Introduction to Complex Analysis

Mathematics 185
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Professor Hongbin Sun
Scribe: Sinho Chewi

Contents

Lecture 1

August 25

1.1 Complex Numbers

Complex numbers correspond to $\mathbb{R}^2$.

Complex numbers are used to solve equations:

- $x^2 = 1$ has the solutions $1, -1$.
- $x^2 = 2$ has the solutions $\sqrt{2}, -\sqrt{2}$.
- $x^2 = -1$ has no solutions in $\mathbb{R}$, since $y^2 > 0$ for $y \in \mathbb{R}$.

Introduce a new "number" i with the property that $i^2 = -1$. Next, invent a new number system from $\mathbb{R}$ and i. $3 \cdot i$ and $2 + i$ are also "numbers" in this number system.

Definition 1.1. A **complex number** is an expression as $z = x + iy$ for $x, y \in \mathbb{R}$.

$$\{\text{all complex numbers}\} = \mathbb{C}$$

$$\text{real numbers } \begin{cases} x = \operatorname{Re} z \text{ is the real part of } z \\ y = \operatorname{Im} z \text{ is the imaginary part of } z \end{cases}$$

1.2 Operations on Complex Numbers

Addition:

$$(2 + 2i) + (1 + 5i) = (2 + 1) + (2 + 5)i$$

Subtraction:

$$\begin{aligned}(2 + 2i) - (1 + 5i) &= (2 - 1) + (2 - 5)i \\ &= 1 - 3i\end{aligned}$$

Addition satisfies:

- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

Multiplication:

$$(2 + 2i) \cdot (1 + 5i) = 2(1 + 5i) + 2i(1 + 5i)$$

$$
\begin{aligned}
&= 2 + 10i + 2i + \overbrace{(-10)}^{2i\cdot 5i} \\
&= -8 + 12i
\end{aligned}
$$

Multiplication satisfies:

- $z_1 \cdot z_2 = z_2 \cdot z_1$
- $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Division:

$$\frac{2+2i}{1+5i} = \frac{(2+2i)(1-5i)}{(1+5i)(1-5i)} = \frac{(2+2i)(1-5i)}{26}$$

The last equality follows from

$$
\begin{aligned}
(1+5i)(1-5i) &= 1 \cdot 1 + \cancel{1 \cdot (-5i)} + \cancel{5i \cdot 1} + 5i \cdot (-5i) \\
&= 1 + 25 = 26
\end{aligned}
$$

If $z = x + iy$, then $\bar{z} = x - iy$ is the conjugation of z. We have that $z \cdot \bar{z} = x^2 + y^2$. Using the conjugate, complex division can be performed:

$$\frac{w}{z} = \frac{w \cdot \bar{z}}{z \cdot \bar{z}} = \frac{w \cdot \bar{z}}{x^2 + y^2}$$

Additionally, the real and imaginary parts of z can be written as

$$
\begin{aligned}
\operatorname{Re} z &= \frac{z + \bar{z}}{2} \\
\operatorname{Im} z &= \frac{z - \bar{z}}{2i}
\end{aligned}
$$

Some other properties that the complex conjugate satisfies:

- $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

1.3 Polar Representation

There is a correspondence:

$$
\begin{gathered}
x + iy \leftrightarrow (x, y) \in \mathbb{R}^2 \\
2 + 2i \leftrightarrow (2, 2) \\
\text{addition/subtraction of complex numbers} \leftrightarrow \text{addition/subtraction of vectors} \\
z \cdot \bar{z} = x^2 + y^2 \leftrightarrow \left\{ \begin{array}{r} \text{modulus of } z \\ |z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}} \end{array} \right\} \\
\left\{ \begin{array}{c} z \mapsto \bar{z} \\ x + iy \mapsto x - iy \end{array} \right\} \leftrightarrow \text{reflection along } x \text{ axis}
\end{gathered}
$$

To understand multiplication, use polar coordinates.

$$(x, y) \leftrightarrow (r, \theta)$$

The change in coordinates is given by

$$\begin{aligned} x &= r\cos\theta \quad & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta \quad & \theta &= \tan^{-1}\frac{y}{x} \end{aligned}$$

A complex number can be written (with multivariable calculus) as

$$z = x + iy = r(cos\theta + i\sin\theta)$$

r is the modulus of z. We can take $\theta \in (-\pi, \pi]$, and $\theta \in (-\pi, \pi]$ is the principle argument of z: $\theta = \operatorname{Arg} z$.

The ordinary arg function is multi-valued:

$$\arg z = \{\operatorname{Arg} z + 2k\pi \mid k \in \mathbb{Z}\}$$

Notation: Write $r(\cos\theta + i\sin\theta)$ as $r \cdot e^{i\theta}$.

Multiplying arbitrary complex numbers in polar coordinates gives

$$\begin{aligned} r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2) \\ &= r_1\cdot r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

In summary, complex multiplication is given by multiplying the modulus and adding the argument.

In the other notation, $r_1 \cdot e^{i\theta_1} \cdot r_2 \cdot e^{i\theta_2} = r_1 \cdot r_2 e^{i(\theta_1+\theta_2)}$. The usage of exponential notation is consistent with $e^x \cdot e^y = e^{x+y}$, $x, y \in \mathbb{R}$.

In particular, $(r(\cos\theta + i\sin\theta))^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$.

1.4 Solving Equations

We can use complex numbers to solve equations. For example, the equation $z^3 = 1$ has one real solution: $z = 1$. Writing $r^3 e^{i3\theta} = 1 \cdot e^{i0}$, then $r = 1$ and $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$. Therefore, there are three complex solutions:

$$\begin{aligned} z &= e^{i0}, e^{i2\pi/3}, e^{i4\pi/3} \\ &= 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$

In general, write $z = re^{i\theta}$. Then if $z^n = \rho e^{i\psi}$, the nth roots are

$$\rho^{1/n} e^{i(\psi/n + 2\pi k/n)}, \quad k = 0, 1, \ldots, n-1$$

because $r^n = \rho$ and $n\theta = \psi + 2k\pi$.

Lecture 2

August 30

2.1 Complex Numbers

Complex numbers are the real numbers with i, where $i^2 = -1$: $z = x + iy$, $x, y \in \mathbb{R}$.

Division is performed by

$$\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}}$$

In polar coordinates, we write

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Using polar coordinates, we can multiply:

$$r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$

To solve $z^n = \rho e^{i\theta_0}$, suppose that $z = re^{i\theta}$. We get $r^ne^{in\theta} = \rho e^{i\theta_0}$. These imply

$$\begin{aligned} r^n &= \rho & r &= \rho^{1/n} \\ n\theta &= \theta_0 + 2k\pi, k \in \mathbb{Z} & \theta &= \frac{\theta_0}{n} + \frac{2k\pi}{n}, k \in \mathbb{Z} \end{aligned}$$

Actually, $k = 0, 1, \ldots, n-1$.

2.2 Polynomials

Consider a polynomial $p(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$, $a_i \in \mathbb{C}$, $a_n \neq 0$. We will show the Fundamental Theorem of Algebra: Any polynomial with complex coefficients $p(z) = a_nz^n + \cdots + a_0$ has a root in $\mathbb{C}$. Moreover, there is a unique factorization $p(z) = a_n(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$.

2.2.1 Polynomials as Functions

View a polynomial as a function $z \mapsto p(z)$. The geometric picture of the Fundamental Theorem of Algebra is that there exists a point (in fact, n points) which maps to 0. Using the Fundamental Theorem of Algebra, we can solve any polynomial equation, say $p(z) = w_0$, by solving $p(z) - w_0 = 0$.

2.3 An Alternative Way to See the Complex Plane

Compare the complex numbers $2 + 2i$ and $1 + 4i$: we can compare their moduli. We say z is "small" if $|z|$ is small or z is close to 0. z is big if $|z|$ is big.

Define the augmented complex plane as $\mathbb{C} \cup \{\infty\} = \bar{\mathbb{C}}$.

2.3.1 Geometric Picture of $\bar{\mathbb{C}}$

We draw a 3-dimensional geometric picture of $\bar{\mathbb{C}}$. The geometric model of $\bar{C}$ will be a sphere:

$$\{(x,y,z) : x^2+y^2+z^2=1\}$$

For any point (x,y,z) on the sphere, we draw a line from $N=(0,0,1)$ through (x,y,z) and associate with (x,y,z) the complex number which is the intersection of the line and the plane $\{(x,y,0)\} \cong \mathbb{C}$. We associate $N=(0,0,1) \sim \infty$. The function is given by

$$\begin{aligned} \text{sphere} \setminus \{N\} &\to \mathbb{C} \\ P &\mapsto (xy\text{-plane}) \cap (\text{line going through } P \text{ and } N) \end{aligned}$$

The southern hemisphere maps to the inside of the unit disk. The northern hemisphere maps to outside of the unit disk. A point close to N maps to a point far from the origin.

A point is represented as $P=(X,Y,Z) \in \text{sphere} - \{N\}$.

A line going through P and N is represented as $(1-t)(0,0,1)+t(X,Y,Z)$. The intersection with the xy-plane is when the third coodinate is $(1-t)+tZ=0$. Using this, we solve for t:

$$t = \frac{1}{1-Z}$$

The intersection point is

$$\left(\frac{X}{1-Z}, \frac{Y}{1-Z}, 0\right)$$

Therefore the mapping is

$$\text{Sphere} \setminus \{N\} \xrightarrow{\text{stereographic projection}} \mathbb{C}$$

$$\{(X,Y,Z) \mid X^2+Y^2+Z^2=1, Z \neq 1\} \to \left\{\left(\frac{X}{1-Z}, \frac{Y}{1-Z}\right)\right\}$$

$$\text{Sphere} \xrightarrow{\text{1-1 correspondence}} \mathbb{C} \cup \{\infty\}$$

The inverse is given by

$$\begin{aligned} \mathbb{C} &\to \text{Sphere} \setminus \{N\} \\ (x,y) &\mapsto \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \end{aligned}$$

2.4 Exponential Function

The exponential function $z \mapsto e^z$ has been defined for

$$\begin{aligned} x+0i &\mapsto e^x \\ 0+yi &\mapsto e^{yi} = \cos y + i \sin y \end{aligned}$$

For $x, x' \in \mathbb{R}$, $e^{x+x'} = e^x \cdot e^{x'}$.

Define:

$$e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$$

We have $e^{x+iy} \cdot e^{x'+iy'} = e^{x+x'+i(y+y')}$. e^x is the modulus and y is the argument. Hence, $e^z = e^{z+2k\pi i}$.

Viewing e^z as a mapping from the complex plane to the complex plane, fixing x and changing y traces out a circle. A different value of x yields a different radius for the circle. Fixing y and changing x traces out a ray from the origin.

2.5 Inverse Function of Exponential Function

For z, we must find w such that $e^w = z$. When $x > 0$, the function is $\log x$. For $z \neq 0$, $z = re^{i\theta}$. Suppose that $w = x + iy$. Then

$$e^w = z \Leftrightarrow e^x \cdot e^{iy} = re^{i\theta}$$

Therefore, we can take

$$\begin{aligned} x &= \log r = \log |z| \\ y &= \theta + 2k\pi, \quad k \in \mathbb{Z} \\ &= \operatorname{Arg} z + 2k\pi \end{aligned}$$

The single-valued inverse is $\operatorname{Log} z = \log |z| + \operatorname{Arg} z$ (where $\operatorname{Arg} z \in (-\pi, \pi]$). The inverse as a multi-valued function is $\log z = \log |z| + \arg z$.

$\operatorname{Log} z$ is not continuous on $\mathbb{C} \setminus \{0\}$, but $\operatorname{Log} z$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$. We can paste infinitely many slit complex planes together to form a Riemann surface.

Lecture 3

September 1

3.1 Review of Mathematics 53/104 & Applications to Complex Numbers

3.1.1 Limits of Sequences

Consider a sequence of real numbers $\{x_n\}$ with $\lim_{n\to\infty} s_n = s$. Then

$$\exists\varepsilon > 0,\ \exists N \text{ s.t. } \forall n > N\ |s - s_n| < \varepsilon$$

Definition 3.1. Let $\{z_n\}$ be a sequence of complex numbers. Then z_n **converges to** z or $\lim z_n = z$ if

$$\exists\varepsilon > 0,\ \exists N \text{ s.t. } \forall n > N\ |z - z_n| < \varepsilon$$

- $\lim z_n = z$ if and only if

$$\begin{aligned}\lim \operatorname{Re}(z_n) &= \operatorname{Re}(z)\\ \lim \operatorname{Im}(z_n) &= \operatorname{Im}(z)\end{aligned}$$

- If $\lim z_n = z$, $\lim w_n = w$, then

$$\begin{aligned}\lim\,(z_n + w_n) &= z + w\\ \lim\,(z_n w_n) &= zw\\ \lim\,(z_n/w_n) &= z/w, \quad \text{if } w \neq 0\\ \lim \overline{z_n} &= \bar{z}\\ \lim |z_n| &= |z|\end{aligned}$$

Cauchy Sequences

$\{z_n \in \mathbb{C}\}$, a sequence of complex numbers, is a **Cauchy sequence** if

$$\forall\varepsilon > 0\ \exists N \text{ s.t. } \forall m, n > N\ |z_m - z_n| < \varepsilon$$

Theorem 3.2. *$\{z_n\}$ converges iff $\{z_n\}$ is a Cauchy sequence.*

Proof. "$\Rightarrow$": Follow the definition.

"$\Leftarrow$": If $\{z_n\}$ is Cauchy, then $|z - z_n| \geq |\operatorname{Re} z_n - \operatorname{Re} z|$ implies that $\{\operatorname{Re} z_n\}$ is Cauchy. Similarly, $\{\operatorname{Im} z_n\}$

is Cauchy. From Mathematics 104, we know that $\operatorname{Re} z_n \to x_0$ and $\operatorname{Im} z_n \to y_0$, so $z_n \to x_0 + iy_0$. □

Theorem 3.3. *Any bounded sequence of complex numbers has a convergent subsequence.*

Proof. If $\{z_n\}$ is bounded, then $\{\operatorname{Re} z_n\}$ and $\{\operatorname{Im} z_n\}$ are bounded. From Mathematics 104, there exists a subsequence such that the real part converges. Apply the theorem again: there exists a further subsequence such that the imaginary part converges. □

3.1.2 Continuous Functions

Consider $f : \mathbb{C} \to \mathbb{C}$ or $f : D \subseteq \mathbb{C} \to \mathbb{C}$. One complex number is mapped to one complex number. The "functions" $\arg z$ and $\log z$ are multi-valued functions, not functions!

$$f : D \to \mathbb{C}$$

f is continuous at $z \in D$ if

- (preferred by textbook author) for any sequence $\{z_n\}$ with $z_n \in D$ and $\lim z_n = z_0$, $\lim f(z_n) = f(z_0)$.
- (equivalent definition) $\forall \varepsilon > 0\, \exists \delta > 0$ s.t. $\forall z \in D$ with $|z - z_0| < \delta, |f(z) - f(z_0)| < \varepsilon$

If f, g are both continuous at z_0, then so are $f + g$, $f - g$, $f \cdot g$, and f/g $(g(z_0) \neq 0)$.

If a function is continuous at a point z_0, then for any line/curve going through z_0, f is continuous on this line.

To prove a function is not continous at a point, we can find a sequence $z_n \to z_0$ but $f(z_n) \not\to f(z_0)$.

Definition 3.4. f is **continuous on a region** $U \subseteq \operatorname{dom}(f)$ if f is continuous on each point in U.

The function $\operatorname{Arg} : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is not continuous on $\mathbb{C} \setminus \{0\}$, but it is continuous on $\mathbb{C} \setminus (-\infty, 0]$. The same is true for $\operatorname{Log} z$.

We can view functions as:

$$f : \mathbb{C} \to \mathbb{C}$$

or

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

where

$$u : \mathbb{R}^2 \to \mathbb{R}$$
$$v : \mathbb{R}^2 \to \mathbb{R}$$

A complex-valued function is made of two real-valued 2-variable functions. f is continuous on U iff u and v are both continuous on U.

- $f(z) = z$ is continuous on $\mathbb{C}$.
- $g(z) = P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + z_0$ is continuous on $\mathbb{C}$.
- $h(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials, is continuous on $\mathbb{C} \setminus \underbrace{\{z \mid Q(z) = 0\}}_{\text{not empty!}}$.
- $j(z) = e^z$ is continuous on $\mathbb{C}$.

3.1.3 Topology of $\mathbb{C}$

Definition 3.5. U is an **open set** if $\forall z \in U, \exists \varepsilon > 0$ s.t. $B(z, \varepsilon) \subseteq U$, where

$$B(z, \varepsilon) = \{w \in \mathbb{C} \mid |w - z| < \varepsilon\}.$$

If f is continuous at z_0, then any sequence $\{z_n\} \to z_0$, $z_n \in \mathrm{dom}(f)$, has $f(z_n) \to f(z_0)$. If $\mathrm{dom}(f)$ is open, we can throw away the condition that $z_n \in \mathrm{dom}(f)$.

Definition 3.6. $U \subseteq \mathbb{C}$ is a **domain** if

1. U is an open set.
2. For any $x, y \in U$ there exists a path from x to y which consists of finitely many line segments (parallel to the x-axis or the y-axis).

Theorem 3.7. *If $h : D \to \mathbb{C}$, $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial y}$ both exist, and*

$$\boldsymbol{\nabla} h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) = (0, 0) = \mathbf{0}$$

on D, then h is a constant on D.

Proof. The condition

$$\frac{\partial h}{\partial x} = 0$$

implies that h is constant on a segment. By induction, $h(x) = h(y)$. $\square$

Definition 3.8. $A \subseteq \mathbb{C}$ is a **closed set** if

- (always prefer this one) for any sequence $\{z_n\}$ with $z_n \in A$ and $\lim z_n = z_0$, then $z_0 \in A$.
- (equivalently) $\mathbb{C} \setminus A$ is open.

Theorem 3.9. *If $f : A \subseteq \mathbb{C} \to \mathbb{R}$, A is closed and bounded (compact), and f is continuous on A, then f is bounded, and its supremum/infimum are attained at some points $z_0, w_0 \in A$.*

3.1.4 Derivatives

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If the limit exists, f is **complex differentiable** at z_0. $f'(z_0)$ is the complex derivative.

- If f is (complex) differentiable at z_0, then it is continuous at z_0.
- $(f + g)'(z) = f'(z) + g'(z)$
- $(f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$
- $(f/g)'(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$
- Chain Rule: $(f \circ g)'(z) = f'(g(z)) \cdot g'(z)$

Lecture 4

September 6

4.1 Complex Differentiation

If D is an open set in $\mathbb{C}$, then $f : D \to \mathbb{C}$ is **complex-differentiable** at $z_0 \in D$ if

$$\underbrace{f'(z_0)}_{\textbf{complex derivative at } z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Example 4.1. Let $f(z) = z$. Then

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1$$

Example 4.2. Let $f(z) = \bar{z}$ $(f(x + iy) = x - iy)$. Then

$$\begin{aligned} f'(z_0) = \lim_{z \to z_0} \frac{\bar{z} - \overline{z_0}}{z - z_0} &\overbrace{=}^{\text{write } z - z_0 = \Delta z} \lim_{\Delta \to 0} \frac{\overline{\Delta z}}{\Delta z} \\ = \lim_{r \to 0} \frac{re^{-i\theta}}{re^{i\theta}} &= \lim_{r \to 0} e^{-2i\theta} \end{aligned}$$

does not exist. Or, take $\Delta z_n = 1/n$. Then

$$\frac{\overline{\Delta z_n}}{\Delta z_n} = 1$$

and if $\Delta w_n = (1/n)i$, then

$$\frac{\overline{\Delta w_n}}{\Delta w_n} = \frac{-(1/n)i}{(1/n)i} = -1$$

Hence,

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

does not exist! $f(z)$ is not complex-differentiable at any point in $\mathbb{C}$. This also implies that

$$\begin{aligned} f(z) &= \operatorname{Re} z \\ g(z) &= \operatorname{Im} z \end{aligned}$$

are not complex-differentiable at any point in $\mathbb{C}$, e.g. by noticing that $\bar{z} = 2\operatorname{Re} z - z$, and z is $\mathbb{C}$-differentiable (so if $\operatorname{Re} z$ is also $\mathbb{C}$-differentiable, then $\bar{z}$ is also $\mathbb{C}$-differentiable). Similarly, $\bar{z} = z - 2i \operatorname{Im} z$.

From last time: $\lim z_n = z_0$ iff

$$\begin{aligned} \lim \operatorname{Re} z_n &= \operatorname{Re} z_0 \\ \lim \operatorname{Im} z_n &= \operatorname{Im} z_0 \end{aligned}$$

Other complex-differentiable functions:

$$\begin{aligned} P(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\ P'(z) &= n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1 \end{aligned}$$

and $P(z)/Q(z)$ is $\mathbb{C}$-differentiable on $\mathbb{C} \setminus \{\text{zero points of } Q\}$.

$$\begin{aligned} f(z) &= e^z \\ f'(z) &= \lim_{z \to z_0} \frac{e^z - e^{z_0}}{z - z_0} = \lim_{\Delta z \to 0} \frac{e^{z_0 + \Delta z} - e^{z_0}}{\Delta z} \\ &= \lim_{\Delta z \to 0} e^{z_0} \frac{e^{\Delta z} - 1}{\Delta z} \end{aligned}$$

Writing $\Delta z = \Delta x + i\Delta y$,

$$\frac{e^{\Delta z} - 1}{\Delta z} = \frac{(e^{\Delta x} \cos \Delta y - 1) + i e^{\Delta x} \sin \Delta y}{\Delta x + i \Delta y}$$

The limit requires some computation. To solve this, we want a closer study of

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

4.2 Cauchy-Riemann Equations

Write $f(z)$ as $f(x, y) = u(x, y) + iv(x, y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

As $\Delta z \to 0$, we consider

$$\begin{aligned} \Delta z &\to 0 \qquad \text{along } x\text{-axis} \\ \Delta z &\to 0 \qquad \text{along } y\text{-axis} \end{aligned}$$

If

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists, take

1. $\Delta z = \Delta x + i0$, $z_0 = x_0 + iy_0$:

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \end{aligned} \tag{4.1}$$

2. $\Delta z = 0 + i\Delta y$:

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i \Delta y} \\ &= \frac{1}{i} \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i \frac{\partial u}{\partial y}(x_0, y_0) \end{aligned} \tag{4.2}$$

Since (4.1) = (4.2), then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at (x_0, y_0). These are the Cauchy-Riemann equations. To memorize them, consider $u = x$ and $v = y$. This tells you which one has a negative sign.

Definition 4.3. Let $D \subseteq \mathbb{C}$ be open. $f : D \to \mathbb{C}$ is **analytic** if

1. f is complex-differentiable at all points in D.
2. f' is continuous on D.

Actually, it will be proven later that 1 $\implies$ 2.

(53: To write $\mathrm{d}f = f_x \,\mathrm{d}x + f_y \,\mathrm{d}y$, we need f_x, f_y to be continuous.)

$$f'(z) = u_x + iv_x \tag{4.3}$$
$$= v_y - iv_y \tag{4.4}$$

C-R equations:

$$u_x = v_y$$
$$u_y = -v_x$$

Theorem 4.4. *$f : D \to \mathbb{C}$ is analytic iff*

(a) writing $f = u + iv$, then the C-R equations hold on D.

(b) u_x, u_y, v_x, v_y are continuous on D.

Proof. $1 + 2 \implies (a) + (b)$: Actually,

$$1 \implies (a) \qquad \text{C-R equations (4.3)}$$
$$2 \implies (b)$$

$(a) + (b) \implies 1 + 2$: We need to show that $f'(z)$ exists, and then $(4.3) + (b) \implies 2$.

$$\lim_{\Delta x + i\Delta y \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta x) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

$$u(x + \Delta x, y + \Delta y) - u(x, y) = u_x \Delta x + u_y \Delta y + o(|\Delta z|)$$
$$v(x + \Delta x, y + \Delta y) - v(x, y) = v_x \Delta x + v_y \Delta y + o(|\Delta z|)$$

because u_x, u_y, v_x, v_y are continuous. Justification:

$$\begin{aligned} u(x + \Delta x, y + \Delta y) - u(x, y) &= u(x + \Delta x, y + \Delta y) - u(x + \Delta x, y) + u(x + \Delta x, y) - u(x, y) \\ &= u_y(x + \Delta x, y + \Delta y')\Delta y + u_x(x + \Delta x', y)\Delta x \\ &= u_y(x, y)\Delta y + u_x(x, y)\Delta x + o(|\Delta z|) \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{f(z+\Delta z)-f(z)}{\Delta z} &= \frac{u_x\Delta x + u_y\Delta y + (\overbrace{v_x}^{-u_y}\Delta x + \overbrace{v_y}^{u_x}\Delta y)i + o(|z|)}{\Delta x + i\Delta y} \\
&= \frac{u_x(\Delta x + i\Delta y) + u_y(\Delta y - i\Delta x) + o(|z|)}{\Delta x + i\Delta y} \\
&= u_x - iu_y + \underbrace{\frac{o(|z|)}{\Delta x + i\Delta y}}_{\to 0} \\
f' &= u_x - iu_y
\end{aligned}$$

□

Consider

$$f(z) = e^z = e^{x+iy} = \underbrace{e^x\cos y}_{u(x,y)} + i\underbrace{e^x\sin y}_{v(x,y)}$$

Then, we can write

$$\begin{aligned}
u_x &= e^x\cos y \qquad & v_x &= e^x\sin y \\
u_y &= -e^x\sin y \qquad & v_y &= e^x\cos y
\end{aligned}$$

The C-R equations hold, so $f(z) = e^z$ is analytic on $\mathbb{C}$. $f'(z) = u_x + iv_x = e^x\cos y + ie^x\sin y = e^z$.

4.2.1 Trigonometric Functions

$$\begin{aligned}
e^{iy} &= \cos y + i\sin y, \qquad & y \in \mathbb{R} \\
\text{formally:} \quad e^{iz} &= \underbrace{\cos z + i\sin z}_{\text{want to define}}, \qquad & z \in \mathbb{C}
\end{aligned}$$

Also,

$$\begin{aligned}
e^{-iy} &= \cos(-y) + i\sin(-y) \\
&= \cos y - i\sin y \\
e^{-iz} &= \cos z - i\sin z
\end{aligned}$$

Define:

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

The trigonometric functions are analytic on $\mathbb{C}$.

4.2.2 Power Functions

We will define z^{1+i} and z^α for $\alpha \in \mathbb{C}$. z^α is a multi-valued function, so we define the principal branch. Since $z = e^{\operatorname{Log} z}$, define

$$\begin{aligned}
z^\alpha &= e^{\alpha\cdot\operatorname{Log} z} \\
z^\alpha &= e^{\alpha\cdot(\operatorname{Log} z + 2\pi i)}
\end{aligned}$$

Lecture 5

September 8

5.1 Analytic Functions

Let f be a $\mathbb{C}$-valued function defined on an open set $U \subseteq \mathbb{C}$. f is **analytic on** U if

1. f is complex-differentiable at any $z \in U$, i.e.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

   exists.

2. $f'(z)$ is continuous on U.

If we write $f(x+iy) = u(x,y) + iv(x,y)$ and if $f'(z)$ exists:

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = u_x + iv_x \\ &= \lim_{\Delta y \to 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y} = v_y - iu_x \end{aligned}$$

Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\ f'(z) &= u_x + iv_x \\ &= v_y - iu_y \end{aligned}$$

f is analytic on U iff

(a) C-R equations hold on U,

(b) u_x, u_y, v_x, v_y are continuous on U.

Example 5.1. Let $f(z) = e^{|z|}$. Does

$$\lim_{\Delta z \to 0} \frac{e^{|z+\Delta z|} - e^{|z|}}{\Delta z}$$

exist? We need some work. We write:

$$f(x+iy) = \underbrace{e^{\sqrt{x^2+y^2}}}_{u(x,y)} + i \underbrace{0}_{v(x,y)}$$

$$v_x = v_y = 0 \underbrace{\implies}_{\text{C-R equations}} u_x = u_y = 0$$

which is a contradiction! $f(z) = e^{|z|}$ is not analytic on any open set $U \subset \mathbb{C}$.

Proposition 5.2. *If $f : U \subseteq \mathbb{C} \to \mathbb{C}$ for an open set U, and if $f(z) \in \mathbb{R}$ for any z and f is analytic on U, then $f'(z) \equiv 0$.*

Proposition 5.3. *If $f : U \subseteq \mathbb{C} \to \mathbb{C}$ with U a domain, if $f(z) \in \mathbb{R}$ for any z, and f is analytic on U, then f is constant.*

Proof.

$$f'(z) = 0 \implies \nabla f = \nabla u + i\nabla v = \mathbf{0} \implies f \text{ is constant}$$

□

If $f(U)$ is a subset of the real line and f is analytic, then f is constant!

What about if $f(U)$ is a subset of a curve? We need some multivariable analysis.

5.2 Inverse Function Theorem

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, with $f(x, y) = (u(x, y), v(x, y))$.

The Jacobian of f is:

$$J_f(x,y) = \begin{pmatrix} \dfrac{\partial u}{\partial x}(x,y) & \dfrac{\partial u}{\partial y}(x,y) \\ \dfrac{\partial v}{\partial x}(x,y) & \dfrac{\partial v}{\partial y}(x,y) \end{pmatrix}$$

Theorem 5.4. *Suppose $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ with D an open set, and suppose that u_x, u_y, v_x, v_y exist and are continuous. If for $(x_0, y_0) \in D$, $\det J_f(x_0, y_0) \neq 0$, then*

1. *There exists an open disk U containing (x_0, y_0) such that f is one-to-one on U.*
2. *$f(U) \subseteq \mathbb{R}^2$ is an open set.*

f is locally a 1-1 correspondence.

1. If $f : \mathbb{R} \to \mathbb{R}$, $f'(x_0) \neq 0$, assume that $f'(x_0) > 0$. Then $f' > 0$ near x_0, which implies that f is strictly increasing in $(x_0 - \varepsilon, x_0 + \varepsilon)$.

If $f = u + iv$ is analytic,

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

so $\det J_f = u_x^2 + v_x^2 = |f'(z)|^2$. Therefore,

$$\det J_f(x,y) = 0 \Leftrightarrow f'(x+iy) = 0$$

Proposition 5.5. *If $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is analytic, where D is a domain, and $f(D)$ does not contain any open sets, then f is constant.*

Proof.
$$f'(z_0) \neq 0 \implies \det J_f(x_0, y_0) \neq 0 \implies f(D) \text{ contains an open set}$$
□

Theorem 5.6 (Inverse Function Theorem for 185)**.** *If $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is analytic, where D is open, if for some $z_0 \in D$, $f'(z_0) \neq 0$, then*

1. *There exists a small open disk U containing z_0 such that f is one-to-one on U.*
2. *$V = f(U) \subseteq \mathbb{R}^2$ is an open set.*
3. *$f : U \to V$ is a 1-1 correspondence. Take $z \in U$ and $w \in V$. For the local inverse of f, $g : V \to U$, g is also analytic and*
$$g'(w) = \frac{1}{f'(g(w))}$$

f is locally a 1-1 correspondence.

Proof. Proof of 3:
$$g'(w) = \lim_{\Delta w \to 0} \frac{g(w + \Delta w) - g(w)}{\Delta w}$$
Take $\Delta w_i \to 0$, and study
$$\frac{g(w + \Delta w_i) - g(w)}{(w + \Delta w_i) - w}$$
Since $w = f(z)$, then $w + \Delta w_i = f(z + \Delta z_i)$, and as $\Delta w_i \to 0$, then $\Delta z_i \to 0$.
$$\begin{aligned} \lim \frac{g(w + \Delta w_i) - g(w)}{w + \Delta w_i - w} &= \lim \frac{\overbrace{g(f(z + \Delta z_i))}^{z+\Delta z_i} - \overbrace{g(f(z))}^{z}}{f(z + \Delta z_i) - f(z)} \\ &= \lim \frac{\Delta z_i}{f(z + \Delta z_i) - f(z)} = \frac{1}{f'(z)} = \frac{1}{f'(g(w))} \end{aligned}$$
□

f is analytic "$\implies$" f^{-1} is analytic has potential problems:

1. f is not a one-to-one function.
2. $f'(z) = 0$ at some point.

Actually, we will later show that if f is one-to-one and analytic, then f' is never 0.

$f(z) = e^z$ is analytic on $\mathbb{C}$. The inverse is $\log z$ or $\operatorname{Log} z$.
$$\{x + iy \mid y \in (-\pi, \pi)\} \underset{\operatorname{Log} z}{\overset{e^z}{\longleftrightarrow}} \mathbb{C} \setminus (-\infty, 0]$$
is a 1-1 correspondence. $\operatorname{Log} z = \operatorname{Log} \sqrt{x^2 + y^2} + i \arctan(y/x)$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$.
$$(\operatorname{Log} w)' = \frac{1}{e^{g(w)}} = \frac{1}{e^{\operatorname{Log} w}} = \frac{1}{w}$$
The principal branch of z^α, $\alpha \in \mathbb{C}$, is $e^{\alpha \operatorname{Log} z}$, defined on $\mathbb{C} \setminus (-\infty, 0]$, is an analytic function.
$$\begin{aligned} (e^{\alpha \operatorname{Log} z})' &= \alpha z^{\alpha - 1} \\ &= \alpha e^{(\alpha - 1) \operatorname{Log} z} \end{aligned}$$

Lecture 6

September 13

6.1 Harmonic Functions

$f : D \subseteq \mathbb{C} \to \mathbb{C}$, where D is open and f is continuous, is analytic if

1. $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$
   exists.
2. $f'(z)$ is continuous on D.

Equivalently, if $f = u + iv$,

(a) $u_x = v_y$ and $u_y = -v_x$.

(b) u_x, u_y, v_x, v_y are continuous.

6.1.1 Algebraic/Analytic Property of $u + iv$

Which functions $u : D \subseteq \mathbb{C} \to \mathbb{R}$ are the real part of an analytic function?

$$\begin{aligned} u(x,y) &= x & f(x+iy) &= x + iy \\ u(x,y) &= e^x \cos y & f(x+iy) &= e^{x+iy} \\ u(x,y) &= x^2 + y^2 & &? \end{aligned}$$

$$u_x = v_y = 2x \qquad - u_y = v_x = -2y$$

Compute the 2nd order partial derivatives.

$$\begin{aligned} (v_y)_x &= 2 \\ (v_x)_y &= -2 \end{aligned}$$

From 53, the second-order partial derivatives should equal to each other.

From the C-R equations:

$$\begin{cases} u_x &= v_y \\ u_y &= -v_x \end{cases} \implies \begin{cases} (u_x)_x &= (v_y)_x \\ (u_y)_y &= (-v_x)_y \end{cases}$$

$v_{yx} = v_{xy}$ if the 2nd-order partial derivatives are continuous, so $u_{xx} + u_{yy} = 0$.

$$\Delta u = 0$$

is the harmonic equation.

Definition 6.1. $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is **harmonic** if u, the 1st-order, and 2nd-order partial derivatives exist and are continuous, and $u_{xx} + u_{yy} = 0$.

Theorem 6.2. *If $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is analytic, $f = u + iv$, and u, v, the 1st-order and 2nd-order partial derivatives exist and are continuous, then u is a harmonic function, and so is v.*

If $f = u + iv$, $-i \cdot f = v - iu$ is analytic.

6.1.2 Harmonic Conjugates

Given a harmonic u, find a v such that $u + iv$ is analytic (if possible).

Example 6.3. Consider $u(x, y) = x^3 - 3xy^2$. Is u harmonic?

$$u_x = 3x^2 - 3y^2 \qquad u_{xx} = 6x$$
$$u_y = -6xy \qquad u_{yy} = -6x$$

u is harmonic.

Find a v with the C-R equations:

$$v_x = -u_y = 6xy \tag{6.1}$$
$$v_y = u_x = 3x^2 - 3y^2 \tag{6.2}$$

$$(6.1) \qquad v(x, y) = \int 6xy \, \mathrm{d}x = 3x^2 y + c(y)$$
$$(6.2) \qquad v_y = 3x^2 - 3y^2 = 3x^2 + c'(y)$$
$$c(y) = -y^3 + c'$$
$$v = 3x^2 y - y^3 + c'$$
$$u + iv = (x + iy)^3 + ic'$$

v is unique up to a constant.

v is a **harmonic conjugate** of u, which means that $u + iv$ is analytic and u, v satisfy the C-R equations.

If v is harmonic, what is a harmonic conjugate of v? $v - iu$ is analytic, so $-u$ is a harmonic conjugate of v.

Consider $u = \operatorname{Log} \sqrt{x^2 + y^2} = \operatorname{Log} |z|$, where $\operatorname{Log} z = \operatorname{Log} |z| + i \operatorname{Arg} z$. Then $v = \operatorname{Arg} z$, but v is not continuous on $\mathbb{C} \setminus \{0\}$. This implies that $u = \log \sqrt{x^2 + y^2}$ has a harmonic conjugate on $\mathbb{C} \setminus (-\infty, 0]$, but does not have a harmonic conjugate $\mathbb{C} \setminus \{0\}$. The harmonic conjugate does not always exist!

The harmonic conjugate always locally exists.

Theorem 6.4. *If $u : D \to \mathbb{R}$, where D is an open disk or an open rectangle, is harmonic, then there exists a harmonic conjugate on D.*

If we are looking for a v such that

$$v_x = -u_y$$
$$v_y = u_x,$$

then we can suppose that $v(x_0, y_0) = 0$. Then

$$\begin{aligned} v(x, y_0) &= \int_{x_0}^{x} v_x(s, y_0)\, \mathrm{d}s \\ v(x, y) &= v(x, y_0) + \int_{y_0}^{y} v_y(x, t)\, \mathrm{d}t \end{aligned}$$

Therefore,

$$v(x, y) = \int_{x_0}^{x} (-u_y)(s, y_0)\, \mathrm{d}s + \int_{y_0}^{y} u_x(x, y)\, \mathrm{d}t$$

Then,

$$\begin{aligned} v_y(x, y) &= 0 + u_x(x, y) \\ v_x(x, y) &= -u_y(x, y_0) + \int_{y_0}^{y} u_{xx}(x, t)\, \mathrm{d}t \\ &= -u_y(x, y_0) - \int_{y_0}^{y} u_{yy}(x, t)\, \mathrm{d}t \\ &= \cancel{-u_y(x, y_0)} - (u_y(x, y) - \cancel{u_y(x, y_0)}) \end{aligned}$$

The C-R equations are satisfied.

6.2 Conformal Mappings

To picture complex functions, we typically look at the image of the horizontal and vertical lines.

For $z_0 \in \mathbb{C}$, consider $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) = z_0$, where $\gamma = x(t) + iy(t)$. The tangent vector of γ at 0 is $\gamma'(0) = x'(0) + iy'(0)$.

If $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is analytic, with $\gamma : [0, 1] \to D$ such that $\gamma(0) = z_0$, then

$$(f \circ \gamma)'(0) = f'(\gamma(0)) \cdot \gamma'(0) = f'(z_0) \cdot \gamma'(0)$$

If $f'(z_0) \neq 0$, $\gamma_1'(0) \neq 0$, $\gamma_2'(0) \neq 0$, then the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$ is equal to the angle between $(f \circ \gamma_1)'(0)$ and $(f \circ \gamma_2)'(0)$.

Lecture 7

September 15

7.1 Conformal Mappings

Let $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = z_0$. If we write $\gamma(t) = x(t) + iy(t)$, then the tangent vector at $t = 0$ is $\gamma'(0) = x'(0) + iy'(0)$.

Lemma 7.1. *If $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is analytic, $\gamma : [0,1] \to D \subseteq \mathbb{C}$, $\gamma(0) = z_0 \in D$, and $f \circ \gamma : [0,1] \to \mathbb{C}$, then $(f \circ \gamma)'(0) = f'(z_0) \cdot \gamma'(0)$.*

Theorem 7.2. *Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be analytic, $z_0 \in D$, such that $f'(z_0) \neq 0$. If we have curves $\gamma_1, \gamma_2 : [0,1] \to D$ such that $\gamma_1(0) = \gamma_2(0) = z_0$, where $\gamma_1'(0)$ and $\gamma_2'(0)$ are non-zero vectors, then the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$ is equal to the angle between $(f \circ \gamma_1)'(0)$ and $(f \circ \gamma_2)'(0)$.*

In this case, we say f is ***conformal*** *at z_0.*

Proof. Consider

$$J_f(z_0) = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix}$$

Using the C-R equations, we have

$$J_f(z_0) = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = |f'(z_0)| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta = \operatorname{Arg} f'(z_0)$. This is angle-preserving. □

If $f : D \to \mathbb{C}$, where D is a domain, where f is conformal at all points in D, then f or $\bar{f}$ is analytic. (Practice Problem)

A conformal mapping is a 1-1 correspondence and also conformal everywhere.

Definition 7.3. $f : D \subseteq \mathbb{C} \to U \subseteq \mathbb{C}$, where D and U are open, is a **conformal mapping** if

- f is continuous and u, v have continuous 1st-order partial derivatives,
- f is a one-to-one map,
- f is conformal at all points in D.

$f^{-1} : U \to D$ is also a conformal mapping.

Example 7.4. Consider the map $z \mapsto \lambda z$, where $\lambda \in \mathbb{C} \setminus \{0\}$. This is a conformal mapping $\mathbb{C} \to \mathbb{C}$.

Example 7.5. Consider the map $z \mapsto z + z_0$, where $z_0 \in \mathbb{C} \setminus \{0\}$. This is a conformal mapping $\mathbb{C} \to \mathbb{C}$.

Example 7.6. The map $z \mapsto 1/z$ is a conformal mapping $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$.

Example 7.7. The map $z \mapsto z^2$, which maps $re^{i\theta} \mapsto r^2 e^{2i\theta}$ is a conformal mapping $\{x + iy \mid x > 0\} \to \mathbb{C} \setminus (-\infty, 0]$.

Example 7.8. The map $z \mapsto e^z$ is a conformal mapping $\{x + iy \mid y \in (A, B)\} \to \{z \mid \operatorname{Arg} z \in (A, B)\}$, where $B - A < 2\pi$, with $-\pi < A, B < \pi$.

Example 7.9. The map $z \mapsto z^{3/2} = e^{(3/2)\operatorname{Log} z}$ is a conformal mapping from the sector $\theta \in (-\pi/6, \pi/6)$ to the sector $\theta \in (-\pi/4, \pi/4)$.

7.2 Fractional Linear Transformations

The maps

$$z \mapsto \lambda z \tag{7.1}$$
$$z \mapsto z + z_0 \tag{7.2}$$
$$z \mapsto \frac{1}{z} \tag{7.3}$$

are all 1-1 correspondences $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, or $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$. The functions are “analytic” on $\bar{\mathbb{C}}$.

Take compositions:

$$z \mapsto \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0$$

The expression above is indeed a composition of the three functions. If $c \neq 0$:

$$z \xrightarrow{(7.1)+(7.2)} cz + d \xrightarrow{(7.3)} \frac{1}{cz + d} \xrightarrow{(7.1)} \frac{(bc - ad)/c}{cz + d} \xrightarrow{+a/c} \frac{az + b}{cz + d}$$

These are called **fractional linear transformations**.

Remarks:

1. $ad - bc \neq 0$? If we have the matrix
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
then, we are saying the determinant is 0. If (a, b) and (c, d) are linearly dependent, then $(az+b)/(cz+d)$ is a constant.

2. Under composition:
$$\frac{az + b}{cz + d} \circ \frac{\alpha z + \beta}{\gamma z + \delta} = \frac{a\dfrac{\alpha z + \beta}{\gamma z + \delta} + b}{c\dfrac{\alpha z + \beta}{\gamma z + \delta} + d}$$
The coefficients come from matrix multiplication:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

3.
$$\left(\frac{az+b}{cz+d}\right)^{-1}$$
is also a FLT.

Observe that
$$\frac{az+b}{cz+d} = \frac{(2a)z+(2b)}{(2c)z+(2d)}$$
Although there are 4 parameters, "essentially" we only need 3 parameters to determine a FLT.

How many points determine a FLT?

Theorem 7.10. *For any three distinct points $z_0, z_1, z_2 \in \bar{\mathbb{C}}$, and three distinct points $w_0, w_1, w_2 \in \bar{\mathbb{C}}$, there exists a unique FLT such that $f(z_n) = w_n$ for $n = 0, 1, 2$.*

Proof by Example. Suppose that $f(0) = 1$, $f(1) = i$, $f(\infty) = -1$. Can we find f?

$$\frac{a\cdot 0+b}{c\cdot 0+d} = 1 \tag{7.4}$$
$$\frac{a\cdot 1+b}{c\cdot 1+d} = i \tag{7.5}$$
$$\frac{a\cdot \infty+b}{c\cdot \infty+d} = -1 \qquad \to \qquad \frac{a}{c} = -1 \tag{7.6}$$

(If $\dfrac{az+b}{cz+d} = \infty$, then $cz+d = 0$.)

(7.4) implies that $b = d$, and (7.6) implies that $a = -c$. (7.5) implies that
$$\underbrace{a}_{-c} + \underbrace{b}_{d} = i(c+d)$$
$$-c+d = i(c+d)$$
$$(1-i)d = (1+i)c$$
Up to scaling, $c = 1-i$ and $d = 1+i$, or $c = 1$ and $d = i$ (and then $a = -1$ and $b = i$).
$$f(z) = \frac{-1\cdot z+i}{1\cdot z+i} = \frac{-z+i}{z+i}$$
□

7.2.1 Geometric Picture

What is the image of the x-axis?

We know that $0, 1, \infty \in x$-axis, with $f(0) = 1$, $f(1) = i$, and $f(\infty) = -1$. The answer is the circle $|z| = 1$. Three points determine a circle.

Theorem 7.11. *All FLTs map lines and circles to lines and circles. (Lines are circles going through ∞. Lines and circles are determined by 3 points in $\bar{\mathbb{C}}$.)*

Proof. For $z \mapsto \lambda z$ and $z \mapsto z + z_0$, the statement holds trivially. $z \mapsto 1/z$ needs some computation. Lines are given by $ax + by = c$ and circles are given by $|z - z_0|^2 = \gamma^2$. $\square$

Lecture 8

September 22

8.1 Line Integrals

Two line integrals were introduced in 53: $\int_\gamma f\,\mathrm{d}s$ and $\int_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y)$. We will discuss the 2nd type only.

$\gamma : [a,b] \to \mathbb{R}^2$, where γ is continuous, is a path. γ is a function, and the curve is the image of γ.

Suppose that $\phi : [a', b'] \to [a,b]$ is an increasing 1-1 correspondence. Then $\gamma \circ \phi : [a', b'] \to \mathbb{R}^2$ is a different function (path), but with the same image as γ.

Any path $\gamma : [a,b] \to \mathbb{R}^2$ can be written as $\gamma(t) = (x(t), y(t))$.

"γ is smooth" means that x, y have as many derivatives as you want. $x', x'', x''', x^{(4)}, \ldots$ exist. A piecewise smooth curve is a concatenation of finitely many smooth paths.

In a closed path, the initial point is the same as the terminal point.

If $\gamma : [a,b] \to \mathbb{R}^2, \mathbb{C}$ is piecewise smooth, and $P, Q : D \to \mathbb{C}$, where $D \supseteq \mathrm{Im}(\gamma)$, then the line integral is

$$\begin{aligned}
\int_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y) &= \lim_{\varepsilon > 0} \sum (P(x(t_i), y(t_i))(x(t_{i+1}) - x(t_i)) + Q(x(t_i), y(t_i))(y(t_{i+1}) - y(t_i))) \\
&= \int_a^b (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t))\,\mathrm{d}t \\
&= \int_{a'}^{b'} (P(\tilde{x}(s), \tilde{y}(s))\tilde{x}'(s) + Q(\tilde{x}(s), \tilde{y}(s))\tilde{y}'(s))\,\mathrm{d}s
\end{aligned}$$

where $a = t_0 < t_1 < \cdots < t_n = b$, $|t_i - t_{i+1}| < \varepsilon$, and $\tilde{\gamma} : [a', b'] \to \mathbb{R}^2$ is a reparameterization of γ. The line integral does not depend on parameterization, but it depends on the orientation (can be negative).

8.2 Green's Theorem

Theorem 8.1 (Green's Theorem). *Let $D \subseteq \mathbb{R}^2$ be bounded, and ∂D be finitely many piecewise smooth closed curves. The orientation of ∂D is such that D lies to the left when walking along ∂D. Let $P, Q : D \cup \partial D \to \mathbb{C}$ be continuous. Then*

$$\int_{\partial D} (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y$$

(In $P\,\mathrm{d}x + Q\,\mathrm{d}y \mapsto P_y\,\mathrm{d}y\,\mathrm{d}x + Q_x\,\mathrm{d}x\,\mathrm{d}y$, to switch the order of $\mathrm{d}y\,\mathrm{d}x$, we need a negative sign.)

8.2.1 Independence of Path

Let $h : D \to \mathbb{C}$ be differentiable (possibly not complex differentiable). Let $\mathrm{d}h = h_x(x,y)\,\mathrm{d}x + h_y(x,y)\,\mathrm{d}y$, and let

$$\begin{aligned}\gamma_1 &: [a_1, b_1] \to D\\ \gamma_2 &: [a_2, b_2] \to D\end{aligned}$$

be two piecewise smooth paths with $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$. Then

$$\begin{aligned}\int_{\gamma_1} \mathrm{d}h &= \int_{\gamma_2} \mathrm{d}h\\ &= \int_a^b (h_x(x_1(t), x_2(t))x_1'(t) + h_y(x_1(t), y_1(t))y_1'(t))\,\mathrm{d}t\\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(h(x_1(t), y_1(t)))\,\mathrm{d}t = h(x_1(b), y_1(b)) - h(x_1(a), x_2(b))\end{aligned}$$

Theorem 8.2. *Let $P, Q : D \to \mathbb{C}$, where D is a domain. If $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is independent of path in D, then $\exists h : D \to \mathbb{C}$ such that $\mathrm{d}h = P\,\mathrm{d}x + Q\,\mathrm{d}y$.*

Proof. Define h: take $z_0 \in D$ and any $z \in D$. Define $h(z) = \int_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y)$, where $\gamma : [a,b] \to D$ such that $\gamma(a) = z_0$ and $\gamma(b) = z$. h is well-defined since the integral is independent of path. Check: $h_x = P$ and $h_y = Q$. Take a special path to z_0. □

Independence of path is equivalent to $P\,\mathrm{d}x + Q\,\mathrm{d}y = \mathrm{d}h$. We call $P\,\mathrm{d}x + Q\,\mathrm{d}y$ an exact form. If $Q_x = P_y$, then $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is a closed form. "Exact" implies "closed" when P, Q are differentiable, because

$$\mathrm{d}h = \underbrace{h_x}_{P}\,\mathrm{d}x + \underbrace{h_y}_{Q}\,\mathrm{d}y$$

so $Q_x = h_{y,x}$ and $P_y = h_{x,y}$. When is the reverse implication true?

Consider

$$\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y$$

defined on $\mathbb{R}^2 \setminus \{0\}$. Then $Q_x = P_y$. We get h by integrating P:

$$\begin{aligned}h(x,y) &= \int \frac{y}{x^2+y^2}\,\mathrm{d}x + c(y)\\ &= \arctan\frac{x}{y} + c(y)\end{aligned}$$

which is not continuous on $\mathbb{R}^2 \setminus \{0\}$.

Theorem 8.3. *Let D be a star-shaped domain, that is, there is some $z_0 \in D$ such that $\forall z \in D$, the straight line segment between z_0 and z lies in D. If $P\,\mathrm{d}x + Q\,\mathrm{d}y$, defined on D, is a closed form, then it is exact.*

Proof. Define $h(z) = \int_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y)$, where γ is the straight line segment from z_0 to z. Let $\tilde{\gamma}$ be the path which starts from z_0, but takes a horizontal path to z at the end of the path. We show that

$$\int_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \int_{\tilde{\gamma}} (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

Then

$$\int_\gamma - \int_{\tilde{\gamma}} = \iint_D \underbrace{(Q_x - P_y)}_{\text{closed form}}\,\mathrm{d}x\,\mathrm{d}y = 0$$

This implies that $h_x = P$. Similarly, $h_y = Q$. □

For any two paths γ_1, γ_2 in D with the same initial and terminal point, we can deform γ_1 to γ_2 continuously.

Lecture 9

September 27

9.1 Line Integrals

Let $\gamma : [a, b] \to \mathbb{C}\ (\mathbb{R}^2)$ be piecewise smooth, where $\gamma(t) = x(t) + iy(t)$. Consider also $P, Q : D \subseteq \mathbb{C} \to \mathbb{C}$, where $D \supseteq \mathrm{Im}(\gamma)$. Then the line integral is $\int_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y)$. $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is defined on $D \subseteq \mathbb{C}$. We showed that the line integral is independent of path on D if and only if $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is an exact form, i.e. $\exists h : D \to \mathbb{C}$ such that $P\,\mathrm{d}x + Q\,\mathrm{d}y = \mathrm{d}h$, which implies that $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is closed, i.e. $Q_x = P_y$.

If D is a star-shaped domain, then closed implies exact. We defined h by $h(z) = \int_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y)$, where γ is the line segment from z_0 to z. Green's Theorem states that

$$\int_\gamma (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \int_{\gamma_1} (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \int_{\gamma_2} (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

Theorem 9.1. *Let $P\,\mathrm{d}x + Q\,\mathrm{d}y$ be a closed form on a domain D. Let $\gamma_s : [0,1] \to D$ be a continuous family of paths, $s \in [0,1]$. If $\gamma_s(0) = A\ \forall s \in [0,1]$ and $\gamma_s(1) = B\ \forall s \in [0,1]$, then*

$$\int_{\gamma_0} (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \int_{\gamma_1} (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

Proof. Consider

$$\Gamma : \underbrace{[0,1]}_{t} \times \underbrace{[0,1]}_{s} \to D$$

where $\Gamma(t, s) = \gamma_s(t)$. Write $\Gamma(t, s) = (X(t, s), Y(t, s))$. Then

$$\int_{\gamma_s} (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \int_0^1 \left(P(\gamma_s(t))\frac{\partial X}{\partial t} + Q(\gamma_s(t))\frac{\partial Y}{\partial t} \right) \mathrm{d}t$$

Take the partial derivative with respect to s: it equals 0, using $P_y = Q_x$, $\gamma_s(0) = A$, $\gamma_s(0) = B$. □

We can rewrite the condition that D is a star-shaped domain to use any $z_0, z_1 \in D$, for any two paths $\gamma_0, \gamma_1 : [0,1] \to D$, with $\gamma_0(0) = \gamma_1(0) = z_0$ and $\gamma_0(1) = \gamma_1(1) = z_1$, such that γ_0 and γ_1 can be connected by a continuous family of paths starting at z_0 and ending at z_1.

9.2 Applications to Harmonic Functions

Given $u : D \to \mathbb{R}$, a harmonic function, can we determine whether a harmonic conjugate v exists on D? Last chapter: If D is a rectangle or a disk, then v exists.

We need

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

or equivalently, find a v such that

$$\begin{aligned} \mathrm{d}v &= v_x\,\mathrm{d}x + v_y\,\mathrm{d}y \\ &= -u_y\,\mathrm{d}x + u_x\,\mathrm{d}y \end{aligned}$$

Check: $-u_y\,\mathrm{d}x + u_x\,\mathrm{d}y$ is a closed form. We need $Q_y = P_x$, which is $u_{xx} = -u_{yy}$. This holds since u is harmonic.

Theorem 9.2. *Let $D \subseteq \mathbb{C}$ be a star-shaped domain and $u : D \to \mathbb{R}$ be harmonic. There exists a harmonic conjugate v of u.*

Let D be a domain and $u : D \to \mathbb{R}$ be harmonic. Local property of u: take $z_0 \in D$. There exists $\rho > 0$ such that $B(z_0, \rho) \subseteq D$. There exists $v : B(z_0, \rho) \to \mathbb{R}$, which is a harmonic conjugate of u. For any $0 < r < \rho$,

$$0 = \int_{|z-z_0|=r} (-u_y\,\mathrm{d}x + u_x\,\mathrm{d}y) \underbrace{=}_{\substack{z=z_0+re^{i\theta} \\ x=x_0+r\cos\theta \\ y=y_0+r\sin\theta}} \int_0^{2\pi} (-u_y(z_0+re^{i\theta})(-r\sin\theta)\,\mathrm{d}\theta + u_x(z_0+re^{i\theta})(r\cos\theta)\,\mathrm{d}\theta)$$

$$= \int_0^{2\pi} r(u_y(z_0+re^{i\theta})\sin\theta + u_x(z_0+re^{i\theta})\cos\theta)\,\mathrm{d}\theta$$

Since $u = u(x, y)$, then $u = u(x_0 + r\cos\theta, y_0 + r\sin\theta)$. Then

$$u_r = u_x \cdot x_r + u_y \cdot y_r = u_x \cos\theta + u_y \sin\theta$$

This implies that

$$0 = \int_0^{2\pi} u_r(z_0+re^{i\theta})\,\mathrm{d}\theta = \frac{\mathrm{d}}{\mathrm{d}r}\left(\int_0^{2\pi} u(z_0+re^{i\theta})\,\mathrm{d}\theta\right)$$

so $\int_0^{2\pi} u(z_0+re^{i\theta})\,\mathrm{d}\theta$ is a constant function of r. In particular, take $r = 0$ and we obtain $2\pi u(z_0)$.

Theorem 9.3 (Mean Value Property)**.** *Let $u : D \to \mathbb{R}$ be harmonic and $B(z_0, \rho) \subseteq D$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,\mathrm{d}\theta = u(z_0) \qquad \forall 0 < r < \rho$$

Theorem 9.4 (Strict Maximal Principle (Real Version))**.** *Let D be a domain, $u : D \to \mathbb{R}$ is harmonic, where $u(z) \le M$ for any $z \in D$. If $\exists z_0 \in D$ such that $u(z) = M$, then $u(z) = M$ for any $z \in D$.*

Proof. Let $\varnothing \neq U = \{z \in D \mid u(z) = M\}$ and $V = \{z \in D \mid u(z) < M\}$. Then $D = U \cup V$ and $U \cap V = \varnothing$. u is continuous, so V is open. 9.3 implies that U is open. Therefore, $V = \varnothing$. □

Theorem 9.5 (Strict Maximal Principle (Complex Version))**.** *Let D be a domain, $f : D \to \mathbb{C}$ (complex-valued) is harmonic. (Analytic functions are harmonic.) If $|f(z)| \le M\ \forall z \in D$ and $\exists z_0 \in D$ such that $|f(z_0)| = M$, then f is a constant function on D, equal to $f(z_0)$.*

Proof. Consider $g(z) = e^{-i \operatorname{Arg} f(z_0)} \cdot f(z)$. g is a $\mathbb{C}$-valued harmonic function with $|g(z)| \leq M$. Also, $g(z_0) = e^{-i \operatorname{Arg} f(z_0)} f(z_0) = |f(z_0)|$. Consider $\operatorname{Re} g$. Then $\operatorname{Re} g$ is a $\mathbb{R}$-valued harmonic function with $\operatorname{Re}(g) \leq M$ and $\operatorname{Re}(g)(z_0) = M$. Therefore, $\operatorname{Re} g \equiv M$ and $\operatorname{Im} g \equiv 0$. □

Theorem 9.6 (Maximal Principle)**.** *If $f : D \to \mathbb{C}$ is harmonic, where D is a bounded domain, and f continuously extends to a function $D \cup \partial D \to \mathbb{C}$, then $|f(z)| \leq M$ on ∂D implies that $|f(z)| \leq M$ on D.*

Lecture 10

September 29

10.1 Recap

If $D \subseteq \mathbb{C}$ is a domain, with two functions $P, Q : D \to \mathbb{C}$, then $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is a form on D. Independence of path is equivalent to $P\,\mathrm{d}x + Q\,\mathrm{d}y = \mathrm{d}h$, an exact form, where $h : D \to \mathbb{C}$. An exact form is a closed form, i.e. $Q_x = P_y$, and the reverse implication is true when D is star-shaped.

Consider $u : D \subseteq \mathbb{C} \to \mathbb{R}$, and take $z_0 \in D$ with $B(z_0, \rho) \subseteq D$. We showed the Mean Value Property:

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\,\mathrm{d}\theta$$

for all $0 < r < \rho$. This implies the maximal principle of harmonic functions (the $\mathbb{R}$-valued version and the $\mathbb{C}$-valued version). (If $u(z_0) \geq u(z)$ for any $z \in D$, then the Mean Value Property implies that $u(z) = u(z_0)$ for all $z \in B(z_0, \rho)$.)

10.2 Complex Line Integral

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and γ be a path in D. We will discuss the complex line integral, $\int_\gamma f(z)\,\mathrm{d}z$, which behaves well if f is analytic.

Writing $z = x + iy$, we define $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$. Then

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &\overset{\text{def}}{=} \int_\gamma f(z)(\mathrm{d}x + i\,\mathrm{d}y) \\
&= \int_\gamma (\underbrace{f(z)}_{P}\,\mathrm{d}x + \underbrace{if(z)}_{Q}\,\mathrm{d}y)
\end{aligned}$$

Example 10.1.

$$\begin{aligned}
\int_{|z|=R} z^m\,\mathrm{d}z \underbrace{=}_{\substack{z=Re^{i\theta} \\ x=R\cos\theta \\ y=R\sin\theta}} &\int_0^{2\pi} (Re^{i\theta})^m (\underbrace{-R\sin\theta\,\mathrm{d}\theta}_{\mathrm{d}x} + i\underbrace{R\cos\theta\,\mathrm{d}\theta}_{\mathrm{d}y}) \\
&= \int_0^{2\pi} R^{m+1} e^{im\theta}(-\sin\theta + i\cos\theta)\,\mathrm{d}\theta \\
&= \int_0^{2\pi} R^{m+1} e^{im\theta} i(\cos\theta + i\sin\theta)\,\mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&= iR^{m+1}\int_0^{2\pi} e^{i(m+1)\theta}\,\mathrm{d}\theta \\
&= \begin{cases} \dfrac{iR^{m+1}}{i(m+1)} e^{i(m+1)\theta}\Big|_{\theta=0}^{2\pi} = 0, & m \neq -1 \\ 2\pi i, & m = -1 \end{cases}
\end{aligned}
$$

We could also write γ as $z(t) = x(t) + iy(t)$.

$$\int_a^b f(z(t)) \cdot \underbrace{z'(t)\,\mathrm{d}t}_{\mathrm{d}z}$$

Then we have

$$\int_{|z|=R} z^m\,\mathrm{d}z \underbrace{=}_{\substack{z=Re^{i\theta} \\ \mathrm{d}z=iRe^{i\theta}\,\mathrm{d}\theta}} \int_0^{2\pi} (Re^{i\theta})^m \cdot iRe^{i\theta}\,\mathrm{d}\theta$$

If $\gamma : [a,b] \to D$, then $a = t_0 < t_1 < \cdots < t_n = b$. The line integral is formally defined as

$$\int_\gamma f(z)\,\mathrm{d}z = \lim_{\max\{t_{k+1}-t_k\}\to 0} \sum_{k=0}^{n-1} f(z(t_k)) \cdot (z(t_{k-1}) - z(t_k))$$

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There is also another line integral:

$$\int_\gamma h(x,y) \underbrace{\mathrm{d}s}_{\text{arc length}} = \lim_{\max\{t_{k+1}-t_k\}\to 0} \sum_{k=0}^{n-1} h(z(t_k)) \cdot |z(t_{k+1}) - z(t_k)|$$

This gives

$$\int_\gamma |f(z)|\,\mathrm{d}s \geq \left|\int_\gamma f(z)\,\mathrm{d}z\right|$$

because $|z_1| + |z_2| + \cdots + |z_n| \geq |z_1 + z_2 + \cdots + z_n|$.

The line integral was computed by

$$\int_\gamma h\,\mathrm{d}s = \int_a^b h(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t$$

We can define

$$
\begin{aligned}
|\mathrm{d}z| &= |\mathrm{d}x + i\,\mathrm{d}y| \\
&= \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} \underbrace{=}_{\text{if curve is given by } x(t)+iy(t)} \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t
\end{aligned}
$$

We write

$$\int_\gamma f(z)\,|\mathrm{d}z| \qquad \text{for} \qquad \int_\gamma f(z)\,\mathrm{d}s$$

so

$$\int_\gamma |f(z)| \cdot |\mathrm{d}z| \geq \left|\int_\gamma f(z)\,\mathrm{d}z\right|$$

10.2.2 ML Estimation

If $M = \sup\{|f(z)| \mid z \in \gamma\}$, then

$$\begin{aligned}\left|\int_\gamma f(z)\,\mathrm{d}z\right| &\le \int_\gamma |f(z)|\,|\mathrm{d}z| \\ &\le M\int_\gamma 1\cdot|\mathrm{d}z| = M\cdot L\end{aligned}$$

where L is the length of γ.

10.3 Primitives

From calculus, $F(x) = \int_0^x f(t)\,\mathrm{d}t \Rightarrow F'(x) = f(x)$.

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. We want to find $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ (F is called a "primitive" of f). Since we always assume that f is continuous, this implies that F is analytic. Write $f = u + iv$ and assume that u, v have continuous 1st-order partial derivatives. By the midterm problem, f is analytic on D.

The function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \bar{z}$ is not analytic, so by the discussion above, there does not exist F such that $F' = f$. Therefore, we will only think about analytic functions and their primitives.

Theorem 10.2. *If $f : D \to \mathbb{C}$ has a primitive $F : D \to \mathbb{C}$, then for $\gamma : [a, b] \to D$ ($z \mapsto \gamma(t)$),*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

In particular, $f(z)\,\mathrm{d}z$ is independent of path.

Proof.

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left((F\circ\gamma)(t)\right)\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a))$$

□

The primitive does not always exist (even for analytic functions).

Example 10.3. The function $1/z$ is defined on $C \setminus \{0\} \to \mathbb{C}$.

$$\oint_{|z|=R} \frac{1}{z}\,\mathrm{d}z = 2\pi i$$

Therefore, there does not exist a primitive for $1/z$.

Proposition 10.4. *f is analytic if and only if $f(z)\,\mathrm{d}z$ is a closed form.*

Proof. Write $f = u + iv$. Then

$$\begin{aligned} f(z)\,\mathrm{d}z &= (u+iv)\,\mathrm{d}x + i(u+iv)\,\mathrm{d}y \\ &= \underbrace{(u+iv)}_{P}\,\mathrm{d}x + \underbrace{(-v+iu)}_{Q}\,\mathrm{d}y\end{aligned}$$

Closed means that $Q_x = P_y$, so $-v_x + iu_x = u_y + iv_y$, which is equivalent to $-v_x = u_y$ and $u_x = v_y$, which are the C-R equations. $\square$

Theorem 10.5. *If D is a star-shaped domain, and $f : D \to \mathbb{C}$ is analytic, then $\exists F : D \to \mathbb{C}$ such that $F_x \,\mathrm{d}x + F_y \,\mathrm{d}y = f(z)\,\mathrm{d}z$. Actually, $F'(z) = f(z)$.*

Proof. We have

$$\begin{aligned} F_x &= u + iv \\ F_y &= i(u + iv) \\ &= -v + iu \end{aligned}$$

Hence, C-R holds for F, and $F' = F_x = f$. $\square$

Lecture 11

October 4

11.1 Complex Line Integrals

$$\int_\gamma f(z)\,\mathrm{d}z \underbrace{=}_{\mathrm{d}z=\mathrm{d}x+i\,\mathrm{d}y} \int_\gamma (f(z)\,\mathrm{d}x + if(z)\,\mathrm{d}y)$$

Theorem 11.1. *f is analytic if and only if $f(z)\,\mathrm{d}z$ is a closed form (by the C-R equations).*

Consider $f : D \to \mathbb{C}$. A **primitive** of f is $F : D \to \mathbb{C}$, such that $F'(z) = f(z)$. If f has a primitive, then f is analytic. If f is analytic and D is a star-shaped domain, then f has a primitive. Define

$$F(z) = \int_{z_0}^{z} f(w)\,\mathrm{d}w$$

where z_0 is a fixed point and the integral is along any path from z_0 to z.

Theorem 11.2 (Cauchy's Theorem)**.** *Let D be a bounded domain with a piecewise smooth boundary. Let $f : D \to \mathbb{C}$ be an analytic function which extends smoothly to ∂D. (In applications of the theorem, $f : U \to \mathbb{C}$ is analytic, with $U = D \cup \partial D$.) Then*

$$\int_{\partial D} f(z)\,\mathrm{d}z = 0$$

Suppose that $f : D \setminus \{z_0\}$ is analytic. Apply Cauchy's Theorem to $D \setminus B(z_0, r)$. Then

$$0 = \int_{\partial(D\setminus B(z_0,r))} f(z)\,\mathrm{d}z = \int_{\partial D} f(z)\,\mathrm{d}z - \int_{|z-z_0|=r} f(z)\,\mathrm{d}z$$

where the last line integral is taken in a counterclockwise orientation. Therefore,

$$\int_{|z-z_0|=r} f(z)\,\mathrm{d}z = \int_{\partial D} f(z)\,\mathrm{d}z$$

The last integral is independent of r as long as $B(z_0, r) \subseteq D$.

Take $f(z)/(z - z_0)$. If f is analytic on D, then $f(z)/(z - z_0)$ is analytic on $D \setminus \{z_0\}$. We get:

$$\begin{aligned}\int_{|z-z_0|=r} \frac{f(z)}{z-z_0}\,\mathrm{d}z &= \int_{\partial D} \frac{f(z)}{z-z_0}\,\mathrm{d}z \\ &= \int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{\cancel{re^{i\theta}}} i(\cancel{re^{i\theta}})\,\mathrm{d}\theta\end{aligned}$$

$$= i \int_0^{2\pi} f(z_0 + re^{i\theta})\,\mathrm{d}\theta = 2\pi i \cdot f(z_0)$$

where $z = z_0 + re^{i\theta}$. The last line follows from the Mean Value Property (f is analytic, so f is harmonic). Here, we assume that u, v have continuous 2nd-order partial derivatives.

$$\int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\,\mathrm{d}z \qquad \text{is independent of } r$$

Let $r \to 0$. We still get the same number:

$$\lim_{r\to 0} i \int_0^{2\pi} f(z_0 + re^{i\theta})\,\mathrm{d}\theta = i \int_0^{2\pi} f(z_0)\,\mathrm{d}\theta = 2\pi i \cdot f(z_0)$$

11.2 Cauchy Integral Formula

Theorem 11.3 (Cauchy Integral Formula)**.** *Let D be a bounded domain with a piecewise smooth boundary. Let $f : D \to \mathbb{C}$ be analytic which extends smoothly to ∂D. Then*

$$\forall z_0 \in D \qquad f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

Written another way:

$$\forall z \in D \qquad f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z}\,\mathrm{d}w$$

The RHS is an analytic function of z on D. Take the complex derivative of z. Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{w-z}\right) &= -\frac{1}{(w-z)^2}\frac{\mathrm{d}}{\mathrm{d}z}(w-z) \\ &= \frac{1}{(w-z)^2} \end{aligned}$$

we have

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^2}\,\mathrm{d}w$$

Looking at the LHS, we do not know if $f'(z)$ is complex differentiable. On the RHS, however, we can compute further complex derivatives of $1/(w-z)^2$. Since the RHS is still complex differentiable, this implies that $f'(z)$ is complex differentiable. Inductively, $f^{(m)}(z)$ exists and

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\,\mathrm{d}w$$

If $f : U \to \mathbb{C}$, where U is open, apply the Cauchy Integral Formula on a small disk D such that $D \cup \partial D \subseteq U$.

Theorem 11.4. *If $f : U \to \mathbb{C}$ is analytic, where U is open, then $f', f'', \ldots$ exist on U and are also analytic. For any $z \in U$,*

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{(w-z)^{m+1}}\,\mathrm{d}w$$

where $B(z,r) \cup \partial B(z,r) \subseteq U$.

This rigorously shows that if f is analytic, then f is harmonic.

11.2.1 Computation

$$\int_{|z|=2} \frac{1}{z^3(z+1)^2}\,\mathrm{d}z$$

If f were analytic on $|z| < 2$, then the line integral would be 0. We know that the function is analytic on $\{z \mid |z| < 2\} - \{0, -1\}$ (or alternatively, $\mathbb{C} - \{0, -1\}$). Apply Cauchy's Theorem:

$$\int_{|z|=2} \frac{\mathrm{d}z}{z^3(z+1)} = \int_{|z|=\varepsilon} \frac{\mathrm{d}z}{z^3(z+1)^2} + \int_{|z+1|=\varepsilon} \frac{\mathrm{d}z}{z^3(z+1)^2}$$

We compute the first integral.

$$\begin{aligned}\int_{|z|=\varepsilon} \frac{\mathrm{d}z}{z^3(z+1)^2} = \int_{|z|=\varepsilon} \frac{1/(z+1)^2}{(z-0)^3}\,\mathrm{d}z &= \frac{2\pi i}{2!}\left(\frac{1}{(z+1)^2}\right)''\Big|_{z=0} \\ &= 6\pi i\end{aligned}$$

since $1/(z+1)^2$ is analytic on $|z| < \varepsilon$. ($3 = 2 + 1$, so take $m = 2$.) Next:

$$\int_{|z+1|=\varepsilon} \frac{\mathrm{d}z}{z^3(z+1)^2} = \int_{|z+1|=\varepsilon} \frac{1/z^3}{(z-(-1))^2}\,\mathrm{d}z = \frac{2\pi i}{1!}\left(\frac{1}{z^3}\right)'\Big|_{z=-1} = -6\pi i$$

where $m = 1$.

Lecture 12

October 6

12.1 Cauchy's Estimate

If D is a bounded domain, and $f : D \to \mathbb{C}$ is analytic, then

1. $\int_{\partial D} f(w)\, \mathrm{d}w = 0$
2. $$f(z) = \frac{1}{2\pi i} \int_{\partial D} \underbrace{\frac{f(w)}{w-z}}_{g(w)} \mathrm{d}w, \qquad z \in D$$

   $g(w)$ is analytic on $D \setminus \{z\}$.
3. $f^{(m)}(z)$ exists for all $z \in D$.
   $$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}} \mathrm{d}w$$

3 implies the Cauchy estimation of $f^{(m)}(z)$:

$$\begin{aligned}
f^{(m)}(z) &= \frac{m!}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{(w-z)^{m+1}} \mathrm{d}w \qquad \text{if } \{w \mid |w-z| \le r\} \in D \\
\left|f^{(m)}(z)\right| &\le \frac{m!}{2\pi} \int_{|w-z|=r} \frac{|f(w)|}{|w-z|^{m+1}} |\mathrm{d}w| \\
&\le \frac{m!}{2\pi} \int_{|w-z|=r} \frac{M}{r^{m+1}} |\mathrm{d}w| \\
&= \frac{m!}{2\pi} \frac{M}{r^{m+1}} 2\pi r = \frac{m!\, M}{r^m}
\end{aligned}$$

if $|f(w)| \le M$ on $|w-z| = r$. This is Cauchy's estimate:

$$\left|f^{(m)}(z)\right| \le \frac{m!\, M}{r^m}$$

From M, a bound for f, we obtain some bound for $f^{(m)}$.

Such a bound is not possible in the real numbers. The function

$$f_n(x) = \frac{1}{n} \sin(n^2 x)$$

is bounded by

$$|f_n| \le \frac{1}{n} \qquad \text{small}$$

but the derivative is bounded by

$$\begin{aligned} f_n' &= n\cos(n^2 x) \\ f_n'(0) &= n \qquad\qquad \text{large} \end{aligned}$$

12.2 Liouville's Theorem

Theorem 12.1 (Liouville's Theorem)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is analytic (an* ***entire*** *function) and also bounded, then f is a constant function.*

Proof. If $\exists M > 0$ such that $|f(z)| \leq M$, then $\forall z \in \mathbb{C}$,

$$|f'(z)| \leq \frac{1!\,M}{r^1} = \frac{M}{r}$$

as long as $\{w \mid |w - z| \leq r\} \subseteq \mathbb{C}$, which is always true. Letting $r \to \infty$, then $f'(z) = 0$, which implies that $f(z)$ is constant. □

12.3 Analytic Functions

$f : D \to \mathbb{C}$ is analytic if

1. $f'(z)$ exists for any $z \in D$,
2. f' is continuous on D.

Actually, 1 implies 2.

12.3.1 Part 1

Theorem 12.2 (Morera's Theorem)**.** *If $f : D \to \mathbb{C}$ is continuous, where D is a domain, if for any closed rectangle $R \subseteq D$,*

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0$$

then f is analytic.

Proof. Find an analytic $F : D \to \mathbb{C}$ such that $F' = f$ (this implies that $F'' = f'$ exists and is continuous). This is equivalent to $\mathrm{d}F(z) = f(z)\,\mathrm{d}z$.

To define F: do the line integral of $f(z)\,\mathrm{d}z$. Consider the line integral on a disk $B \subseteq D$.

$$F(z) = \int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

where γ_0 and γ_1 differ by a rectangular path. To show that F is analytic, use the Cauchy-Riemann equations: $F = U + iV$ and $f = u + iv$.

$$\begin{aligned} F_x &= \lim_{\Delta x \to 0} \frac{F(z + \Delta x) - F(z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_{x_0}^{x_0 + \Delta x} f(t + iy)\,\mathrm{d}t - \int_{x_0}^{x} f(t + iy)\,\mathrm{d}t}{\Delta x} \\ &= f(x + iy) = u + iv \\ F_y &= \lim_{\Delta y \to 0} \frac{F(z + \Delta y) - F(z)}{\Delta y} \underbrace{=}_{\gamma_1 \text{ formula}} if(x + iy) = i(u + iv) \end{aligned}$$

so

$$\begin{aligned} F' = F_x &= U_x + iV_x = u + iv \\ F_y &= U_y + iV_y = -v + iu \end{aligned}$$

which imply

$$\begin{aligned} U_x &= V_y = u \\ U_y &= -V_x = -v \end{aligned}$$

which are all continuous. The C-R equations hold for F, which implies that F is analytic. □

12.3.2 Part 2

Theorem 12.3 (Goursat's Theorem)**.** *If $f : D \to \mathbb{C}$ is continuous, if $f'(z)$ exists for any $z \in D$, then f' is continuous on D.*

Proof. Use Morera's Theorem, 12.2, and then show that $\int_{\partial R} f(z)\,\mathrm{d}z = 0$.

Suppose that $\int_{\partial R} f(z)\,\mathrm{d}z = A \neq 0$ for some closed rectangle $R \subseteq D$. There exists R_1 ("$= R/2$") such that

$$\left| \int_{\partial R_1} f(z)\,\mathrm{d}z \right| \geq \frac{|A|}{4}$$

Then there exists R_2 ("$= R/2^2$") such that

$$\left| \int_{\partial R_2} f(z)\,\mathrm{d}z \right| \geq \frac{|A|}{4^2}$$

Continue inductively. There exists R_n ("$= R/2^n$") such that

$$\left| \int_{\partial R_n} f(z)\,\mathrm{d}z \right| \geq \frac{|A|}{4^n}$$

Then $\bigcap_{n=1}^\infty R_n = \{z_0\}$. We want to find a contradiction on $\int_{\partial R_n} f(z)\,\mathrm{d}z$ for large n.

In R_n, any two points are $\leq \sqrt{2}L/2^n$ apart. We know that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

For $\varepsilon = |A|/(8L^2)$, there exists n such that $\forall z \in R_n$,

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \leq \varepsilon$$

Then, we estimate the integral.

$$\begin{aligned} \frac{|A|}{4^n} &\leq \left| \int_{\partial R_n} f(z)\,\mathrm{d}z \right| = \left| \int_{\partial R_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\,\mathrm{d}z \right| \\ &\leq \int_{\partial R_n} |f(z) - f(z_0) - f'(z_0)(z - z_0)| \cdot |\mathrm{d}z| \\ &\leq \int_{\partial R_n} \varepsilon |z - z_0| \cdot |\mathrm{d}z| \leq \int_{\partial R_n} \sqrt{2}\frac{L}{2^n}\varepsilon\,|\mathrm{d}z| \leq \sqrt{2}\frac{L}{2^n} \cdot 4\frac{L}{2^n}\frac{|A|}{8L^2} = \frac{|A|}{\sqrt{2} \cdot 4^n} \end{aligned}$$

□

12.3.3 Morera's Theorem

If $f : D \to \mathbb{C}$ is continuous, with $\int_{\partial R} f(z)\,\mathrm{d}z = 0$ for any closed rectangle $R \subseteq D$, then f is analytic.

Example 12.4. Let $g : D \to \mathbb{C}$ be analytic, with $g(z_0) = 0$, for some $z_0 \in D$. Define a new function

$$f(z) = \begin{cases} \dfrac{g(z) - g(z_0)}{z - z_0}, & \text{if } z \in D \setminus \{z_0\} \\ g'(z_0), & z = z_0 \end{cases}$$

f is analytic on $D \setminus \{z_0\}$ and continuous on D. This implies that f is analytic on D (by Morera's Theorem, 12.2).

Consider a rectangle $R \subseteq D$. If $z_0 \notin R$, then $\int_{\partial R} f(z)\,\mathrm{d}z = 0$. If $z_0 \in \partial R$, then push the rectangle up by a distance of ε to R_ε. Since f is continuous,

$$\int_{\partial R} f(z)\,\mathrm{d}z = \lim_{\varepsilon \to 0} \int_{\partial R_\varepsilon} f(z)\,\mathrm{d}z = 0$$

If $z_0 \in R$, then we can divide R into R_1 and R_2, then

$$\int_{\partial R} f(z)\,\mathrm{d}z = \int_{\partial R_1} f(z)\,\mathrm{d}z + \int_{\partial R_2} f(z)\,\mathrm{d}z = 0$$

The same argument applies if $f : D \to \mathbb{C}$ is continuous and analytic on $D \setminus \mathbb{R}$. Then f is analytic on $\mathbb{R}$.

Lecture 13

October 11

13.1 Power Series

Suppose that $f : \{|z - z_0| < r\} \to \mathbb{C}$ is analytic. Our main goal is to write f as a power series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

We will review and develop series of complex numbers, and sequences and series of (analytic) functions.

Let $a_k \in \mathbb{C}$. Then the series of complex numbers $\sum_{k=0}^{\infty} a_k$ converges if $\lim_{n\to\infty} \sum_{k=0}^{n} a_k$ exists.

$\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} \mathrm{Re}(a_k)$ and $\sum_{k=0}^{\infty} \mathrm{Im}(a_k)$ both converge.

If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Theorem 13.1 (Comparison Test). *If $r_k \in \mathbb{R}$, $r_k \geq 0$ and $\sum_{k=0}^{\infty} r_k$ converges, if $a_k \in \mathbb{C}$, $|a_k| \leq r_k$, then $\sum_{k=0}^{\infty} a_k$ converges and*

$$\left| \sum_{k=0}^{\infty} a_k \right| \leq \sum_{k=0}^{\infty} r_k$$

Proof. $|\mathrm{Re}(a_k)| \leq |a_k| \leq r_k$, so $\sum \mathrm{Re}(a_k)$ converges.

$|\mathrm{Im}(a_k)| \leq |a_k| \leq r_k$, so $\sum \mathrm{Im}(a_k)$ converges. Therefore, $\sum a_k$ converges.

$$\left| \sum_{k=0}^{n} a_k \right| \leq \sum_{k=0}^{n} |a_k| \leq \sum_{k=0}^{n} r_k$$

Then, let $n \to \infty$. □

$\sum a_k$ **absolutely converges** if $\sum |a_k|$ converges.

An important sequence r_k that we will use is $\sum_{k=0}^{\infty} s^k = 1/(1-s)$, for $s \in (0,1)$. Another convergent series is $\sum_{k=0}^{\infty} 1/k^s$, where $s > 1$.

13.2 Sequences of Functions

Let $f_n : D \subseteq \mathbb{C} \to \mathbb{C}$ be a sequence of functions.

$\{f_n\}$ **pointwisely converges** to $f : D \to \mathbb{C}$ if for all $z \in D$, $\lim_{n\to\infty} f_n(z) = f(z)$, i.e.

$$\forall z \in D\ \forall \varepsilon > 0\ \exists N \text{ such that } \forall n > N\ |f_n(z) - f(z)| < \varepsilon$$

N depends on ε, z.

$\{f_n\}$ **uniformly converges** (N only depends on ε) to $f : D \to \mathbb{C}$ if

$$\forall \varepsilon > 0\ \exists N \text{ such that } \forall n > N\ \forall z \in D\ |f_n(z) - f(z)| < \varepsilon$$

If $f_n \xrightarrow{\text{uniformly}} f$ on D, then $f_n \xrightarrow{\text{pointwise}} f$ on D.

$\sum_{k=0}^{\infty} f_k$ pointwise converges on D if $\sum_{k=0}^{n} f_k$ pointwise converges on D.

$\sum_{k=0}^{\infty} f_k$ uniformly converges on D if $\sum_{k=0}^{n} f_k$ uniformly converges on D.

Theorem 13.2 (Weierstrass M-Test)**.** *Suppose $M_k \geq 0$ and $\sum_{k=0}^{\infty} M_k$ converges. For $f_k : D \to \mathbb{C}$, if $|f_k(z)| \leq M_k$ for all $z \in D$, then $\sum_{k=0}^{\infty} f_k$ uniformly converges on D.*

Usually, we will use the case where $M_k = s^k$ for $s \in (0, 1)$.

Theorem 13.3. *If $f_n : D \subseteq \mathbb{C} \to \mathbb{C}$, where $f_n \xrightarrow{\text{uniformly}} f$ on D,*

1. *If every f_n is continuous on D, then f is continuous on D.*
2. *If the f_n are continuous on D, $\gamma : [a, b] \to D$ is piecewise smooth, then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\, \mathrm{d}z = \int_\gamma f(z)\, \mathrm{d}z$$

Proof. 1. The proof in 104 works for all functions from metric spaces to metric spaces.

2. If L is the length of γ,

$$\forall \varepsilon > 0\ \exists N\ \forall n > N\ \forall z \in D\ |f_n(z) - f(z)| < \frac{\varepsilon}{L}$$

For $n > N$,

$$\left|\int_\gamma f_n(z)\, \mathrm{d}z - \int_\gamma f(z)\, \mathrm{d}z\right| \leq \int_\gamma |f_n(z) - f(z)| \cdot |\mathrm{d}z| \leq \frac{\varepsilon}{L} \cdot L = \varepsilon$$

□

Suppose that $f_n \xrightarrow{\text{uniformly}} f$ on $D \subseteq \mathbb{R} \to \mathbb{C}$.

1. f_n' exists does not imply that f' exists.
2. f_n', f' exist does not imply that $f_n' \to f'$ pointwise.

Consider $f_n = (1/n)\sin(n^2 x)$. Then $f_n \xrightarrow{\text{uniformly}} f$, where $f(x) \equiv 0$. Here, $f_n' = n\cos(n^2 x)$, and f_n' does not converge pointwise to f'.

Theorem 13.4. *If $f_n : D \to \mathbb{C}$ is analytic, where D is an open set in $\mathbb{C}$, $f_n \xrightarrow{\text{uniformly}} f$ on D, then*

1. *f' also exists, i.e. f is analytic on D.*
2. *$f_n^{(m)} \xrightarrow{\text{uniformly}} f^{(m)}$ on W, where W is a slightly smaller set, for any $m > 0$. There exists $\delta > 0$*

such that $\forall w \in W\ \forall z \notin D\ |w - z| > \delta$.

Proof. 1. We cannot use real analysis methods. Use Morera's Theorem: $\forall R \subseteq D$, where R is a closed rectangle, if $\int_{\partial R} f(z)\,\mathrm{d}z = 0$, then f is analytic.

$$\int_{\partial R} f(z)\,\mathrm{d}z = \lim_{n\to\infty} \int_{\partial R} f_n(z)\,\mathrm{d}z = 0$$

where the first equality uses $f_n \xrightarrow{\text{uniformly}} f$ and the second equality follows because f_n is analytic.

2. Uniform continuity says $\forall \varepsilon > 0\ \exists N$ such that $\forall n > N\ \forall z \in D\ |f_n(z) - f(z)| < \varepsilon$. Apply Cauchy's Integral Formula:

$$\begin{aligned}
\left|f_n^{(m)}(z_0) - f^{(m)}(z_0)\right| &= \left|\frac{m!}{2\pi i}\int_{|z-z_0|=\delta} \frac{f_n(z) - f(z)}{(z-z_0)^{m+1}}\,\mathrm{d}z\right| \\
&\le \frac{m!}{2\pi}\int_{|z-z_0|=\delta} \frac{|f_n(z) - f(z)|}{|z-z_0|^{m+1}} \cdot |\mathrm{d}z| \\
&\le \frac{m!}{2\pi}\frac{\varepsilon}{\delta^{m+1}} 2\pi\delta = \frac{m!}{\delta^m}\varepsilon
\end{aligned}$$

∎

We need $\{|z - z_0| \le \delta\} \subseteq D$.

$f_n \xrightarrow{\text{normally}} f$ on D if for any closed disk $B \subseteq D$, $f_n \xrightarrow{\text{uniformly}} f$ on B. If we use the hypothesis that $f_n \xrightarrow{\text{normally}} f$ in the above theorem, then we can replace the second statement by $f_n^{(m)} \xrightarrow{\text{normally}} f^{(m)}$ on D (for any $m > 0$).

Lecture 14

October 13

14.1 Power Series

Let $f : \{|z - z_0| < r\} \to \mathbb{C}$ be analytic. Today, we will write $f(z)$ as a power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$.

A **power series** is a series $\sum_{k=0}^{\infty} a_k z^k$, $a_k \in \mathbb{C}$. For which $z \in \mathbb{C}$ does a power series converge?

Theorem 14.1. *There exists R (the **radius of convergence**), $0 \leq R \leq +\infty$, such that $\sum_{k=0}^{\infty} a_k z^k$ absolutely converges pointwise on $\{|z| < R\}$.*

$\sum_{k=0}^{\infty} a_k z^k$ diverges for any $z \in \mathbb{C}$, $|z| > R$.

$\sum_{k=0}^{\infty} a_k z^k$ uniformly converges on $\{|z| < r\}$, for any $r < R$.

If $R = 0$, then the power series converges only at $z = 0$.

If $R = +\infty$, then the power series converges on $\mathbb{C}$, then the power series uniformly converges on $\{|z| < r\}$, for all $r > 0$.

Proof. If $\sum_{k=0}^{\infty} a_k z^k$ converges for $z \in \mathbb{C}$, then $\lim_{k\to\infty} \left|a_k z^k\right| = 0$, which implies that $\left|a_k \cdot z^k\right|$ is bounded.

Let $R = \sup\left\{r \mid \left\{\left|a_k r^k\right|\right\}_{k=0}^{\infty} \text{ is bounded}\right\}$. We consider two cases.

1. $|z| < R$
2. $|z| > R$

1. If $|z| < R$, then there exists r such that $|z| < r < R$. Then

$$\left|a_k z^k\right| = \underbrace{\left|a_k r^k\right|}_{\text{bounded } \leq M, \text{ any } k} \cdot \left(\frac{|z|}{r}\right)^k \leq M \cdot \left(\frac{|z|}{r}\right)^k$$

   Then $\sum (|z|/r)^k$ converges, because $|z|/r < 1$, which implies that $\sum a_k z^k$ absolutely converges on $\{|z| < R\}$ and uniformly converges on $\{|z| < r\}$ for $r < R$.

2. If $|z| > R$, then $\left\{\left|a_k z^k\right|\right\}_{k=0}^{\infty}$ is not bounded.

$\square$

$\sum_{k=0}^{\infty} a_k z^k$ uniformly converges on $\{|z| < r\}$ for any $r < R$. This means:

$$\left\{\sum_{k=0}^{n} a_k z^k\right\}_{n=0}^{\infty} \xrightarrow{\text{uniformly}} \sum_{k=0}^{\infty} a_k z^k \quad \text{on } \{|z| < r\}$$

The finite series are polynomials, which are analytic. Therefore, the power series is also analytic on $\{|z| < R\}$.

In general: $R = 1/(\limsup_{k\to\infty} |a_k|^{1/k})$.

For special $\sum a_k z^k$, $R = 1/(\lim_{k\to\infty} |a_k|^{1/k})$. Alternatively,

$$R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$$

if the limit exists.

Example 14.2. Take $\sum_{k=0}^{\infty} z^k/k = -\operatorname{Log}(1-z)$. Here, $R = 1$. If $z = 1$, then the power series diverges. If $z = -1$, then the power series converges. If $|z| = 1$, for $z \neq \pm 1$, the power series converges.

Example 14.3. Take $\sum_{k=0}^{\infty} z^k/k^2$. Here, $R = 1$. $\left|z^k/k^2\right| \leq 1/k^2$ if $|z| < 1$, so the power series uniformly converges on $\{|z| \leq 1\}$ and is analytic on $\{|z| < 1\}$.

For an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on $\{|z| < R\}$,

$$\begin{aligned} f'(z) &= \sum_{k=1}^{\infty} k a_k z^{k-1} \\ f''(z) &= \sum_{k=2}^{\infty} k(k-1) a_k z^{k-2} \\ &\vdots \end{aligned}$$

If $f_n \xrightarrow{\text{uniformly}} f$ on $\{|z| < R\}$, then $f_n^{(m)} \xrightarrow{\text{uniformly}} f^{(m)}$ on $\{|z| < r\}$ for any $r < R$. For all of these derivatives, the radius of convergence is R. We can also see

$$f^{(m)}(0) = m! \cdot a_m$$

Suppose $f : \{|z| < R\} \to \mathbb{C}$ is analytic. One candidate for a power series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

In 104, we studied remainders:

$$f(z) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$$

We estimated remainders by using special properties of f.

For analytic functions, we have the Cauchy integral formula.

$$f(z) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{f(\xi)}{\xi - z} \, \mathrm{d}\xi$$

for $|\xi| < r < R$. We will try to write the integral as a power series.

We have the conditions $|\xi| = r$ and $|z| < r$. We want to write $1/(\xi - z)$ in a way similar to

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots$$

for $|w| < 1$. Therefore,

$$\frac{1}{\xi - z} = \frac{1}{\xi}\frac{1}{1 - z/\xi} \underbrace{=}_{|z/\xi|<1} \frac{1}{\xi}\sum_{k=0}^{\infty}\left(\frac{z}{\xi}\right)^k$$

Plug it in.

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\xi|=r}\sum_{k=0}^{\infty}\frac{f(\xi)}{\xi^{k+1}}z^k \, \mathrm{d}\xi \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|\xi|=r}\frac{f(\xi)}{\xi^{k+1}}\,\mathrm{d}\xi\right) z^k \end{aligned}$$

We need the fact that

$$\frac{f(\xi)}{\xi}\sum_{k=0}^{\infty}\left(\frac{z}{\xi}\right)^k$$

uniformly converges on $|\xi| = r$. Fix z and use the Weierstrass M-test.

Therefore,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

where

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\int_{|\xi|=r}\frac{f(\xi)}{\xi^{k+1}}\,\mathrm{d}\xi \\ &= \frac{1}{k!}f^{(k)}(0) \end{aligned}$$

for $|z| < r < R$. Also,

$$\int_{|\xi|=r}\frac{f(\xi)}{\xi^{k+1}}\,\mathrm{d}\xi = \int_{|\xi|=r'}\frac{f(\xi)}{\xi^{k+1}}\,\mathrm{d}\xi$$

so we can take any $r < R$. Hence, $\sum_{k=0}^{\infty} a_k z^k$ for any $|z| < R$, where

$$a_k = \frac{f^{(k)}(0)}{k!}$$

Let us return to

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots$$

On the RHS, we need $|w| < 1$. On the LHS, we only need $w \neq 1$, so there is a difference between analytic functions and power series.

In 104, consider

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

The expression on the LHS makes sense for $x \in \mathbb{R}$. The expression on the RHS makes sense only for $|x| < 1$.

In 185,

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

On the LHS, $z \in \mathbb{C} \setminus \{i, -i\}$. On the RHS, $|z| < 1$. Then, we can only define the power series of $1/(1+z^2)$ on the disk.

Lecture 15

October 18

15.1 Power Series

Last time, we proved that (almost) all analytic functions are power series.

Theorem 15.1. *If $f : \{|z - z_0| < \rho\} \to \mathbb{C}$ is analytic, then $f(z)$ has a power series expansion*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

on $\{|z - z_0| < \rho\}$ and

$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{k+1}} \, \mathrm{d}w$$

for any $r \in (0, \rho)$.

Example 15.2. Consider the power series at 0 of

$$f(z) = \frac{1}{1 + z^2}$$

We can use the geometric series

$$\frac{1}{1 - w} = 1 + w + w^2 + \cdots$$

to write

$$\frac{1}{1 + z^2} = 1 - z^2 + z^4 - \cdots = \sum_{k=0}^{\infty} (-1)^k z^{2k}$$

The radius of convergence is 1.

Example 15.3. Consider $f(z) = e^z$. Then $f^{(k)}(z) = e^z$, so $f^{(k)}(0) = 1$. Therefore,

$$\begin{aligned} e^z &= 1 + \frac{1}{1!} z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad \text{(convention: } 0! = 1) \end{aligned}$$

Here,

$$R = \lim_{k\to\infty} \frac{a_k}{a_{k+1}} = \lim_{k\to\infty} \frac{1/k!}{1/(k+1)!} = \lim_{k\to\infty} (k+1) = \infty$$

Initially, we defined $e^{i\theta} = \cos\theta + i\sin\theta$. Take the power series:

$$\begin{aligned} e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} &= \underbrace{\sum_{k \text{ even}} (-1)^{k/2} \frac{\theta^k}{k!}}_{k=2n} + \underbrace{\sum_{k \text{ odd}} i(-1)^{(k-1)/2} \frac{\theta^k}{k!}}_{k=2n+1} \\ &= \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!}}_{\cos\theta} + i \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}}_{\sin\theta} \end{aligned}$$

$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ is determined by $a_0, a_1, \ldots, a_n, \ldots$ $(f(z_0), f(z_0)/1!, \ldots, f^{(n)}(z_0)/n!, \ldots)$.

Proposition 15.4. *If $f, g : \{|z-z_0| < \rho\} \to \mathbb{C}$ are analytic functions, with $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k = 0, 1, 2, \ldots,$ then $f = g$.*

Proposition 15.5. *If D is a domain, $f, g : D \to \mathbb{C}$ are analytic, if for some $z_0 \in D$, $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k = 0, 1, 2, \ldots,$ then $f \equiv g$ on D.*

Proof. f and g have power series expansions on $\{|z-z_0| < \rho\} \subseteq D$. Therefore, $f = g$ on $\{|z-z_0| < \rho\}$. Write

$$\begin{aligned} U &= \{z \in D \mid f^{(k)}(z) = f^{(k)}(z), k = 0, 1, \ldots\} \\ V &= \{z \in D \mid \exists n \text{ s.t. } f^{(n)}(z) \neq g^{(n)}(z)\} \\ D &= U \cup V \\ U \cap V &= \varnothing \end{aligned}$$

U is open: use 15.4. V is open: use $f^{(n)} - g^{(n)}$ is continuous. $U \neq \varnothing$ implies that $V = \varnothing$. □

In 104, there was a function

$$f(z) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$f^{(n)}(x)$ exists and is continuous, and $f^{(n)}(0) = 0$ for all n.

Consider e^{-1/z^2}, defined on $\mathbb{C} \setminus \{0\}$. If we define

$$f(z) = \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

then f is not continuous. On the y-axis, $f(iy) = e^{-1/(iy)^2} = e^{1/y^2} \to \infty$ as $y \to 0$.

We can predict the radius of convergence. 15.1 implies that $R \geq \rho$.

Example 15.6. Consider the power series of

$$\frac{(z^6 - z^3 + 1)\cancel{(z^3+1)}}{(z^3-1)\cancel{(z^3+1)}} = \frac{z^9+1}{z^6-1}$$

at 0. $z^6 - 1$ has zeros at $1, e^{i\pi/3}, e^{i(2\pi/3)}, -1, e^{i(4\pi/3)}, e^{i(5\pi/3)}$, which lie on $\{|z| = 1\}$. The function is analytic on the open disk, so we can conclude that $R = 1$.

Consider the power series at -2. Here, $R = \left|-2 - e^{i(2\pi/3)}\right| = \sqrt{3}$.

15.2 Analytic at ∞

Consider $f(z) = 1/z$, which is defined on $\mathbb{C} \setminus \{0\}$. As $z \to \infty$, $f(z) \to 0$. Define "$f(\infty) = 0$". We have $1/z : \infty \leftrightarrow 0$. More generally, $\{|z| > R\} \leftrightarrow \{0 < |z| < 1/R\}$.

Let D be a domain which contains $\{|z| > R\}$ (a neighborhood of ∞). Take a function $f : D \to \mathbb{C}$. Define $g(w) = f(1/w)$, defined on $\{0 < |z| < 1/R\}$. f is continuous at ∞ if we can extend g to $\{|z| < 1/R\}$ such that g is continuous (give a value to $g(0)$ such that g is continuous).

Consider

$$\begin{aligned} f(z) &= \frac{z^2 + z - 1}{4z^2 - z + 5} \\ g(w) &= f\left(\frac{1}{w}\right) = \frac{(1/w)^2 + (1/w) - 1}{4(1/w)^2 - (1/w) + 5} \cdot \frac{w^2}{w^2} = \frac{1 + w - w^2}{4 - w + 5w^2} \end{aligned}$$

$g(w)$ is not defined for $w = 0$, but we can see that $g(0)$ should be $1/4$. For any function $P(z)/Q(z)$ with $\deg P \leq \deg Q$, we can perform the same procedure.

Consider

$$\begin{aligned} f(z) &= e^z \\ g(w) &= f\left(\frac{1}{w}\right) = e^{1/w} \end{aligned}$$

If $w = x$, then as $x \to 0^+$, we have $g(w) \to \infty$. As $x \to 0^-$, $g(w) \to 0$. Therefore, f is not continuous at ∞.

15.2.1 Analytic at ∞

If $f : \{|z| > R\} \to \mathbb{C}$ is analytic, $g(w) = f(1/w)$ is analytic on $\{0 < |z| < 1/R\}$. If g is continuous at 0, then g is continuous on $\{|z| < 1/R\}$. Therefore, g is analytic on $\{|z| < 1/R\}$.

If g is analytic on $\{|w| < 1/R\}$, we can write

$$\begin{aligned} g(w) &= \sum_{k=0}^{\infty} b_k w^k \\ f(z) &= g\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{b_k}{z^k} \end{aligned}$$

If z is big, then b_k/z^k is small. This is the power series of f at ∞, as long as f is analytic on $\{|z| > R\}$ and continuous at ∞.

What are the b_k? One way to get b_k is to use the formula of the power series of g. Another way:

$$\int_{|z|=\rho} f(z) z^l \, \mathrm{d}z$$

for $\rho > R$.

$$f(z) \cdot z^l = \sum_{k=0}^{\infty} b_k z^{l-k}$$

so we have

$$\oint_{|z|=\rho} \sum_{k=0}^{\infty} b_k z^{l-k} \, \mathrm{d}z$$

and the only non-zero contribution is for $l - k = -1$. This yields the formula

$$b_k = \frac{1}{2\pi i} \oint_{|z|=\rho} f(z) z^{k-1} \, \mathrm{d}z$$

Lecture 16

October 20

16.1 Manipulation of Power Series

If $f, g : \{|z - z_0| < \rho\} \to \mathbb{C}$ are analytic, we can write $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ and $g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$.

$f + g, f - g, f \cdot g : \{|z - z_0| < \rho\} \to \mathbb{C}$ are also analytic. The power series for $f + g$ is

$$(f + g)(z) = \sum_{k=0}^{\infty} (a_k + b_k)(z - z_0)^k$$

Similarly, we can write down a power series for $f \cdot g$:

$$(f \cdot g)(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \cdot \sum_{l=0}^{\infty} b_l (z - z_0)^l$$

$$(f \cdot g)(z) = \sum_{m=0}^{\infty} \underbrace{c_m}_{=\sum_{k=0}^{m} a_k b_{m-k}} (z - z_0)^m$$

The definition of the power series is

$$\lim_{n\to\infty} \sum_{m=0}^{n} c_m (z - z_0)^m = f(z) \cdot g(z)$$

We can write

$$f(z) = \lim_{K\to\infty} \sum_{k=0}^{K} a_k (z - z_0)^k$$

$$g(z) = \lim_{L\to\infty} \sum_{l=0}^{L} b_l (z - z_0)^l$$

$$f(z) \cdot g(z) = \lim_{K,L\to\infty} \sum_{k=0}^{K} a_k (z - z_0)^k \cdot \sum_{l=0}^{L} b_l (z - z_0)^l$$

They are not the same thing! It is not easy to show $\lim_{n\to\infty} \sum_{m=0}^{n} c_m (z - z_0)^m = f(z) \cdot g(z)$ by 104.

Instead, we will show $c_m = (f \cdot g)^{(m)}(z_0)/m!$ (using the fact that the functions are analytic).

$$(f \cdot g)^{(m)}(z_0) = \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(z_0) \cdot g^{(m-k)}(z_0) \qquad \text{(induction)}$$

$$= \sum_{k=0}^{m} \frac{m!}{k!\,(m-k)!} k!\, a_k (m-k)!\, b_{m-k} = m! \sum_{k=0}^{m} a_k b_{m-k} = m! \cdot c_m$$

Now, consider $(f/g)(z_0)$. Assume that $g(z_0) \neq 0$ on a small disk $\{|z - z_0| < \varepsilon\}$.

$$\frac{f}{g}(z) = f(z) \cdot \frac{1}{g}(z)$$

$$\frac{1}{g}(z) = \frac{1}{b_0 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots} = \frac{1}{b_0} \cdot \frac{1}{1 + \frac{b_1}{b_0}(z-z_0) + \frac{b_2}{b_0}(z-z_0)^2 + \cdots}$$

$$= \frac{1}{b_0} \cdot \left(1 - \left(\frac{b_1}{b_0}(z-z_0) + \frac{b_2}{b_0}(z-z_0)^2 + \cdots\right) + (\cdots)^2 - (\cdots)^3 + \cdots\right)$$

The denominator is small as long as $|z - z_0|$ is small.

16.2 Zeros of Analytic Functions

Let $f : D \to \mathbb{C}$ be an analytic function. We will study $\{z \in D \mid f(z) = 0\}$. The zeros form a closed set (by continuity). For example, $x \cdot \sin(1/x)$ has the zero set $\{1/(n\pi)\} \cup \{0\}$.

Assume that f is not constantly 0. $z_0 \in D$ is a zero (point) of f if $f(z_0) = 0$. Consider the power series $f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$. Then $a_0 = f(z_0) = 0$. If $a_1 \neq 0$, then the order of z_0 is 1. More generally, the **order of** z_0 is the number $N \in \mathbb{Z}_+$ such that $a_0 = \cdots = a_{N-1} = 0$ and $a_N \neq 0$.

If all $a_n = 0$, then $f(z) = 0$ on a small disk $\{|z - z_0| < \varepsilon\}$. Use the fact that D is a domain to conclude that $f(z) = 0$ on D. If D is a domain, $f \not\equiv 0$, then the order of a zero point of f is defined.

Suppose that $z_0 \in D$ is an order 5 zero point of f. Then

$$\begin{aligned} f(z) &= \underbrace{a_5}_{\neq 0}(z-z_0)^5 + a_6(z-z_0)^6 + \cdots \\ &= (z-z_0)^5 \cdot \underbrace{(a_5 + a_6(z-z_0) + a_7(z-z_0)^2 + \cdots)}_{\text{analytic function}} \\ &= (z-z_0)^5 \cdot h(z), \qquad h(z_0) = a_5 \neq 0 \end{aligned}$$

On a small disk $\{|z - z_0| < \varepsilon\}$, there are no other zero points of f, since: $h(z_0) \neq 0$ and h is continuous imply that $\exists \varepsilon > 0$ such that h is never 0 on $\{|z - z_0| < \varepsilon\}$. Therefore

$$\underbrace{f(z)}_{\neq 0} = \underbrace{(z-z_0)^5}_{\neq 0}\underbrace{h(z)}_{\neq 0} \qquad \text{on} \qquad \{0 < |z - z_0| < \varepsilon\}$$

Theorem 16.1. *Let $f : D \to \mathbb{C}$ be analytic, where D is a domain, not constantly 0. Then the zero point set $Z = \{z \in D \mid f(z) = 0\}$ is isolated (i.e. there does not exist a sequence $\{z_n\}$, $z_n \in Z$, such that $\lim z_n = z_\infty \in Z$). More rigorously, $\forall z \in Z\ \exists \varepsilon > 0$ such that $\{|z - z_0| < \varepsilon\} \cap Z = \{z_0\}$.*

We know that $\cos^2\theta + \sin^2\theta = 1$ for $\theta \in \mathbb{R}$. Now, consider $f(z) = \cos^2 z + \sin^2 - 1$. Is $f(z)$ constantly 0? Yes, because the zero point set must be isolated if $f \not\equiv 0$. Without this technique, we would have to compute

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots$$

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$$

After computation, we would conclude $\cos^2 z + \sin^2 z - 1 = 0$.

Corollary 16.2. *Let $f : D \to \mathbb{C}$ be analytic, not constantly* 0*. If $E \subseteq D$ is closed and bounded, then $E \cap Z$ is a finite set.*

Warning: Z itself might be infinite. $\{z_n\}$ can converge to ∂D for $z_n \in Z$.

Consider the analytic function

$$\sin \frac{1}{1-z} : \{|z| < 1\} \to \mathbb{C}$$

Then $Z = \{1 - 1/(n\pi) \mid n \in \mathbb{Z}_+\}$. Then $1 - 1/(n\pi) \to 1 \in D$ as $n \to \infty$.

16.2.1 Zero at ∞

Suppose that $f : \{|z| > R\} \to \mathbb{C}$ is analytic and continuous at ∞ (so it is analytic at ∞) if $g(w) = f(1/w)$ can be continuously extended to $\{|z| < 1/R\}$ from $\{0 < |z| < 1/R\}$.

$$f(\infty) = g(0)$$

∞ is a zero of f if $g(0) = 0$. ∞ is an order N zero point of f if 0 is an order N zero point of g.

If $g(w) = b_0 + b_1 w + \cdots$, $g(w) \neq 0$ on $\{0 < |w| < \varepsilon\}$, then $f(z) = b_0 + b_1/z + b_2/z^2 + \cdots$ and $f(z) \neq 0$ on $\{|z| > 1/\varepsilon\}$.

Corollary 16.3. *If $f, g : \{|z| > R\} \to \mathbb{C}$ are analytic, both analytic at ∞, if $\exists z_n \to \infty$ with every $z_n \in \{|z| > R\}$ such that $f(z_n) = g(z_n)$ for all n, then $f = g$.*

We do need f, g to be analytic at ∞. Otherwise, take $f = \sin z$ and $g = 0$. (f is not analytic at ∞.) Here $\sin(n\pi) = 0$, and $n\pi \to \infty$.

Lecture 17

October 25

17.1 Laurent Series

In Chapter V, we studied power series. For $f : \{|z - z_0| < \rho\} \to \mathbb{C}$, an analytic function, then we have $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$, $|z| < \rho$.

$$a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{k+1}} \, \mathrm{d}w \tag{17.1}$$

for all $0 < r < \rho$.

In Chapter VI, we study analytic functions $f : \{\sigma < |z - z_0| < \rho\} \to \mathbb{C}$, for $0 \leq \sigma < \rho \leq +\infty$. For example, we could take $\sigma = 0$ or $\rho = +\infty$. For $\sigma = 0$ and $\rho = +\infty$, the function is analytic on $\mathbb{C} \setminus \{0\}$.

If $f : \{\sigma < |z - z_0| < \rho\} \to \mathbb{C}$ is analytic, it may be $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ with radius of convergence $\geq \rho$. (For example, $f(z) = 1/(z - z_0)$. Another possibility is

$$\sum_{k=0}^{\infty} a_k(z - z_0)^k + \underbrace{\sum_{l=-\infty}^{-1} b_l(z - z_0)^l}_{\sum_{l=1}^{\infty} b_l \cdot 1/(z-z_0)^l}$$

which is analytic on $\{\sigma < |z - z_0| < \rho\}$, if the radii of convergence are correct.

We want to show that this is always the case, that is $f : \{\sigma < |z - z_0| < \rho\} \to \mathbb{C}$ is analytic implies that $f(z) = \sum_{k=-\infty}^{+\infty} a_k(z - z_0)^k$.

The formula $a_k = f^{(k)}(z_0)/k!$ does not work for Chapter VI, but the formula

$$a_k = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{k+1}} \, \mathrm{d}w$$

still works in Chapter VI.

17.1.1 Complex Line Integral

To prove the original formula, we used the Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w - z} \, \mathrm{d}w$$

for $r > |z|$.

$f(z)$ cannot be written as a single complex line integral here because f is not analytic in the interior of the disk. Take r_1, r_2 such that $\sigma < r_1 < |z| < r_2 < \rho$. Use the Cauchy Integral Formula:

$$f(z) = \frac{1}{2\pi i}\left(\int_{|w-z_0|=r_2} \frac{f(w)}{w-z}\,\mathrm{d}w - \int_{|w-z_0|=r_1} \frac{f(w)}{w-z}\,\mathrm{d}w\right) \tag{17.2}$$

for any z such that $r_1 < |z| < r_2$. Write

$$f_0(z) = \frac{1}{2\pi i}\int_{|w-z_0|=r_1} \frac{f(w)}{w-z}\,\mathrm{d}w$$

which is analytic on $z \in \{|z-z_0| \neq r_1\}$. If $|z-z_0| < r_1$, it is not interesting in our setting. For z such that $|z-z_0| > r_1$, $f_0(z)$, $f_0(z)$ is analytic on $\{|z-z_0| > r_1\}$. Write

$$f_1(z) = \frac{1}{2\pi i}\int_{|w-z_0|=r_2} \frac{f(w)}{w-z}\,\mathrm{d}w$$

which is analytic on $\{|z-z_0| < r_2\}$. Then $f(z) = f_1(z) - f_2(z)$ on the annulus, where $f_0(z)$ is analytic for $|z-z_0| > r_1$ and $f_1(z)$ is analytic for $|z-z_0| < r_2$. This is the **Laurent decomposition**. f_1 has a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$. As $z \to \infty$, $f_0(z) \to 0$, so f_1 has a power series, $\sum_{k=1}^{\infty} b_k \cdot 1/(z-z_0)^k$.

Write (17.2) as a bi-infinite power series of $z-z_0$ for $r_1 < |z-z_0| < r_2$. For $|w-z_0| = r_2$,

$$\frac{1}{w-z} = \frac{1}{\underbrace{(w-z_0)}_{|\cdot|=r_2} - \underbrace{(z-z_0)}_{|\cdot|<r_2}} = \frac{1}{w-z}\cdot\frac{1}{1-\underbrace{\dfrac{z-z_0}{w-z_0}}_{|\cdot|<1}} = \frac{1}{w-z_0}\cdot\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k$$

For $|w-z_0| = r_1$,

$$\frac{1}{w-z} = \frac{1}{\underbrace{(w-z_0)}_{|\cdot|=r_1} - \underbrace{(z-z_0)}_{|\cdot|>r_1}} = \frac{1}{z-z_0}\frac{1}{1-\dfrac{w-z_0}{z-z_0}} = \frac{1}{z-z_0}\sum_{l=0}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^l$$

Then, we obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left(\sum_{k=0}^{\infty}\left(\int_{|w-z_0|=r_1}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w\right)\cdot(z-z_0)^k \right.\\ &\qquad \left. \underbrace{+\sum_{l=0}^{\infty}\left(\int_{|w-z_0|=r_2} f(w)(w-z_0)^l\,\mathrm{d}w\right)(z-z_0)^{-l-1}}_{k=-l-1,\,l=-k-1}\right) \\ &= \frac{1}{2\pi i}\left(\sum_{k=0}^{\infty}\left(\int_{|w-z_0|=r_2}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w\right)(z-z_0)^k + \sum_{k=-\infty}^{-1}\left(\int_{|w-z_0|=r_1}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w\right)(z-z_0)^k\right) \end{aligned}$$

We are computing the complex line integral of $f(w)/(w-z_0)^{k+1}$, which is analytic on $\{\sigma < |z-z_0| < \rho\}$.

$$\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w$$

is independent of r as long as $r \in (\sigma, \rho)$. Our final formula is

$$f(z) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{|w-z_0|=r}\frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w\right)(z-z_0)^k$$

for any $r \in (\sigma, \rho)$, any $z \in \{\sigma < |z-z_0| < \rho\}$. This is the **Laurent expansion**.

Example 17.1. $1/(z(z-1)(z-2))$ is analytic on $\mathbb{C} \setminus \{0, 1, 2\}$. Consider the annulus centered at 0, where the function is analytic.

1. $\{0 < |z| < 1\}$
2. $\{1 < |z| < 2\}$
3. $\{2 < |z| < \infty\}$

We write

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

1. On $\{0 < |z| < 1\}$:

$$\begin{aligned}\frac{1}{z}\left(\frac{1}{1-z} - \frac{1}{2}\frac{1}{1-z/2}\right) &= \frac{1}{z}\left(\sum_{k=0}^{\infty} z^k - \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k\right) = \frac{1}{z}\sum_{k=0}^{\infty}\left(1 - \frac{1}{2^{k+1}}\right)z^k \\ &= \frac{1}{2z} + \sum_{k=0}^{\infty}\left(1 - \frac{1}{2^{k+1}}\right)z^k\end{aligned}$$

2. $\{1 < |z| < 2\}$:

$$\begin{aligned}\frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) &= \frac{1}{z}\left(-\frac{1}{2}\frac{1}{1-z/2} - \frac{1}{z}\frac{1}{1-1/z}\right) \\ &= \frac{1}{z}\left(-\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k - \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k\right)\end{aligned}$$

3. $\{|z| > 2\}$:

$$\begin{aligned}\frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right) &= \frac{1}{z^2}\left(\frac{1}{1-2/z} - \frac{1}{1-1/z}\right) \\ &= \frac{1}{z^2}\left(\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k - \sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k\right)\end{aligned}$$

   There are only negative powers since $f(\infty) = 0$.

Special/Important Case. Let $f : \{0 < |z - z_0| < \rho\} \to \mathbb{C}$ be analytic. Then 0 is a singularity. There are three possibilities:

1. f can be extended to $\{|z - z_0| < \rho\}$ analytically
2. $f \to \infty$ as $z \to z_0$
3. essential singularity

Lecture 18

November 1

18.1 Isolated Singularities

Last Tuesday: if $f : \{\sigma < |z - z_0| < \rho\} \to \mathbb{C}$ is analytic, then $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ and

$$a_k = \frac{1}{2\pi i} \int_{|w-z_0|=r} f(w)(w-z_0)^{-k-1}\, \mathrm{d}w$$

for any $r \in (\sigma, \rho)$.

We will use the above theory to study isolated singularities. We take $\sigma = 0$, so we consider the disk minus a point, $\{|z - z_0| < \rho\} \setminus \{z_0\}$. Suppose that $f : \{0 < |z - z_0| < \rho\} \to \mathbb{C}$ is analytic. Then z_0 is an **isolated singularity**.

Suppose that z_0 is an isolated singularity.

1. Trivial case: Actually f is defined on $\{|z - z_0| < \rho\}$. Then we say z_0 is a **removable singularity**. f extends to $\{|z - z_0| < \rho\} \to \mathbb{C}$ analytically (a continuous extension of f), so $f(k) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$, so $a_{-1}, a_{-2}, \ldots$ vanish. For example, $(\sin z)/z$ has a removable singularity at 0. The Laurent series is $\sum_{k=0}^{\infty} a_k (z - z_0)^k$. If f has z_0 as a removable singularity, then $f(z) \to f(z_0)$ as $z \to z_0$.

Theorem 18.1 (Riemann's Theorem for Removable Singularities)**.** *If f is bounded on a smaller disk $\{0 < |z - z_0| < \varepsilon\}$, then z_0 is a removable singularity.*

Proof. Take any $k < 0$ and check $a_k = 0$.

$$\begin{aligned} |a_k| &\le \frac{1}{2\pi} \int_{|w-z_0|=r} |f(w)| \cdot \left|(w-z_0)^{-k-1}\right| \cdot |\mathrm{d}w| \\ &\le \frac{1}{2\pi} M \cdot r^{-k-1} \cdot 2\pi r \\ &= M r^{-k} \end{aligned}$$

for $r < \varepsilon$. Let $r \to 0$ and we get $a_k = 0$. □

2. Less trivial case: some $a_k \neq 0$ for $k < 0$. For example, take $(\cos z)/z$. At $z = 0$, $\cos z = 1$ and $z = 0$. The power series is

$$\frac{1 - z^2/2 + z^4/24 - \cdots}{z} = \frac{1}{z} - \frac{z}{2} + \frac{z^3}{24} - \cdots$$

In this case, we can write

$$f(z) = \frac{g(z)}{(z - z_0)^N}$$

where $g : \{|z - z_0| < \rho\} \to \mathbb{C}$ is analytic. Assume that $g(z_0) \neq 0$. z_0 is a **pole** of f of order N.

Since $g(z)$ is analytic on the disk, $g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k$ with $b_0 \neq 0$. The Laurent expansion of f is

$$f(z) = \underbrace{\frac{b_0}{(z-z_0)^N} + \frac{b_1}{(z-z_0)^{N+1}} + \cdots + \frac{b_{N-1}}{z - z_0}}_{\text{principal part of } f \text{ at } z_0} + \underbrace{b_N + b_{N+1}(z - z_0) + \cdots}_{\text{power series/analytic function}}$$

There are only finitely many $a_k \neq 0$ with $k < 0$. N is the order of z_0 as a pole. $f -$ principal part is analytic on $\{|z - z_0| < \rho\}$.

In our less trivial case, z_0 is a pole and

$$f(z) = \frac{g(z)}{(z - z_0)^N}$$

where g is analytic on $\{|z - z_0| < \rho\}$. The Laurent expansion is $\sum_{k=-N}^{\infty} a_k (z - z_0)^k$.

Theorem 18.2. *Let $f : \{0 < |z - z_0| < \rho\} \to \mathbb{C}$ be analytic. z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. If z_0 is a pole, then $|f(z)| \to \infty$.

If $|f(z)| \to \infty$, then we must prove that there are finitely many $a_k \neq 0$, $k < 0$. Take

$$h(z) = \frac{1}{f(z)}$$

which is analytic on $\{0 < |z - z_0| < \varepsilon\}$. $|h(z)| \to 0$ as $z \to z_0$, so z_0 is a removable singularity of h (not constantly 0). Therefore,

$$h(z) = (z - z_0)^N k(z)$$

with $k(z_0) \neq 0$.

$$f(z) = \frac{1}{h(z)} = \frac{1/k(z)}{(z - z_0)^N}$$

□

Example 18.3. Consider

$$\frac{ze^z}{(z^2+1)^2} = \frac{ze^z}{(z+i)^2 (z-i)^2}$$

which is analytic everywhere except i and $-i$, which are poles of order 2. The principal part at i is

$$\frac{?}{(z-i)^2} + \frac{?}{(z-i)}$$

where the first coefficient is given by

$$\left(\frac{ze^z}{(z+i)^2} \right) \bigg|_{z=i} = -\frac{i}{4} e^i$$

and the second coefficient is given by

$$\left(\frac{ze^z}{(z+i)^2} \right)' \bigg|_{z=i}$$

Definition 18.4. A **meromorphic function** on $\mathbb{C}$ (or a domain D) is a function f on $\mathbb{C}$ (D) which is analytic on $D - \{\text{isolated singularities}\}$ such that all singularities are poles.

If we have $f(z)/g(z)$, where $f, g : D \to \mathbb{C}$ (D is a domain) are analytic, $g \not\equiv 0$, then f/g is a meromorphic function.

3. Non-trivial case. $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$, where $a_k \neq 0$ for infinitely many $k < 0$. Here, z_0 is an **essential singularity**.

In our non-trivial case, z_0 is an essential singularity and the Laurent expansion is $\sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$. It is not clear how to write this case in terms of analytic functions.

Example 18.5. Consider $e^{1/z}$. We could also consider $\sin(1/z)$ or $\cos(1/z)$, but they are the same.

The power series expansion is

$$e^w = 1 + w + \frac{w^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{w^k}{k!}$$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}$$

$$= \sum_{k=-\infty}^{0} \frac{1}{(-k)!} z^k$$

As $z \to 0$, $e^{1/z} \not\to \infty$. Take $z_n = 1/(ni)$ and $\left|e^{1/z_n}\right| = 1$, so it is not a pole. $e^{1/z}$ is not bounded, because we can take $z_n = 1/n$ and $e^{1/z_n} = e^n \to \infty$, so the singularity is not removable.

Theorem 18.6. *If z_0 is an essential singularity of f, then for any $w_0 \in \mathbb{C}$, $\exists z_n \to z_0$ such that $f(z_n) \to w_0$.*

The values of f are **dense** in $\mathbb{C}$ "everywhere".

Proof. Consider

$$g(z) = \frac{1}{f(z) - w_0}$$

If g is not defined on any small punctured disk $\{0 < |z - z_0| < \varepsilon\}$, then we can find a sequence $z_n \to z_0$ with $f(z_n) = w_0$. Otherwise, if g is defined on $\{0 < |z - z_0| < \varepsilon\}$, then z_0 is not a removable singularity of g, so $1/(f(z) - w_0)$ is not bounded, which implies that there exists $z_n \to z_0$ such that $f(z_n) \to w_0$. $\square$

Lecture 19

November 3

19.1 Isolated Singularities

Last time, we said that if $f : \{0 < |z - z_0| < \rho\} \to \mathbb{C}$ is analytic, then z_0 is an isolated singularity.

For a removable singularity, f extends to $\{|z - z_0| < \rho\}$ analytically, $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$. **Theorem**: z_0 is a removable singularity of f if and only if f is bounded near z_0.

For a pole, we can write

$$f(z) = \frac{g(z)}{(z - z_0)^N}$$

where $g : \{|z - z_0| < \rho\} \to \mathbb{C}$ is analytic, $g(z_0) \neq 0$. The Laurent expansion is $f(z) = \sum_{k=-N}^{\infty} a_k (z - z_0)^k$, $a_{-N} \neq 0$. **Theorem**: z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$. z_0 is a pole of order N. The principal part of z_0 is $\sum_{k=-N}^{-1} a_k (z - z_0)^k$.

For an essential singularity, the Laurent expansion is $\sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k$. **Theorem**: For any $\varepsilon > 0$, the image of $\{0 < |z - z_0| < \varepsilon\}$ under f is dense in $\mathbb{C}$.

19.2 Singularity at ∞

Suppose that $f : D \to \mathbb{C}$ is analytic, where D is a "neighborhood" of ∞ (we require $D \supseteq \{|z| > R\}$).

Recall the definition of f being continuous/analytic at ∞: Consider $g(w) = f(1/w)$. Then g is defined on $\{0 < |z| < 1/R\}$. $w = 0 \leftrightarrow 1/w = \infty$. f is analytic at ∞ if and only if 0 is a removable singularity of g if and only if ∞ is a removable singularity of f. We have

$$g(w) = \sum_{k=0}^{\infty} b_k w^k$$

$$f(z) = g\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{b_k}{z^k}$$

$$f(\infty) = b_0 + \underbrace{\frac{b_1}{\infty}}_{=0} + \underbrace{\frac{b_2}{\infty}}_{=0} + \cdots = b_0$$

All of the powers are negative. If z is large, then b_k/z^k is small. Example: consider polynomials.

∞ is a pole of f if and only if 0 is a pole of g.

$$g(w) = \frac{b_{-k}}{w^k} + \cdots + \frac{b_{-1}}{w} + b_0 + b_1 w + \cdots$$

$$f(z) = \underbrace{b_{-k}z^k + \cdots + b_{-1}z + b_0}_{\text{principal part of } f \text{ at } \infty} + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$$

∞ is an essential singularity of f if and only if 0 is an essential singularity of g. We have $\sum_{k=-\infty}^{+\infty} a_k z^k$, so there are infinitely many non-vanishing positive power terms. As an example, consider e^z, $\sin z$, or $\cos z$.

Example 19.1. Consider

$$\frac{1}{1-z}$$

If $z = \infty$, then the function goes to 0. We can write the power series as $1 + z + z^2 + \cdots$, but the power series expansion is only valid in the unit disk. We need an expansion on the complement of a disk:

$$\begin{aligned} \frac{1}{1-z} &= -\frac{1}{z}\frac{1}{1-1/z} \\ &= -\frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) \end{aligned}$$

19.3 Meromorphic Functions

We have been considering $f : D \subseteq \mathbb{C} \to \mathbb{C}$. We could also consider $f : D \subseteq \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, but we prefer to think about $f : D \setminus \{\text{isolated singularities}\} \to \mathbb{C}$.

f is a **meromorphic function** on $D \subseteq \mathbb{C} \cup \{\infty\}$ if f is defined and analytic on $D \setminus \{\text{isolated singularities}\}$ and all singularities are poles. If z_0 is a pole, then $1/f(z)$ has z_0 as a removable singularity.

A meromorphic function $D \to \mathbb{C}$ is equivalent to an "analytic function" $D \to \mathbb{C} \cup \{\infty\}$. Today, we consider a meromorphic function $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$.

1. If ∞ is an isolated singularity, then it is either removable or a pole, so f is analytic on $\{|z| > R\}$ for some $R > 0$. (All singularities lie in a disk.)
2. Since the singularities are isolated, there are only finitely many singularities.

Proposition 19.2. *If f is a meromorphic function on $\mathbb{C} \cup \{\infty\}$, then f has only finitely many singularities.*

The singularities are: $\infty, z_1, z_2, \ldots, z_n$.

With knowledge of singularities, we can find f. The singularities tell us the principal parts of f.

$$\begin{aligned} P_\infty(z) &= b_0 + b_1 z + \cdots + b_m z^m \\ P_j(z) &= \frac{a_1}{(z-z_j)} + \cdots + \frac{a_{N_j}}{(z-z_j)^{N_j}} \end{aligned}$$

P_∞ is analytic on $\mathbb{C}$, and P_j is analytic on $\mathbb{C} \setminus \{z_j\}$, with $P(\infty) = 0$.

Claim: $f(z) = P_\infty(z) + \sum_{j=1}^n P_j(z)$.

Proof. Consider $f - P_j(z)$. f is analytic on $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$ and P_j is analytic on $\mathbb{C} \setminus \{z_j\}$. Therefore, this function is analytic on $(\mathbb{C} \setminus \{z_1, z_2, \ldots, z_n\}) \cup \{z_j\}$. Therefore, $f - \sum_{j=1}^n P_j(z)$ is analytic on $\mathbb{C}$.

We have that $f(z) - \sum_{j=1}^n P_j(z)$ is analytic on $\mathbb{C}$. Then $h(z) = f(z) - \sum_{j=1}^n P_j(z) - P_\infty(z)$ is analytic on $\mathbb{C}$. We know

$$(f(z) - P_\infty(z))\Big|_{z=\infty} = 0$$

because

$$f(z) = \underbrace{b_m z^m + \cdots + b_0}_{P_\infty(z)} + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \cdots$$

and $P_j(\infty) = 0$, so $h(\infty) = 0$ ($h(z) \to 0$ as $z \to \infty$). This implies that h is bounded, so h is a constant function: $h \equiv 0$. $\square$

P_∞ is a polynomial and P_j is the quotient of two polynomials.

Theorem 19.3. *Any meromorphic function on $\mathbb{C} \cup \{\infty\}$ is a rational function, i.e.* $\dfrac{\textit{polynomial}}{\textit{polynomial}}$.

If we write a rational function as

$$f(z) = P_\infty(z) + \sum_{j=1}^{n} P_j(z)$$

this is called the **partial fraction decomposition**. We can use this to solve integrals such as

$$\int \frac{x^4 + 2x + 5}{(x-1)(x+1)^2}\, \mathrm{d}x = \int \left(\underbrace{\text{polynomial}}_{P_\infty} + \underbrace{\frac{?}{x-1}}_{P_1} + \underbrace{\frac{?}{x+1} + \frac{?}{(x+1)^2}}_{P_2} \right) \mathrm{d}x$$

with the principal parts at 1 and -1.

Lecture 20

November 8

20.1 Residue Theorem

Recall Cauchy's Theorem: If $D \subseteq \mathbb{C}$ is a domain with a bounded, piecewise smooth boundary ∂D, $f : D \to \mathbb{C}$ is analytic and extends continuously to ∂D, then

$$\int_{\partial D} f(z)\,\mathrm{d}z = 0$$

Now, suppose that $D \subseteq \mathbb{C}$ is a domain, bounded with a piecewise smooth boundary, and we have that $f : D - \{z_1, \ldots, z_n\} \to \mathbb{C}$ is analytic and extends smoothly to ∂D (z_k are isolated singularities).

$$\int_{\partial D} f(z)\,\mathrm{d}z = ?$$

Take $\varepsilon > 0$ such that $\{|z - z_j| < \varepsilon\} \cap \{|z - z_k| < \varepsilon\} = \varnothing$ and $\{|z - z_j| < \varepsilon\} \subseteq D$. We will work on $D_\varepsilon = D - (\bigcup_{j=1}^{n} \{|z - z_j| < \varepsilon\})$. f is analytic on D_ε, and by Cauchy's Theorem,

$$\int_{\partial D_\varepsilon} f(z)\,\mathrm{d}z = 0$$

Therefore,

$$\int_{\partial D} f(z)\,\mathrm{d}z = \sum_{j=1}^{n} \int_{|z - z_j| = \varepsilon} f(z)\,\mathrm{d}z$$

In the Laurent expansion centered at z_j,

$$\begin{aligned}
f(z) &= \sum_{k=-\infty}^{\infty} a_k (z - z_j)^k \\
a_k &= \frac{1}{2\pi i} \int_{|z - z_j| = \varepsilon} f(z)(z - z_j)^{-k-1}\,\mathrm{d}z \\
a_{-1} &= \frac{1}{2\pi i} \int_{|z - z_j| = \varepsilon} f(z)\,\mathrm{d}z \overset{\text{def}}{=} \operatorname{Res}[f(z), z_j]
\end{aligned}$$

This is the **residue of f at the isolated singularity** z_j. Hence,

$$\int_{\partial D} f(z)\,\mathrm{d}z = \sum_{j=1}^{n} \int_{|z - z_j| = \varepsilon} f(z)\,\mathrm{d}z = 2\pi i \left(\sum_{j=1}^{n} \operatorname{Res}[f(z), z_j] \right)$$

Theorem 20.1 (Residue Theorem). *Let $D \subseteq \mathbb{C}$ be a domain, bounded, with a piecewise smooth boundary. Let $f : D - \{z_1, \ldots, z_n\} \to \mathbb{C}$ be analytic, extending to ∂D, where z_k are the isolated singularities.*

$$\int_{\partial D} f(z)\,\mathrm{d}z = 2\pi i \left(\sum_{j=1}^{n} \mathrm{Res}[f(z), z_j] \right)$$

The residue of f at z_j $\mathrm{Res}[f(z), z_j]$ is the coefficient of $(z - z_j)^{-1}$ in the Laurent expansion.

To find $\int_{\partial D} f(z)\,\mathrm{d}z$,

1. Find all singularities (usually only finitely many).
2. Find all residues $\mathrm{Res}[f(z), z_j]$.

To find the residues:

Method 0: Write down the Laurent expansion (sometimes, this is very complicated).

Method 0.5: z_j is a removable singularity.

$$\mathrm{Res}[f(z), z_j] = 0$$

Method 1: z_j is a simple pole of f.

$$f(z) = \frac{a_{-1}}{z - z_j} + a_0 + a_1(z - z_j) + \cdots$$

$$\lim_{z \to z_j} (z - z_j) f(z) = a_{-1} = \mathrm{Res}[f(z), z_j]$$

or $(z - z_j) f(z)|_{z = z_j}$.

Method 2: Suppose

$$f(z) = \frac{g(z)}{h(z)}$$

where g, h are analytic near z_j. Assume $g(z_j) \neq 0$ and z_j is a simple zero of h.

$$\begin{aligned} \mathrm{Res}[f(z), z_j] &= \lim_{z \to z_j} (z - z_j) \frac{g(z)}{h(z)} = g(z_j) \lim_{z \to z_j} \frac{(z - z_j)}{h(z) - h(z_j)} \\ &= \frac{g(z_j)}{h'(z_j)} \end{aligned}$$

since

$$\lim_{z \to z_j} \frac{h(z) - h(z_j)}{z - z_j} = h'(z_j)$$

Method 3: z_j is a double pole of $f(z)$.

$$\begin{aligned} f(z) &= \frac{a_{-2}}{(z - z_j)^2} + \frac{a_{-1}}{z - z_j} + a_0 + \cdots \\ (z - z_j)^2 f(z) &= a_{-2} + a_{-1}(z - z_j) + a_0(z - z_j)^2 + \cdots \\ [(z - z_j)^2 f(z)]' &= a_{-1} + 2a_0(z - z_j) + \cdots \\ \mathrm{Res}[f(z), z_j] &= \lim_{z \to z_j} [(z - z_j)^2 f(z)]' \end{aligned}$$

Suppose

$$f(z) = \frac{g(z)}{h(z)}$$

where g, h are analytic near z_j, $g(z_j) \neq 0$, and z_j is a double pole of h.

$$\begin{aligned}\operatorname{Res}[f(z), z_j] &= \lim_{z \to z_j} \left((z - z_j)^2 \frac{g(z)}{h(z)} \right)' \\ &\neq \frac{g(z)}{h''(z)}\end{aligned}$$

which is fairly complicated.

20.2 Applications of Residue Theorem

We will compute $\int_a^b f(x)\,\mathrm{d}x$, where x is a real variable and f is a real-valued function in calculus.

Step 1: Using f as a hint, "guess" some analytic function $g(z)$.

Step 2: Using f and $[a, b]$, "guess" some domain D_R, where R is some parameter.

Step 3: Use the Residue Theorem: $\int_{\partial D_R} g(z)\,\mathrm{d}z = 2\pi i \sum \operatorname{Res}$.

Step 4: Do some simplification/limiting process. Get a relation between $\int_a^b f(x)\,\mathrm{d}x$ and $\int_{\partial D_R} g(z)\,\mathrm{d}z$.

Example 20.2.

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^4}\,\mathrm{d}x$$

In calculus, we can write

$$\frac{1}{1 + x^4} = \frac{1}{(1 + x^2)^2 - (\sqrt{2}x)^2} = \frac{ax + b}{x^2 + \sqrt{2}x + 1} + \frac{cx + d}{x^2 - \sqrt{2}x + 1}$$

and the integral is maybe something like $\ln(x^2 + 2\sqrt{x} + 1) + \tan^{-1}(\cdot)$.

Step 1:

$$g(z) = \frac{1}{1 + z^4}$$

is analytic $\mathbb{C} \setminus \{4 \text{ points}\} \to \mathbb{C}$.

Step 2: Take the bounded domain D_R:

$$D_R = \{z \in \mathbb{C} \mid |z| < R, \operatorname{Im} z > 0\}$$

Step 3: Use the Residue Theorem. $1 + z^4 = 0$ when $z^4 = -1 = e^{i\pi}$, so $z = e^{i\pi/4}, e^{i(3\pi/4)}, e^{i(5\pi/4)}, e^{i(7\pi/4)}$. Only $e^{i\pi/4}, e^{i(3\pi/4)} \in D_R$.

$$\begin{aligned}\int_{\partial D_R} \frac{1}{1 + z^4}\,\mathrm{d}z &= 2\pi i \left(\operatorname{Res}\left[\frac{1}{1 + z^4}, e^{i\pi/4} \right] + \operatorname{Res}\left[\frac{1}{1 + z^4}, e^{i(3\pi/4)} \right] \right) \\ &= 2\pi i \left(\frac{1}{4z^3}\Big|_{z = e^{i\pi/4}} + \frac{1}{4z^3}\Big|_{z = e^{i(3\pi/4)}} \right) \\ &= 2\pi i \cdot \frac{1}{4} (\underbrace{e^{-i(3\pi/4)}}_{\cos(3\pi/4) - i\sin(3\pi/4)} + \underbrace{e^{-i\pi/4}}_{\cos(\pi/4) - i\sin(\pi/4)}) = 2\pi i \cdot \frac{1}{4} \cdot (-2i \sin(\pi/4)) \\ &= \frac{\sqrt{2}\pi}{2}\end{aligned}$$

Since $e^{i\pi/4}$ and $e^{i(3\pi/4)}$ are simple poles, we applied Method 2 with

$$\frac{g(z)}{h(z)} = \frac{1}{1+z^4}$$

and $(1+z^4)' = 4z^3$.

Step 4: Let $R \to \infty$.

$$\frac{\sqrt{2}\pi}{2} = \int_{\partial D_R} \frac{1}{1+z^4}\,\mathrm{d}z = \int_{-R}^{R} \frac{1}{1+x^4}\,\mathrm{d}x + \int_{\gamma_R} \frac{1}{1+z^4}\,\mathrm{d}z$$
$$\frac{\sqrt{2}\pi}{2} = \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x + \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{1+z^4}\,\mathrm{d}z$$

$$\left|\int_{\gamma_R} \frac{1}{1+z^4}\,\mathrm{d}z\right| \leq \underbrace{\frac{1}{R^4-1}}_{M} \cdot \underbrace{\pi R}_{L} \to 0$$

as $R \to \infty$ by the ML estimation, since $|z| = R$ on the circle, with $\operatorname{Im} z > 0$. Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x = \frac{\sqrt{2}}{2}\pi$$

Lecture 21

November 10

21.1 Residue Theorem

If $D \subseteq \mathbb{C}$ is a domain which is bounded, with a piecewise smooth boundary (so the integral $\int_{\partial D} \cdots$ exists), and $f : D - \{z_1, \ldots, z_n\} \to \mathbb{C}$ is analytic (where $\{z_1, \ldots, z_n\}$ are isolated singularities) and extends to ∂D continuously (so $\int_{\partial D} f(z)\, \mathrm{d}z$ exists), then

$$\int_{\partial D} f(z)\, \mathrm{d}z = 2\pi i \left(\sum_{j=1}^{n} \mathrm{Res}[f(z), z_j] \right)$$

where $\mathrm{Res}[f(z), z_j]$ is the complex number which is the coefficient of $(z - z_j)^{-1}$ in the Laurent expansion of f centered at z_j.

With the Residue Theorem, we can compute definite integrals in calculus. Last lecture, we computed $\int_{-\infty}^{\infty} 1/(1 + x^4)\, \mathrm{d}x$, which is an integral along a line. With the Residue Theorem, we calculate the integral along loops.

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^4}\, \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1 + x^4}\, \mathrm{d}x$$

We found that

$$\begin{aligned}
\underbrace{\frac{\sqrt{2}\pi}{2}}_{\text{constant independent of } R} &= \lim_{R \to \infty} \int_{\partial D_R} \frac{1}{1 + z^4}\, \mathrm{d}z \\
&= \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1 + x^4}\, \mathrm{d}x + \lim_{R \to \infty} \underbrace{\int_{\gamma_R} \frac{1}{1 + z^4}\, \mathrm{d}z}_{\leq 1/(R^4 - 1) \cdot \pi R \to 0} \\
&= \int_{-\infty}^{\infty} \frac{1}{1 + x^4}\, \mathrm{d}x
\end{aligned}$$

where γ_R is a half-circle.

We can also solve the integral $\int_{-\infty}^{\infty} P(x)/Q(x)\, \mathrm{d}x$ with the conditions

1. P, Q are both polynomials.
2. $Q(z) \neq 0$ for any $z \in \mathbb{R}$.
3. $\deg Q \geq \deg P + 2$.

The last two conditions guarantee that the improper integral exists.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x &= 2\pi i \sum_{\text{zeros of } Q \text{ in upper half-plane}} \operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_j\right] \\ &= 2\pi i \sum_{\text{zeros of } Q \text{ in lower half-plane}} \operatorname{Res}\left[\frac{P(z)}{Q(z)}, z_j\right]\end{aligned}$$

The answer should be a real number.

We will compute $\int_{-\infty}^{\infty} \cos x/(1+x^2)\,\mathrm{d}x = A$. Consider $\int_{-R}^{R} \cos x/(1+x^2)\,\mathrm{d}x$. Then

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \underbrace{=}_{z=x+iy} \frac{e^{-y+ix}+e^{y-ix}}{2}$$

but since $e^{i\theta} = \cos\theta + i\sin\theta$, then the dependence is $\sim e^{|z|}/2$, so $\cos z/(1+z^2)$ is big on the arc. Try something else. Compute: $\int_{-\infty}^{\infty} e^{ix}/(1+x^2)\,\mathrm{d}x = A+iB$, and we know that $e^{ix} = \cos x + i\sin x$. On the upper half-plane, $\left|e^{iz}\right| = \left|e^{i(x+iy)}\right| = \left|e^{-y+ix}\right| = e^{-y} \le 1$.

Compute:

$$\int_{\partial D_R} \frac{e^{iz}}{1+z^2}\,\mathrm{d}z = 2\pi i \operatorname{Res}\left[\frac{e^{iz}}{1+z^2}, i\right] = 2\pi i \frac{e^{iz}}{(z^2+1)'}\Big|_{z=i} = \frac{\pi}{e}$$

$$\begin{aligned}\frac{\pi}{e} &= \lim_{R\to\infty} \int_{\partial D_R} \frac{e^{iz}}{1+z^2}\,\mathrm{d}z \\ &= \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,\mathrm{d}x + \lim_{R\to\infty} \underbrace{\int_{\gamma_R} \frac{e^{iz}}{1+z^2}\,\mathrm{d}z}_{\le 1/(R^2-1)\cdot\pi R\to 0} \\ \frac{\pi}{e} &= \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{1+x^2}\,\mathrm{d}x\end{aligned}$$

where γ_R is a half-circle. We used the fact that $\left|e^{iz}\right| \le 1$, and the ML estimate as $R\to\infty$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,\mathrm{d}x &= \frac{\pi}{e} \\ \int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,\mathrm{d}x &= 0\end{aligned}$$

which can be seen because $\sin x/(1+x^2)$ is an odd function.

We cannot use $e^{iz}/(1+z^2)$ if we use the lower half-plane, since $\left|e^{iz}\right| \gg 1$ on the arc. Instead, use $e^{-iz}/(1+z^2)$, so $\left|e^{-iz}\right| \le 1$ if $\operatorname{Im} z \le 0$.

Next, we will compute $\int_0^{\infty} x^{1/3}/(1+x^3)\,\mathrm{d}x$. We can use the analytic function $z^{1/3}/(1+z^3)$, where we define $z^{1/3} = e^{(1/3)\operatorname{Log} z}$, which is analytic on $\mathbb{C}\setminus(-\infty,0]$. The domain D_R consists of the sector with angle $2\pi/3$, for which the denominator is $1+(ze^{i(2\pi/3)})^3$.

$$\int_{\partial D_R} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z = \int_0^R \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x - \int_0^R \frac{(xe^{i(2\pi/3)})^{1/3}}{1+(xe^{i(2\pi/3)})^3}\,\mathrm{d}(xe^{i(2\pi/3)}) + \underbrace{\int_{\gamma_R} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z}_{\le R^{1/3}/(R^3-1)\cdot(2\pi/3)R\to 0}$$

$$= (1 - e^{i(8\pi/9)}) \int_0^R \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x + \underbrace{\int_{\gamma_R} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z}_{\to 0}$$

$$\int_0^\infty \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x = \frac{\lim_{R\to\infty} \int_{\partial D_R} z^{1/3}/(1+z^3)\,\mathrm{d}z}{1 - e^{i(8\pi/9)}}$$

since $(xe^{i(2\pi/3)})^{1/3} = x^{1/3}e^{i(2\pi/9)}$ and $(xe^{i(2\pi/3)})^3 = x^3$.

$$\begin{aligned}\int_{\partial D_R} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z &= 2\pi i \operatorname{Res}\left[\frac{z^{1/3}}{1+z^3}, e^{i\pi/3}\right] \\ &= 2\pi i \left(\frac{z^{1/3}}{3z^2}\right)\Big|_{z=e^{i\pi/3}} \\ &= \frac{2\pi}{3} i e^{-i(5/9)\pi} = \frac{2\pi}{3} e^{-i\pi/18}\end{aligned}$$

since $i = e^{i\pi/2}$.

$$\int_{-\infty}^\infty \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x = \frac{2\pi}{3}\frac{e^{-i\pi/18}}{1+e^{-i\pi/9}} = \frac{2\pi}{3}\frac{1}{e^{i(1/18)\pi} + e^{-i(1/18)\pi}} = \frac{\pi}{3\cos(\pi/18)}$$

since the denominator is $2\cos(\pi/18)$.

Consider $\int_0^{2\pi} (1+\sin\theta)/(2+\cos\theta)\,\mathrm{d}\theta$. Since $[0, 2\pi]$ is the unit circle, we can use $z = e^{i\theta}$ with $\theta \in [0, 2\pi]$, so the integral is the complex line integral on $\{|z| = 1\}$. With this change of variables, $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta$, and $ie^{i\theta} = iz$. Therefore, $\mathrm{d}\theta = \mathrm{d}z/(iz)$, and we use $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$. Also, $e^{i\theta} = z$ and $e^{-i\theta} = 1/z$. We now have the integral

$$\begin{aligned}\int_{|z|=1} \frac{1 + (z - 1/z)/(2i)}{2 + (z+1/z)/2}\frac{\mathrm{d}z}{iz} &= \int_{|z|=1} \frac{2z - i(z^2-1)}{4z + (z^2+1)}\frac{\mathrm{d}z}{iz} \\ &= \frac{1}{i}\int_{|z|=1} \frac{-iz^2 + 2z + i}{z(z^2+4z+1)}\,\mathrm{d}z\end{aligned}$$

(We multiplied the numerator and denominator by $2z$.) The singularities are at $0, -2+\sqrt{3}, -2-\sqrt{3}$. Only the first two singularities lie in the unit disk.

$$\frac{2\pi i}{i}\left(\operatorname{Res}[\cdots, 0] + \operatorname{Res}[\cdots, -2+\sqrt{3}]\right) = \frac{2\pi}{\sqrt{3}}$$

Lecture 22

November 15

22.1 Key Hole Domains

Recall the Residue Theorem: If $D \subseteq \mathbb{C}$ is a domain with a bounded, piecewise smooth boundary ($\int_{\partial D}$ exists) and $f : D - \{z_1, \ldots, z_n\} \to \mathbb{C}$ (where $z_1, \ldots, z_n$ are isolated singularities) is analytic and extends continuously to ∂D ($\int_{\partial D} f(z)\,\mathrm{d}z$ exists), then $\int_{\partial D} f(z)\,\mathrm{d}z = 2\pi i \cdot \sum_{j=1}^{n} \operatorname{Res}[f(z), z_j]$.

Last lecture, we computed $\int_0^\infty x^{1/3}/(1+x^3)\,\mathrm{d}x$. We considered $z^{1/3}/(1+z^3)$, which is analytic on $\mathbb{C} \setminus (-\infty, 0]$ (since we defined $z^{1/3} = e^{(1/3)\operatorname{Log} z}$). On the real line from 0 to R, $z^{1/3}/(1+z^3) = |z|^{1/3}/(1+|z|^3)$. On the ray of angle $2\pi/3$, $z^{1/3}/(1+z^3) = |z|^{1/3} e^{i(2\pi/9)}/(1+|z|^3)$. Finally,

$$\int_{\partial D_R} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z = (1 - e^{i(8\pi/9)}) \int_0^R \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x + \underbrace{\int_{\gamma_R} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z}_{\leq R^{1/3}/(R^3-1)\cdot 2\pi R/3 \to 0 \text{ as } R\to\infty}$$

Hence,

$$\begin{aligned} \lim_{R\to\infty} \int_{\partial D_R} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z &= (1 - e^{i(8\pi/9)}) \int_0^\infty \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x \\ &= 2\pi i \operatorname{Res}\left[\frac{z^{1/3}}{1+z^3}, e^{i\pi/3}\right] = 2\pi i \frac{z^{1/3}}{(1+z^3)'}\Big|_{z=e^{i\pi/3}} = \frac{2\pi i}{3} e^{-i(5\pi/9)} \end{aligned}$$

since $e^{i(\pi/3)}$ is a simple pole. Therefore, we obtain the answer $\pi/(3\cos(\pi/18))$.

Take the integral $\int_0^\infty x^{1/3}/(1+x^3)\,\mathrm{d}x$. We have the function $z^{1/3}/(1+z^3)$. We can use a new domain which is $2/3$ of a disk. Another possible domain is the entire disk, but we cut out the positive half of the horizontal axis. Define $z^{1/3} = e^{(1/3)(\log|z| + i \arg z)}$ where $\arg z \in [0, 2\pi)$. We also need a key hole of size ε at the origin. If $D = \{|z| \leq R\} \setminus [0, R]$, then ∂D is not the union of closed curves.

$$\int_{\partial D_{R,\varepsilon}} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z = \int_\varepsilon^R \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x - \int_\varepsilon^R \frac{e^{i(2\pi/3)} x^{1/3}}{1+x^3}\,\mathrm{d}x + \underbrace{\int_{\gamma_R} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z}_{\leq R^{1/3}/(R^3-1)\cdot 2\pi R \to 0} - \underbrace{\int_{\gamma_\varepsilon} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z}_{\varepsilon^{1/3}/(1-\varepsilon^3)\cdot 2\pi\varepsilon \to 0}$$

as $R \to \infty$ and $\varepsilon \to 0$, by the ML-estimation. Let $R \to \infty$ and $\varepsilon \to 0$:

$$\begin{aligned} \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\partial D_{R,\varepsilon}} \frac{z^{1/3}}{1+z^3}\,\mathrm{d}z &= (1 - e^{i(2\pi/3)}) \int_0^\infty \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x \\ &= 2\pi i \sum_{k=0}^{2} \operatorname{Res}\left[\frac{z^{1/3}}{1+z^3}, e^{i((2k+1)\pi/3)}\right] \end{aligned}$$

$$\begin{aligned}&= 2\pi i \sum_{k=0}^{2} \frac{z^{1/3}}{3z^2}\Big|_{z=e^{i((2k+1)\pi/3)}} \\ &= \frac{2\pi i}{3}[e^{-i(5\pi/9)} + e^{-i(5\pi/3)} + e^{-i(25\pi/9)}]\end{aligned}$$

since we have simple poles. After some computation:

$$\int_0^\infty \frac{x^{1/3}}{1+x^3}\,\mathrm{d}x = \frac{2}{3\sqrt{3}}\pi\left(1 - \cos\left(\frac{\pi}{9}\right)\right)$$

The sector that we used previously only works for special cases. The key hole domain works for more general functions with branches, such as $\operatorname{Log} z$ and $z^{1/n}$. For example, consider

$$\int_0^\infty \frac{x^{1/4}}{(x+1)(x+2)}\,\mathrm{d}x = \sqrt{2}\pi(2^{1/4} - 1)$$

Use the key hole domain $D_{R,\varepsilon}$ with $R \to \infty$ and $\varepsilon \to 0$.

$$(1 - e^{i\pi/2})\int_0^\infty \frac{x^{1/4}}{(x+1)(x+2)}\,\mathrm{d}x = 2\pi i\left(\operatorname{Res}\left[\frac{z^{1/4}}{(z+1)(z+2)}, -1\right] + \operatorname{Res}\left[\frac{z^{1/4}}{(z+1)(z+2)}, -2\right]\right)$$

22.2 Fractional Residues

Consider

$$\int_0^\infty \frac{1-\cos x}{x^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{1-\cos x}{x^2}\,\mathrm{d}x$$

Note that

$$\begin{aligned}1 - \cos x &= 1 - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \\ &= \frac{x^2}{2} - \frac{x^4}{24} + \cdots\end{aligned}$$

We would like to integrate on the half-circle, which contains $\int_{-R}^{R}(1-\cos x)/x^2\,\mathrm{d}x$ and a contribution that is hopefully small. However, the function $(1-\cos z)/z^2$ is not small on the circular arc. Instead, replace $\cos z \mapsto e^{iz}$. The function $(1-e^{iz})/z^2$ is analytic on the upper half-plane, and

$$|1 - \underbrace{e^{iz}}_{=e^{-y+ix}}| \leq 2$$

However,

$$\begin{aligned}1 - e^{iz} &= 1 - \left(1 + iz + \frac{(iz)^2}{2} + \cdots\right) \\ &= -iz + \cdots\end{aligned}$$

so 0 is a simple pole. The function cannot be extended continuously to 0, so we cannot apply the Residue Theorem to the half-disk! Consider the domain $D_{R,\varepsilon}$, where the large disk has radius R and we remove a small disk of radius ε from the origin.

$$0 = \int_{\partial D_{R,\varepsilon}} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z = \int_\varepsilon^R \frac{1-e^{ix}}{x^2}\,\mathrm{d}x + \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,\mathrm{d}x + \int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z - \int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z$$

since there are no singularities in $D_{R,\varepsilon}$. By our choice of numerator,

$$\left|\frac{1-e^{iz}}{z^2}\right| \leq \frac{2}{R^2}$$

so the integral over γ_R goes to 0 as $R \to \infty$.

$$0 = \int_\varepsilon^R \frac{(1-e^{ix}) + (1-e^{-ix})}{x^2}\,\mathrm{d}x + \underbrace{\int_{\gamma_R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z}_{\to 0} + \int_{\gamma_\varepsilon} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z$$

where the last integral is on the half-circle. If we integrate over the full circle:

$$\begin{aligned}\int_{|z|=\varepsilon} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z &= 2\pi i \cdot \mathrm{Res} \\ &= 2\pi i(-i) \\ &= 2\pi\end{aligned}$$

The last integral should be π. Then

$$\int_0^\infty \frac{2-2\cos x}{x^2}\,\mathrm{d}x = \pi$$

so the original integral should be $\pi/2$.

Theorem 22.1 (Fractional Residue Theorem)**.** *Suppose that z_0 is a simple pole of f and suppose we have $C_\varepsilon = \{|z-z_0| = \varepsilon, \arg(z-z_0) \in [\theta_0, \theta_0+\alpha]\}$ where θ_0, α are fixed numbers. Then*

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,\mathrm{d}z = \alpha i \cdot \mathrm{Res}[f(z), z_0]$$

Proof. Let $f(z) = g(z) + a_{-1}/(z-z_0)$, which is analytic near z_0. Then $\int_{C_\varepsilon} g(z)\,\mathrm{d}z \to 0$ by the ML-estimation, and as $\varepsilon \to 0$, $\int_{C_\varepsilon} a_{-1}/(z-z_0)\,\mathrm{d}z = \alpha i \cdot a_{-1}$. □

Lecture 23

November 17

23.1 Logarithmic Integral

Residue Theorem: Let $D \subseteq \mathbb{C}$ be a domain which is bounded, with a piecewise smooth boundary. Let $f : \{z_1, \ldots, z_n\} \to \mathbb{C}$ be analytic, extending continuously to ∂D. Then

$$\int_{\partial D} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \operatorname{Res}[f(z), z_j]$$

Let $f : D - \{z_1, \ldots, z_n\} \to \mathbb{C}$ be analytic, with f not constantly zero. Consider $f'(z)/f(z)$, defined and analytic on $D - \{z_1, \ldots, z_n\} - \{z \in D \mid f(z) = 0\}$. (Potentially, $\{z \in D \mid f(z) = 0\}$ is an infinite set.)

Add more conditions:

1. z_j are all poles of f. f is a meromorphic function on D. (There are no zeros among the z_j.)
2. f extends analytically to ∂D. f is analytic on D' (slightly bigger than D) and f' exists on ∂D. (There are no zeros on ∂D.)

Then, $f'(z)/f(z)$ is defined on $D \cup \partial D -$ finitely many points.

For the function $f'(z)/f(z)$, let $\gamma : [a, b] \to D -$ finitely many points be smooth. Consider $\int_\gamma f'(z)/f(z)\,\mathrm{d}z$. Then $(\operatorname{Log} w)' = 1/w$, so $(\operatorname{Log} f(z))' = (1/f(z)) \cdot f'(z)$, so $\int_\gamma f'(z)/f(z)\,\mathrm{d}z = \int_\gamma \mathrm{d} \operatorname{Log} f(z)$.

"$\operatorname{Log} f(\gamma(t))$" represents infinitely many compelx numbers which differ by $2\pi i \cdot n$. Fix a complex number for $\operatorname{Log} f(\gamma(a))$ (where $f(\gamma(a)) \in \mathbb{C}$), with $f(\gamma(a)) \neq 0$. Then, we continuously define $\operatorname{Log} f(\gamma(t))$ on $[a, b]$ and get $\operatorname{Log} f(\gamma(b))$. For $|t - a| < \varepsilon$, $|f(\gamma(t)) - f(\gamma(a))|$ is small. Choose $\operatorname{Log} f(\gamma(t))$ close to $\operatorname{Log} f(\gamma(a))$.

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \operatorname{Log} f(\gamma(b)) - \operatorname{Log} f(\gamma(a))$$

$\operatorname{Log} f(\gamma(t))$ is *continuously* defined on $[a, b]$. $\operatorname{Log} f(\gamma(b)) - \operatorname{Log} f(\gamma(a))$ is determined by $\gamma(b), \gamma(a)$, up to $2\pi i \cdot \mathbb{Z}$. For a curve that intersects itself, $\gamma(r) = \gamma(s)$, it is possible for $\operatorname{Log} f(\gamma(r)) \neq \operatorname{Log} f(\gamma(s))$.

Example 23.1. Take $f(z) = z$, so $f'/f = 1/z$. Consider $\gamma : [0, 1] \to \mathbb{C}$, with $\gamma(t) = e^{2\pi i t}$. Then

$$\operatorname{Log} f(\gamma(t)) = \operatorname{Log} e^{2\pi i t}$$

Fix $\operatorname{Log} f(\gamma(0)) = 0$. Then $\operatorname{Log} f(\gamma(t)) = 2\pi i t$ and $\operatorname{Log} f(\gamma(1)) = 2\pi i$.

Most interesting application: γ is a closed curve, i.e. $\gamma(a) = \gamma(b)$, $f(\gamma(a)) = f(\gamma(b))$. Then

$$\begin{aligned}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z &= \operatorname{Log} f(\gamma(b)) - \operatorname{Log} f(\gamma(a)) \\ &= i(\arg f(\gamma(b)) - \arg f(\gamma(a))) \\ &= 2\pi i \cdot (\text{an integer})\end{aligned}$$

since $\operatorname{Log} z = \log|z| + i\arg z$, where $\arg z$ is the only ambiguity. We can read this integer from a picture, as the number of times that $f \circ \gamma$ goes around the origin in a counterclockwise direction.

The logarithmic integral equals $2\pi i$ times the number of times that $f \circ \gamma$ goes around the origin.

Let f be a meromorphic function on D, and let $D' \cup \partial D' \subseteq D$, with $f(z)$ never 0 on $\partial D'$ and does not have a pole on $\partial D'$. $f'(z)/f(z)$ is defined on D'.

Theorem 23.2.

$$\int_{\partial D'} \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i(\#\ \textit{of zeros of}\ f - \#\ \textit{of poles of}\ f)$$

counted with multiplicity.

Proof. We know that the integral equals $2\pi i \cdot \sum_{\text{singularities in } D'} \operatorname{Res}[f'/f, z_j]$. For example, suppose that z_0 is an order N zero. Then

$$\begin{aligned}f(z) &= \underbrace{a_N}_{\neq 0}(z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + \cdots \\ f'(z) &= N \cdot a_N (z - z_0)^{N-1} + (N+1) \cdot a_{N+1}(z - z_0)^N + \cdots\end{aligned}$$

Therefore,

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_0} \frac{N a_N + (N+1)a_{N+1}(z - z_0) + \cdots}{a_N + a_{N+1}(z - z_0) + \cdots}$$

where $a_N \neq 0$. We have found

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)}, z_0\right] = N$$

□

More generally: suppose that D is a domain and f is a meromorphic function on D which extends analytically to ∂D and never vanishes on D.

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D} \frac{f'}{f}\,\mathrm{d}z &= N_0 - N_\infty \\ &= \#\text{ of times } f(\partial D) \text{ (with the correct orientation) goes around the origin}\end{aligned}$$

Example 23.3. Take $f(z) = z^5 + 2z^3 - z^2 + z + 1$. Find the number of zeros on $\{\operatorname{Re} z > 0\}$.

1. $f(z)$ has no zero on $\{|z| \geq 3\}$.
2. $f(z)$ has no zero on the y-axis.

$$f(iy) = (y^2 + 1) + (y^5 - 2y^3 + y)i$$

3. Use the geometric picture of $f(\partial D)$. Assume that $R \gg 0$.

$$f(-iR) \approx (-iR)^5 = -iR^5$$

and $\arg f(-iR) = -\pi/2$. Around a half-circle of radius R, the behavior is as z^5, which changes by 5π.

$$f(iR) \approx (iR)^5 = iR^5$$

so $\arg f(iR) \approx -\pi/2 + 5\pi$. Along the imaginary axis, $f(it)$ has $\mathrm{Re} > 0$, so in order to go from $-\pi/2 + 5\pi \to -\pi/2 + 2n\pi$, it must be that we have $-\pi/2 + 4\pi = \arg f(-iR)$, the terminal point.

$$\frac{\arg f(\gamma(b)) - \arg f(\gamma(a))}{2\pi} = 2$$

Lecture 24

November 22

24.1 Argument Principle

Recall the Argument Principle: if D is a bounded domain with a piecewise smooth boundary, f is a meromorphic function on D that extends to be analytic on ∂D, $f(z) \neq 0$ for all $z \in \partial D$ (there exists a bigger open set $D' \supseteq D \cup \partial D$ where f is meromorphic on D' and analytic on a neighborhood of ∂D), then

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}\,\mathrm{d}z = \underbrace{N_0}_{\#\text{ of zeros of } f \text{ in } D} - \underbrace{N_\infty}_{\#\text{ of poles of } f \text{ in } D}$$

counted with multiplicity.

We can also write the integral as

$$\frac{1}{2\pi i}\int_{\partial D}\mathrm{d}(\log f(z)) \underbrace{=}_{\text{imaginary part}} \frac{1}{2\pi}\int_{\partial D}\mathrm{d}(\arg f(z)) = \#\text{ of times } f(\partial D) \text{ goes around the origin}$$

Define $\log f(z)$ or $\arg f(z)$ continuously on each component of ∂D, except at one point. At this point, we obtain another number which differs from $\log f(z)$ by $2\pi i \cdot n$.

Applications: We can count the number of zeros of an *analytic function* in a certain domain (last lecture, homework).

24.1.1 More Applications

Theorem 24.1 (Rouche's Theorem)**.** *Let D be a bounded domain with a piecewise smooth boundary. Let f, h be analytic functions on $D \cup \partial D$ (a neighborhood of $D \cup \partial D$). If $|f(z)| > |h(z)|$ on ∂D, then the number of zeros of f in D equals the number of zeros of $f + h$ in D.*

Proof. Consider the sun, earth, and moon. $|f| > |h|$ on ∂D means $\mathrm{dist}(\mathrm{sun}, \mathrm{moon}) > \mathrm{dist}(\mathrm{earth}, \mathrm{moon})$. If the earth goes around the sun one time, then so does the moon.

We will compare $(2\pi)^{-1}\int_{\partial D}\mathrm{d}(\arg f(z))$ and $(2\pi)^{-1}\int_{\partial D}\mathrm{d}(\arg{(f+h)(z)})$. Both of these quantities are integers! Start from $z_0 \in \partial D$. Fix $\arg f(z_0), \arg{(f+h)(z_0)}$ such that

$$|\arg{(f+h)(z_0)} - \arg f(z_0)| < \pi/2$$

Continuously define $\arg f(z), \arg{(f+h)(z)}$ on ∂D. Then

$$|\arg{(f+h)(z)} - \arg f(z)| < \frac{\pi}{2}$$

We find that

$$\int_{\partial D} \mathrm{d}(\arg f(z)) = \arg f(\underbrace{z_0}_{\text{as terminal point}}) - \arg f(\underbrace{z_0}_{\text{as initial point}})$$

$$\int_{\partial D} \mathrm{d}(\arg{(f+h)(z)}) = \arg{(f+h)(z_0)} - \arg{(f+h)(z_0)}$$

Therefore,

$$\left| \int_{\partial D} \mathrm{d}(\arg f(z)) - \int_{\partial D} \mathrm{d}(\arg{(f+h)(z)}) \right| < \pi$$

$$\left| \frac{1}{2\pi} \int_{\partial D} \mathrm{d}(\arg f(z)) - \frac{1}{2\pi} \int_{\partial D} \mathrm{d}(\arg{(f+h)(z)}) \right| < \frac{1}{2}$$

(The proof assumes that ∂D is one circle.) ☐

Example 24.2. Find the number of zeros of $p(z) = 2z^5 + 6z - 1$ on $\{|z| < 1\}$ and $\{1 < |z| < 2\}$.

1. Use Rouche's Theorem 24.1. On $\{|z| = 1\}$, $p = f + h$, with $|f| > |h|$. Here, take

   $$(\underbrace{6z}_{|\cdot|=6}) + (\underbrace{2z^5 - 1}_{|\cdot| \le 2+1=3<6})$$

   The number of zeros of p in $\{|z| < 1\}$ is the same as the number of zeros of $6z$ in $\{|z| < 1\}$, which is 1.

2. $\left|2z^5 + 6z\right| > 1$ on $\{|z| = 1 \text{ or } 2\}$ (this needs some work). Alternatively, we find the number of zeros in $\{|z| < 2\}$. On $\{|z| = 2\}$,

   $$\underbrace{2z^5}_{|\cdot|=2\cdot 2^5=64} + (\underbrace{6z - 1}_{|\cdot| \le 6\cdot 2+1=13})$$

   The number of zeros of p in $\{|z| < 2\}$ is the number of zeros of $2 \cdot z^5$ in $\{|z| < 2\}$, which is 5. The number of zeros in $\{1 < |z| < 2\} = 4$.

24.2 Number of Zeros Under Uniform (Normal) Convergence

Theorem 24.3 (Hurwitz's Theorem)**.** *Let $f_k : D \to \mathbb{C}$ be analytic and $\{f_k\}$ uniformly (normally) converges to $f : D \to \mathbb{C}$. Suppose z_0 is an order N zero of f (f is not constant). Then, $\exists \rho > 0$ such that for any large k, f_k has N zeros in $\{|z - z_0| < \rho\}$ and these zeros go to z_0 as $k \to \infty$.*

Proof. $f(z) = (z - z_0)^N \cdot g(z)$, where $g(z_0) \neq 0$. Then $\exists \rho > 0$ such that $f(z) \neq 0$ in $\{0 < |z - z_0| \le \rho\}$ (we need $g \neq 0$ in $\{|z - z_0| \le \rho\}$). Then

$$\inf \{|f(z)| \mid |z - z_0| = \rho\} = \alpha > 0$$

Write $f_k = f + (f_k - f)$. If k is large, $|f_k - f| < \alpha$ on $\{|z - z_0| = \rho\}$. By Rouche's Theorem 24.1, the number of zeros of f_k equals the number of zeros of f in $\{|z - z_0| < \rho\}$. This also holds for any $r \in (0, \rho)$. Let $\rho \to 0$ and the zeros of f_k (N such zeros) go to z_0. □

Definition 24.4. $f : D \to \mathbb{C}$ is **univalent** if f is analytic and one-to-one.

Theorem 24.5. *Suppose that $f_k : D \to \mathbb{C}$ are univalent functions, and $\{f_k\}$ uniformly converges to $f : D \to \mathbb{C}$. Then, f is either univalent or constant.*

For example, consider $D = \{|z| < 1\}$, $f_k(z) = z/k$. Then $f_k \xrightarrow{\text{uniformly}} 0$.

Proof. Consider z_1, z_2 with $f(z_1) = f(z_2) = w_0$. Then $f(z) - w_0$ has zeros in small disks around z_1 and z_2. If f is not constantly zero, then $f_k - w_0$ has zeros in the disks. Therefore, f_k is not one-to-one, which is impossible. □

Lecture 25

November 29

25.1 Univalent Functions

Rouche's Theorem: Let D be a domain which is bounded, with a piecewise smooth boundary. Consider $f, h : D \to \mathbb{C}$ be analytic, extending to be analytic on ∂D. If $|f(z)| > |h(z)|$ for $z \in \partial D$, then the number of zeros of f in D equals the number of zeros of $f + h$ (and $f - h$, $f + e^{i\theta}h$) in D.

A univalent function is analytic and one-to-one.

Hurwitz's Theorem: If f_k are univalent, $f_k \xrightarrow{\text{uniformly}} f$ on D, f is either univalent or constant.

Recall: If $f : D \to \mathbb{C}$ is analytic, $z_0 \in D$, $f'(z_0) \neq 0$ (as a multivariable real-valued function), then the Inverse Function Theorem says that f is one-to-one on a small disk $D(z_0, \varepsilon)$ (locally-one-to-one). If f is one-to-one, then $f'(z_0) \neq 0$ only has to hold for most points in D.

Theorem 25.1. *If $f : D \to \mathbb{C}$ is univalent, then for any $z_0 \in D$, $f'(z_0) \neq 0$.*

Proof. Rough Idea: If $f'(z_0) = 0$, then for $g(z) = f(z) - f(z_0) = a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$, z_0 is an order 2 zero of $g(z)$. What about $g(z) - \varepsilon$, which is $f(z) - (f(z_0) + \varepsilon)$? Does this also have 2 zeros?

Rigorous Proof: Assume $f'(z_0) = 0$ for some $z_0 \in D$. Take $\varepsilon > 0$ such that $\{|z - z_0| \leq \varepsilon\} \subseteq D$. Then

$$\frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} \frac{(f(z) - f(z_0))'}{f(z) - f(z_0)} \, \mathrm{d}z = \text{order of zero of } f(z) - f(z_0) \text{ at } z_0 = N \geq 2$$

Choose δ such that $\{|w - f(z_0)| \leq \delta\}$ is within $f(\{|z - z_0| = \varepsilon\})$, e.g. by choosing

$$0 < \delta < \inf \{|f(z) - f(z_0)| \mid |z - z_0| \leq \varepsilon\}$$

For all $w \in \{|w - f(z_0)| \leq \delta\}$, there exists N zeros of $f(z) - w$ in $\{|z - z_0| \leq \varepsilon\}$, by applying Rouche's Theorem to $(f(z) - f(z_0)) - (w - f(z_0))$ ($|f(z) - f(z_0)| > |f(z_0) - w|$ on $\{|z - z_0| = \varepsilon\}$). There is a order N zero of $f(z) - w_0$ in the disk, which implies that $f'(z) = 0$ at this point. Since there are infinitely points in $\{|w - f(z)| \leq \delta\}$, there are infinitely many zeros of $f'(z)$ in $\{|z - z_0| \leq \varepsilon\}$. If an analytic function has infinitely many zeros in a closed, bounded region, then $f' \equiv 0$, so f is constant, which is a contradiction! $\square$

If $f : D \to \mathbb{C}$ is univalent, let $f^{-1} : f(D) \subseteq \mathbb{C} \to D \subseteq \mathbb{C}$ be the inverse function ($f(D)$ is the image). f' is never zero implies that f^{-1} is also analytic and univalent (using multivariable analysis), so $(f^{-1})'$ exists!

25.2 Open Mapping

A univalent function attains each complex number ≤ 1 time.

For $f : D \to \mathbb{C}$, and $\{|z - z_0| \leq \varepsilon\} \subseteq D$, let

$$\begin{aligned} N(w) &= \#\text{ of } \{z \mid |z - z_0| \leq \varepsilon, f(z) = w\} \text{ with multiplicity} = \#\text{ of zeros of } f(z) - w \text{ in } \{|z - z_0| \leq \varepsilon\} \\ &= \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} \frac{f'(z)}{f(z) - w} \, \mathrm{d}z \end{aligned}$$

We need $w \notin f(\{|z - z_0| = \varepsilon\})$.

$N(w)$ is a locally constant function on $\{w \mid w \notin f(\{|z - z_0| = \varepsilon\})\}$. Compare $f(z) - w_1$ v.s. $f(z) - w_2$.

$$f(z) - w_1 = \underbrace{f(z) - w_2}_{\geq \delta \text{ on } \{|z-z_0|=\varepsilon\}} + w_2 - w_1$$

where δ depends on w_2. As long as $|w_2 - w_1| < \delta$, $N(w_1) = N(w_2)$. $N(f(z_0)) \geq 1$ implies that $\exists \delta > 0$ such that $\forall w \in \{w \mid |w - f(z_0)| < \delta\}$, $N(w) \geq 1$.

Theorem 25.2 (Open Mapping Theorem)**.** *If $f : D \to \mathbb{C}$ is analytic and not constant, where D is an open set, then $f(D)$ is an open set in $\mathbb{C}$.*

Proof. $\forall w_0 \in f(D)\ \exists z_0 \in D$ such that $f(z_0) = w_0$. $N(w_0) \geq 1$ implies that there exists $\delta > 0$ such that for all $w \in \{|w - w_0| < \delta\}$, $N(w) \geq 1$, so $w \in f(D)$. $f(D)$ is open. □

(This is true if f' is never zero by the Inverse Function Theorem.)

Alternative Proof of Open Mapping Theorem 25.2. Let $f : D \to \mathbb{C}$ be analytic, $w_0 \in f(D)$, and write $w_0 = f(z_0)$ for $z_0 \in D$. If $f'(z_0) \neq 0$, then the theorem is true by the Inverse Function Theorem in multivariable analysis. If $f'(z_0) = 0$, then, for $N \geq 2$ and $a_N \neq 0$,

$$\begin{aligned} f(z) - f(z_0) &= a_N(z - z_0)^N + a_{N+1}(z - z_0)^{N+2} + \cdots \\ &= (z - z_0)^N \cdot \underbrace{g(z)}_{g(z_0)\neq 0} \\ &= \Big((z - z_0) \cdot g^{1/N}(z)\Big)^N \end{aligned}$$

Choose a branch near z_0. We can view f as the composition of $(z-z_0)g^{1/N}(z)$ (analytic) and $w^N + f(z_0)$. Take the derivative.

$$\begin{aligned} ((z - z_0) \underbrace{h(z)}_{=g^{1/N}})'\Big|_{z=z_0} &= (h(z) + (z - z_0) \cdot h'(z))\Big|_{z=z_0} \\ &= h(z_0) = g^{1/N}(z_0) \\ &\neq 0 \end{aligned}$$

The function w^N also maps disks to disks. □

Let $f : \{|z - z_0| \leq \varepsilon\} \to \mathbb{C}$ be analytic. If $f'(z_0) \neq 0$, then f is locally one-to-one. For all w in the image $f(\{|z - z_0| \leq \varepsilon\})$, we can find the formula for $f^{-1}(w)$:

$$\frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} \frac{zf'(z)}{f(z) - w} \, \mathrm{d}z$$

Lecture 26

December 1

Lecturer: Professor Lin Lin

26.1 Riemann Mapping Theorem

A conformal map with $D, V \subseteq \mathbb{C}$ is one-to-one, onto, and angle-preserving.

In the homework, we proved that a function f is angle-preserving if and only if f or $\bar{f}$ is analytic. We are only interested in the case when f is analytic.

Last class: if f is analytic and one-to-one (univalent), then $f' \neq 0$. $f^{-1} : V \to D$ is analytic, which implies that it is conformal. We will prove that given any region D, we can find an analytic function f that maps the region to $\mathbb{D} = \{z : |z| < 1\}$.

Example 26.1. Let us find a mapping $D = \{z : |z| < R\} \to \mathbb{D}$. Here, the mapping is

$$f(z) = \frac{z}{R}$$

Example 26.2. Take $D = \{z : \operatorname{Im} z > 0\}$. We can use a Möbius transform.

$$f(z) = \frac{i - z}{i + z}$$

$$\begin{aligned} f(0) &= 1 \\ f(1) &= \frac{i-1}{i+1} \\ f(\infty) &= -1 \\ f(i) &= 0 \end{aligned}$$

This function indeed maps D to the interior of the unit disk.

Example 26.3. Consider the sector $D = \{z : 0 < \arg z < \theta\}$. Use the function $z^{\pi/\theta} = e^{(\pi/\theta) \operatorname{Log} z}$ to map the region to the upper-half plane, and then compose this with the previous mapping:

$$f(z) = \frac{i - z^{\pi/\theta}}{i + z^{\pi/\theta}}$$

When $\theta = 2\pi$,

$$\begin{aligned} D &= \{z : 0 < \arg z < 2\pi\} \\ &\equiv \mathbb{C} \setminus [0, \infty) \end{aligned}$$

Here, our mapping is

$$f(z) = \frac{i - z^{1/2}}{i + z^{1/2}}$$

Example 26.4. Consider the domain $D = \{z : -1 < \operatorname{Im} z < 1\}$ and the map e^z.

$$\begin{aligned} |e^z| = \left|e^{x+iy}\right| = e^x \in (0, \infty) \\ \arg e^z = y \in (-1, 1) \end{aligned}$$

Therefore, e^z maps the region to a sector. We compose the map $z \mapsto e^z$ with $w \mapsto e^{(\pi/2)(\operatorname{Log} w + i)}$ and the map $u \mapsto (i - u)/(i + u)$.

$$f(z) = \frac{i - e^{(\pi/2)(z+i)}}{i + e^{(\pi/2)(z+i)}}$$

This not always doable:

Example 26.5. Let $D = \mathbb{C}$. $f : D \to \mathbb{D}$ does not exist, by the Liouville Theorem: if f is analytic on $\mathbb{C}$ and bounded, then f is constant.

Example 26.6. Take $D = \mathbb{D} \setminus \{0\}$. There is a topological obstruction: we start with a contour in $\mathbb{D}$, $C = \{z : |z| = 1/2\}$, such that the origin lies within the pre-image. Shrink the contour to $rC = \{z : |z| = r/2\}$. Take the limit as $r \to \infty$, and the contour shrinks to $\{0\}$. However, $f^{-1}(0)$ exists, and by a continuity argument, $f^{-1}(0)$ must be 0.

Definition 26.7. $D \subseteq \mathbb{C}$ is a **simply connected domain** if any loop can deformed to a point. More precisely, for any $\gamma : [a, b] \to D$, $\gamma(a) = \gamma(b)$, there exists $\Gamma : [a, b] \times [0, 1] \to D$ such that Γ is continuous, $\Gamma(s, 0) = \gamma(s)$, $\Gamma(s, 1) = x \in D$, and $\Gamma(a, t) = \Gamma(b, t)$, $\forall 0 \le t \le 1$.

For example, to deform the contour from 26.6 into 0, take $\Gamma : [0, 2\pi] \times [0, 1] \to D$ with

$$\begin{aligned} \Gamma(s, 0) &= \frac{1}{2} e^{i\theta} \\ \Gamma(s, t) &= \frac{1 - t}{2} e^{i\theta} \end{aligned}$$

Theorem 26.8 (Riemann Mapping)**.** *Let $D \subseteq \mathbb{C}$ be simply connected, $D \neq \mathbb{C}$. Then, there exists an analytic function $f : D \to \mathbb{D}$ which is conformal.*

Idea of Proof.

1. There exists analytic $h : D \to V \subseteq \mathbb{D}$. Use $D \neq \mathbb{C}$. Find $a \in \mathbb{C}$, $a \notin D$. If $B(a, \varepsilon) \subseteq \mathbb{C} \setminus D$, then the map $\varepsilon/(z - a)$ works. Here, $|z - a| > \varepsilon\ \forall z \in D$, then $|\varepsilon/(z - a)| < 1$, so $V \subseteq \mathbb{D}$. Otherwise, choose $a \in \mathbb{C} \setminus D$ and a single branch of $g(z) = \sqrt{z - a}$ (this uses the assumption that D is simply connected). If $w_0 \in g(D)$, then $B(w_0, \varepsilon) \subseteq g(D)$, which means that $B(-w_0, \varepsilon) \notin g(D)$. Then, we can use the composition of this with the previous case.

2. There are too many degrees of freedom! Pick *any* $z_0 \in D$. There exists an analytic function $l : D \to V \subseteq \mathbb{D}$, with $l(z_0) = 0$, $l'(z_0) > 0$. We only need to find $K : V \to \mathbb{D}$, with $K(h(z_0)) = 0$

and $K'(h(z_0)) \cdot h'(z_0) > 0$. Take $l = K(h(z))$. To do this, find a Möbius transform.

3. Find the mapping that maximizes $l'(z_0)$, $f(z)$. This is a conformal mapping $D \to \mathbb{D}$. This requires a compactness argument in function space. If the maximizer is not what we need, then $f(D) \neq \mathbb{D}$. Choose $a \in \mathbb{D} \setminus f(D)$. Then, we can find a map $K : f(D) \to \tilde{V} \subseteq \mathbb{D}$ with $K(0) = 0$, $K'(0) > 1$ (technical point). Then,

$$(K \circ f)(z_0) = K(f(z_0)) = 0$$
$$(K \circ f)'(z_0) = K'(0) \cdot f'(z_0) > f'(z_0)$$

□