MATH 2418: Linear Algebra

Solution to Assignment 1

Due `January 20, 2016` Term `Spring, 2016`

1. Suppose that the augmented matrix for a linear system has been reduced by row operations into the following matrix. For each of the matrices, i) determine whether or not the given matrix is in **reduced row echelon form**, whether or not in **row echelon form**; ii) solve each of the linear system. (Use x, y, z, u, v for unknowns if necessary)

$$a)\qquad A=\begin{bmatrix}1&0&0&-1\\0&1&0&0\\0&0&1&2\end{bmatrix}$$

$$b)\qquad B=\begin{bmatrix}1&3&0&1&-3&0\\0&0&1&2&5&0\\0&0&0&0&0&1\end{bmatrix}$$

$$c)\qquad C=\begin{bmatrix}1&5&0&0&-2&3\\0&0&1&0&4&-5\\0&0&0&1&2&6\\0&0&0&0&0&0\end{bmatrix}$$

$$d)\qquad D=\begin{bmatrix}1&7&0&0&0\\0&1&1&4&0\\0&0&0&0&0\end{bmatrix}$$

Solution: a) A is in reduced row echelon form and hence in row echelon form. The solution of the corresponding system of linear equations is $x=-1$, $y=0$, $z=2$.

b) B is in reduced row echelon form and hence in row echelon form. From the last row we know that the corresponding system of linear equations is inconsistent and has no solution.

c) C is in reduced row echelon form and hence in row echelon form. There are three leading ones corresponding to the variables x, z, u. Let $y=s$, $v=t$ where s, t are arbitrary real numbers. Then the solution is

$$\begin{cases}x=3-5s+2t\\y=s\\z=-5-4t\\u=6-2t\\v=t.\end{cases}$$

d) D is not in reduced row echelon form since there is a nonzero entry above the second leading one. But D is in row echelon form. The reduced row echelon form of D is

$$\begin{bmatrix}1&0&-7&-28&0\\0&1&1&4&0\\0&0&0&0&0\end{bmatrix}.$$

Let $z = s$, $u = t$ where s, t are arbitrary real numbers. Then the solution is

$$\begin{cases} x = 7s + 28t \\ y = -s - 4t \\ z = s \\ u = t. \end{cases}$$

2. Solve the following system of linear equations using Gauss-Jordan elimination.

$$\begin{cases} x + 2y - z = 2, \\ 2x + 5y + 2z = -1, \\ 7x + 17y + 5z = -1. \end{cases}$$

Solution: We re-write the system of linear equations in the matrix form $A\mathbf{u} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 2 \\ 7 & 17 & 5 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

Then the corresponding augumented matrix is

$$[A : b] = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & 2 & -1 \\ 7 & 17 & 5 & -1 \end{bmatrix}.$$

By the elementary row operations on $[A : \mathbf{b}]$ we have

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & 2 & -1 \\ 7 & 17 & 5 & -1 \end{bmatrix} \xRightarrow[R_3 - 7R_1]{R_2 + (-2)R_1} \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & -5 \\ 0 & 3 & 12 & -15 \end{bmatrix}$$
$$\xRightarrow[R_3 + (-3)R_2]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -9 & 12 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 0 & -9 & 12 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is the reduced echelon form of the augmented matrix $[A : \mathbf{b}]$. Let $z = t$, where t is an arbitrary number. The solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9t + 12 \\ -4t - 5 \\ t \end{bmatrix}.$$

3. Solve the following system of linear equations using Gauss-Jordan elimination.

$$\begin{cases} 10z + x = 5, \\ 3x + y - 4z = -1, \\ 4x + y + 6z = 1. \end{cases}$$

Solution: We re-write the system of linear equations in the matrix form $A\mathbf{u} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 10 \\ 3 & 1 & -4 \\ 4 & 1 & 6 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}.$$

Then the corresponding augmented matrix is

$$[A : \mathbf{b}] = \begin{bmatrix} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & 1 & 6 & 1 \end{bmatrix}.$$

By the elementary row operations on $[A : \mathbf{b}]$ we have

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & 1 & 6 & 1 \end{bmatrix} &\xrightarrow[R_3-4R_1]{R_2+(-3)R_1} \begin{bmatrix} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 1 & -34 & -19 \end{bmatrix} \\ &\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & -3 \end{bmatrix} \\ &\xrightarrow{R_3/-3} \begin{bmatrix} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Then

$$\begin{bmatrix} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a row echelon form of the augmented matrix $[A : \mathbf{b}]$. From the last row of the row echelon form we know that the corresponding system of linear equations is inconsistent and has no solution.

4. Find all possible values of $k \in \mathbb{R}$ so that the system of linear equations

$$\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + 3x_2 + kx_3 = 3 \\ x_1 + kx_2 + 3x_3 = 2 \end{cases}$$

has

i) a unique solution;

ii) no solution;

iii) infinitely many solutions.

Note : The notation $k \in \mathbb{R}$ means that k is **in** the set $\mathbb{R}$ of all real numbers. **We will always use this notation throughout the materials of this course.**

Solution: We re-write the system of linear equations in the matrix form $A\mathbf{u} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Then the corresponding augumented matrix is

$$[A : \mathbf{b}] = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{bmatrix}.$$

By the elementary row operations on $[A : \mathbf{b}]$ we have

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & k & 3 \\ 1 & k & 3 & 2 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2 + (-2)R_1} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & k+2 & 1 \\ 0 & k-1 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3-(k-1)R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & -k-3 & 0 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & k^2+k-6 & k-2 \end{bmatrix} =: B.$$

Then from the last row of B we know that

(a) if $k^2 + k - 6 = (k+3)(k-2) \neq 0$, then we can have a leading one in the third row by reducing B into

$$\begin{bmatrix} 1 & 0 & -k-3 & 0 \\ 0 & 1 & k+2 & 1 \\ 0 & 0 & 1 & \frac{1}{k+3} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{k+3} \\ 0 & 0 & 1 & \frac{1}{k+3} \end{bmatrix}.$$

Therefore, if $k \neq 2$ and $k \neq -3$, the system of linear equations has a unique solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{k+3} \\ \frac{1}{k+3} \end{bmatrix}.$$

(b) If $k^2 + k - 6 = (k+3)(k-2) = 0$ then $k = -3$ or $k = 2$. Suppose that $k = -3$. Then B becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

the last row of which corresponding to an inconsistent equation $0 = 5$. Therefore, if $k = -3$, the system of linear equations has no solution.

(c) If $k = 2$, then B becomes

$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which corresponds to the system of linear equations

$$\begin{cases} x_1 - 5x_3 = 0 \\ x_2 + 4x_3 = 1 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5t \\ 1 - 4t \\ t \end{bmatrix}, \; t \in \mathbb{R}.$$

5. True or False.

(a) **T** **F**: If a matrix is in reduced row echelon form, it is also in row echelon form.

(b) **T** **F**: Every matrix has a unique row echelon form.

(c) **T** **F**: If a linear system has more unknowns than equations, then it has infinitely many solutions.

(d) **T** **F**: A homogeneous system is always consistent.

(e) **T** **F**: A homogeneous linear system with five unknowns and two nonzero rows in reduced echelon form has three free variables.

(f) **T** **F**: A homogeneous linear system with more unknowns than equations has infinitely many solutions.

(g) **T** **F**: If a linear system has the trivial solution, then it has no other solutions.

(h) **T** **F**: A linear system of two equations with two unknowns always has a unique solution.

(i) **T** **F**: The Gauss-Jordan elimination procedure requires only two elementary row operations.

(j) **T** **F**: Wilhelm Jordan popularized the method of elimination successfully used by Carl Friedrich Gauss for solving important systems of linear equations.