# Lecture 6. Independence and Conditional Probability

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1. Total Probability

2. Bayes Rule

# The "Divide and Conquer Method"

Let $B_1, B_2, \ldots, B_k$ be disjoint and exhaustive events, i.e. $(B_i \cap B_j = \emptyset$ for $i \neq j$, and $\cup_{j=1}^{k} B_j = \Sigma$ Then for any event $A$

$$P(A) = \sum_{j=1}^{k} P(A \cap B_j) = \sum_{j=1}^{k} P(A|B_j)P(B_j)$$

This is called **the Law of Total Probability**. It allows us to compute the probability of a complicated event from knowledge of probabilities of simpler events.

# The Chess Example

During the chess tournament, there are three types of opponents for a certain player:

- $P(Type1) = 0.5,\ P(Win|Type1) = 0.3$
- $P(Type2) = 0.25,\ P(Win|Type2) = 0.4$
- $P(Type3) = 0.25,\ P(Win|Type3) = 0.5$

What is probability of player winning?

Solution Using the formula of total probability:

$$P(A) = \sum_{j=1}^{3} P(A \cap B_j)$$

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Solution Using the formula of total probability:

$$
\begin{aligned}
P(A) &= \sum_{j=1}^{3} P(A \cap B_j) = \sum_{j=1}^{3} P(A|B_j)P(B_j) \\
&= P(Type1) \times P(Win|Type1) \\
&+ P(Type2) \times P(Win|Type2) \\
&+ P(Type3) \times P(Win|Type3)
\end{aligned}
$$

Solution Using the formula of total probability:

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\begin{aligned}
P(A) &= \sum_{j=1}^{3} P(A \cap B_j) = \sum_{j=1}^{3} P(A|B_j)P(B_j) \\
&= P(Type1) \times P(Win|Type1) \\
&+ P(Type2) \times P(Win|Type2) \\
&+ P(Type3) \times P(Win|Type3) \\
&= 0.5 \times 0.3 + 0.25 \times 0.4 + 0.25 \times 0.5 \\
&= 0.375.
\end{aligned}
$$

## Thomas Bayes (1702–1761)

Note that $P(A|B)$ does not need to coincide with $P(B|A)$. Now, since $A \cap B = B \cap A$ and thus $P(A \cap B) = P(B)P(A|B) = P(B \cap A) = P(A)P(B|A)$, solving for $P(B|A)$ gives

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)}.$$

Using the formula on total probability, we can apply Bayes rule to a general case when we observe $B_1, B_2, \ldots, B_k$ disjoint and exhaustive events (and even to a case when $k = \infty$):

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}, \quad j = 1, 2, \ldots, k.$$

# The Radar Example

Recall that $B$ is event that the aircraft is flying above and $A$ is the event that the aircraft is detected by the radar. What is the probability that an aircraft is actually there, given that the radar indicates a detection? Recall $P(B) = 0.05$, $P(A|B) = 0.99$, $P(A|B^c) = 0.1$.

<u>Solution</u> Using Bayes rule:

$$P(\text{there is an aircraft} \quad | \quad \text{radar detects it}) = P(B|A)$$

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Solution Using Bayes rule:

$$\begin{aligned} P(\text{there is an aircraft} \quad | \quad \text{radar detects it}) &= P(B|A) \\ &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} \end{aligned}$$

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Solution Using Bayes rule:

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\begin{aligned}
P(\text{there is an aircraft} \quad | \quad \text{radar detects it}) &= P(B|A) \\
&= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} \\
&= \frac{0.99 \times 0.05}{0.99 \times 0.05 + 0.1 \times 0.95} \\
&= 0.3426.
\end{aligned}
$$