Lecture 15 and 16. Stochastic Processes

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Stochastic Process: Definition

A **stochastic process** (also called **random process**) is a random variable that also depends on time.

Thus, stochastic processes are used to model random experiments that evolve in time:

- Received sequence/waveform at the output of a communication channel
- Packet arrival times at a node in a communication network
- Thermal noise in a resistor
- Scores of an NBA team in consecutive games
- Daily price of a stock
- Winnings or losses of a gambler
- Earth movement around a fault line

Stochastic Process: Definition

More formally, a **stochastic process** is therefore a function of two arguments, $X(t, \omega)$, where:

- $t \in \mathrm{T}$ is time, with T being a set of possible times, usually $[0, \infty)$, (∞, ∞), $\{0, 1, 2, \ldots\}$, or $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$;
- $\omega \in \Omega$, as before, is an outcome of an experiment, with Ω being the whole sample space.
- Values of $X(t, \omega)$ are called **states**. We denote the set of all states by S.

Stochastic Process: Definition

- At any fixed time t, we observe a random variable $X_t(\omega)$, a function of a random outcome.

- On the other hand, if we fix ω, we obtain a function of time $X_\omega(t)$. This function is called a **realization**, a **sample path**, or a **trajectory** of a process $X(t, \omega)$.

  Here $X_\omega(t)$ is also called a **time series** process.

Stochastic Process: Classification via States

- Stochastic process $X(t, \omega)$ is **discrete-state** if variable $X_t(\omega)$ is discrete for each time t.
- Stochastic process $X(t, \omega)$ is a **continuous-state** if $X_t(\omega)$ is continuous.

Stochastic Process: Classification via States

- Stochastic process $X(t, \omega)$ is **discrete-time** process if the set of times T is is discrete, that is, it consists of separate, isolated points.
- Stochastic process $X(t, \omega)$ is **continuous-time** process if the set of times T is is a connected, possibly unbounded interval.

Stochastic Process: Examples

Example. Find the index set and the state space of daily hits of STAT3341 ELearning discussion forum. Classify this stochastic process.

Solution. Here the index set $T = 1, 2, \ldots$ are days and are countable. Hence, the process is discrete-time.

Similarly, the state space $S = 0, 1, 2, \ldots$ are hits and also are countable. Hence, the process is discrete-state.

Stochastic Process: Examples

Example. Find the index set and the state space of daily stock prices. Classify this stochastic process.

Solution. Here the index set $T = 1, 2, \ldots$ are days and are countable. Hence, the process is discrete-time.

Now, the state is in $ and potentially any number from the interval $(0, \infty)$, hence $S = (0, \infty)$. Thus, the process is continuous-state.

Stochastic Process: Examples

Example. What is about wind speed in Dallas?

Stochastic Process: Examples

Example. What is about wind speed in Dallas?

Answer. It changes continuously and never jumps from one value to another.

Hence, it is continuous-state $(0, +\infty)$ and continuous-time: from Earth creation till Earth exists, i.e. almost like this $(-\infty, \infty)$.

Counting Process

Definition A stochastic process X is counting if $X(t)$ is the number of items counted by the time t.

As time passes, one can count additional items, therefore, sample paths of a counting process are always **non-decreasing**.

Also, counts are nonnegative integers, $X(t) \in \{1, 2, 3, \dots\}$. Hence, counting processes are **discrete state**.

Example. Number of gray hair accumulated over years...

Markovian Property

We will discuss it in more details later but basic definitions will be useful.

Definition A stochastic process X_t is a Markov process if $P(future|past, present) = P(future|present)$. I.e., future depends on the past only through present.

Or more generally, if

$$P[X(t) \in A|X(t_1) \in A_1, \ldots, X(t_n) \in A_n] = P[X(t) \in A|X(t_n) \in A_n],$$

for $t_1 < \ldots < t_n < t$ and any sets $A, A_1, \ldots, A_n$.

Markovian Property – Example

Example. We can model economic mobility in a society as a Markov process, where a person's probability that they will be in a certain economic class is only dependent on their parents' economic class.

Bernoulli Process

We start from one of the simplest stochastic processes, namely, **Bernoulli** process:

- The Bernoulli process is an infinite sequence $X_1, X_2, \ldots$ of i.i.d. Bern(p) random variables.
- The outcome from a Bernoulli process is an infinite sequence of 0s and 1s.
- A Bernoulli process is often used to model occurrences of random events; $X_n = 1$ if an event occurs at time n, and 0, otherwise.

The simplest and familiar example is tossing a coin. Each tossing, we record the result and form a sequence $(H, H, T, H, \ldots)$. This is a Bernoulli process.

Bernoulli Process

Three associated random processes of interest are:

- Binomial counting process: The number of events in the interval [1, n]
- Arrival time process: The time of event arrivals
- Interarrival time process: The time between consecutive event arrivals

Independence of the Bernoulli trials implies that this process is **memoryless**. Given that the probability p is known, past outcomes provide no information about future outcomes.

If the process is infinite, then from any point the future trials constitute a Bernoulli process identical to the whole process, i.e. it is a so-called the **fresh-start property**.

Binomial Process

Once again, let us consider a sequence of independent Bernoulli trials with probability of success p and count "successes". We can now introduce a bit more sophisticated process.

Definition. Binomial process $X(n)$ is the number of successes in the first n independent Bernoulli trials, where $n = 0, 1, 2, \ldots$.

Binomial process $X(n)$ is

- **discrete-time**
- **discrete-space**
- **counting**
- **Markovian**.

Verify these properties as an exercise at home.

Binomial vs. Bernoulli

The Bernoulli process W_n can be obtained from the Binomial counting process by simple differencing:

$$W_n = X_n - X_{n-1},$$

where $X_0 = 0$ and for $n = 1, 2, \ldots$.

Note that here X_0 is our fixed initial value before any trials.

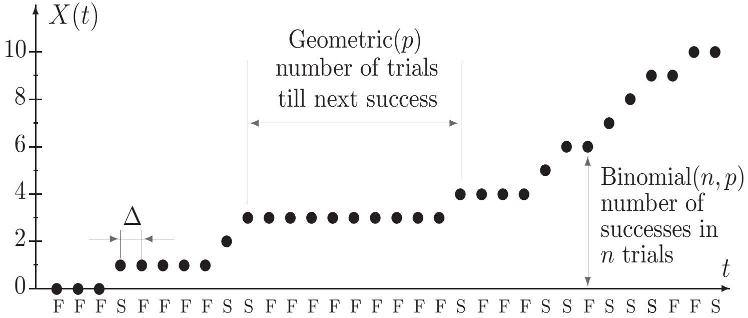
Binomial Process – More Notations

We denote number of successes in n trials by $X(n)$ and number of trials between consecutive successes by Y.

Note that

- the distribution of $X(n)$ at any time n is Binomial(n, p),
- the number of trials Y between two consecutive successes is Geometric(p).

Binomial process



Here S and F are success and failure respectively.

Binomial Process – Relation to Frames

The "time" variable n actually measures the number of trials. It is not expressed in minutes or seconds. However, it can be related to real time.

Let Bernoulli trials occur at equal time intervals, e.g. every Δ seconds. This time interval Δ is called a **frame**.

Then we can compute that n trials occur during time $t = n\Delta$.

Binomial Process – Relation to Frames

Hence, the value of the process at time t has Binomial distribution with parameters $n = t/\Delta$ and p.

The expected number of successes during t seconds is therefore

$$E\left\{X\left(\frac{t}{\Delta}\right)\right\} = \frac{t}{\Delta}p,$$

which amounts to

$$\lambda = \frac{p}{\Delta}$$

successes per second.

We say that λ above is the **arrival rate**.

Binomial Process – Summarizing

- λ is the **arrival rate**
- Δ is the **frame** size
- p is probability of arrival (success) during one frame (trial)
- $X(t/\Delta)$ is number of arrivals by the time t
- T is **interarrival** time. Since interarrival period consists of a Geometric number of frames Y, each frame taking Δ seconds, the interarrival time $T = Y\Delta$.

  Hence, T is a rescaled geometric r.v., and we get

  $$E(T) = E(Y\Delta) = E(Y)\Delta = \Delta/p$$

  and

  $$Var(T) = Var(Y)\Delta^2 = \frac{1-p}{p^2}\Delta^2.$$

Binomial Process – Summarizing

These concepts, arrival rate and interarrival time, deal with modeling arrivals of jobs, messages, customers, and so on with a Binomial counting process.

The key assumption in such models is that no more than 1 arrival is allowed during each Δ-second frame.

If this assumption appears unreasonable, and two or more arrivals can occur during the same frame, one has to model the process with a smaller Δ.

Packet Arrive Example

Example. Packet arrivals at a node in a communication network can be modeled by a Bernoulli process with $p = 0.09$.

- What is the probability that 3 packets arrive in the interval $[1, 20]$, 6 packets arrive in $[1, 40]$ and 12 packets arrive in $[1, 80]$?

Packet Arrive Example

Solution. Let $X(n)$ be the number of packets arriving in interval $[1, n]$. Note that $X(n)$ is a Binomial counting process with probability of success p. First, we want to find the following probability $P\{X(20) = 3, X(40) = 6, X(80) = 12\}$, which is equal to

$$P\{X(20) = 3, X(40) - X(20) = 3, X(80) - X(40) = 6\}.$$

So, now we can work with increments, and by the fresh start property of the Bernoulli process, we notice that the number of events in the interval $[k+1, n+k]$, $n \geq 1$, $k \geq 1$ is identical to that of $[1, n]$, i.e., X_n and $(X_{k+n} - X_k)$ are identically distributed. By the independence property of the Bernoulli process this is equal to

$P\{X(20) = 3\} \times P\{X(40) - X(20) = 3\} \times P\{X(80) - X(40) = 6\}$.

Packet Arrive Example

Now, by the fresh start property of the Bernoulli process

$$P\{X(40) - X(20) = 3\} = P\{X(20) = 3\},$$
$$P\{X(80) - X(40) = 6\} = P\{X(40) = 6\}.$$

Thus,

$$\begin{aligned} P\{X(20) &= 3, X(40) = 6, X(80) = 12\} \\ &= (P\{X(20) = 3\})^2 \times P\{X(40) = 6\} \\ &= \left[C(20,3)(0.09)^3(0.91)^{17}\right] \times \left[C(40,6)(0.09)^6(0.91)^{34}\right] \\ &= (0.1607)^2 \times 0.0826 = 0.0021. \end{aligned}$$

Supercomputer Example

Example. Tasks are sent to a supercomputer at an average rate of 6 tasks per minute. Their arrivals are modeled by a binomial counting process with 2-second frames.

1. Compute the probability of more than 2 tasks sent during 10 seconds?
2. Compute the probability of more than 20 tasks sent during 100 seconds.

Supercomputer Example

Solution. We have $\lambda = 6/min = 0.1/sec$ and $\Delta = 2$. Hence, $p = \lambda\Delta = 0.1 \times 2 = 0.2$.

First, notice that 10 seconds represents $10/\Delta = 10/2 = 5$ frames. The number of tasks during 5 frames has Binomial distribution with $n = 5$ and $p = 0.2$. Hence,

$$\begin{aligned} P\{X > 2\} &= 1 - P(X \leq 2) \\ &= 1 - (P(X = 0) + P(X = 1) + P(X = 2)) \\ &= \sum_{k=0}^{2} C(5, k)p^k(1 - p)^{n-k} \\ &= 0.0579. \end{aligned}$$

Supercomputer Example

The number of tasks during 100 seconds, i.e. 50 frames, has Binomial distribution with $n = 50$ and $p = 0.2$.

For large n, we can use normal approximation with $\mu = np = 10$ and $\sigma = \sqrt{np(1-p)} = 2.83$:

$$\begin{aligned} P\{X > 2\} &= P\left(\frac{X-\mu}{\sigma} > \frac{20-\mu}{\sigma}\right) \\ &= P\{Z > 3.53\} = 0.0002. \end{aligned}$$

You can improve normal approximation by using the continuity correction but we skip this step here.

Poisson Process

Going from discrete time to continuous time, Poisson process is the limiting case of a Binomial counting process as $\Delta \to 0$.

Definition. Poisson process is a continuous-time counting process obtained from a Binomial counting process when its frame size Δ decreases to 0 while the arrival rate λ remains constant.

Poisson process is Markovian. (Show it as an exercise at home.)

Poisson Process

Indeed, let $X(t)$ be a Binomial process. Then, $X(t)$ is number of events during time t or in other words, number of events during $n = t\Delta$ frames.

When $\Delta \to 0$,

$$X(t) = Binomial(n = \frac{t}{\Delta}; p) \to Poisson(\lambda t).$$

Hence,

$$E(X(t)) = \lambda t, \quad Var(X(t)) = \lambda t.$$

Poisson Process

The **interarrival** time T becomes a random variable with the exponential c.d.f..

Indeed, since $T = Y\Delta$ and $t = n\Delta$,

$$\begin{aligned} F_T(t) &= P(T \leq t) = P(Y \leq n) = 1 - (1-p)^n \\ &= 1 - \left(1 - \frac{\lambda t}{n}\right)^n \to 1 - e^{\lambda t}. \end{aligned}$$

Here in the 3rd equality we used the fact that Y has a Geometric distribution with p and in the last relationship, we used the "Euler limit", i.e.

$$(1 + x/n)^n \to e^x, \quad as \;\; n \to \infty.$$

Poisson Process

Finally, the time T_k of the k-th arrival is the sum of k Exponential interrarival times that has Gamma(k, λ) distribution. Thus, we can use the Gamma-Poisson formula:

$$P\{T_k \leq t\} = P\{k\mathrm{-th\ \ arrival\ \ before\ \ time\ \ t}\} = \mathrm{P\{X(t) \geq k\}},$$

where T_k is Gamma(k, λ) and $X(t)$ is Poisson(λt).

Similarly,

$$P\{T_k > t\} = P\{X(t) < k\}.$$

Messages

Example. Messages arrive at an electronic message center at random times, with an average of 9 messages per hour.

1. What is the probability of receiving at least five messages during the next hour?
2. What is the probability of receiving exactly five messages during the next hour?

Messages

Solution. There is no restriction on the number of arrived messages arriving at random times with the given arrival rate. Hence, we can Poisson process to model arrival of messages.

Let X be the number of messages during the next hour. Then X has Poisson distribution with rate $\lambda = 9/hour$. Hence,

$$P\{X \geq 5\} = 1 - P\{X < 5\} = 1 - \sum_{k=0}^{4} e^{-9}\frac{9^x}{x!} = 1 - 0.055 = 0.945$$

Similarly,

$$P\{X = 5\} = e^{-9}\frac{9^5}{5!} = 0.061.$$