PACKET 7

(9.4) Models of Population Growth,
(10.1) Plane Curves and Parametric Equations,
(10.2) Parametric Equations and Calculus

(9.4) Models of Population Growth

1. Suppose that a population develops according to the logistic differential equation (limited growth model.)

$$\frac{dP}{dt} = 2P - 0.001P^2$$,where t represents years.

a) What is the carrying capacity? (You can get this directly from the DE, but it will also be the positive equilibrium solution.)
b) For what values of P is the population increasing? Give your answer in interval notation.
c) At what value of P is the population increasing the *fastest*, i.e., where the slope is the greatest? Note: this occurs at inflection points!
d) For what values of P will P remain the same, i.e., what are the equilibrium solutions?
e) Draw a coordinate system with P vs. t, and sketch the curves in part (d).
f) On this same coordinate system, make a rough sketch of curves for each of the following initial populations (initial conditions)

   $P(0) = 100$

   $P(0) = 3000$

   In the long term, what level does the population seem to approach?
g) Recall that the logistic population above includes a term due to birth rate, $2P$, and a term due to competition, $-0.001P^2$. Suppose that, in addition to the terms on the right side of the DE above, there is a term due to immigration. Specifically, suppose the population increases by an additional 5250 per year, due to immigration. Write the new DE, which includes this term.
h) Find the new nonnegative equilibrium solution for your equation in part (g).
i) Make a rough sketch of the solution curves for each of the initial populations:

   $P(0) = 0$

   $P(0) = 7000$

   In the long term, what level does the population seem to approach?

j) If $P(10) = 100$, find $P'(10)$. Hint: you don't need to solve for $P(t)$!

k) Extra Problem: If time, solve your equation from part (g) with $P(0)$=7000 (and P > 3500), and compute

$\lim_{t\to\infty} P(t)$.

l) The solution to the general logistic equation $\dfrac{dP}{dt} = kP\left(1 - \dfrac{1}{M}P\right)$ with initial condition $P(0) = P_0$ is

$$P(t) = \frac{M}{1 + Ae^{-kt}} \text{ where } A = \frac{M - P_0}{P_0}.$$

If $\dfrac{dP}{dt} = 2P - 0.001P^2$ and $P(0) = 100$, after how many years will the population reach 80% of its carrying capacity? Round your answer to two decimal digits.

Solution:

a) Rewriting the DE in standard form,

$$\frac{dP}{dt} = 2P(1 - \frac{0.001}{2}P)$$

$$\frac{dP}{dt} = 2P(1 - \frac{1}{2000}P)$$

so, the carrying capacity must be $K = 2000$.

b) We know from the logistic equation, that the population P will increase to the carrying capacity when $P < K$ and will decrease to the carrying capacity when $P > K$. Therefore, when $0 < P < 2000$, P will increase.

c)

This occurs when $\dfrac{dP}{dt}$

is maximum, or, equivalently, when the right side

$f(P) = 2P - 0.001P^2$

is maximum, so locating the critical point to maximize the function,

$$f'(P) = 2 - 0.002P = 0$$

$$P = 1000$$

Therefore, when P = 1000,

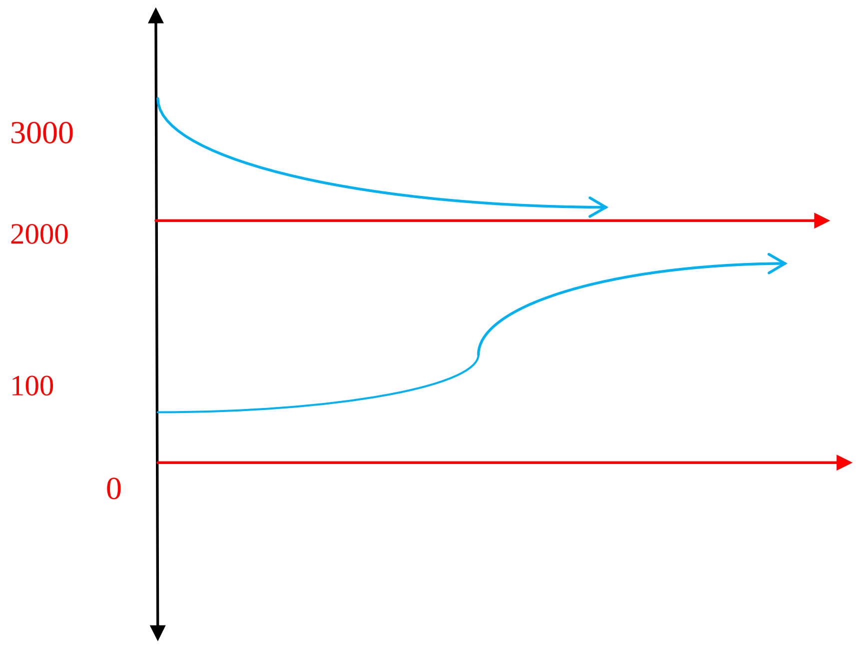
$\dfrac{dP}{dt}$

will be maximum.

(Of course, we know from the logistics equation that, in general, P will increase the fastest when it is half the carrying capacity: 2000/2=1000.

d) $P = 0$ and $P = 2000$.

e) and f)



g) $\dfrac{dP}{dt} = 2P - 0.001P^2 + 5250$

h) Let the right hand side equal $f(P)$:

$$f(P) = 2P - 0.001P^2 + 5250$$
$$f(P) = -0.001P^2 + 2P + 5250$$

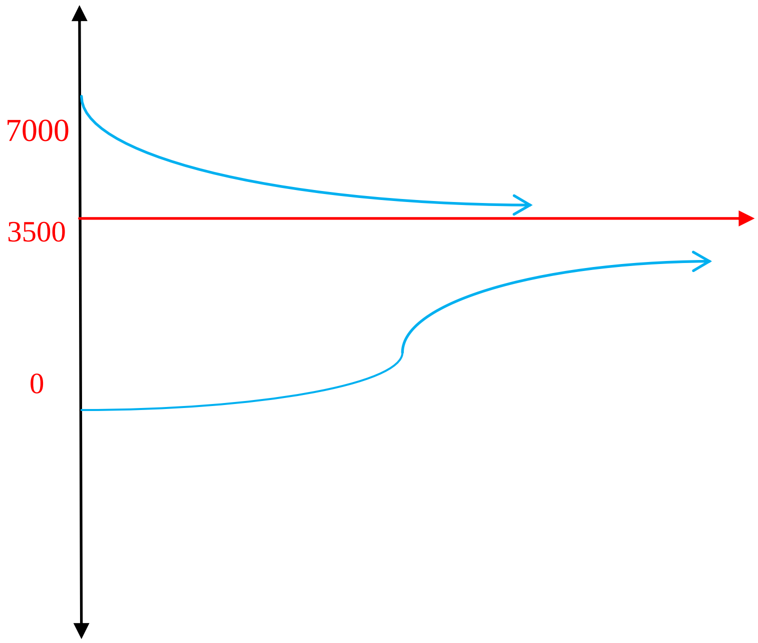
Applying the quadratic formula for the roots, we have

$$\frac{-2 \pm \sqrt{4 - 4(-0.001)(5250)}}{2(-0.001)} = 3500, -1500$$

Therefore, $\dfrac{dP}{dt} = 2P - 0.001P^2 + 5250 = 0$

when $P = 3500$ and $P = -1500$. The only relevant solution is $P = 3500$. The population will now asymptotically approach this solution instead of 2000, as in the logistics model.

i)



j) If $P(10) = 100$, find $P'(10)$. Hint: you don't need to solve for $P(t)$!

Note that $\frac{dP}{dt} = 2P - 0.001P^2$ means that $P'(t) = 2P(t) - 0.001(P(t))^2$. In other words, the derivative of P at any point t (in the domain) is equal to twice P at that same point t less 0.001 times the square of P at that same point. Therefore,

$$P'(10) = 2P(10) - 0.001(P(10))^2 = 2(100) - 0.001(100)^2 = 190.$$

k) (next page)

$$\frac{dP}{dt} = 2P - 0.001P^2 + 5250$$

$$\int \frac{dP}{-0.001P^2 + 2P + 5250} = \int dt$$

$$-\int \frac{dP}{(P+1500)(P-3500)} = \int dt$$

$$\int \left(\frac{\frac{1}{5000}}{P+1500} - \frac{\frac{1}{5000}}{P-3500} \right) dP = \int dt$$

$$\frac{1}{5000}\ln|P+1500| - \frac{1}{5000}\ln|P-3500| = t + C$$

$$\frac{1}{5000}\ln\left|\frac{P+1500}{P-3500}\right| = t + C$$

$$\frac{1}{5000}\ln\left(\frac{P+1500}{P-3500}\right) = t + C, \quad P > 3500$$

$$\ln\left(\frac{P+1500}{P-3500}\right) = 5000t + C$$

$$\frac{P+1500}{P-3500} = e^C e^{5000t}$$

$$\frac{P+1500}{P-3500} = Ke^{5000t}$$

$$P + 1500 = PKe^{5000t} - 3500Ke^{5000t}$$

$$P - PKe^{5000t} = -3500Ke^{5000t} - 1500$$

$$P = \frac{-3500Ke^{5000t} - 1500}{1 - Ke^{5000t}}$$

$$P = \frac{3500Ke^{5000t} - 1500}{Ke^{5000t} - 1}$$

where partial fraction expansion had been performed in the fourth line. Applying the initial condition, we have

$$7000 = \frac{3500K - 1500}{K - 1}$$

$$7000K - 7000 = 3500K - 1500$$

$$3500K = 5500$$

$$K = \frac{11}{7}$$

So that

$$P(t) = \frac{3500\left(\frac{11}{7}\right)e^{5000t} - 1500}{\left(\frac{11}{7}\right)e^{5000t} - 1}$$

Indeed, $\lim_{t\to\infty} P(t) = 3500$, which is what we expected from our knowledge of the equilibrium solutions.

l) The solution to the general logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{1}{M}P\right)$ with initial condition

$P(0) = P_0$ is

$P(t) = \frac{M}{1 + Ae^{-kt}}$ where $A = \frac{M - P_0}{P_0}$.

If $\frac{dP}{dt} = 2P - 0.001P^2$ and $P(0) = 100$, after how many years will the population reach 80% of its carrying capacity? Round your answer to two decimal digits.

We are given that $P_0 = 100$, $M = 2000$, and $k = 2$, so we want to know how long it will take before $P = 0.8M = 0.8(2000) = 1600$.

Also, $A = \frac{2000 - 100}{100} = 19$, so that $P(t) = \frac{M}{1 + Ae^{-kt}}$ means $1600 = \frac{2000}{1 + 19e^{-2t}}$.

We then solve for *t:*

$$(1 + 19e^{-2t})(1600) = 2000$$

$$1600 + 30400e^{-2t} = 2000$$

$$30400e^{-2t} = 400$$

$$e^{-2t} = \frac{400}{30400}$$

$$-2t = \ln\left(\frac{400}{30400}\right)$$

$$t = -\frac{1}{2}\ln\left(\frac{400}{30400}\right) \approx 2.16\ yrs$$

(10.1) Parametric Curves and Parametric Equations

In the predator-prey model of the last section, the solution of the pair of differential equations is a set of two functions:

$$x = f(t)$$
$$y = g(t)$$

This set of two equations, which relates two dependent variables (in this case x and y) to the same independent variable, t in this case, is referred to as a set of **parametric equations**. In general, for the predator-model, one cannot find these functions.

Parametric equations, if known, can often be sketched individually, or in rectangular form, i.e., as x vs. y. Also, parametric equations often have a specified domain.

Before finding parametric curves for parametric equations, let's find graphs for some basic Cartesian equations (rectangular equations).
Sketch the following Cartesian equations:

1) $y = \ln x$
2) $y = e^x$
3) $y = \sqrt{x}$
4) $y = (x+2)^2 - 6$
5) $x = (y+2)^2 - 6$
6) $\dfrac{(x+2)^2}{9} + \dfrac{(y-4)^2}{16} = 1$
7) $\dfrac{(x+2)^2}{16} + \dfrac{(y-4)^2}{16} = 1$

For each set of parametric equations below,

a) Eliminate the parameter to find a Cartesian equation of the curve.
b) Sketch the curve and indicate with arrows the direction in which the curve is traced as the parameter increases.

8) $x = t + 2$
$y = -t^2 + 3$
$-1 \le t \le 2$

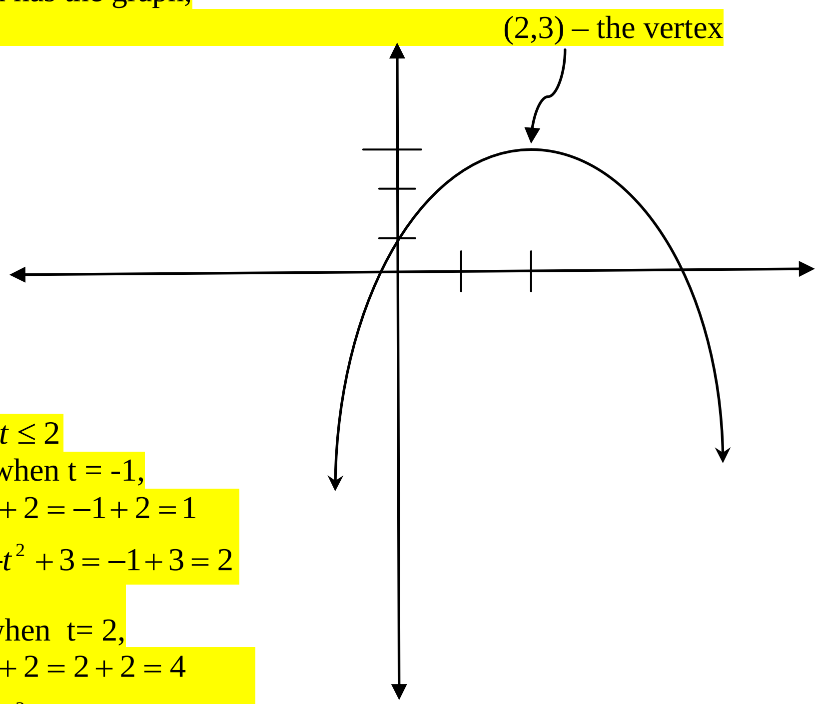
Solution:

$$x = t + 2$$
$$t = x - 2$$

so that

$$y = -t^2 + 3$$
$$y = -(x-2)^2 + 3$$

which has the graph,



Since

$$-1 \le t \le 2$$

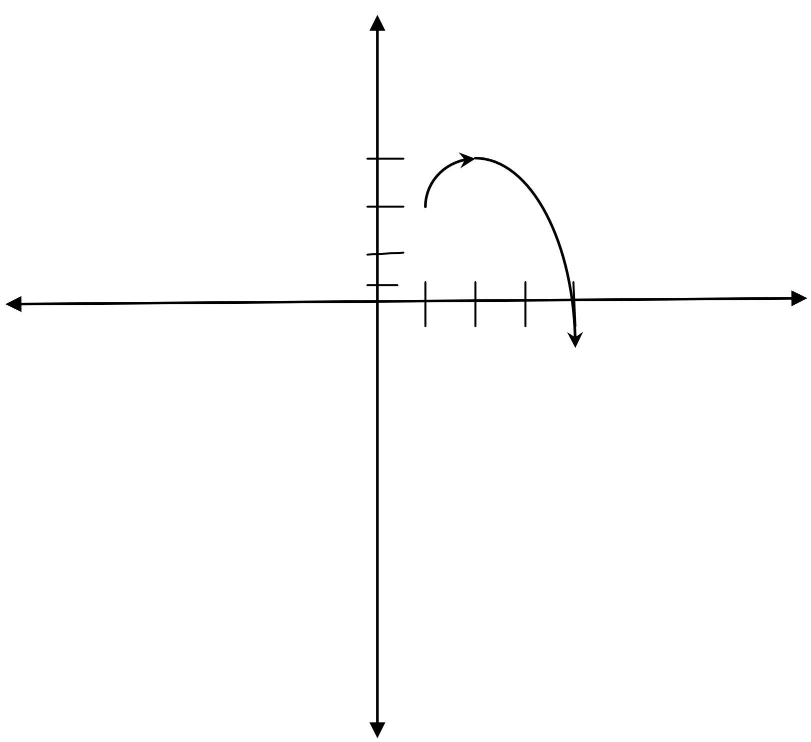
then when t = -1,

$$x = t + 2 = -1 + 2 = 1$$
$$y = -t^2 + 3 = -1 + 3 = 2$$

and when t= 2,

$$x = t + 2 = 2 + 2 = 4$$
$$y = -t^2 + 3 = -4 + 3 = -1$$

Therefore, the curve starts at point (1,2) and ends at (4,-1):



9) $x = 3\sin t - 2$
$y = 2\cos t + 3$
$0 \le t \le \pi$

Solution:

$x = 3\sin t - 2 \qquad\qquad y = 2\cos t + 3$

$x + 2 = 3\sin t \qquad\qquad y - 3 = 2\cos t$

$\dfrac{x+2}{3} = \sin t$ and $\dfrac{y-3}{2} = \cos t$

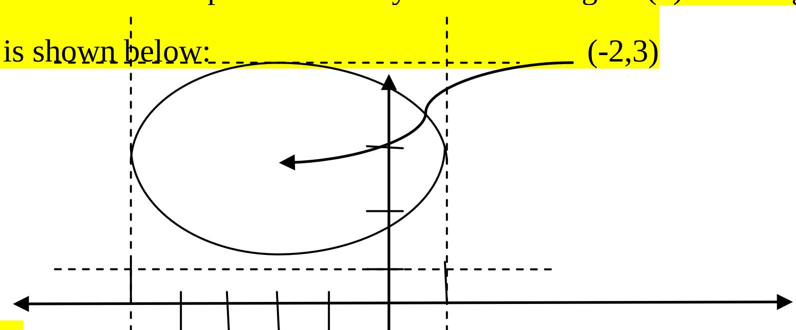
Since
$\sin^2 t + \cos^2 t = 1$

then

$$\left(\frac{x+2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$$

$$\frac{(x+2)^2}{3^2} + \frac{(y-3)^2}{2^2} = 1$$

This is an ellipse with center (-2,3); the major axis runs parallel to the x-axis with length 2(3)=6 and the minor axies runs parallel to the y-axis with length 2(2)=4. The graph for the rectangular equation is shown below:



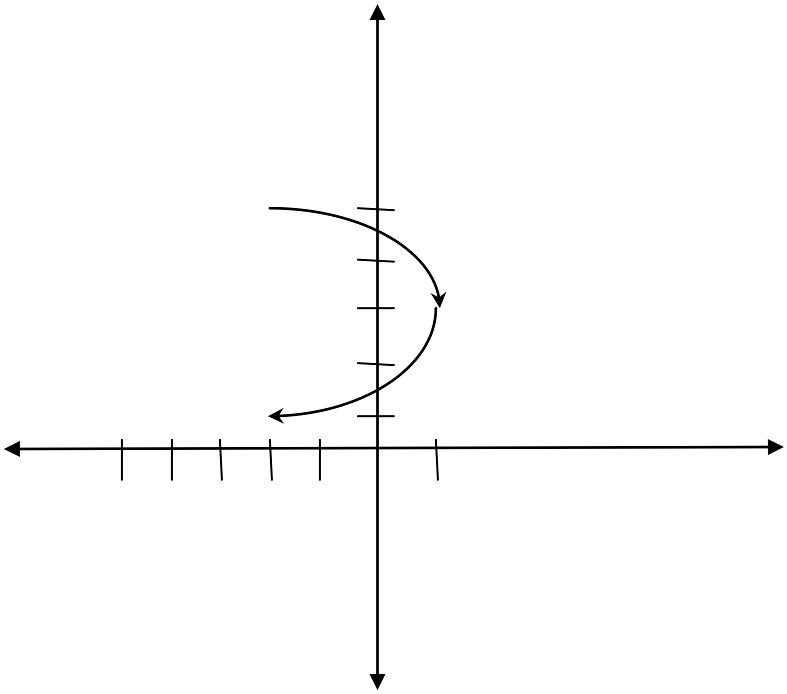
Endpoints:

$t = 0: \quad x = 3\sin 0 - 2 = -2; \quad y = 2\cos 0 + 3 = 5$

$t = \pi: \quad x = 3\sin \pi - 2 = -2; \quad y = 2\cos \pi + 3 = 1$

The direction is downward because, as t increases from 0, cos$t$ decreases from 1, and, therefore, y decreases.
So the starting point is (-2,5) and the "finish" is at (-2,1).

The plane curve is shown below



10)

$$x = 3 - \cos^2(2t)$$
$$y = \sin(2t)$$
$$0 \le t \le \frac{\pi}{4}$$

Solution:

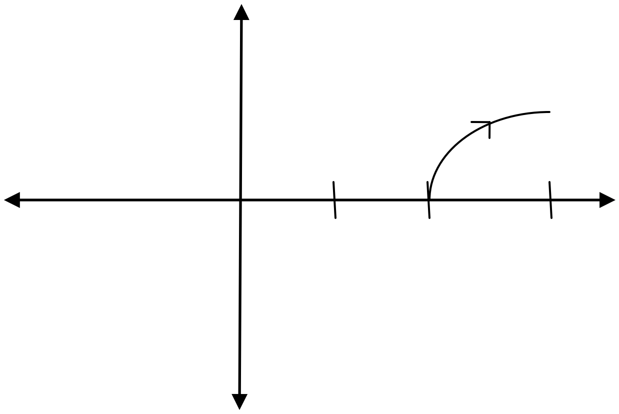
$\sin^2(2t) = y^2$ and

$\cos^2(2t) = 3 - x$ so that

$1 = \sin^2(2t) + \cos^2(2t) = y^2 + 3 - x$ and

$$1 = y^2 + 3 - x$$
$$x = y^2 + 2$$

At $t = 0$, $(x,y) = (2,0)$, and at $t = \frac{\pi}{4}$, $(x,y) = (3,1)$, so that the plane curve (parametric curve) is



11) $\begin{aligned} x &= e^{-2t} - 3 \\ y &= 4t + 1 \end{aligned}$

Solution:

Solving the first equation for *t*, we have

$x + 3 = e^{-2t}$

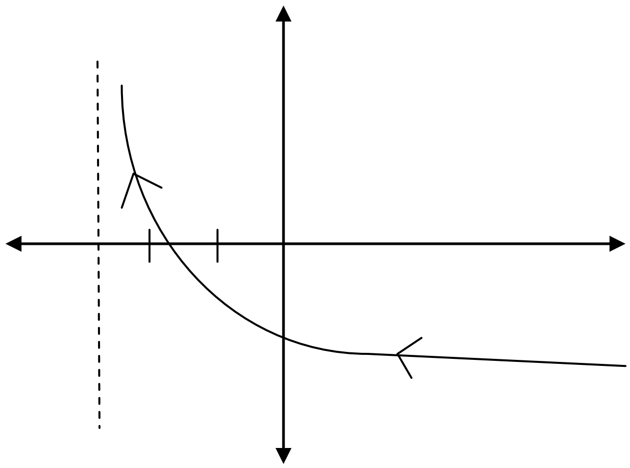
$\ln(x+3) = -2t$

$t = -\frac{1}{2}\ln(x+3)$

Substituting this result into the second equation in the problem, we have

$y = -2\ln(x+3) + 1$

Note that, as *t* increases, *x* decreases, and vice versa. A sketch of the parametric curve is given below.



12) $\begin{aligned} x &= e^{t} + e^{-t} \\ y &= e^{t} - e^{-t} \end{aligned}$

(Don't sketch this one, but do you know the name of this graph?)

Solution:

Squaring both sides of each equation, we have

$x^2 = e^{2t} + 2 + e^{-2t}$ and $y^2 = e^{2t} - 2 + e^{-2t}$

If we subtract the second equation form the first, we have

$x^2 - y^2 = e^{2t} + 2 + e^{-2t} - \left(e^{2t} - 2 + e^{-2t}\right) = 4$, so that

$x^2 - y^2 = 4$ The graph is a hyperbola!

Alternately, adding the two original equations, we have

$x + y = 2e^{t}$

Subtracting the equation, we also have

$x - y = 2e^{-t}$

From the first equation, we note that $\dfrac{x+y}{2} = e^{t}$

Using the second equation, we note that

$\dfrac{x-y}{2} = e^{-t}$ and $\dfrac{2}{x-y} = e^{t}$

Therefore, $\dfrac{2}{x-y} = \dfrac{x+y}{2}$, and finally $4 = x^2 - y^2$.

13) a) Suppose an ant walks in a straight line, starting at point (1,-2) at time t = 0, and finishing at point (-3,4) at t = 10 (assume t is in seconds and that the ant walks with constant speed). Write a set of parametric equations that represents this path (parametric curve). Hint: you may assume that the parametric equations take the form

$$x = at + b$$
$$y = ct + d$$

where a,b,c, and d are constants.. (In Math 2415, you learn why the equations must take this form.)

b)Now, suppose that the same ant turns around at time t = 10 and walks back to the original point in half the time it took to walk the same path in the other direction. Find parametric equations that represent this new path.

Solution:

a) At point (1,-2), t must be 0, so substituting these into the equations, we have:

$$1 = 0 + b$$
$$-2 = 0 + d$$

So that $b = 1$, and $d = -2$.

At point (-3,4), t must be 10, so substituting these into the equations, we have:

$$-3 = 10a + 1$$
$$4 = 10c - 2$$

so that $a = -\dfrac{2}{5}$, and $c = \dfrac{3}{5}$. The final answer is

$$x = -\frac{2}{5}t + 1$$
$$y = \frac{3}{5}t - 2$$

b) Now, suppose that the same ant turns around at time $t = 10$ and walks back to the original point in half the time it took to walk the same path in the other direction. Find parametric equations that represent this new path.

Solution:

At $t = 10$, $x = -3$ and $y = 4$, so that

$$-3 = 10a + b$$
$$4 = 10c + d$$

At $t = 15$, $x = 1$ and $y = -2$, so that

$$1 = 15a + b$$
$$-2 = 15c + d$$

We now have four equations with four unknowns. Looking at the two equations

$$-3 = 10a + b$$
$$1 = 15a + b$$

if we subtract the second from the first, we have $-4 = -5a$, so that $a = \frac{4}{5}$ and, substituting this value into the last equation above, we have $1 = 15\frac{4}{5} + b$, so that $b = -11$. Similarly, subtracting the second equation from the first in the set

$$4 = 10c + d$$
$$-2 = 15c + d$$

we have $6 = -5c$, so that $c = -\frac{6}{5}$, and $-2 = 15\left(-\frac{6}{5}\right) + d$, so that $d = 16$. The final answer is

$$x = \frac{4}{5}t - 11$$
$$y = -\frac{6}{5}t + 16$$

14) Find a set of parametric equations that traces the graph

$$x^2 - 4x + y^2 + 6y = 3$$

a) counterclockwise on $0 \le t \le 2\pi$, starting at point (2,1) when $t = 0$.
b) counterclockwise on $0 \le t \le 2\pi$, starting at point (6,-3) when $t = 0$.

c) Now find another set of parametric equations for each of your answers to parts (a) and (b).

Hint: it will help to use the sine and cosine functions in your answer.

Solution:

a) Completing the square, we notice that the graph is that of a circle:

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 3 + 4 + 9$$
$$(x-2)^2 + (y+3)^2 = 16$$

This is a circle with radius 4, centered at (2,-3).

We note that movement around a circle involves oscillatory functions, so sine or cosine may be appropriate functions for the parametric representation. We let

$$x - 2 = 4\sin t$$
$$y + 3 = 4\cos t$$

so that

$$x = 4\sin t + 2$$
$$y = 4\cos t - 3$$

These equations are consistent with the Cartesian equation because, if we substitute these functions into the left side of the equation, we have

$$(4\sin t)^2 + (4\cos t)^2 = 16\sin^2 t + 16\cos^2 t = 16(1) = 16$$

We now must check to see if we are at point (2,1) when $t = 0$:

$$x = 4\sin 0 + 2 = 2$$
$$y = 4\cos 0 - 3 = 1$$

Next, we must check to see if these parametric equations represent a counterclockwise rotation. Starting at $t = 0$, we are at the top point on the circle. If we note that

$$\left.\frac{dx}{dt}\right|_{t=0} = 4\cos t\Big|_{t=0} = 4 > 0$$

then x is increasing, which indicates clockwise motion.

Therefore, to reverse the direction, we simply replace t with $-t$ in the equations:

$$x - 2 = 4\sin(-t) = -4\sin t$$
$$y + 3 = 4\cos(-t) = 4\cos t$$

So that

$$x = -4\sin t + 2$$
$$y = 4\cos t - 3$$

b) Noting that we could reverse the roles of sine and cosine, and still satisfy the circle equation, let

$$x = 4\cos t + 2$$
$$y = 4\sin t - 3$$

Checking the point at $t = 0$, we have

$$x = 4\cos 0 + 2 = 6$$
$$y = 4\sin 0 - 3 = -3$$

which is the desired point. This is the right-most point on the circle.

We note that

$\left.\frac{dy}{dt}\right|_{t=0} = 4\cos t\Big|_{t=0} = 4 > 0$, which indicates a counterclockwise direction, so the final

answer is indeed

$$x = 4\cos t + 2$$
$$y = 4\sin t - 3$$

c) Note that the answers to (a) and (b) are not unique. There are an infinite number of answers. For example, we could also use

$$x = -4\sin(2t) + 2$$
$$y = 4\cos(2t) - 3$$

for part (a) and

$$x = 4\cos(2t) + 2$$
$$y = 4\sin(2t) - 3$$

for part (b).

These equations will cover the same circle in the same directions – just faster.

(10.2) Parametric Equations and Calculus

Before we find tangent lines for parametric curves, let's find tangent lines for graphs of Cartesian (rectangular) equations.

1) Find the equation for the line tangent to the curve at the given value of *x*:
$y = (x+2)^2 - 6; \quad x = 3$

2) Find the concavity of $y = (x+2)^2 - 6$ at $x = 3$.

Now, let's find a tangent line for a parametric curve.

3) Find an equation of the tangent line to the curve at the point corresponding to the given value of the parameter.

$$x = t - t^{-1} \quad y = 1 + t^2 \qquad t = 1$$

Solution:

$$\frac{dx}{dt} = 1 + t^{-2} \quad \text{and} \quad \frac{dy}{dt} = 2t$$

so

$$\frac{dy}{dx} = \frac{2t}{1+t^{-2}} \text{ and } \left.\frac{dy}{dx}\right|_{t=1} = \frac{2(1)}{1+1} = 1$$

Therefore, for the point-slope equation,

$y - y_1 = m(x - x_1)$,

$$x_1 = 1 - 1^{-1} = 0 \quad (x @ t = 1)$$
$$y_1 = 1 + 1^2 = 2 \quad (y @ t = 1)$$
$$m = 1$$

so that

$y - 2 = 1(x - 0)$
or

$y = x + 2$

4) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward and concave downward? Give your answers in interval notation.

$$x = t + \ln t \qquad y = t - \ln t$$

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-\frac{1}{t}}{1+\frac{1}{t}} = \frac{1-\frac{1}{t}}{1+\frac{1}{t}} \cdot \frac{t}{t} = \frac{t-1}{t+1}$$

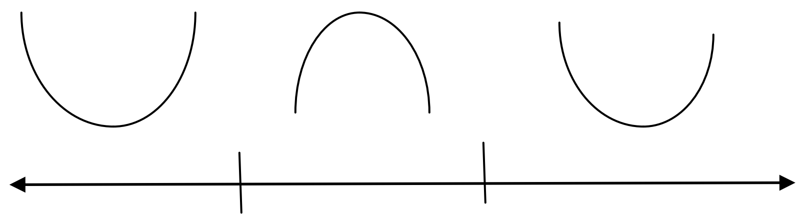
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{t-1}{t+1}\right]}{1+\frac{1}{t}} = \frac{\frac{1(t+1)-(t-1)(1)}{(t+1)^2}}{1+\frac{1}{t}} = \frac{\frac{t+1-t+1}{(t+1)^2}}{1+\frac{1}{t}} = \frac{\frac{2}{(t+1)^2}}{\frac{t+1}{t}} = \frac{2t}{(t+1)^3}$$

Therefore, the second derivative

$$\frac{d^2y}{dx^2} = \frac{2t}{(t+1)^3},$$

is zero or does not exist at t = 0 and t = -1. These two values divide the real line into 3 intervals, so we can evaluate the second derivative at test points, e.g. t = -2, -1/2 and 1, and, performing a sign analysis, we find

$$\left.\frac{d^2y}{dx^2}\right|_{t=-2} = \frac{2(-2)}{(-2+1)^3} > 0 \qquad \left.\frac{d^2y}{dx^2}\right|_{t=-1/2} < 0 \qquad \left.\frac{d^2y}{dx^2}\right|_{t=1} > 0$$



-1 0
-2 -1/2 1

Therefore, the curve is concave up on $(-\infty,-1)\cup(0,\infty)$ and down on $(-1,0)$.

5) Consider the parametric curve.

$$x = 2t^3 + 3t^2 - 12t \qquad y = t^3 + 3t^2 + 1$$

a) Find the points on the curve where the tangent is horizontal or vertical.

Solution:

$$\frac{dx}{dt} = 6t^2 + 6t - 12$$

so, we find the roots:

$$\begin{aligned}
&6t^2 + 6t - 12 = 0\\
&6(t^2 + t - 2) = 0\\
&6(t+2)(t-1) = 0\\
&t = -2, t = 1
\end{aligned}$$

and $\frac{dy}{dt} = 3t^2 + 6t$

so

$3t^2 + 6t = 0$

$3t(t+2) = 0$

$t = 0, t = -2$

From this information alone, since $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$ when $t = 1$, then the point of vertical tangency is

$$x = 2(1) + 3(1) - 12(1) = -7$$
$$y = 1 + 3(1) + 1 = 5$$
$$(x, y) = (-7, 5)$$

Also, from this information alone, since $\frac{dy}{dt} = 0$ but $\frac{dx}{dt} \neq 0$ when $t = 0$, then the point of horizontal tangency is

$$(x, y) = (0, 1)$$

b) There is one point where the tangent slope is indeterminate. Find the value of the slope at this point.

Solution:

Note that $\frac{dy}{dt} = \frac{dx}{dt} = 0$ when t = -2.

We then compute the limit at this point:

$$\lim_{t \to -2} \frac{dy}{dx} = \lim_{t \to -2} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \lim_{t \to -2} \frac{3t(t+2)}{6(t+2)(t-1)} = \lim_{t \to -2} \frac{3t}{6(t-1)} = \frac{1}{3}$$

6) The parametric curve represented by the equations

$x = t - t^{-1} \qquad y = 1 + t^2$

will cross the same point (0,2) at two different values of the parameter t.

a) Find these two values of t.

b) At each of these values of t, what is the slope of the line tangent to the parametric curve?

Solution:

$0 = t - \frac{1}{t}$

$0 = t^2 - 1$

$t^2 = 1$

$t = \pm 1$

To confirm, if $t = \pm 1$, then $y = 1 + 1 = 2$.

Therefore, $\frac{dy}{dx} = \frac{2t}{1 + t^{-2}}$ and $\left.\frac{dy}{dx}\right|_{t=1} = \frac{2(1)}{1+1} = 1$

and

$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{2(-1)}{1+1} = -1$

We can also find the arc length for parametric curves, as well as surfaces of revolution.

7) Suppose that the curve $y = \ln x; \quad 1 \le x \le 2$ can also be represented by the parametric equations $x = t^2; \quad y = 2\ln t$. Set up, but do not evaluate, all three integrals that can be can be used to compute the arc length. Fill in the blanks in the brackets.

a) $\int_{[\ ]}^{[\ ]} [\qquad\qquad\qquad] dx$

b) $\int_{[\ ]}^{[\ ]} [\qquad\qquad\qquad] dy$

c) $\int_{[\ ]}^{[\ ]} [\qquad\qquad\qquad] dt$

Solution:

a) $\int_1^2 \sqrt{1 + \left(\frac{d}{dx}\ln x\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\, dx$

b) If $x = 1$, then $y = \ln 1 = 0$, and if $x = 2$, $y = \ln 2$. Also, $x = e^y$, so that

$$\int_0^{\ln 2} \sqrt{1+\left(\frac{d}{dy}e^y\right)^2}\, dy = \int_0^{\ln 2} \sqrt{1+e^{2y}}\, dy$$

c) Substituting 1 and 2 for x in $x = t^2$ to find t leads to positive and negative values of t. To determine the unique values, we must substitute the corresponding values of y into $y = 2\ln t$ to find the t values: if $y = 0$, then $0 = 2\ln t$, and $t = 1$. If $y = \ln 2$, then

$$\ln 2 = 2\ln t$$
$$\frac{1}{2}\ln 2 = \ln t$$
$$\ln \sqrt{2} = \ln t$$
$$\sqrt{2} = t$$

Therefore,

$$\int_1^{\sqrt{2}} \sqrt{\left(\frac{d}{dt}t^2\right)^2 + \left(\frac{d}{dt}2\ln t\right)^2}\, dt = \int_1^{\sqrt{2}} \sqrt{4t^2 + \frac{4}{t^2}}\, dt$$

8) Find the exact length of the curve.

$$x = (t+3)^2 \qquad y = \frac{2}{3}(t+3)^3 \quad -3 \le t \le 1$$

Solution:

Since

$$\frac{dx}{dt} = 2(t+3)$$
$$\frac{dy}{dt} = 2(t+3)^2$$

$$
\begin{aligned}
&\int_{-3}^{1}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \\
&=\int_{-3}^{1}\sqrt{\left(2(t+3)\right)^2+\left(2(t+3)^2\right)^2}\,dt \\
&=\int_{-3}^{1}\sqrt{4(t+3)^2+4(t+3)^4}\,dt \\
&=\int_{-3}^{1}2(t+3)\sqrt{1+(t+3)^2}\,dt \\
&=\frac{2}{3}(1+(t+3)^2)^{3/2}\Big|_{-3}^{1} \\
&=\frac{2}{3}\left(17^{3/2}-1\right)
\end{aligned}
$$

where u-substitution was applied in computing antiderivative.

9) Find the exact length of the curve.

$$x=e^{2t}\sin t \qquad y=e^{2t}\cos t \qquad 0\le t\le 1$$

Solution:

$$\int_0^1 \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{\left(2e^{2t}\sin t + e^{2t}\cos t\right)^2 + \left(2e^{2t}\cos t - e^{2t}\sin t\right)^2}\, dt$$

$$= \int_0^1 \sqrt{4e^{4t}\sin^2 t + 4e^{4t}\sin t\cos t + e^{4t}\cos^2 t + 4e^{4t}\cos^2 t - 4e^{4t}\sin t\cos t + e^{4t}\sin^2 t}\, dt$$

$$= \int_0^1 \sqrt{4e^{4t}\sin^2 t + e^{4t}\cos^2 t + 4e^{4t}\cos^2 t + e^{4t}\sin^2 t}\, dt$$

$$= \int_0^1 \sqrt{5e^{4t}\sin^2 t + 5e^{4t}\cos^2 t}\, dt$$

$$= \int_0^1 \sqrt{5e^{4t}\left(\sin^2 t + \cos^2 t\right)}\, dt$$

$$= \int_0^1 \sqrt{5e^{4t}}\, dt$$

$$= \sqrt{5}\int_0^1 e^{2t}\, dt$$

$$= \frac{\sqrt{5}}{2}\left(e^2 - 1\right)$$

Recall that one form of the integral that represents the surface area of revolution of the graph of $y = f(x)$ on $a \le x \le b$ is given by $\int_a^b 2\pi r\, ds = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$. Again, this is one version of the integral. You can use a modified version of this integral to solve the next problem.

10) a) Find the surface area generated by rotating the given curve about the *x*-axis.

$x = 3t - t^3 \qquad y = 3t^2 \qquad$ between points (0,0) and (2,3)

Solution:

First, we need to find the values of *t* corresponding to the points. If $y = 0$, then $0 = 3t^2$ which implies that $t = 0$. If $y = 3$, then $3 = 3t^2$ implies that $t = \pm 1$. To determine which value of *t* is relevant, we substitute each in to $x = 3t - t^3$. For $t = -1$, $x = -3 + 1 = -2$, but, for $t = 1$, $x = 3 - 1 = 2$, which is the desired value.

Therefore, we need $0 \le t \le 1$.

Since
$\frac{dx}{dt} = 3 - 3t^2$ and $\frac{dy}{dt} = 6t$,

then we have

$$\int_0^1 2\pi\, y\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$
$$= \int_0^1 2\pi(3t^2)\sqrt{(6t)^2 + (3 - 3t^2)^2}\, dt$$
$$= 6\pi\int_0^1 t^2\sqrt{36t^2 + 9 - 18t^2 + 9t^4}\, dt$$
$$= 6\pi\int_0^1 t^2\sqrt{9t^4 + 18t^2 + 9}\, dt$$
$$= 6\pi\int_0^1 t^2(3)\sqrt{t^4 + 2t^2 + 1}\, dt$$
$$= 18\pi\int_0^1 t^2\sqrt{(t^2 + 1)^2}\, dt$$
$$= 18\pi\int_0^1 t^2(t^2 + 1)\, dt$$
$$= 18\pi\int_0^1 (t^4 + t^2)\, dt$$
$$= 18\pi\left[\frac{t^5}{5} + \frac{t^3}{3}\right]\Bigg|_0^1$$
$$= 18\pi\left(\frac{1}{5} + \frac{1}{3}\right)$$
$$= \frac{48\pi}{5}$$

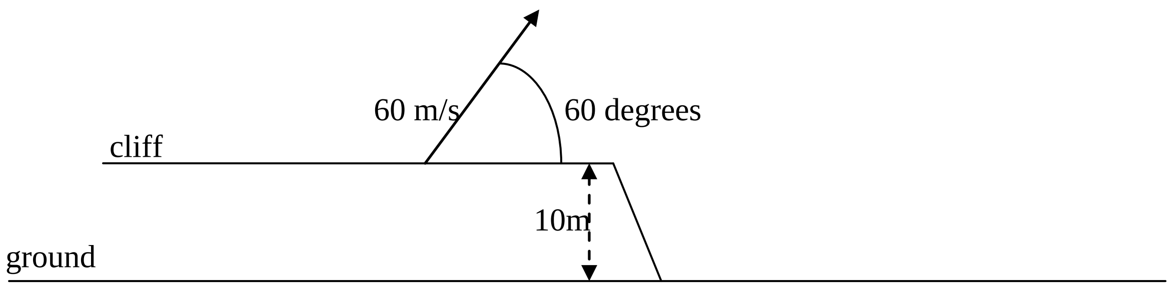
b)Set up, but do not evaluate, an integral that represents the same portion of the parametric curve,
$x = 3t - t^3$ $\quad y = 3t^2$ $\quad$ between points (0,0) and (2,3),
rotated about the y-axis.

Solution:

$$\int_0^1 2\pi\, x\sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2}\,dt$$
$$=\int_0^1 2\pi(3t-t^3)\sqrt{(6t)^2+\left(3-3t^2\right)^2}\,dt$$

Extra Problems

1) Suppose a cannonball is shot from a 10-meter high cliff at a speed of 60 m/s at an angle of 60 degrees from horizontal.



a) Find parametric equations for the trajectory. Let *x* represent the horizontal displacement, let *y* represent the vertical displacement, and let *t* be the time parameter. Assume that there is no wind, so that the horizontal velocity is constant, and that the vertical acceleration is a constant -9.8 m/s, due to gravity. Also, let the initial values of *x* and *y* be 0 and 10, respectively.

b) When does the cannonball hit the ground? Round your answer to two decimal digits.

c) How far has the cannonball traveled when it hits the ground? (This is the entire distance covered – not just the horizontal distance.) Set up the integral, only, but explain how you would evaluate the integral.

d) At what angle (from the horizontal) is the cannonball descending when *t* = 10? Round your answer to two decimal digits.

Solution:

a) We know that $\frac{dx}{dt}=60\cos 60^\circ=30$ for the entire time that the cannonball is in the air. Integrating both sides, we have $x=30t+C_1$

Since $x = 0$ when $t = 0$, then $0=30(0)+C_1$, and, therefore, $C_1=0$, so that $x=30t$.

We also know that $\frac{d^2y}{dt^2}=-9.8$. Integrating both sides of the equation, we have

$\frac{dy}{dt} = -9.8t + C_2$. Since $\frac{dy}{dt} = 60\sin 60^\circ = 30\sqrt{3}$ when t =0, then $30\sqrt{3} = -9.8(0) + C_2$, so that $C_2 = 30\sqrt{3}$, and $\frac{dy}{dt} = -9.8t + 30\sqrt{3}$. Integrating both sides of the last equation leads to $y = -4.9t^2 + 30\sqrt{3}t + C_3$. Since y = 10 when t = 0, we have $10 = -4.9(0) + 30\sqrt{3}(0) + C_3$, so that
$C_3 = 10$, and $y = -4.9t^2 + 30\sqrt{3}t + 10$. In summary, the parametric equations are

$$x = 30t$$
$$y = -4.9t^2 + 30\sqrt{3}t + 10$$

b) The cannonball hits the ground when y = 0, so that $0 = -4.9t^2 + 30\sqrt{3}t + 10$. We apply the quadratic formula to find the roots:

$$t_{1,2} = \frac{-30\sqrt{3} \pm \sqrt{2700 - 4(-4.9)(10)}}{2(-4.9)}$$
$$t_1 = -.19, t_2 = 10.79$$

The answer that make physical sense is $t = 10.79$, in seconds .

c) The distance traveled can be represented by the following integral.

$$\int_0^{10.79} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$
$$= \int_0^{10.79} \sqrt{\left(-9.8t + 30\sqrt{3}\right)^2 + (30)^2}\, dt$$
$$= \int_0^{10.79} \sqrt{96.04t^2 - 294\sqrt{3}t + 2700 + 900}\, dt$$
$$= \int_0^{10.79} \sqrt{96.04t^2 - 294\sqrt{3}t + 3600}\, dt$$

One would complete the square and then apply trigonometric substitution to compute this integral.

d) $\frac{dy}{dx} = \frac{-9.8(10) + 30\sqrt{3}}{30} = \frac{30\sqrt{3} - 98}{30}$

$\arctan\left(\frac{30\sqrt{3} - 98}{30}\right) = -56.91$ degrees from horizontal