Discrete Mathematics for Computing

UTD

Ch 1.4 Predicates and Quantifiers

- Motivation:
  1. $x + 2 = 3$.
  2. Computer y is not functioning properly.

Are these sentences propositions?

Neither T or F, values of variables are not specified

Predicates

- How can Propositions be created?

$$x \quad > \quad 3$$

- Denoted by P(x)
- When x has a value, P(x) is a proposition
- Truth value 'T' – true, 'F' - false

Predicates

- Let P(x) denote the statement x > 3
- What is the value of P(4)?

  P(4), x = 4

  4 > 3 **T**

- What is the value of P(2)?
- P(2), x = 2

  2 > 3 **F**

Predicates

- Let Q(x,y) denote x = y + 3
- What is the value of Q(1,2)?

  set x = 1, y = 2 in Q(x, y)

  1 = 2 + 3 **F**

- What is the value of Q(4,1)?

  set x = 1, y = 2 in Q(x, y)

  4 = 1 + 3 **T**

Predicates

- Let $A(c, n)$ denote the statement "Computer c is connected to network n", where c represents a computer, n represents a network.
- Suppose that computer MATH1, is connected to network CAMPUS2 but not to network CAMPUS1. What are the values of *A*(MATH1, CAMPUS1) and *A*(MATH1, CAMPUS2)?

MATH1 not connected to CAMPUS1

A(MATH1, CAMPUS1) is **F**

MATH1 is connected to CAMPUS2

A(MATH1, CAMPUS2) is **T**

Predicates

- Example: Let Q(x, y) denote the statement "x is the capital of y". What are the truth values?
- Q (Denver, Colorado)

  This is **T**

- Q (Detroit, Michigan)

  This is **F**, since Lansing is the capital

Predicates

- Example: State the value of x after the statement "if P(x) then x := 1" is executed, where P(x) is the statement "x > 1", if the value of x when this statement is reached is
- x = 0

  Condition is **F**, x = 0
- x = 2

  Condition is **T**, x = 1

Quantifiers

- Quantification: Another method to create a proposition from a propositional function
- Predicate “T” over a range of elements
- “all”, “some”, “many”, “none”, “few”
- Predicate Calculus

Quantifiers

- Classification

  Universal Quantification – Predicate is T for every element under consideration

  Existential Quantification – There is one or more element under consideration for which the predicate is T

Quantifiers

- The Universal Quantifier

  - P(x) is true for all values of x in a particular "domain"

  - Denoted by $\forall$

  - $\forall x P(x)$ "for all *x*P(*x*)"

  - CounterExample: An element for which P(x) is false

Quantifiers

- Example: Let Q(x) be the statement "x < 2". What is the truth value of the quantification $\forall x Q(x)$ given the domain consists of all real numbers?

Q(x) is not true for every number

Q(4) is F, x = 4 is a counterexample

$\therefore \forall x Q(x)$ is **F**

Quantifiers

- Example: Let P(x) be the statement "$x^2 < 10$". What is the truth value of the quantification $\forall x P(x)$ given the domain consists of the positive integers not exceeding 4?

Domain, x = 1, 2, 3, 4

P(4) is F, $x^2 = 16 < 10$ is a counterexample

$\therefore \forall x P(x)$ is **F**

Quantifiers

- Example: Let P(x) be the statement "$x^2 \geq x$". What is the truth value of the quantification $\forall x P(x)$ given the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

$$x^2 \geq x \text{ iff } x^2 - x = x(x - 1) \geq 0 \text{ , } x \leq 0 \text{ or } x \geq 1$$

$\Rightarrow \forall x P(x)$ is **F** for $0 < x < 1$ if domain is all real numbers.
$(1/2)^2 \not\geq 1/2$ is a counterexample

$\Rightarrow$ **T** if domain is all integers, no integers x, $0 < x < 1$.

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Quantifiers

- Existential Quantification

"There exists an element x in the domain such that P(x)"

Denoted by $\exists x P(x)$

Existential Quantifier - $\exists$

Quantifiers

- Example: Let P(x) be the statement "$x^2 > 10$". What is the truth value of the quantification

$\exists x P(x)$ given the domain consists of the positive integers not exceeding 4?

Domain, x = 1, 2, 3, 4

P(4), $x^2 = 16 > 10$

$\therefore \quad \exists x P(x)$ is T

Quantifiers

Statement	When True?	When False?
$\forall x P(x)$		
$\exists x P(x)$		

Quantifiers

Statement	When True?	When False?
$\forall x P(x)$	**P(*x*) is true for every *x***	
$\exists x P(x)$		

Quantifiers

Statement	When True?	When False?
$\forall x P(x)$	P(x) is true for every x	**There is an *x* for which P(*x*) if false.**
$\exists x P(x)$		

Quantifiers

Statement	When True?	When False?
$\forall x P(x)$	P(*x*) is true for every *x*	There is an *x* for which P(*x*) if false.
$\exists x P(x)$	**There is an *x* for which P(*x*) is true.**	

Quantifiers

Statement	When True?	When False?
$\forall x P(x)$	P(*x*) is true for every *x*	There is an *x* for which P(*x*) if false.
$\exists x P(x)$	There is an *x* for which P(*x*) is true.	**P(*x*) is false for every *x***

Quantifiers with Restricted Domains

- Restricted Domains

  - Abbreviated Notation

  - Included after the quantifier

  Example: $\forall x < 0(x^2 > 0)$

  "For every real number x with $x < 0$, $x^2 > 0$".

  "The square of a negative real number is positive."

Quantifiers with Restricted Domains

- Example: $\forall y \neq 0 (y^3 \neq 0)$

  "For every real number y with $y \neq 0$, we have $y^3 \neq 0$".

  "The cube of every nonzero real number is nonzero."

- Example: $\exists z > 0 (z^2 = 2)$

  "There exists a real number z with $z > 0$, such that $z^2 = 2$".

  "There is a positive square root of 2."

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Quantifiers

- Universal Quantifier, Existential Quantifier

  Higher precedence than all other logical operators
- Bound variable - if quantifier used
- Free variable otherwise

  $\forall y(y+z=2)$ variable y bound by $\forall$

  z is free

Quantifiers

- Example: Determine the truth value of each of these statements if the domain for each variable consists of all real numbers.
- $\exists x(x^3 = -1)$

$$x = -1, (-1)^3 = -1$$

This is **T**

Quantifiers

- Determine the truth value of each of these statements if the domain for each variable consists of all real numbers.
- $\forall x((-x)^2 = x^2)$

Since $(-x)^2 = ((-1)^2 x^2) = x^2$

This is **T**

Quantifiers

- Determine the truth value of each of these statements if the domain for each variable consists of all real numbers.
- $\forall x(x^2 \neq x)$

F, not true for $x = 1$

or $x = 0$

Quantifiers

- Example: Let N(x) be the statement "x has visited North Dakota," where the domain consists of students in your school. Express each of these quantifications in English.
- $\exists x N(x)$

There exists a student in the school who has visited North Dakota.

Some student in the school has visited North Dakota.

Quantifiers

- Example: Let N(x) be the statement “x has visited North Dakota,” where the domain consists of students in your school. Express the given quantification in English.
- $\neg \exists x N(x)$

There does not exist a student in the school who has visited North Dakota.

No student in the school has visited North Dakota.

Quantifiers

- Example: Let N(x) be the statement “x has visited North Dakota,” where the domain consists of students in your school. Express the below quantification in English.

- $\forall x \neg N(x)$

All students in the school have not visited North Dakota.

Quantifiers

- Example: Translate these statements into English where C(x) is "x is a comedian" and F(x) is "x is funny" and the domain consists of all people.
- $\forall x(C(x) \rightarrow F(x))$

For every x, if x is a comedian, then x is funny.

Every comedian is funny.

Quantifiers

- Example: Translate these statements into English where C(x) is "x is a comedian" and F(x) is "x is funny" and the domain consists of all people.
- $$\exists x(C(x) \wedge F(x))$$

There exists an x in the domain such that x is a comedian and x is funny.

Some comedians are funny.

English Statements

- Example: Express the statement "Every student in your class has a cellular phone." using predicates and quantifiers and logical connectives.
- Introduce variable x: "For every student x in your class, x has a cellular phone."
- Let P(x) be the statement "x has a cellular phone." For domain *x* consisting of students in a class,

$$\forall x P(x)$$

English Statements

- For domain x consisting of all people
- Introduce variable x: "For every person x, if person x is a student in your class, x has a cellular phone."
- Let C(x) be the statement "person x is in your class."

  Let P(x) be "x has a cellular phone."

$$\forall x(C(x) \rightarrow P(x))$$

Logical Equivalences

- Two statements S and T

$$S \equiv T$$

  - Iff same truth value
  - Predicates used
  - Domain used

Logical Equivalences

- Example: $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

1. If $\forall x(P(x) \wedge Q(x))$ is T, then $\forall x P(x) \wedge \forall x Q(x)$ is T
2. If $\forall x P(x) \wedge \forall x Q(x)$ is T, then $\forall x(P(x) \wedge Q(x))$ is T

1. Suppose that $\forall x(P(x) \wedge Q(x))$ is T.

=> If variable 'a' is in the domain, then $P(a) \wedge Q(a)$ is T.

$\therefore$ $P(a)$ is T and $Q(a)$ is T.

$\because$ $P(a)$ and $Q(a)$ are T for every element in the domain, we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both T.

$\therefore$ $\forall x P(x) \wedge \forall x Q(x)$ is T.

Logical Equivalences

2. Suppose that $\forall x P(x) \wedge \forall x Q(x)$ is T.

=> $\forall x P(x)$ is T and $\forall x Q(x)$ is T.

∴ If variable a is in the domain, then $P(a)$ is T and $Q(a)$ is T.

=> For all a, $P(a) \wedge Q(a)$ is T.

=> $\forall x (P(x) \wedge Q(x))$ is T.

From 1. and 2. $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

De Morgan's Laws for Quantifiers

- De Morgan's Laws are

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

- When the domain of a predicate P(*x*) consists of m elements, where m is a positive integer, then the above rules of negating quantified predicates are exactly the same as De Morgan's Laws in Propositional Logic.

De Morgan's Laws for Quantifiers

Negation	Equivalent st.	When true?	When false?
$\neg \forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x that P(x) is false.	For all x P(x) is true.
$\neg \exists x\, P(x)$	$\forall x\, \neg P(x)$	For all x P(x) is false.	There is an x that P(x) is true.

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English Statements

- Find the negation of the statement

"All people like pizza."

Write propositional function $C(x)$: x likes pizza

Logical expression: $\forall x C(x)$, domain consists of all people

Using De Morgan's law for quantifiers

$$\neg \forall x C(x) \equiv \exists x \neg C(x)$$

Rewrite in English: Some people do not like pizza.

English Statements

- Find the negation of the statement

$$\forall x(x^2 \neq x)$$

Using De Morgan's law for quantifiers

$$\neg\forall x\left(x^2 \neq x\right)$$
$$\equiv \exists x\neg(x^2 \neq x)$$
$$\equiv \exists x(x^2 = x)$$