MATH 2418: Linear Algebra

Assignment 9

Due: March 30, 2016 | Term: Spring, 2016

Suggested problems(do not turn in): Section 4.8: 1, 3, 5, 7, 9, 13, 15. Section 4.9: 13, 15, 17, 19, 21, 23, 39.

1. [10 points] (2 points each) Use matrix multiplication to:

   (a) Find the reflection of (a,b) about the y-axis.

   Solution:

   $$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$$

   (b) Find the reflection of (a,b,c) about the xz-plane.

   Solution:

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -b \\ c \end{bmatrix}$$

   (c) Find the orthogonal projection of (a,b) onto the the y-axis.

   Solution:

   $$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

   (d) Find the orthogonal projection of (a,b,c) onto the yz-plane.

   Solution:

   $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix}$$

   (e) Find the image of the nonzero vector $\mathbf{v} = (v_1, v_2)$ when it is rotated about the origin through a negative angle $-\alpha$.

   Solution:

   $$\begin{bmatrix} cos(-\alpha) & -sin(-\alpha) \\ sin(-\alpha) & cos(-\alpha) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 cos(-\alpha) - v_2 sin(-\alpha) \\ v_1 sin(-\alpha) + v_2 cos(-\alpha) \end{bmatrix} = \begin{bmatrix} v_1 cos\alpha + v_2 sin\alpha \\ -v_1 sin\alpha + v_2 cos\alpha \end{bmatrix}$$

2. [10 points] Consider the following matrices, where R is the reduced row-echelon form of A:

$$A = \begin{bmatrix} 2 & 0 & -4 & 1 & 3 \\ 1 & 3 & 1 & 3 & 7 \\ 0 & 2 & 2 & -2 & 0 \\ -1 & 1 & 3 & 4 & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

Determine the following values without completing the solution to any system of equations.

(a) What is $\text{rank}(A)$?

Solution:
Count the leading ones in R to get $\text{rank}(A) = 3$.

(b) What is $\text{rank}(A^{\mathrm{T}})$?

Solution:
The rank of the transpose, A^{T}, is the same as for A: $\text{rank}(A^{\mathrm{T}}) = \text{rank}(A) = 3$.

(c) What is $\text{nullity}(A)$?

Solution:
The nullity of A is the number of columns of A minus $\text{rank}(A) = 5-3 = 2$, so $\text{nullity}(A) = 2$.

(d) What is $\text{nullity}(A^{\mathrm{T}})$?

Solution:
The number of columns of A^{T} minus $\text{rank}((^{\mathrm{T}}A)$ is the number of rows of A minus the rank of A: $\text{nullity}(A^{\mathrm{T}}) = 4 - \text{rank}(A) = 4 - 3 = 1$. Note that this is the same as the number of zero rows in R. Again, $\text{nullity}(A^{\mathrm{T}}) = 1$.

(e) What is the dimension of the row space of A?

Solution:
We have $\dim(\text{row}(A)) = \text{rank}(A) = 3$.

(f) What is the dimension of the row space of A^{T}?

Solution:
Again, $\dim(\text{row}(A^{\mathrm{T}})) = \text{rank}(A^{\mathrm{T}}) = 3$.

(g) What is the dimension of the column space of A?

Solution:
We have $\dim(\text{col}(A)) = \text{rank}(A) = 3$.

(h) What is the dimension of the column space of A^{T}?

Solution:
Again, $\dim(\mathrm{col}(A^{\mathrm{T}})) = \mathrm{rank}(A^{\mathrm{T}}) = 3$.

(i) What is the dimension of the null space of A?

Solution:
We have $\dim(\mathrm{null}(A)) = \mathrm{nullity}(A) = 2$.

(j) What is the dimension of the null space of A^{T}?

Solution:
This time we get $\dim(\mathrm{col}(\mathrm{null}(A^{\mathrm{T}}))) = \mathrm{nullity}(A^{\mathrm{T}}) = 1$.

3. [10 points] (5 + 5) In R^3 the **orthogonal projections** onto the x-axis, y-axis and z-axis are

$$T_1(x, y, z) = (x, 0, 0), \qquad T_2(x, y, z) = (0, y, 0), \qquad T_3(x, y, z) = (0, 0, z),$$

respectively.

(a) Find the standard matrices for T_1, T_2 and T_3.

Solution:

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}; T_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Show that if $T : R^3 \to R^3$ is one of these orthogonal projections, then for every vector $\mathbf{v} \in R^3$, $T(\mathbf{v})$ and $(\mathbf{v} - T(\mathbf{v}))$ are orthogonal.

Solution:
Consider, for example, T_1. If $\mathbf{v} = (v_1, v_2, v_3)$, $T_1(\mathbf{v}) = (v_1, 0, 0)$. Moreover,

$$T_1(\mathbf{v}) \cdot (\mathbf{v} - T_1(\mathbf{v})) = (v_1, 0, 0) \cdot (0, v_2, v_3) = 0.$$

4. [10 points] Find the rank (5 points) and nullity (5 points) of the standard matrix for T, where

$$T : \mathbb{R}^2 \to \mathbb{R}^3, \quad T(x_1, x_2) = (x_1 + 3x_2,\, x_1 - x_2,\, x_1).$$

Solution:

$T\,(x_1, x_2) = \begin{bmatrix} x_1 + 3x_2 \\ x_1 - x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; the standard matrix is $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$.

Its reduced row echelon form is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) $\operatorname{rank}(A) = 2$ **(b)** $\operatorname{nullity}(A) = 0$. Note that these conclusions could have been derived from a row echelon form.

5. [10 points] True or False.

(a) **T** **F**: In a square matrix A, $\text{nullity}(A) = \text{nullity}(A^{\mathrm{T}})$.

Solution:

True. Suppose A is an $n \times n$ square matrix with rank r. Then the nullity of A would be $n - r$, as would the nullity of A^{T}.

(b) **T** **F**: The nullity of an $m \times n$ matrix is at most m.

Solution:

False. The nullity of A is limited by the number of its columns, not the number of its rows. A suitable counterexample would be a 3×6 matrix A with rank 2, and thus nullity 4, which is greater than the number of rows, 3.

(c) **T** **F**: If A has more rows than columns, the nullity of A^{T} is less than the nullity of A.

Solution:

False. Suppose A is $m \times n$, so that $m > n$. If $\text{rank}(A) = r$ then $\text{nullity}(A) = n - r$ and $\text{nullity}(A^{\mathrm{T}}) = m - r$. From $m > n$ we get $\text{nullity}(A) = n - r < m - r = \text{nullity}(A^{\mathrm{T}})$..

(d) **T** **F**: If V is a subspace of $\mathbb{R}^n$ and W is a subspace of V then $V^{\perp}$ is a subspace of $W^{\perp}$.

Solution:

True. Any vector $\mathbf{v}$ in $V^{\perp}$ orthogonal to all vectors in V, and that includes all vectors in its subspace W, so $\mathbf{v}$ is orthogonal to all vectors in W, and thus is in $W^{\perp}$. As this is true for any such vector $\mathbf{v}$, we must have that $V^{\perp}$ is a subspace of $W^{\perp}$.

(e) **T** **F**: The kernel of the matrix transform $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is the same as the null space of the corresponding $m \times n$ matrix A.

Solution:

True. $\text{kernel}(T_A) = \{\mathbf{x} \in \mathbb{R}^n \mid T_A(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\} = \text{null}(A)$.