DS-GA 1018: Homework 2

Due Monday October 15$^{\text{th}}$ at 5:00 pm

Problem 1 (20 points): Consider a causal AR(2) process of the form:

$$X_t = \phi_2 X_{t-2} + W_t, \tag{1}$$

with $0 < \phi_2 < 1$ and $W_t \sim \mathcal{N}(0, \sigma_w^2)$.

i. (6 points): Assume that we have observations $\{x_1, x_2\}$. Derive the mean and variance of a future observation x_t with $t > 2$. (*Hint: you'll need your solutions from Problem 4 of Homework 1.*)

ii. (2 points): What is the mean for $t = 3$ and $t = 4$. Explain the intuition behind this result.

iii. (2 points): What is the covariance for $t = 3$ and $t = 4$. Explain the intuition behind this result.

iv. (2 points): What is the mean and variance of x_t as $t \to \infty$? Explain the intuition behind this result.

v. (6 points): Assume that we have observations x_1. Derive the mean vector and covariance matrix of a future set of observations $\{x_t, x_{t+1}\}$ with $t > 1$.

vi. (2 points): What is the mean vector and covariance matrix of $\{x_t, x_{t+1}\}$ for $t = 2$? Explain the intuition behind this result.

Problem 2 (10 points): Consider the generalization of ARCH(1) model given by:

$$R_t = \delta + Y_t \tag{2}$$
$$Y_t = \sigma_t W_t, \quad W_t \sim N(0,1) \tag{3}$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2, \tag{4}$$

where $\alpha_0 > 0$, $1 > \alpha_1 > 0$, and δ is a constant value.

i. (2 points): Derive the mean μ_{R_t}.

ii. (6 points): Derive the covariance $\gamma_{R_t,R_{t+h}}$.

iii. (2 points): Is R_t a (weak) stationary process? Justify your answer quantitatively.

Problem 3 (10 points): Consider the latent space model we presented in class defined by:

$$\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{w}_t \tag{5}$$
$$\mathbf{x}_t = \mathbf{C}\mathbf{z}_t + \mathbf{v}_t \tag{6}$$

where the latent space $\mathbf{z}$ is has dimension d and the data $\mathbf{x}$ has dimension n. Our noise is being drawn from $\mathbf{w}_t \sim \mathcal{N}(0, \mathbf{Q})$ and $\mathbf{v}_t \sim \mathcal{N}(0, \mathbf{R})$.

i. (4 points): Assume that, $n = d$ and that we have $C = \alpha\mathbb{I}$ and $R = \sigma_v^2\mathbb{I}$. Write the mean $\mu_{t|t} = \mu_{\mathbf{z}_t|\mathbf{z}_{t-1},\mathbf{x}_t}$ and covariance $\Sigma_{t|t}$ in terms of the mean $\mu_{t|t-1} = \mu_{\mathbf{z}_t|\mathbf{z}_{t-1}}$ and covariance $\Sigma_{t|t-1}$. Simplify as much as possible.

ii. (2 points): What happens in the limit $\sigma_v \to 0$? Demonstrate the answer quantitatively and explain the intuition behind this limit.

iii. (2 points): What happens in the limit $\sigma_v \to \infty$? Demonstrate the answer quantitatively and explain the intuition behind this limit.

iv. (2 points): What happens in the limit $\alpha \to 0$? Demonstrate the answer quantitatively and explain the intuition behind this limit.

Problem 4 (5 points): Consider a modified version of our latent space model that depends on a set of **observed** values $\mathbf{y}_t$ as follows:

$$\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}\mathbf{y}_t + \mathbf{w}_t, \tag{7}$$

otherwise all the other components are identical to those described in Problem 3.

i. (5 points): Derive how this new process changes the filtering step of our Kalman filtering.