MATH 2418: Linear Algebra

Assignment 6

Due March 2, 2016 Term Spring, 2016

Recommended Text Book Problems (do not turn in): [Sec 4.2: # 1, 3, 9, 11, 19]; [Sec 4.3: # 3, 11, 13, 15, 17, 19];

1. Which of the followings are subspaces of $\mathbb{R}^3$? Show all of your work to receive full credit:

   (a) $W = \{(x, y, z) : x, y, z \in \mathbb{R}; x = y + z\}$.

   (b) $V = \{(x, y, 0) : x, y \in \mathbb{R}\}$

   (c) $U = \{(1, 1, z) : z \in \mathbb{R}\}$.

   Solution:
   (a) $W = \{(x, y, z) : x, y, z \in \mathbb{R};\ x = y + z\} = \{(y + z,\ y,\ z) : y, z \in \mathbb{R}\}$
   Let $\mathbf{u} = (y + z,\ y,\ z), \quad \mathbf{v} = (y' + z',\ y',\ z') \in W$ and a, b be any scalars. Then

$$\begin{aligned} a\mathbf{u} + b\mathbf{v} &= a(y + z, y, z) + b(y' + z',\ y',\ z') \\ &= (ay + az,\ ay,\ az) + (by' + bz',\ by',\ bz') \\ &= (ay + az + by' + bz',\ ay + by',\ az + bz') \\ &= \big((ay + by') + (az + bz'),\ ay + by',\ az + bz'\big) \in W. \end{aligned}$$

   So W is a subspace of $\mathbb{R}^3$.

   (b) Let $\mathbf{u} = (x, y, 0), \mathbf{v} = (x',\ y'\ , 0) \in V$ and a, b be any scalars. Then

$$\begin{aligned} a\mathbf{u} + b\mathbf{v} &= a(x,\ y,\ 0) + b(x',\ y'\ 0) \\ &= (ax,\ ay,\ 0) + (bx' + by',\ 0) \\ &= (ax + bx',\ ay + by',\ 0) \in V. \end{aligned}$$

   So V is a subspace of $\mathbb{R}^3$.

   (c) Let $\mathbf{u} = (1, 1, z),\ \mathbf{v} = (1, 1, z') \in U$. But, then $\mathbf{u} + \mathbf{v} = (1, 1, z) + (1, 1,\ z') = (2,\ 2,\ z + z') \notin U$. So U is not a subspace of $\mathbb{R}^3$.

2. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = (1,6,4), \mathbf{v}_2 = (2,4,-1), \mathbf{v}_3 = (-1,2,5); \text{ and } \mathbf{w}_1 = (1,-2,-5), \mathbf{w}_2 = (0,8,9).$$

Prove that, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

Solution: We will prove that every vector in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$, and every vector in the set $\{\mathbf{w}_1, \mathbf{w}_2\}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$\mathbf{v}_1 = (1,6,4) = (1,-2,-5) + (0,8,9) = \mathbf{w}_1 + \mathbf{w}_2$
$\mathbf{v}_2 = (2,4,-1) = 2(1,-2,-5) + (0,8,9) = 2\mathbf{w}_1 + \mathbf{w}_2$
$\mathbf{v}_3 = (-1,2,5) = (-1)(1,-2,-5) = (-1)\mathbf{w}_1 + (0)\mathbf{w}_2$
Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

Again,
$\mathbf{w}_1 = (1,-2,-5) = (-1)(1,6,4) + (2,4,-1) = (-1)\mathbf{v}_1 + \mathbf{v}_2 + (0)\mathbf{v}_3$
$\mathbf{w}_2 = (0,8,9) = 2(1,6,4) - (2,4,-1) = 2\mathbf{v}_1 + (-1)\mathbf{v}_2 + (0)\mathbf{v}_3$
So $\mathbf{w}_1, \mathbf{w}_2 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Therefore $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

3. Let $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ be a matrix transformation, the multiplication by the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $\mathbf{u}_1 = (0,1,1), \mathbf{u}_2 = (2,-1,1), \mathbf{u}_3 = (1,1,-2)$ be vectors in $\mathbb{R}^3$. Determine if $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ spans $\mathbb{R}^2$? Show all of your work to receive full credit.

Solution:

$T_A(\mathbf{u}_1) = A\mathbf{u}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$T_A(\mathbf{u}_2) = A\mathbf{u}_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$T_A(\mathbf{u}_3) = A\mathbf{u}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Now, we will check if any vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as a linear combination of $T_A(\mathbf{u}_1),\ T_A(\mathbf{u}_2),\ T_A(\mathbf{u}_3)$.
i.e. if we can find scalars x_1, x_2, x_3 such that : $x_1 T_A(\mathbf{u}_1) + x_2 T_A(\mathbf{u}_2) + x_3 T_A(\mathbf{u}_3) = \mathbf{b}$.

i.e.
$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The augmented matrix is $\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -2 & 3 & b_2 \end{bmatrix} \Rightarrow [\frac{-1}{2} R_2] \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & \frac{-3}{2} & \frac{-b_2}{2} \end{bmatrix}$
which corresponds to a consistent linear system. So, we can determine x_1, x_2, x_3 such that :

$$x_1 T_A(\mathbf{u}_1) + x_2 T_A(\mathbf{u}_2) + x_3 T_A(\mathbf{u}_3) = \mathbf{b}$$

for any $\mathbf{b} \in \mathbb{R}^2$. Hence $\{T_A(\mathbf{u}_1),\ T_A(\mathbf{u}_2),\ T_A(\mathbf{u}_3)\}$ spans $\mathbb{R}^2$.

4. Determine if the following set of vectors are linearly independent or dependent.

   (a) $(-3,0,4),\ (5,-1,2),\ (1,1,3)$ in $\mathbb{R}^3$.

   (b) $\cos 2x,\ 3\sin^2 x,\ -4\cos^2 x$ in the space $F(-\infty,\infty)$ of all real valued functions defined on $(-\infty,\infty)$.

   (c) $1+3x+3x^2,\ x+4x^2,\ 5+6x+3x^2$ in P_2, the vector space of all polynomials of degree ≤ 2.

Solution:
(a) Suppose $k_1(-3,0,4)+k_2(5,-1,2)+k_3(1,1,3)=\mathbf{0}$ in $\mathbb{R}^3$.
The corresponding linear system is

$$\begin{aligned} -3k_1+5k_2+k_3 &= 0 \\ -k_2+k_3 &= 0 \\ 4k_1+2k_2+3k_3 &= 0 \end{aligned}$$

The determinant of the coefficient matrix:

$$\begin{vmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{vmatrix} = -3(-3-2)+4(5+1) = 15+24 = 39 \neq 0.$$

Therefore the linear system has only the trivial solution $k_1=k_2=k_3=0$. Hence the vectors $(-3,0,4),\ (5,-1,2),\ (1,1,3$
in $\mathbb{R}^3$.

(b) We know $\cos 2x = \cos^2 x - \sin^2 x = \left(\frac{-1}{4}\right)\left(-4\cos^2 x\right)+\left(\frac{-1}{3}\right)\left(3\sin^2 x\right)$.
So $\cos 2x$ is a linear combination of $3\sin^2 x$ and $-4\cos^2 x$. Therefore $\cos 2x,\ 3\sin^2 x,\ -4\cos^2 x$ are linearly dependent.

(c) Suppose $a\left(1+3x+3x^2\right)+b\left(x+4x^2\right)+c\left(5+6x+3x^2\right)=0$ in P_2.
i.e. $(a+5c)+(3a+b+6c)x+(3a+4b+3c)x^2=0$ in P_2(meaning that a zero polynomial).
So

$$\begin{aligned} a \qquad + 5c &= 0 \\ 3a + \quad b + 6c &= 0 \\ 3a + 4b + 3c &= 0 \end{aligned}$$

The determinant of the coefficient matrix:

$$\begin{vmatrix} 1 & 0 & 5 \\ 3 & 1 & 6 \\ 3 & 4 & 3 \end{vmatrix} = 1(3-24)+5(12-3) = -21+45 = 24 \neq 0.$$

So the coefficient matrix is invertible and the linear system has only the trivial solution $a=b=c=0$. Hence $1+3x+3x^2,\ x+4x^2,\ 5+6x+3x^2$ are linearly independent in P_2.

5. (a) Determine if $(2,-2,0),(2,-1,4),(2,7,-6)$ lie on the same plane in $\mathbb{R}^3$.
   (b) Determine if $(-1,2,3),(2,-4,-6),(-7,14,21)$ lie on the same line on $\mathbb{R}^3$.

Solution:
(a) Suppose $a(2,-2,0)+b(2,-1,4)+c(2,7,-6)=\mathbf{0}$ in $\mathbb{R}^3$.
The corresponding linear system is:

$$\begin{aligned} 2a+2b+2c&=0\\ -2a-b+7c&=0\\ 4b-6c&=0 \end{aligned}$$

The determinant of coefficient matrix is :

$$\begin{vmatrix} 2 & 2 & 2\\ -2 & -1 & 7\\ 0 & 4 & -6 \end{vmatrix} = 2(6-28)+2(-12-8)=-44-40=-84\neq 0.$$

So the system has only the trivial solution $a=b=c=0$ and therefore the vectors $(2,-2,0)$, $(2,-1,4)$, $(2,7,-6)$ are linearly independent, so do not lie on same plane.

(b) Since $(2,-4,-6)=(-2)(-1,2,3)$ and $(-7,14,21)=7(-1,2,3)$, they are multiple of eachother, so they lie on the same line.

6. Use the **Wronskian** $W(x)$ to check if the following vectors are linearly independent in $F(-\infty,\infty)$.

   (a) $2,\ 2x+3,\ x^2-1.$

   (b) $5e^x,\ e^x\sin x,\ e^x\cos x.$

Solution:

(a) $W(x) = \begin{vmatrix} 2 & 2x+3 & x^2-1 \\ 0 & 2 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 8 \neq 0.$

So $2,\ 2x+3,\ x^2-1$ are linearly independent in $F(-\infty,\infty)$.

(b) $W(x) = \begin{vmatrix} 5e^x & e^x\sin x & e^x\cos x \\ 5e^x & e^x(\sin x+\cos x) & e^x(\cos x-\sin x) \\ 5e^x & 2e^x\cos x & -2e^x\sin x \end{vmatrix} = 5e^x.e^x.e^x \begin{vmatrix} 1 & \sin x & \cos x \\ 1 & \sin x+\cos x & \cos x-\sin x \\ 1 & 2\cos x & -2\sin x \end{vmatrix}$

$= 5e^{3x} \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -2\sin x & -2\cos x \end{vmatrix} = 5e^{3x}(1)(-2\cos^2 x - 2\sin^2 x) = -10e^{3x} \neq 0.$

Therefore $5e^x,\ e^x\sin x,\ e^x\cos x$ are linearly independent in $F(-\infty,\infty)$.

7. True or False.

(a) **T** **F**: Let A and B be two subsets of a vector space V such that $\text{Span}\{A\} = \text{Span}\{B\}$, then $A = B$.

(b) **T** **F**: Let A be an $m \times n$ matrix, then the solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

(c) **T** **F**: Let A be an $m \times n$ matrix, then the solution set of $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathbb{R}^n$.

(d) **T** **F**: If the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent then $\{2\mathbf{u}, 3\mathbf{v}\}$ is also linearly independent.

(e) **T** **F**: If three vectors in $\mathbb{R}^3$ are linearly dependent, then they must lie on the same line.

(f) **T** **F**: The vectors $(-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)$ in $\mathbb{R}^3$ are linearly dependent.

Reasons:

(a) $A = \{(1, 0), (0, 1)\}$ and $B = \{(2, 0), (0, 2)\}$ are two subsets of $\mathbb{R}^2$ such that $\text{Span}\{A\} = \text{Span}\{B\}$, but $A \neq B$.

(b) Let W denote the solution set of $A\mathbf{x} = \mathbf{0}$. Suppose $\mathbf{u}, \mathbf{v}$ be any two vectors in W, then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Now, let a, b be any scalars, then $A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}$. Hence $a\mathbf{u} + b\mathbf{v} \in W$. Therefore W is a subspace of $\mathbb{R}^n$.

(c) Let W denote the solution set of $A\mathbf{x} = \mathbf{b}$. Suppose $\mathbf{u}, \mathbf{v}$ be any two vectors in W, then $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$. But then $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = \mathbf{b}+\mathbf{b} = 2\mathbf{b}$. So $\mathbf{u}+\mathbf{v} \notin W$ for $\mathbf{b} \neq \mathbf{0}$. Hence W is not a subspace.

(d) Suppose $a(2\mathbf{u}) + b(3\mathbf{v}) = \mathbf{0} \Rightarrow (2a)\mathbf{u} + (3b)\mathbf{v} = 0$.
But $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. So $2a = 0$ and $3b = 0 \Rightarrow a = 0$ and $b = 0$. Hence $\{2\mathbf{u}, 3\mathbf{v}\}$ is also linearly independent.

(e) The vectors $(1, 0, 0), (0, 1, 0)$ and $(1, 1, 0)$ are linearly dependent in $\mathbb{R}^3$ but do not line on same line.

(f) Set of 4 vectors in $\mathbb{R}^3$ is linearly dependent.