# MATH 2418: Linear Algebra

**Assignment 4**

Due: February 10, 2016 Term: Spring, 2016

**Recommended Text Book Problems (do not turn in):** [Section 1.7: #1, 3, 5, 7, 9, 11, 13, 19, 23, 25, 27, 29, 31 ]; [Section 1.8: #1, 3, 5, 7, 9, 13, 15, 19, 21, 23, 27, 29, 31].

1. (a) [2 points] Let $A = \begin{bmatrix} 4-x & 0 & x^2-25 \\ 0 & 2x^2+1 & 0 \\ 0 & 0 & x-5 \end{bmatrix}$. Find all values of $x \in \mathbb{R}$ for which $A$ is diagonal.

   **Solution:** We must have $x^2 - 25 = 0$, hence $x = \pm 5$.

   (b) [4 points] Let $B = \begin{bmatrix} 1 & 0 & x \\ 0 & x^2+4 & 0 \\ 0 & 0 & x-x^3 \end{bmatrix}$. Find all values of $x \in \mathbb{R}$ for which $B$ is diagonal and invertible.

   **Solution:** We must have $x = 0$ and $x - x^3 \neq 0$, hence $x = 0$ and $x \neq 0$. Therefore there is no $x$ for which $B$ is diagonal and invertible.

   (c) [4 points] Let $A = \begin{bmatrix} x^2-x & 0 & 0 \\ 0 & x^2-9 & 0 \\ 0 & 0 & x^3-8 \end{bmatrix}$. Find all values of $x \in \mathbb{R}$ for which $A$ is singular, i.e., non-invertible.

   **Solution:** A diagonal matrix is invertible if and only if all the entries in the main diagonal are non-zero. So $A$ is singular if and only if one of the entries in the main diagonal are zero. Now,

$$x^2 - x = x(x-1) = 0 \iff x = 0 \text{ or } 1,$$
$$x^2 - 9 = (x-3)(x+3) = 0 \iff x = \pm 3,$$
$$x^3 - 8 = (x-2)\underbrace{(x^2+2x+2)}_{(x+1)^2+1>0} = 0 \iff x = 2.$$

   Therefore, $A$ is singular if $x = 0, 1, 2, \pm 3$.

2. (a) [4 points] Let $Q = \begin{bmatrix} x & x^2-25 & 1 \\ x^2-y & 2y & 4 \\ y & x-5 & 3+y \end{bmatrix}$. Find all values of $(x, y) \in \mathbb{R}^2$ for which $Q$ is symmetric.

**Solution:** We must have $y = 1$ and $x-5 = 4$, hence $(x, y) = (9, 1)$. But then $x^2-y \neq x^2-25$. Therefore, there are no $(x, y) \in \mathbb{R}^2$ for which $Q$ is symmetric.

(b) [6 points] Let $A = \begin{bmatrix} 3 & a+2b+c & 3a-2c \\ 1 & 8 & b+2c \\ -4 & 7 & -2 \end{bmatrix}$. Find all values of $(a, b, c) \in \mathbb{R}^3$ such that $A$ is symmetric.

**Solution:** $A$ is symmetric if and only if $a, b, c$ satisfy the following equations:

$$\left.\begin{array}{l} a+2b+c=1 \\ 3a-2c=-4 \\ b+2c=7 \end{array}\right\} \iff \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}.$$

Applying Elementary Row Operations to the augmented matrix,

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & -2 & -4 \\ 0 & 1 & 2 & 7 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -6 & -5 & -7 \\ 0 & 1 & 2 & 7 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 7 \\ 0 & -6 & -5 & -7 \end{bmatrix}$$

$$\xrightarrow{R_3+6R_1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 7 & 35 \end{bmatrix} \xrightarrow{R_3/7} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow[R_1-R_3]{R_2-2R_3} \begin{bmatrix} 1 & 2 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

Therefore, $A$ is symmetric if and only if $(a, b, c) = (2, -3, 5)$.

3. (a) [23 points] Consider the transformation $F(< x_1, x_2 >) =< 4x_1,\ -5x_2,\ x_1 - 2x_2,\ 8x_1 - 4x_2 >$. Is it linear? Find the domain and codomain of $F$.

**Solution:** Yes, $F$ is a linear transformation because $F(\alpha\mathbf{x} + \mathbf{y}) = \alpha F(\mathbf{x}) + F(\mathbf{y})$, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. The domain of $F$ is $\mathbb{R}^2$ because it takes values $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ as an input. The codomain of $F$ is $\mathbb{R}^4$ because the output has four components.

(b) [4 points] Let $Q : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with $Q(\mathbf{e}_1) = (1, 2, 3)$ and $Q(\mathbf{e}_2) = (-3, -1, 4)$. Find $Q(< 5, -3 >)$. (Hint: recall that $\mathbf{e}_1$ and $\mathbf{e}_2$ form the standard basis for $\mathbb{R}^2$.)

**Solution:**

$$\begin{aligned} Q(< 5, -3 >) &= Q(5\mathbf{e}_1 - 3\mathbf{e}_2) \\ &= 5Q(\mathbf{e}_1) - 3Q(\mathbf{e}_2) \\ &= 5(1, 2, 3) - 3(-3, -1, 4) \\ &= (5, 10, 15) + (9, 3, -12) = (14, 13, 3) \end{aligned}$$

(c) [4 points] Consider the linear transformation

$$T(x_1, x_2, x_3) = (2x_1 - 3x_3,\ 5x_2 + 7x_3,\ 9x_1 - 4x_2 + x_3,\ 8x_2 - 6x_3).$$

Find the standard matrix for $T$.

**Solution:**

$$T(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 - 3x_3 \\ 5x_2 + 7x_3 \\ 9x_1 - 4x_2 + x_3 \\ 8x_2 - 6x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 5 & 7 \\ 9 & -4 & 1 \\ 0 & 8 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

So the standard matrix for $T$ is $\begin{bmatrix} 2 & 0 & -3 \\ 0 & 5 & 7 \\ 9 & -4 & 1 \\ 0 & 8 & -6 \end{bmatrix}$.

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that

$$T(2,1) = (2,-3,5), \quad \text{and} \quad T(1,1) = (4,7,2).$$

(a) [2 points] Find $T(-6,-3)$.

**Solution:**

$$T(-6,-3) = T(-3(2,1)) = -3T(2,1) = -3(2,-3,5) = (-6,9,-15).$$

(b) [3 points] Find $T(3,2)$.

**Solution:**

$$T(3,2) = T((2,1)+(1,1)) = T(2,1) + T(1,1) = (2,-3,5) + (4,7,2) = (6,4,7).$$

(c) [5 points] Find $T(3,-2)$.

**Solution:** We first would like to write $(3,-2)$ as a linear combination of $(2,1)$ and $(1,1)$. So we set

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \iff \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

We can solve the linear system easily:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and so

$$T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = 5\,T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) - 7\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 5 \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} - 7 \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -18 \\ -64 \\ 11 \end{bmatrix}.$$

5. [10 points] True or False.

(a) **T** **F**: If $A$ and $B$ are both diagonal $n \times n$ matrices, then so is $AB$.

**Solution: True**. Straightforward calculations.

(b) **T** **F**: If $A$ and $B$ are both symmetric $n \times n$ matrices, then so is $AB$.

**Solution: False**. $A$ and $B$ have to commute, i.e., $AB = BA$ so that $AB$ is symmetric: See Theorem 1.7.3. Take, e.g.,

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \implies AB = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}.$$

(c) **T** **F**: If $A$ and $B$ are $n \times n$ matrices such that $A + B$ is symmetric, then $A$ and $B$ are also symmetric.

**Solution: False**. Take, e.g.,

$$A = \begin{bmatrix} 3 & -5 \\ 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ -9 & 6 \end{bmatrix} \implies A + B = \begin{bmatrix} 7 & -2 \\ -2 & 8 \end{bmatrix}.$$

(d) **T** **F**: If $A$ and $B$ are $n \times n$ matrices such that $A + B$ is upper triangular, then $A$ and $B$ are also upper triangular.

**Solution: False**. Take, e.g.,

$$A = \begin{bmatrix} 3 & -5 \\ 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 8 \\ -7 & 6 \end{bmatrix} \implies A + B = \begin{bmatrix} 7 & 3 \\ 0 & 8 \end{bmatrix}.$$

(e) **T** **F**: For any diagonal matrix $A$, the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

**Solution: False**. If there is a zero in the main diagonal, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, e.g., for $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we have $(x_1, x_2) = (0, t)$ with $t \in \mathbb{R}$.

(f) **T** **F**: For *every* linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{0}) = \mathbf{0}$.

**Solution: True**. Recall that a linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ satisfies $T(c\,\mathbf{x}) = c\,T(\mathbf{x})$ for any $c \in \mathbb{R}$. Particularly, setting $c = 0$ gives $T(\mathbf{0}) = \mathbf{0}$.

(g) **T** **F**: If $T_A \colon \mathbb{R}^3 \to \mathbb{R}^5$ is the matrix transformation associated with a matrix $A$, then $A$ is a $3 \times 5$ matrix.

**Solution: False**. It should be $5 \times 3$, not $3 \times 5$.

(h) **T** **F**: If a matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ satisfies $T_A(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x}$ in $\mathbb{R}^n$, then $A$ is the $m \times n$ zero matrix.

**Solution: True**. Recall that the standard matrix $A$ for a matrix transformation $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$A = [T_A(\mathbf{e}_1) \,|\, \dots \,|\, T_A(\mathbf{e}_n)];$$

but then $T_A(\mathbf{e}_i = \mathbf{0}$ for $i = 1, \dots, n$ because of the assumption, and so $A$ becomes the $m \times n$ zero matrix.

(i) **T** **F**: There is at least one linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ for which $T(3\mathbf{x}) = 5T(\mathbf{x})$ for **some** vector $\mathbf{x}$ in $\mathbb{R}^n$.

**Solution: True**. Take $\mathbf{x} = \mathbf{0}$: $T(3\mathbf{x}) = T(\mathbf{0}) = \mathbf{0}$ whereas $5T(\mathbf{x}) = 5T(\mathbf{0}) = \mathbf{0}$.

(j) **T** **F**: If the matrix transformation $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ associated with a matrix $A$ satisfies $T_A(\mathbf{x}) = T_A(-\mathbf{x})$ for **every** vector $\mathbf{x}$ in $\mathbb{R}^n$, then $A$ is the $m \times n$ zero matrix.

**Solution:** **True**. Based on the assumption, for **any** vector $\mathbf{x}$ in $\mathbb{R}^n$, we have

$$T_A(\mathbf{x}) = T_A(-\mathbf{x}) = -T_A(\mathbf{x});$$

but then this implies that $T_A(\mathbf{x}) = \mathbf{0}$ for **any** vector $\mathbf{x}$ in $\mathbb{R}^n$. Combined with the result from (h), this implies that $A$ is the $m \times n$ zero matrix.