# Discrete Mathematics for Computing

# Ch 9.1 Relations and Their Properties

- Motivation

Relationships between set elements occur in many contexts

What are some of the relationships?

Any business and its telephone number

An employee and his or her salary

Computer Science

Program and a variable

Computer Language and a valid statement in the language

# Relations and Their Properties

- Ordered Pairs of two elements

- Most direct way to express a relationship between elements of two sets

- Set of ordered pairs – **binary relations**
- Let A and B be sets, binary relation from A to B is a subset of a cartesian product A x B

# Relations and Their Properties

- Binary relation from A to B - set R of ordered pairs

- first element of each ordered pair comes from A

- second element comes from B

Example: Let $A = \{a,b,c\}$ and $B = \{1,2,3\}$.

$R = \{(a,1),(b,2),(c,2)\}$

example of a relation from A to B

# Relations and Their Properties

- Binary Relation Notation:

$$a \, R \, b \Leftrightarrow (a, b) \in R$$

$$a \, \cancel{R} \, b \Leftrightarrow (a, b) \notin R$$

(a, b) belongs to R => a is related to b by R

# Relations and Their Properties

- Example: Let A be the set of all cities, and let B be the set of the 50 states in the United States of America. Define the relation R by specifying (a, b) belongs to R if city a is in state b.

- A = set of all cities
  B = set of the 50 states in the USA

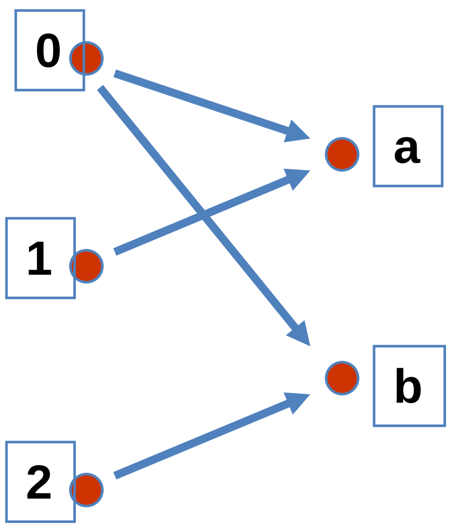
Relation R - (a, b) belongs to R if city a is in state b

$$\left.\begin{array}{l} (Boulder, Colorado) \\ (Bangor, Maine) \\ (Ann\ Arbor, Michigan) \\ (Cupertino, California) \\ Red\ Bank, New\ Jersey) \end{array}\right\} \text{ are in } R.$$

# Relations and Their Properties

- Example: Let A = {0, 1, 2}, B = {a, b}
- Relation R = { (0, a), (0, b), (1, a), (2, b) }

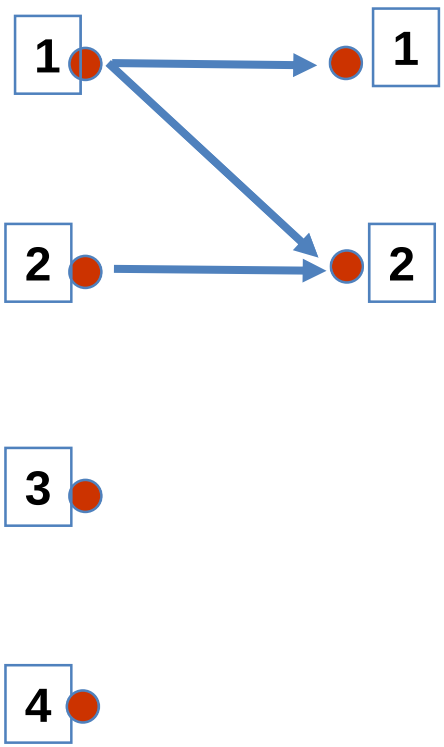
- $R \subseteq A \times B$, Graphical representation – arrows represent ordered pairs, table showing (marking) the ordered pairs of R




| R | a | b |
|---|---|---|
| 0 | X | X |
| 1 | X | |
| 2 | | X |

# Relations and Their Properties

- Example: Let A = {1, 2, 3, 4}, B = {1, 2}
- Relation R = {(a, b) | a divides b}
- R = {(1,1), (1,2), (2,2)}




| R | 1 | 2 |
|---|---|---|
| 1 | X | X |
| 2 | | X |
| 3 | | |
| 4 | | |

## Relations and Their Properties

- Functions as Relations
- Function f from a set A to a set B -> assigns exactly one element of B to each element of A
- Graph of f - set of ordered pairs (a, b) such that $b = f(a)$
- Subset of A x B $\Rightarrow$ it is a relation from A to B

# Relations and Their Properties

- Relations on a Set
- Relations from a set ‘A’ to itself

- A relation on a set ‘A’ is a relation from ‘A’ to ‘A’

Example: Let A = set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?

Since (a, b) is in R - if and only if a and b are positive integers not exceeding 4 such that a divides b

R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

# Relations and Their Properties

- Example: How many relations are there on a set with 'n' elements?
- A relation on a set A is a subset of A x A
- A x A = $n^2$ elements, if A has n elements
- Set with m elements = $2^m$ subsets
- A x A = $2^{n^2}$ subsets, n elements for each set A
- $2^{3^2} = 2^9 = 512$ relations on the set with 3 elements {a, b, c}

# Relations and Their Properties

- Properties of Relations
- Several properties – classify relations on a set
- Reflexive
- A relation R on a set A is called reflexive

  - if (a, a) € R for every element a € A
- Relation R on the set A is reflexive if

$$\forall a((a,a) \in R)$$

domain is set of all elements in A

# Properties of Relations

- Example a): Consider the following relations on {1, 2, 3, 4}

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

Which of these relations are reflexive?

# Properties of Relations

- $R_3$ and $R_5$: reflexive $\Leftarrow$ both contain all pairs of the form (a, a): (1,1), (2,2), (3,3) & (4,4)
- $R_1$, $R_2$, $R_4$ and $R_6$: not reflexive $\Leftarrow$ does not contain all of these ordered pairs. (3,3) is not in any of these relations.

**$R_1$ = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}**
**$R_2$ = {(1,1), (1,2), (2,1)}**
**$R_3$ = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}**
**$R_4$ = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}**
**$R_5$ = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}**
**$R_6$ = {(3,4)}**

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## Properties of Relations

- Symmetric and Antisymmetric
- A relation R on a set is called symmetric

  - if (b, a) € R

  - whenever (a, b) € R, for all a, b € A

$$\forall a \forall b((a,b) \in R) \rightarrow (b,a) \in R)$$

- A relation R on a set A such that for all a, b € A

  - if (a, b) € R and (b, a) € R

  - then a = b is called antisymmetric

$$\forall a \forall b(((a,b) \in R \wedge (b,a) \in R) \rightarrow (a = b))$$

# Properties of Relations

- Example: Which of the relations from example (a) are symmetric and which are antisymmetric?

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$
$R_2 = \{(1,1), (1,2), (2,1)\}$
$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,3), (4,4)\}$
$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$
$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$
$R_6 = \{(3,4)\}$

$R_2$ & $R_3$: symmetric $\Leftarrow$ each case (b, a) belongs to the relation whenever (a, b) does.
For $R_2$: check that both (1,2) & (2,1) belong to the relation
For $R_3$: it is necessary to check that both (1,2) & (2,1) belong to the relation.

# Properties of Relations

- None of the other relations is symmetric: find a pair (a, b) so that it is in the relation but (b, a) is not.
- $R_4$, $R_5$ and $R_6$: antisymmetric $\Leftarrow$ for each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation
- None of the other relations is antisymmetric.: find a pair (a, b) with $a \neq b$ so that (a, b) and (b, a) are both in the relation.

# Properties of Relations

- Transitive
- A relation R on a set A is called transitive
- if whenever (a, b) $\in$ R and (b, c) $\in$ R

- then (a, c) $\in$ R, for all a, b, c $\in$ R

$$\forall a \forall b \forall c(((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$$

# Properties of Relations

- Example: Which of the relations in example (a) are transitive?
- $R_4$ , $R_5$ & $R_6$ : transitive $\Leftarrow$ verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation, $R_4$ transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and (3,1) , (4,1) and (4,2) belong to $R_{4,}$ Same reasoning for $R_5$ and $R_6$
- $R_1$ : not transitive $\Leftarrow$ (3,4) and (4,1) belong to $R_1$, but (3,1) does not
- $R_2$ : not transitive $\Leftarrow$ (2,1) and (1,2) belong to $R_2$, but (2,2) does not
- $R_3$ : not transitive $\Leftarrow$ (4,1) and (1,2) belong to $R_3$, but (4,2) does not

**$R_1$ = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}**
**$R_2$ = {(1,1), (1,2), (2,1)}**
**$R_3$ = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}**
**$R_4$ = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}**
**$R_5$ = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}**
**$R_6$ = {(3,4)}**

# Properties of Relations

- Combining Relations
- Example: Let A = {1, 2, 3} and B = {1, 2, 3, 4, }.
- The relations $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain:
- $R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$
- $R_1 \cap R_2 = \{(1,1)\}$
- $R_1 - R_2 = \{(2,2), (3,3)\}$
- $R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

## Properties of Relations

- Composite
- Let R be a relation from a set A to a set B
- S a relation from B to a set C
- The composite of R and S is the relation

- consisting of ordered pairs (a, c), where $a \in A$, $c \in C$

- and for which there exists an element $b \in B$

- such that $(a, b) \in R$ and $(b, c) \in S$

Denote the composite of R and S by $S \circ R$

# Properties of Relations

- Example: What is the composite of the relations R and S where R is the relation from {1,2,3} to {1,2,3,4} with R = {(1,1), (1,4), (2,3), (3,1), (3,4)} and S is the relation from {1,2,3,4} to {0,1,2} with S = {(1,0), (2,0), (3,1), (3,2), (4,1)}?

- $S \circ R$ => construct using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S
- For example => the ordered pairs (2,3) in R and (3,1) in S produce the ordered pair (2,1) in $S \circ R$
- Computing all the ordered pairs in the composite

  $S \circ R$ = {(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)}

# The Powers of a Relation

- The powers of a relation R are recursively defined from the definition of a composite of two relations.
- Let $R$ be a relation on the set $A$. The powers $R^n$, for $n = 1$, 2, 3, … are defined recursively by:

$$R^1 = R$$

$$R^{n+1} = R^n \circ R$$

So:

$$R^2 = R \circ R$$

$$R^3 = R^2 \circ R = (R \circ R) \circ R \text{ etc.}$$

# The Powers of a Relation

- Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$
- Find the powers $R^n$ , where $n = 1, 2, 3, 4, \ldots$

$$R^1 = R = \{(1,1), (2,1), (3,2), (4,3)\}$$

$$R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

$$R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$

$$R^5 = R^4 \circ R = \{(1,1), (2,1), (3,1), (4,1)\}$$