Discrete Math for Computing

UTD

Ch 2.2 Set Operations

- Two sets can be combined in different ways

  - SET OPERATIONS

Example: Set of math majors

Set of computer science majors

a) Set of students who are math majors or computer science majors

b) Set of students who are joint majors in mathematics and computer science

Union

- Let A and B be sets
- The union of the sets A and B

  denoted by A U B

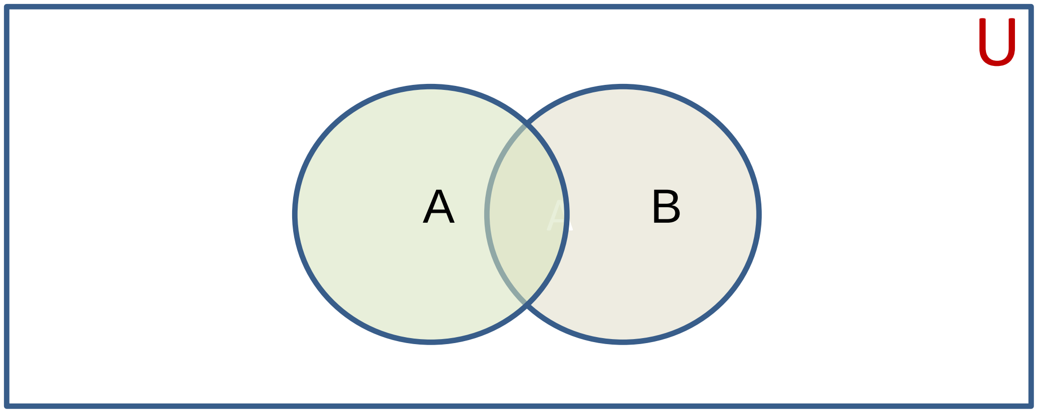
  set that contains those elements that are either in A or in B or in both
- An element x belongs to the union of the sets A and B if and only if x belongs to A

  or x belongs to B

  A U B = {x | x € A v x € B}

Union

- Venn Diagram



- Example:

What is the union of the sets {1, 2, 4, 6} and {1, 3, 4, 7}?

{1, 2, 3, 4, 6, 7}

Intersection

- Let A and B be sets
- The intersection of the sets A and B

  denoted by $A \cap B$

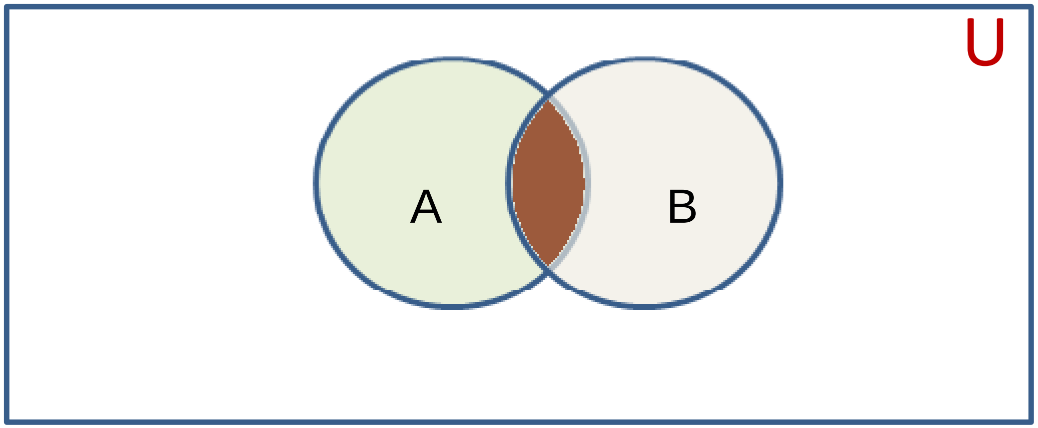
  set that contains those elements in both A and B
- An element x belongs to the intersection of the sets A and B if and only if x belongs to A

  and x belongs to B

  $A \cap B = \{x \mid x € A \wedge x € B\}$
- Two sets are disjoint if intersection is Ø

Intersection

- Venn Diagram



- Example:

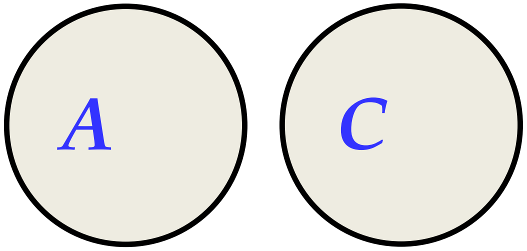
What is the intersection of the sets {1, 2, 3, 5} and {1, 3, 6}?

{1, 3}

Set Operations

- Two sets are called disjoint if their intersection is the empty set

- $A = \{1,3,5\}$, $B = \{1,2,3\}$, $C = \{6,7,8\}$
- Are A and B disjoint? NO
- Are A and C are disjoint? YES



Set Operations

- Cardinality of the union of two sets A and B

  $|A \cup B| = |A| + |B| - |A \cap B|$

- Principle of inclusion-exclusion
- Important technique used in enumeration

Set Operations

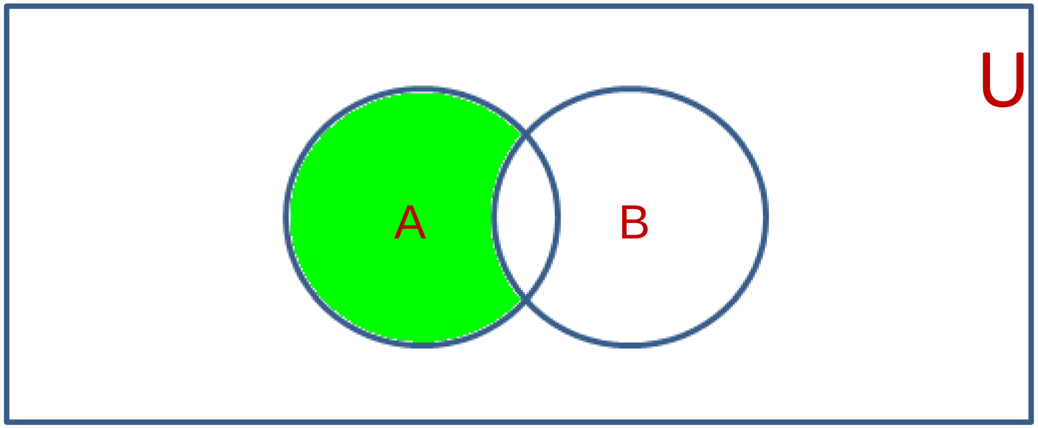
- Difference of two sets A and B
- Set containing those elements that are in A but not in B
- Denoted by A – B
- Complement of B with respect to A

An element x belongs to the difference of A and B if and only if x € A and x ~~€~~ B

A – B = {x | x € A ^ x ~~€~~ B}

Set Operations

- Venn Diagram



Example: What is the difference of the sets

{1, 3, 5} - {1, 2, 3}

{5}

{1, 2, 3} - {1, 3, 5}

{2}

Set Operations

- Example: Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$ b) $A \cap B$

c) $A - B$ d) $B - A$

a) $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$

b) $A \cap B = \{3\}$

c) $A - B = \{1, 2, 4, 5\}$

d) $B - A = \{0, 6\}$

Set Operations

- Complement of a set
- Let ‘U’ be the universal set
- The complement of a set A

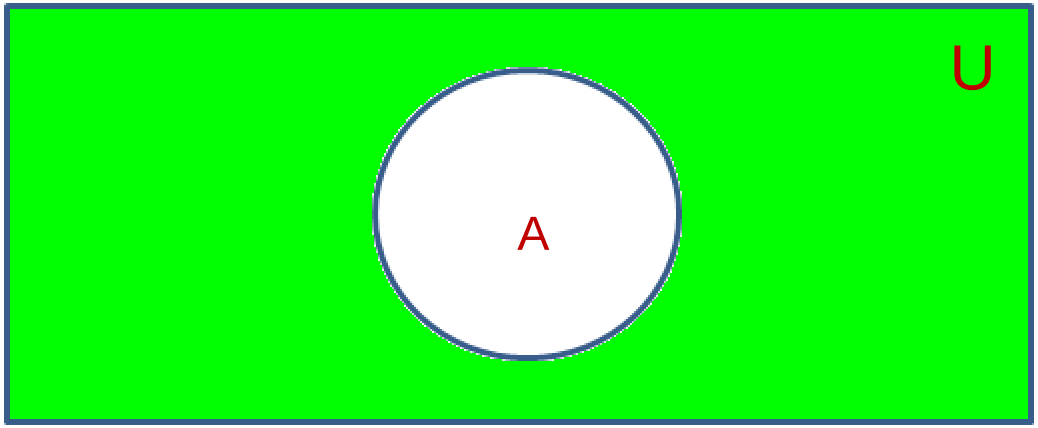
  complement of A with respect to U
- Denoted by $\bar{A}$

  An element x belongs to $\bar{A}$ if and only if

  $x \notin A$

Set Operations

- Venn Diagram



- Example: Let A be the set of positive integers greater than 10. What is the complement?

$$\bar{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Database Query

- The query that returns students that their GPA is more than grade B and they are either computer science or mathematics major.

A: students that their GPA is more than grade B

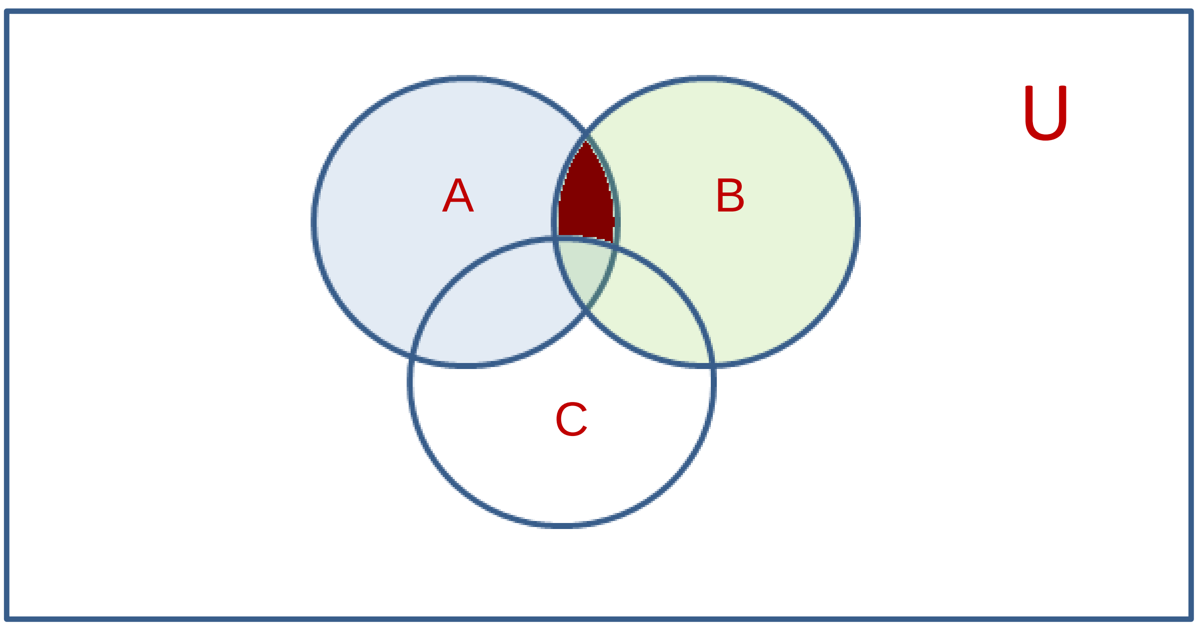
B: students that are computer science major.

C: students that are Mathematics major.

- $A \cap (B \cup C)$

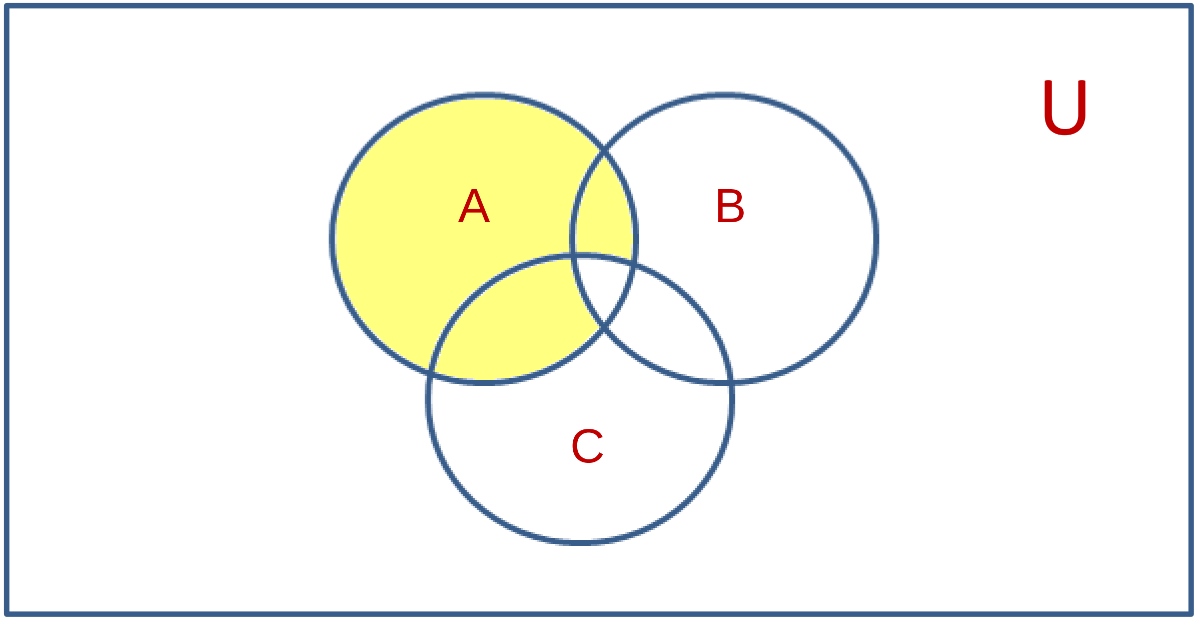
Set Operations

- Draw Venn Diagrams for the combination of the sets A, B, and C.
- $A \cap (B - C)$



Set Operations

- Draw Venn Diagrams for each of these combinations of the sets A, B, and C.
- $(A \cap \bar{B}) U (A \cap \bar{C})$



Set Operations

- Example: What can you say about the sets A and B if we know that:
- A U B = A?

Set B does not add any elements to A, therefore all the elements of B were already in A. $\therefore B \subseteq A$

- A – B = A?

Since A-B contains all the elements of A, none of the elements of set A are in set B. Thus sets A and B are disjoint. $\therefore A \cap B = \varnothing$

Set Identities

- Page 130, Table 1
- De Morgan's Laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Set Identities

$A \cup \varnothing = A$	Identity
$A \cap U = A$	
$A \cup U = U$	Domination
$A \cap \varnothing = \varnothing$	
$A \cup A = A$	Idempotent
$A \cap A = A$	
$\overline{\overline{A}} = A$	Double Complement

Set Identities

$A \cup B = B \cup A$	Commutative
$A \cap B = B \cap A$	
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative
$A \cap (B \cap C) = (A \cap B) \cap C$	
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
$A \cup (A \cap B) = A$	Absorption
$A \cap (A \cup B) = A$	
$\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's
$\overline{A \cap B} = \overline{A} \cup \overline{B}$	

Set Identities

- Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$, by definition of complement

$= \{x \mid \neg(x \in (A \cap B))\}$, by definition of does not belong symbol

$= \{x \mid \neg(x \in A \wedge x \in B)\}$, by definition of intersection

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$, first De Morgan law logical equivalences

$= \{x \mid x \notin A \vee x \notin B\}$, by definition of does not belong symbol

$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$, by definition of complement

$= \{x \mid x \in \overline{A} \cup \overline{B}\}$, by definition of union

$= \overline{A} \cup \overline{B}$, by meaning of set builder notation

Set Identities

- Example: Prove the first distributive law

$A \cap (B \cup C) = (A \cap B) U (A \cap C)$ for all sets A, B, and C

To prove, show that each side is a subset of the other side

1. Suppose that $x \in A \cap (B \cup C)$. Then, $x \in A$ and $x \in (B \cup C)$

By definition of union, it follows that

$x \in A$, and $x \in B$ or $x \in C$ (or both)

=> $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$

By definition of intersection, $x \in A \cap B$ or $x \in A \cap C$

By definition of union, we conclude that $x \in (A \cap B) U (A \cap C)$

$$\therefore A \cap (B \cup C) \subseteq (A \cap B) U (A \cap C)$$

Set Identities

2. Suppose that $(A \cap B) U (A \cap C) \subseteq A \cap (B \cup C)$

By definition of union, $x \in (A \cap B)$ or $x \in (A \cap C)$

By definition of intersection,

$x \in A$ and $x \in B$ or $x \in A$ and $x \in C$

=> $x \in A$, and $x \in B$ or $x \in C$

By definition of union, $x \in A$ and $x \in B \cup C$

By definition of intersection, $x \in A \cap (B \cup C)$

We conclude that $(A \cap B) U (A \cap C) \subseteq A \cap (B \cup C)$

This completes the Proof of the Identity

Set Identities

- Proved using 'Membership Tables'
- '1' – element is in a set
- '0' – element is not in a set

Similarity between truth tables and membership tables

Set Identities

- Example: Use a membership table to show that

$$A \cap (B \cup C) \quad = \quad (A \cap B) U (A \cap C)$$

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) U (A \cap C)$
1	1	1	1	**1**	1	1	**1**
1	1	0	1	**1**	1	0	**1**
1	0	1	1	**1**	0	1	**1**
1	0	0	0	**0**	0	0	**0**
0	1	1	1	**0**	0	0	**0**
0	1	0	1	**0**	0	0	**0**
0	0	1	1	**0**	0	0	**0**
0	0	0	0	**0**	0	0	**0**

Generalized Unions and Intersections

- Unions, intersections of sets satisfy associative laws
- Do not have to use parentheses to indicate which operation comes first

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C

$A \cap B \cap C$ contains those elements that are in all of the sets A, B, C

Generalized Unions and Intersections

- Example: Let A = {0, 2, 4, 6, 8}, B = {0, 1, 2, 3, 4}, and C = {0, 3, 6, 9}. What are $A \cup B \cup C$ and $A \cap B \cap C$?

$$A \cup B \cup C$$

= {0, 1, 2, 3, 4, 6, 8, 9}

$$A \cap B \cap C$$

= {0}

Generalized Unions and Intersections

- Union of a collection of sets
- Set that contains those elements that are members of at least one set in the collection

Notation

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

denotes union of sets A_1, A_2, …, A_n

Generalized Unions and Intersections

- Intersection of a collection of sets
- Set that contains those elements that are members of all the sets in the collection

Notation

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

denotes intersection of sets A_1, A_2, …, A_n

Generalized Unions and Intersections

- Let A_i = {i, i+1, i+2,...}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\}$$

Generalized Unions and Intersections

- To denote the union of sets $A_1, A_2, \ldots, A_n$

$$A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots = \bigcup_{i=1}^{\infty} A_i$$

Intersection is denoted by

$$A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots = \bigcap_{i=1}^{\infty} A_i$$

Generalized Unions and Intersections

- Let $A_i = \{1,2,3,...,i\}$ for $i = 1, 2, 3,...$

Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1,2,3,\ldots,i\} = \{1,2,3,\ldots\} = \mathbf{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1,2,3,\ldots,i\} = \{1\}$$

Computer Representations of Sets

- Store elements of set in unordered way
- Time consuming

  operations - union, intersection

  searching for elements
- Arbitrary ordering of elements

  of the universal set ‘U’

  Subset A of U ordering = $a_1, a_2, a_3, a_4, \ldots, a_n$

  if a_i belongs to A, ‘1’ else ‘0’

Computer Representations of Sets

- Example: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

  What bit strings represent the subset of all odd integers in U, all even integers in U?

  All odd integers = {1, 3, 5, 7, 9}

  10 1010 1010

  All even integers = {2, 4, 6, 8, 10}

  01 0101 0101

Computer Representations of Sets

- Example: Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

  What is the bit string for the complement of the subset of all odd integers?

  All odd integers 10 1010 1010

  Complement 01 0101 0101

  {2, 4, 6, 8, 10}

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Computer Representations of Sets

- Example: What is the union and intersection of the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} using bit strings?

Union:

$$11\ 1110\ 0000 \vee 10\ 1010\ 1010$$

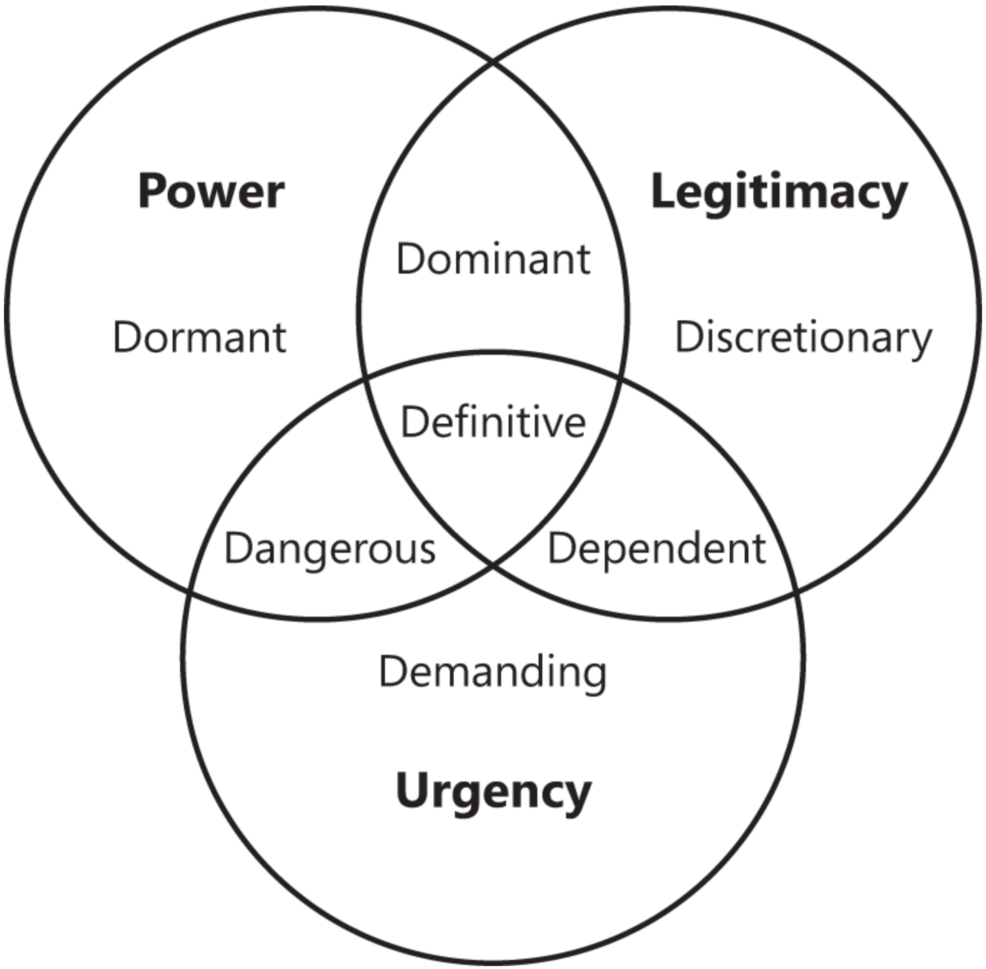
$$11\ 1110\ 1010 = \{1, 2, 3, 4, 5, 7, 9\}$$

Intersection:

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010$$

$$10\ 1010\ 0000 = \{1, 3, 5\}$$

Practical Example



Stakeholder Analysis

Practical Example

Find a subset of {31,27,15,11,7,5} with a sum that equals 39.

Summing a Subset