Lecture 17. Markov Chains

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1. Markov Chains. One Step Transition Probability Matrix

Markovian Property

Recall that the definition of Markovian process that we discussed last week:

Definition A stochastic process X_t is a Markov process if $P(future|past, present) = P(future|present)$. I.e., future depends on the past only through present.

Or more generally, if

$$P[X(t) \in A|X(t_1) \in A_1, \ldots, X(t_n) \in A_n] = P[X(t) \in A|X(t_n) \in A_n],$$

for $t_1 < \ldots < t_n < t$ and any sets $A, A_1, \ldots, A_n$.

History Excurse

Due to a well-developed theory and a number of simple techniques available for Markov processes, it is important to know whether the process is Markov or not.

The idea of Markov dependence was proposed and developed by Andrei Markov (1856-1922) who was a student of P. Chebyshev at St. Petersburg University in Russia.

Andrei Markov (1856-1922).

Markov Chains

Definition. A **Markov chain** is a discrete-time, discrete-state Markov stochastic process.

Since time is discrete, we can define the time index $T = 0, 1, 2, \ldots$. Similarly, since states are discrete, without loss of generality, we enumerate them as $1, 2, \ldots$.

We can look at a Markov chain as a random sequence

$$X(0), X(1), X(2), \ldots$$

Markov Chains

The Markov property means that only the value of $X(t)$ matters for predicting $X(t+1)$, so the conditional probability

$$
\begin{aligned}
p_{ij}(t) &= P\{X(t+1) = j | X(t) = i, X(t-1) = h, X(t-2) = g, \ldots\} \\
&= P\{X(t+1) = j | X(t) = i\}
\end{aligned}
$$

depends on i, j, and t only.

Definition. Probability $p_{ij}(t)$ is called a **transition probability**. Probability

$$p_{ij}^{(h)}(t) = P\{X(t+h) = j | X(t) = i\}$$

of moving from state i to state j after h transitions is an h-**step transition probability**.

Markov Chains

Definition. A Markov chain is **homogeneous**, or **stationary** if all its transition probabilities are independent of t.

Being **homogeneous** means that transition from i to j has the same probability at any time. Then

$$p_{ij}^{(h)}(t) = p_{ij}^{(h)}, \quad h = 1, 2, \ldots$$

In other words, here probabilities depend on elapsed time, not absolute time.

Homogeneous Markov Chains – Summarizing

- At time epochs $n = 1, 2, 3, \ldots$ the process changes from one state i to another state j with probability p_{ij}.
- Hence, we can write the one-step **transition** matrix $P = (p_{ij}, i, j \in S)$ where S is a set of states.
- Example: a frog hopping on 3 rocks. Put $S = \{1, 2, 3\}$. Then, the one-step **transition** matrix can be defined, for example, as follows:

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 5/8 & 1/8 & 1/4 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

Notice that sum of across rows is always 1. Is it by chance?

Homogeneous Markov Chains – Visualizing

Example. Again consider our frog example where frog hops on 3 rocks. We know that $S = \{1, 2, 3\}$ and the one-step **transition** matrix is:

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 5/8 & 1/8 & 1/4 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

We can gain some insight by drawing a picture: