MATH 2418: Linear Algebra

Assignment 2

Due January 27, 2016 Term Spring, 2016

Recommended Text Book Problems (do not turn in): [Section 1.3: #1, 3, 5, 7, 9, 11, 13, 15, 17]; [Section 1.4: #1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 25].

1. For given matrices A and B, determine whether or not AB and $A-2B$ exist. If yes, then find AB and $A-2B$.

$$a) \quad A = \begin{bmatrix} 3 & -5 & 0 & -3 \\ 2 & 1 & 0 & 0 \\ 8 & 6 & 1 & 11 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$b) \quad A = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 4 & -2 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & -1 & 5 \\ 1 & 4 & 2 \end{bmatrix}$$

$$c) \quad A = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$$

$$d) \quad A = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$$

Solution: a) A is 3×4 and B is 3×4. So AB is not defined.

$$A-2B = \begin{bmatrix} 3 & -5 & 0 & -3 \\ 2 & 1 & 0 & 0 \\ 8 & 6 & 1 & 11 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 0 & -3 \\ 2 & 1 & 0 & 0 \\ 8 & 6 & 1 & 11 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & -6 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 2 & 12 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 & 3 \\ 2 & -7 & 0 & 0 \\ 8 & 6 & -1 & -1 \end{bmatrix}.$$

b) A is 3×3 and B is 3×3. So AB is defined and $AB = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 4 & -2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 0 \\ 3 & -1 & 5 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 21 & -7 \\ 6 & -8 & 16 \\ 1 & 8 & 12 \end{bmatrix}$

$$A-2B = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 4 & -2 \\ 3 & 2 & 1 \end{bmatrix} - 2\begin{bmatrix} -2 & 2 & 0 \\ 3 & -1 & 5 \\ 1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -7 & 4 \\ -4 & 6 & -12 \\ 1 & -6 & -3 \end{bmatrix}$$

c) $AB = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} = 25$. $A-2B$ does not exist because dimensions of A and $2B$ do not match.

d) $AB = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 14 & -7 & 21 \end{bmatrix}$. $A-2B$ does not exist (dimensions do not match).

2. (a) Let f be the quadratic polynomial of x, y defined by $f(x,\,y) = 2x^2 + 3y^2 - 4xy$. Verify that

$$f(x,\,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Solution: The expressions can be verified by using matrix multiplication directly – show your work.

(b) Let g be the quadratic polynomial of x, y and z defined by $g(x,\,y,\,z) = x^2+3y^2+5z^2-4xy-6yz+8xz$. Verify that

$$g(x,\,y,\,z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & -3 \\ 4 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Solution: The expressions can be verified by using matrix multiplication directly – show your work.

(c) Find all possible values of $k \in \mathbb{R}$ so that

$$\begin{bmatrix} 1 & 2 & k \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix} = 0.$$

Solution: Using matrix multiplication we obtain from

$$\begin{bmatrix} 1 & 2 & k \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ k \end{bmatrix} = 0,$$

which leads to $k^2 + 12k - 7 = 0$. Hence we have

$$k = \frac{-12 \pm \sqrt{172}}{2} = -6 \pm \sqrt{43}.$$

3. a) Let

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -3 & -7 \\ 7 & -3 & -2 & 9 \end{bmatrix}.$$

Find $(AB)_{13}$ and $(AB)_{24}$ without computing the whole matrix of AB.

b) Let C be a 100×200 matrix whose row 5 (5-th row) contains all -2's and row 51 contains all 2's, D a 200×201 matrix whose column 100 is all -2's, and column 101 is all 3's. Does CD exist? If yes, what is the value of the entry $(CD)_{5,101}$?

Solution:

a)

$$\begin{aligned}(AB)_{13} &= (\text{row 1 of } A) \times (\text{column 3 of } B) \\ &= \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \end{bmatrix} = -12. \\ (AB)_{24} &= (\text{row 2 of } A) \times (\text{column 4 of } B) \\ &= \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} -7 \\ 9 \end{bmatrix} = 1.\end{aligned}$$

b) Since C is 100×200 and D is 200×201, CD exists.

$$\begin{aligned}(CD)_{5,101} &= (\text{row 5 of } C) \times (\text{column 101 of } D) \\ &= \begin{bmatrix} -2 & -2 & \cdots & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ \vdots \\ 3 \end{bmatrix} \\ &= \underbrace{(-6) + (-6) + \cdots + (-6)}_{\text{200 terms}} \\ &= -1200.\end{aligned}$$

4. Find the inverses of the following matrices if they exist.

$$a) \qquad A = \begin{bmatrix} 3 & -5 \\ -1 & 1 \end{bmatrix}$$

$$b) \qquad B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \theta \in \mathbb{R}$$

$$c) \qquad C = \begin{bmatrix} \frac{1}{2}(e^{-x} - e^{x}) & \frac{1}{2}(e^{-x} + e^{x}) \\ \\ \frac{1}{2}(e^{-x} + e^{x}) & \frac{1}{2}(e^{-x} - e^{x}) \end{bmatrix}, \quad x \in \mathbb{R}.$$

Solution: a) By the formula of the inverse of 2×2 matrices, we have

$$\begin{aligned} A^{-1} &= \frac{1}{3 \cdot 1 - (-1) \cdot (-5)} \begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix} \\ &= \frac{1}{-2} \begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & -\frac{5}{2} \\ \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}. \end{aligned}$$

b) We have

$$\begin{aligned} B^{-1} &= \frac{1}{\cos\theta\cos\theta - (-\sin\theta) \cdot \sin\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \frac{1}{1} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \end{aligned}$$

c) We have

$$\begin{aligned} C^{-1} &= \frac{1}{\frac{1}{4}(e^{-x} - e^{x})^2 - \frac{1}{4}(e^{-x} + e^{x})^2} \begin{bmatrix} \frac{1}{2}(e^{-x} - e^{x}) & -\frac{1}{2}(e^{-x} + e^{x}) \\ \\ -\frac{1}{2}(e^{-x} + e^{x}) & \frac{1}{2}(e^{-x} - e^{x}) \end{bmatrix} \\ &= \frac{1}{-1} \begin{bmatrix} \frac{1}{2}(e^{-x} - e^{x}) & -\frac{1}{2}(e^{-x} + e^{x}) \\ \\ -\frac{1}{2}(e^{-x} + e^{x}) & \frac{1}{2}(e^{-x} - e^{x}) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2}(e^{-x} - e^{x}) & \frac{1}{2}(e^{-x} + e^{x}) \\ \\ \frac{1}{2}(e^{-x} + e^{x}) & -\frac{1}{2}(e^{-x} - e^{x}) \end{bmatrix}. \end{aligned}$$

5. True or False.

(a) **T** **F**: Let A be a square matrix. Then $\text{tr}(A) = \text{tr}(A^T)$.

T **F**.

(b) **T** **F**: Let A and B be square matrices. Then $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$.

T **F**.

(c) **T** **F**: If A is of size 1×2, and B is of size 2×1. Then $\text{tr}(AB) = \text{tr}(BA)$.

T **F**.

(d) **T** **F**: If A and B are invertible, then we have $((AB)^T)^{-1} = (A^{-1})^T(B^{-1})^T$.

T **F**.

(e) **T** **F**: If A and B are square matrices of the same size then $A^2 + 2AB + B^2 = (A+B)^2$.

T **F**: $AB \neq BA$.

(f) **T** **F**: Let A, B and C be matrices of the same size. If A is invertible and $AB = AC$, then $B = C$.

T **F**: multiply both sides by A^{-1}.

(g) **T** **F**: Let A and I be $n \times n$ matrices. If $A^2 + 3A + I = 0$, then $A^{-1} = A + 3I$.

T F: $I = -A^2 - 3A = A(-A-3I) = (-A-3I)A$. Hence $A^{-1} = -A - 3I$.

(h) **T** **F**: Let A, B and C be matrices of size $k \times n$, $n \times l$ and $l \times s$, respectively. Then ABC is a matrix with size $s \times k$.

T F.

(i) **T** **F**: There exists a 2×2 matrix A such that $A^2 = 0$ but $A \neq 0$, where 0 is the 2×2 zero matrix.

T **F**: For example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$. Verify that $A^2 = 0$.

(j) **T** **F**: There exists a 2×2 matrix A such that $A^2 = I$ but $A \neq \pm I$, where I is the 2×2 identity matrix.

T **F**: For example, consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \pm I$. Verify that $A^2 = I$.