Discrete Mathematics for Computing

UTD

Ch 9.6 Partial Orderings

- A relation R on a set S is called a partial ordering or *partial order* if it is:

  reflexive

  antisymmetric

  transitive

- A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R).

Example

- Let R be a relation on set A. Is R a partial order?

  $A = \{1, 2, 3, 4\}$

  $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

Example

So, given

$A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2),$
$(2,3), (2,4), (3,3), (3,4), (4,4)\}$

R is a partial order, and

(A, R) is a poset.

Example

- Is the "$\geq$" relation a partial ordering on the set of integers?

Partial Orderings

Example: Show that the divisibility relation (|) is a partial ordering on the set of integers.

Comparable / Incomparable

- In a poset the notation $a \preccurlyeq b$ denotes $(a, b) \in R$
  - The "less than or equal to" ($\leq$)is just an example of partial ordering
- The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
- The elements a and b of a poset $(S, \preccurlyeq)$ are called *incomparable* if neither $a \preccurlyeq b$ nor $b \preccurlyeq a$.

Comparable / Incomparable

- In the poset $(\mathbf{Z}^+, |)$:
  - Are 3 and 9 comparable?
  - Are 5 and 7 comparable?

Total Order

- "Partial ordering" - pairs of elements may be incomparable.
- If every two elements of a poset $(S, \preccurlyeq)$ are comparable, then S is called a *totally ordered* or *linearly ordered* set and $\preccurlyeq$ is called a *total order* or *linear order*.
- A totally ordered set is also called a *chain*.

Total Order

- Is the poset $(\mathbf{Z}, \leq)$ is totally ordered?

  Every two elements of $\mathbf{Z}$ are comparable; that is, $a \leq b$ or $b \leq a$ for all integers.

- Is poset $(\mathbf{Z}^{+}, |)$ is totally ordered?
- It contains elements that are incomparable; for example $5 \mid 7$.

Lexicographic Order

- We say that (a_1, a_2) is *less than* (b_1, b_2) – that is, (a_1, a_2) ⓪ (b_1, b_2) – either if:

  a_1 ⓪ b_1 , or

  $a_1 = b_1$ and a_2 ⓪ b_2

Lexicographic Order

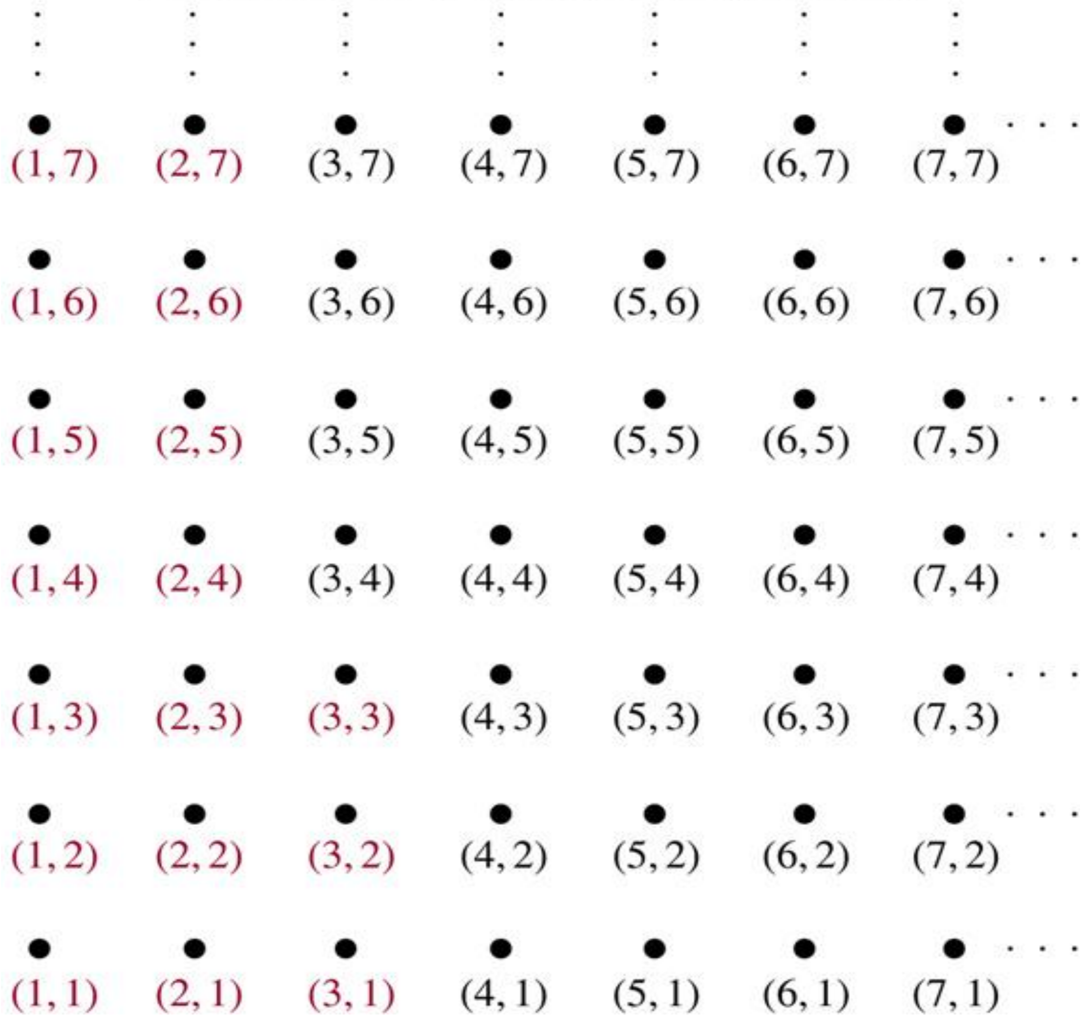
- In the poset ($\mathbf{Z} \times \mathbf{Z}$, $\preccurlyeq$),

  is (3, 5) ⓪ (4, 8)?

  is (3, 8) ⓪ (4, 5)?

  is (4, 9) ⓪ (4, 11)?

Lexicographic Order



The ordered pairs in red are all less than (3,4).

Lexicographic Order

Consider the strings consisting of lowercase characters.

Let n_A be the number of characters in string A and n_B be the number of characters in string B. Let n be the smaller of the two values.

For $i = 1$ to n, compare character A_i with B_i:

- If A_i matches B_i, and $n_A = n_B$, then A = B.
- If A_i matches B_i, but $n_A < n_B$, then A < B.
- If A_i matches B_i, but $n_B < n_A$, then B < A.
- If, for some $i \leq n$, character A_i comes before B_i in the alphabet, then A < B.

Lexicographic Order

Those are the actual rules by which words are listed in order in the dictionary.

discreet ⓪ *discrete*, because these strings differ in the 7th position, and *e* ⓪ *t*.

discreet ⓪ *discreetness*, because these strings agree for the first 8 characters (the length of the shorter string), but the second string has more letters.

Finally, *discrete* ⓪ *discretion*, because these strings differ in the 8th position, and *e* ⓪ *i*.

Hasse Diagram

- Hasse diagram - graphical representation of a poset.
- Since a poset is by definition reflexive and transitive (and antisymmetric), the graphical representation for a poset can be compacted.
- For example, why do we need to include loops at every vertex?
- Since it's a poset, it *must* have loops there.

Constructing a Hasse Diagram

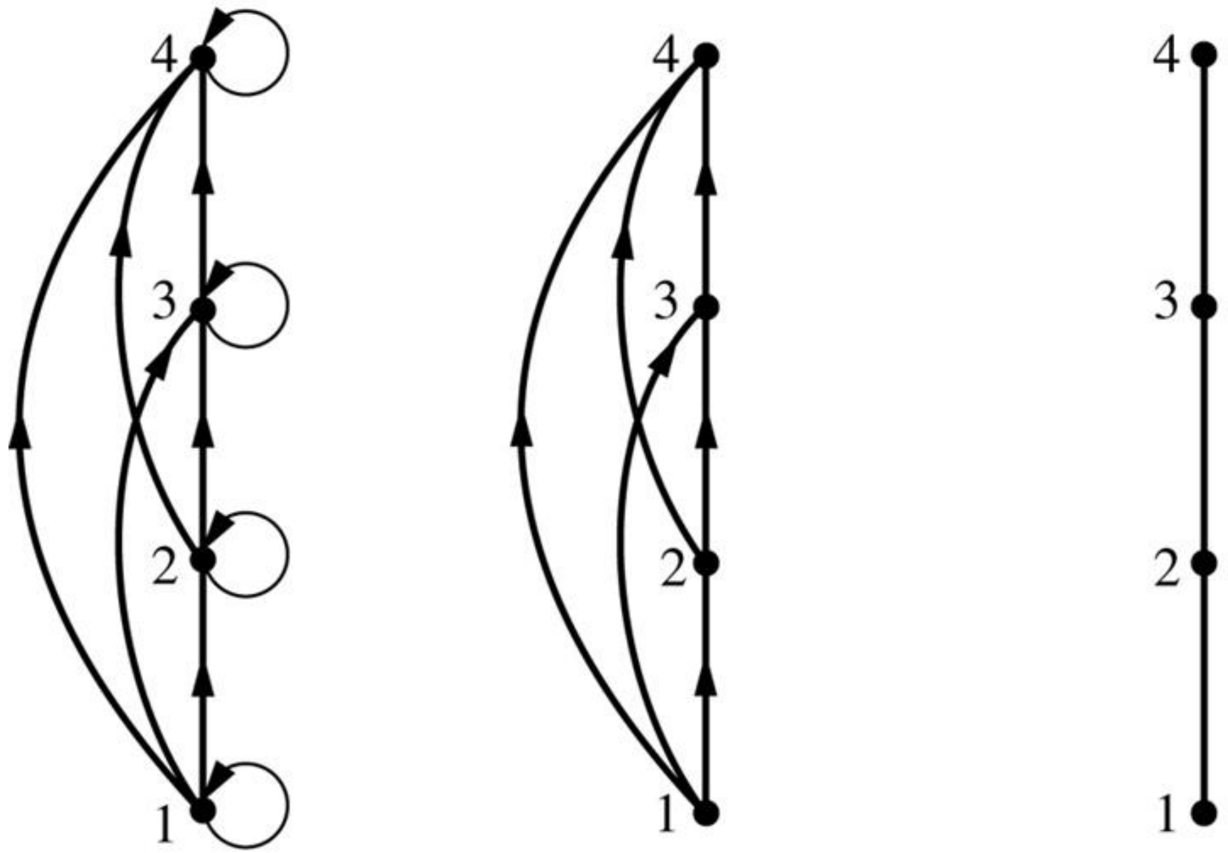
- Start with the digraph of the partial order.
- Remove the loops at each vertex.
- Remove all edges that *must* be present because of the transitivity.
- Arrange each edge so that all arrows point up.
- Remove all arrowheads.

Example

- Construct the Hasse diagram for $(\{1, 2, 3\}, \leq)$

Hasse Diagram Example

Steps in the construction of the Hasse diagram for ({1, 2, 3, 4}, ≤)



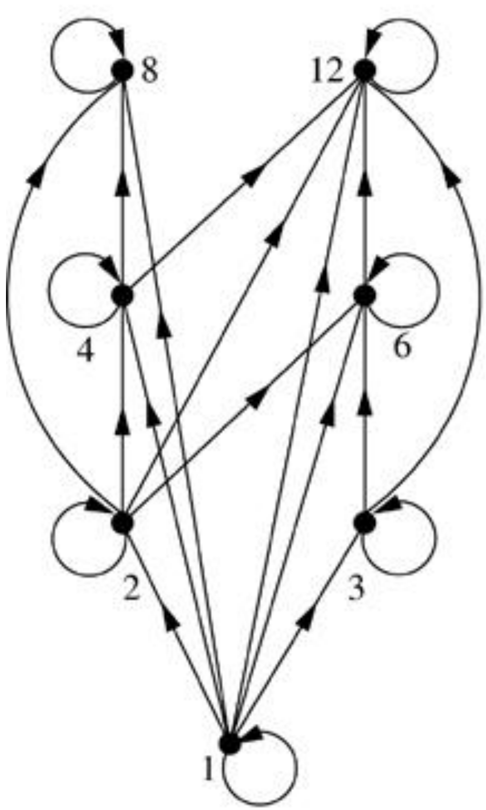
(a) (b) (c)

Example

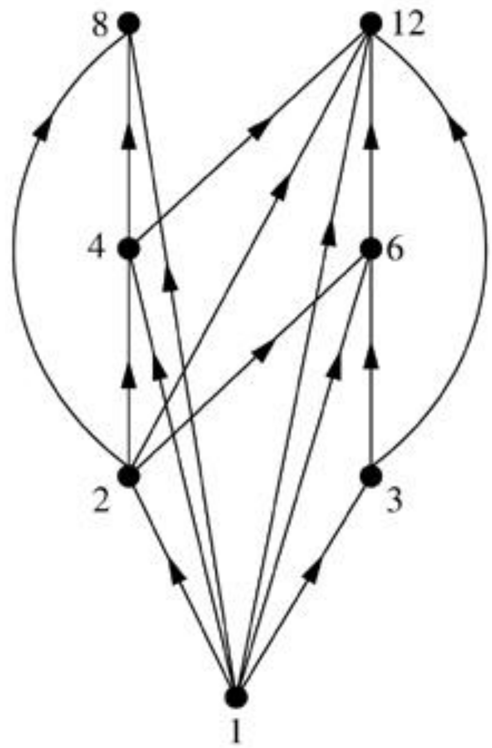
- Construct the Hasse diagram for $(\{1, 2, 3, 4\}, |)$

Hasse Diagram Example

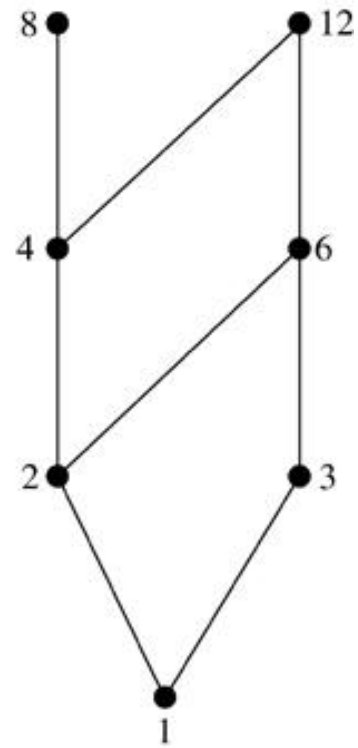
Steps in the construction of the Hasse diagram for ({1, 2, 3, 4, 6, 8, 12}, |)



(a)



(b)



(c)

Example

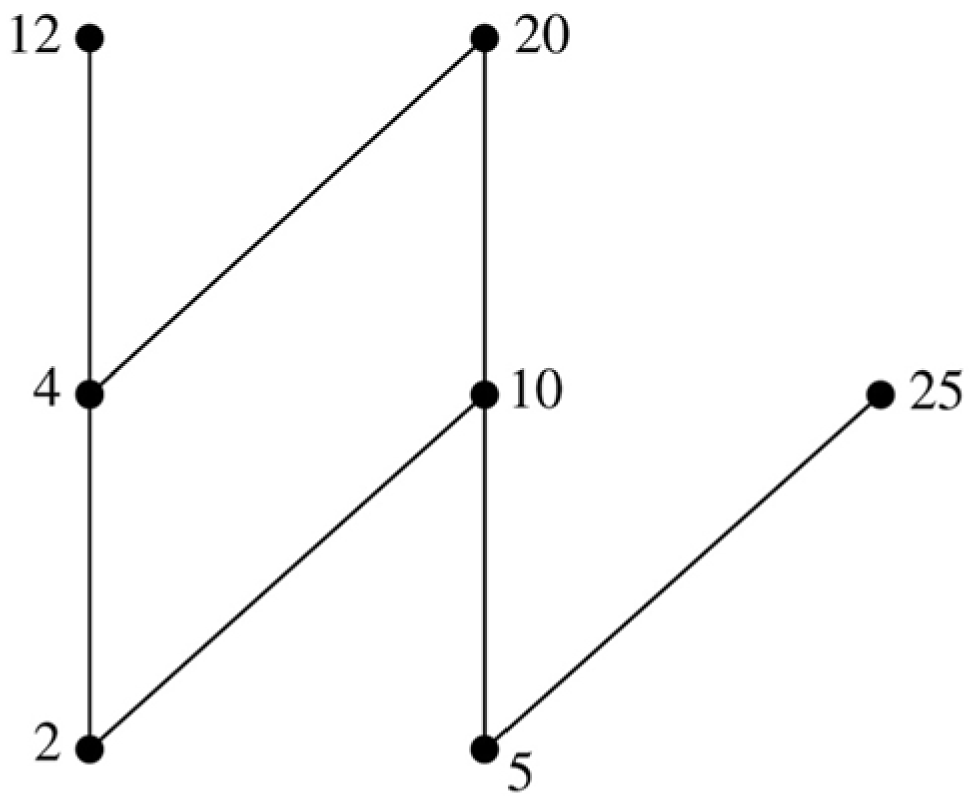
- Construct the Hasse diagram for the power set $P(\{1, 2, 3, 4\})$

Hasse Diagram Terminology

- Let $(S, \preccurlyeq)$ be a poset.
- a is *maximal* in $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \preccurlyeq b$. (top of the Hasse diagram)
- a is *minimal* in $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \preccurlyeq a$. (bottom of the Hasse diagram)

Hasse Diagram Terminology

Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, |) are maximal? Which are minimal?

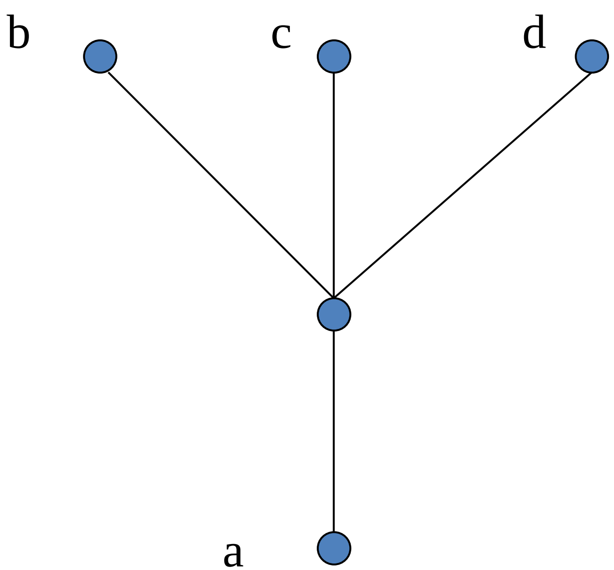


Hasse Diagram Terminology

- Let $(S, \preccurlyeq)$ be a poset.
- a is the *greatest element* of $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$...
  - It must be unique
- a is the *least element* of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.
  - It must be unique

Hasse Diagram Terminology

- Does the poset represented by this Hasse diagram have a *greatest element*? If so, what is it? Does it have a *least element*? If so, what is it?



Hasse Diagram Terminology

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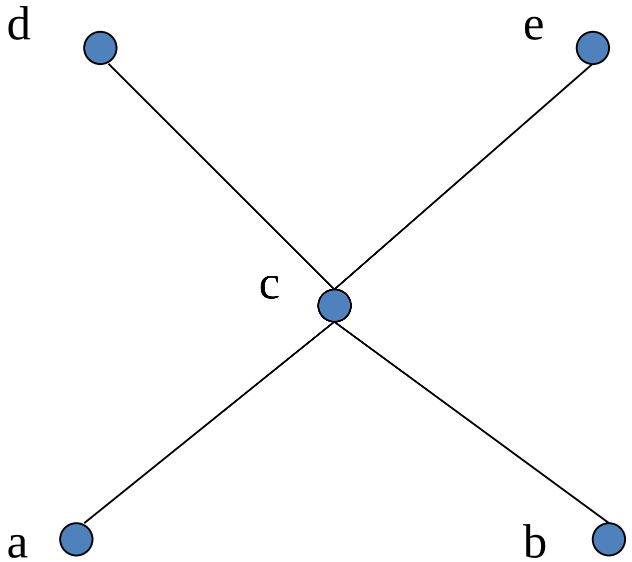

Hasse Diagram Terminology

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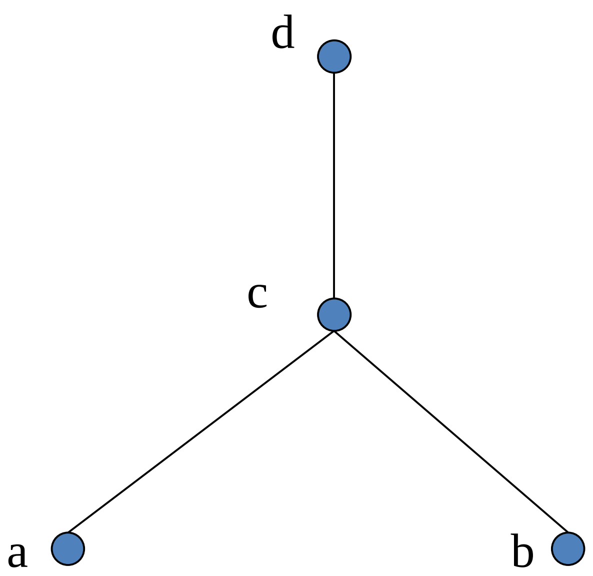
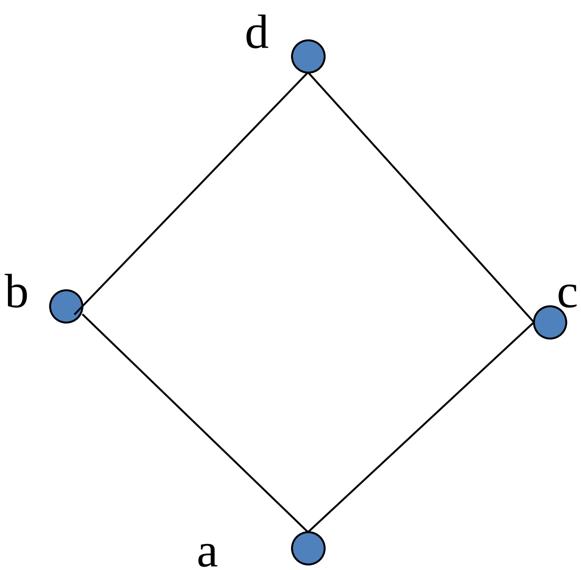


Hasse Diagram Terminology

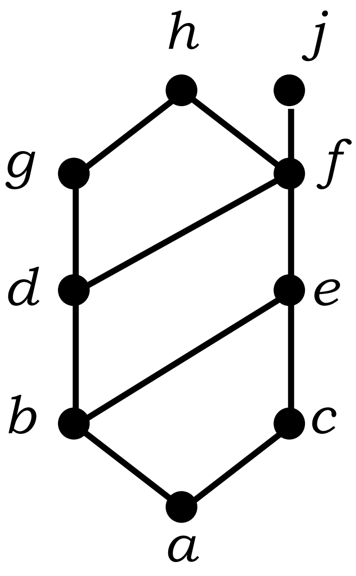
- Does the poset represented by this Hasse diagram have a *greatest element*? If so, what is it? Does it have a *least element*? If so, what is it?



Hasse Diagram Terminology

- Let A be a subset of $(S, \preccurlyeq)$.
- If $u \in S$ such that $a \preccurlyeq u$ for all $a \in A$, then u is called an *upper bound of A.*
- If $l \in S$ such that $l \preccurlyeq a$ for all $a \in A$, then l is called a *lower bound of A.*
- If x is an upper bound of A and $x \preccurlyeq z$ whenever z is an upper bound of A, then x is called the *least upper bound* of A.
  - It must be unique
- If y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A, then y is called the *greatest lower bound* of A.
  - It must be unique

Example



Maximal:

Minimal:

Greatest element:

Least element:

Upper bound of $\{a,b,c\}$:

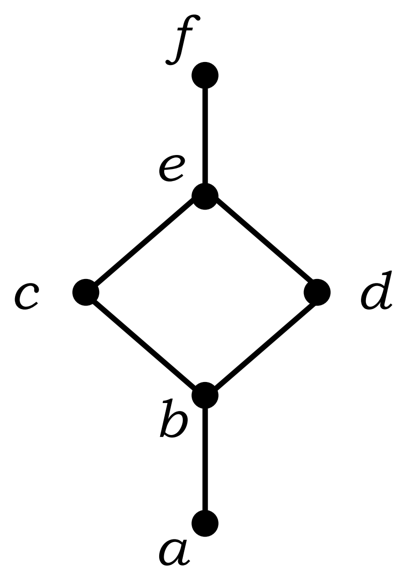
Least upper bound of $\{a,b,c\}$:

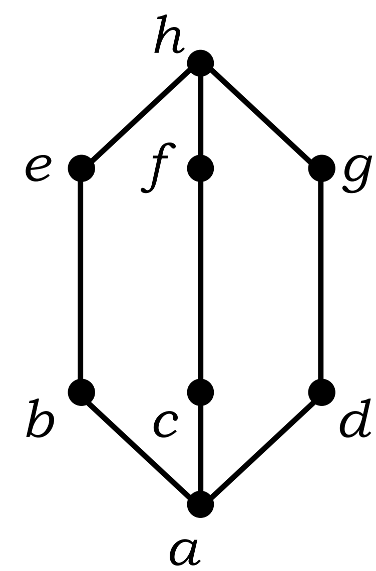
Lower bound of $\{a,b,c\}$:

Greatest lower bound of $\{a,b,c\}$:

Lattices

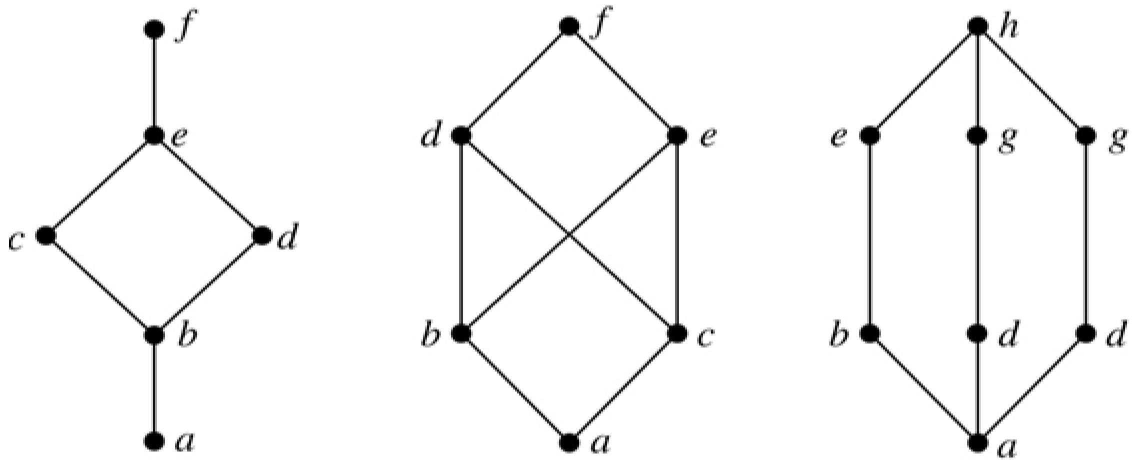
- A *lattice* is a partially ordered set in which every pair of elements has both a *least upper bound* and *greatest lower bound*.





Lattice example

- Are the following three posets *lattices*?



(a) (b) (c)

Conclusion

In this chapter we have studied:

- Relations and their properties
- How to represent relations
- Closures of relations
- Equivalence relations
- Partial orderings