# Discrete Math for Computing

# Ch 2.3 Functions

- Assign to each element of a set
  a particular element of a second set
- Assignment is called 'FUNCTION'
- Math and Computer Science

sequences, strings, recursive functions

- Also called Mappings, Transformations

# Functions

- Let A and B be nonempty sets
- A function 'f' from A to B is an assignment of exactly one element of B to each element of A
- $f(a) = b$
- If f is a function from A to B

$$f : A \rightarrow B$$

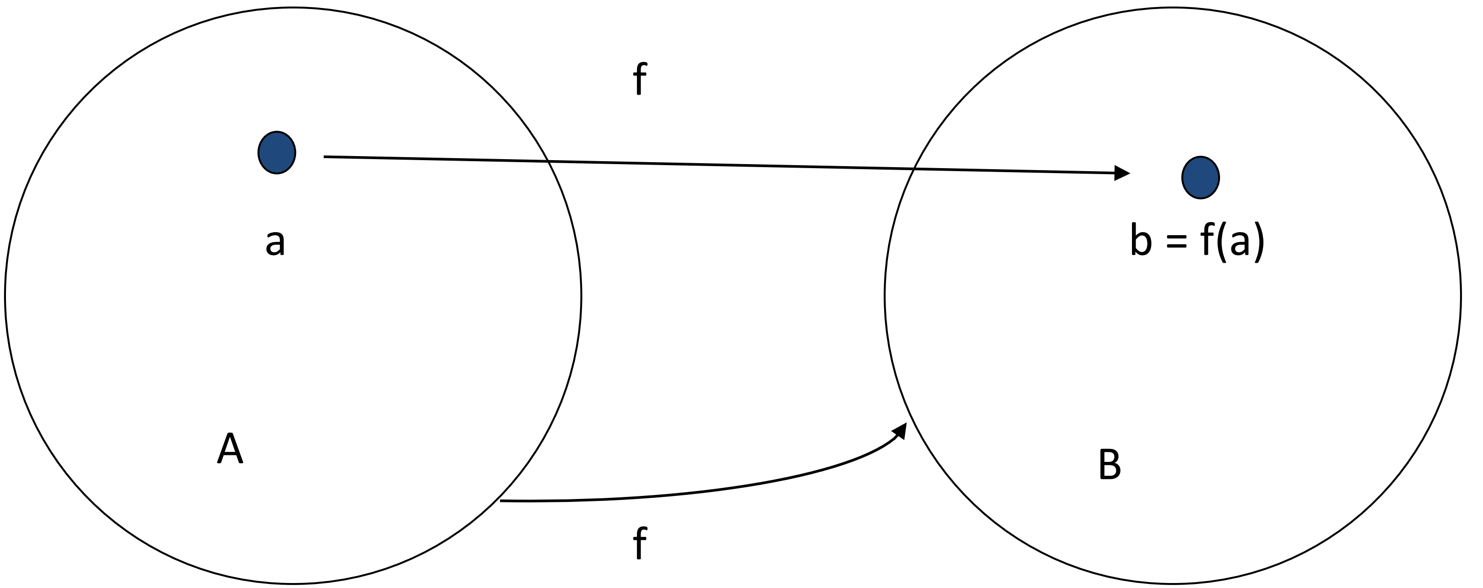
# Functions

- ‘f’ is a function from A to B

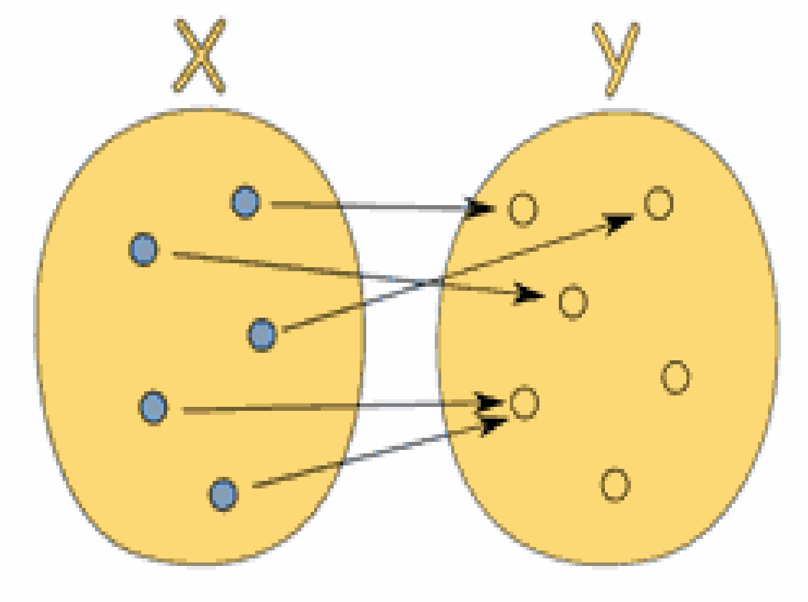
  $f: A \rightarrow B$

  f maps A to B
- A is the “domain” of ‘f’
- B is the “codomain” of ‘f’
- f(a) = b, b is image of a, a is preimage of b
- Range of a Function: The range of f is the set of all values that the function takes in the domain.

# Functions

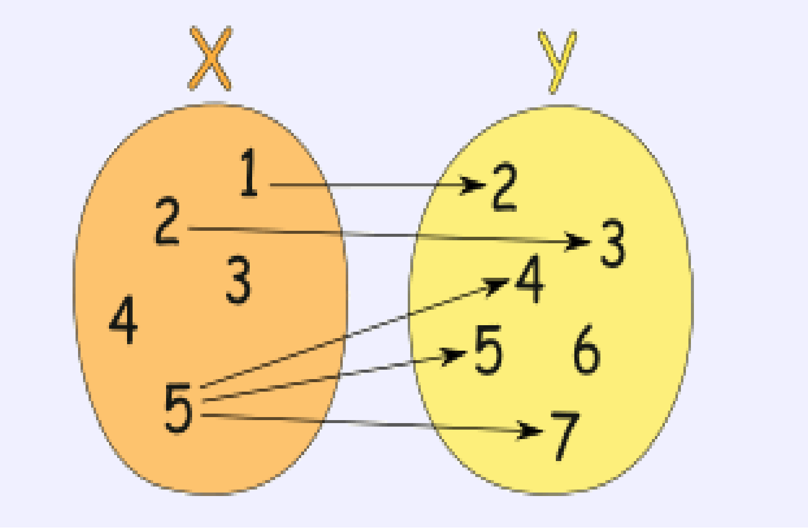


# Functions



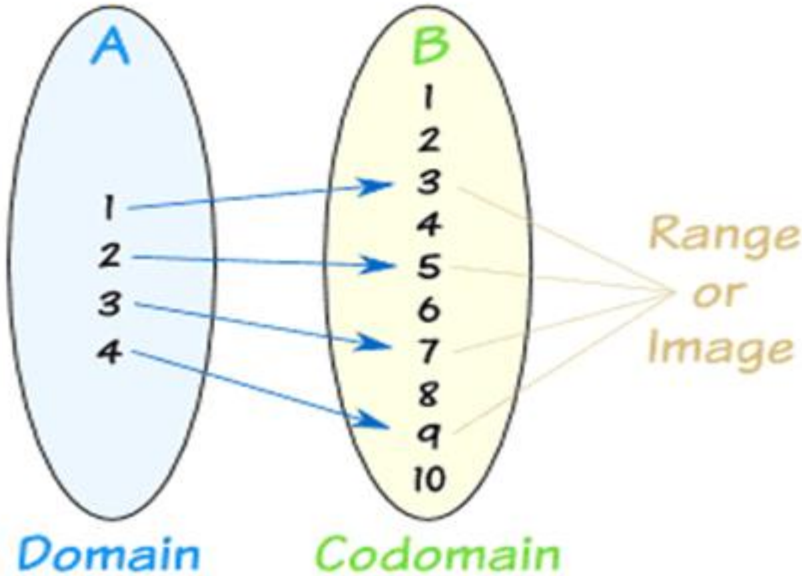

Every element in X has one value in Y
Therefore a function

# Functions




Value "3" in X has no relation in Y
Value "4" in X has no relation in Y
Value "5" is related to more than one value in Y
Therefore not a function

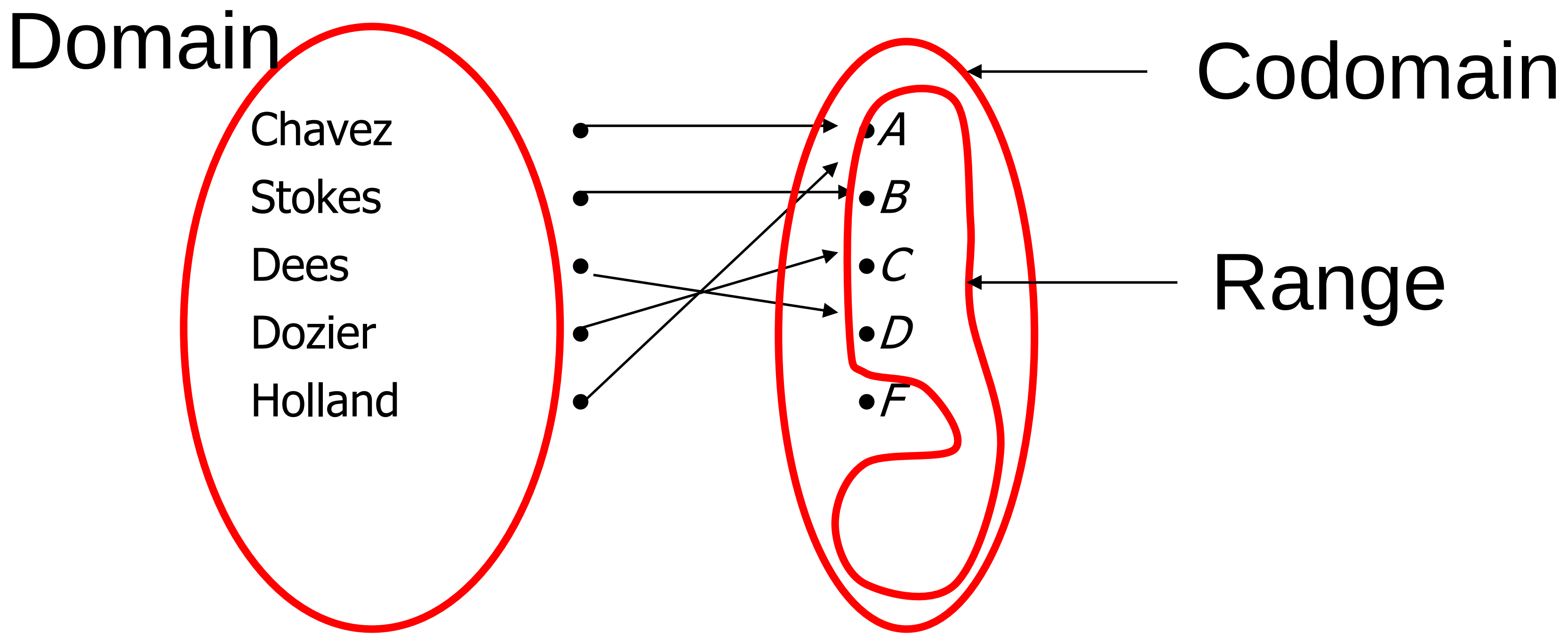
# Functions




Domain: {1, 2, 3, 4}
Codomain: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
Range: {3, 5, 7, 9}

# Example

- Suppose that each student in a class is assigned a letter grade from the set $\{A, B, C, D, F\}$. Let $g$ be the function that assigns a grade to a student.



# Functions

- Example:

Java programming language

```
int floor (float real)
{
        …
}
```

domain of the floor function - set of real numbers

codomain - set of integers

# Functions

- Consider a function $f : Z \rightarrow Z$ that assigns the square of an integer to this integer.
- How can you write this function?

$$f(x) = x^2$$

- What is the domain of $f$ ? set of integers
- What is the codomain of $f$ ? set of integers
- What is the range of $f$ ?

set of nonnegative integers {0,1,4,9,..}

# Functions

- Example: Is f a function from **Z** to **R** if

  $f(n) = 1/(n^2 - 4)$

  f(2), f(-2) are not defined, division by zero

  Therefore not a function.

- Is f a function from **Z** to **R** if

  $f(n) = \sqrt{n^2+1}$

  For all integers, well defined real numbers.

  Therefore a function.

# Functions

- Example: Let R be the relation consisting of ordered pairs (Sarah, 22), (Jake, 23), (Stevens, 22), (Bob, 24) where each pair consists of a graduate student and the age of this student. What is the function that this relation determines? What is the domain, codomain, and range of this function?

Defines function f where f(Sarah) = 22, f(Jake) = 23, f(Stevens) = 22, and f(Bob) = 24

Domain: {Sarah, Jake, Stevens, Bob}

Codomain: The set of positive integers

Range: {22, 23, 24}

UTD

# Functions

- Two functions are equal

  - same domain

  - same codomain

  - map elements of their common domain

  - to the same elements in their common codomain

# Functions

- Two real valued functions with the same domain can be added and multiplied.
- Let $f_1$ and $f_2$ be functions from R to R.
- Then $f_1 + f_2$ and $f_1 f_2$ are also functions from R to R
- Specify their values at x

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \qquad (1)$$

$$(f_1 f_2)(x) = f_1(x) f_2(x) \qquad (2)$$

UTD

# Functions

- Example: Let $f_1$ and $f_2$ be the functions from R to R such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?
- $(f_1 + f_2)(x)$

$$= f_1(x) + f_2(x), \text{ definition of sum of functions}$$

$$= x^2 + (x - x^2)$$

$$= x$$

- $(f_1 f_2)(x)$ $= x^2 (x - x^2)$, definition of product of functions

$$= x^3 - x^4$$

# Functions

- Let f be a function from the set A to the set B
- S be the subset of A
- Image of S under the function f is the subset of B that consists of the images of the elements of S
- Image of S is denoted by f(S)

$$f(S) = \{t \mid \exists s \in S (t = f(s))\}$$

# Functions

- Example: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. What is the image of the subset $S$ of $A = \{b, c, d\}$

- The image of the subset $S = \{b, c, d\}$ is the set

$$f(S) = \{1, 4\}$$

# Functions

- A function f: one-to-one, or injective
- if and only if f(a) = f(b) implies that a = b for all a and b in the domain of f
- No value in the range is used by more than one value in the domain.

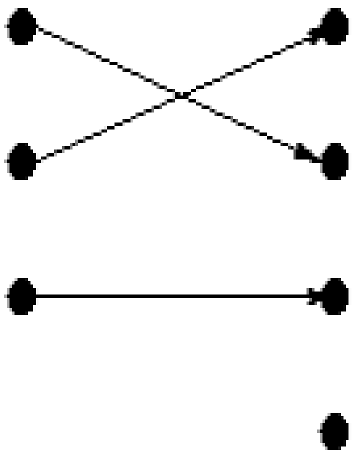
- In other words

$$\forall x\ \forall y\ (f(x) = f(y) \rightarrow x = y),$$

or using the contrapositive

$$\forall x\ \forall y\ (x \neq y \rightarrow f(x) \neq f(y))$$

# Onto Functions

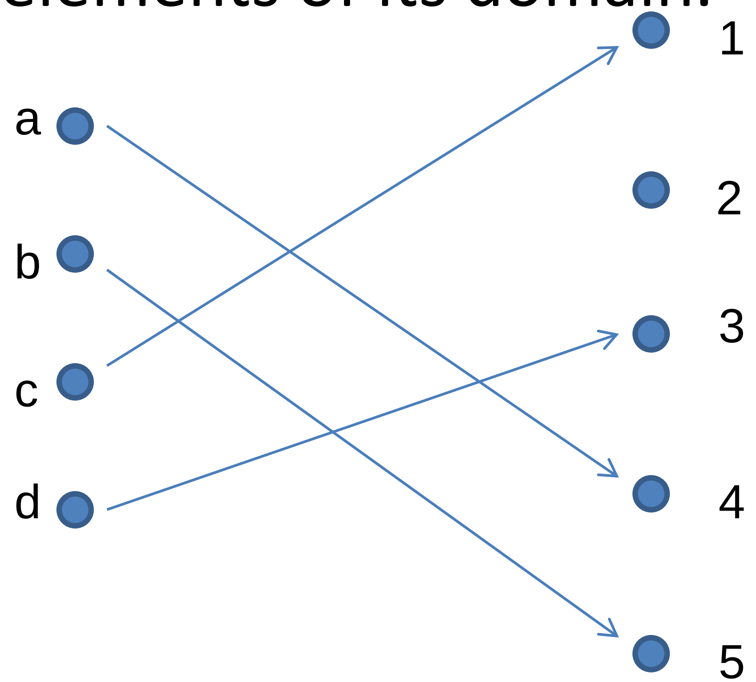


A one-to-one function

# Functions

- Example: Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with f(a) = 4, f(b) = 5, f(c) = 1, and f(d) = 3 is one-to-one.

The function f is one-to-one because f takes on different values at the four elements of its domain.



# Functions

- Example: Determine whether the function $f(n) = n^2 + 1$ is one-to-one from **Z** to **Z**.

At $n = 3$, $f(3) = f(-3)$

$= 10$

Therefore the function is not one-to-one

# Functions

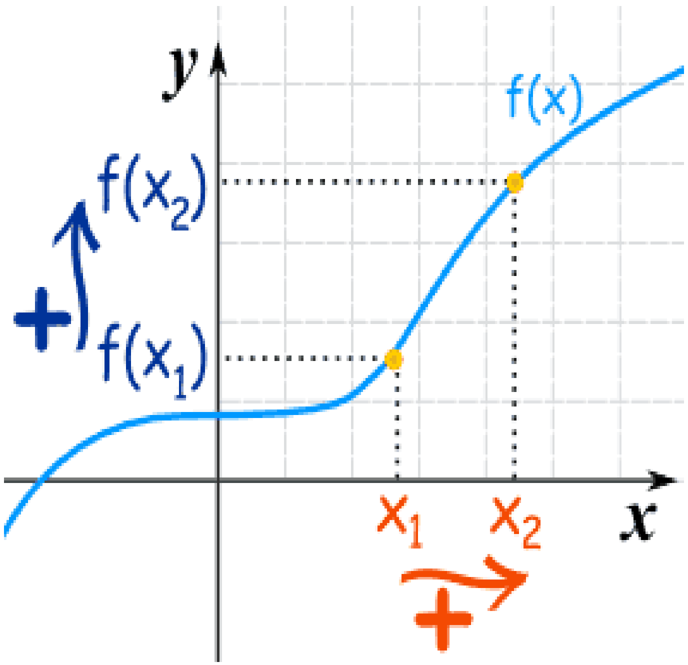
- Example: Is the function $f(x) = x^2$ from the set of integers to the set of integers one-to-one?
  - $1^2 = (-1)^2$ but $1 \neq -1$
  - NO
- Is the function $f(x) = x + 1$ one-to-one?
  - $(x + 1) \neq (y + 1)$ only when $x \neq y$
  - YES

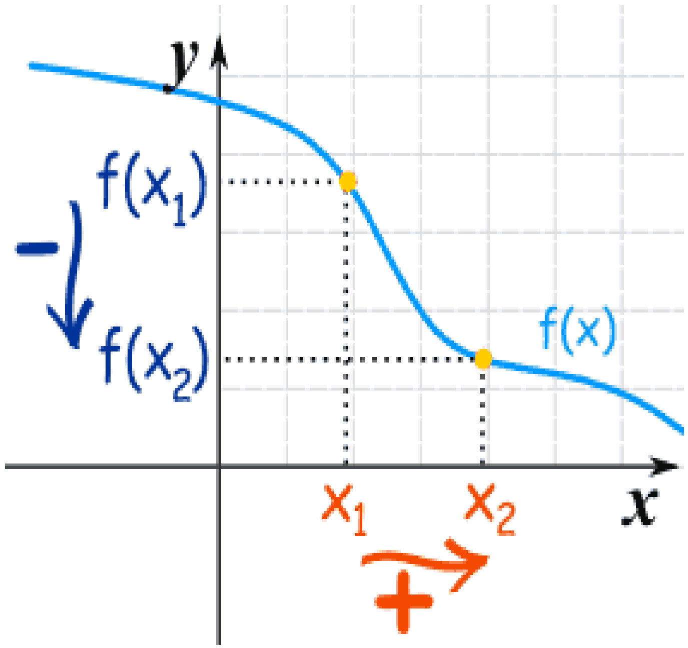
# Functions

- A function f whose domain and codomain are subsets of the set of real numbers
- Increasing if $f(x) \leq f(y)$, x and y in the domain of f
- Strictly increasing if $f(x) < f(y)$
- Decreasing if $f(x) \geq f(y)$
- Strictly decreasing if $f(x) > f(y)$
- Strictly decreasing or strictly increasing must be one-to-one

# Functions

- Increasing, Decreasing



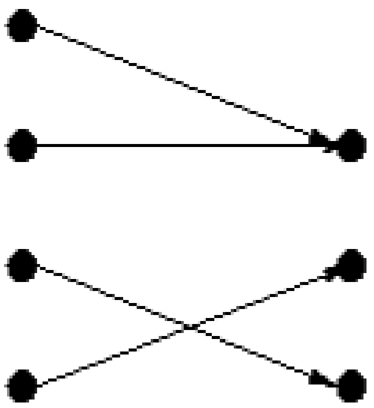



# Functions

- Every member x of the codomain is the image of some element of the domain.
- 'onto' functions
- A function f from A to B – onto or surjective
- if and only if for every element b € B, there is an element a € A with f(a) = b

- In other words, $\forall y\ \exists x\ (f(x) = y)$
- Codomain = range!

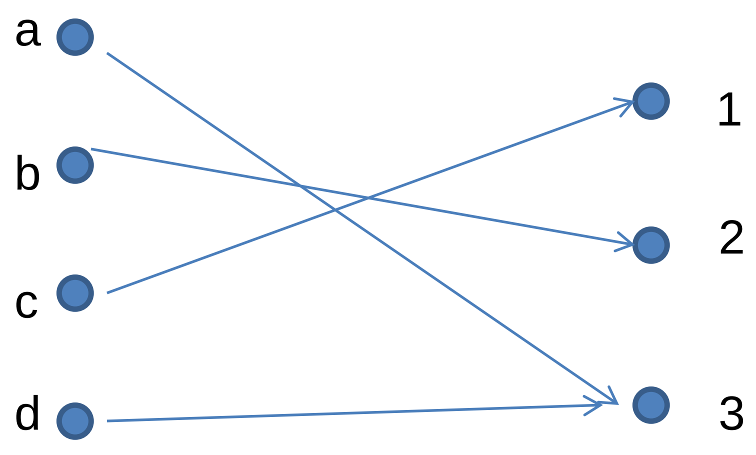
# Onto Functions



An onto function

# Functions

- Example: Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f(a) = 3, f(b) = 2, f(c) = 1, and f(d) = 3. Is f an onto function?
- All three elements of the codomain are images of elements in the domain, f is onto.



# Functions

- Example: Is the function $f(n) = n^2 + 1$ onto from **Z** to **Z**?

- The range cannot include any negative integer

  $n^2 + 1$ is always positive

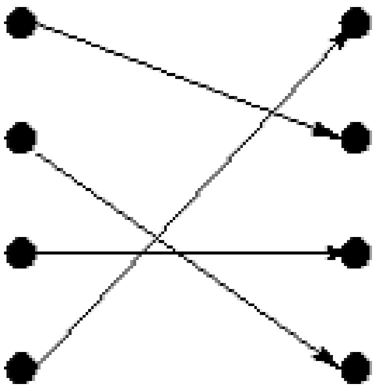
  Therefore the function is not onto

# Onto Functions

- Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?
  - Is it true that $\forall y \exists x (x^2 = y)$?
  - -1 is one of the possible values of $y$, but there does not exists an $x$ such that $x^2 = -1$
  - NO
- Is the function $f(x) = x + 1$ onto?
  - Is it true that $\forall y \exists x (x + 1 = y)$?
  - For every $y$, some $x$ exists such that $x = y - 1$.
  - YES

# Functions

- A function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto



A bijection

# Functions

- Identity Function

Let A be a set

The identity function on A is the function $\iota_A : A \rightarrow A$, where $\iota_A(x) = x$ for all x € A.

Function that assigns each element to itself

Function $\iota_A$ is one-to-one and onto

Identity function is a bijection

# Functions

- Example: Let f be the function from {a, b, c, d} to {1, 2, 3, 4} with f(a) = 4, f(b) = 2, f(c) = 1, and f(d) = 3. Is f a bijection?

Domain: {a, b, c, d}

Codomain: {1, 2, 3, 4}

a) It is one-on-one, no two values in the domain are assigned the same function value

b) It is onto, all four elements of the codomain are images of elements in the domain

Function f is a bijection.

# Functions

- Example: Is the function $f(x) = x^2 + 1$ a bijection from **R** to **R**?

- Range is the set of real numbers greater than or equal to 1
- It is not all of **R**, not an injection
- Therefore not a bijection

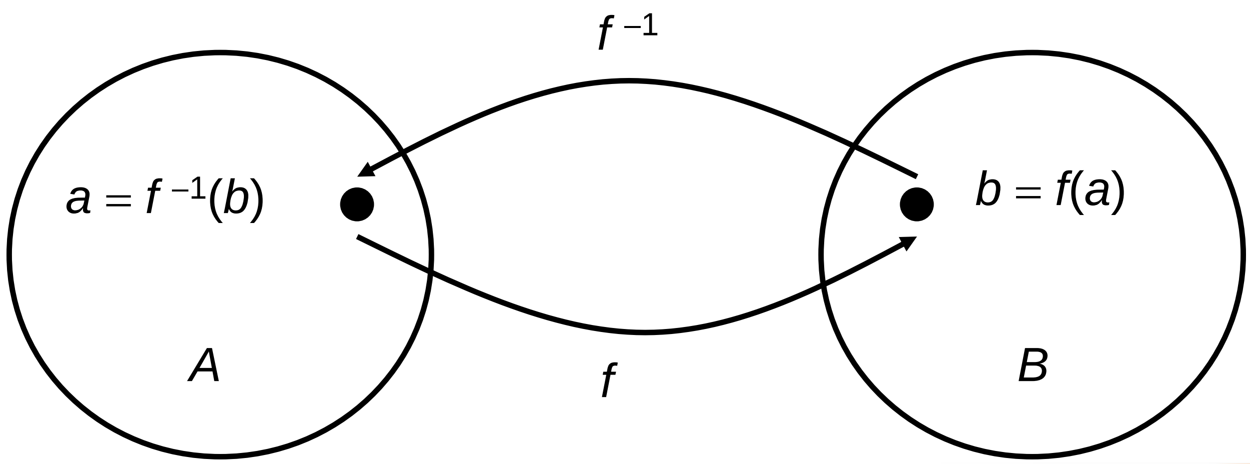
# Inverse Functions

- Inverse Function of f:

  Function that assigns to an element b belonging to B the unique element a in A such that f(a) = b.

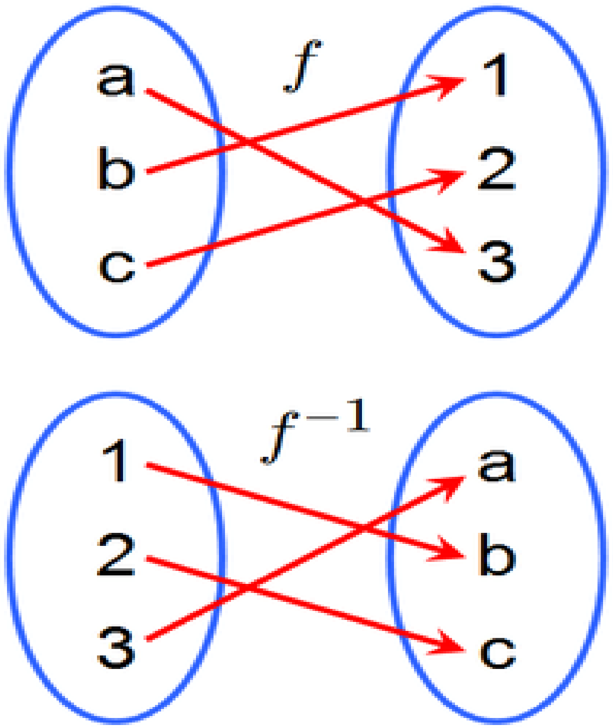
  Denoted by $f^{-1}$

  $f^{-1}(b) = a$ when $f(a) = b$



# Functions

- Inverse Functions



# F needs to be bijection

- If $f$ is not a bijection (one-to-one correspondence)
  - $f$ is not injective (one-to-one)
  - $f$ is not surjective (onto)

- Why can't we invert such a function?

  We cannot assign to each element $b$ in the codomain a unique element $a$ in the domain such that $f(a) = b$, because:
  - For some $b$ there is either
    - More than one $a$
    - No such $a$

# Inverse Functions

- Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ be a function with $f(x) = x + 1$
- Is $f$ invertible? Is $f$ a bijection?
  - Is $f$ one-to-one? YES
  - Is $f$ onto? YES
  - So $f$ is a one-to-one correspondence and is therefore invertible.
- Then, what is its inverse?

  $f(y) = y - 1$

# Functions

- Example: Let f be the function from {a, b, c} to {1, 2, 3} such that f(a) = 2, f(b) = 3, and f(c) = 1. Is f invertible and if yes, what is its inverse?

  Domain: {a, b, c}

  Codomain: {1, 2, 3}

- Function f is a one-to-one correspondence, invertible
- Inverse function reverses the correspondence of f,

  $f^{-1}(1) = c$, $f^{-1}(2) = a$, $f^{-1}(3) = b$

  No two values in the domain are assigned the same function value

# Composition

- Let g be a function from the set A to the set B
- Let f be a function from the set B to the set C
- Composition of the functions f and g, denoted by f o g, is defined by:

$$(f \circ g)(a) = f(g(a))$$

- f o g – function that assigns to the element a of A the element assigned by f to g(a)
- First apply the function g to a to obtain g(a), then apply the function f to the result g(a) to obtain the composition

# Functions

- Example: Let $g$ be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let $f$ be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of $f$ and $g$, and what is the composition of $g$ and $f$?

$f \circ g =$ $(f \circ g)(a) = f(g(a)) = f(b) = 2$

$(f \circ g)(b) = f(g(b)) = f(c) = 1$

$(f \circ g)(c) = f(g(c)) = f(a) = 3$

What about $g \circ f$?

Not defined, range of $f$ is not a subset of the domain of $g$

# Functions

- Graphs of Functions

Pictorial representations

Let f be a function from the set A to the set B

The graph of the function f is the set of ordered pairs

$$\{(a, b) \mid a \text{ € } A \text{ and } f(a) = b\}$$

UTD

# Functions

- Example: Display the graph of the function $f(x) = 2x + 1$ from the set of integers to the set of integers.
- The graph of f is the set of ordered pairs of the form $(x, 2x + 1)$, where n is an integer.

$x = 0, \quad y = 1$

$x = 1, \quad y = 3$

$x = -1, \quad y = -1$

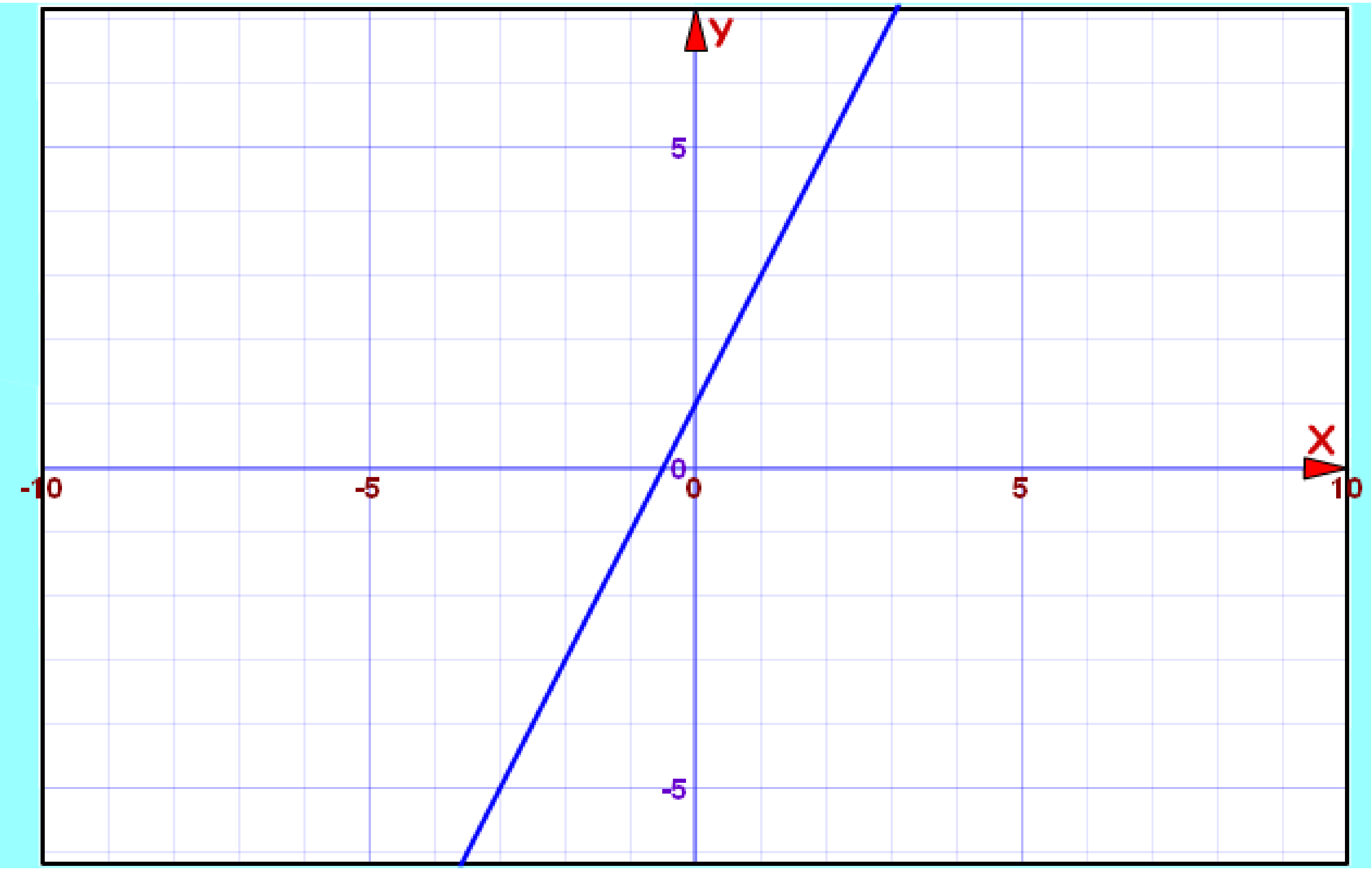
$x = 2, \quad y = 5$

$x = -2, \quad y = -3$

$x = 3, \quad y = 7$

$x = -3, \quad y = -5$

# Functions

- 

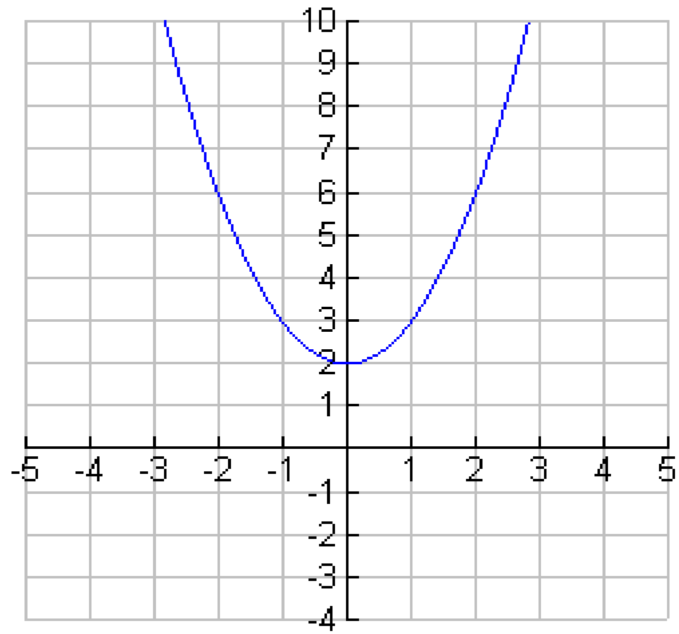
# Functions

- Example: Display the graph of the function f(x) = $x^2$+ 1, domain is real numbers.

$$y = x^2 + 2$$

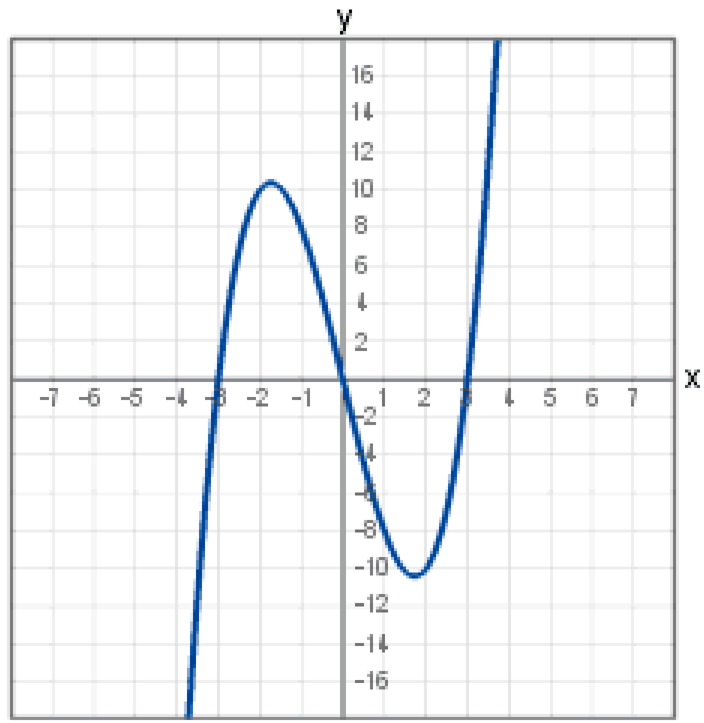
Domain: $\mathbb{R}$ (all real numbers)

Range: $y \geq 2$



# Functions

- Example: Display the graph of the function $f(x) = x^3 - 9x$, domain is real numbers.



# Functions

- Some Important Functions

Let x be a real number.

Floor Function: Rounds x down to the closest integer less than or equal to x

Ceiling Function: Rounds x up to the closest integer greater than or equal to x

Usage: Analysis of the number of steps used by procedures to solve problems

# Functions

- Floor function: Assigns to the real number x

  The largest integer that is less than or equal to x

- Ceiling function: Assigns to the real number x

  The smallest integer that is greater than or equal to x

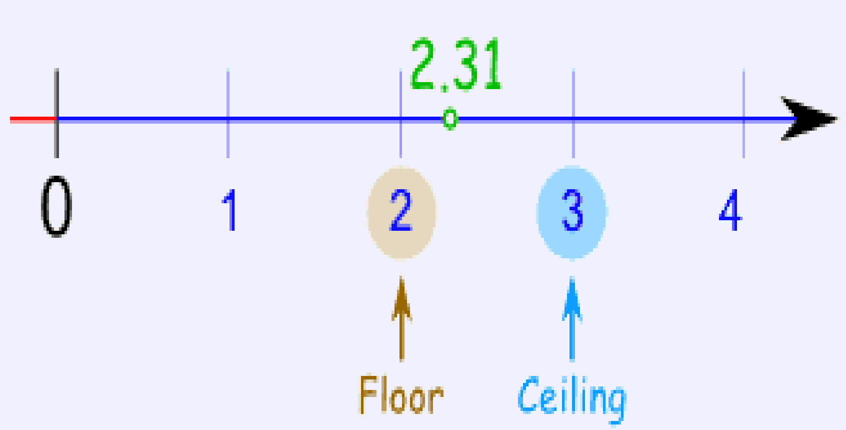
Denoted by $\lfloor x \rfloor$ $\lceil x \rceil$

floor(x) ceil(x)

# Functions

- Example:

Example: What is the floor and ceiling of 2.31?




The Floor of 2.31 is **2**

The Ceiling of 2.31 is **3**

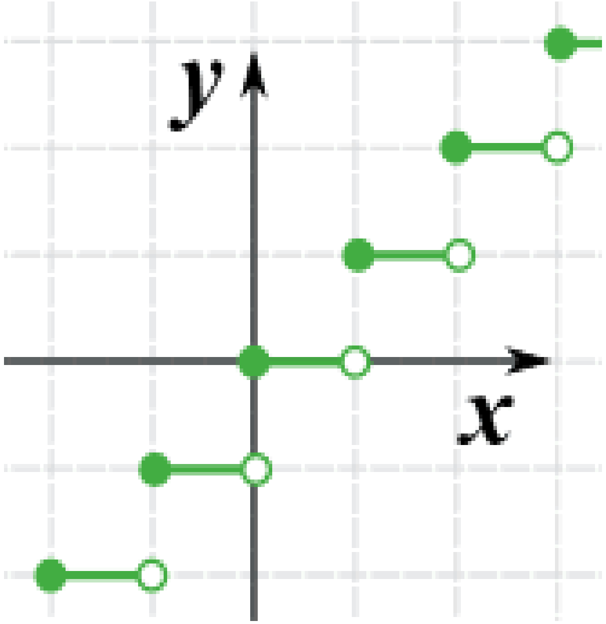
# Functions

- Example:

| x | Floor | Ceiling |
|---|---|---|
| -1.1 | -2 | -1 |
| 0 | 0 | 0 |
| 1.01 | 1 | 2 |
| 2.9 | 2 | 3 |
| 3 | 3 | 3 |

# Functions

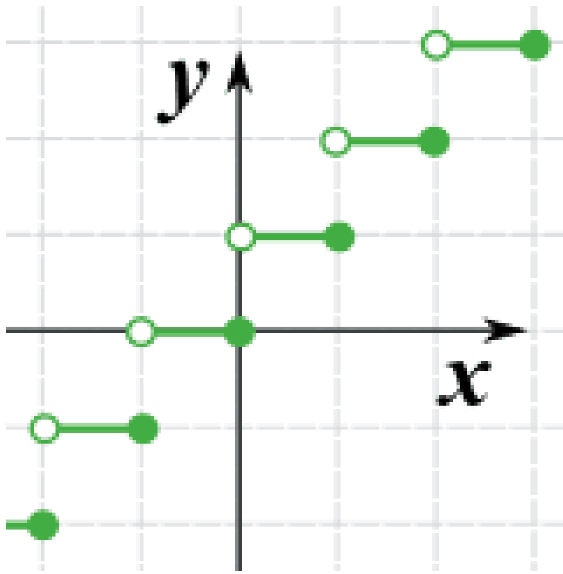
- Graph: Step function, solid dot – includes
- open dot – does not include, x = 2, y = 2




The Floor Function

# Functions

x = 1, y = 1




The Ceiling Function

# Practical Examples

- Functional Programming Languages

Haskell

Scheme

PERL – Scripting language