Discrete Mathematics for Computing

UTD

Ch 9.4 Closures of Relations

- Definition: The *closure* of a relation *R* with respect to property P is the relation obtained by adding the minimum number of ordered pairs to *R* to obtain property P.
- Properties: reflexive, symmetric, and transitive

Relational closures

- Three types we will study
  - Reflexive
    - Easy
  - Symmetric
    - Easy
  - Transitive
    - Hard

Example: Reflexive closure

- $A=\{1, 2, 3\}$
- $R=\{(1,1), (1,2), (2,1), (3,2)\}$

- Is R reflexive? Why?

- What pairs do we need to make it reflexive?
  (2,2), (3,3)

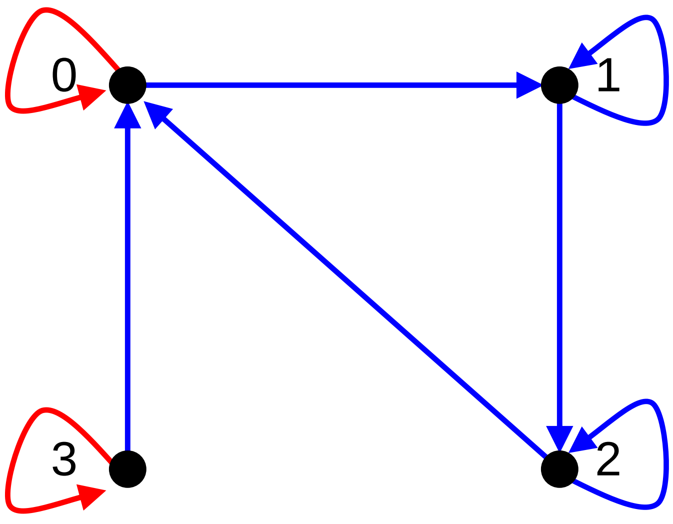
Reflexive closure of R= $\{(1,1), (1,2), (2,1), (3,2)\} \cup \{(2,2), (3,3)\}$ is reflexive.

Reflexive Closure

- In terms of the digraph representation
  - Add loops to all vertices
- In terms of the 0-1 matrix representation
  - Put 1's on the diagonal

Reflexive Closure

- Let R be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the reflexive closure of R?
- We add all pairs of edges (a,a) that do not already exist



We add edges:
(0,0), (3,3)

Example: Symmetric closure

- $A=\{1, 2, 3\}$
- $R= \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)\}$

- Is R symmetric?

- What pairs do we need to make it symmetric?

  (2,1) and (1,3)
- Symmetric closure of $R = \{(1,1), (1,2), (2,2), (2,3), (3,1), (3,2)\} \cup \{(2,1), (1,3)\}$

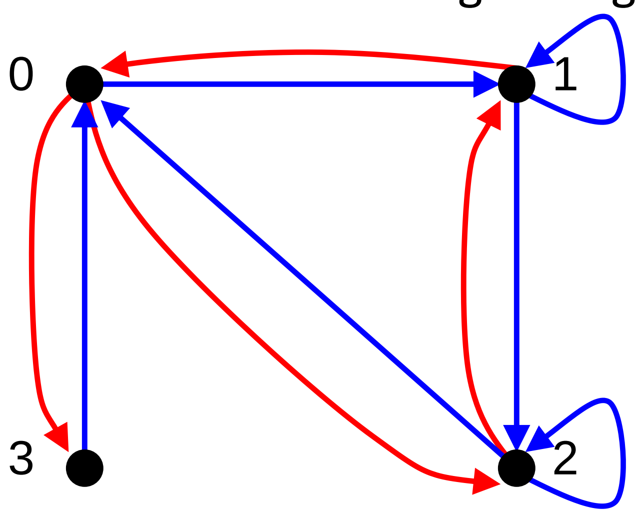
Symmetric Closure

- Can be constructed by taking the union of a relation with its inverse.
- In terms of the digraph representation
  - Add arcs in the opposite direction
- In terms of the 0-1 matrix representation
  - Add 1's to the pairs across the diagonals that differ in value.

$$
\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}
\quad \Rightarrow \quad
\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & 1 & 0 \end{matrix}
$$

Symmetric Closure

- Let *R* be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the symmetric closure of *R*?
- We add all pairs of edges (*a*,*b*) where (*b*,*a*) exists
  - We make all “single” edges into anti-parallel pairs



We add edges:
(0,2), (0,3)
(1,0), (2,1)

Example: Transitive closure

- $A = \{1, 2, 3, 4\}$
- $R = \{(1,3), (1,4), (2,1), (3,2)\}$

- Is R transitive?

  What pairs do we need to make it transitive?

  (1,2), (2, 3), (2, 4), and (3, 1)
- Is R now transitive?

  Adding the pairs does not produce a transitive relation – contains (3, 1) and (1, 4) but does not contain (3, 4)

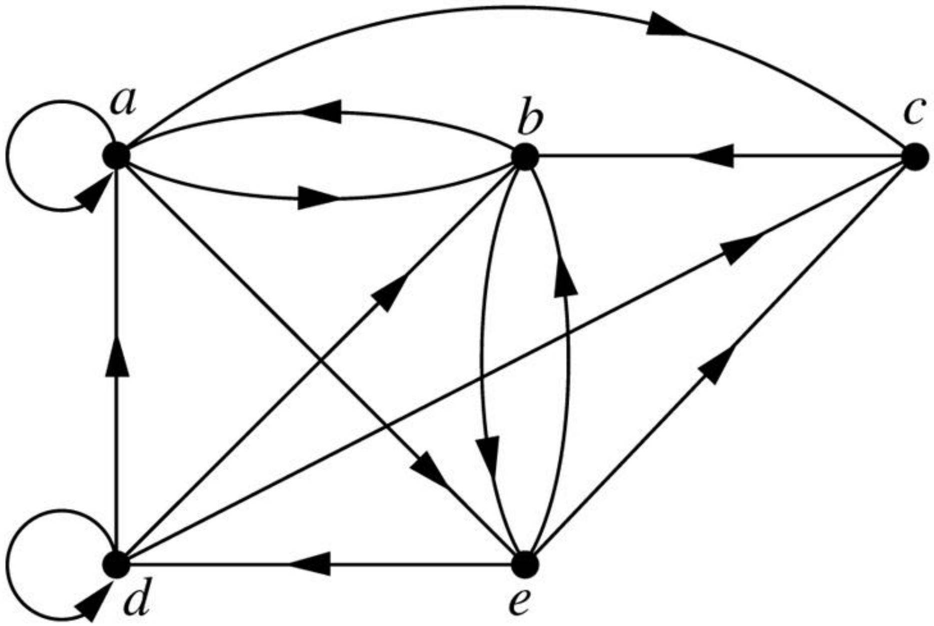
Definition of "path"

- A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer and $x_0 = a$ and $x_n = b$.
- Note that this is just a sequence of edges where the terminal vertex of one edge is the same as the initial vertex in the next edge in the path.

Definition of "path"

- In informal terms, a path from *a* to *b* in the digraph *G*
- sequence of one or more edges starting at *a* and ending at *b*.

Paths in directed graphs



Is there a path from *a* to *d*?

Yes: *a*, *c*, *b*, *e*, *d*

A path in a directed graph is obtained by traversing along edges in the same direction as indicated by the arrowhead on the edge.

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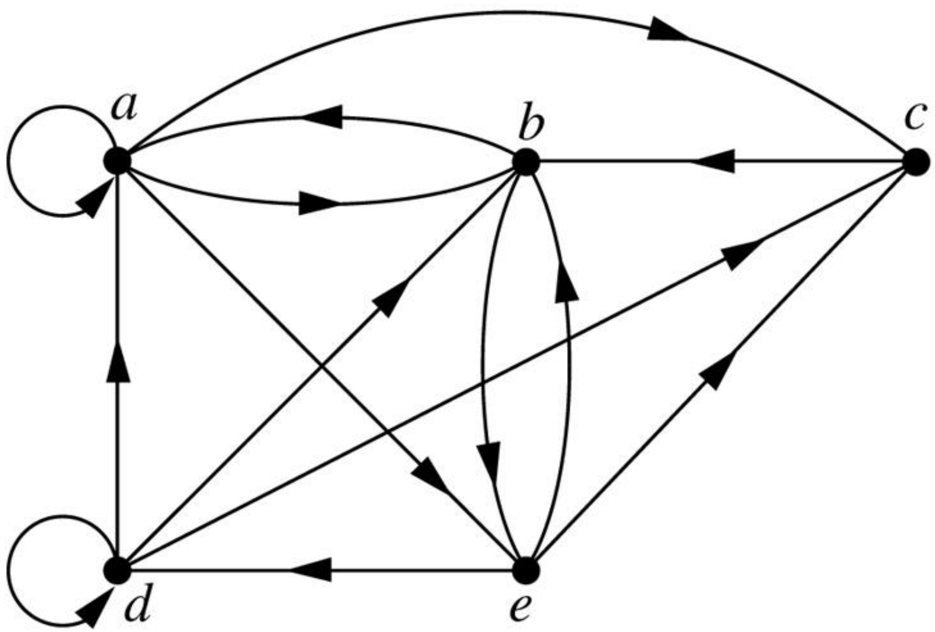
More about paths

- A path in a directed graph is denoted by $x_0, x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ and has length n
- The length of a path is the number of edges in the path.
- The empty set of edges can be thought of as a path from *a* to *a*.
- A path of length ≥ 1 that begins and ends at the same vertex is called a *circuit* or *cycle*.
- A path in a digraph can pass through a given vertex more than once.
- An edge in a digraph can occur more than once in a path.

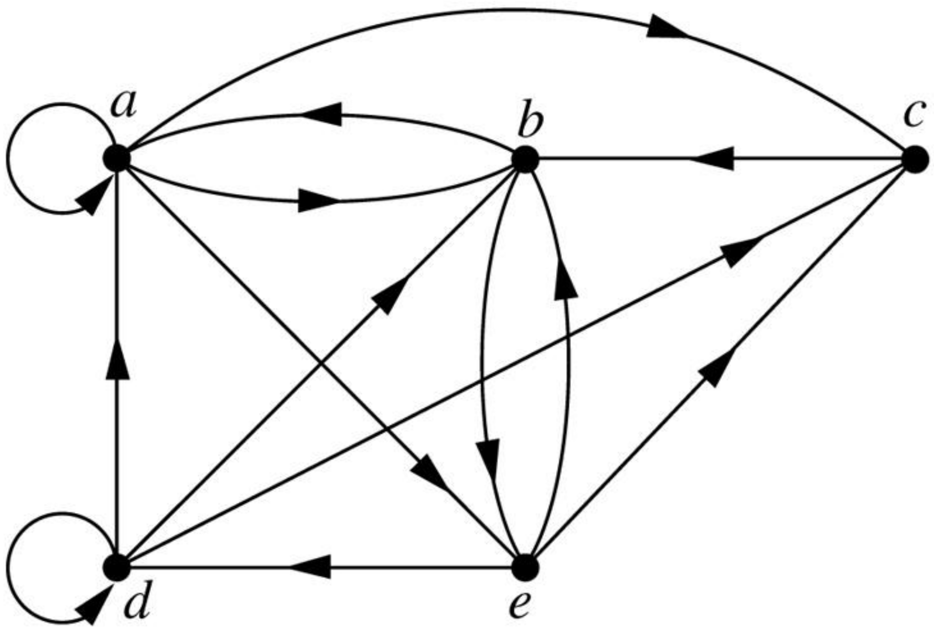
Paths in directed graphs



Possible paths:
1) a, b, e, d
2) a, e, c, d, b
3) b, a, c, b, a, a, b
4) d, c
5) c, b, a
6) e, b, a, b, a, b, e

Which of these paths are in the directed graph?
What are the lengths of these paths?
Which of these paths are circuits?

Paths in directed graphs

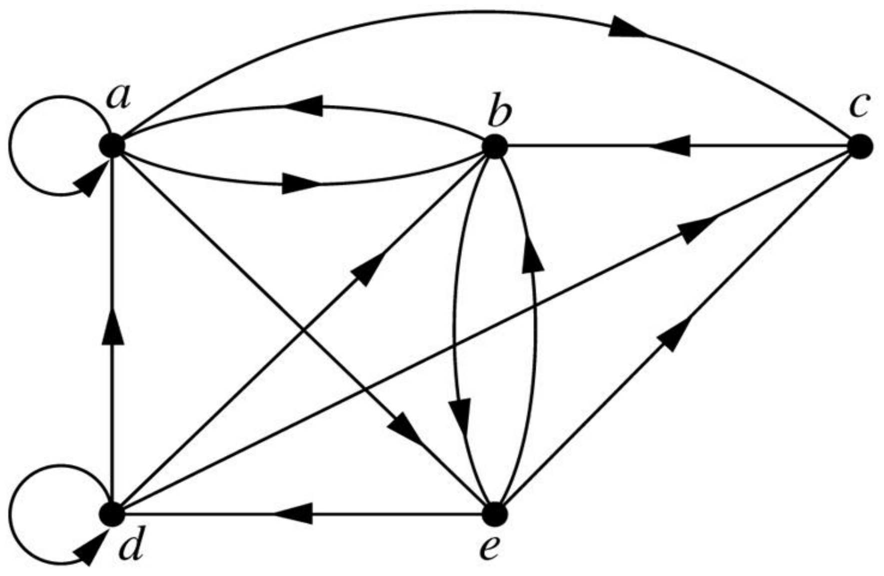


Possible paths:

1) a, b, e, d
2) a, e, c, d, b
3) b, a, c, b, a, a, b
4) d, c
5) c, b, a
6) e, b, a, b, a, b, e

1) is a path, of length 3
2) is not a path, because there is no edge (c, d)
3) is a path, of length 6
4) is a path, of length 1

Paths in directed graphs



Possible paths:

1) a, b, e, d
2) a, e, c, d, b
3) b, a, c, b, a, a, b
4) d, c
5) c, b, a
6) e, b, a, b, a, b, e

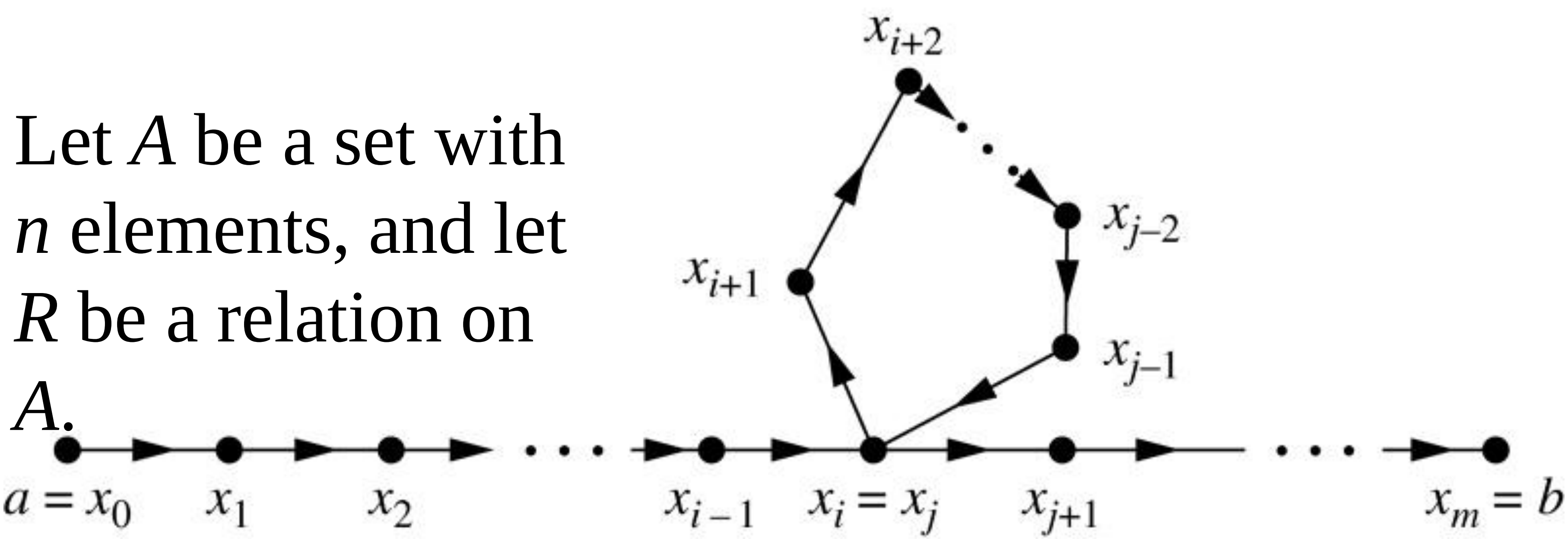
5) is a path, of length 2
6) is a path, of length 6
3) and 6) are circuits because each one begins and ends at the same vertex

Transitive Closure

- In terms of the digraph representation, finding the transitive closure of a relation
- Equivalent to determining which pairs of vertices in the corresponding digraph are connected by a path.

Transitive Closure

Let A be a set with n elements, and let R be a relation on A.



If there is a path of length at least 1 from a to b, then there is such a path with length not exceeding n.

Connectivity Relation

- Let R be a relation on a set.
- Then the *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length ≥ 1 from a to b in R.
- Since R^n consists of the pairs (a,b) such that there is a path of length n from a to b, this means that R^* is the union of all the sets R^n.

$$R^* = \bigcup_{k=1}^{n} R^k$$

Transitive Closure

- The transitive closure of a relation R equals the connectivity relation R^*.

Transitive Closure

- In terms of the matrix representation, to form the transitive closure:
  - Warshall's algorithm finds the transitive closure in $2n^3$ bit operations.

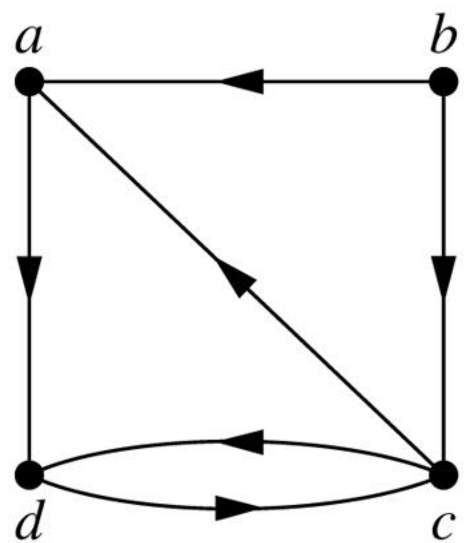
Warshall's Algorithm

```
procedure Warshall (M_R : n × n  0-1 matrix)
W := M_R
for k := 1 to n do
  {   for i := 1 to n do
      {   for j := 1 to n do
              w_ij := w_ij ∨ ( w_ik ∧ w_kj )
      }
  }
```

At termination, $\mathbf{W} := [w_{ij}]$ is M_{R^*}

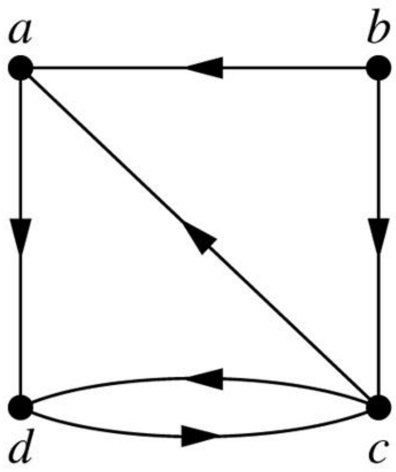
Warshall's Algorithm

- Let R be the relation with the following directed graph:



- The elements of the set are *a*, *b*, *c*, and *d*, which are represented by vertices v_1, v_2, v_3, and v_4, respectively, of the digraph. There are $n = 4$ vertices.

Warshall's Algorithm



$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_0 is the zero-one matrix representation of relation *R*.

Warshall's Algorithm

$$W_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_1 is the zero-one matrix representation of relation R_1. W_1 has 1 as its $(i, j)^{th}$ entry if there is a path from vertex v_i to vertex v_j that has only $v_1 = a$ as an interior vertex.

Warshall's Algorithm

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

W_2 is the zero-one matrix representation of relation R_2. W_2 has 1 as its $(i, j)^{th}$ entry if there is a path from vertex v_i to vertex v_j that has only $v_1 = a$ and/or $v_2 = b$ as an interior vertex.

Warshall's Algorithm

$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

W_3 is the zero-one matrix representation of relation R_3. W_3 has 1 as its $(i, j)^{th}$ entry if there is a path from vertex v_i to vertex v_j that has only $v_1 = a$, $v_2 = b$, and/or $v_3 = c$ as an interior vertex.

Warshall's Algorithm

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

W_4 is the zero-one matrix representation of relation R_4. W_4 has 1 as its $(i, j)^{th}$ entry if there is a path from vertex v_i to vertex v_j that has only $v_1 = a$, $v_2 = b$, $v_3 = c$, and/or $v_3 = d$ as an interior vertex.

Warshall's Algorithm

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

We have examined all paths of length $n = 4$. We know we do not have to examine any paths that are longer than $|v|$. So W_4 is the matrix of the transitive closure.