Lecture 11. Continuous distributions and densities

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Continuous Random Variables

Recall that a **continuous** random variable is a r.v. that can take on an **uncountable** number of values.

Example. Suppose the experiment is measuring the time to complete a computing task, and let X be total time taken.

This is a **continuous** random variable. Indeed, since time is continuous, the range of values that X can take is a continuum, and therefore uncountable.

Continuous vs. Discrete

To help understand the distinction between countable and uncountable sets, keep in mind:

- The set of all integer numbers is countable.
- The set of all real numbers is uncountable (a continuum).

Continuous Random Variables: Integration vs. Summing Up

How do we describe probabilities of various computing times in the previous example?

Idea. For discrete r.v., we sum a probability mass function (PMF) over points in a set to find its probability.

For continuous r.v., we cannot sum over possible outcomes because they are uncountable. However, we can **integrate** a probability density over a set to find its probability.

Probability Density Function

A continuous r.v. X can be specified by a **probability density function** $f_X(x)$ (pdf) such that, for any event A,

$$P\{X \in A\} = \int_A f(x)dx.$$

For example, for $A = (a, b]$, the probability can be computed as

$$P\{X \in (a, b]\} = \int_a^b f(x)dx.$$

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For a continuous r.v. X, $P\{X = x\} = 0$.

Probability Density Function – Properties

Properties of $f_X(x)$:

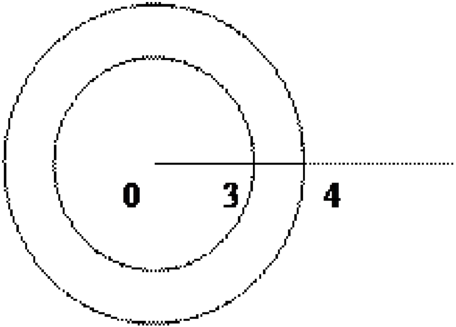
1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

Note that $f(x)$ should not be interpreted as the probability that $X = x$.

In fact, $f(x)$ is **not** a probability measure, e.g., it can be > 1.

Random Sampling – Path to Uniform Distribution

Suppose that we randomly choose a point within a circle of radius 4.



What is the probability that it is within distance 3 from the center?

Random Sampling – Path to Uniform Distribution

Obviously, all the points within the inner circle will do (these are the favorable cases).

However, there is an infinity of them, and the number of possible points (that is, all the points in the larger circle) is also infinite.

The problem is that we cannot divide an infinite number by an infinite number.

The solution is clear: We need to do is to divide the area of the inner circle by that of the outer circle. I.e.,

$$P(point\ \ is\ \ within\ \ distance\ \ 3\ \ from\ \ center) = \frac{9\pi}{16\pi} = \frac{9}{16}.$$

Uniform Distribution – where do you meet it in real world?

In general, when a point is randomly chosen in an interval (length), an area, or a volume, we have a continuous **uniform distribution**,

and the probability of an event is obtained by dividing the favorable length, area, or volume by the total length, area, or volume.

Many real life situations can be modeled in this way.

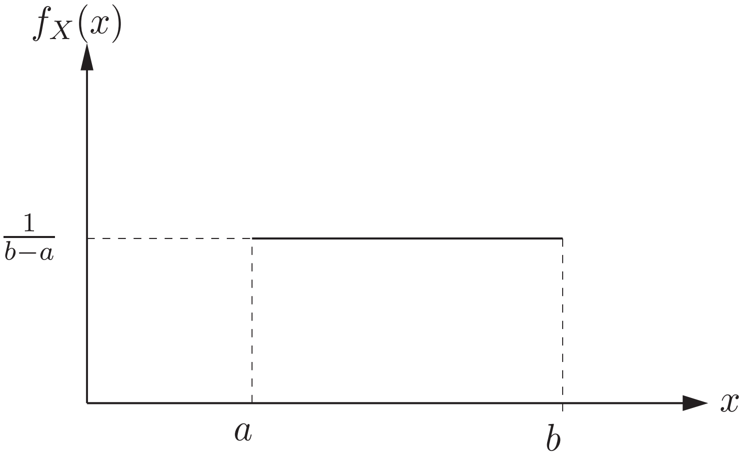
Uniform Distribution – Definition

The probability density function (pdf) of the continuous uniform distribution over the interval $[a, b]$ is:

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

If a r.v. X follows a uniform distribution, we typically denote it as $X \sim U[a, b]$.

Uniform r.v. is commonly used to model quantization noise and finite precision computation error (roundoff error).


$f_X(x)$
$\frac{1}{b-a}$
a
b
x

Uniform Distribution – Properties

Properties.

- Range

$$a \leq x \leq b$$

- Expectation

$$E(X) = \frac{a+b}{2}$$

- Variance

$$\mathrm{var}(X) = \frac{(b-a)^2}{12}$$

- Standard Deviation

$$\mathrm{std}(X) = \frac{b-a}{\sqrt{12}}$$

Example: Arrive on time

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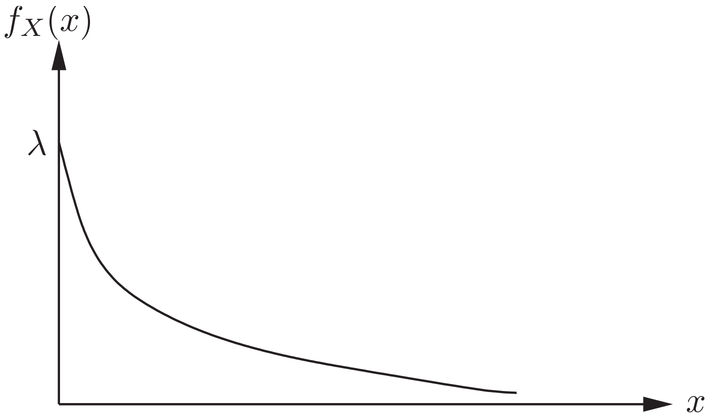
Exponential Distribution – Definition

The probability density function (pdf) of the **exponential** distribution for $\lambda > 0$ is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

If a r.v. X follows an exponential distribution, we typically denote it as $X \sim Exp(\lambda)$.

Exponential r.v. is commonly used to model inter-arrival time in a queue, i.e., time between two consecutive packet or customer arrivals, service time in a queue, and lifetime of a particle, etc.


$f_X(x)$
λ
x

Exponential Distribution – "No Memory"

We say that the exponential distribution "loose memory", or "memoryless".

Suppose that an exponential variable T represents waiting time. Memoryless property means that the fact of having waited for t minutes gets "forgotten", and it does not affect the future waiting time.

Regardless of the event $T > t$, when the total waiting time exceeds t, the remaining time still has an exponential distribution with the same parameter λ.

To see why, you can also check the pdf plot of the exponential distribution.

Exponential Distribution – Properties

Properties.

- Range

$$x > 0$$

- Expectation

$$E(X) = \frac{1}{\lambda}$$

- Variance

$$\mathrm{var}(X) = \frac{1}{\lambda^2}$$

- Standard Deviation

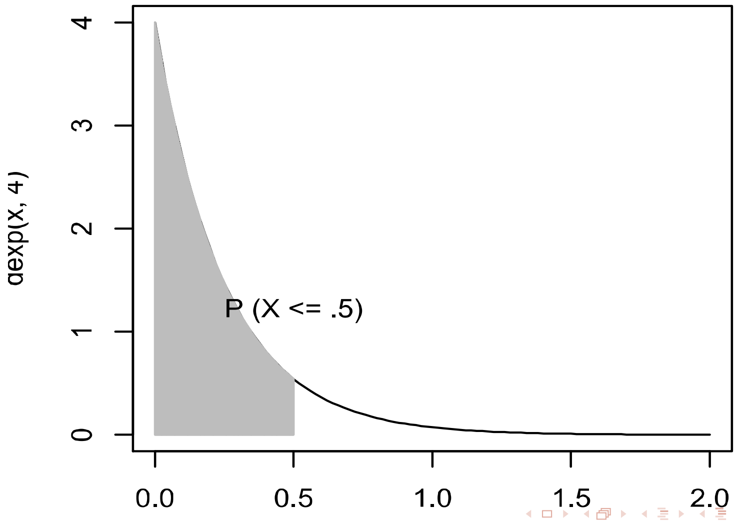
$$\mathrm{std}(X) = \frac{1}{\lambda}$$

Example: Waiting Time

Example. Suppose that jobs arrive every 15 seconds on average, $\lambda = 4$ per minute.

What is the probability of waiting less than or equal to 30 seconds, i.e 0.5 min?

Example: Waiting Time



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Solution. We are interested in

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$$

Example: Waiting Time

Solution. We are interested in

$$\begin{aligned} P(T \leq 0.5) &= \int_0^{0.5} 4e^{-4t}\,dt \\ &= \left[-e^{-4t}\right]_{t=0}^{0.5} \\ &= 1 - e^{-2} \\ &= 0.86 \end{aligned}$$