Lecture 12. Continuous distributions and densities

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1. Continuous Random Variables (see Lecture 11)
2. Uniform Distribution (see Lecture 11)
3. Exponential Distribution (see Lecture 11)
4. Gamma Distribution
5. Normal Distribution
6. The Normal approximation to the Binomial

Gamma Distribution – Definition

This distribution is used to model total waiting time of a procedure that consists of α independent stages, each stage with a waiting time having a distribution $Exp(\lambda)$.

The probability density function (pdf) of the **Gamma** distribution with parameters $\alpha > 0$ and $\lambda > 0$ is:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad x > 0.$$

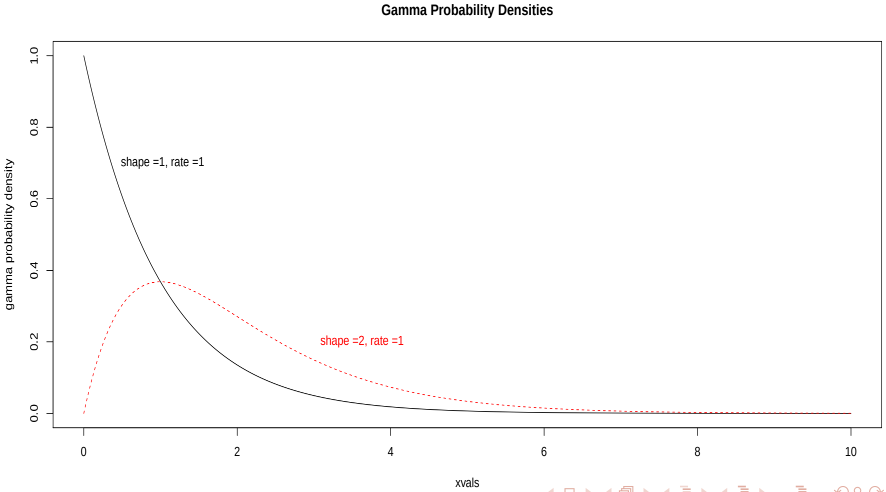
Here λ is called the *rate*, or *frequency* parameter, and α is called the *shape* parameter; $\Gamma(\alpha)$ is the Gamma function, i.e. an integral (see section 4.2.3 and 12.3.5).

Gamma Distribution – Definition

If a r.v. X follows a Gamma distribution, we denote it as $X \sim Gamma(\alpha, \lambda)$.

Remark. When α is an integer, the gamma random variable can be represented as the sum of α independent and identically exponentially distributed random variables with parameter λ.

It follows that $\Gamma(1, \lambda) \equiv Exp(\lambda)$.


Gamma Probability Densities
shape =1, rate =1
shape =2, rate =1
gamma probability density
xvals
0.0
0.2
0.4
0.6
0.8
1.0
0
2
4
6
8
10

Gamma Distribution – Properties

Properties.

- Range

$$x > 0$$

- Expectation

$$E(X) = \frac{\alpha}{\lambda}$$

- Variance

$$\mathrm{var}(X) = \frac{\alpha}{\lambda^2}$$

- Standard Deviation

$$\mathrm{std}(X) = \frac{\sqrt{\alpha}}{\lambda}$$

Example: Space Shuttle

Example. Engineers designing the next generation of space shuttles plan to include two fuel pumps – one active, the other in reserve.

If the primary pump malfunctions, the second is automatically brought on line.

Suppose a typical mission is expected to require that fuel be pumped for at most 50 hours. According to the manufacturer's specifications, pumps are expected to fail once every 100 hours.

What are the chances that such a fuel pump system would not remain functioning for the full 50 hours?

Example: Space Shuttle

Solution. We are given that the average number of failures in a 100-hour interval is 1, and hence the rate $\lambda = 1/100$. Since there are two pumps, i.e. $\alpha = 2$.

Let Y denote the time elapsed until the 2nd pump breaks down (i.e. both pumps are broken). Then, the probability density function of Y is:

$$f(x) = \frac{1/100^2}{\Gamma(2)} x^{2-1} e^{-x/100} = \frac{1}{10000} x^1 e^{-x/100}, \quad x > 0.$$

Hence,

$$P(Y < 50) = \int_0^{50} \frac{1}{10000} x^1 e^{-x/100} dx.$$

Example: Space Shuttle

We will integrate it by parts, i.e. $\int u dv = uv - \int v du$. Let $u = x$ and $dv = e^{-\frac{x}{100}} dx$.

Thus, $du = dx$ and $v = -100e^{-\frac{x}{100}}$.

Example: Space Shuttle

Now, we obtain

$$
\begin{aligned}
P(x < 50) &= \frac{1}{10000}\left\{uv - \int vdu\right\} \\
&= \frac{1}{10000}\left\{[-100xe^{-\frac{x}{100}}]_{x=0}^{50} - \int_0^{50} (-100)e^{-\frac{x}{100}}\,dx\right\}
\end{aligned}
$$

Example: Space Shuttle

Now, we obtain

$$\begin{aligned}
P(x<50) &= \frac{1}{10000}\left\{uv-\int vdu\right\} \\
&= \frac{1}{10000}\left\{[-100xe^{-\frac{x}{100}}]_{x=0}^{50}-\int_0^{50}(-100)e^{-\frac{x}{100}}\,dx\right\} \\
&= \frac{1}{10000}\left\{-5000e^{-\frac{1}{2}}+\int_0^{50}100e^{-\frac{x}{100}}\,dx\right\} \\
&= \frac{1}{10000}\left\{-5000e^{-\frac{1}{2}}+100[(-100)e^{-\frac{x}{100}}]_{x=0}^{50}\right\} \\
&= \frac{1}{10000}\left\{-5000e^{-\frac{1}{2}}-10000[e^{-\frac{1}{2}}-1]\right\}=1-\frac{3}{2}e^{-\frac{1}{2}}.
\end{aligned}$$

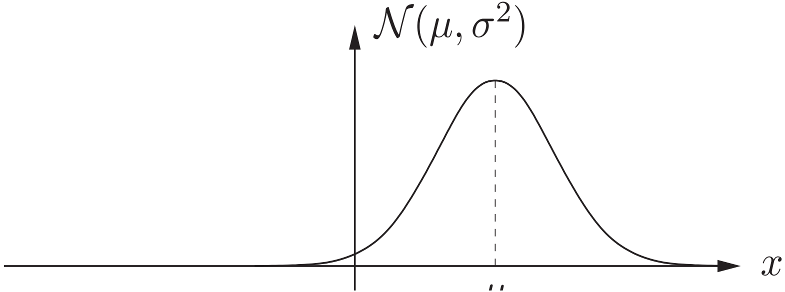
Normal, or Gaussian Distribution – Definition

The probability density function (pdf) of the **normal**, or **Gaussian** distribution with mean μ and variance σ^2 is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

If a r.v. X follows a normal distribution, we denote it as $X \sim N(\mu, \sigma^2)$.

Normal r.v.s are frequently encountered in nature, e.g., thermal and shot noise in electronic devices are normally distributed, and very frequently used in modelling various social, biological, and other phenomena.


$\mathcal{N}(\mu, \sigma^2)$
x

Normal Distribution – Properties

Properties.

- Range

$$-\infty < x < \infty$$

- Expectation

$$E(X) = \mu$$

- Variance

$$\mathrm{var(X)} = \sigma^2$$

- Standard Deviation

$$\mathrm{std(X)} = \sigma$$

Normal Distribution – Properties

Properties.

- $$P(\mu - \sigma \leq X \leq \mu + \sigma) = 0.68$$
- $$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 0.95$$
- $$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 0.997$$
- If $X \sim N(\mu, \sigma^2)$, then $Y = a + bX$ is also normal with mean $a + b\mu$ and variance $b^2\sigma^2$.

Standard Normal Distribution – A special Case of mean 0 and variance 1

Definition. Normal distribution with *standard parameters* mean $\mu = 0$ and standard deviation $\sigma = 1$ is called **Standard Normal Distribution**.

Notations:

- We typically denote **standard normal** random variable by Z.
- The **standard normal** probability density function is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty.$$

- The **standard normal** cumulative density function is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \qquad -\infty < x < \infty.$$

Normal Distribution – Reading a Table

Let X be a normal random variable with mean μ and variance σ^2. We standardize X by defining a new random variable Z given by

$$Z = \frac{X - \mu}{\sigma}.$$

Note that Z is standard normal r.v., i.e. $Z \sim N(0,1)$. So, mean of Z is ?

Normal Distribution – Reading a Table

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Normal Distribution – Reading a Table

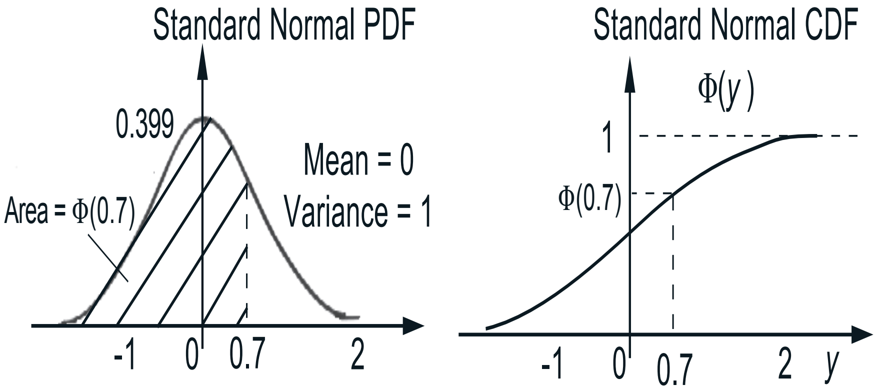
Let X be a normal random variable with mean μ and variance σ^2. We standardize X by defining a new random variable Z given by

$$Z = \frac{X - \mu}{\sigma}.$$

Note that Z is standard normal r.v., i.e. $Z \sim N(0, 1)$. So, mean of Z is 0 and variance of Z is 1.

This fact allows us to calculate the probability of any event defined in terms of X: we redefine the event in terms of Z, and then use the standard normal table.

Normal Distribution – Reading a Table



Example: Snowfall

Example. The annual snowfall at a particular geographic location is modeled as a normal random variable with a mean of $\mu = 60$ inches, and a standard deviation of $\sigma = 20$.

What is the probability that this years snowfall will be at least 80 inches?

Example: Snowfall

Solution. Let X be the snow accumulation, viewed as a normal random variable, and let

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 60}{20}$$

be the corresponding standard normal random variable. We want to find

$$\begin{aligned} P(X \geq 80) &= P\left(\frac{X - 60}{20} \geq \frac{80 - 60}{20}\right) \\ &= P\left(Z \geq \frac{80 - 60}{20}\right) = P(Z \geq 1) = 1 - \Phi(1), \end{aligned}$$

where Φ is the CDF of the standard normal.

Example: Snowfall

Solution. We read the value $\Phi(1)$ from the table:

$$\Phi(1) = 0.8413.$$

Hence,

$$P(X \geq 80) = 1 - 0.8413 = 0.1587.$$

Example: Candidates Scoring

Example. The marks of 500 candidates in an examination are normally distributed with a mean of 45 marks and a standard deviation of 20 points.

If 20% of candidates obtain a distinction by scoring x points or more, estimate the value of x.

Example: Candidates Scoring

Solution. We have $X \sim N(45, 20^2)$, and we want to find x such that $P(X > x) = 0.2$.

This implies that we want to find x such that $P(X < x) = 0.8$.

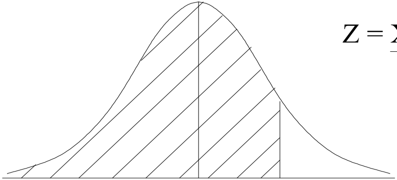
Example: Candidates Scoring

$X \sim N(45, 400)$

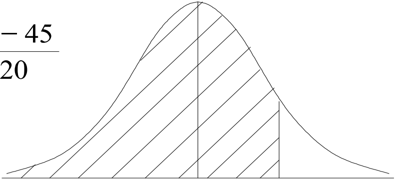
$Z \sim N(0, 1)$

$P(X < x) = 0.8$

$P(Z < 0.84) = 0.8$



$$Z = \frac{X - 45}{20}$$



Example: Candidates Scoring

Solution. By standardizing, we get

$$P(X < x) = P\left(\frac{X-45}{20} < \frac{x-45}{20}\right) = P\left(Z < \frac{x-45}{20}\right) = 0.8$$

.

From the table, we know that $P(Z < 0.84) \approx 0.84$. Hence,

$$\frac{x-45}{20} \approx 0.84.$$

Thus, $x \approx 45 + 20 \times 0.84 = 61.8$.

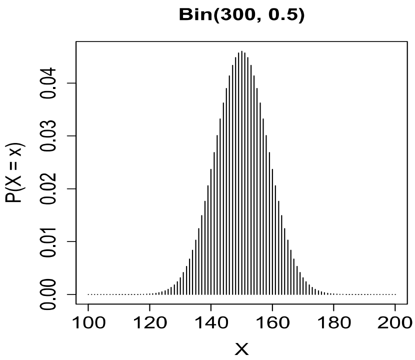
Normal Approximation to the Binomial Distribution

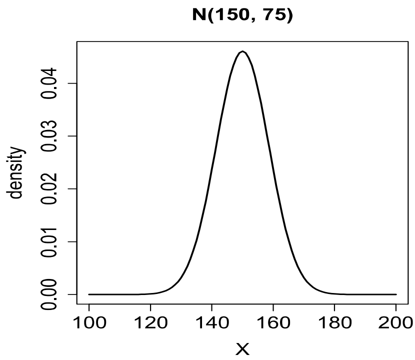
Under certain conditions we can use the normal distribution to approximate the Binomial distribution.

This can be very useful when we need to sum up a large number of Binomial probabilities to calculate the probability that we want.

Normal Approximation to the Binomial Distribution

For example, compare Binomial $(300, 0.5)$ and $N(150, 75)$ distributions which both have the same mean and variance.





Normal Approximation to the Binomial Distribution

In general, if $X \sim Binomial(np)$ then its mean $\mu = np$ and its variance $\sigma^2 = npq$ where $q = 1 - p$.

Then, for large n and p not too small or too large

$$X \sim N(np, npq).$$

As a rule of thumb, The farther p is from $1/2$, the larger n needs to be for the approximation to work.

Thus, as a rule of thumb, only use the approximation if $np \geq 10$ and $n(1-p) \geq 10$.

Example: Senior Men

Example. Suppose that you know that 10% of men above 65 y.o. are bald.

what is the probability that fewer than 100 in a random sample of 818 men are bald?

Example: Senior Men

Solution. Here mean $\mu = np = 818 * 0.1 = 81.8$ and standard deviation $\sigma = \sqrt{np(1-p)} = \sqrt{818 \times 0.1 \times 0.9} = 8.5802.$

Hence, by standardization

$$Z = \frac{n - \mu}{\sigma} = \frac{100 - 81.8}{8.58} = 2.12.$$

From the normal table we find that 98.3% of the time there will be fewer than 100 bald men.