$$

The following integrals are among the types you might see after trigonometric substitution.
Find the integrals:

10) $\displaystyle\int \frac{\tan\theta}{\sec\theta}\,d\theta$

Ans: $\displaystyle\int \frac{\tan\theta}{\sec\theta}\,d\theta = \int \sin\theta d\theta = -\cos\theta + C$

11) $\displaystyle\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$

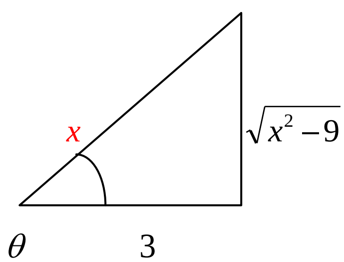
Ans: $\displaystyle\int \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int \cos^{-2}\theta\sin\theta d\theta = -\int u^{-2}du = \ldots = \frac{1}{\cos\theta} + C$

Use of the Pythagorean Theorem, along with a right triangle, will help one rewrite the function of θ, resulting from trig substitution, in terms of *x*.

12) Given that $x = 3\sec\theta$, use a right triangle, along with the Pythagorean Theorem, to find the following in terms of *x*:

a) $\cos\theta$
b) $\sin\theta$
c) $\tan\theta$
d) $\csc\theta$
e) $\cot\theta$

Solution: Since $\dfrac{x}{3} = \sec\theta$, then we define the lengths of the sides of the right triangle below,



which leads to:

a) Ans. $\dfrac{3}{x}$

b) Ans. $\dfrac{\sqrt{x^2-9}}{x}$

c) Ans. $\dfrac{\sqrt{x^2-9}}{3}$

d) Ans. $\dfrac{x}{\sqrt{x^2-9}}$

e) Ans. $\dfrac{3}{\sqrt{x^2-9}}$

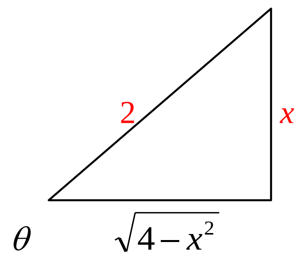
Now, find these integrals using trig substitution:

13) $\int \dfrac{x^2}{\sqrt{4-x^2}}\,dx$. After you make your substitution, and simplify, you will need to apply a technique discussed in the last section (7.2).

Ans: let $x = 2\sin\theta$

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx$$
$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} 2\cos\theta d\theta$$
$$= 4\int \sin^2\theta d\theta$$
$$= 4\int \left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)$$
$$= 2\theta - \sin(2\theta) + C$$
$$= 2\theta - 2\sin\theta\cos\theta + C$$

At this point, using the substitution equation $\dfrac{x}{2} = \sin\theta$ on a right triangle with acute angle θ,



we have

$$2\theta - 2\sin\theta\cos\theta + C$$
$$= 2\arcsin\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2}$$

<u>An alternate problem:</u>

$$\int \sqrt{9-x^2}\,dx$$

Ans: $\frac{9}{2}\arcsin\left(\frac{x}{3}\right)+\frac{x\sqrt{9-x^2}}{2}+C$

14) $\int \frac{dx}{\left(9x^2+1\right)^{3/2}}$

Ans: Use $3x=\tan\theta$

$$dx=\frac{1}{3}\sec^2\theta d\theta$$

$$\int \frac{dx}{\left(9x^2+1\right)^{3/2}}$$
$$=\frac{1}{3}\int \frac{\sec^2\theta}{\sec^3\theta}d\theta$$
$$=\frac{1}{3}\int \cos\theta d\theta$$
$$==\frac{1}{3}\sin\theta+C$$
$$=\ldots$$
$$=\frac{x}{\sqrt{9x^2+1}}+C$$

An alternate problem:

$$\int \frac{\sqrt{4x^2+1}}{x^4}dx$$

Ans: $-\frac{8}{3}\left(\frac{\sqrt{1+4x^2}}{2x}\right)^3+C$

15) $\int_{\frac{8}{3}}^{\frac{4\sqrt{2}}{3}} \frac{\sqrt{9x^2-16}}{x}dx$

Ans: Use $3x = 4\sec\theta$

$$dx = \frac{4}{3}\sec\theta\tan\theta d\theta$$

$$\int_{\frac{8}{3}}^{\frac{4\sqrt{2}}{3}} \frac{\sqrt{9x^2-16}}{x}dx$$

$$= \ldots$$

$$= 4\int_{\pi/3}^{\pi/4} \tan^2\theta d\theta$$

$$= 4\int_{\pi/3}^{\pi/4} \left(\sec^2\theta - 1\right)d\theta$$

$$= 4\left(\tan\theta - \theta\right)\Big|_{\pi/3}^{\pi/4}$$

$$= 4 - 4\sqrt{3} + \frac{\pi}{3}$$

An alternate problem:

$$\int_{\frac{3}{2\sqrt{2}}}^{\frac{3}{2}} \frac{\sqrt{16x^2-9}}{x}dx$$

Ans: $3\sqrt{3} - 3 - \frac{\pi}{4}$

Extra Problems

More trigonometric integrals:

1) $\int \cos^4(3x)dx$

Ans: (let $\cos^4(3x) = \cos^2(3x)\cos^2(3x)$ and apply the power reduction formulas)

$$\frac{3}{8}x + \frac{1}{12}\cos(6x) + \frac{1}{96}\cos(12x) + C$$

2) $\int (2-3\sin x)^2 dx$

Ans:

$$\int (2-3\sin x)^2 dx$$

$$= \int (4-12\sin x+9\sin^2 x)dx$$

$$= \int \left(4-12\sin x+9\frac{1-\cos(2x)}{2} \right) dx$$

$$= \int \left(\frac{17}{2}-12\sin x-\frac{9}{2}\cos(2x) \right) dx$$

$$= \frac{17}{2}x+12\cos x-\frac{9}{4}\sin(2x)+C$$

3) $\int \tan^6(x)dx$

Ans: (factor out $\tan^2(x)$ repeatedly, and change to $\sec^2(x)-1$)

$$\frac{1}{5}\tan^5 x-\frac{1}{3}\tan^3 x-\tan x+x+C$$

4) $\int \sin^5(x)\cos^4(x)dx$

(factor out $\sin(x)$ and convert the cosines into sines)

Ans: $-\dfrac{\cos^5 x}{5}-\dfrac{2\cos^7 x}{7}-\dfrac{\cos^9 x}{9}+C$

5) $\int \sin(3x)\sin(5x)dx$

(apply product-to-sum trig identity)

Ans: $\dfrac{1}{4}\cos(2x)-\dfrac{1}{16}\cos(8x)+C$

6) $\int \sec^6 x dx$ (Hint: factor out $\sec^2 x$)

Ans: $\dfrac{1}{5}\tan^5 x+\dfrac{2}{3}\tan^3 x-+\tan x+C$

7) $\int \sec^3 x dx$

Ans:

$$\int \sec^3 x dx$$
$$= \int \sec x \sec^2 x dx$$
$$= \sec x \tan x - \int \tan x \sec x \tan x dx \quad (int\_by\_parts)$$
$$= \sec x \tan x - \int \sec x(\sec^2 x - 1) dx$$
$$= \sec x \tan x - \int (\sec^3 x - \sec x) dx$$
$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$
$$= \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x|$$
$$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x|$$
$$\int \sec^3 x dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C$$

8) $\int_0^{\pi} \frac{\sin^3 x}{\cos^2 x} dx$

Ans:

$$\int \frac{\sin^3 x}{\cos^2 x} dx$$
$$= \int \frac{\sin^2 x \sin x}{\cos^2 x} dx$$
$$= \int \frac{(1-\cos^2 x)\sin x}{\cos^2 x} dx$$
$$= \int \frac{\sin x - \cos^2 x \sin x}{\cos^2 x} dx$$
$$= \int \left(\frac{\sin x}{\cos^2 x} - \sin x \right) dx$$
$$= \int \left(\cos^{-2} x \sin x - \sin x \right) dx$$

Let $u = \cos x$ in the first integral, etc… Final answer: -4.

More trigonometric substitution:

1) $\int \frac{\sqrt{x^2+9}}{x^4}dx$

Ans: Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta d\theta$, and

$$\begin{aligned}
&\int \frac{\sqrt{x^2+9}}{x^4}dx \\
&= \int \frac{\sqrt{9\tan^2\theta+9}}{81\tan^4\theta}3\sec^2\theta\, d\theta \\
&= \int \frac{3\sqrt{\tan^2\theta+1}}{81\tan^4\theta}3\sec^2\theta\, d\theta \\
&= \frac{1}{9}\int \frac{\sqrt{\tan^2\theta+1}}{\tan^4\theta}\sec^2\theta\, d\theta \\
&= \frac{1}{9}\int \frac{\sqrt{\sec^2\theta}}{\tan^4\theta}\sec^2\theta\, d\theta \\
&= \frac{1}{9}\int \frac{\sec^3\theta}{\tan^4\theta}d\theta \\
&= \frac{1}{9}\int \frac{1}{\cos^3\theta}\frac{\cos^4\theta}{\sin^4\theta}d\theta \\
&= \frac{1}{9}\int \sin^{-4}\theta\cos\theta d\theta \\
&= \frac{1}{9}\frac{\sin^{-3}\theta}{-3}+C \\
&= -\frac{1}{27}\frac{1}{\sin^3\theta}+C \\
&= -\frac{1}{27}\left(\frac{\sqrt{x^2+9}}{x}\right)^3+C
\end{aligned}$$

2) $\int \sqrt{4-9x^2}dx$

Ans: Use $3x = 2\sin\theta$ to yield $\frac{2}{3}\arcsin\left(\frac{3x}{2}\right)+\frac{x\sqrt{4-9x^2}}{2}+C$

3) $\int x^5\sqrt{4x^2-9}dx$ (note: one can either perform a u-substitution with $u = 4x^2-9$ to find this integral or trigonometric substitution). First, use trig sub, then try to apply u-sub.

Ans: First, let $2x = 3\sec\theta$ to yield,

$$\frac{1}{128}\left(\frac{2}{7}\left(4x^2-9\right)^{7/2}+\frac{36}{5}\left(4x^2-9\right)^{5/2}+\frac{162}{3}\left(4x^2-9\right)^{3/2}\right)+C$$

Alternately, let $u = 4x^2 - 9$. Then, $du = 8xdx$, and

$$\int x^5\sqrt{4x^2-9}\,dx$$

$$= \int x^2x^2\sqrt{4x^2-9}\,xdx$$

$$= \int\left(\frac{u+9}{4}\right)\left(\frac{u+9}{4}\right)\sqrt{u}\,\frac{1}{8}du$$

$$= \frac{1}{128}\int\left(u^{5/2}+18u^{3/2}+81u^{1/2}\right)du$$

$$= \ldots$$

$$= \frac{1}{128}\left(\frac{2}{7}\left(4x^2-9\right)^{7/2}+\frac{36}{5}\left(4x^2-9\right)^{5/2}+\frac{162}{3}\left(4x^2-9\right)^{3/2}\right)$$

4) $\displaystyle\int \frac{x^2}{\left(4-x^2\right)^{3/2}}dx$

Ans: Let $x = 2\sin\theta$ to yield $\displaystyle\frac{\sqrt{4-x^2}}{x} - \arcsin\left(\frac{x}{2}\right)+C$

5) The following integral must be evaluated by completing the square, and then applying trigonometric substitution. Evaluate this integral:

$$\int \frac{dx}{\sqrt{x^2+2x-3}}$$

Ans:

$$\int \frac{dx}{\sqrt{x^2+2x-3}}$$

$$= \int \frac{dx}{\sqrt{x^2+2x+1-1-3}}$$

$$= \int \frac{dx}{\sqrt{(x+1)^2-4}}$$

Now, let $x+1 = 2\sec\theta$. Then, $dx = 2\sec\theta\tan\theta d\theta$, and

$$\int \frac{2\sec\theta\tan\theta\, d\theta}{\sqrt{4\sec^2\theta - 4}}$$

$$= \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta - 1}} d\theta$$

$$= \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}} d\theta$$

$$= \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x+1}{2} + \frac{\sqrt{(x+1)^2 - 4}}{2}\right| + C$$

6) $\int \sqrt{4x^2 - 9}dx$

Ans: let $4x = 3\sec\theta$

$$\int \sqrt{4x^2 - 9}dx = \ldots$$

$$= 3\int \tan^2\theta\sec\theta d\theta$$

$$= 3\int \left(\sec^2\theta - 1\right)\sec\theta d\theta$$

$$= 3\int \left(\sec^3\theta - \sec\theta\right)d\theta$$

Note:

$$\int \sec^3\theta d\theta =$$

$$= \int \sec\theta\sec^2\theta d\theta$$

$$= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta d\theta \text{ (int } by\ \ parts)$$

$$= \sec\theta\tan\theta - \int (\sec^2 x - 1)\sec\theta d\theta$$

$$= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta d\theta \text{ (int } by\ \ parts)$$

$$= \sec\theta\tan\theta - \int \sec^3 x d\theta + \int \sec x d\theta$$

Then, solve for $\int \sec^3\theta d\theta = \dfrac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} + C$

Therefore,

$$\int \sqrt{4x^2 - 9}\,dx = \frac{3\left(\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right)}{2} - 3\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{3}{2}\left[\sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|\right] + C$$

$$= \frac{3}{2}\left[\frac{4x}{3}\frac{\sqrt{4x^2 - 9}}{3} - \ln\left|\frac{4x}{3} + \frac{\sqrt{4x^2 - 9}}{3}\right|\right] + C$$