Lecture 3. Basic Probability Rules

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Definition of Probability

The simplest situation is when we have a finite number n of possible *equally likely* outcomes. Then, calculating probability reduces to counting, i.e.

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}.$$

This a very **rare** situation. Instead, we shall use a more general definition below.

The probability of event E, denoted by $P(E)$, is defined by the proportion of times that E occurs in a infinite sequence of separate trials, performed independently and under the same conditions. Hence,

$$P(E) = \lim_{n\to\infty} \frac{\text{times E happens}}{n}.$$

Axioms of Probability

1. for any event E, $P(E) \geq 0$.
2. $P(S) = 1$
3. For any sequence of disjoint events $E_1, E_2, \ldots$,

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i).$$

Properties of Probability

1. $0 \leq P(E) \leq 1$, i.e. probability is always between 0 and 100%
2. $P(E^c) = 1 - P(E)$. Hence, $P(\emptyset) = 0$.
3. For any two events E_1 and E_2,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

Properties of Probability: contd

4 $P(\text{some possible outcome}) = 1$, i.e. one of possible outcomes must happen.

Example 2. Suppose that there are two persons behind the door, one male and one female. So we have two possible outcomes M or F. One of the two persons comes in to the classroom, it can be him or her, but one outcome **must** happen for sure.

From these three rules, everything else can be derived. Kolmogorov (Russia, 1903-1987) and Levy (France, 1886-1971) were the greatest mathematicians of the 20th century who shaped the modern probability theory and its axiomatics.

Drawing from the Box

Example 3. During the early part of the Vietnam war, males could be exempt from serving in the military (and being sent to war) by attending college. Eventually this practice was ruled unfair (to people who could not afford college), so the college exemption was eliminated (from C. Wetzel).
In 1969 the Selective Service organized a lottery to decide which boys would be drafted into the military. They decided to draft people by randomly selecting people by their birth dates as follows:

1. A piece of paper with each calendar day was put it in a capsule, and the capsule was dropped into a large bin set up like a hamster treadmill.
2. The Head of the Selective Service stuck his hand into the bin and mixed things up for a few seconds, and then the bin was spun around for a few minutes.
3. Blind-folded, the Selective Service Head reached in and pulled out a capsule–that date was assigned draft number 1, the next date was draft 2, until the last one was draft 366 (including leap days).

Drawing from the Box: contd

Republican Alexander Pirnie, R-NY, draws the first capsule in the lottery drawing held on Dec. 1, 1969. The capsule contained the date, Sept. 14.

Suppose one draws two dates from the box (without replacement). What is the probability that both are in December? To answer this we must count all the ways in which one can make two draws. There exists

$$366 \times 365 = 133590$$

ways to draw two different dates. Out of these,

$$31 \times 30 = 930$$

are both in December. Hence, the probability that both dates are in December is

$$P(\text{two December dates}) = \frac{930}{133590} = 0.006961599.$$

The other way to see it is to note that there are $31/365$ ways to draw December on the first draw, and after removing that date there are $30/364$ chances for December on the second draw. Then

$$P(\text{two December dates}) = \frac{31}{366} \times \frac{30}{365} = 0.006961599.$$

Vietnam Draft Challenge

Some individuals noticed that those people with low numbers tended to be people born in the latter months of the year. In fact, the correlation between day of birth (1-366) and draft number was - 0.28.

Would this worry you?

What correlation should be if the lottery is fair?

Vietnam Draft Challenge

Thus, the lottery was challenged in court for unfairness. Despite the statistical evidence and the plausible flaws you hopefully identified, the judge ruled the lottery to be fair and refused to have it redone. However, the military did correct these flaws in subsequent years.

The Addition Rule

Suppose you want to know the probability that event A or event B happens? Then we can use the formula (known as the **union of events**)

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

The Addition Rule

- If A and B are incompatible (or mutually exclusive), then what is $P(A \text{ and } \mathrm{B}) = ?$

The Addition Rule

- If A and B are incompatible (or mutually exclusive), then what is $P(A \text{ and } \mathrm{B}) = 0$

Example 4. Consider a single draw from a deck of cards. Let A be the event of getting a king and B be the event of getting a queen. What is the probability of A **or** B?

Solution Using the **union** formula

$$P(A \text{ or } \mathrm{B}) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A \ and \ B)}$$

Example 4. Consider a single draw from a deck of cards. Let A be the event of getting a king and B be the event of getting a queen. What is the probability of A **or** B?

Solution Using the **union** formula

$$P(A \text{ or } \mathrm{B}) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A \text{ and } B)} = \frac{4}{52} + \frac{4}{52} - \frac{0}{52} = \frac{2}{13}.$$

Example 5. Consider a single draw from a deck of cards. Let A be the event that the card is a king and B be the event that the card is black. What is the probability of A **or** B?

Solution Using the **union** formula

$$P(A \text{ or } \mathrm{B}) = P(A) + P(B) - P(A \text{ and } \mathrm{B})$$

Example 5. Consider a single draw from a deck of cards. Let A be the event that the card is a king and B be the event that the card is black. What is the probability of A **or** B?

Solution Using the **union** formula

$$
\begin{aligned}
P(A \text{ or } \mathrm{B}) &= P(A) + P(B) - P(A \text{ and } \mathrm{B}) \\
&= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} \\
&= \frac{7}{13}.
\end{aligned}
$$

Example 6. Consider **two** draws from a deck of cards **with replacement**. Let A be the event that the 1st card is a king and B be the event that the 2rd card is a king. What is the probability of A **or** B?

Solution Using the **union** formula

$$P(A \text{ or } \mathrm{B}) = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A \text{ and } B)} =$$

Example 6. Consider **two** draws from a deck of cards **with replacement**. Let A be the event that the 1st card is a king and B be the event that the 2rd card is a king. What is the probability of A **or** B?

Solution Using the **union** formula

$$P(A \text{ or } \mathrm{B}) = \mathrm{P(A)+P(B)-P(A \text{ and } B)} = \frac{4}{52}+\frac{4}{52}-\frac{4}{52}\times\frac{4}{52} = 0.14792.$$