Discrete Mathematics for Computing

UTD

Ch 9.3 Representing Relations

- Represent Binary Relations

  Ch 9.1 - Ordered Pairs, Table

- Two alternative methods

  - zero-one matrices: representation of relations in computer programs

  - pictorial representation: directed graphs

Matrices

- A *matrix* is a rectangular array of numbers.
- An $m \times n$ ("m by n") matrix has exactly m horizontal rows, and n vertical columns.
- An $n \times n$ matrix is called a *square* matrix, whose *order* is n.

$$\begin{bmatrix} 2 & 3 \\ 5 & -1 \\ 7 & 0 \end{bmatrix}$$

a 3×2 matrix

Row and Column Order

- The rows in a matrix are usually indexed 1 to *m* from top to bottom. The columns are usually indexed 1 to *n* from left to right. Elements are indexed by row, then column.

$$\mathbf{A} = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Matrix Equality

- Two matrices **A** and **B** are equal iff they have the same number of rows, the same number of columns, and all corresponding elements are equal.

$$\begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix} \neq \begin{bmatrix} 3 & 2 & 0 \\ -1 & 6 & 0 \end{bmatrix}$$

Matrix Sums

- The *sum* **A**+**B** of two matrices **A**, **B** (which **must** have the same number of rows, and the same number of columns) is the matrix (also with the same shape) given by adding corresponding elements.
- **A**+**B** = $[a_{i,j}+b_{i,j}]$

$$\begin{bmatrix} 2 & 6 \\ 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 3 \\ -11 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ -11 & -5 \end{bmatrix}$$

Matrix Products

- For an $m \times k$ matrix **A** and a $k \times n$ matrix **B**, the *product* **AB** is the $m \times n$ matrix:

$$\mathbf{AB} = \mathbf{C} = [c_{i,j}] \equiv \left[\sum_{\ell=1}^{k} a_{i,\ell} b_{\ell,j} \right]$$

- *i.e.*, element (i,j) of **AB** is given by the vector *dot product* of the <u>*i*th row of **A**</u> and the <u>*j*th column of **B**</u> (considered as vectors).
- Note: Matrix multiplication is **not** commutative!

Matrix Product Example

- An example matrix multiplication to practice in class:

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -5 & -1 \\ 3 & -2 & 11 & 3 \end{bmatrix}$$

Powers of Matrices

If **A** is an $n \times n$ square matrix and $p \geq 0$, then:

- $\mathbf{A}^p \equiv \underbrace{\mathbf{AAA} \cdots \mathbf{A}}_{p \text{ times}} \quad (\mathbf{A}^0 \equiv \mathbf{I}_n)$
- Example:

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$$

Boolean Products

- Let $\mathbf{A}=[a_{ij}]$ be an $m\times k$ zero-one matrix, & let $\mathbf{B}=[b_{ij}]$ be a $k\times n$ zero-one matrix,
- The *boolean product* of **A** and **B** is like normal matrix $\times$, but using $\vee$ instead + in the row-column "vector dot product."

$$\mathbf{A}\odot\mathbf{B}=\mathbf{C}=[c_{ij}]=\left[\bigvee_{\ell=1}^{k} a_{i\ell}\wedge b_{\ell j}\right]$$

Arithmetic/Boolean Products

$$\mathbf{AB} = \mathbf{C} = [c_{i,j}] \equiv \left[\sum_{\ell=1}^{k} a_{i,\ell} b_{\ell,j} \right]$$

$$\mathbf{A} \odot \mathbf{B} = \mathbf{C} = [c_{ij}] = \left[\bigvee_{\ell=1}^{k} a_{i\ell} \wedge b_{\ell j} \right]$$

Arithmetic/Boolean Products

An example of Boolean multiplication.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Boolean Powers

- For a square zero-one matrix **A**, and any $k \geq 0$, the *kth Boolean power of **A*** is simply the Boolean product of k copies of **A**.
- $$\mathbf{A}^{[k]} \equiv \underbrace{\mathbf{A} \odot \mathbf{A} \odot \ldots \odot \mathbf{A}}_{k \text{ times}}$$

Matrix review

- We will only be dealing with zero-one matrices
  - Each element in the matrix is either a 0 or a 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- These matrices will be used for Boolean operations
  - 1 is true, 0 is false

Matrix transposition

- Given a matrix **M**, the transposition of **M**, denoted **M**^{t}, is the matrix obtained by switching the columns and rows of **M**

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\mathbf{M}^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$\mathbf{M}^t = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

- In a "square" matrix, the main diagonal stays unchanged

Matrix join

- A *join* of two matrices performs a Boolean OR on each relative entry of the matrices
  - Matrices must be the same size
  - Denoted by the or symbol: $\vee$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Matrix meet

- A *meet* of two matrices performs a Boolean AND on each relative entry of the matrices
  - Matrices must be the same size
  - Denoted by the or symbol: $\wedge$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Relation Matrix

Let R be a relation from A to B

$A=\{a, b, c\}$

$B=\{d, e\}$

$R=\{(a, d), (b, e), (c, d)\}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Note that A is represented by the rows and B by the columns in the matrix.

Cell$_{ij}$ in the matrix contains a 1 iff a_i is related to b_j.

Relations using matrices

- List the elements of sets *A* and *B* in a particular order
  - Order doesn't matter, but we'll generally use ascending order
- Create a matrix

$$\mathbf{M}_R = [m_{ij}]$$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Relations using matrices

- Consider the relation of who is enrolled in which class
  - Let A = { Alice, Bob, Claire, Dan }
  - Let B = { CS101, CS201, CS202 }
  - R = { (*a*,*b*) | person *a* is enrolled in course *b* }

	CS101	CS201	CS202
Alice	X		
Bob		X	X
Claire			
Dan		X	X

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Relations using matrices

- What is it good for?
  - It is how computers view relations
    - A 2-dimensional array
  - Very easy to view relationship properties
- We will generally consider relations on a single set
  - In other words, the domain and co-domain are the same set
  - And the matrix is square

Relation Matrix

Which ordered pairs are present in a relation from *A* to *B*?

A={*a*, *b*, *c*}

B={*d*, *e*, *f*}

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

R = {(a,e), (b,d), (b,f), (c,f)}

Relation Matrices and Properties

- A relation matrix can be used to determine whether the relation has various properties
  - *Reflexive*
  - *Symmetric*
  - *Antisymmetric*
  - *Transitive*

Reflexivity

- Consider a reflexive relation:
  - One which every element is related to itself
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If the center (main) diagonal is all 1's, a relation is reflexive

Irreflexivity

- Consider a irreflexive relation:
  - One which every element is *not* related to itself
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If the center (main) diagonal is all 0's, a relation is irreflexive

Symmetry

- Consider an symmetric relation *R*
  - One which if *a* is related to *b* then *b* is related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- If, for *every* value, it is equal to the value in its transposed position, then the relation is symmetric

Asymmetry

- Consider an asymmetric relation:
  - One which if *a* is related to *b* then *b* is *not* related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- If, for every value and the value in its transposed position, if they are not both 1, then the relation is asymmetric
- An asymmetric relation must also be irreflexive
- Thus, the main diagonal must be all 0's

Antisymmetry

- Consider an antisymmetric relation:
  - One which if *a* is related to *b* then *b* is *not* related to *a* unless *a*=*b* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- If, for every value and the value in its transposed position, if they are not both 1, then the relation is antisymmetric
- The center diagonal can have both 1's and 0's

Transitivity

- Consider a transitive relation:
  - One which if *a* is related to *b* and *b* is related to *c* then *a* is related to *c* for all (*a,b*), (*b,c*) and (*a,c*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- If, for every spot (*a*,*b*) and (*b*,*c*) that each have a 1, there is a 1 at (*a*,*c*), then the relation is transitive

Representing Relations

Example: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution:

Since all the diagonal elements of this matrix are equal to 1, R is reflexive

The lower left triangle of the matrix = the upper right triangle

M_R is symmetric, R is symmetric

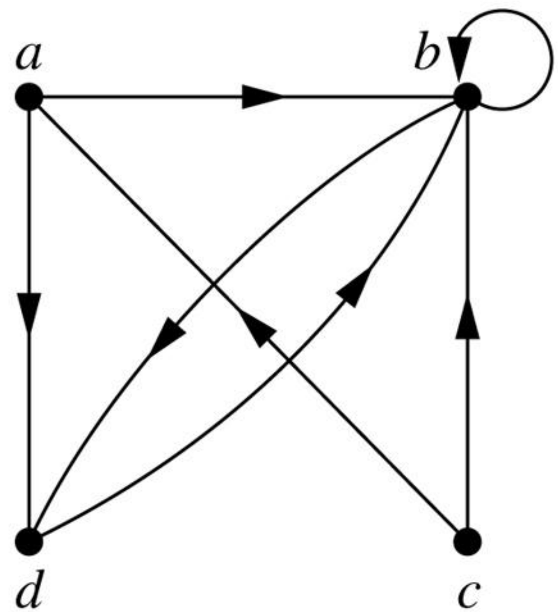
$M_{23} = M_{32} = 1$, R is not antisymmetric

Representing Relations

- Representing relations using digraphs
- A directed graph, or digraph

  - consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs)
- The vertex 'a' is called the initial vertex of the edge (a, b)
- The vertex 'b' is called the terminal vertex of this edge

Representing Relations Using Digraphs



This is a digraph with:

V = {a, b, c, d}

E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}

Note that edge (*b*, *b*) is represented using an arc from vertex *b* back to itself - *loop*

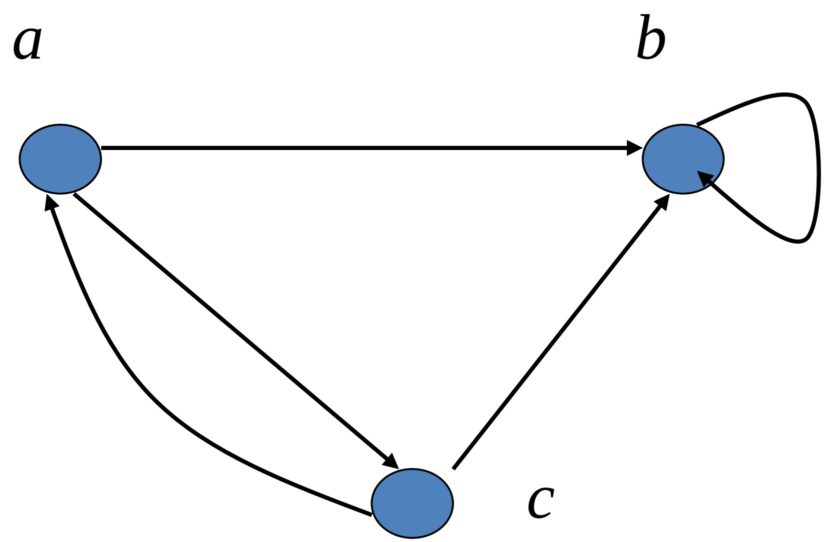
Example

- Let R be a relation on set A

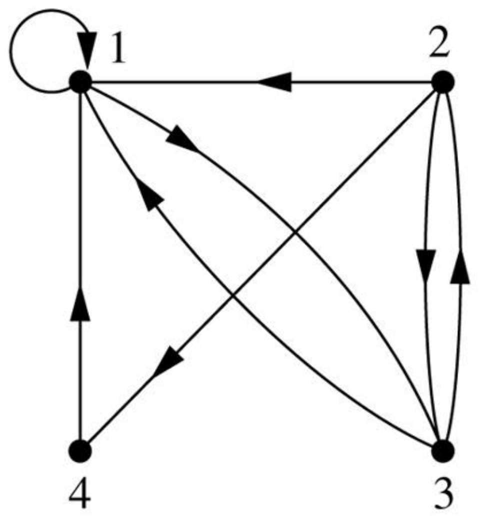
  $A=\{a, b, c\}$

  $R=\{(a, b), (a, c), (b, b), (c, a), (c, b)\}.$

- Draw the digraph that represents R



Example



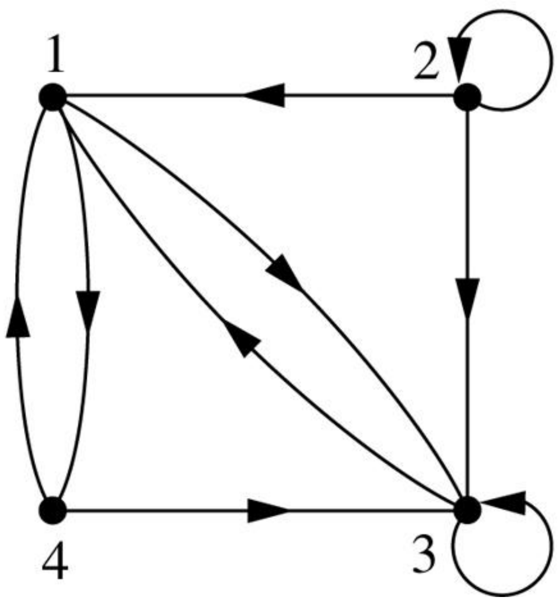
What are the ordered pairs in the relation R represented by the directed graph to the left?

This digraph represents the relation

$R = \{(1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)\}$

on the set $\{1, 2, 3, 4\}$.

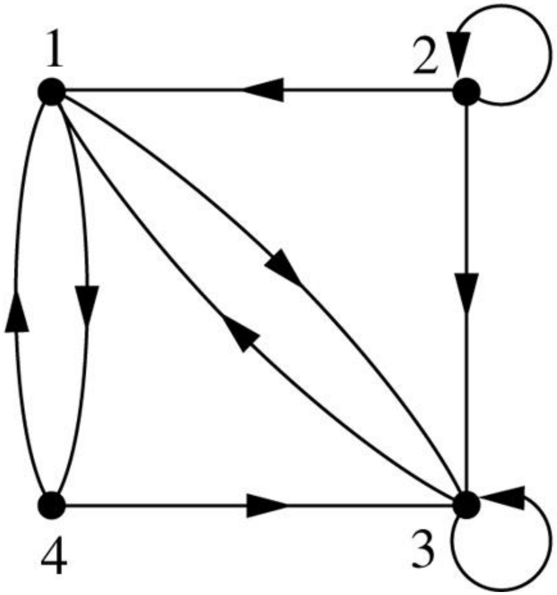
Example



What are the ordered pairs in the relation R represented by the directed graph to the left?

R = {(1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,1), (4,3)}

Example



According to the digraph representing *R*:

- is (4,3) an ordered pair in *R*?
- is (3,4) an ordered pair in *R*?
- is (3,3) an ordered pair in *R*?

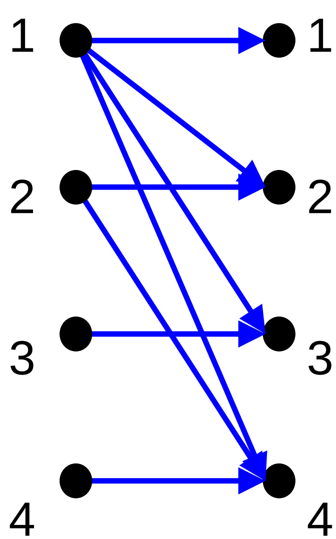
(4,3) is an ordered pair in *R*

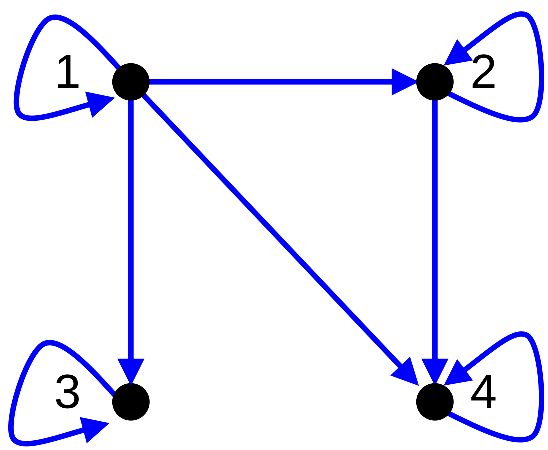
(3,4) is <u>not</u> an ordered pair in *R* – no arrowhead pointing from 3 to 4

(3,3) is an ordered pair in *R* – loop back to itself

Representing relations using directed graphs

- A directed graph consists of:
  - A set *V* of vertices (or nodes)
  - A set *E* of edges (or arcs)
  - If (*a*, *b*) is in the relation, then there is an arrow from *a* to *b*
- Will generally use relations on a single set
- Consider our relation *R* = { (*a*,*b*) | *a* divides *b* }
- Old way:



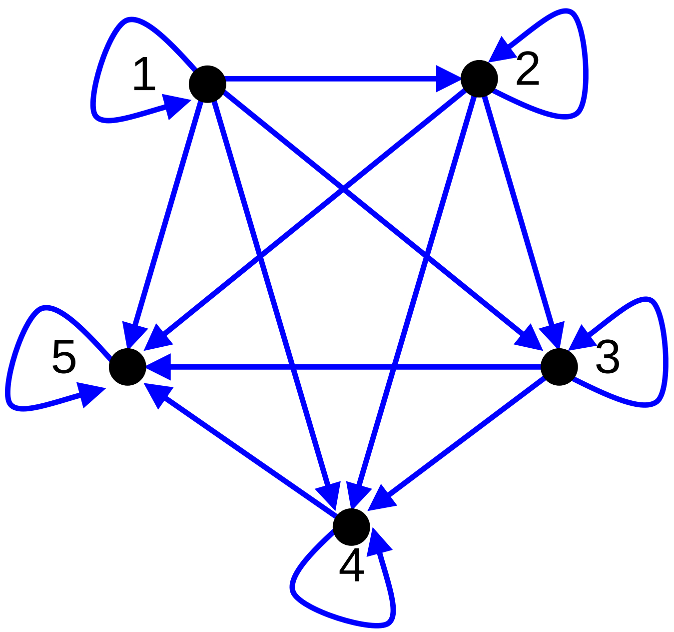


Relation Digraphs and Properties

- A relation digraph can be used to determine whether the relation has various properties
  - *Reflexive*
  - *Symmetric*
  - *Antisymmetric*
  - *Transitive*

Reflexivity

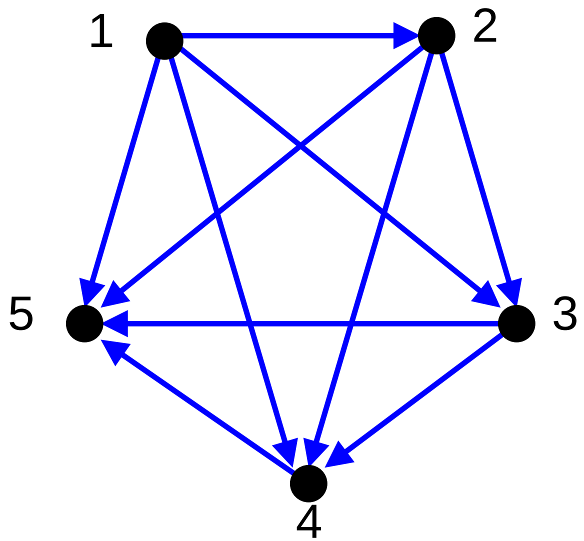
- Consider a reflexive relation:
  - One which every element is related to itself
  - Let A = { 1, 2, 3, 4, 5 }



If every node has a loop, a relation is reflexive

Irreflexivity

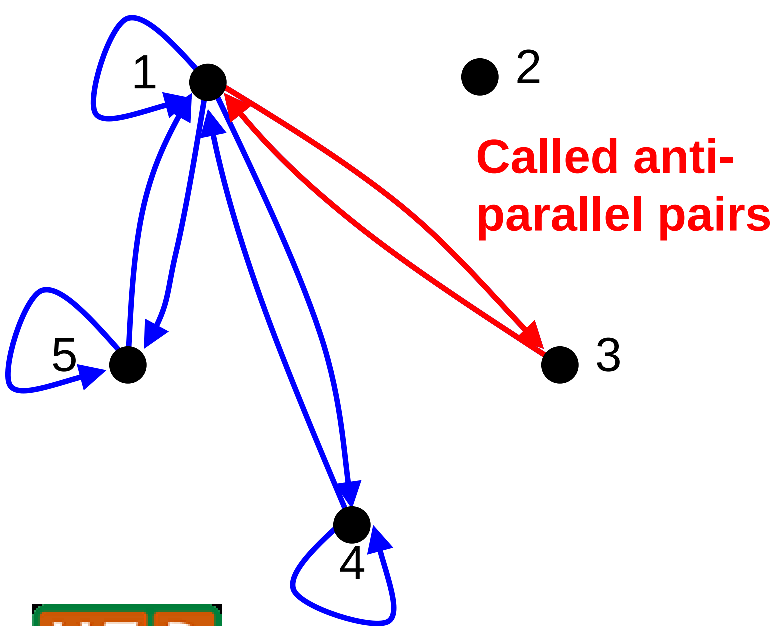
- Consider a irreflexive relation:
  - One which every element is *not* related to itself
  - Let A = { 1, 2, 3, 4, 5 }



If every node does *not* have a loop, a relation is irreflexive

Symmetry

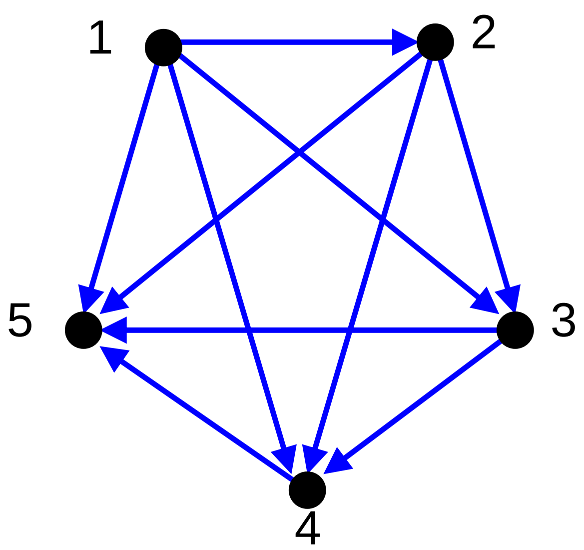
- Consider an symmetric relation *R*
  - One which if *a* is related to *b* then *b* is related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }



- If, for every edge, there is an edge in the other direction, then the relation is symmetric
- Loops are allowed, and do not need edges in the "other" direction

Asymmetry

- Consider an asymmetric relation:
  - One which if *a* is related to *b* then *b* is *not* related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

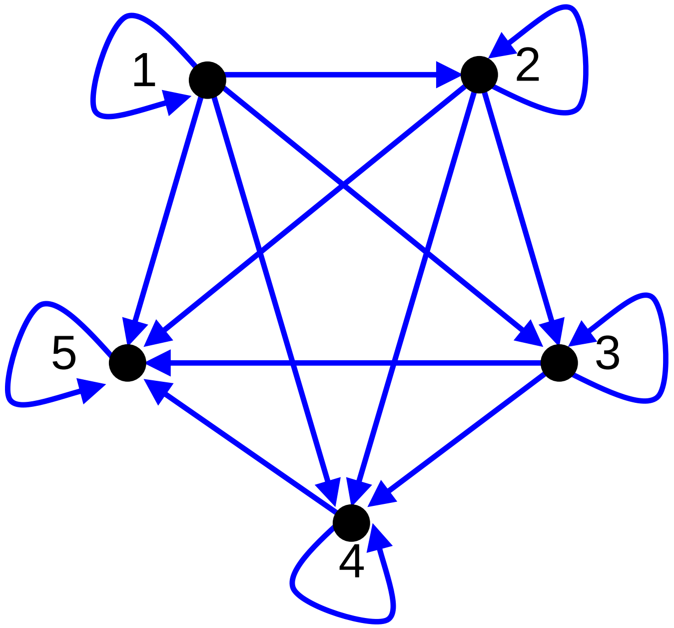


- A digraph is asymmetric if:

1. If, for every edge, there is *not* an edge in the other direction, then the relation is asymmetric
2. Loops are *not* allowed in an asymmetric digraph

Antisymmetry

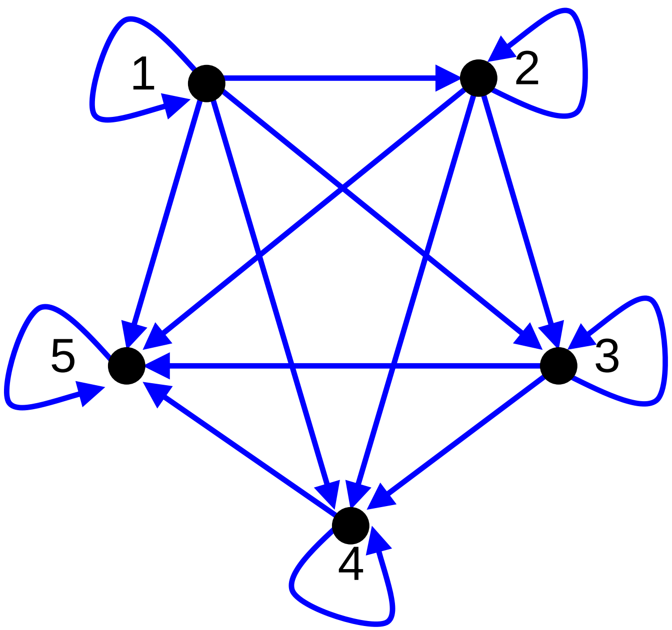
- Consider an antisymmetric relation:
  - One which if *a* is related to *b* then *b* is *not* related to *a* unless *a*=*b* for all (*a,b*)
  - Let A = { 1, 2, 3, 4, 5 }



- If, for every edge, there is *not* an edge in the other direction, then the relation is antisymmetric
- Loops are allowed in the digraph

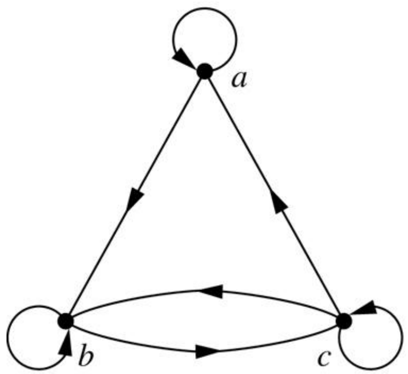
Transitivity

- Consider an transitive relation:
  - One which if *a* is related to *b* and *b* is related to *c* then *a* is related to *c* for all (*a*,*b*), (*b*,*c*) and (*a*,*c*)
  - Let A = { 1, 2, 3, 4, 5 }



- A digraph is transitive if there is a edge from *a* to *c* when there is a edge from *a* to *b* and from *b* to *c*

Example



(a) Directed graph of *R*

According to the digraph representing *R*:

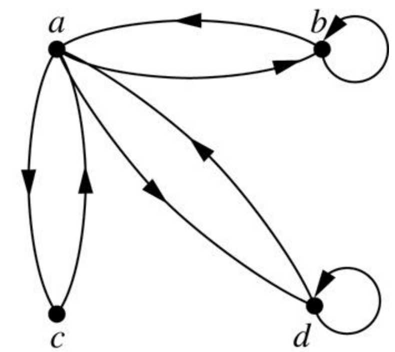
- is *R* reflexive?
- is *R* symmetric?
- is *R* antisymmetric?
- is *R* transitive?

- R is reflexive – there is a loop at every vertex
- R is not symmetric – there is an edge from *a* to *b* but not from *b* to *a*
- R is not antisymmetric – there are edges in both directions connecting *b* and *c*
- R is not transitive – there is an edge from *a* to *b* and an edge from *b* to *c*, but not from *a* to *c*

Example

According to the digraph representing *S*:

- is *S* reflexive?
- is *S* symmetric?
- is *S* antisymmetric?
- is *S* transitive?



(b) Directed graph of *S*

- S is not reflexive – there aren't loops at every vertex
- S is symmetric – for every edge from one distinct vertex to another, there is a matching edge in the opposite direction
- S is not antisymmetric – there are edges in both directions connecting *a* and *b*
- S is not transitive – there is an edge from *c* to *a* and an edge from *a* to *b*, but not from *c* to *b*