Discrete Math for Computing

UTD

Ch 2.4 Sequences and Summations

- What is a sequence?

A sequence is a discrete structure used to represent an ordered list.

Sequences and Summations

- A sequence is a function from a subset of the set integers (usually {0,1,2,...} or {1,2,3,...}) to a set S
- a_n = term of a sequence

  or image of the integer n

$\{a_n\}$ = Represents a sequence

1, 2, 4, 5, 8 = finite sequence

1, 4, 16, 32,….,120,… = infinite sequence

Sequences and Summations

- Example: Consider the sequence $\{a_n\}$ where $a_n = 1/n$. What are the terms in this sequence?

The terms in this sequence are:

$a_1, a_2, a_3, a_4, a_5, \ldots$

$= 1, 1/2, 1/3, 1/4, \ldots$

Sequences and Summations

- **Geometric Progression**
- Sequence of the form:

  $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$

- 'a' is the initial term
- 'r' is the common ratio
- 'a' and 'r' are real numbers

  $f(x) = ar^x$

Sequences and Summations

- Example: Is $\{(-1)^n\}$ a geometric progression?

1,-1,1,-1,...

Yes, a=1 and r=-1

- Example: Is $\{2(5)^n\}$ a geometric progression?

2,10,50,250,...

Yes, a=2 and r=5

- Example: Is $\{6(1/3)^n\}$ a geometric progression?

6,2,2/3,2/9,...

Yes, a=6 and r=1/3

Sequences and Summations

- Example: The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2.5^n$, and $d_n = 6.(1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and -1; 2 and 5; and 6 and 1/3 respectively, if we start with $n = 0$. What are the terms?

Sequence $\{b_n\}$, $a = 1$, $r = -1$

Sequence $\{c_n\}$, $a = 2$, $r = 5$

Sequence $\{d_n\}$, $a = 6$, $r = 1/3$

List of terms $b_0, b_1, b_2, b_3, b_4, \ldots$ begins with $1, -1, 1, -1, 1, \ldots$

List of terms $c_0, c_1, c_2, c_3, c_4, \ldots$ begins with $2, 10, 50, 250, 1250, \ldots$

List of terms $d_0, d_1, d_2, d_3, d_4, \ldots$ begins with $6, 2, 2/3, 2/9, 2/27, \ldots$

Sequences and Summations

- **Arithmetic Progression**
- Sequence of the form

  $a, a + d, a + 2d, \ldots, a + nd, \ldots$
- 'a' is the initial term
- 'd' is the common difference
- 'a' and 'd' are real numbers

  $f(x) = dx + a$

Sequences and Summations

- Example: Is {-1 + 4n} an arithmetic progression?

-1, 3, 7, 11,...

Yes, a=-1 and d=4

- Example: Is {7 - 3n} an arithmetic progression?

7, 4, 1, -2,...

Yes, a=7 and d=-3

Sequences and Summations

- Example: The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\} = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to -1 and 4, and 7 and -3 respectively, if we start with at n = 0. What are the terms?

  Sequence $\{s_n\}$ 'a' = -1, d = 4

  Sequence $\{t_n\}$ 'a' = 7, d = -3

- List of terms $s_0, s_1, s_2, s_3, s_4, \ldots$ begins with -1, 3, 7, 11, 15,…
- List of terms $t_0, t_1, t_2, t_3, t_4, \ldots$ begins with 7, 4, 1, -2, -5,…

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Sequences and Summations

- **Strings**

  Sequences of the type $a_1, a_2, a_3, \ldots, a_n$
- Finite sequences denoted by $a_1a_2a_3\ldots a_n$

  Length of string S - number of terms in this string

  Empty String - string has no terms
- Example: Bit strings, finite sequences of bits

Sequences and Summations

- **Special Integer Sequences**
- Finding a formula or rule for finding a formula or a general rule for constructing the terms of a sequence

  GOAL: Identify the sequence

  Page 162, Table 1 – Useful sequences

Sequences and Summations

- Example: Find a formula for the sequence 5, 11, 17, 23, 29, 35, 41, 47, 53, 59.
- Each of the first 10 terms after the first in the sequence are obtained after adding 6 to the previous term
- nth term : $5 + 6(n-1) = 6n - 1$
- Arithmetic Progression: $a = 5$, $d = 6$

Sequences and Summations

Example: Find formula for the following sequence

1, -1, 1, -1, 1, ...

It is a geometric progression with a=1,r=-1

nth term: $\{(-1)^{n-1}\}$

Sequences and Summations

- **Summations**
- Sum of terms $a_m, a_{m+1},\ldots,a_n$ is denoted by the notation

$$\sum_{j=m}^{n} a_j \quad \text{or} \quad \sum\nolimits_{j=m}^{n} a_j \quad \text{or} \quad \sum\nolimits_{1\le j\le n} a_j$$

to represent $a_m + a_{m+1} + \ldots + a_n$

Variable j : index of summation, arbitrary choice

Index of summation : lower limit m, upper limit n

Summation: Large uppercase Greek letter sigma Σ

Sequences and Summations

- Example: Express the sum of the first 100 terms of the sequence {1/n} for n=1,2,3,... .

$$\sum_{k=1}^{100} 1/k$$

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Sequences and Summations

- Example: What is the value of $\sum_{j=1}^{5} j^2$?

- $\sum_{j=1}^{5} j^2$

  = 1 + 4 + 9 + 16 + 25

  = 55

Sequences and Summations

- Example: What is the value of $\sum_{k=50}^{100} k^2$?
- $\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$

Formulas: Table 2, page 157

$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$$

$$\sum_{k=50}^{100} k^2 = \frac{100.101.201}{6} - \frac{49.50.99}{6} = 338{,}350 - 40{,}425$$

$$= 297{,}925$$

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Sequences and Summations

- **Double Summation:** Nested loops
- Example: What is the value of $\sum_{i=1}^{4}\sum_{j=1}^{3} ij$?
- $\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i+2i+3i)$

  $= \sum_{i=1}^{4}(6i)$

  $= 6 + 12 + 18 + 24$

  $= 60$

Sequences and Summations

- **Cardinality**
- The sets A and B have the same cardinality
- if and only if there is a one-to-one correspondence from A to B
- A set that is either finite or has the same cardinality as the set of positive integers is called countable
- A set that is not countable is called uncountable

Sequences and Summations

- Example: Determine whether the given set is countable. If it is countable, exhibit a one-to-one correspondence between the set of natural numbers and the set.

The set of integers greater than 10

- The integers in the set are 11, 12, 13, 14, …
- List the numbers to establish the desired correspondence $n \leftrightarrow (n + 10)$. $\therefore$ The set is countable.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 11 & 12 & 13 & 14 \ldots \end{array}$$