Discrete Mathematics for Computing

Ch 10.2: Graph Terminology and Special Types of Graphs

- Basic Terminology

  Goal: Introduce graph terminology in order to further classify graphs

- Definition 1: Two vertices u and v in an undirected graph G

  - are called adjacent (or neighbors) in G
  - if {u,v} is an edge of G
  - If e = {u,v}, the edge e is called incident with the vertices u and v
  - The edge e is also said to connect u and v
  - The vertices u and v are called endpoints of the edge {u,v}

Graph Terminology and Special Types of Graphs

- Degree of a vertex in an undirected graph

- Number of edges incident with it

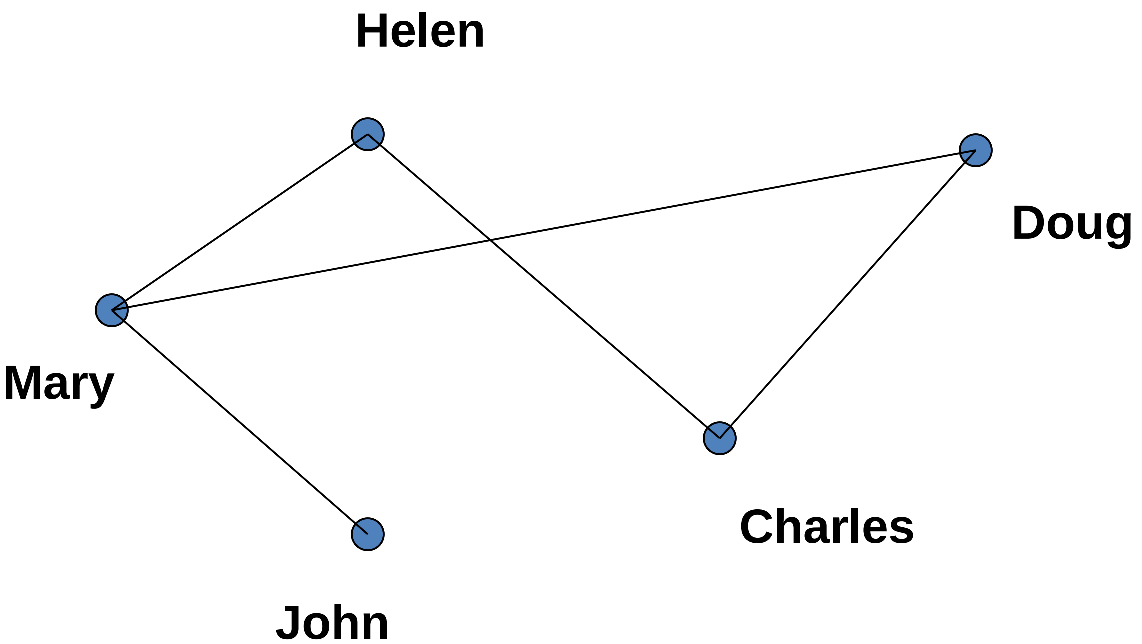
- Loop at a vertex contributes twice to the degree of that vertex

- Degree of vertex ‘v’

  deg(v)

Graph Terminology and Special Types of Graphs

- Example: What are the degrees of acquaintance graph?

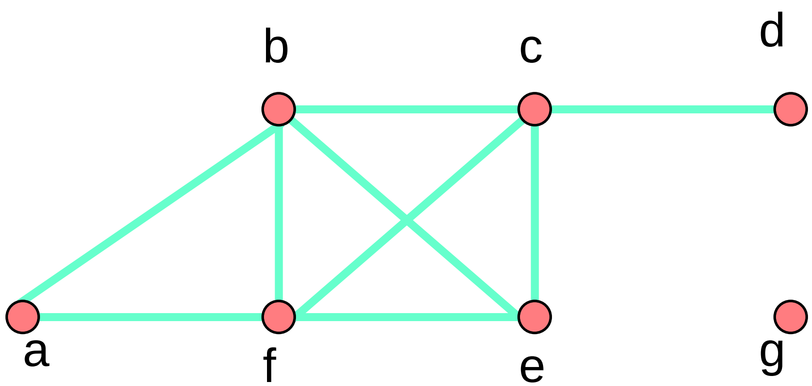


Solution: deg (Mary) = 3, deg(John) = 1, deg(Charles)=2

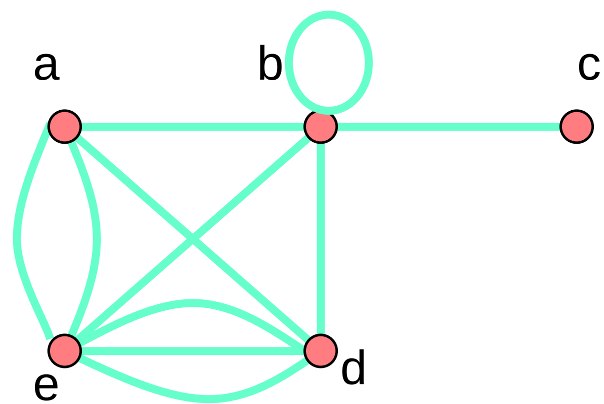
deg(Doug)= 2 and deg(Helen) = 2

Graph Terminology and Special Types of Graphs

- Example: What are the degrees of the vertices in the graphs G and H?



G



H

Solution:

$$In\ G \begin{cases} deg(a) = 2 \\ deg(b) = deg(c) = deg(f) = 4 \\ deg(d) = 1 \\ deg(e) = 3 \\ deg(g) = 0 \end{cases} \qquad In\ H \begin{cases} deg(a) = 4 \\ deg(b) = deg(e) = 6 \\ deg(c) = 1 \\ deg(d) = 5 \end{cases}$$

Graph Terminology and Special Types of Graphs

- Theorem 1: The handshaking theorem

Let G = (V,E) be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} \deg(v).$$

Sum of the degrees of the vertices is twice the number of edges

- Corollary: An undirected graph has an even number of vertices of odd degree.

$$2e = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

Graph Terminology and Special Types of Graphs

- Example: How many edges are there in a graph with ten vertices each of degree 6 ?

Solution:

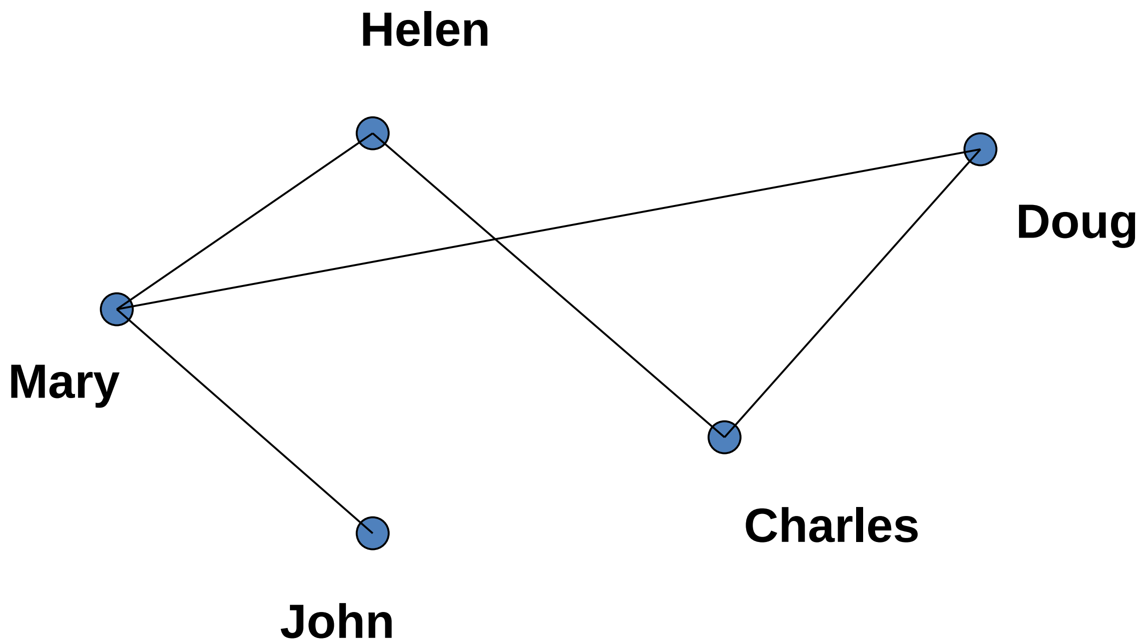
Since the sum of the degrees of the vertices is

$6 * 10 = 60$

$\Rightarrow 2e = 60$

Therefore, $e = 30$

Graph Terminology and Special Types of Graphs



- deg (Mary) = 3, deg(John) = 1, deg(Charles)=2, deg(Doug)= 2 and deg(Helen) = 2.

  The sum of all degrees is 10

  There are 5 edges.

  There are two vertices with odd degree.

Graph Terminology and Special Types of Graphs

- Definition 3: When (u,v) is an edge of the graph G with directed edges

- u is said to be adjacent to v

- and v is said to be adjacent from u

- The vertex u is called the initial vertex of (u,v)

- and v is called the terminal or end vertex of (u,v)

- The initial vertex and terminal vertex of a loop are the same

Adjacent Vertices in Directed Graphs

(u, v)

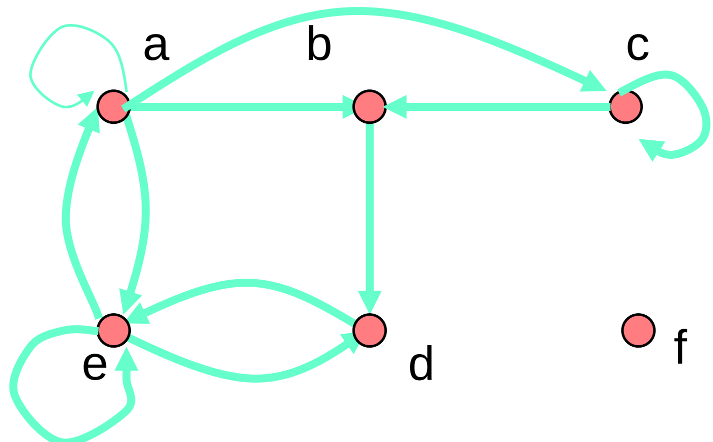
initial vertex　　terminal vertex

Degree of a Vertex

- *In-degree* of a vertex *v*
  - The number of vertices *adjacent to v* (the number of edges with *v* as their terminal vertex
  - Denoted by $\deg^{-}(v)$
- *Out-degree* of a vertex *v*
  - The number of vertices *adjacent from v* (the number of edges with *v* as their initial vertex)
  - Denoted by $\deg^{+}(v)$
- A loop at a vertex contributes 1 to both the in-degree and out-degree.

Graph Terminology and Special Types of Graphs

- Example: Find the in-degree and the out-degree of each vertex in the graph G.



The in-degree of G are: $deg^-(a) = 2$, $deg^-(b) = 2$, $deg^-(c) = 2$, $deg^-(d) = 2$, $deg^-(e) = 3$, and $deg^-(f) = 0$

The out-degree of G are: $deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$, $deg^+(d) = 1$, $deg^+(e) = 3$, and $deg^+(f) = 0$

Theorem 3

- The sum of the in-degrees of all vertices in a digraph = the sum of the out-degrees = the number of edges.
- Let $G = (V, E)$ be a graph with directed edges. Then:

$$\sum_{v\in V} \deg^-(v) = \sum_{v\in V} \deg^+(v) = |E|$$

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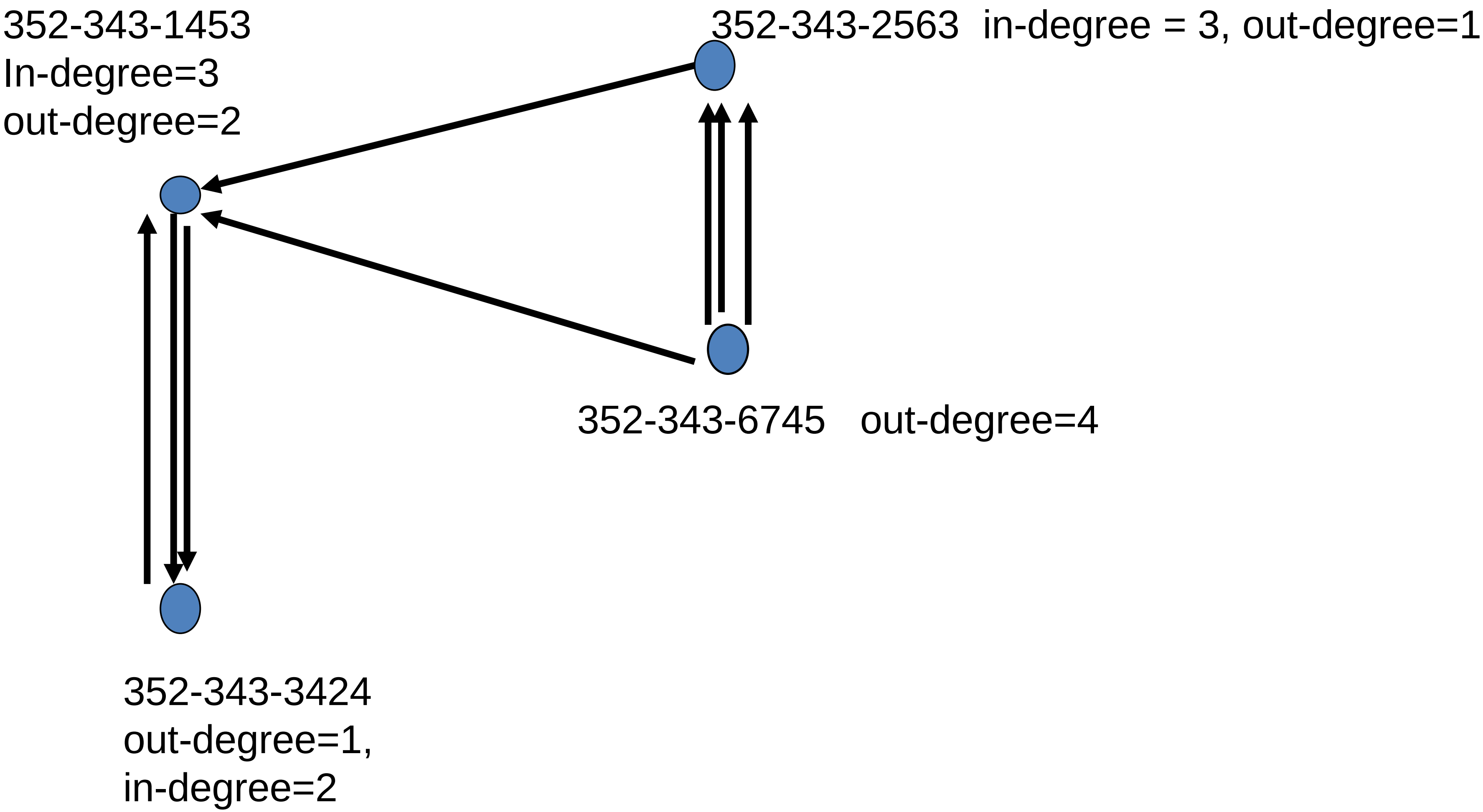
Graph Terminology and Special Types of Graphs



Graph Terminology and Special Types of Graphs

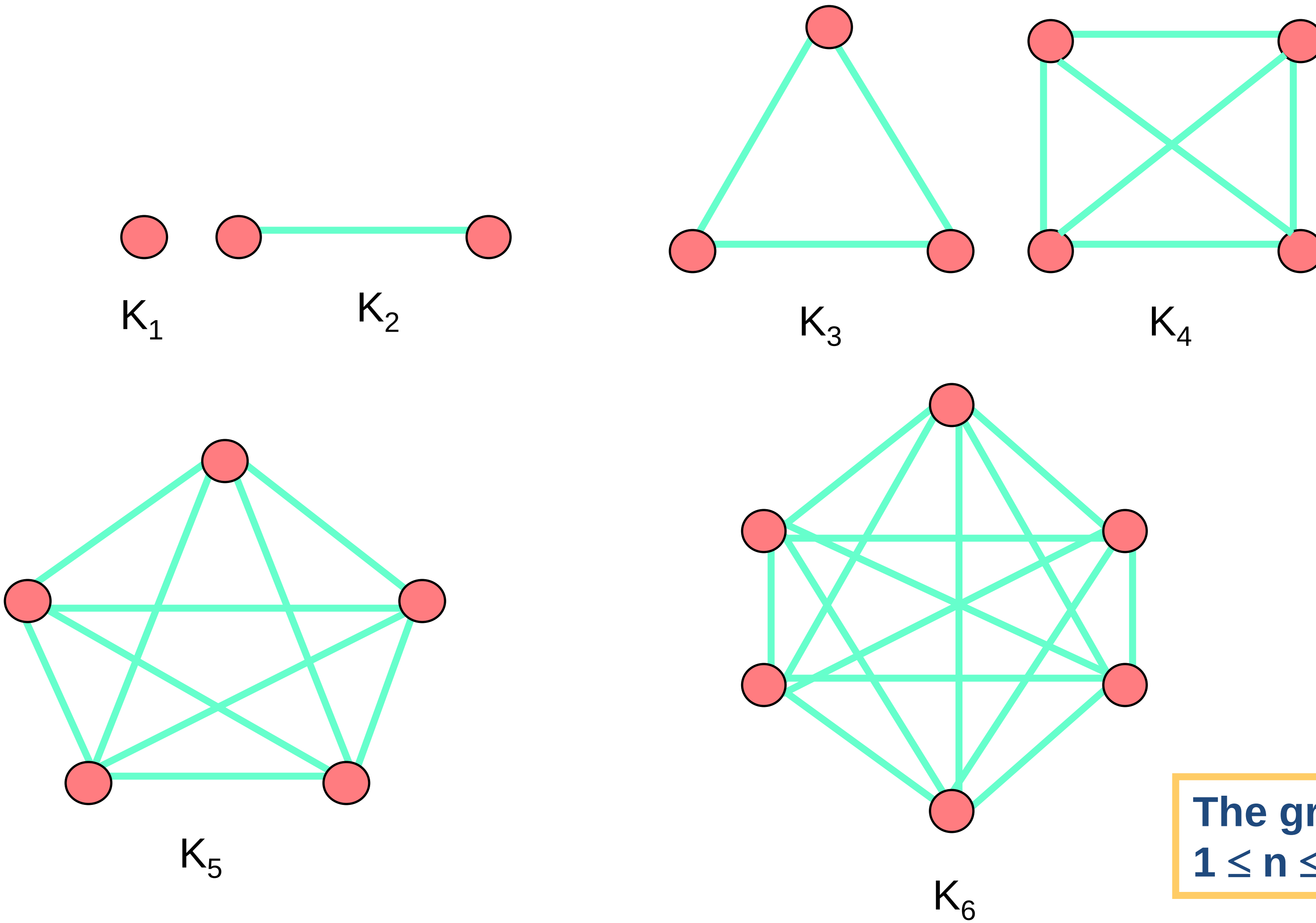
- Some special simple graphs

  Complete graph - They are denoted by K_n

  They contain exactly one edge

  between each pair of distinct vertices

Graph Terminology and Special Types of Graphs



The graphs K_n for $1 \leq n \leq 6$

Graph Terminology and Special Types of Graphs

- Cycles

They are denoted by $C_n (n \geq 3)$

- they consist of 'n' vertices $v_1, v_2, \ldots, v_n$

- and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_n, v_{n-1}\}$ and $\{v_n, v_1\}$

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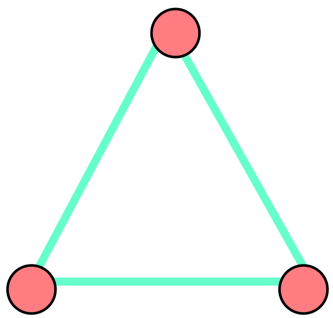
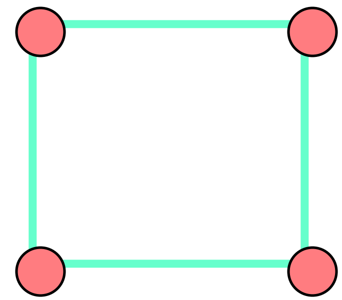
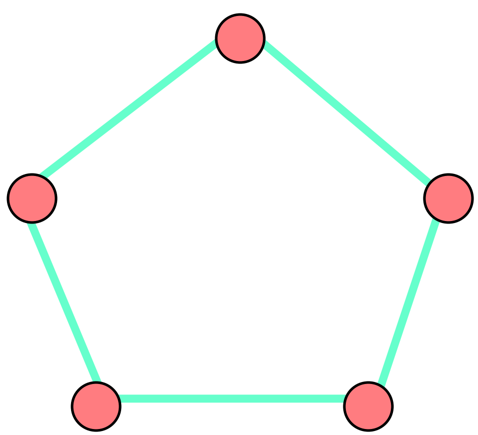
Graph Terminology and Special Types of Graphs



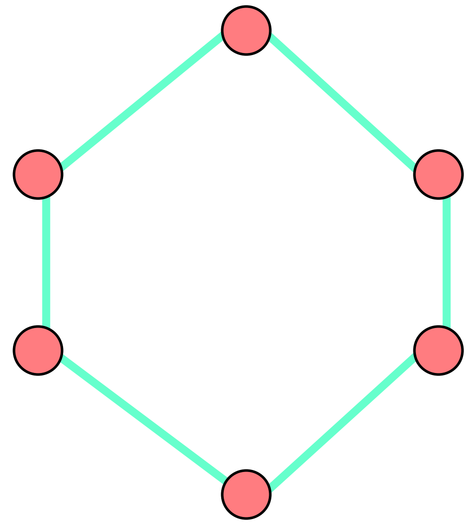
C_3



C_4



C_5



C_6

The cycles C_3, C_4, C_5 & C_6

Graph Terminology and Special Types of Graphs

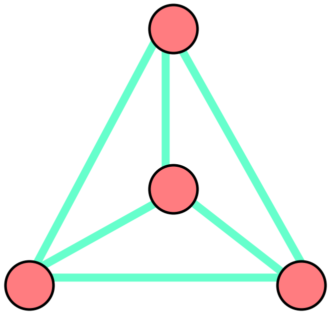
- Wheels

They are denoted by W_n

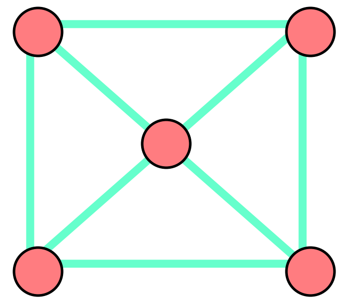
-obtained by adding a vertex to the graphs C_n

- and connect this vertex to all vertices

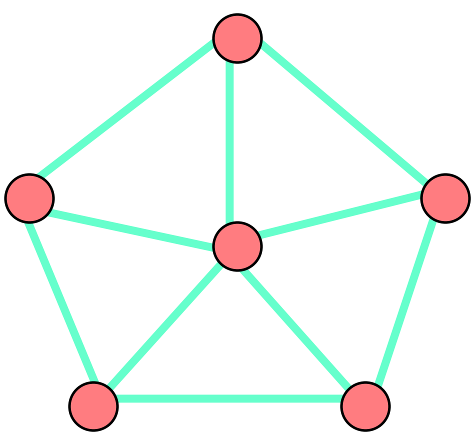
Graph Terminology and Special Types of Graphs



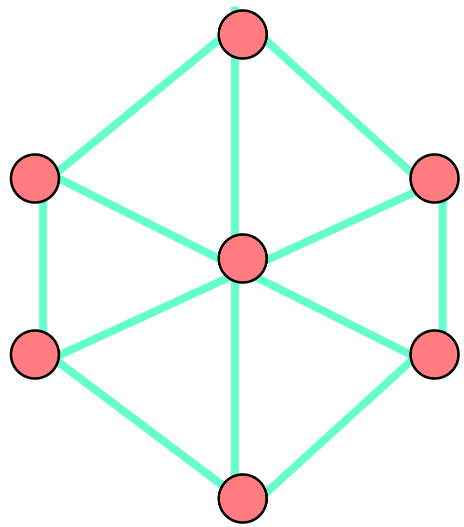
W_3



W_4



W_5



W_6

The Wheels W_3, W_4, W_5 & W_6

Graph Terminology and Special Types of Graphs

- n-cubes

-They are denoted by Q_n

- they are graphs that have vertices

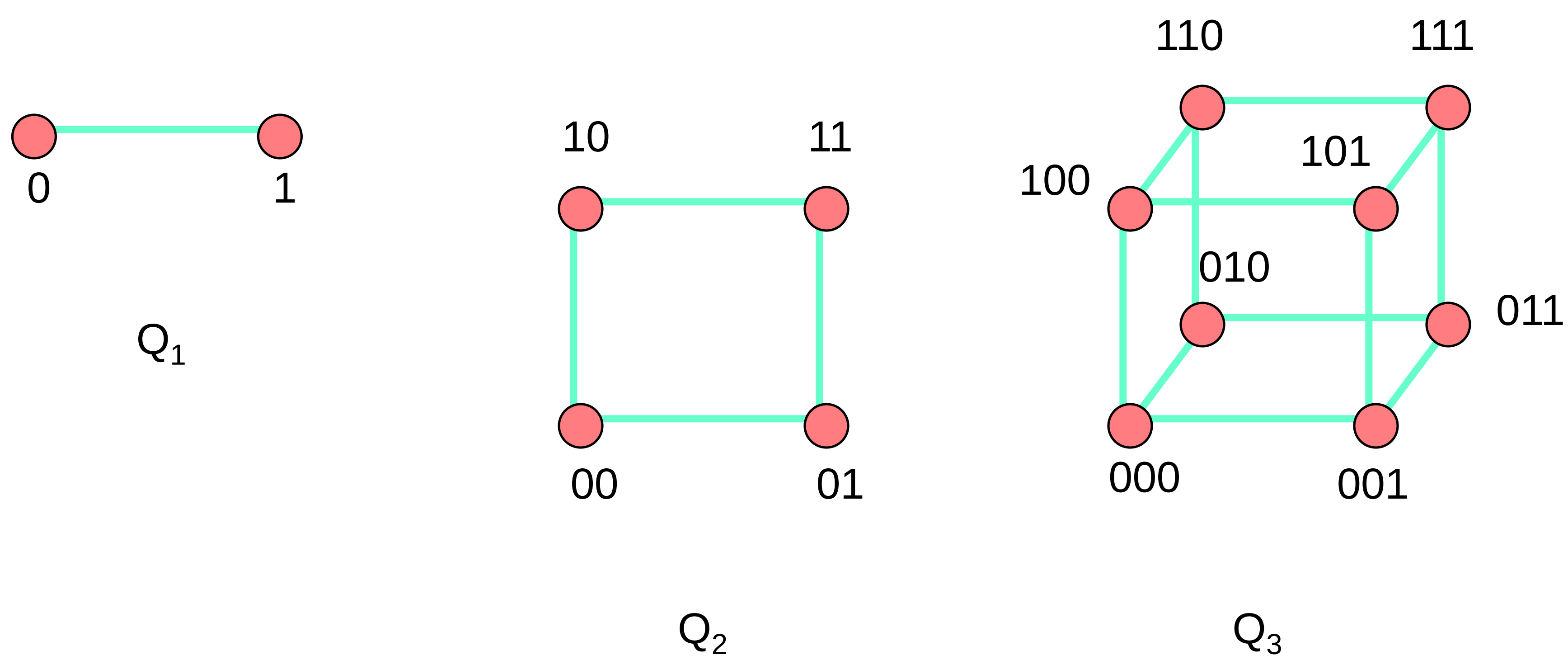
- representing the 2^n bit strings of length n

Two vertices are adjacent if and only if

- the bits strings that they represent differ

- in exactly one bit position

Graph Terminology and Special Types of Graphs



The n-cube Q_n for n = 1, 2, and 3.

Graph Terminology and Special Types of Graphs

- Bipartite graph

  Definition 5: A simple graph is called bipartite

  - if its vertex set V can be partitioned

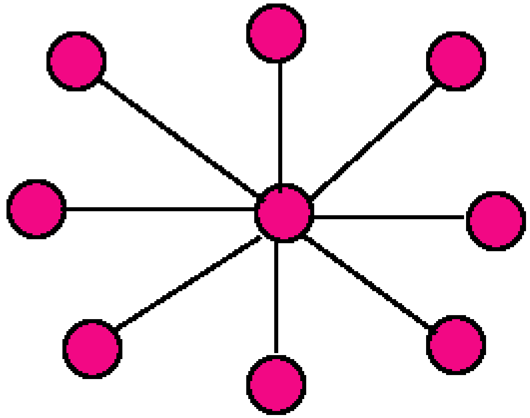
  - into 2 disjoint sets V_1 and V_2

  - such that every edge in the graph

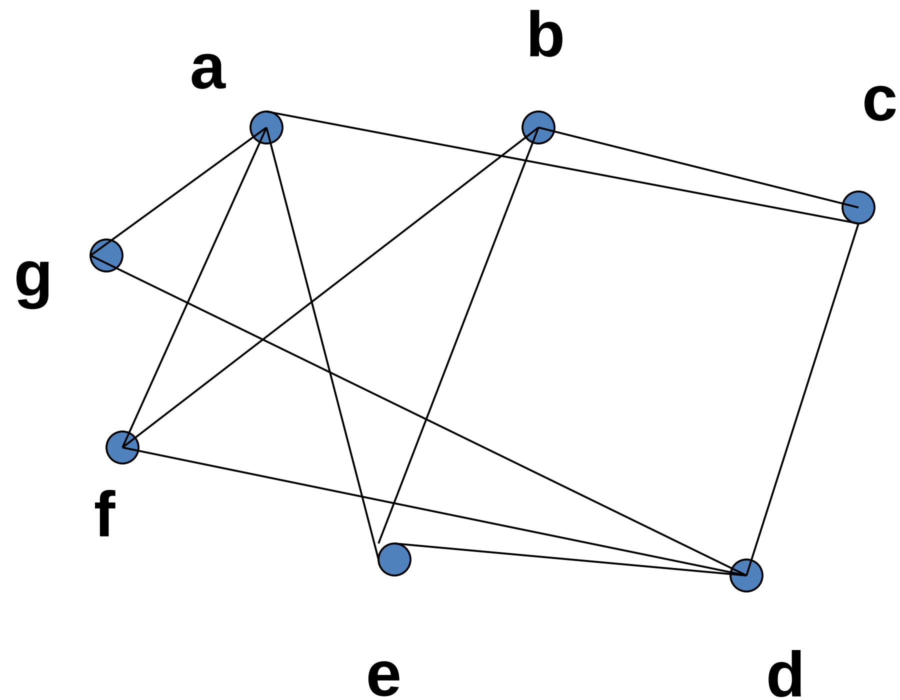
  - connects a vertex in V_1 and a vertex in V_2

  - so that no edge in G connects either 2 vertices in V_1 or 2 vertices in V_2

Graph Terminology and Special Types of Graphs



$K_{1,8}$

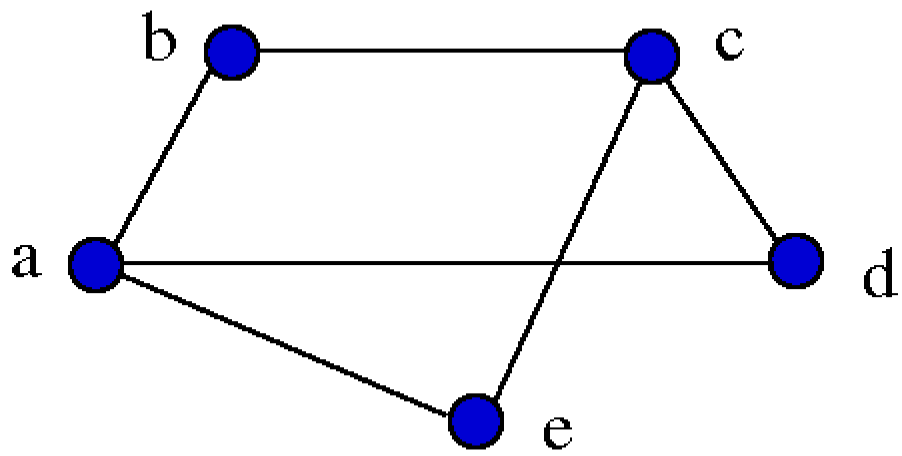


{a, b, d }, {c, e, f, g}

$K_{3,4}$

Graph Terminology and Special Types of Graphs

- Is the following graph bipartite?

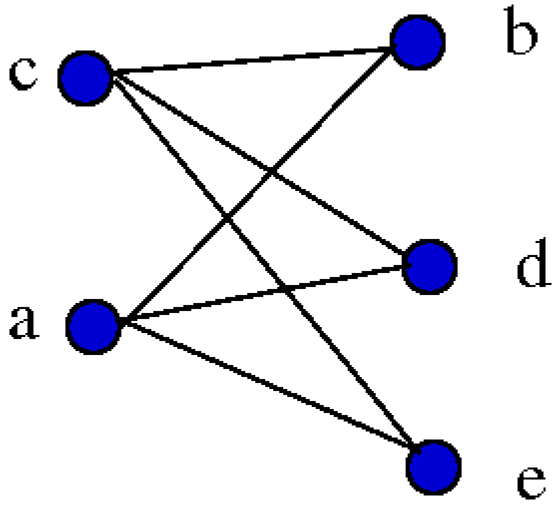


If a is in V1, then, b, d, e must be in V2

Then, c is in V1 and there is no inconsistency

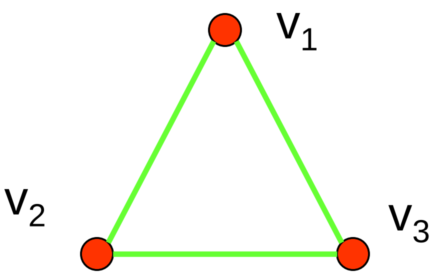
Graph Terminology and Special Types of Graphs

- So we can rearrange the graph as follows, bipartite



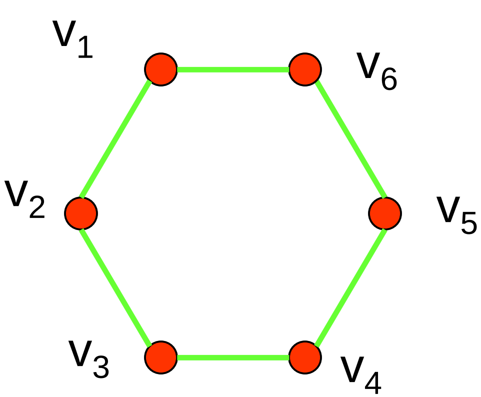
Graph Terminology and Special Types of Graphs

- Example: Is C_3 bipartite?

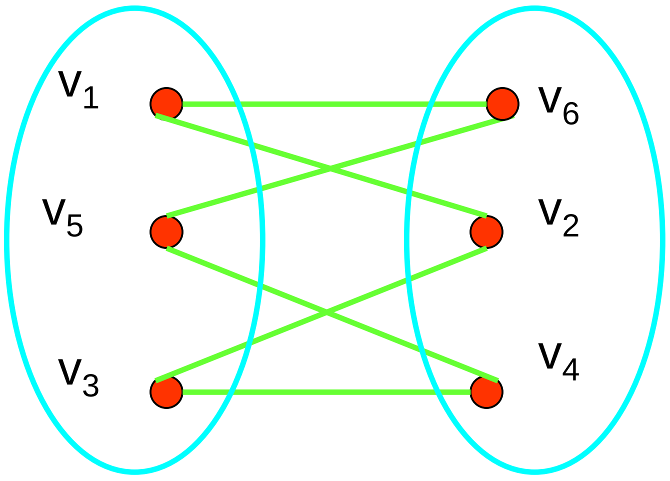


No, because there is no way to partition the vertices into two sets so that there are no edges with both endpoints in the same set.

- Example: Is C_6 bipartite?



Yes, because we can display C_6 like this:



Graph Terminology and Special Types of Graphs

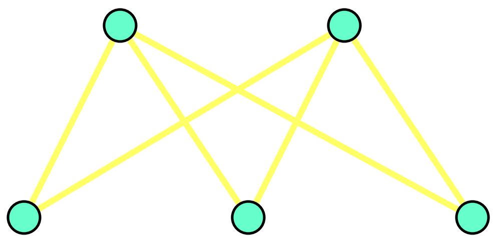
- Complete bipartite graphs

- they are denoted by $K_{m,n}$

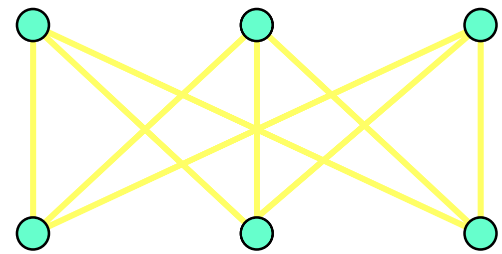
- Their vertices set is partitioned into 2 subsets of m and n vertices, respectively

- every vertex of the first set is connected to every vertex of the second set

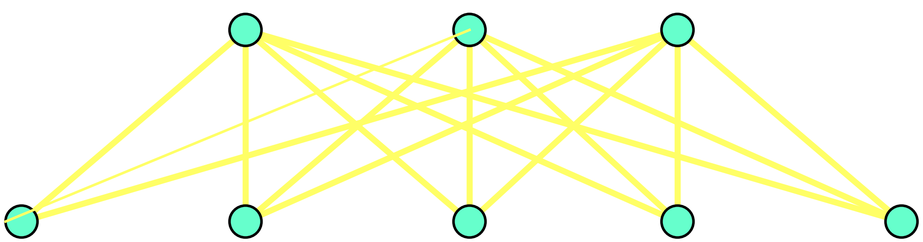
Graph Terminology and Special Types of Graphs



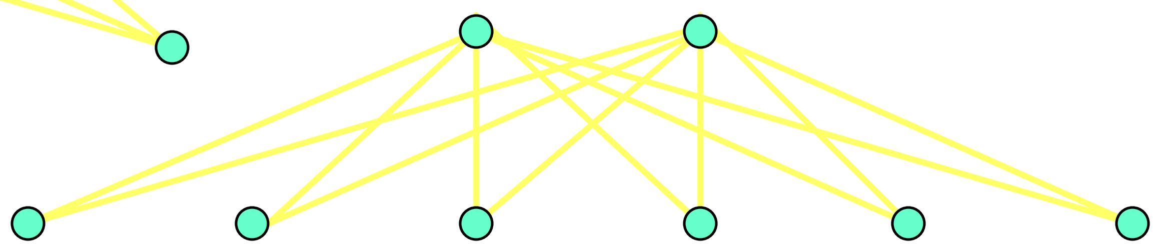
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$

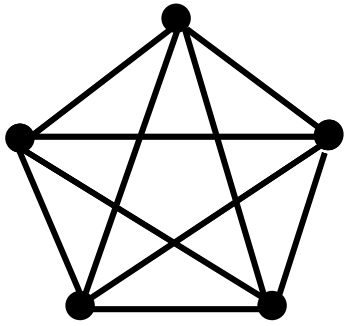


$K_{2,6}$

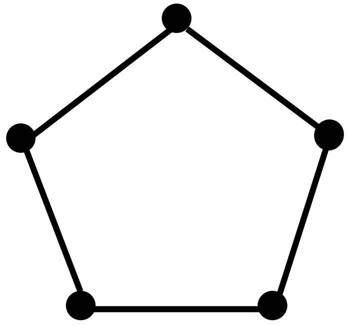
Some complete bipartite graphs

Subgraph

- A *subgraph* of a graph $G = (V,E)$ is a graph $H = (W,F)$ where $W \subseteq V$ and $F \subseteq E$.



K_5

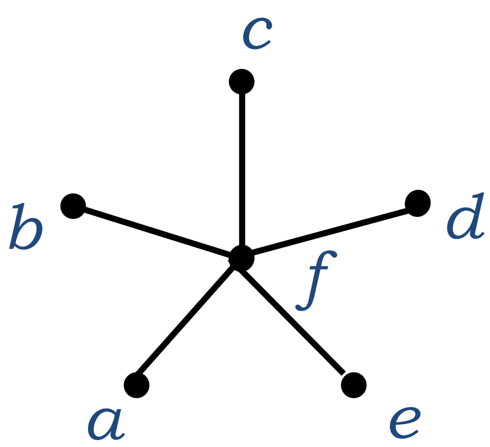


C_5

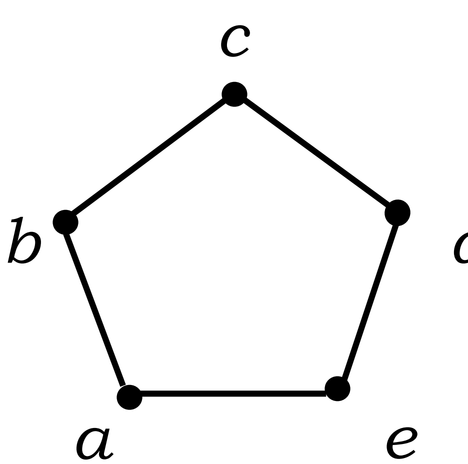
Is C_5 a subgraph of K_5?

Union

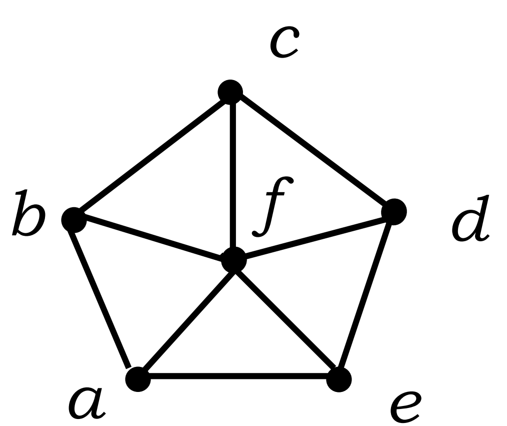
- The *union* of 2 simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The union is denoted by $G_1 \cup G_2$.



S_5



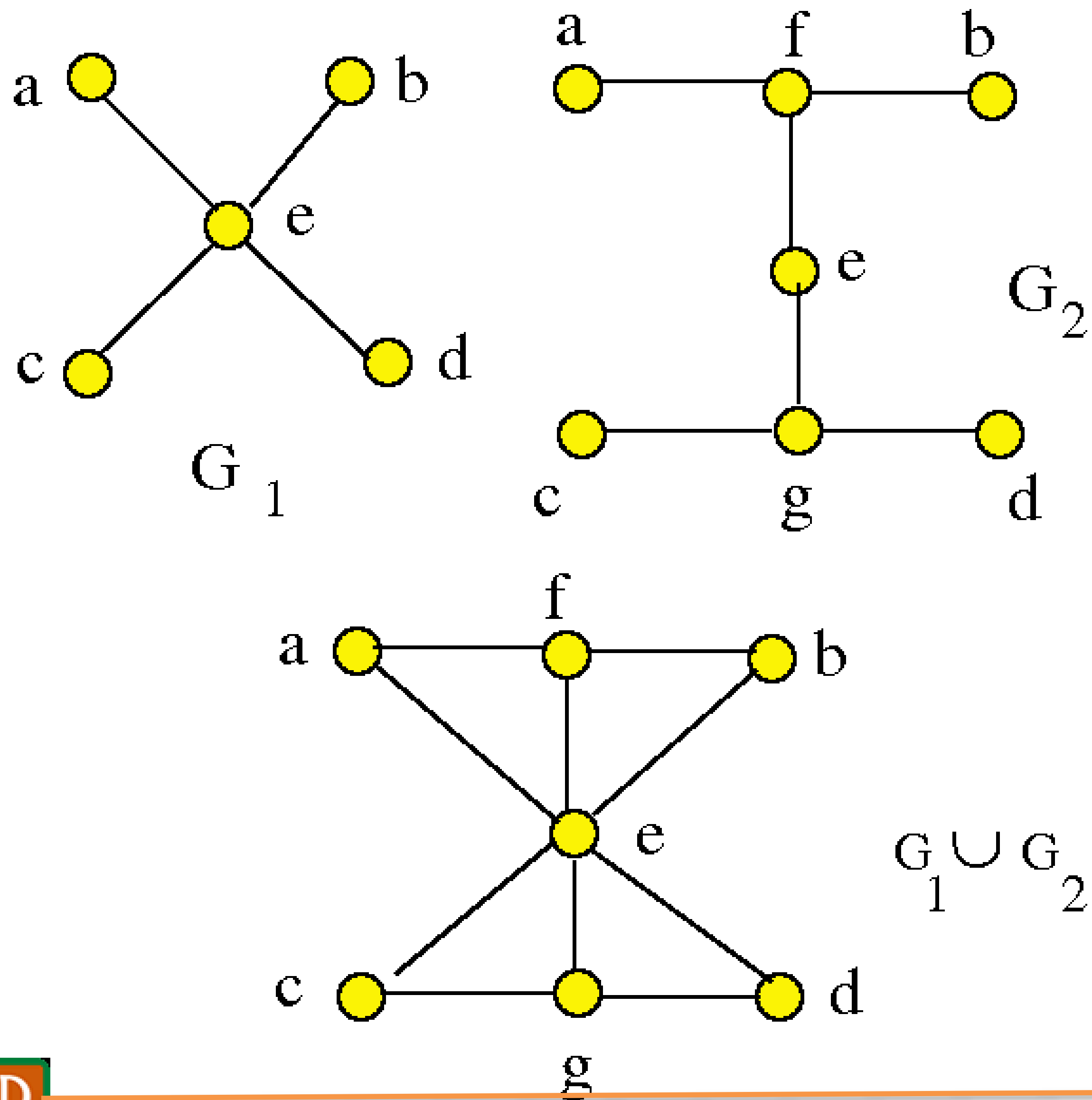
C_5



W_5

$$S_5 \cup C_5 = W_5$$

Graph Terminology and Special Types of Graphs



Graph Terminology and Special Types of Graphs

- Some applications of special types of graphs
- Local area network

Goal:

- Connecting computers as well as peripheral devices in a building using a local area network topology
- Some of these networks are based on a star topology, where all devices are connected to a central control device
- The star topology is equivalent to a $K_{1,n}$ complete bipartite graph

Graph Terminology and Special Types of Graphs

- Other local area networks use a ring topology $\Leftrightarrow$ C_n graphs

- Finally, the hybrid topology which is equivalent to a W_n graph is also used

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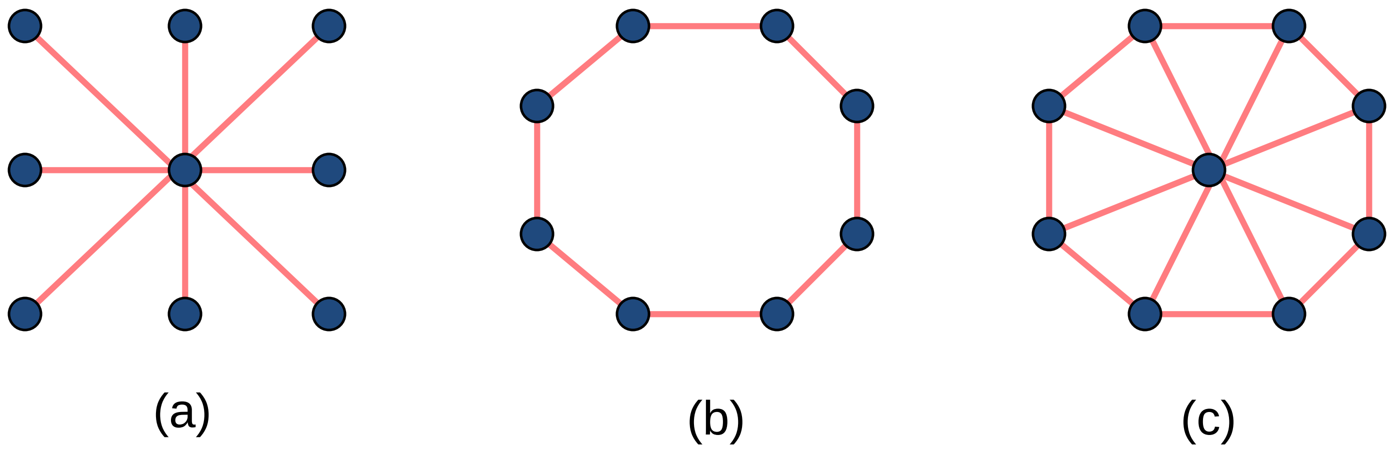
Graph Terminology and Special Types of Graphs



Star, ring, and hybrid topologies for local area networks