Discrete Mathematics for Computing

UTD

Ch 1.5 Nested Quantifiers

- Two Quantifiers P and Q are nested
  - One is within the scope of the other
  - Propositional Function
  - Math and Computer Science

Nested Quantifiers

- $\forall x \exists y (x + y = 0)$

$$\forall x Q(x)$$

where Q(x) is $\exists y P(x, y)$

where P(x,y) is $x + y = 0$

Nested Quantifiers

- Commutative law for Addition of real numbers

$$\forall x \forall y (x + y = y + x)$$

- For all real numbers x and y,

$$x + y = y + x$$

Nested Quantifiers

- Associative law for Addition of real numbers

$$\forall x \forall y \forall z (x + (y + z) = (x + y) + z)$$

For all real numbers x, y and z,

$$x + (y + z) = (x + y) + z$$

Nested Quantifiers

- Additive Inverse

$$\forall x \exists y (x + y = 0)$$

For every real number x, there is a real number y

$$x + y = 0$$

Nested Quantifiers

- Translate into English the statement

$$\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$$

- For real numbers x and y, if x is positive and y is negative, then xy is negative.
- The product of a positive real number and a negative real number is always a negative real number.

Order of Quantifiers

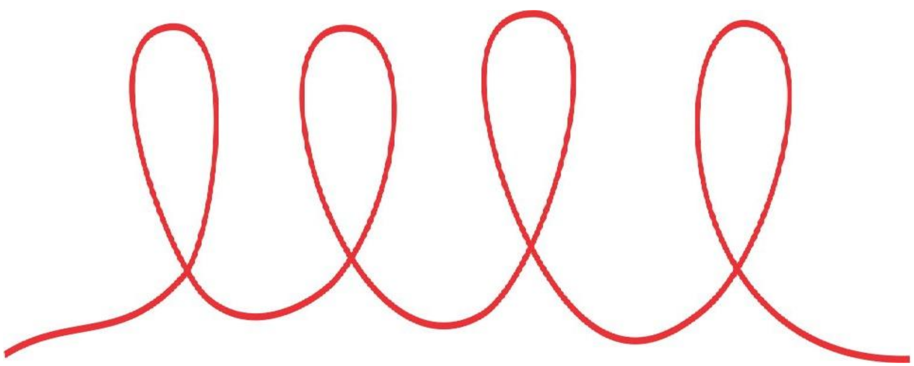
- Mathematical Statements

  - Multiple Quantifications of Propositional Functions

  - Involve more than one variable

  - Order of Quantifiers is relevant

Order of Quantifiers



```
for ( i = 0; i < rows; i++) {
    for ( j = 0; j < columns; j++) {
        System.out.print(aryNumbers[i][j] + " ");
    }
    System.out.println( "" );
}
```

Nested Quantifiers

- Quantification as Loops

  - Nested Loops

$$\forall x \forall y P(x, y)$$

- Loop through the values of x

- For each value of x, loop through the values of y.

Nested Quantifiers

- $\exists x \forall y P(x, y)$

  - We loop through values of *x*
  - Until we find an *x* for which P(x,y) is always **T**
  - when we loop through all values of *y*
  - If we never hit such an *x*, **F**

Nested Quantifiers

$\forall x \forall y P(x, y)$

When true?

P(x,y) is true for every pair x,y.

When false?

There is a pair x, y for which P(x,y) is false.

$\forall x \exists y P(x, y)$

When true?

For every x there is a y for which P(x,y) is true.

When false?

There is an x such that P(x,y) is false for every y.

Nested Quantifiers

$\exists x \forall y P(x, y)$

When true?

There is an x for which P(x,y) is true for every y.

When false?

For every x there is a y for which P(x,y) is false.

$\exists x \exists y P(x, y)$

When true?

There is a pair x, y for which P(x,y) is true.

When false?

P(x,y) is false for every pair x, y.

Order of Quantifiers

- Example: Let P(x,y) be the statement $x+y=y+x$. What are the truth values of the quantification $\forall x \forall y P(x,y)$ where the domain for all variables consists of all real numbers?

The quantification $\forall x \forall y P(x,y)$ denotes the proposition

"For all real numbers x, for all real numbers y, $x+y=y+x$."

P(x,y) is T for all real numbers x and y,

=> $\forall x \forall y P(x,y)$ is **T**

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Order of Quantifiers

- Example: Let Q(x,y) be the statement x + y = 0. What are the truth values of the quantification $\exists y \forall x Q(x, y)$ where the domain for all variables consists of all real numbers?

The quantification $\exists y \forall x Q(x, y)$ denotes the proposition "There is a real number y such that for every real number x, Q(x,y)."

=> $\exists y \forall x Q(x, y)$ is **F**

Order of Quantifiers

- Example: Let Q(x,y) be the statement $x + y = 0$. What are the truth values of the quantification $\forall x \exists y Q(x, y)$ where the domain for all variables consists of all real numbers?

The quantification $\forall x \exists y Q(x, y)$ denotes the proposition " For every real number x, there is a real number y such that Q(x,y)."

$\Rightarrow \quad \forall x \exists y Q(x, y)$ is **T**

ORDER OF QUANTIFIERS IMPORTANT!

Translating Mathematical Statements

- Example: Translate the statement into logical expressions.

"The sum of two positive integers is always positive."

Rewrite: "For every two integers, if these integers are positive, then the sum of these integers is positive."

Introduce variables x & y: "For all positive integers x and y, x + y is positive."

$\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$, domain all integers.

$\forall x \forall y(x + y > 0)$, domain all positive integers.

Translating Mathematical Statements

- Example: Translate the statement into logical expressions.

"Every real number except zero has a multiplicative inverse."

Translate to a logical expression: "For every real number x, if $x \neq 0$, then there is a real number y such that $xy = 1$."

$$\forall x\, ((x \neq 0) \rightarrow \exists y\, (xy = 1)),\ \text{domain real numbers}$$

Translating Mathematical Statements

- Example: Translate the statement into logical expressions using nested quantifiers.

"There is a woman who has taken a flight on every airline in the world."

Let P(w,f) be "w has taken f"

Let Q(f,a) be "f is a flight on a"

$$\exists w \forall a \exists f (P(w, f) \wedge Q(f, a))$$

Domain for 'w' consists of all the women in the world, domain for 'f' all the airplane flights, domain for 'a' all airlines.

Translating Mathematical Statements

- Example: What is the negation of the following statement

$$\forall x \exists y (x = -y)$$

$\neg \forall x\, P(x)$ where $P(x) = \exists y\, (x = -y)$

$\equiv \exists x\, \neg P(x)$

$\equiv \exists x\, \neg \exists y (x = -y)$

$\equiv \exists x\, \forall y\, \neg(x = -y)$

$\equiv \exists x\, \forall y\, (x \neq -y)$

Negating Nested Quantifiers

- Example: Use quantifiers to express the statement that “There does not exist a woman who has taken a flight on every airline in the world.”

Use De Morgan’s laws for Quantifiers, move the negation

$$\neg\exists w \forall a \exists f\,(P(w,f) \wedge Q(f,a))$$

$$\equiv \forall w \neg\forall a \exists f\,(P(w,f) \wedge Q(f,a))$$

$$\equiv \forall w \exists a \neg\exists f\,(P(w,f) \wedge Q(f,a))$$

$$\equiv \forall w \exists a \forall f \neg(P(w,f) \wedge Q(f,a))$$

$$\equiv \forall w \exists a \forall f\,(\neg P(w,f) \vee \neg Q(f,a))$$

For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline.

Practical Example

- Prolog - Based on first-order predicate logic

- Developed in 1970 by Colmerauer & Roussel (Marseilles) and Kowalski (Edinburgh) + others.
- Used in Artificial Intelligence, databases, expert systems.
- Program = a bunch of axioms
- Run your program by:
  - Enter a series of facts and declarations
  - Pose a query
  - System tries to prove your query by finding a series of inference steps