MATH 2418: Linear Algebra

Assignment 7

Due `March 9, 2016` Term `Spring, 2016`

Recommended Text Book Problems (do not turn in): [Sec 4.4: # 1, 7, 13, 17, 19, 21, 25]; [Sec 4.5: # 3, 5, 7, 9, 13, 17, 19];

1. (a) Prove that $\mathcal{B} = \{(0,1,0),(2,2,0),(3,3,3)\}$ is a basis for $\mathbb{R}^3$.
   (b) Write the coordinate vector of $(5,-12,3)$ with respect to basis $\mathcal{B}$ of $\mathbb{R}^3$.

Solution: (a) Suppose $x_1(0,1,0)+x_2(2,2,0)+x_3(3,3,3) = \mathbf{0}$ in $\mathbb{R}^3$.
Which gives the homogeneous linear system:

$$\begin{cases} 2x_2+3x_3=0 \\ x_1+2x_2+3x_3=0 \\ 3x_3=0 \end{cases} \tag{1}$$

Consider the determinant of the coefficient matrix: $\begin{vmatrix} 0 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = (-1)(6-0) = -6 \neq 0$. So the vectors in $\mathcal{B}$ are linearly independent.
Since $\begin{vmatrix} 0 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} \neq 0$, the matrix equation $\begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has a unique solution for every $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$. i.e. $x_1(0,1,0)+x_2(2,2,0)+x_3(3,3,3) = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^3$. Hence $\mathcal{B}$ spans $\mathbb{R}^3$. Therefore $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(b) Let $(5,-12,3) = x_1(0,1,0)+x_2(2,2,0)+x_3(3,3,3)$
i.e.

$$\begin{cases} 2x_2+3x_3=5 \\ x_1+2x_2+3x_3=-12 \\ 3x_3=3 \end{cases} \tag{2}$$

From the third equation $x_3 = 1$. From the first equation $2x_2 = 5-3x_3 = 5-3 = 2 \Rightarrow x_2 = 1$. From the second equation $x_1 = -12-2x_2-3x_3 = -12-2(1)-3(1) = -17$
So $(5,-12,3) = (-17)(0,1,0)+1(2,2,0)+1(3,3,3)$.

Therefore $\begin{bmatrix} 5 \\ -12 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -17 \\ 1 \\ 1 \end{bmatrix}$

2. Let M_3^T be the vector space of all 3×3 symmetric matrices. Find a basis for M_3^T.

Solution: We know $M_3^T = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} : a_{ij} = a_{ji} \right\} = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \right\}$.

Suppose $\epsilon_{ij} : 1 \leq i, j \leq 3$ be a 3×3 matrix with 1 as an entry in $(i, j)^{th}$ position and 0s elsewhere. Consider the following subset $\mathcal{B}$ of M_3^T consisting of the following vectors:

$$\epsilon_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \epsilon_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \epsilon_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\epsilon_{12} + \epsilon_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \epsilon_{13} + \epsilon_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \epsilon_{23} + \epsilon_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now any $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \in M_3^T$ can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} +$$
$$a_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = a_{11}\epsilon_{11} + a_{22}\epsilon_{22} + a_{33}\epsilon_{33} + a_{12}(\epsilon_{12} + \epsilon_{21}) + a_{13}(\epsilon_{13} + \epsilon_{31}) + a_{23}(\epsilon_{23} + \epsilon_{32}).$$

Hence $\mathcal{B} = \{\epsilon_{11}, \ \epsilon_{22}, \ \epsilon_{33}, \ (\epsilon_{12} + \epsilon_{21}), \ (\epsilon_{13} + \epsilon_{31}), \ (\epsilon_{23} + \epsilon_{32})\}$ spans M_3^T.

Next, suppose $k_1\epsilon_{11} + k_2\epsilon_{22} + k_3\epsilon_{33} + k_4(\epsilon_{12} + \epsilon_{21}) + k_5(\epsilon_{13} + \epsilon_{31}) + k_6(\epsilon_{23} + \epsilon_{32}) = \mathbf{0}$ in M_3^T, for some scalars $k_1, k_2, k_3, k_4, k_5, k_6$. i.e.

$$k_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + k_5 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + k_6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} k_1 & k_4 & k_5 \\ k_4 & k_2 & k_6 \\ k_5 & k_6 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 0.$$

Hence $\mathcal{B}$ is linearly independent. So $\mathcal{B}$ is a basis for M_3^T.

3. (a) Find a basis for the solution space of the given homogeneous linear system. State the dimension.

$$\begin{cases} 3x_1 + x_2 + x_3 + x_4 = 0, \\ 5x_1 - x_2 + x_3 - x_4 = 0. \end{cases}$$

Solution: The augmented matrix is: $\begin{bmatrix} 3 & 1 & 1 & 1 & \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix} \Rightarrow [R_2 - 2R_1] \begin{bmatrix} 3 & 1 & 1 & 1 & \\ -1 & -3 & -1 & -3 & 0 \end{bmatrix}$

$\Rightarrow [R_1{+}3R_2, (-1)R_2] \begin{bmatrix} 0 & -8 & -2 & -8 & 0 \\ 1 & 3 & 1 & 3 & 0 \end{bmatrix} \Rightarrow [R_1 \Leftrightarrow R_2] \begin{bmatrix} 1 & 3 & 1 & 3 & 0 \\ 0 & -8 & -2 & -8 & 0 \end{bmatrix} \Rightarrow [\frac{R_2}{-8}] \begin{bmatrix} 1 & 3 & 1 & 3 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$.

Here x_3 and x_4 are free variables. Let $x_3 = r$ and $x_4 = s$, then from the second row $x_2 = \frac{-x_3}{4} - x_4 = \frac{-r}{4} - s$, and from the first row $x_1 = -3x_2 - x_3 - 3x_4 = \frac{3r}{4} + 3s - r - 3s = \frac{-r}{4}$. Hence the solution is:

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{-r}{4} \\ \frac{-r}{4} - s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} \frac{-1}{4} \\ \frac{-1}{4} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. So the solution space is spanned by $\mathcal{B} = \left\{ \begin{bmatrix} \frac{-1}{4} \\ \frac{-1}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$,

which consists of two vectors not multiple of eachother, so is linearly independent. So, $\mathcal{B}$ is a basis for the solution space.

(b) Find a basis for the subspace $W = \{(a, b, c, d) : d = a + b, c = a - b\}$ of $\mathbb{R}^4$. State the dimension.

Solution: Here $W = \{(a, b, c, d) : d = a + b, c = a - b\} = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : d = a + b, c = a - b \right\} = \left\{ \begin{bmatrix} a \\ b \\ a - b \\ a + b \end{bmatrix} \right\} =$

$\left\{ a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

So W is spanned by $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$, a set of two vectors not multiple of eachother, so is linearly independent.

Hence $\mathcal{B}$ is a basis of W.

4. (a) Let $\mathbf{v}_1 = (1, -1, 3)$, $\mathbf{v}_2 = (2, 2, 1)$. Find the standard basis vector of $\mathbb{R}^3$ that can be added to $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for $\mathbb{R}^3$. Show all of your work to receive full credit.

Solution: We start with adding all 3 standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and decide which one actually works.

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow [R_2{+}R_1, R_3{-}3R_1] \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 1 & 0 & 0 \\ 0 & -5 & -3 & 0 & 1 & 0 \end{bmatrix} \Rightarrow [R_2{+}R_3] \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & 1 & 0 \\ 0 & -5 & -3 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow [(-1)R_2, R_3 - 5R_2] \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 7 & -5 & -4 & 0 \end{bmatrix} \Rightarrow [\tfrac{R_3}{7}] \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & \frac{-5}{7} & \frac{-4}{7} & 0 \end{bmatrix}.$$

This tells us that whenever we solve the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{e}_1 + x_4\mathbf{e}_2 + x_5\mathbf{e}_3 = \mathbf{0}$ in $\mathbb{R}^3$, the variables x_4 and x_5 will be free and so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ will be linearly dependent. But, if we add only e_1 the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ will be linearly independent. Since $\dim(\mathbb{R}^3) = 3$, any set of three linearly independent vectors forms a basis. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ is a basis for $\mathbb{R}^3$. Note that adding $\mathbf{e}_2$ or $\mathbf{e}_3$ also works in this case.

(b) Let $\mathbf{v}_1 = (0, 1, 1, 0)$, $\mathbf{v}_2 = (2, 2, 2, 0)$, $\mathbf{v}_3 = (0, 0, 0, 4)$, $\mathbf{v}_4 = (4, -3, -3, -3)$. Find a basis for the $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Solution: We will reduce the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ down to a linearly independent subset, which will be a basis of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

$$\begin{bmatrix} 0 & 2 & 0 & 4 \\ 1 & 2 & 0 & -3 \\ 1 & 2 & 0 & -3 \\ 0 & 0 & 4 & -3 \end{bmatrix} \Rightarrow [R_1 \Leftrightarrow R_2] \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 2 & 0 & 4 \\ 1 & 2 & 0 & -3 \\ 0 & 0 & 4 & -3 \end{bmatrix} \Rightarrow [R_3{-}R_1] \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 \end{bmatrix} \Rightarrow [R_3 \Leftrightarrow R_4] \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$[\tfrac{R_3}{4}] \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{-3}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. So $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

5. True or False.

(a) **T** **F**: The set $S = \{(1,1),(2,3),(3,4)\}$ is linearly independent.

(b) **T** **F**: Every set consisting of 100 vectors that span $\mathbb{R}^{100}$ is a basis for $\mathbb{R}^{100}$.

(c) **T** **F**: The coordinate vector of $\mathbf{x} \in \mathbb{R}^n$ with respect to the standard basis of $\mathbb{R}^n$ is $\mathbf{x}$.

(d) **T** **F**: If $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \mathbf{v}_2 \cdots, \mathbf{v}_n\}$ is a basis of V.

(e) **T** **F**: There exists a basis of $M_{2\times 2}$ (the vector space of all 2×2 matrices) consisting of invertible matrices.

Reasons:

(a) Set of 3 vectors in $\mathbb{R}^2$ can not be linearly independent.

(b) Any set S of n vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$ if S spans $\mathbb{R}^n$ or S is linearly independent.

(c) $\mathbf{x} = (x_1, x_2, \cdots x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$.

(d) The vectors also need to be linearly independent. For example $\mathbb{R}^2 = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}$ but $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right\}$ is not linearly independent, so is not a basis for $\mathbb{R}^2$.

(e) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ are four invertible, linearly independent 2×2 matrices, so form a basis for $M_{2\times 2}$.