MATH 2418: Linear Algebra

Assignment 3

Due February 3, 2016 Term Spring, 2016

Recommended Text Book Problems (do not turn in): [Section 1.5: #1, 3, 5, 7, 9, 11, 13, 19, 23, 25, TF]; [Section 1.6: #1, 5, 9, 13, 15, 19, 21, TF].

1. Determine if the given matrix is elementary. If it is elementary, find the corresponding row operation and an elementary matrix that will restore the original matrix to the identity matrix.

a) [2 pt] $B = \begin{bmatrix} 1 & 0 \\ 0 & \pi \end{bmatrix}$

Solution: B is an elementary matrix corresponding to the elementary row operation $(\pi)R2$. The elementary matrix $E = \begin{bmatrix} 1 & 0 \\ 0 & 1/\pi \end{bmatrix}$ is such that $BE = EB = I_2$.

b) [2 pt] $C = \begin{bmatrix} 4 & 2 \\ 0 & 7 \end{bmatrix}$

Solution: C is not an elementary matrix.

c) [2 pt] $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Solution: D is not an elementary matrix.

d) [2 pt] $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Solution: A is an elementary matrix corresponding to the elementary row operation $R3 - R1$. The elementary matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is such that $AE = EA = I_3$.

e) [2 pt] $G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution: G is not an elementary matrix.

2. Let $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 2 & -1 & 4 \end{bmatrix}$. Use row operations to determine the inverse of A if it exists.

a) [4 pt] Write elementary matrices corresponding to the following steps in Gauss-Jordan elimination process.

1) $R2 + (-1)R1$ $\qquad E_1 =$

2) $R3 + (-2)R1$ $\qquad E_2 =$

3) $R1 + (-1)R2$ $\qquad E_3 =$

4) $R3 + (3)R2$ $\qquad E_4 =$

Solution: $E1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $E3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$

b) [4 pt] Apply elementary matrices $E_1, \ldots, E_4$ to the matrix $[A : I_3] = \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{bmatrix}$ to determine if A^{-1} exists.

Solution:

1) $R2 + (-1)R1$ $\qquad E_1[A : I_3] = \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 2 & -1 & 4 & 0 & 0 & 1 \end{bmatrix}$

2) $R3 + (-2)R1$ $\qquad E_2E_1[A : I_3] = \begin{bmatrix} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{bmatrix}$

3) $R1 + (-1)R2$ $\qquad E_3E_2E_1[A : I_3] = \begin{bmatrix} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -3 & -6 & -2 & 0 & 1 \end{bmatrix}$

4) $R3 + (3)R2$ $\qquad E_4E_3E_2E_1[A : I_3] = \begin{bmatrix} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -5 & 3 & 1 \end{bmatrix}$

We see that the "**reduced row echelon form of** A" $= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$, hence A is not invertible.

c) [1 pt] If possible, write A as a product of elementary matrices. If not, explain why.

Solution: In part c) we showed that the "**reduced row echelon form of** A" $\neq I_3$. It follows from theorem 1.5.3 that A cannot be written as a product of elementary matrices.

d) [1 pt] If possible, write A^{-1} as a product of elementary matrices. If not, explain why.

Solution: A^{-1} does not exist, hence A^{-1} cannot be written as a product of elementary matrices.

3. Let $A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. Use row operations to determine the inverse of A if it exists.

a) [4 pt] Write elementary matrices corresponding to the following steps in Gauss-Jordan elimination process.

1) $R3 + (-1)R1$ $\qquad E_1 =$

2) $(-1)R2$ $\qquad E_2 =$

3) $R1 + R2$ $\qquad E_3 =$

4) $R3 + (-2)R2$ $\qquad E_4 =$

5) $R1 + R3$ $\qquad E_5 =$

6) $R2 + (-1)R3$ $\qquad E_6 =$

Solution: $E1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, $E5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

b) [4 pt] Apply elementary matrices $E_1, \ldots, E_6$ to the matrix $[A : I_3] = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ to determine if A^{-1} exists.

Solution:

1) $R3 + (-1)R1$ $\qquad E_1[A : I_3] = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{bmatrix}$

2) $(-1)R2$ $\qquad E_2E_1[A : I_3] = \begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{bmatrix}$

3) $R1 + R2$ $\qquad E_3E_2E_1[A : I_3] = \begin{bmatrix} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{bmatrix}$

4) $R3 + (-2)R2$ $\qquad E_4E_3E_2E_1[A : I_3] = \begin{bmatrix} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{bmatrix}$

5) $R1 + R3$ $\qquad E_5E_4E_3E_2E_1[A : I_3] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{bmatrix}$

6) $R2 + (-1)R3$ $\qquad E_6E_5E_4E_3E_2E_1[A : I_3] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{bmatrix}$

We see that the reduced row echelon form of A is I_3, hence A is invertible.

c) [1 pt] If possible, write A^{-1} as a product of elementary matrices. If not, explain why.

Solution: By part b) and theorem 1.5.3:

$$A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -3 & -1 \\ -1 & 2 & 1 \end{bmatrix} = E_6 E_5 E_4 E_3 E_2 E_1$$

d) [1 pt] If possible, write A as a product of elementary matrices. If not, explain why.

Solution: By parts b), c) and theorem 1.5.3:

$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$

4. (6+4 pts) 1) Solve the following system of linear equations $Qx = b$, where

$$Q = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

by determining the inverse of the coefficient matrix Q and then using matrix multiplication.

2) Is it true that b is a linear combination of the columns of Q? Justify your answer.

Solution: 1) Observe that if $E_{Q1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix corresponding to row operation $R2 + (-2)R1$, and if $E_{Q2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is an elementary matrix corresponding to row operation $R3 + R1$, then

$$E_{Q2}E_{Q1}Q = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 2 & 3 \end{bmatrix} = E_1 A,$$

where A is the matrix from the previous problem, and E_1 is the first elementary matrix from part a) of the same problem. Hence we can use elementary matrices $E_1, \ldots, E_6$ from part a) of the previous problem to see that

$$I_3 = (E_6E_5E_4E_3E_2)(E_1A) = (E_6E_5E_4E_3E_2)(E_{Q2}E_{Q1}Q)$$

It follows that

$$Q^{-1} = E_6E_5E_4E_3E_2E_1E_{Q2}E_{Q1} = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix}$$

Next we solve the system using matrix multiplication

$$x = Q^{-1}b = \begin{bmatrix} 0 & 1 & 1 \\ 5 & -3 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -6 \end{bmatrix}$$

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2) Q can be partitioned as

$$Q = [c_1 : c_2 : c_3].$$

Then we have

$$b = [c_1 : c_2 : c_3]x.$$

By 1) we have

$$\begin{aligned} b =& [c_1 : c_2 : c_3] \begin{bmatrix} 1 \\ 10 \\ -6 \end{bmatrix} \\ =& (1)c_1 + (10)c_2 + (-6)c_3. \end{aligned}$$

That is

$$\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + (10) \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} + (-6) \begin{bmatrix} -2 \\ -5 \\ 5 \end{bmatrix}.$$

5. (10 pts) Verify whether or not the matrix $b = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ is a linear combination of the matrices $A_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$.

Solution: We need to find scalars x_1, x_2 and x_3 such that

$$x_1 A_1 + x_2 A_2 + x_3 A_3 = b$$

which is equivalent to

$$[A_1 : A_2 : A_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b.$$

That is,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix},$$

whose solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -16 & 4 & 3 \\ -2 & 1 & 0 \\ 7 & -2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 36 \\ 4 \\ -15 \end{bmatrix}. \end{aligned}$$

6. (3+3+1 pts) 1) Use matrix partition to explain that if the system of linear equations $A_{m\times n}x = b_{m\times 1}$ is consistent, that is, $A_{m\times n}x = b_{m\times 1}$ has a solution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, where $x_1, x_2, \cdots, x_n$ are scalars, then $b_{m\times 1}$ is a linear combination of the columns of $A_{m\times n}$;

2) Explain that if $b_{m\times 1}$ is a linear combination of the columns of $A_{m\times n} = [c_1 : c_2 : \cdots : c_n]$, i.e.,

$$b_{m\times 1} = x_1c_1 + x_2c_2 + \cdots + x_nc_n$$

where $x_1, x_2, \cdots, x_n$ are scalars, then the system of linear equations $A_{m\times n}x = b_{m\times 1}$ is consistent.

3) Is it true that the system of linear equations $A_{m\times n}x = b_{m\times 1}$ is consistent if and only if $b_{m\times 1}$ is a linear combination of the columns of $A_{m\times n}$?

Solution: 1) Let $A_{m\times n}$ be partitioned into columns. That is, $A_{m\times n} = [c_1 : c_2 : \cdots : c_n]$. If the system of linear equations $A_{m\times n}x = b_{m\times 1}$ has a solution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, where $x_1, x_2, \cdots, x_n$ are scalars, then we have

$$[c_1 : c_2 : \cdots : c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b_{m\times 1},$$

which leads to

$$b_{m\times 1} = x_1c_1 + x_2c_2 + \cdots + x_nc_n.$$

Namely, $b_{m\times 1}$ is a linear combination of the columns of $A_{m\times n}$.

2) if $b_{m\times 1}$ is a linear combination of the columns of $A_{m\times n} = [c_1 : c_2 : \cdots : c_n]$, i.e.,

$$b_{m\times 1} = x_1c_1 + x_2c_2 + \cdots + x_nc_n$$

where $x_1, x_2, \cdots, x_n$ are scalars, then we have

$$b_{m\times 1} = x_1c_1 + x_2c_2 + \cdots + x_nc_n = [c_1 : c_2 : \cdots : c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

which leads to

$$A_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b_{m\times 1}.$$

Namely $A_{m\times n}x = b_{m\times 1}$ has a solution $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and is consistent.

3) By 1) and 2), the system of linear equations $A_{m\times n}x = b_{m\times 1}$ is consistent if and only of $b_{m\times 1}$ is a linear combination of the columns of $A_{m\times n}$.