DS-GA 3001 001 | Lecture 2

Reinforcement Learning

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DS-GA 3001 RL Curriculum

Reinforcement Learning:

- Introduction to Reinforcement Learning
- **Multi-armed Bandit**
- Dynamic Programming on Markov Decision Process
- Model-free Reinforcement Learning
- Value Function Approximation (Deep RL)
- Policy Function Approximation (Actor-Critic)
- Planning from a Model of the Environment (AlphaZero)
- Aligning AI systems to Human Preferences (ChatGPT)
- Examples of Industrial Applications
- Advanced Topics

Multi-armed Bandit

Last lecture:

- What is Reinforcement Learning?
- Key components of Reinforcement Learning
- Introduction to the Gym Python library

Today:

- **Multi-armed Bandit with action values**
- **Upper Confidence Bound**
- **Bayesian Bandit model**

Multi-armed Bandit with action values

What is Multi-armed Bandit?

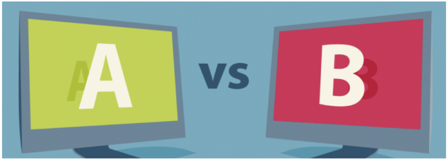
What is Multi-armed Bandit?

The Multi-armed Bandit problem

- Reinforcement learning uses data it receives by acting to evaluate actions, which creates a need to explore (correct actions are not given)
- A Bandit is a RL problem involving learning to act in only one situation: 1 state, k possible actions
- No sequential structure, past actions do not influence the future: the distribution of reward r_t given a_t is identical and independent across time

What is Multi-armed Bandit?

Example of Multi-armed Bandit problem

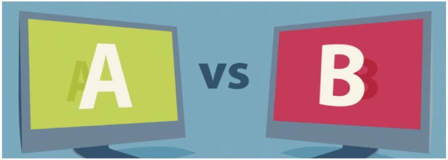


	action	$ return
Week 1	A	
Week 2	B	
Week 3	?	

What action would you take, A or B?

What is Multi-armed Bandit?

Example of Multi-armed Bandit problem



	action	$ return
Week 1	A	
Week 2	B	
Week 3	B	
Week 4	?	

How about now?

Exploration vs. Exploitation

Online decision-making involves a fundamental choice:

Exploitation:

Maximize performance using current knowledge

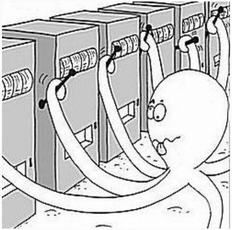
Exploration:

Increase knowledge

- The best strategy may involve short-term sacrifices
- The agent needs gather enough information to make the best overall decisions

Multi-armed Bandit Formalism

Problem Statement:



- The agent is faced repeateadly with a choice among k different actions (”*arms*”)
- At each step t the agent selects an action a_t
- After each choice it receives a numerical reward r_t that depends on the action selected
- The distribution $p(r|a)$ is assumed to be fixed, but unknown
- Goal is to maximize cumulative reward:

$$\sum_{i=1}^{t} r_i$$

Exploit knowledge with action value

The value of action a is the expected reward for a:

$$q(a) \doteq \mathbb{E}(r|a) = \sum_{r \in (R)} p(r|a) \times r = \lim_{t \to +\infty} \frac{1}{t} \sum_{i=1}^{t} r_i | a$$

- An estimate is the average of the sampled rewards:

$$q_t(a) \doteq \frac{\textit{sum of rewards when a taken prior to t}}{\textit{number of times a taken prior to t}}$$

- With an estimate of $q(a)$, we can select an action:

Greedy policy: $a_t \doteq \arg\max_a q_t(a)$

Incremental implementation

The agent can learn online with a moving average:

For $a = a_t$:
$$q_t(a) = \frac{1}{t}\sum_{i=1}^{t} r_i | a$$

$$q_t(a) = q_{t-1}(a) + \frac{1}{t}\left(r_t - q_{t-1}(a)\right)$$

$\forall\, a \neq a_t$:
$$q_t(a) = q_{t-1}(a)$$

For non-stationary problems, the agent can *track* q(a):

$$q_t(a) = q_{t-1}(a) + \alpha\left(r_t - q_{t-1}(a)\right)$$

Explore new actions with ϵ-greedy

The agent must explore to learn q-values

- Greedy selection always exploits current knowledge on q-values to maximize reward, it never explore

- Alternative: Behave greedily most of the time, but every once in a while select a random action

- **ϵ-greedy algorithm:**
  - Select random action (explore) with $p = \epsilon$
  - Select greedy action (exploit) with $p = 1 - \epsilon$

- ϵ-greedy ensures all actions can be sampled indefinitely:

$$\lim_{t \to +\infty} q_t(a) = q(a)$$

Practice: k-armed Bandit Algorithm

k-armed Bandit both evaluates $q(a)$ and improves a:

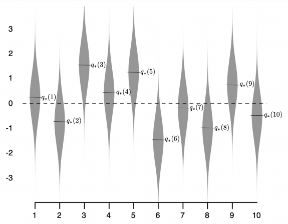
```
Initialize, for a = 1 to k:
  q(a) = 0
  n(a) = 0

Loop forever:
  a  = random action  with p = epsilon
  or = argmax q(a)    with p = 1 - epsilon
  Execute a, observe r
  n(a) = n(a) + 1
  q(a) = q(a) + 1/n(a) * (r - q(a))
```

Case Study: 10-armed testbed

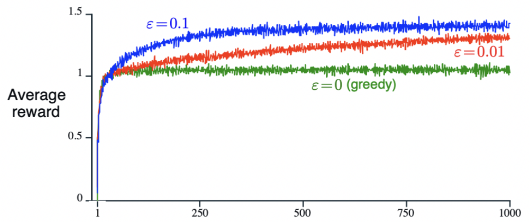
k-armed Bandit problem

Distribution of $q(a)$ vs. action a



Average performance of ϵ-greedy

Average reward over 2000 runs vs. time step



* Sutton and Barto, 1998

Total regret L_t

Analyzing regret in Multi-armed Bandit

- How can we reason about the exploration trade off?
- The (true) optimal value is: $v_* = \max_a q(a)$
- Regret is the opportunity loss for action taken at t: $v_* - q(a_t)$
- Thus the best trade-off between exploration and exploitation is the one that minimizes total regret L_t:

$$L_t = \sum_{i=1}^{t} (v_* - q(a_i))$$

Action Regret Δ_a

Analyzing regret in Multi-armed Bandit

- The action regret Δ_a for an action a is the difference between the optimal value and the true value of a:

$$\Delta_a = (v_* - q(a))$$

- L_t can be defined by action regrets and action counts:

$$L_t = \sum_{i=1}^{t} (v_* - q(a_i)) = \sum_{a \in (A)} N_t(a)(v_* - q(a)) = \sum_{a \in (A)} N_t(a)\Delta_a$$

- Thus the best trade-off between exploration and exploitation is one that ensures small count for actions with large regret
- The agent cannot measure regret directly, but regret can be used to analyze different RL algorithms on solved problems

Upper Confidence Bound

Explore new actions with UCB

Upper Confidence Bound (UCB)

- For each action value $q(a)$, compute an upper confidence $u_t(a)$ such that $q(a) \leq q_t(a) + u_t(a)$
- Select action that maximizes this Upper Confidence Bound:

$$a_t = \underset{a \in (A)}{\arg\max} \left[q_t(a) + u_t(a) \right]$$

where:

$$u_t(a) = c \sqrt{\frac{\ln t}{N_t(a)}}$$

- The UCB algorithm can achieve logarithmic regret (demo out of scope). In contrast, ϵ-greedy has linear regret.

Explore new actions with UCB

Upper Confidence Bound (UCB)

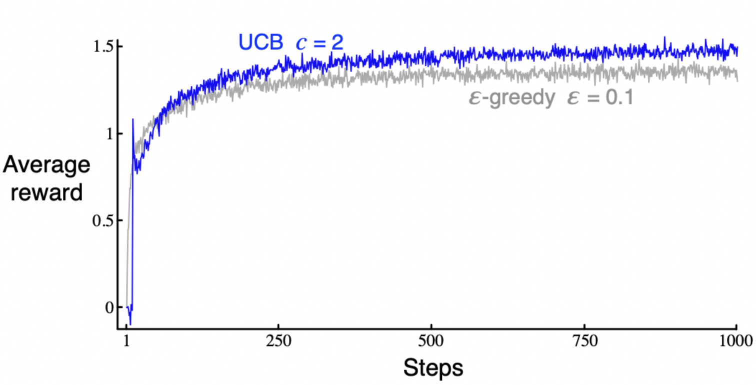
$$a_t = \underset{a \in (A)}{\arg\max} \left[q_t(a) + c \sqrt{\frac{\ln t}{N_t(a)}} \right]$$

- Uncertainty depends on number of times an action is selected:
  - **Small N_t(a)** $\Rightarrow$ **Large** $u_t(a) \Leftarrow$ Estimated q-value uncertain
  - **Large N_t(a)** $\Rightarrow$ **Small** $u_t(a) \Leftarrow$ Estimated q-value is accurate
- UCB favors an action because its estimated q-value is high, or because it has not been explored a lot relative to time elapsed
- UCB guarantees all actions will be explored without the need to manually predefine an ϵ-schedule

Case Study: 10-armed testbed

Average performance of ϵ-greedy and UCB algorithms

Average reward over 2000 runs vs. time step



Bayesian Bandit

The Bandit Model

The Bandit Model

A Bandit model is a reward transition function:

$$p\,(r\,|\,a) = p\,(r_{t+1} = r\,|\,a_t = a) \;\;\Leftrightarrow\;\; r(a) = \mathbb{E}(r,a)$$

$$\text{where}\;\; \mathbb{E}(r\,|\,a) = \sum_{r \in (R)} p(r\,|\,a) \times r = \lim_{t \to +\infty} \frac{1}{t} \sum_{i=1}^{t} r_i\,|\,a$$

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- **Bandit model-based algorithm:** (Expectation model)

$$\hat{r}_t(a) = \hat{r}_{t-1}(a) + \alpha\,(r_t - \hat{r}_{t-1}(a))$$

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- **Bandit model-based algorithm:** (Expectation model)

$$\hat{r}_t(a) = \hat{r}_{t-1}(a) + \alpha\,(r_t - \hat{r}_{t-1}(a))$$

- **Bandit value-based algorithm:**

$$q_t(a) = q_{t-1}(a) + \alpha\,(r_t - q_{t-1}(a))$$

...Identical?

The Bayesian Bandit Model

Bayesian Bandit models the full distribution of rewards:

- Bayesian Bandit tracks a parameterized distribution function of expected reward $p_\theta(\mathbb{E}(r) \mid a)$, called likelihood function
- Select action based on $p_\theta(\mathbb{E}(r) \mid a)$ e.g., using UCB
- Execute a
- Use reward observed to update posterior distributions of θ:

$$p_t(\theta|r) \propto p_\theta(\mathbb{E}(r) \mid a) \times p_{t-1}(\theta|r)$$

- For example, $\theta = (\mu, \sigma)_a$ if $p(\mathbb{E}(r)|\theta, a)$ are Gaussian distributions

The Bayesian Bandit Model

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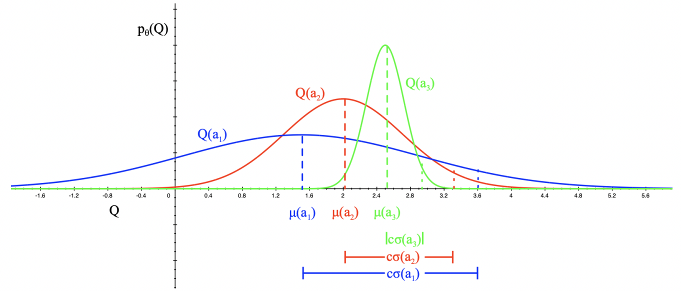
- Select action based on $p_\theta(\mathbb{E}(r) \mid a)$ e.g., using UCB
- Execute a
- Use reward observed to update posterior distributions of θ:

$$(\mu_t, \sigma_t)_a \propto p_{\mu,\sigma}(\mathbb{E}(r) \mid a) \times (\mu_{t-1}, \sigma_{t-1})_a$$

Example: Bayesian Bandit with UCB

Apply UCB to a Bayesian Bandit model:



- Define Gaussian likelihood function: $p_\theta(\mathbb{E}(r) \mid a) = p_\theta(q(a))$ with mean $\mu_t(a)$ and standard deviation $\sigma_t(a)$ for each action
- Select greedy action with UCB: $a_t = \arg\max_a \left(q_t(a) + c\sigma_t(a)\right)$
- Adjust $\mu_t(a)$ and $\sigma_t(a)$ for a_t based on r_t actually observed

Bandit with Thompson Sampling

Bayesian model with Probability Matching:

- Instead of selecting actions from q-values with highest mean according to $p_\theta(q(a))$ with ϵ-greedy or UCB, Thompson sampling explicitly samples q-values from $p_\theta(q(a))$
- Thompson sampling selects action a according to probability that $q(a)$ is the maximum given the data sampled so far:

$$\pi_t(a) \doteq p\left(q(a) = \max_{a'} q(a') \mid \text{history}_{t-1}\right)$$

$$\pi_t(a) = \mathbb{E}\left(\mathcal{I}\left(q_t(a) = \max_{a'} q_t(a')\right) \mid \text{history}_{t-1}\right)$$

$$\pi_t(a) \simeq \frac{1}{t-1}\sum_{i=1}^{t-1}\left(\mathcal{I}\left(q_i(a) = \max_{a'} q_i(a')\right)\right)$$

$$\text{where} \quad \mathcal{I}(\text{True}) = 1\,, \quad \mathcal{I}(\text{False}) = 0$$

Bandit with Thompson Sampling

Thompson Sampling:

- Sample q-value for each action from $p_\theta(q(a))$
- Update $\pi_t(a)$
- Select action from $\pi_t(a)$
- Execute a
- Use reward observed to update parameters θ

Toward Sequential RL and MDP...

Information State Space Bandit Model

- Bayesian Bandit tracks an evolving probability distribution of reward, which can be considered an *information state* s_t
- Each action a_t causes a transition to a new state s_{t+1} (by adding information), which is a sequential RL problem
- The tree of possible chains of events grows extremely rapidly, so approximate RL methods (lectures 5-7) are required

Toward Sequential RL and MDP...

Contextual Bandits

- Bandit with more than one state...
- If context on distinctive states are given to the agent, it can learn actions and values specific to each state
- In this case, the actions selected may depend on the state, but they do not affect which next states can be accessed later
- This is a simplified case of more general sequential RL problem where actions may affect next states and thus future possible rewards

Policy Gradient in Multi-armed Bandit

Policy Gradient Bandit

Can we learn a policy without learning values?

- Yes we can!
- Define a parameterized function $\pi_\theta(a) : a \mapsto p_\theta(a)$ and learn parameters θ that maximize a performance measure $J_{\pi_\theta}(a)$
- $\pi_\theta(a)$ can be arbitrary (just need distinguish possible actions)
- $J_{\pi_\theta}(a)$ can also be arbitrary (e.g. "always turn right in a maze")
- If $J_{\pi_\theta}(a)$ is unknown, it needs to be learned... It is often defined based on $q_t(a) =$ the *critic* in Actor-Critic algorithms:

$$\theta = \theta + \alpha \nabla_\theta \, q(a)$$

- **Out of scope for today (covered in depth in lecture 6)**

Today's Takeaways

Bandits is an RL problem where there is only one state

- **The fundamental problem is to balance exploration and exploitation to behave optimally**
- **To balance exploration and exploitation, the agent can use ϵ-greedy** which is a simple but efficient way to do it
- **...or UCB** which explicitly measures uncertainty to balance exploration and exploitation
- **...or a parameterized policy** with arbitrary objective measure J_θ
- **For example, J_θ can be the q-values parameterized by their means and standard deviations**, themselves updated based on the sampled rewards, as done in Thompson Sampling

Thank you!