Discrete Mathematics for Computing

UTD

Ch 9.5 Equivalence Relations

- A relation on set A is called an *equivalence relation* if it is:

  reflexive,

  symmetric, and

  transitive

Equivalence Relations

- Two elements *a* and *b* that are related by an equivalence relation are said to be *equivalent*.
- We use the notation

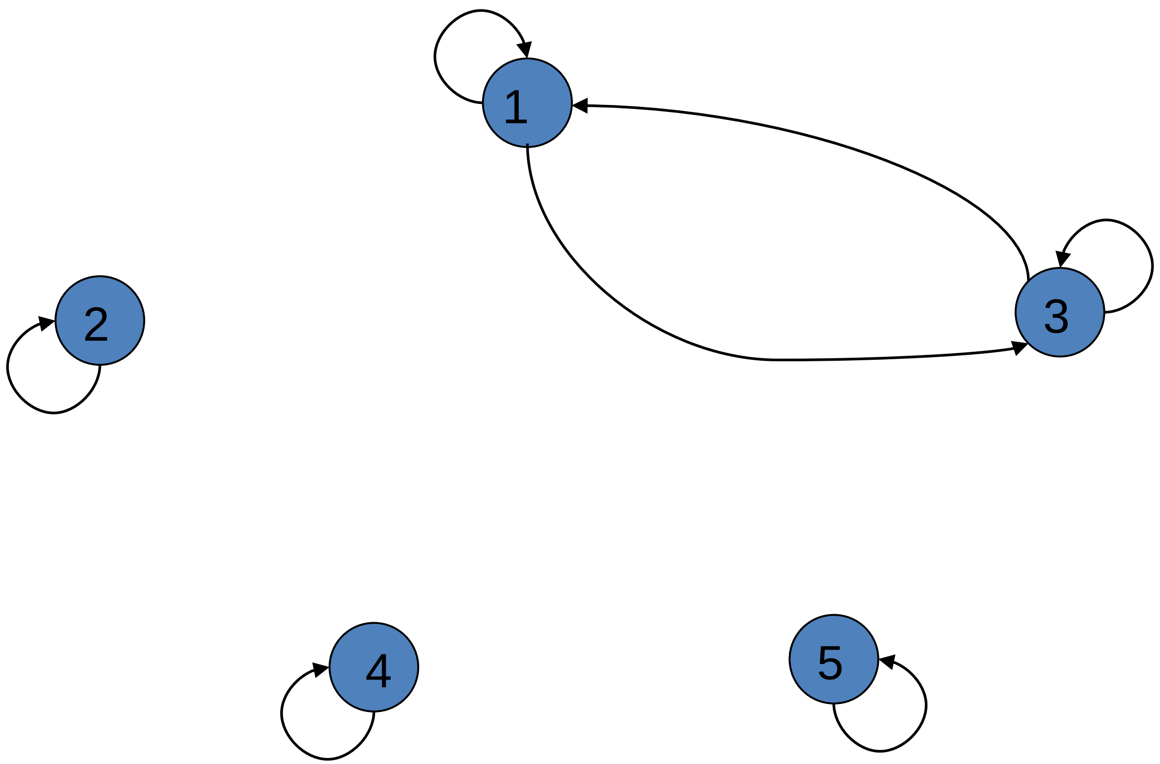
  $$a \sim b$$

  to denote that *a* and *b* are equivalent elements with respect to a particular equivalence relation.

Example

- Let R be a relation on set A, where $A = \{1, 2, 3, 4, 5\}$ and $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1)\}$
- Is R an equivalence relation?
- We can solve this by drawing a relation digraph:
  - *Reflexive* – there must be a loop at every vertex.
  - *Symmetric* - for every edge between two distinct points there must be an edge in the opposite direction.
  - *Transitive* - if there is an edge from x to y and an edge from y to z, there must be an edge from x to z.

Example



Is R an equivalence relation? *yes*

Example – Congruence modulo *m*

- Let $R = \{(a, b) \mid a \equiv b \pmod{m}\}$ be a relation on the set of integers and m be a positive integer > 1.

  Is R an equivalence relation?

  - *Reflexive* – is it true that $a \equiv a \pmod{m}$}? *yes*
  - *Symmetric* – is it true that if $a \equiv b \pmod{m}$ then $b \equiv a \pmod{m}$? *yes*
  - *Transitive* - is it true that if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $a \equiv c \pmod{m}$? *yes*

Example

- Suppose that R is the relation on the set of strings of English letters such that aRb iff $l(a) = l(b)$, where $l(x)$ is the length of the string x.
- Is R an equivalence relation?

Example

- Since $l(a) = l(a)$, then aRa for any string a. So R is <u>reflexive</u>.
- Suppose aRb, so that $l(a) = l(b)$. Then it is also true that $l(b) = l(a)$, which means that bRa. Consequently, R is <u>symmetric</u>.
- Suppose aRb and bRc. Then $l(a) = l(b)$ and $l(b) = l(c)$. Therefore, $l(a) = l(c)$ and so aRc. Hence, R is <u>transitive</u>.
- Therefore, R is an equivalence relation.

Equivalence Classes

- Let R be a equivalence relation on set A.
- The set of all elements that are related to an element a of A is called the *equivalence class* of a.
- The equivalence class of a with respect to R is:

$$[a]_R = \{s \mid (s, a) \in R\}$$

  – When only one relation is under consideration, we will just write $[a]$.

Equivalence Class

- Let R be the relation on the set of integers such that aRb iff $a = b$ or $a = -b$. We can show that this is an equivalence relation.

- The equivalence class of element a is

  $[a] = \{a, -a\}$

- Examples:

  $[7] = \{7, -7\}$ $[-5] = \{5, -5\}$

  $[0] = \{0\}$

Equivalence Example

- Consider the equivalence relation R on set A. What are the equivalence classes?

  $A = \{1, 2, 3, 4, 5\}$

  $R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1)\}$

- Just look at the aRb relationships. Which elements are related to which?

  $[1] = \{1, 3\}$ $[2] = \{2\}$

  $[3] = \{3, 1\}$ $[4] = \{4\}$

  $[5] = \{5\}$

UTD

Equivalence Example

Example: What are the equivalences classes for congruence modulo 4?

Solution:

The equivalence class of 0 contains all the integers a such that

$a \equiv 0 \pmod 4$. Hence, the equivalence class of 0 for this relation is

$$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod 4$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

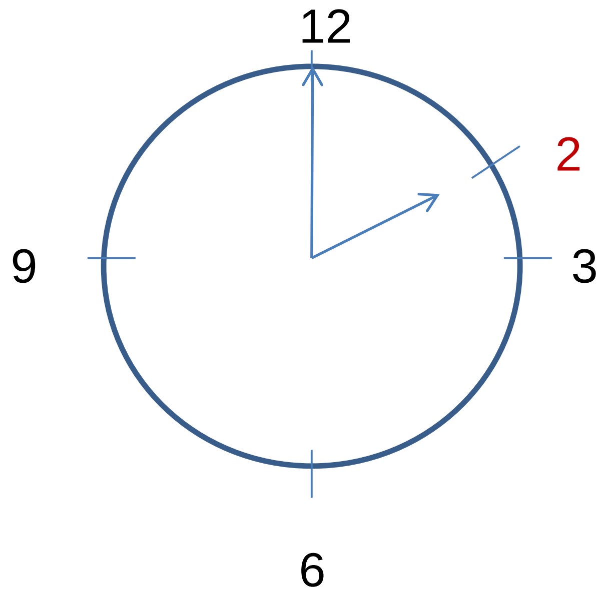
$$[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

Similarly, $[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$

and $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

So there are four equivalence classes for congruence modulo 4.

Equivalence Example



$[1]_{12}$ = {1, 13, 25, ... } = {1+ 12n: n € N} ,
$[2]_{12}$ = {2, 14, 26, ... } = {2+ 12n: n € N} , ...

where *N* is the set of natural numbers

A useful theorem about classes

- Let R be an equivalence relation on a set A. These statements for *elements* a and b of A are equivalent:

$$aRb$$

$$[a] = [b]$$

$$[a] \cap [b] = \varnothing$$

A useful theorem about classes

- More importantly:

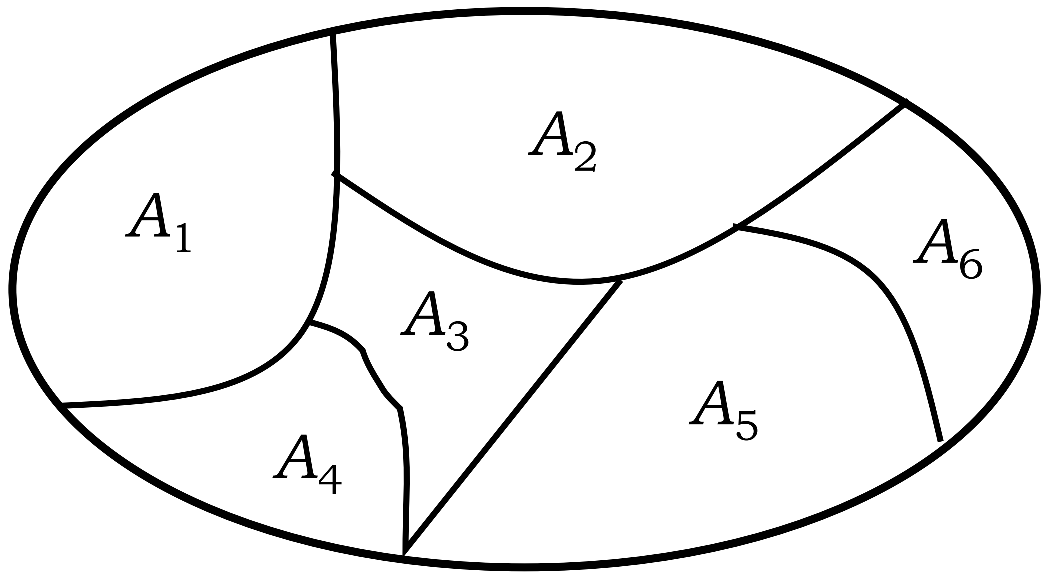
  Equivalence classes are EITHER

  - **equal** or
  - **disjoint**

Partitions

- A *partition* of a set *A* divides *A* into non-overlapping subsets:
  - A partition of a set A is a collection of disjoint nonempty subsets of A that have A as their union.

<u>Example 1</u>



Set *A*

Partitions

- A *partition* of a set A divides A into non-overlapping subsets:
- <u>Example</u>

$S = \{a, b, c, d, e, f\}$

$S_1 = \{a, d, e\}$

$S_2 = \{b\}$

$S_3 = \{c, f\}$

$P = \{S_1, S_2, S_3\}$

P is a partition of set S

Example

If $S = \{1, 2, 3, 4, 5, 6\}$, then

$A_1 = \{1, 3, 4\}$

$A_2 = \{2, 5\}$

$A_3 = \{6\}$

form a partition of S, because:

– these sets are disjoint

– the union of these sets is S.

Example

If $S = \{1, 2, 3, 4, 5, 6\}$, then

$A_1 = \{1, 3, 4, 5\}$

$A_2 = \{2, 5\}$

$A_3 = \{6\}$

do not form a partition of S, because:

- these sets are not disjoint (5 occurs in two different sets)

Example

If $S = \{1, 2, 3, 4, 5, 6\}$, then

$A_1 = \{1, 3\}$

$A_2 = \{2, 5\}$

$A_3 = \{6\}$

do not form a partition of S, because:

- the union of these sets is not S (since 4 is not a member of any of the subsets, but is a member of S).

Example

If $S = \{1, 2, 3, 4, 5, 6\}$, then

$A_1 = \{1, 3, 4\}$

$A_2 = \{2, 5\}$

$A_3 = \{6, 7\}$

do not form a partition of S, because:

- the union of these sets is not S (since 7 is a member of set A_3 but is not a member of S).

Constructing an Equivalence Relation from a Partition

Given set $S = \{1, 2, 3, 4, 5, 6\}$ and a partition of S,

$A_1 = \{1, 2, 3\}$

$A_2 = \{4, 5\}$

$A_3 = \{6\}$

then we can find the ordered pairs that make up the equivalence relation R produced by that partition.

Constructing an Equivalence Relation from a Partition

The subsets in the partition of S,

$A_1 = \{1, 2, 3\}$

$A_2 = \{4, 5\}$

$A_3 = \{6\}$

are the equivalence classes of R. This means that the pair $(a,b) \in R$ iff a and b are in the same subset of the partition.

UTD

Constructing an Equivalence Relation from a Partition

Let's find the ordered pairs that are in R:

$A_1 = \{1, 2, 3\} \rightarrow (1,1), (1,2), (1,3), (2,1),$

$(2,2), (2,3), (3,1), (3,2), (3,3)$

$A_2 = \{4, 5\} \rightarrow (4,4), (4,5), (5,4), (5,5)$

$A_3 = \{6\} \rightarrow (6,6)$

So R is just the set consisting of all these ordered pairs:

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$

UTD