Lecture 4 and 5. Independence and Conditional Probability

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Independent Events

We say that events $E_1, \ldots, E_n$ are independent if they occur independently of each other, i.e., occurrence of one event does not affect the probabilities of others.

Formally, we say that $E_1, \ldots, E_n$ are independent, if

$$P(E_1 \text{ and } \ldots \text{ and } \mathrm{E_n}) = \mathrm{P}(\mathrm{E_1} \bigcap \ldots \bigcap \mathrm{E_n}) = \mathrm{P}(\mathrm{E_1}) \ldots \mathrm{P}(\mathrm{E_n})$$

Recall that

- If A and B are incompatible (or mutually exclusive), then what is $P(A \text{ and } \mathrm{B}) = ?$

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- Now, if A and B are independent, then what is $P(A \text{ and } \mathrm{B}) =?$

Recall that

- If A and B are incompatible (or mutually exclusive), then what is $P(A \text{ and } \mathrm{B}) = 0$
- Now, if A and B are independent, then what is $P(A \text{ and } \mathrm{B}) = \mathrm{P(A)P(B)}$

Does it mean that mutually exclusive are independent?

Re-consider Example 6 from Lecture 3

Example 6. Consider **two** draws from a deck of cards **with replacement**. Let A be the event that the 1st card is a king and B be the event that the 2rd card is a king. What is the probability of A **or** B?

Solution Using the **union** formula

$$P(A \text{ or } \mathrm{B}) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A \ and \ B)} =$$

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Solution Using the **union** formula

$$P(A \text{ or } \mathrm{B}) = \mathrm{P(A)+P(B)-P(A \text{ and } B)} = \frac{4}{52}+\frac{4}{52}-\frac{4}{52}\times\frac{4}{52} = 0.14792.$$

Notice that here we in fact used our intrinsic understanding that A and B are independent.

Conditional Probability

Conditional probability of event A given event B is the probability that A occurs when B is known to occur.

Conditional probability of A given B is

$$P(A|B) = \frac{P(A \bigcap B)}{P(B)}.$$

Example 7. A CS instructor gave his class two tests. 25% of the class passed both tests and 42% of the class passed the first test. What percent of those who passed the first test also passed the second test?

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Solution Let A denote the success on the 1st test and B denote the success on the 2nd test.

Then using the **conditional probability** formula

$$P(B|A) = \frac{P(A \text{ and } \mathrm{B})}{P(A)} = \frac{0.25}{0.42} = 0.6$$

Example 8. The probability that it is Thursday and that a student is absent is 0.03. What is the probability that a student is absent given that today is Thursday?

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Solution Since there are 5 school days in a week, the probability that it is Thursday is 0.2. Then using the **conditional probability** formula

$$P(\text{Absent}|\text{Thursday}) = \frac{\text{P(Absent and Thursday)}}{\text{P(Thursday)}} = \frac{0.03}{0.20} = 0.15$$

Example 9. At UTD, the probability that a student takes STAT3341 and CS3340 is 0.087. The probability that a student takes STAT3341 is 0.68. What is the probability that a student takes CS3340 given that the student is taking STAT3341?

Solution Since there are 5 school days in a week, the probability that it is Thursday is 0.2. Then using the **conditional probability** formula

$$P(\mathrm{CS3340}|\mathrm{STAT3341}) = \frac{\mathrm{P(CS3340\ \ and\ \ STAT3341)}}{\mathrm{P(STAT3341)}} = \frac{0.087}{0.68} = 0.13$$

Example 10. Roll a die twice. What is the probability of a six on the second throw if the first throw was a six?

Solution Let the event A be rolling a six on the second trial and the event B be rolling a six on the first trial. Then

$$P(A|B) = \frac{P(A \text{ and } \mathrm{B})}{P(B)}$$

Example 10. Roll a die twice. What is the probability of a six on the second throw if the first throw was a six?

Solution Let the event A be rolling a six on the second trial and the event B be rolling a six on the first trial. Then

$$P(A|B) = \frac{P(A \text{ and } \mathrm{B})}{P(B)} = \frac{1/36}{1/6} = \frac{1}{6}.$$

So,

$$P(A|B) = P(A)!$$

Independent Events

If $P(A|B) = P(A)$, then A is **independent** of B. When events are independent, the occurrence of one does not influence the chance of the occurrence of the other.

Hence, using the conditional probability rule, we arrive to the same formula as before. Namely, if A and B are independent, then $P(A \text{ and } \mathrm{B}) = \mathrm{P(A)} \times \mathrm{P(B)}$.

If A is independent of B, then B is independent of A. Why?

Independent Events: More Examples

Example 11. Consider a noisy binary communication channel, where 0 or 1 is sent and 0 or 1 is received. Assume that 0 is sent with probability 0.2, and 1 is sent with probability 0.8. Assume that this binary channel is used to send two bits independently.

The channel is noisy. If a 0 is sent, a 0 is received with probability 0.9, and if a 1 is sent, a 1 is received with probability 0.975.

What is the probability that both bits are in error?

Solution First, define the two events

$$A = \{0 \;\; is \;\; sent\} = \{(0,1),(0,0)\},$$

$$B = \{0 \;\; is \;\; received\} = \{(0,0),(1,0)\}.$$

Thus, from the problem conditions, we get $P(A) = 0.2$, $P(A^c) = 0.8$, $P(B|A) = 0.9$ and $P(B^c|A^c) = 0.975$.

Define the two events

- $E_1 = \{First \;\; bit \;\; is \;\; in \;\; error\}$
- $E_2 = \{Second \;\; bit \;\; is \;\; in \;\; error\}$

Since the bits are sent independently, the probability that both are in error is

$$P(E_1 \;\; and \;\; E_2) = P(E_1) \times P(E_2).$$

Hence,

$$E_1 = (A \cap B^c) \cup (A^c \cap B).$$

Then, we get

$$\begin{aligned} P(E_1) &= P(A \cap B^c) + P(A^c \cap B) \\ &= P(A)P(B^c|A) + P(A^c)P(B|A^c) \\ &= 0.2 \times 0.1 + 0.8 \times 0.025 = 0.04. \end{aligned}$$

Why do we have $(A \cap B^c) \cup (A^c \cap B) = P(A \cap B^c) + P(A^c \cap B)$?

What is the probability $P(E_2) = ?$.

Hence,

$$E_1 = (A \cap B^c) \cup (A^c \cap B).$$

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Why do we have
$P((A \cap B^c) \cup (A^c \cap B)) = P(A \cap B^c) + P(A^c \cap B)$?

The probability $P(E_2) = P(E_1)$ (by symmetry).

Pascal and Fermat laid out the basic rules of probability in a series of letters. One of the most famous consequences of that discussion was Pascal's Wager. Pascal's Wager define two events:

- $A = \{\text{God exists}\}$
- $A^c = \{\text{God does not exists}\}$

A^c is called a complement of A, i.e. the event that A does not happen. From the formula $P(A \text{ or } B) = P(A) + P(B)$ for incompatible events A and B, we can easily see why $P(A^c) = 1 - P(A)$.

Pascal also defined a pay-off matrix:

	Bet on A	Bet on A^c
A is **true**	∞ reward	∞ loss
A^c is **true**	finite misery	finite reward

So Pascal argued that in terms of gambling, the optimal strategy for this game is to believe in God, no matter how small the probability $P(A)$ might be.