Elementary Differential Topology

Mathematics 141
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Contents

Lecture 1

August 24

1.1 Differential Topology

We will focus on smooth spaces such as a circle in $\mathbb{R}^2$, instead of spaces with singularities (such as a heart shape, which has sharp corners). The main objects of study are differentiable manifolds.

Example 1.1. Points are differentiable manifolds.

Example 1.2. The unit sphere defined by the set of solutions to the equation $x^2 + y^2 + z^2 = 1$ is a surface in $\mathbb{R}^3$ which is a differentiable manifold. A torus, or a higher genus torus, is an example of another differentiable manifold which is not defined by equations.

The course will be divided into two parts.

Local theory: If X is a differentiable manifold, around each point x, there exists a map $\phi : U \to V$, where U is an open set in $\mathbb{R}^k$ and V is an open set in X containing x. For example, consider the unit circle $S^1 \subset \mathbb{R}^2$ and the point $x = (0, 1)$. We can map the interval $(-\varepsilon, \varepsilon) \xrightarrow{\phi} \mathbb{R}^2$ with $\phi(t) = (t, \sqrt{1 - t^2})$.

TX is the tangent space of X. We will consider smooth maps $f : X \to Y$, which can be seen as a generalization of smooth maps $\mathbb{R}^n \to \mathbb{R}^m$.

First half: How *df* dictates the behavior of f locally. (Munkres, Chapters 1 & 2)

Second half: Intersection theory. Consider the intersection of a parabola with a line. There can be two intersection points, one intersection point, or no intersection points. If we ignore the case of one intersection point (which can be thought of as the limiting case of two intersection points as the intersection points approach each other), then the number of intersection points are equal modulo 2. We can then give the two curves orientations, and so a parabola which intersects a line at two points will be assigned 1 at one intersection point and -1 at the other intersection point.

Transversality: If we count intersection points correctly, then it will give us an invariant up to *homotopy*. Generically, for any two differentiable manifolds, one can deform (via a homotopy) a bit so that we have transversal intersections.

Example 1.3. Suppose we are interested in the number of fixed points of $f : X \to X$. If we plot $X \times X$, the diagonal is the set of points (x, x), and the graph of f is $\{(x, f(x)) \mid x \in X\}$, and the fixed points of f are the intersections of the graph of f and the diagonal.

Example 1.4. Consider a function $f : X^n \to Y^n$. We are interested in the *degree* of f, which is how much f wraps X around Y (for example, the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$). This can also be reduced to a problem in intersection theory.

Morse theory: Study a differentiable manifold X by studying functions $X \to \mathbb{R}$. For example, for the unit sphere $x^2 + y^2 + z^2 = 1$, study the function $f(x, y, z) = z$ and its critical points (at the top and bottom of the sphere). If we instead consider the torus, the critical points can be saddle points. As we pass through critical points, the sublevels of the manifold change. If we deform a sphere, the number of critical points may change, but counting critical points in the correct way gives an invariant of differentiable manifolds.

Morse homology: Vector spaces provide a richer algebraic invariant.

1.2 Topology of $\mathbb{R}^n$

Munkres, Section 1.3.

$\mathbb{R}^n$ has the standard metric $d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$. d gives $\mathbb{R}^n$ the structure of a metric space.

Definition 1.5. (X, d) is a **metric space** if $d : X \times X \to \mathbb{R}$ satisfies:

- $d(x, y) > 0$, unless $x = y$, in which case $d(x, x) = 0$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq d(x, y) + d(y, z)$.

In a metric space (X, d), for each $x_0 \in X$, $\varepsilon > 0$, we can define $B_\varepsilon(x_0) = \{x \in X \mid d(x, x_0) < \varepsilon\}$.

Definition 1.6. $U \subseteq X$ is **open** if $\forall x_0 \in U$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq U$.

Definition 1.7. $K \subseteq X$ is **closed** if $X - K$ is open.

Definition 1.8. $\{x_n\} \to x_0$ if for all open sets U, $x_0 \in U$, there exists N such that for $n > N$, $x_n \in U$.

Properties:

(a) If $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets in X, then $\bigcup_{\alpha \in A} U_\alpha$ is also open.

(b) For a finite collection, $\bigcap_{i=1}^n U_i$ is open.

Definition 1.9. Given two metric spaces X and Y, a map f is **continuous at** x_0 if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that whenever $\delta_X(x, x_0) < \delta$, $d_Y(f(x), f(x_0)) < \varepsilon$.

Definition 1.10. f is **continuous in** X if for all open sets V in Y, $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X.

Consider the case when $Y = \mathbb{R}^n$, that is, $f : X \to \mathbb{R}^n$. We can write $f = (f_1, \ldots, f_n)$, where $f_i(x)$ is the ith coordinate of $f(x)$.

Proposition 1.11. *f is continuous if and only if each function f_i is continuous.*

Proof. $f_i = \pi_i \circ f$, where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the ith projection. So, if f is continuous, then f_i is also continuous.

Suppose each f_i is continuous. Given $\varepsilon > 0$, $\exists \delta$ such that whenever $d_X(x, x_0) < \delta$,

$$|f_i(x) - f_i(x_0)| < \frac{\varepsilon}{\sqrt{n}}$$

for all $i = 1, \dots, n$. Then,

$$\|f(x) - f(x_0)\| = \sqrt{\sum_{i=1}^{n} (f_i(x) - f_i(x_0))^2} < \sqrt{n \cdot \frac{\varepsilon^2}{n}} = \varepsilon.$$

Hence, f is continuous. $\square$

Lecture 2

August 29

2.1 More Metric Space Topology

Let (X, d) be a metric space. If U is open, we can write $U = \bigcup_{x \in U} B_{\varepsilon_x}(x)$, where ε_x is small enough so that $B_{\varepsilon_x}(x) \subseteq U$. K is closed if $X \setminus K$ is open.

Definition 2.1. Given a subset $A \subseteq X$, a **limit point** x of A is a point such that for all $\varepsilon > 0$, $(B_\varepsilon(x) \cap A) \setminus \{x\} \neq \varnothing$.

Definition 2.2. The **closure of** A is $\overline{A} = A \cup \{\text{limit points of } A\}$.

Proposition 2.3. *$\overline{A}$ is the smallest closed set containing A, i.e.*

$$\overline{A} = \bigcap_{\substack{K \text{ closed} \\ A \subseteq K}} K.$$

Proof. (a) $\overline{A} \subseteq K$ for any K closed and $A \subseteq K$. Indeed, if $x \in K^{\mathsf{c}}$ (which is open), then $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \cap K = \varnothing$. Hence, $B_\varepsilon(x) \cap A = \varnothing$. Thus, $x \notin \overline{A}$. Therefore, $\overline{A} \subseteq K$.

(b) $\overline{A}$ is closed. If $x \in \overline{A}^{\mathsf{c}}$, then x is not in A or a limit point of A. So, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \varnothing$. Then, this implies $B_\varepsilon(x) \cap \overline{A} = \varnothing$. Hence, $\overline{A}^{\mathsf{c}}$ is open and therefore $\overline{A}$ is closed. $\square$

Corollary 2.4. *If A is closed, then $\overline{A} = A$.*

2.1.1 Compactness

Definition 2.5. Given a subset $K \subseteq X$, an **open cover** $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of K is a collection of open sets $U_\alpha \subseteq X$ such that $K \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$.

Definition 2.6. $K \subseteq X$ is **compact** if for every open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of K, there exists a finite subcover $U_1, \ldots, U_n$ of K, i.e. $\bigcup_{i=1}^n U_i \supseteq K$.

Example 2.7 (Counter-Example of Compactness)**.** Let $X = \mathbb{R}$, $K = (0, 1)$. Take the collection

$\{U_n\}_{n\in\mathbb{N}}$, where $U_n = (0, 1-1/n)$. Then, $\bigcup_{n\in\mathbb{N}} U_n = K$, but there is no finite subcover.

Example 2.8 (Counter-Example of Compactness)**.** Let $K = \mathbb{R}\setminus\{0\}$. Take the collection $\{U_n\}_{n\in\mathbb{N}}$, where $U_n = \mathbb{R}\setminus[-1/n, 1/n]$. Again, $\bigcup_{n\in\mathbb{N}} U_n \supseteq K$, but there is no finite subcover.

We will see that $[0,1]$ is a compact subset.

Proposition 2.9. *(a) If $K \subseteq X$ is compact, then K is closed and bounded.*

(b) If X is compact, then any closed subset $K \subseteq X$ is also compact.

Proof. (a) Compactness implies closedness and boundedness.

Boundedness: If not, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq K$ such that $d(0, x_n) \xrightarrow{n\to\infty} \infty$, where 0 is a reference point in X. Now, for each $x \in K$, $U_x = B_\varepsilon(x)$, where $\varepsilon > 0$ is fixed. $K \subseteq \{U_x\}_{x\in K}$, but K is compact, so there is a finite list $x_1^*, \ldots, x_m^*$ such that $K \subseteq B_\varepsilon(x_1^*) \cup \cdots \cup B_\varepsilon(x_m^*)$, which contradicts the existence of the sequence $\{x_n\}_{n\in\mathbb{N}}$.

Closedness: We need to show that K^c is open in X. Given $x \in K^\mathsf{c}$, consider the following open cover of K: $U_\varepsilon = X \setminus \overline{B_\varepsilon(x)}$, and $\bigcup_{\varepsilon>0} U_\varepsilon = X \setminus \{x\} \supseteq K$. K is compact, so there exists $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $\bigcup_{i=1}^n U_{\varepsilon_i} \supseteq K$. Say $\varepsilon_1 = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then, $U_{\varepsilon_1} \supseteq K$ (check that if $\varepsilon_1 < \varepsilon_2$, then $U_{\varepsilon_1} \supseteq U_{\varepsilon_2}$). In particular, $B_{\varepsilon_1}(x) \cap K = \varnothing$. Thus, K^c is open and hence K is closed.

(b) Let us say that $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover of K. Since K is closed, $U = K^\mathsf{c}$ is open in X. Then, $\{U_\alpha\}_{\alpha\in\mathcal{A}} \cup \{U\}$ is an open cover of X. X is compact, which implies that there is a finite subcover $U_1 \cup \cdots \cup U_n \cup U = X$. Hence, $U_1 \cup \cdots \cup U_n \supseteq K$. Hence, K is compact.

□

Proposition 2.10. *If $f : X \to Y$ is a continuous map between two metric spaces and $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.*

Theorem 2.11. *Suppose $f : X \to \mathbb{R}$ is a continuous function and assume that X is compact. Then, f has a minimum and a maximum point.*

Proof. X is compact and f is continuous, so $f(X) \subseteq \mathbb{R}$ is compact, which implies that $f(X)$ is closed and bounded. That means that $\exists x_{\max} \in X$ such that $f(x_{\max}) = \sup f(X)$. Also, for some $x_{\min} \in X$, $f(x_{\min}) = \inf f(X)$. □

Theorem 2.12 (Heine-Borel Theorem)**.** *In the case $X = \mathbb{R}^n$, K is compact if and only if K is closed and bounded.*

Proof. Suppose K is closed and bounded. Boundedness implies that there exists a closed interval $[a,b]$ such that $K \subseteq [a,b]^n$. It is sufficient to show that $[a,b]^n$ is compact since

$$K \subseteq [a,b]^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in [a,b]\}$$

and K is closed. In Homework 2, you will show that if X and Y are compact, then $X \times Y$ are also compact. Hence, it is enough to show $[a,b]$ is compact. Suppose not; then, there is an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of $[a,b]$ that has no finite subcover. If we take $a = 0$, $b = 1$, then split the interval into $[0, 1/2]$ and $[1/2, 1]$. At least one of these must have no finite subcover from $\{U_\alpha\}_{\alpha\in\mathcal{A}}$. Say that $[0, 1/2]$ has no finite subcover. Again, one of $[0, 1/4]$, $[1/4, 1/2]$ has no finite subcover from $\{U_\alpha\}_{\alpha\in\mathcal{A}}$. Let $I_0 = [0, 1]$, $I_1 = [0, 1/2]$, and so on. As we keep splitting, the interval I_n has length $1/2^n$, and $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$. The completeness of $\mathbb{R}$ says that $\bigcap_{k=0}^{\infty} I_k \neq \varnothing$, so let $x_0 \in \bigcap_{k=0}^{\infty} I_k$. Notice that x_0 has to be in some U_{α_0}, which is open. So, there exists $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq U_{\alpha_0}$. However, $B_\varepsilon(x_0)$ contains I_n when $n \gg N$. Hence, $I_n \subseteq U_{\alpha_0}$, which is a contradiction, since from the construction, I_n does not have a finite subcover from $\{U_\alpha\}_{\alpha\in\mathcal{A}}$. Thus, $[0, 1]$ is compact. $\square$

Lecture 3

August 31

3.1 Derivatives

Munkres, Chapter 2.

If $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in U$, then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Theorem 3.1 (Mean Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ such that f is differentiable on (a, b) and continuous on $[a, b]$, then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Sketch of the Proof. Say $f(b) = f(a)$. We need to show that there exists $c \in (a, b)$ such that $f'(c) = 0$. Check that if $c \in (a, b)$ is a maximum or a minimum of f, then $f'(c) = 0$. □

Now, suppose that $U \subseteq \mathbb{R}^m$ is open and $f : U \to \mathbb{R}$.

Definition 3.2. Given $x_0 \in U$ and v is some vector in $\mathbb{R}^n$, then the **directional derivative of f at x_0 along v** is

$$f'(x_0, v) = \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}.$$

Example 3.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$f(x, y) = \begin{cases} x^2y/(x^4 + y^2), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

f is not continuous at $(x, y) = (0, 0)$. If we take $x = y = t \to 0$,

$$f(t, t) = \frac{t^3}{t^4 + t^2} = \frac{t}{t^2 + 1} \to 0 = f(0, 0).$$

If we let $x = t, y = t^2 \xrightarrow{t\to 0} (0,0)$,

$$f(t,t) = \frac{t^4}{t^4+t^4} = \frac{1}{2} \not\to f(0,0).$$

However, if

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

then:

$$\begin{aligned}
f'((0,0),v) &= \lim_{h\to 0} \frac{f(hv) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{(hv_1)^2(hv_2)/((hv_1)^4 + (hv_2)^2) - 0}{h} \\
&= \lim_{h\to 0} \frac{h^3 v_1^2 v_2}{h(h^4 v_1^4 + h^2 v_2^2)} \\
&= \lim_{h\to 0} \frac{v_1^2 v_2}{h^2 v_1^4 + v_2^2} = \begin{cases} v_1^2/v_2, & v_2 \neq 0 \\ 0, & v_2 = 0 \end{cases}
\end{aligned}$$

All directional derivatives of f at $(0,0)$ exist.

Recall that graph $f = \{(x, f(x)) \mid x \in U\} \subseteq \mathbb{R}^m \times \mathbb{R}$.

Definition 3.4. f is **differentiable at** x_0 if there exists an $1 \times m$ matrix B such that the limit

$$\lim_{\|h\|\to 0} \frac{f(x_0 + h) - f(x_0) - Bh}{\|h\|} = 0,$$

and B is called the **derivative of** f **at** x_0.

Theorem 3.5. *If f is differentiable at x_0 then all directional derivatives of f at x_0 exist and are equal to*

$$f'(x_0, v) = \underbrace{B}_{1\times m}\underbrace{v}_{m\times 1}.$$

Notation: B is denoted by $Df(x_0)$ or df_{x_0}.

Proof of 3.5. If $h = cv$, then

$$\begin{aligned}
f'(x_0, v) &= \lim_{c\to 0} \frac{f(x_0 + cv) - f(x_0)}{c} \\
&= \lim_{\|h\|\to 0} \underbrace{\frac{f(x_0 + h) - f(x_0) - Bh}{c}}_{\to 0} + \frac{Bh}{c} \\
&= Bv,
\end{aligned}$$

since $h/c = v$. □

Notation:

$$\frac{\partial f}{\partial x_i}(x_0) = f'(x_0, e_i).$$

Theorem 3.6. *If $f : U \to \mathbb{R}$ is such that $\dfrac{\partial f}{\partial x_i}$ is continuous in U for all $i = 1, \ldots, m$, then f is differentiable in U.*

Proof. Check that if

$$B = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}(x_0) & \ldots & \dfrac{\partial f}{\partial x_m}(x_0) \end{bmatrix},$$

then

$$\lim_{\|h\| \to 0} \frac{f(x_0 + h) - f(x_0) - Bh}{\|h\|} = 0.$$

Write

$$\begin{aligned} f(x_0 + h) - f(x_0) = (f(x_0 + h_1 e_1) - f(x_0)) + (f(x_0 + h_1 e_1 + h_2 e_2) - f(x_0 + h_1 e_1)) + \cdots \\ + (f(x_0 + h) - f(x_0 + h_1 e_1 + \cdots + h_{m-1} e_{m-1})). \end{aligned}$$

By the MVT 5.3, there exists $c_1 \in (0, h_1)$ such that

$$f(x_0 + h_1 e_1) - f(x_0) = h_1 \frac{\partial f}{\partial x_1}(x_0 + c_1 e_1).$$

Similarly, there exists $c_1 \in (0, h_2)$ such that

$$f(x_0 + h_1 e_1 + h_2 e_2) - f(x_0 + h_1 e_1) = h_2 \frac{\partial f}{\partial x_2}(x_0 + h_1 e_1 + c_2 e_2).$$

Now, using the continuity of $\dfrac{\partial f}{\partial x_i}$ at x_0 and letting

$$h = \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} \to 0,$$

then

$$f(x_0 + h) - f(x_0) - Bh = \sum_{i=1}^{m} h_i \left(\frac{\partial f}{\partial x_i}(x_0 + h_1 e_1 + \cdots + c_i e_i) - \frac{\partial f}{\partial x_i}(x_0) \right).$$

Hence,

$$\lim_{\|h\| \to 0} \frac{f(x_0 + h) - f(x_0) - Bh}{\|h\|} = 0.$$ □

Definition 3.7. C^1 is the set of continuous functions whose partial derivatives are continuous (by 3.6, these functions are differentiable). C^r is the set of continuous functions whose partial derivatives are in C^{r-1}.

Theorem 3.8. *If $f \in C^2$, then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Proof. The statement allows us to consider $f : \mathbb{R}^2 \to \mathbb{R}$ in the two variables (x_1, x_2). Let us look at the following:

$$\begin{aligned} u(h_1, h_2) &= \underbrace{f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2)}_{\phi_{a_1+h_1,a_2}(h_2)} - \underbrace{(f(a_1, a_2 + h_2) - f(a_1, a_2))}_{\phi_{a_1,a_2}(h_2)} \\ &= \phi_{a_1+h_1,a_2}(h_2) - \phi_{a_1,a_2}(h_2). \end{aligned}$$

There exists $c_2 \in (0, h_2)$ such that

$$\phi_{a_1+h_1,a_2}(h_2) = h_2 \frac{\partial f}{\partial x_2}(a_1 + h_1, a_2 + c_2).$$

Similarly, there exists $c_2' \in (0, h_2)$ such that

$$\phi_{a_1,a_2}(h_2) = h_2 \frac{\partial f}{\partial x_2}(a_1, a_2 + c_2').$$

Hence,

$$u(h_1, h_2) = h_2 \left(\frac{\partial f}{\partial x_2}(a_1 + h_1, a_2 + c_2) - \frac{\partial f}{\partial x_2}(a_1, a_2 + c_2') \right). \qquad \square$$

Lecture 4

September 5

4.1 Interchanging the Order of Partial Differentiation

Last time: $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at x_0 if there exists $Df(a)$, a $n \times m$ matrix (a linear map) such that

$$\lim_{\|h\| \to 0} \frac{f(a+h) - f(a) - Df(a)h}{\|h\|} = 0.$$

Theorem: If f is continuous, the partial derivatives $\dfrac{\partial f}{\partial x_i}$ exist and are also continuous, then f is differentiable.

We say $f \in C^1$ if Df exists and all of the entries of Df are continuous. Similarly, $f \in C^r$ if $Df \in C^{r-1}$. If $f \in C^\infty$, we call it a **smooth** map.

Theorem: If $f \in C^2$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Proof of 3.8. Consider $x = a$. It is sufficient to consider f as a two-variable function $f(x_1, x_2)$. Say $a = (a_1, a_2)$. Consider $u(h,k) = f(a_1 + h, a_2 + k) - f(a_1 + h, a_2) - f(a_1, a_2 + k) + f(a_1, a_2)$. Define $\phi(h) = f(a_1 + h, a_2 + k) - f(a_1 + h, a_2)$. So, notice

$$\begin{aligned} \phi(h) &= f(a_1 + h, a_2 + k) - f(a_1 + h, a_2), \\ \phi(0) &= f(a_1, a_2 + k) - f(a_1, a_2), \end{aligned}$$

so

$$u(h,k) = \phi(h) - \phi(0).$$

There exists $c_1 \in (0, h)$ so that

$$\begin{aligned} &= h\phi'(c_1) \\ &= h\Big(\frac{\partial f}{\partial x_1}(a_1 + c_1, a_2 + k) - \frac{\partial f}{\partial x_1}(a_1 + c_1, a_2)\Big) \end{aligned}$$

and there exists $c_2 \in (0, k)$ so that

$$= hk \frac{\partial^2 f}{\partial x_2 \partial x_1}(a_1 + c_1, a_2 + c_2).$$

Now, writing

$$u(h, k) = f(a_1 + h, a_2 + k) - f(a_1, a_2 + k) - f(a_1 + h, a_2) + f(a_1, a_2)$$

there exists $d_1 \in (0, h)$, $d_2 \in (0, k)$ so that

$$= hk \frac{\partial^2 f}{\partial x_1 \partial x_2}(a_1 + d_1, a_2 + d_2).$$

As $(h, k) \to (0, 0)$, then $(c_1, c_2) \to (0, 0)$, $(d_1, d_2) \to (0, 0)$, i.e. $(a_1 + c_1, a_2 + c_2) \to (a_1, a_2)$ and $(a_1 + d_1, a_2 + d_2) \to (a_1, a_2)$. Hence, by continuity of $\frac{\partial^2 f}{\partial x_1 \partial x_2}$ and $\frac{\partial^2 f}{\partial x_2 \partial x_1}$, we have

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(a) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(a). \qquad \square$$

4.2 Linear Maps

$L : \mathbb{R}^m \to \mathbb{R}^n$ is **linear** if

- $L(x + y) = L(x) + L(y)\ \forall x, y \in \mathbb{R}^m$,
- $L(\alpha x) = \alpha L(x)\ \forall x \in \mathbb{R}^m, \alpha \in \mathbb{R}$.

To determine L, it is sufficient to know $Le_1, \dots, Le_m$. The matrix representation is a $n \times m$ matrix,

$$L = \begin{bmatrix} Le_1 & Le_2 & \cdots & Le_m \end{bmatrix}.$$

If $x = x_1 e_1 + \cdots + x_m e_m$, then

$$\begin{aligned} L(x) &= L(x_1 e_1 + \cdots + x_m e_m) \\ &= x_1 L(e_1) + \cdots + x_m L(e_m) \\ &= \begin{bmatrix} Le_1 & \cdots & Le_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \\ &= \underbrace{Lx}_{\text{matrix-vector multiplication}}. \end{aligned}$$

Given linear maps $L_1 : \mathbb{R}^m \to \mathbb{R}^n$ and $L_2 : \mathbb{R}^n \to \mathbb{R}^p$, then $L_2 \circ L_1 : \mathbb{R}^m \to \mathbb{R}^p$ is also linear. Its matrix representation is a $p \times m$ matrix,

$$L_2 \circ L_1 = \underbrace{\overbrace{L_2}^{p\times n} \overbrace{L_1}^{n\times m}}_{\text{matrix multiplication}}.$$

If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map, $f(x) = Lx$, L is an $n \times m$ matrix. $Df(a) = L$.

$$\lim_{\|h\| \to 0} \frac{f(a + h) - f(a) - Df(a)h}{\|h\|} \overset{f=L}{=} \lim_{\|h\| \to 0} \frac{Lh - Df(a)h}{\|h\|} = \big(L - Df(a)\big) \frac{h}{\|h\|} = 0.$$

So, $h/\|h\|$ is a unit vector in $\mathbb{R}^m$ (i.e. in the unit sphere S^{m-1}). Since $L - Df(a)$ is linear and the action on S^{m-1} is the 0 map, $Df(a) = L$.

4.3 Chain Rule

Theorem 4.1 (Chain Rule)**.** *Suppose $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ are open. Given $f : A \to \mathbb{R}^n$, $g : B \to \mathbb{R}^p$ and $a \in A$ such that $f(a) \in B$, assume that f is differentiable at a and g is differentiable at $f(a)$. Then, $g \circ f$ is differentiable at a with*

$$D(g \circ f)(a) = \overbrace{Dg\big(f(a)\big)}^{p \times n} \overbrace{Df(a)}^{n \times m}.$$

Proof. We want to show

$$\lim_{\|h\| \to 0} \frac{(g \circ f)(a+h) - (g \circ f)(a) - Dg(f(a))Df(a)h}{\|h\|} = 0.$$

$(g \circ f)(a+h) - (g \circ f)(a) = g(f(a+h)) - g(f(a))$, so denote $\Delta h = f(a+h) - f(a)$. Because f is differentiable at a, f is continuous at a. Hence, as $\|h\| \to 0$, $\|\Delta h\| \to 0$. Since g is differentiable at $b = f(a)$, if we define

$$G(k) = \begin{cases} \dfrac{g(b+k) - g(b) - Dg(b)k}{\|k\|}, & \text{when } k \neq 0 \\ 0, & \text{when } k = 0 \end{cases}$$

then G is continuous at $k = 0$ (by differentiability of g at b).

$$\frac{g(b+k) - g(b)}{\|k\|} = G(k) + Dg(b)\frac{k}{\|k\|}$$

and the norm is bounded by some M when $\|k\|$ is small, since G is continuous at b and S^{m-1} is compact so $Dg(b)(S^{m-1})$ is bounded. For $f(a) = b$,

$$\begin{aligned} \frac{(g \circ f)(a+h) - (g \circ f)(a) - Dg(f(a))Df(a)h}{\|h\|} &= \frac{g(b + \Delta h) - g(b) - Dg(b)Df(a)h}{\|h\|} \\ &= \frac{(G(\Delta h) + Dg(b)\Delta h/\|\Delta h\|)\|\Delta h\| - Dg(b)Df(a)h}{\|h\|} \\ &= \underbrace{G(\Delta h)}_{\xrightarrow{h \to 0} G(0)} \underbrace{\frac{\|\Delta h\|}{\|h\|}}_{\text{bounded}} + Dg(b)\frac{\Delta h}{\|h\|} - Dg(b)Df(a)\frac{h}{\|h\|}. \end{aligned}$$

Since f is differentiable at a, if we define

$$F(h) = \begin{cases} \dfrac{f(a+h) - f(a) - Df(a)h}{\|h\|}, & \text{when } h = 0 \\ 0, & \text{when } h = 0 \end{cases}$$

then F is continuous at $h = 0$ (by differentiability of f at a). So,

$$\frac{\Delta h}{\|h\|} = \frac{f(a+h) - f(a)}{\|h\|} = F(h) + Df(a)\frac{h}{\|h\|}$$

and again the norm is bounded by some M when $\|h\|$ is small. Hence, as $\|h\| \to 0$,

$$G(\Delta h)\frac{\Delta h}{\|h\|} \to 0 \qquad \text{(by the Squeeze Theorem).}$$

Also,

$$Dg(b)\frac{\Delta h}{\|h\|} = Dg(b)\Big(F(h) + Df(a)\frac{h}{\|h\|}\Big)$$

and hence

$$Dg(b)\frac{\Delta h}{\|h\|} - Dg(b)Df(a)\frac{h}{\|h\|} = Dg(b)\underbrace{\quad F(h) \quad}_{\xrightarrow{h\to 0} F(0)=0} \xrightarrow{h\to 0} 0.$$

Hence,

$$\lim_{\|h\|\to 0} \frac{(g\circ f)(a+h) - (g\circ f)(a) - Dg(f(a))Df(a)h}{\|h\|} = 0.$$

By definition, $g \circ f$ is differentiable at a with $D(g\circ f)(a) = Dg(f(a))Df(a)$. □

4.4 Inverse Function Theorem

Consider the special case $f : A \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$ is open, and $g : B \to \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$ is open, and g is the inverse of f, i.e. $g \circ f = \mathrm{id}$.

Remark: It is not true that if f is differentiable at a, then g is differentiable at $f(a)$.

Example 4.2. If $f(x) = x^3$, $g(x) = x^{1/3}$, g is not differentiable at $f(0) = 0$.

Proposition 4.3. *Assume that g is differentiable at $f(a)$. Then, $Dg(f(a)) = (Df(a))^{-1}$.*

Proof. By the Chain Rule 4.1, $g \circ f = \mathrm{id}$ implies

$$\begin{aligned} D(g\circ f) &= \mathrm{id} \\ Dg\big(f(a)\big)Df(a) &= \mathrm{id} \implies Dg\big(f(a)\big) = \big(Df(a)\big)^{-1}. \end{aligned}$$

□

Theorem 4.4 (Inverse Function Theorem)**.** *Let $A \subseteq \mathbb{R}^n$ be open, $f : A \to \mathbb{R}^n$, $f \in C^r$. Assume that $Df(a)$ is non-singular for some $a \in A$. Then, there exists an open set U' around a and an open set V' around $f(a)$ such that $f : U' \to V'$ is bijective with inverse $g : V' \to U'$ in C^r.*

Lecture 5

September 7

5.1 Proof of Inverse Function Theorem

Theorem: Let $A \subseteq \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^n$ be a C^r map. Let $a \in A$ such that $Df(a)$ is *non-singular*. Then, there exists an open set $U \subseteq A$, $a \in U$, such that $f : U \to V$, where $V \subseteq \mathbb{R}^n$ is open and $f(a) \in V$, has an inverse $g : V \to U$ and g is C^r.

Example 5.1. Consider the map $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2$, where $\mathbb{R}_+ = (0, +\infty)$, given by:

$$(r, \theta) \overset{f}{\mapsto} \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$$

Let $a = (r_0, \theta_0)$.

$$Df(a) = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta_0 & -r_0 \sin\theta_0 \\ \sin\theta_0 & r_0\cos\theta_0 \end{bmatrix}$$

is non-singular because the determinant is $r_0((\cos\theta_0)^2 + (\sin\theta_0)^2) = r_0 > 0$.

Remark: The IFT 4.4 gives a local picture, not a global one. Here, f may fail to be one-to-one (as in the example).

Definition 5.2. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear. L is **non-singular** if $\det L \neq 0$.

Properties: The following are equivalent.

- $\det L \neq 0$.
- $\{Le_1, \ldots, Le_n\}$ are linearly independent.
- L has an inverse L^{-1}.
- $\ker L = \{0\}$, i.e. $Lx = 0$ iff $x = 0$.

Proof of 4.4. Step 1: f is locally one-to-one, i.e. there exists open $U \subseteq A$, $a \in U$, such that $f|_U$ is a one-to-one map. I want to show that there exists $\alpha > 0$ and an open set $U \subseteq A$, $a \in U$, such that for all $x_1, x_2 \in U$, $|f(x_1) - f(x_2)| \geq \alpha|x_1 - x_2|$. Here, we will use the MVT 5.3. So first, since $Df(a)$ is non-singular and $f \in C^r$, we also have that Df is non-singular in a neighborhood U of a. If we have a non-singular matrix L,

$$\frac{\|Lh\|}{\|h\|} > \alpha > 0, \qquad h \neq 0,$$

since $L(\{\text{set of unit vectors}\})$ is a bounded set in $\mathbb{R}^n$ and so there must be a minimum norm in the set $L(\{\text{unit vectors}\})$. Since L is non-singular, there exists $\alpha > 0$ such that $\|L(\text{unit vector})\| \geq \alpha$. Here, $Df(a)$ is non-singular, so there exists $\alpha > 0$ such that $\|Df(a)(\text{unit vector})\| \geq \alpha$. We can choose U open, $a \in U$, small enough such that

$$\forall x \in U, \qquad \|Df(x)(\text{unit vector})\| \geq \frac{\alpha}{2}. \tag{5.1}$$

Now, apply the MVT 5.3. We get, for all $x_1, x_2 \in U$,

$$\frac{\|f(x_1) - f(x_2)\|}{\|x_1 - x_2\|} \geq \frac{\alpha}{2}.$$

Hence, f is one-to-one in U.

Step 2: f is locally onto, i.e. there exists open $U \subseteq A$, $a \in U$, and open $V \subseteq \mathbb{R}^n$, $f(a) \in V$, such that $f : U \to V$ is onto. By Step 1, we can choose U such that $f|_U$ is one-to-one. You can think of $U = B_\varepsilon(a)$. Consider $B_{\varepsilon/2}(a)$ so that $f|_{\overline{B_{\varepsilon/2}(a)}}$ is one-to-one. Notice that the boundary $S_{\varepsilon/2}(a)$ of $\overline{B_{\varepsilon/2}(a)}$ is compact, and hence $f(S_{\varepsilon/2}(a))$ is compact in $\mathbb{R}^n$. Moreover, $f|_{\overline{B_{\varepsilon/2}(a)}}$ is one-to-one, $f(a) \notin f(S_{\varepsilon/2}(a))$, and hence $\operatorname{dist}(f(S_{\varepsilon/2}(a)), f(a)) \geq \delta > 0$. Let $V = B_{\delta/2}(f(a))$ and we claim that for all $c \in V$, there exists $x_0 \in U$ such that $f(x_0) = c$. The idea is that x_0 is the minimizer of $\|f(x) - c\|^2$, where $x \in U$. $\overline{B_{\varepsilon/2}(a)}$ is compact. Hence, $g(x) = \|f(x) - c\|^2$ has a minimum x_0 in $\overline{B_{\varepsilon/2}(a)}$. The minimum can be in one of the two cases.

- x_0 is in the boundary of $\overline{B_{\varepsilon/2}(a)}$. Then, $\|f(x_0) - f(a)\| > \delta$ $(\operatorname{dist}(f(S_{\varepsilon/2}(a)), f(a)) \geq \delta)$ and $\|c - f(a)\| < \delta/2$. Hence, $\|f(x_0) - c\| \geq \|f(x_0) - f(a)\| - \|f(a) - c\| > \delta/2$. Notice that this is contradictory because $a \in B_{\varepsilon/2}(a)$, $\|f(a) - c\| < \delta/2 < \|f(x_0) - c\|$. So, x_0 is not the minimizer if x_0 is on the boundary.
- $g(x) = \|f(x) - c\|^2$, and we must have $Dg(x_0) = 0$.

$$Dg(x_0) = 2Df(x_0)\big(f(x_0) - c\big) = 0 \overset{Df(x_0) \text{ is non-singular}}{\Longrightarrow} f(x_0) - c = 0 \implies f(x_0) = c.$$

Step 1: f is locally one-to-one.

Step 2: f is locally onto.

Step 3: So far, we showed that there exists open $U \subseteq A$, $a \in U$, and open $V \subseteq \mathbb{R}^n$ such that $f(a) \in V$ and $f|_U : U \to V$ is a bijective map. Hence, there exists an inverse $g : V \to U$ of f. First, check that g is continuous. Then, check that g is differentiable and the derivative is continuous, i.e. $g \in C^1$. Then, iterate using $f \in C^r$ to get $g \in C^r$.

g is continuous. $g : V \to U$ and notice that f is an open map, i.e. whenever $\tilde{U} \subseteq U$ is open, $f(\tilde{U})$ is open in V (this is exactly what we proved in Step 1 and Step 2). So, if $\tilde{U} \subseteq U$ is open, $g^{-1}(\tilde{U})$ is open because g is the inverse of f. This is exactly the condition for g to be continuous in V. ☐

Theorem 5.3 (MVT)**.** *Suppose U is convex and f is differentiable in U. Given $x_1, x_2 \in U$, there exists c on the segment $[x_1, x_2]$ such that $f(x_1) - f(x_2) = Df(c)(x_1 - x_2)$.*

Lecture 6

September 12

6.1 Proof of Inverse Function Theorem

Theorem (Inverse Function Theorem): Given $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$, where A is open and f is C^r, and $a \in A$ such that $Df(a)$ is non-singular, then there exists open $U \subseteq A$, $a \in U$, and open $V \subseteq \mathbb{R}^n$, $f(a) \in V$, such that $f : U \to V$ is bijective with inverse $g : V \to U$ also in C^r.

Proof of 4.4, Continued. Step 1: There exists open $U \subseteq A$, $a \in U$, such that $f|_U$ is one-to-one. The main idea is the MVT 5.3. There exists $c \in [x_1, x_2]$ such that $f(x_1) - f(x_2) = Df(c)(x_1 - x_2)$. So, $\|f(x_1) - f(x_2)\| = \|Df(c)(x_1 - x_2)\|$.

Size of $Df(c)$: If $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then let $\operatorname{size} L = \min_{\|x\|=1} \|Lx\|$. If L is non-singular, then $\operatorname{size} L > \alpha > 0$. $Df(a)$ is non-singular, then there exists $\alpha > 0$ so that $\operatorname{size} Df(a) > \alpha > 0$. Df is continuous, so there exists open convex U, $a \in U$, such that $\operatorname{size} Df(x) > \alpha/2 > 0$ for $x \in U$, so

$$\|f(x_1) - f(x_2)\| \geq \frac{\alpha}{2}\|x_1 - x_2\|,$$

which implies that if $f(x_1) = f(x_2)$, then

$$\frac{\alpha}{2}\|x_1 - x_2\| = 0,$$

i.e., $x_1 = x_2$.

Step 2: Given open $U \subseteq A$, $a \in U$, there exists $V \subseteq \mathbb{R}^n$, $f(a) \in V$, such that for all $c \in V$, there exists $x_0 \in U$ such that $f(x_0) = c$. We will show that $V = B_{\delta/2}(f(a))$ satisfies the desired property. Given $c \in V$, there exists $x_0 \in B_{\varepsilon/2}(a) \subseteq U$ such that $f(x_0) = c$. Define $h : \overline{B_{\varepsilon/2}(a)} \to \mathbb{R}$ by $h(x) = \|f(x) - c\|^2$. Since $\overline{B_{\varepsilon/2}(a)}$ is compact, there exists x_0 such that $h(x_0)$ is a minimum.

Case 1: If $x_0 \in B_{\varepsilon/2}(a)$, then $Dh(x_0) = c$, so $Df(x_0)(f(x_0) - c) = 0$. Since $Df(x_0)$ is non-singular (by the choice of ε), we have $f(x_0) = c$.

Case 2: If $x_0 \in \operatorname{bd} \overline{B_{\varepsilon/2}(a)}$, we need to show that if $c \in B_{\delta/2}(f(a))$, the Triangle Inequality tells you that $h(x_0) > h(a)$.

$$\begin{aligned} \sqrt{h(x_0)} &= \|f(x_0) - c\| \\ &\geq \|f(x_0) - f(a)\| - \|f(a) - c\| \end{aligned}$$

$$\begin{aligned} &\geq \delta - \frac{\delta}{2} \\ &= \frac{\delta}{2} > \|f(a) - c\| \\ &= \sqrt{h(a)}. \end{aligned}$$

Step 3: Take open $U \subseteq A$, $a \in U$, in Step 1 (i.e. for all $x \in U$, $Df(x)$ has size $> \alpha/2$). Let $V = f(U)$. We claim V is open. Given $y_0 = f(x_0) \in f(U)$, we need to show that there exists δ such that $B_\delta(y_0) \subseteq f(U)$. This is exactly Step 2 with "a" replaced by "x_0". So, $V = f(U)$ is open. $f : U \to V$ is a bijection with inverse $g : V \to U$.

Step 4: g is continuous (i.e. $g \in C^0$), i.e. if $\tilde{U} \subseteq U$ is open, then $g^{-1}(\tilde{U})$ is open in V, i.e. if $\tilde{U} \subseteq U$ is open, then $f(\tilde{U})$ is open in V. This is exactly what we showed in Step 3. Hence, g is continuous.

Step 5: $g \in C^r$. First, check that g is differentiable with $Dg = (Df)^{-1}$. Given $y_0 \in V$, we need to show

$$\lim_{\|k\| \to 0} \frac{g(y_0 + k) - g(y_0) - (Df)^{-1}(x_0)k}{\|k\|} = 0.$$

Because g is continuous, $g(y_0 + k) = x_0 + \Delta k$, where $\|\Delta k\| \to 0$ as $\|k\| \to 0$. This is equivalent to

$$\lim_{\|k\| \to 0} Df(x_0)\Big(\frac{x_0 + \Delta k - x_0 - (Df)^{-1}(x_0)k}{\|k\|}\Big) = Df(x_0)0,$$

so equivalently, we need to show

$$\lim_{\|k\| \to 0} \frac{Df(x_0)\Delta k - k}{\|k\|} = 0.$$

Therefore,

$$\lim_{\|k\| \to 0} \frac{Df(x_0)\Delta k - k}{\|\Delta k\|} \frac{\|\Delta k\|}{\|k\|} = \lim_{\|k\| \to 0} \frac{Df(x_0)\Delta k - (f(x_0 + \Delta k) - f(x_0))}{\|\Delta k\|} = 0,$$

since $k = f(x_0 + \Delta k) - f(x_0)$, $\|\Delta k\| \to 0$ as $\|k\| \to 0$, and

$$\frac{\|\Delta k\|}{\|k\|} = \frac{\|x_0 + \Delta k - x_0\|}{\|f(x_0 + \Delta k) - f(x_0)\|} \overset{\text{Step 1}}{\leq} \frac{2}{\alpha} \qquad \text{is bounded.}$$

Hence, the limit is 0 and thus g is differentiable at y_0 with $Dg(y_0) = (Df(x_0))^{-1}$. Since the entries of the inverse can be written as a smooth function of the entries of the original matrix, $f \in C^r \implies g \in C^r$. □

6.2 Implicit Function Theorem

Suppose $f : U \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^n$, where U is open, and the first k dimensions correspond to x and the last n dimensions correspond to y. We are interested in the level set $f^{-1}(0)$. If

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix},$$

then $f^{-1}(0) = \{(x, y) : f_1(x, y) = \cdots = f_n(x, y) = 0\}$. For example, if $f(x, y) = x^2 + y^2 - 1$, $f : \mathbb{R}^2 \to \mathbb{R}$, then $f^{-1}(0)$ is the unit circle. Is the circle the graph of some $\{y = g(x)\}$? Locally, at $(0, 1)$, $y = \sqrt{1 - x^2}$. Locally, at $(1, 0)$, although the circle is not the graph of any $\{y = g(x)\}$, it is the graph of $\{x = h(y)\}$.

Concretely, $x = \sqrt{1-y^2}$.

Theorem 6.1 (Implicit Function Theorem)**.** *If $f : U \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^n$ is C^r, where U is open, $(x_0, y_0) \in U$ and $(x_0, y_0) \in f^{-1}(0)$, then if*

$$\underbrace{\frac{\partial f}{\partial y}(x_0, y_0)}_{n\times n \ matrix}$$

is non-singular, there exists a C^r map $g : A \subseteq \mathbb{R}^k \to \mathbb{R}^n$, where A is open and $x_0 \in A$, such that $f(x, g(x)) = 0$ and $g(x_0) = y_0$, i.e. the level set $f^{-1}(0)$ is the graph of $y = g(x)$ near (x_0, y_0).

Proof. Next time. □

For now, we will compute $\frac{\partial g}{\partial x}$ in terms of Df given that g exists. Since $f(x, g(x)) = 0$, consider

$$\mathbb{R}^k \xrightarrow{G} \mathbb{R}^{k+n} \xrightarrow{f} \mathbb{R}^n,$$

where $x \overset{G}{\mapsto} (x, g(x))$. Notice that $f \circ G = 0$. The Chain Rule 4.1 implies that

$$\begin{aligned} D(f \circ G)(x_0) &= 0, \\ Df\big(G(x_0)\big)DG(x_0) &= 0, \end{aligned}$$

with

$$DG(x_0) = \begin{bmatrix} \mathrm{id}_k \\ Dg(x_0) \end{bmatrix}$$

where id_k is the $k \times k$ matrix $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$. So,

$$\begin{aligned} 0 = Df\big(G(x_0)\big)DG(x_0) &= \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \mathrm{id}_k \\ Dg(x_0) \end{bmatrix} \\ &= \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) Dg(x_0). \end{aligned}$$

If $\frac{\partial f}{\partial y}(x_0, y_0)$ is non-singular, then

$$Dg(x_0) = -\Big(\frac{\partial f}{\partial y}(x_0, y_0)\Big)^{-1} \frac{\partial f}{\partial x}(x_0, y_0).$$

Lecture 7

September 14

7.1 Proof of Implicit Function Theorem

Theorem (Implicit Function Theorem): Let $f : A \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^n$ be C^r, where A is open, x denotes the first k coordinates, and y denotes the last n coordinates. Consider the level set $S = \{f = 0\}$ and let $(x_0, y_0) \in S$ (i.e., $f(x_0, y_0) = 0$). Assume that $\dfrac{\partial f}{\partial y}(x_0, y_0)$ is non-singular, where $\dfrac{\partial f}{\partial y}$ is the $n \times n$ matrix given by

$$Df = {}_n\left\{ \overbrace{\begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix}}^{k+n} \right. .$$

Then, there exists open $U \subseteq \mathbb{R}^k$, $x_0 \in U$, and a C^r-map $g : U \to \mathbb{R}^n$ such that S near (x_0, y_0) can be written as $y = g(x)$.

Example 7.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2 - 1$, $Df(x, y) = \begin{bmatrix} 2x & 2y \end{bmatrix}$. If $y_0 \neq 0$, then

$$\frac{\partial f}{\partial y}(x_0, y_0) = 2y_0 \neq 0.$$

Example 7.2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^3$, $Df(x, y) = \begin{bmatrix} 2x & -3y^2 \end{bmatrix}$. Then,

$$\frac{\partial f}{\partial y}(x, y) = -3y^2,$$

so at $y = 0$,

$$\frac{\partial f}{\partial y}(0, 0) = 0.$$

Still, $\{f = 0\}$ can be written as the graph of

$$y = \underbrace{\sqrt[3]{x^2}}_{g(x)},$$

but g is not differentiable at $x = 0$.

Proof of 6.1. Consider $F : A \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^{k+n}$ given by $F(x, y) = (x, f(x, y))$.

$$DF = \begin{bmatrix} \mathrm{id}_k & 0 \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{bmatrix}.$$

Since $\dfrac{\partial f}{\partial y}(x_0, y_0)$ is a $n \times n$ non-singular matrix, $DF(x_0, y_0)$ is a $(k+n) \times (k+n)$ non-singular matrix:

$$\det DF(x_0, y_0) = \det \frac{\partial f}{\partial y}(x_0, y_0) \neq 0.$$

The Inverse Function Theorem 4.4 implies that there exists open $\tilde{U} \subseteq A$, $(x_0, y_0) \in \tilde{U}$, and open $W \subseteq \mathbb{R}^{k+n}$, $(x_0, f(x_0, y_0)) \in W$, such that $F : \tilde{U} \to W$ is bijective, with a C^r-inverse $G : W \to \tilde{U}$. Here, we can choose $\tilde{U} = U \times V$, where U is open in $\mathbb{R}^k$, V is open in $\mathbb{R}^n$, $x_0 \in U$, and $y_0 \in V$. Now consider $g : U \to S \to \mathbb{R}^n$, where $g(x)$ is the y-coordinate of $G(x, 0)$. Since G is in C^r, g is also in C^r, and $\tilde{U} \cap S$ is exactly $\{y = g(x)\}$ where $x \in U$. $\square$

7.2 Smooth Manifolds

Definition 7.3. Let $X \subseteq \mathbb{R}^N$ is any subset. A map $f : X \to \mathbb{R}^n$ is **smooth** if for every $x_0 \in X$, there exists U open in $\mathbb{R}^N$ containing x_0 and an **extension** $\tilde{f} : U \to \mathbb{R}^n$ (i.e., $\tilde{f}|_{U \cap X} = f$) such that $\tilde{f}$ is smooth.

Definition 7.4. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. $f : X \to Y$ is a **diffeomorphism** if f is a bijective map and $f : X \to \mathbb{R}^m$ is a smooth map and the inverse $f^{-1} : Y \to X \subseteq \mathbb{R}^n$ is also a smooth map.

Definition 7.5. A **k-dimensional manifold** $X \subseteq \mathbb{R}^N$ is locally diffeomorphic to an open set in $\mathbb{R}^k$, i.e., $\forall x \in X$, there exists open $V \subseteq \mathbb{R}^N$, $x \in V$, and a diffeomorphism $\phi : U \subseteq \mathbb{R}^k \to V \cap X$, where U is open.

Example 7.6. Let $X = \{x^2 + y^2 - 1 = 0\}$. The notation for a circle is S^1. Given $(x_0, y_0) \in X$ in the lower half of the circle, we claim that there exists $\phi : (x_0 - \varepsilon, x_0 + \varepsilon) \subseteq \mathbb{R}^1 \to X \cap V$ and ϕ is a diffeomorphism. Take

$$\phi(t) = (t, -\sqrt{1 - t^2}).$$

ϕ is a diffeomorphism because each coordinate function is smooth. Also, the inverse

$$\phi^{-1} : X \cap V \to (x_0 - \varepsilon, x_0 + \varepsilon)$$

can be extended as the projection map $(x, y) \mapsto x$.

Example 7.7. Let $X = \{x^2 + y^2 + z^2 - 1 = 0\} \subseteq \mathbb{R}^3$. The notation is S^2. X is a manifold.

Lecture 8

September 19

8.1 Smooth Manifolds

Definition: $X \subseteq \mathbb{R}^N$ is a **(smooth)** k**-dimensional manifold** if for all $x \in X$, there exists open $\tilde{V} \subseteq \mathbb{R}^N$ and open $U \in \mathbb{R}^k$, $0 \in U$, and a diffeomorphism

$$\phi : U \to \underbrace{\tilde{V} \cap X}_{V}$$

such that $\phi(0) = x$. We call V an open set around x in X. This is called a **parameterization** of X near the point x.

Example 8.1. If we consider $X = S^1$, consider a parameterization around the point

$$(x_0, y_0) = \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big).$$

One parameterization is

$$\begin{aligned} \phi_1 : \Big(\frac{1}{\sqrt{2}} - \delta, \frac{1}{\sqrt{2}} + \delta\Big) &\to X = S^1 \\ u &\mapsto (u, \sqrt{1-u^2}). \end{aligned}$$

Another parameterization is

$$\begin{aligned} \phi_2 : \Big(\frac{1}{\sqrt{2}} - \delta, \frac{1}{\sqrt{2}} - \delta\Big) &\to X = S^1 \\ v &\mapsto (\sqrt{1-v^2}, v^2). \end{aligned}$$

The **transition map** is $h_{1,2} : U_2 \to U_1$ given by $h_{1,2}(v) = \phi_1^{-1} \circ \phi_2(v) = \phi_1^{-1}((\sqrt{1-v^2}, v)) = \sqrt{1-v^2}$.

The transition map is defined only in $\phi_2^{-1}(V_1 \cap V_2)$. So, $h_{1,2} : \phi_2^{-1}(V_1 \cap V_2) \to \phi_1^{-1}(V_1 \cap V_2)$ is a diffeomorphism.

Definition 8.2. $Z \subseteq X$ is a **manifold** if $Z \subseteq \mathbb{R}^N$ is a manifold.

Example 8.3. If $X = S^2$, then Z, the equator of the sphere, is a submanifold.

If $X \subseteq \mathbb{R}^{N_1}$, $X_2 \subseteq \mathbb{R}^{N_2}$ are smooth k_1- and k_2-dimensional manifolds, then $X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} = \mathbb{R}^{N_1+N_2}$ is a (k_1+k_2)-dimensional manifold.

Example 8.4. If $X_1 \subseteq \mathbb{R}^2$, $X_2 \subseteq \mathbb{R}^2$, $X_1 = X_2 = S^1$, then $X_1 \times X_2 \subseteq \mathbb{R}^4$ is a torus.

8.1.1 Tangent Spaces

Given $X \subseteq \mathbb{R}^N$, a k-dimensional manifold, we can define T_xX, where $x \in X$, by first parameterizing a neighborhood V around x of X, $\phi : U \to V$, and the defining $T_xX = \operatorname{image} D\phi(0)$ (where we think of $D\phi(0)$ as a linear map $\mathbb{R}^k \to \mathbb{R}^N$), $\phi(0) = x$.

Example 8.5. Consider the point (x_0, y_0) on the unit circle with parameterization

$$\phi : (x_0 - \delta, x_0 + \delta) \to V$$

given by $\phi(u) = (u, \sqrt{1-u^2})$. Then,

$$D\phi(x_0) = \begin{bmatrix} 1 \\ -\dfrac{x_0}{\sqrt{1-x_0^2}} \end{bmatrix}$$

so

$$\begin{aligned} T_{(x_0,y_0)}X &= \operatorname{image} D\phi(x_0) \\ &= \operatorname{span}\{\text{columns of } D\phi(x_0)\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -\dfrac{x_0}{\sqrt{1-x_0^2}} \end{bmatrix}\right\}. \end{aligned}$$

One has to check that the definition is independent of the choice (ϕ, U, V). Say that (ϕ_1, U_1, V_1) and (ϕ_2, U_2, V_2) are chosen. Here, we can assume $V_1 = V_2$ (or else you can consider $V_1 \cap V_2$ and smaller open sets in U_1 and U_2). Hence:

$$\begin{tikzcd} U_1 \arrow[d, bend right, "h_{2,1}"'] \arrow[dr, "\phi_1"] & \\ U_2 \arrow[u, "h_{1,2}"'] \arrow[r, "\phi_2"] & V \end{tikzcd}$$

The transition maps $h_{1,2}$ and $h_{2,1}$ are diffeomorphisms.

$$\begin{aligned} h_{1,2} &= \phi_1^{-1} \circ \phi_2 \\ h_{2,1} &= \phi_2^{-1} \circ \phi_1 \end{aligned}$$

So,

$$\begin{aligned} \phi_2 &= \phi_1 \circ h_{1,2} \\ \phi_1 &= \phi_2 \circ h_{2,1} \end{aligned}$$

and by the Chain Rule 4.1 we obtain

$$\begin{aligned} D\phi_2(0) &= D\phi_1\big(h_{1,2}(0)\big) Dh_{1,2}(0) \\ &= D\phi_1(0) \quad \underbrace{Dh_{1,2}(0)}_{\text{invertible } k\times k \text{ matrix}} \quad . \end{aligned}$$

This tells you that $\operatorname{image} D\phi_2(0) = \operatorname{image} D\phi_1(0)$. The definition of T_xX is independent of (ϕ, U, V). Also, $D\phi(0)$ is injective (Homework 4) and hence T_xX is a k-dimensional subspace in $\mathbb{R}^N$.

Next we can talk about smooth maps $f : X \to Y$ and discuss Df.

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "Dh_{1,2}(0)"] \arrow[rr, bend right, "D\phi_2(0)"'] & \mathbb{R}^k \arrow[r, "D\phi_1(0)"] & \mathbb{R}^N \end{tikzcd}$$

8.2 Review

If $\mathbb{R}^m \xrightarrow{f} \mathbb{R}$, f is continuous if one of these conditions holds.

1. If $\{x_n\} \to x_0 \in \mathbb{R}^m$, then $\{f(x_n)\} \to f(x_0)$.
2. For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|x - x_0\| < \delta$, then $\|f(x) - f(x_0)\| < \varepsilon$.
3. If U is open in $\mathbb{R}$, then $f^{-1}(U)$ is open in $\mathbb{R}^m$.

Note that the existence of the partial derivatives does not imply that f is differentiable. As an example, the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

has partial derivatives at the origin, but it is not differentiable.

For $L : \mathbb{R}^n \to \mathbb{R}^n$, if L is invertible, then there exists $\alpha > 0$ such that $\|L(x)\| \geq \alpha \|x\|$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is such that $Df(x_0)$ is non-singular, then there exists open U containing x_0, $\alpha > 0$, such that

$$\|f(x_1) - f(x_2)\| \geq \alpha \|x_1 - x_2\|$$

for all $x_1, x_2 \in U$. The MVT 5.3 only applies for the case $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$:

$$f(x_1) - f(x_2) = Df(c)(x_1 - x_2), \qquad c \in [x_1, x_2],$$

but this is *not* true for $\mathbb{R}^n \to \mathbb{R}$. Instead, write

$$f(x_1) - f(x_2) = \begin{bmatrix} f_1(x_1) - f_1(x_2) \\ f_2(x_1) - f_2(x_2) \\ \vdots \\ f_n(x_1) - f_n(x_2) \end{bmatrix}$$

and so

$$\begin{aligned} f_1(x_1) - f(x_2) &= Df_1(c_1)(x_1 - x_2) \\ f_2(x_1) - f(x_2) &= Df_2(c_2)(x_1 - x_2) \\ &\vdots \end{aligned}$$

and so

$$f(x_1) - f(x_2) = \underbrace{\begin{bmatrix} Df_1(c_1) \\ Df_2(c_2) \\ \vdots \\ Df_n(c_n) \end{bmatrix}}_{\approx Df(x_0)} (x_1 - x_2).$$

Note that we can define $\|L\| = \max_{\|x\|=1} \|L(x)\|$.

Lecture 9

September 26

9.1 Tangent Spaces

Let $X \subseteq \mathbb{R}^N$ be a k-dimensional manifold and $x \in X$, with open sets $V_1 \subseteq X$, $V_2 \subseteq X$ containing x. Let $\phi_1 : \mathbb{R}^k \to V_1$ and $\phi_2 : \mathbb{R}^k \to V_2$ be diffeomorphisms. Then, ϕ_1 is called a **parameterization** and ϕ_1^{-1} is called a **coordinate function**. The **transition map** is $h_{2,1} = \phi_2^{-1} \circ \phi_1$. The transition map is a diffeomorphism because we also have the inverse $h_{1,2} = \phi_1^{-1} \circ \phi_2$.

Definition 9.1. For each $x \in X$, we choose (U, V, ϕ) such that $\phi(0) = x$. Define

$$T_x X = \text{image}\big(D\phi(0) : \mathbb{R}^k \to \mathbb{R}^N\big).$$

In Guillemin-Pollack, $D\phi(0)$ is denoted $d\phi_0$. In our old notation, we have $D\phi(0)(v)$. In the new notation, we have $d\phi_0(v)$.

Notice that $T_x X$ is well-defined, i.e., independent of the choice of (U, V, ϕ).

Example 9.2. If $X = \mathbb{R}^N$, $x \in X$, then $T_x X \subseteq \mathbb{R}^N$ is a subspace. In fact, $T_x X = \mathbb{R}^N$. The parameterization is $\phi : \mathbb{R}^N \xrightarrow{\text{id}} X$, so $d\phi_x = \text{id} : \mathbb{R}^N \to \mathbb{R}^N$. Thus, $T_x X = \text{image}\, d\phi_x = \mathbb{R}^N$.

Example 9.3. Let $X = \{z^2 - x^2 - y^2 = 1\} \subset \mathbb{R}^3$. Then, $z^2 = x^2 + y^2 + 1$. Consider the point $x = (1, 1, \sqrt{3})$. Let us compute $T_x X$. Consider an open set U near $(1, 1)$ and open $V \subseteq X$ containing $x = (1, 1, \sqrt{3})$ with $\phi : U \to V$, $(u, v) \mapsto (u, v, \sqrt{1 + u^2 + v^2})$. Note that $\phi(1, 1) = x$. So,

$$\begin{aligned} d\phi_{(1,1)} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ u/\sqrt{1+u^2+v^2} & v/\sqrt{1+u^2+v^2} \end{bmatrix}_{(1,1)} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}. \end{aligned}$$

Thus,

$$T_x X = \text{image}\, d\phi_{(1,1)} = \text{span}\{\text{column vectors of } d\phi_{(1,1)}\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/\sqrt{3} \end{bmatrix} \right\}.$$

Notice that $T_x X$ is a linear subspace (in particular $0 \in T_x X$). So, the tangent plane we usually draw is the translation of T_xX along the vector $x \in \mathbb{R}^N$.

Definition 9.4. The **tangent space of** X is $TX = \bigcup_{x\in X}(\{x\} \times T_xX) \subseteq X \times \mathbb{R}^N \subseteq \mathbb{R}^N \times \mathbb{R}^N$. Each point in TX has the form (x, v) where $x \in X$, $v \in T_xX$.

Example 9.5. Let $X = \mathbb{R}^N$. For all $x \in X$, $T_xX = \mathbb{R}^N$. So,

$$TX = \mathbb{R}^{2N} = \{(x, v) \mid x \in X, v \in T_xX\} = \{(x, v) \mid x \in \mathbb{R}^N, v \in \mathbb{R}^N\}.$$

Example 9.6. Let $X = S^1 \subset \mathbb{R}^2$. What is TX? At each point x, there is a canonical vector v_x which spans the tangent space T_xX in the counterclockwise direction.

$$\begin{aligned} TX &= \{(x, v) \mid x \in S^1, v \in T_xX \subset \mathbb{R}^2\} \\ &= \{(x, tv_x) \mid x \in S^1, v_x \text{ is my canonical vector}, t \in \mathbb{R}\}. \end{aligned}$$

We can identify this via a diffeomorphism as

$$\begin{aligned} &= \{(x, t) \mid x \in S^1, t \in \mathbb{R}\} \\ &= S^1 \times \mathbb{R} \\ &= \text{cylinder}. \end{aligned}$$

Theorem 9.7. *TX is a smooth manifold of $\mathbb{R}^N \times \mathbb{R}^N$ with dimension $2\dim X$.*

Remark: Usually $TX \ncong X \times \mathbb{R}^k$.

Example 9.8. $TS^2 \neq S^2 \times \mathbb{R}^2$. In particular, any smooth vector field on S^2 must vanish at some point.

9.2 Derivatives of Maps

Definition 9.9. Given a smooth map $f : X \to Y$, we can define $df_x : T_xX \to T_{f(x)}Y$ by

$$\begin{tikzcd} T_xX \arrow[r, "df_x := d\psi_0 \circ dh_0 \circ d\phi_0^{-1}"] & T_{f(x)}Y \\ T_0U \arrow[u, "d\phi_0", "\cong"'] \arrow[r, "d(\psi^{-1}\circ f\circ\phi)_0"] & T_0V \arrow[u, "\cong", "d\psi_0"'] \end{tikzcd}$$

where $h = \psi^{-1} \circ f \circ \phi$, the maps $\phi : U \to X$, $\psi : V \to Y$ are parameterizations of X and Y at x and $f(x)$ respectively, and $U \subseteq \mathbb{R}^{\dim X}$, $V \subseteq \mathbb{R}^{\dim Y}$ are open.

Note that df_x is a linear map. It is a good exercise to check that df_x is independent of the parameterizations (this is just a consequence of the Chain Rule 4.1).

Example 9.10. For $S^2 = \{x^2 + y^2 + z^2 = 1\}$, let $f(x, y, z) = z$ be a map from S^2 to $\mathbb{R}$. Then, $\phi : U \to V$ is a parameterization, where U is an open set in $\mathbb{R}^2$ around $(1/2, -1/2)$ and V is an open set around $x = (1/2, -1/2, 1/\sqrt{2})$, given by $(u, v) \mapsto (u, v, \sqrt{1 - u^2 - v^2})$. In U, f can be written as $h : U \to \mathbb{R}$ given by $(u, v) \mapsto \sqrt{1 - u^2 - v^2}$. Let us calculate $df_{(1/2,-1/2,1/\sqrt{2})}$. We compute

$$dh_{(1/2,-1/2)} = \begin{bmatrix} -u/\sqrt{1 - u^2 - v^2} & -v/\sqrt{1 - u^2 - v^2} \end{bmatrix}_{(1/2,-1/2)} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

Also,

$$d\phi_{(1/2,-1/2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -u/\sqrt{1-u^2-v^2} & -v/\sqrt{1-u^2-v^2} \end{bmatrix}_{(1/2,-1/2)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

Consider $e_1 + e_2$.

$$dh_{(1/2,-1/2)}(e_1 + e_2) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0.$$

On the sphere, it is the vector

$$\begin{bmatrix} 1 \\ 0 \\ -\sqrt{2}/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

This vector is tangent to the level circle $\{z = 1/\sqrt{2}\}$ on S^2. So,

$$df_{(1/2,-1/2,1/\sqrt{2})} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0.$$

Lecture 10

September 28

10.1 Derivatives of Smooth Maps

Let $\dim X = k$ and $\dim Y = \ell$. Also, let $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^\ell$ be open. Let $\phi(0) = x$ and $\psi(0) = y$, where $f(x) = y$, so the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\psi^{-1}\circ f\circ\phi"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Then $df_x : T_xX \to T_{f(x)}Y$ makes the following diagram commute:

$$\begin{tikzcd} T_xX \arrow[r, dashed, "df_x := d\psi_0\circ dh_0\circ d\phi_0^{-1}"] & T_{f(x)}Y \\ \mathbb{R}^k \cong T_0U \arrow[u, "d\phi_0", "\cong"'] \arrow[r, "dh_0"] & T_0V \cong \mathbb{R}^\ell \arrow[u, "\cong", "d\psi_0"'] \end{tikzcd}$$

Last time, we considered $f : S^2 \to \mathbb{R}$ which maps $(x, y, z) \mapsto z$ at $p = (1/2, -1/2, 1/\sqrt{2})$. Then,

$$df_p : T_pS^2 \to T_{f(p)}\mathbb{R} \cong \mathbb{R}.$$

Example 10.1 (Stereographic Projection)**.** Define the map $\pi : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2$ given by projecting the point p onto $\mathbb{R}^2$ along the line joining p to the point $(0, 0, 1)$ on the sphere. If $p = (x, y, z) \in S^2$, then

$$\pi(p) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Consider $d\pi_p : T_pS^2 \to T_{\pi(p)}\mathbb{R}^2 \cong \mathbb{R}^2$. From the horizontal and vertical circles passing through p, I get two vectors v_1, v_2. What are $d\pi_p(v_1)$ and $d\pi_p(v_2)$? The image of the horizontal circle on S^2 passing through p is a circle in $\mathbb{R}^2$ centered at the origin passing through $\pi(p)$, and the vector $d\pi_p(v_1)$ is tangent to this circle at $\pi(p)$. If we choose another point p' which is farther north on S^2, then the image of the horizontal circle passing through p' is a much larger circle in $\mathbb{R}^2$, so the vector $d\pi_{p'}(v_1)$ will have a greater length. The image of the vertical circle passing through p is a line in $\mathbb{R}^2$ passing through the origin and $\pi(p)$. Thus, $d\pi_p(v_2)$ is a vector in the direction $-\pi(p)$.

Theorem 10.2 (Chain Rule)**.** *If we have*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

then

$$\boxed{d(g \circ f)_x = dg_{f(x)} \circ df_x}.$$

Proof. Look at:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\ U \arrow[u, "\phi"] \arrow[r, "\psi^{-1}\circ f\circ\phi"] & V \arrow[u, "\psi"] \arrow[r, "\eta^{-1}\circ g\circ\psi"] & W \arrow[u, "\eta"] \end{tikzcd}$$

Let $i = \psi^{-1} \circ f \circ \phi$ and $j = \eta^{-1} \circ g \circ \psi$. Then,

$$\begin{aligned} dg_{f(x)} \circ df_x &= (d\eta_0 \circ dj_0 \circ d\psi_0^{-1}) \circ (d\psi_0 \circ di_0 \circ d\phi_0^{-1}) = d\eta_0 \circ dj_0 \circ di_0 \circ d\phi_0^{-1} \\ &= d\eta_0 \circ d(j \circ i)_0 \circ d\phi_0^{-1} = d(g \circ f)_0 \end{aligned}$$

by the Chain Rule 4.1 applied to the maps defined on open sets in Euclidean spaces. □

Theorem 10.3 (Inverse Function Theorem)**.** *Suppose $f : X \to Y$ is smooth whose derivative at x_0, $df_{x_0} : T_{x_0}X \xrightarrow{\cong} T_{f(x_0)}Y$, is an isomorphism. Then, f is a local diffeomorphism near x_0, i.e., there exist open $U \subseteq X$, $V \subseteq Y$, with $x_0 \in U$, $f(x_0) \in V$, such that $f|_U : U \to V$ is a diffeomorphism.*

Proof. Diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \tilde{U} \arrow[u, "\phi"] \arrow[r, "h"] & \tilde{V} \arrow[u, "\psi"'] \end{tikzcd}$$

Then, $df_{x_0} = d\psi_0 \circ dh_0 \circ d\phi_0^{-1}$. By assumption, df_{x_0} is an isomorphism. So, we have:

$$\begin{tikzcd} T_{x_0}X \arrow[r, "df_{x_0}"] & T_{f(x_0)}Y \\ T_0\tilde{U} \arrow[u, "d\phi_0", "\cong"'] \arrow[r, "dh_0"] & T_0\tilde{V} \arrow[u, "\cong", "d\psi_0"'] \end{tikzcd}$$

Therefore, dh_0 is an isomorphism. By the Inverse Function Theorem 4.4 applied to h, h is a local diffeomorphism near 0. So, now choose a smaller $U' \subseteq \tilde{U}$, $V' \subseteq \tilde{V}$, with $0 \in U'$ and $0 \in V'$ such that $h : U' \to V'$ is a diffeomorphism. Define $U = \phi(U')$, $V = \psi(V')$, so $x_0 \in U$, $f(x_0) \in V$, and $f : U \to V$ is a diffeomorphism. □

Note that in

$$\begin{tikzcd} X \arrow[d, bend left, "\phi^{-1}"] \\ U \arrow[u, bend left, "\phi"] \end{tikzcd}$$

we have $\phi(u_1, \ldots, u_k) \in X$. In the book, whenever it says "let $(x_1, \ldots, x_k)$ be a coordinate system near x_0", it means $(x_1, \ldots, x_k)$ comes from the **coordinate functions** of some parameterization ϕ.

The Inverse Function Theorem 10.3 is equivalent to the following statement. Given $f : X \to Y$, where $\dim X = \dim Y = k$, such that df_{x_0} is an isomorphism, then there are coordinate systems near x_0, $(x_1, \ldots, x_k)$, and $f(x_0)$, $(y_1, \ldots, y_k)$, such that when we write f in these coordinate systems, f is the identity map $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k)$. Thus,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\uparrow & & \uparrow\psi \\ U' & \xrightarrow{h} & V \end{array}$$

where h is a diffeomorphism, can be changed to the following diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\uparrow & & \uparrow\psi\circ h \\ U' & \xrightarrow{\text{id}} & U' \end{array}$$

where $\psi \circ h$ is a diffeomorphism.

10.2 Immersions & Submersions

So far, we have considered $f : X \to Y$ where $\dim X = \dim Y$. Now we will consider other cases.

- $\underline{\dim X < \dim Y}$. We will consider special types of f known as immersions.
- $\underline{\dim X > \dim Y}$. We will consider special types of f known as submersions.

The *local model* for an immersion is

$$\mathbb{R}^k \to \mathbb{R}^{k+\ell}$$
$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$$

and the *local model* for a submersion is

$$\mathbb{R}^{k+\ell} \to \mathbb{R}^k$$
$$(x_1, \dots, x_{k+\ell}) \mapsto (x_1, \dots, x_k).$$

Definition 10.4. $f : X \to Y$, where $\dim X = k$, $\dim Y = k + \ell$, is an **immersion at** x_0 if

$$df_{x_0} : T_{x_0}X \to T_{f(x_0)}Y$$

is injective.

Definition 10.5. f is an **immersion** if it is an immersion at every $x_0 \in X$.

Example 10.6. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, and consider a map f which maps $\mathbb{R}$ to a smooth curve in $\mathbb{R}^2$. Then, df_{x_0} maps non-zero vectors in $\mathbb{R}$ to the (non-zero) velocity vector of the curve, so it is injective, and thus f is an immersion.

Example 10.7. Suppose that f maps $\mathbb{R}$ to a curve in $\mathbb{R}^2$ which has a cusp at x_0. As we approach the cusp, the velocity has to be decreasing, so $df_{x_0} = 0$, so f is not an immersion.

Example 10.8. Consider $X = S^1$, $Y = \mathbb{R}^2$, where the circle is mapped to a figure eight. Here, f is *not* injective because the intersection point of the figure eight has two pre-images, but it is still an immersion.

Lecture 11

October 3

11.1 Local Immersion Theorem

Inverse Function Theorem: Given $X \xrightarrow{f} Y$, assume that $x_0 \in X$ is such that $df_{x_0} : T_{x_0}X \to T_{f(x_0)}Y$ is an isomorphism. Then, f is a local diffeomorphism near x_0.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

where $\phi(0) = x_0$, $\psi(0) = f(x_0)$, $g = \psi^{-1} \circ f \circ \phi$ is a diffeomorphism. We can also write the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi", "\cong"'] \arrow[r, "\mathrm{id}"] & U \arrow[u, "\cong", "\psi \circ g"'] \end{tikzcd}$$

f is **locally equivalent** to id near x_0.

Definition: $X \xrightarrow{f} Y$ is an **immersion** at x_0 if $df_{x_0} : T_{x_0}X \to T_{f(x_0)}Y$ is injective.

Example 11.1. A map $f : \mathbb{R} \to \mathbb{R}^2$ which maps the real line to a smooth curve in $\mathbb{R}^2$ is an immersion.

Example 11.2. A map $f : \mathbb{R} \to \mathbb{R}^2$ which maps the real line to a curve with a cusp is not an immersion at the cusp.

Example 11.3. The map $f : S^1 \to \mathbb{R}^2$ which maps the circle to a figure eight is an immersion.

Theorem 11.4. *Suppose f is an immersion at $x_0 \in X$, $\dim X = k$, $\dim Y = n$. Then, there exist parameterizations U and V of X and Y near x_0 and $f(x_0)$ such that f, when written in these coordinates, gives the map*

$$i(x_1, \dots, x_k) = (x_1, \dots, x_k, \underbrace{0, \dots, 0}_{n-k}).$$

Proof. $k \le n$. Let

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

with $\phi(0) = x_0$, $\psi(0) = f(x_0)$. Notice that $dg_0 : T_0U \cong \mathbb{R}^k \to T_0V \cong \mathbb{R}^n$ is injective (since df_{x_0} is injective). Then,

$$dg_0 = \begin{bmatrix} dg_0(e_1) & \cdots & dg_0(e_k) \end{bmatrix}$$

has linearly independent column vectors. If I do a change of basis of coordinates of V (i.e., just compose it with a suitable linear transformation) such that the first k coordinates are the column vectors $dg_0(e_1), \dots, dg_0(e_k)$, then with respect to the new coordinates,

$$dg_0 = \begin{bmatrix} \mathrm{id}_k \\ 0 \end{bmatrix}.$$

Now, define the following map: $G : U \times \mathbb{R}^{n-k} \to V$ given by

$$G(x, x_{k+1}, \dots, x_n) = g(x) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k+1} \\ \vdots \\ x_n \end{bmatrix}.$$

Then,

$$\begin{aligned} dG_0 &= \begin{bmatrix} dg_0 & 0 \\ \vdots & \mathrm{id}_{n-k} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{id}_k & 0 \\ 0 & \mathrm{id}_{n-k} \end{bmatrix} = \mathrm{id}_n \end{aligned}$$

is an isomorphism. By the IFT 10.3, G is a local diffeomorphism near

$$0 = (\underbrace{0}_{\in U}, \underbrace{0}_{\in \mathbb{R}^{n-k}}).$$

I can choose a small neighborhood $\tilde{U} \times B_\delta^{n-k}(0)$, where $\tilde{U}$ is an open set in $\mathbb{R}^k$ near 0, and

$$G : \tilde{U} \times B_\delta^{n-k}(0) \to \tilde{V} \subseteq V$$

is a diffeomorphism. Now, we have:

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ \tilde{U} \arrow[u] \arrow[rr] \arrow[dr, "x \mapsto (x,0)"'] & & \tilde{V} \arrow[u, "\psi"'] \\ & \tilde{U} \times B_\delta^{n-k}(0) \arrow[ur, "\cong"', "G"] & \end{tikzcd}$$

which we can change to:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \tilde{U} \arrow[u, "\tilde{\phi}", "\cong"'] \arrow[r, "i"] & \tilde{U} \times B_\delta^{n-k}(0) \arrow[u, "\cong", "\psi\circ G"'] \end{tikzcd}$$

Hence, f in these coordinates is the map i. □

Corollary 11.5. *If f is an immersion at $x_0 \in X$, then f is an immersion at $x \in U$ for some open set U near x_0. Equivalently, the set of points x where f is an immersion at x is an open set in X.*

11.2 Embeddings

Suppose now that f is an immersion, i.e., $\forall x \in X$, f is an immersion at x. We want to look at the image $f(X)$. In the example mapping S^1 to the figure eight, notice that even though locally f is nice, f is not an injection.

Q: What if f is injective? Is $f(X)$ a submanifold of Y?

Let $f : X = \mathbb{R} \to Y = \mathbb{R}^2$ where the image is a curve which visits any neighborhood of a specific point infinitely many times. Here, $f(X)$ is a manifold $\cong \mathbb{R}^1$, but it is not a submanifold of Y. We need a condition at ∞ for the map f so that $f(X)$ is a submanifold of Y.

Definition: $Z \subseteq Y$ is a **submanifold** if $\forall z_0 \in Z$, there is an open set $U \subseteq Y$ such that $U \cap Z$ is diffeomorphic to an open set in $\mathbb{R}^k$.

Example 11.6. Take the torus $\mathbb{R}^2/\{x \sim x+1, y \sim y+1\}$. If we consider a map $\mathbb{R} \to$ Torus from a line with irrational slope, the image of this injective immersion is a dense subset of the torus, so it is not a submanifold of the torus.

Definition 11.7. $f : X \to Y$ is **proper** if $\forall K \subseteq Y$ compact, $f^{-1}(K)$ is also compact.

Definition 11.8. An **embedding** $f : X \to Y$ is a proper injective immersion.

Theorem 11.9. *If f is an embedding $X \xrightarrow{f} Y$, then $f(X)$ is a submanifold of Y.*

Proof. Given $z_0 \in f(X)$, since f is injective, $\exists! x_0 \in X$ such that $f(x_0) = z_0$. By the Local Immersion Theorem 11.4, in a neighborhood of x_0 in X, f is locally equivalent to the inclusion. We claim that if the open set $U \subseteq Y$ containing x_0 is sufficiently small, $U \cap f(X)$ is diffeomorphic to an open set in $\mathbb{R}^{\dim X}$. Otherwise, $\exists x_n \in X$ that goes to ∞ such that $f(x_n) \to z_0$, but by properness, x_0 must be in a compact set in X and so $x_n \to x_0$ and when $n \gg N$, x_n is in the neighborhood of x_0. So, $U \cap f(X)$ has to be diffeomorphic to an open set in $\mathbb{R}^{\dim X}$. □

Lecture 12

October 5

12.1 Immersions & Embeddings

The map $f : X \to Y$, $\dim X = m$, $\dim Y = n$, $m < n$, is an **immersion** if $df_{x_0} : T_{x_0}X \to T_{f(x_0)}Y$ is injective.

Local Immersion Theorem: Near x_0, f is **locally equivalent** to the inclusion map $\mathbb{R}^m \to \mathbb{R}^n$ which maps $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

where $\phi(0) = x_0$, $\psi(0) = f(x_0)$.

In particular, f is locally injective near x_0, but globally, f may not be injective.

Definition: An **embedding** $f : X \to Y$ is a proper injective immersion. A map is **proper** if $\forall K \subseteq Y$ compact, then $f^{-1}(K)$ is compact in X.

Theorem: If $f : X \to Y$ is an embedding, then $f(X)$ is a submanifold of Y.

12.2 Submersions

Definition 12.1. The map $f : X \to Y$, where $\dim X = k + n$ and $\dim Y = n$, is a **submersion at** x_0 if $df_{x_0} : T_{x_0}X \to T_{f(x_0)}Y$ is surjective.

In other words, n columns of the matrix of df_{x_0} are linearly independent, i.e., there exists a $n \times n$ submatrix obtained from selecting n columns, which is an invertible $n \times n$ matrix.

Example 12.2. Consider the map $\mathbb{R}^{k+n} \xrightarrow{\pi} \mathbb{R}^n$ given by $(x_1, \ldots, x_{k+1}, \ldots, x_{k+n}) \mapsto (x_{k+1}, \ldots, x_{k+n})$. We see that

$$d\pi_x = \begin{bmatrix} 0 & \mathrm{id}_n \end{bmatrix}$$

is surjective, so π is a submersion.

Theorem 12.3. *If f is a submersion at x_0, then near x_0, f is locally equivalent to π, i.e., there exist parameterizations $U \xrightarrow{\phi} X$, $V \xrightarrow{\psi} Y$, $\phi(0) = x_0$, $\psi(0) = f(x_0)$, such that:*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\pi"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Corollary 12.4. *1. If f is a submersion at x_0, then f is onto for some neighborhood of $f(x_0)$.*

2. If f is a submersion at x_0, then f is a submersion at every point near x_0.

Proof of 12.3. Choose any parameterization:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Here, by assumption, $dg_0 : T_0U \to T_0V$ is surjective. First, we make a change of basis for U so that dg_0 in this new basis is $\begin{bmatrix} 0 & \mathrm{id}_n \end{bmatrix}$. Pick $v_1, \ldots, v_k$ to be a basis of $\ker dg_0$, and also choose $v_{k+1}, \ldots, v_n$ such that $dg_0(v_{k+i}) = e_i$ in T_0V $(1 \le i \le n)$. Use this basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for U and check that $dg_0 = \begin{bmatrix} 0 & \mathrm{id}_n \end{bmatrix}$. Now, we construct $G : U \to \mathbb{R}^{k+n}$ by

$$G(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+n}) = \big(x_1, \ldots, x_k, g(x_1, \ldots, x_{k+n})\big).$$

If we compute dG_0, then it is

$$dG_0 = \begin{bmatrix} \mathrm{id}_k & 0 \\ 0 & \mathrm{id}_n \end{bmatrix}.$$

This implies that dG_0 is an isomorphism. By the IFT 10.3, we obtain that G is a local diffeomorphism near 0, i.e.,

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ U \arrow[u, "\phi"'] \arrow[rr, "g"] \arrow[d, "\cong"', "G"] & & V \arrow[u, "\psi"'] \\ \tilde{U} \subseteq \mathbb{R}^{k+n} \arrow[urr, "\pi"] \arrow[uu, dashed, bend left, "\phi\circ G^{-1}"] & & \end{tikzcd}$$

where we choose U and V to be smaller according to the IFT 10.3. Note that $g = \pi \circ G$. We can modify the diagram to:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U' \arrow[u, "\phi\circ G^{-1}"] \arrow[r, "\pi"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

where $\pi(x_1, \ldots, x_{k+n}) = (x_{k+1}, \ldots, x_{k+n})$. $\square$

Definition 12.5. $y \in Y$ is a **regular value** of f if for all $x \in f^{-1}(y)$, f is a submersion at x.

Theorem 12.6. *If $y \in Y$ is a regular value of f, then $f^{-1}(y)$ is a k-dimensional submanifold in X.*

Proof. Note that $\pi(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (x_{k+1}, \dots, x_n)$ in local coordinates and

$$\pi^{-1}(y) = \{(x_1, \dots, x_k, y) \mid (x_1, \dots, x_k) \in \mathbb{R}^k\}. \qquad \square$$

Remark: This is the manifold version of the Implicit Function Theorem 6.1.

Example 12.7. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by $f(x_1, \dots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$. Then,

$$df_x = \begin{bmatrix} 2x_1 & \cdots & 2x_{n+1} \end{bmatrix} : \underbrace{T_x\mathbb{R}^{n+1}}_{\cong \mathbb{R}^{n+1}} \to \underbrace{T_{f(x)}\mathbb{R}}_{\mathbb{R}}.$$

So, df_x is surjective if and only if not every x_i is 0, so f is a submersion everywhere except at $x = 0$. Then, $f^{-1}(1)$ is the unit sphere S^n. Notice that 1 is a *regular value* of f. Indeed, if $x \in S^n$, then $x \neq 0$. Thus, f is a submersion at x. Hence, S^n is a submanifold of $\mathbb{R}^{n+1}$.

Example 12.8. Let $\mathrm{M}(n) = \{n \times n \text{ matrices}\} = \mathbb{R}^{n \times n}$. Then, consider $\mathrm{O}(n) \subseteq \mathrm{M}(n)$, where

$$\mathrm{O}(n) = \{A \in \mathrm{M}(n) \mid AA^\mathsf{T} = \mathrm{id}_n\}.$$

For example, when $n = 1$, $\mathrm{M}(1) = \mathbb{R}$ and $\mathrm{O}(n) = \{\pm 1\}$.

When $n = 2$,

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{A^\mathsf{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so

$$\begin{aligned} \mathrm{O}(2) &= \{(a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 = 1,\ c^2 + d^2 = 1,\ ac + bd = 0\} \\ &= S^1 \sqcup S^1. \end{aligned}$$

In other words,

$$\mathrm{O}(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} : \theta \in S^1 \right\} \cup \left\{ \begin{bmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : \theta \in S^1 \right\}.$$

We will show that $\mathrm{O}(n) \subseteq \mathrm{M}(n) \cong \mathbb{R}^{n \times n}$ is a submanifold. $\mathrm{O}(n)$ is a Lie group. Take $\mathrm{M}(n) \xrightarrow{f} \mathrm{M}(n)$ given by $A \mapsto AA^\mathsf{T}$, so $f(A) = AA^\mathsf{T}$. So, $\mathrm{O}(n) = f^{-1}(\mathrm{id}_n)$. You want to show that id_n is a regular value of f. Notice that $\forall A \in \mathrm{M}(n)$, AA^T is symmetric, because $(AA^\mathsf{T})^\mathsf{T} = (A^\mathsf{T})^\mathsf{T} A^\mathsf{T} = AA^\mathsf{T}$. The image of f is contained in $\{n \times n \text{ symmetric matrices}\}$, which is a submanifold of $\mathrm{M}(n)$. So, we must change the map to $\mathrm{M}(n) \xrightarrow{f} \mathrm{S}(n)$, where $\mathrm{S}(n)$ is the set of symmetric matrices.

Lecture 13

October 10

13.1 More on Submersions

13.1.1 Review

$X \subseteq \mathbb{R}^N$ is an n-dimensional **manifold** if for all $x \in X$, there exists an open set U in X ($U = \tilde{U} \cap X$, where $\tilde{U}$ is open in $\mathbb{R}^N$) that is diffeomorphic to $\mathbb{R}^n$.

$Z \subseteq X$ is a **submanifold of** X if $Z \subseteq X$ and $Z \subseteq \mathbb{R}^N$ is a manifold.

$Z \subseteq \mathbb{R}^N$ is a manifold if for all $z \in Z$, there exists an open set V in Z ($V = \tilde{V} \cap Z$, where $\tilde{V}$ is open in $\mathbb{R}^N$) diffeomorphic to $\mathbb{R}^k$. $V' = \tilde{V} \cap X$ is an open set in X. Hence, Z is a submanifold of X if for all $z \in Z$, there exists open V' in X such that $V = V' \cap Z$ is diffeomorphic to $\mathbb{R}^k$.

Next HW: If Z is a submanifold of X, then there exists a parameterization around each $z \in Z$, $\phi : U \to X$, such that Z in these local coordinates is given by $\{x_{k+1} = \cdots = x_n = 0\}$.

Last time.
Definition: f is a **submersion at** x if df_x is surjective.

Theorem: If f is a submersion at x, then:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\psi} \\ U & \xrightarrow{\pi} & V \end{array}$$

where $\dim X = k + n$, $\dim Y = n$, and $\pi(x_1, \ldots, x_{k+n}) = (x_{k+1}, \ldots, x_{k+n})$.

Definition: $y \in Y$ is a **regular value of** f if f is a submersion at every point $x \in f^{-1}(y)$.

Theorem: If y is a regular value of f, then $Z = f^{-1}(y)$ is a submanifold of X. Z has **codimension** ($\dim X - \dim Z$) equal to $n = \dim Y$ and $T_z Z \subseteq T_z X$ is the kernel of $df_z : T_z X \to T_y Y$.

Proof. Notice that if $z \in Z$, by the Local Submersion Theorem 12.3:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\psi} \\ U & \xrightarrow{\pi} & V \end{array}$$

where $\phi(0) = z$, $\psi(0) = y$. So, under these coordinates, $\phi^{-1}(Z) = \pi^{-1}(0)$ and of course this is a submanifold in U. Also, $T_0 \phi^{-1}(Z) = \ker d\pi_0$. Hence, $T_z Z = \ker df_z$.

13.1.2 Orthogonal Group

Example 13.1. $\mathrm{O}(n) = \{A \in \mathrm{M}(n) \mid AA^\intercal = \mathrm{id}\}$, where $\mathrm{M}(n)$ is the set of $n \times n$ entries, $\mathrm{M}(n) = \mathbb{R}^{n \times n}$. Notice that the image of f is in the set of symmetric matrices $\mathrm{S}(n)$. Thus, $\mathrm{S}(n) = \{B \in \mathrm{M}(n) \mid B = B^\intercal\}$. Check that $(AA^\intercal)^\intercal = (A^\intercal)^\intercal A^\intercal = AA^\intercal$, so $AA^\intercal \in \mathrm{S}(n)$. So here, in order for f to be potentially submersive, write $f : \mathrm{M}(n) \to \mathrm{S}(n)$. In fact, $\mathrm{S}(n) = \mathbb{R}^{n(n+1)/2}$. We need to check that id is a regular value of f. Given $A \in \mathrm{O}(n) = f^{-1}(\mathrm{id})$, check that df_A is surjective. Consider $df_A(B)$, where $B \in T_A \mathrm{M}(n) = \mathrm{M}(n)$. By the definition of the directional derivative in the direction B at the point A,

$$\begin{aligned} df_A(B) &= \lim_{s \to 0} \frac{f(A+sB) - f(A)}{s} \\ &= \lim_{s \to 0} \frac{(A+sB)(A^\intercal + sB^\intercal) - AA^\intercal}{s} \\ &= \lim_{s \to 0} \frac{AA^\intercal + sAB^\intercal + sBA^\intercal + s^2 BB^\intercal - AA^\intercal}{s} = AB^\intercal + BA^\intercal. \end{aligned}$$

Suppose $C \in T_{\mathrm{id}} \mathrm{S}(n) = \mathrm{S}(n)$. We want to show that there exists $B \in \mathrm{M}(n)$ such that $df_A(B) = C$, i.e., $AB^\intercal + BA^\intercal = C$. We can also write

$$AB^\intercal + BA^\intercal = (BA^\intercal)^\intercal + BA^\intercal = \frac{1}{2}(C + C^\intercal).$$

Hence, we choose B such that

$$BA^\intercal = \frac{1}{2}C.$$

Since $AA^\intercal = \mathrm{id}$, $(A^\intercal)^{-1} = A$, so we solve for B.

$$B = \frac{1}{2}C(A^\intercal)^{-1} = \frac{1}{2}CA.$$

What we just showed is that if $A \in f^{-1}(\mathrm{id})$, then df_A is surjective. So, id is a regular value of f and $f^{-1}(\mathrm{id})$ is a submanifold of $\mathrm{M}(n)$.

$$f : \underbrace{\mathrm{M}(n)}_{\dim = n^2} \to \underbrace{\mathrm{S}(n)}_{\dim = n(n+1)/2}$$

so $\mathrm{O}(n) = f^{-1}(\mathrm{id})$ has codimension in $\mathrm{M}(n)$ equal to

$$\dim \mathrm{S}(n) = \frac{n(n+1)}{2}.$$

Hence, it has dimension $n^2 - n(n+1)/2 = n(n-1)/2$.

13.1.3 Geometrical Interpretation

Given $X \times Y$, $T_{(x,y)}(X \times Y) = T_x X \times T_y Y$. Let

$$\begin{aligned} \Gamma_f &= \operatorname{graph} f \\ &= \big\{\big(x, f(x)\big) \bigm| x \in X\big\}. \end{aligned}$$

If $Z_y = X \times \{y\}$, then $\Gamma_f \cap Z_y = \{(x, y) \mid x \in f^{-1}(y)\}$. Thus, $T_{(x,y)}\Gamma_f = \{(v, df_x(v)) \mid v \in T_x X\}$. Then, f is a submersion at x_1 if and only if $T_{(x_1,y)}\Gamma_f = \{(v, df_{x_1}(v)) \mid v \in T_{x_1} X\}$ satisfies

$$T_{(x_1,y)}(X \times Y) = T_{(x_1,y)}\Gamma_f + T_{(x_1,y)} Z_y.$$

Definition 13.2. Suppose M is a manifold and X and Y are submanifolds of M. We say $X \cap Y$ **transversely at** p if $T_pX + T_pY = T_pM$.

We can have $T_pM = T_pX + T_pY$ even if $T_pX \cap T_pY \neq \{0\}$.

Notation: We will use the notation $X \pitchfork Y$.

Definition 13.3. Let $f : X \to Y$ and $Z \subseteq Y$ be a submanifold of Y. Let $x \in f^{-1}(Z)$. We say f is **transversal to** Z **at** x if:

$$\boxed{\text{image}\, df_x + T_zZ = T_zY}$$

In the case that f is an embedding so that $f(X)$ is a submanifold in Y, then f is transversal to Z at x if and only if $f(X) \pitchfork Z$ at $z = f(x)$.

Lecture 14

October 12

14.1 Transversality

Definition: If Y is a manifold and $X \subseteq Y$ is a submanifold, $Z \subseteq Y$ is a submanifold, say $p \in X \cap Z$. Then we say $X \cap Z$ **tranversally at** p if $T_pX + T_pZ = T_pY$ where

$$T_pX + T_pZ = \{v \in T_pY \mid v = w_1 + w_2,\ w_1 \in T_pX,\ w_2 \in T_pZ\}.$$

We write $X \pitchfork Z$ at p.

Definition: If $f : X \to Y$ and $Z \subseteq Y$ is a submanifold, we say $f \pitchfork Z$ if for all $x \in f^{-1}(Z)$ we have $\operatorname{image} df_x + T_{f(x)}Z = T_{f(x)}Y$.

In the case $Z = \{y\}$, if $x \in f^{-1}(\{y\})$, then $f \pitchfork Z$ means $\operatorname{image} df_x + \{0\} = T_yY$, i.e., df_x is surjective. Hence, y is a regular value of f. In the case when f is an embedding, $f(X)$ is a submanifold of Y with tangent space at $f(x)$ exactly $\operatorname{image} df_x$. Thus, the condition becomes $T_{f(x)}f(X) + T_{f(x)}Z = T_{f(x)}Y$. Hence, if f is an embedding, $f(X) \pitchfork Z$.

Theorem 14.1. *If $f \pitchfork Z$, then $f^{-1}(Z)$ is a submanifold in X with* $\operatorname{codim}_X f^{-1}(Z) = \operatorname{codim}_Y Z$.

Proof. Given $x_0 \in f^{-1}(Z)$, then $f(x_0) = z_0 \in Z$. One needs to check that there exists open $\tilde{U} \subseteq X$, with $x \in \tilde{U}$, such that $\tilde{U} \cap f^{-1}(Z)$ is diffeomorphic to some open set in some $\mathbb{R}^\ell$. Denote

$$k = \operatorname{codim}_Y Z := \dim Y - \dim Z.$$

Since $Z \subseteq Y$ is a submanifold, there exists $\tilde{V} \subseteq Y$, $z_0 \in \tilde{V}$, with coordinates $(y_1, \ldots, y_k, \ldots, y_n)$ such that $Z = \{y_1 = \cdots = y_k = 0\}$. Consider $\tilde{V} \xrightarrow{\pi} \mathbb{R}^k$ given by $\pi(y) = (y_1, \ldots, y_k)$, which is a submersion.

Claim: $\pi \circ f$ is a submersion at x_0, i.e., $d(\pi \circ f)_{x_0}$ is surjective. By the Chain Rule 10.2,

$$\operatorname{image} d(\pi \circ f)_{x_0} = d\pi_{z_0}(\operatorname{image} df_{x_0}).$$

Since π is a submersion at z_0, $d\pi_{z_0}(T_{z_0}Y) = T_0\mathbb{R}^k$. Since we assume $f \pitchfork Z$, $\operatorname{image} df_{x_0} + T_{z_0}Z = T_{z_0}Y$. Applying $d\pi_{z_0}$,

$$\begin{aligned} d\pi_{z_0}(\operatorname{image} df_{x_0} + T_{z_0}Z) &= d\pi_{z_0}(T_{z_0}Y) \\ &= T_0\mathbb{R}^k. \end{aligned}$$

Since $\tilde{V} \cap Z = \pi^{-1}(0)$, $d\pi_{z_0}(T_{z_0}Z) = \{0\}$. Thus, $d\pi_{z_0}(\operatorname{image} df_{x_0}) = T_0\mathbb{R}^k$, i.e., $d(\pi \circ f)_{x_0}$ is surjective.

Hence,

$$\begin{aligned}(\pi \circ f)^{-1}(0) &= f^{-1}\big(\pi^{-1}(0)\big)\\ &= f^{-1}(\tilde{V} \cap Z)\\ &= \tilde{U} \cap f^{-1}(Z)\end{aligned}$$

where $\tilde{U} = f^{-1}(\tilde{V})$ is open. Hence, by the Pre-Image Theorem 12.6, $f^{-1}(Z)$ is a submanifold in X.

$$\begin{aligned}\operatorname{codim}_X f^{-1}(Z) &= \dim \mathbb{R}^k\\ &= k\\ &= \operatorname{codim}_Y Z.\end{aligned}$$

□

Corollary 14.2. *In the case when f is an embedding, if $f \pitchfork Z$, $f^{-1}(Z)$ is a submanifold in X which is diffeomorphic to $f(X) \cap Z$.*

So, suppose X, Z are submanifolds of Y. If $X \pitchfork Z$, then $X \cap Z$ is a submanifold in Y. The inclusion $i : X \to Y$ is an embedding.

Example 14.3. Let

$$\begin{aligned}Y &= \mathbb{R}^3,\\ X &= \{x^2 + y^2 + z^2 = a\},\\ Z &= \{x^2 + y^2 - z^2 = 1\}.\end{aligned}$$

If $a < 1$, $X \cap Z = \varnothing$. For what values of a does $X \pitchfork Z$? A: When $a > 1$. Suppose $p = (x_0, y_0, z_0) \in X \cap Z$. We must check $T_pX + T_pZ = T_p\mathbb{R}^3$.

$$\begin{aligned}T_pX &= \ker d\big((x,y,z) \mapsto x^2 + y^2 + z^2\big)_p\\ &= \ker \underbrace{\begin{bmatrix}2x_0 & 2y_0 & 2z_0\end{bmatrix}}_{n_X}.\end{aligned}$$

So, $n_X \perp T_pX$. Also,

$$\begin{aligned}T_pZ &= \ker d\big((x,y,z) \mapsto x^2 + y^2 - z^2\big)_p\\ &= \ker \underbrace{\begin{bmatrix}2x_0 & 2y_0 & -2z_0\end{bmatrix}}_{n_Z},\end{aligned}$$

and $n_Z \perp T_pZ$. Thus, $T_pX + T_pZ = T_p\mathbb{R}^3$ is equivalent to $n_X \nparallel n_Z$.

$$\begin{bmatrix}2x_0\\ 2y_0\\ 2z_0\end{bmatrix} \nparallel \begin{bmatrix}2x_0\\ 2y_0\\ -2z_0\end{bmatrix} \overset{(x_0,y_0)\neq(0,0)}{\iff} z_0 \neq 0.$$

So, $X \pitchfork Z$ when $a > 1$. When $a = 1$, $X \cap Z = \{(x, y, z = 0) \mid x^2 + y^2 = 1\}$ and in this case, $X \not\pitchfork Z$.

Given any closed set C in $\mathbb{R}$, there exists $f : \mathbb{R} \xrightarrow{\text{smooth}} \mathbb{R}$ such that $f^{-1}(0) = C$. If we look at graph f, we see that in general, the intersection of smooth manifolds can be very complicated.

14.2 Deformations

Definition 14.4. Given smooth $f_0 : X \to Y$, $f_1 : X \to Y$, a **homotopy** $f_0 \overset{f_t}{\sim} f_1$ is a smooth map $F : X \times I \to Y$, where $I = [0, 1]$, such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.

Next time, we will show that if $f_0 \sim f_1$, $f_1 \sim f_2$, then $f_0 \sim f_2$.

Lecture 15

October 17

15.1 Homotopy

Definition: Given maps $f_0, f_1 : X \to Y$, $f_0 \overset{f_t}{\sim} f_1$ is a **homotopy** if there exists a smooth map $F : X \times I \to Y$, where $I = [0, 1]$, such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Then, $f_t(x) := F(x, t)$. Here, $f_t : X \to Y$.

Example 15.1. If $f_0 = \text{id}$, $f_1(x) = 2x$, then take $F(x, t) = (t+1)x$. Then, $F(x, 0) = x$ and $F(x, 1) = 2x$.

Proposition 15.2. *(a)* $f \sim f$.

(b) $f \sim g \implies g \sim f$.

(c) $f \sim g,\ g \sim h \implies f \sim h$.

Proof. (a) $f \sim f$: Take $f_t = f$.

(b) $f \sim g$, so there exists $F : X \times I \to Y$. To show that $g \sim f$, take $G : X \times I \to Y$ given by $G(x, t) = F(x, 1 - t)$.

(c) We are given $f \overset{F}{\sim} g$, $g \overset{G}{\sim} h$, where $F : X \times I \to Y$ and $G : X \times I \to Y$. Define $H : X \times I \to Y$ by:

$$H(x, t) = \begin{cases} F(x, 2t) & \text{when } t \leq \dfrac{1}{2} \\ G(x, 2t - 1), & \text{when } t > \dfrac{1}{2} \end{cases}$$

Check H is continuous, but it may not be smooth. Therefore, we must modify H. In an interval near $t = 1/2$, let H be defined as $H(x, t) = g(x)$, and then for F, reparameterize t so that the speed near $t = 1/2$ decays rapidly to 0.

□

15.1.1 Stable Properties

Definition 15.3. A property (P) is **stable** if given $f : X \to Y$ that has property (P), then for any homotopy $F = \{f_t\}$, there exists $\varepsilon > 0$ such that whenever $0 \leq t < \varepsilon$, f_t also has property (P).

Theorem 15.4. *Suppose X is compact. Then, the following properties are stable:*

(a) immersion;

(b) submersion;

(c) $\pitchfork Z$ where $Z \subseteq Y$ is a closed submanifold;

(d) embedding.

Example 15.5. If $X = \mathbb{R}_+ = (0, \infty)$, then X is not compact. Take $f : X \to Y = \mathbb{R}$ with $f(x) = 1/x$, which is an immersion. However,

$$f_t(x) = \frac{1}{x} + tx$$

is not an immersion for all $t > 0$.

Example 15.6. Here is a concrete example when X is not compact. Let $X = \mathbb{R}$, $Y = \mathbb{R}$. If we map X to a smooth curve in $\mathbb{R}^2$ and project the curve onto the horizontal axis, such that the smooth curve passes y_0 once and returns closer and closer to the point y_0 forever, then y_0 is a regular value of f, but if we consider $f_t(x) = f(x) - t$, then for all $\varepsilon > 0$, there exists $\tilde{t} < \varepsilon$ such that y_0 is not a regular value of $f_{\tilde{t}}$.

Example 15.7. Let $f : X = \mathbb{R} \to \mathbb{R}^2$, $f \pitchfork Z$. Take $Z = (-1, 1) \times \{0\}$ and $Y = \mathbb{R}^2$, so that $f(X)$ oscillates infinitely often in the interval $(-1, 1)$, with the oscillations decaying to 0. Now, for $f_t : X \to \mathbb{R}^2$ given by $f_t(x) = f(x) + (0, t)$, the stability of $\pitchfork Z$ does not hold.

Proof of 15.4. (a) Starting with an immersion $f : X \to Y$ and a homotopy $F = \{f_t\}$, notice that $F : X \times I \to Y$ is smooth, so dF is continuous.

$$dF_{(x,t)} = \begin{bmatrix} d(f_t)_x & \dfrac{\partial F}{\partial t}(x, t) \end{bmatrix}$$

Given $x_0 \in X$, at time $t = 0$, $df_{x_0} : T_{x_0}X \to T_{f(x_0)}Y$ is injective. Since dF is continuous, there exists $U_{x_0} \times [0, \varepsilon_{x_0})$ such that $d(f_t)_x$ is injective for $(x, t) \in U_{x_0} \times [0, \varepsilon_{x_0})$. Now, $\{U_{x_0}\}_{x_0 \in X}$ is an open cover of X, and X is compact so there exists $\{U_{x_i}\}_{i=1}^m$ such that $X = \bigcup_{i=1}^m U_{x_i}$. So, we let $\varepsilon = \min\{\varepsilon_{x_1}, \ldots, \varepsilon_{x_m}\}$. Then, for all $(x, t) \in X \times [0, \varepsilon)$, there exists U_{x_i} such that $(x, t) \in U_{x_i} \times [0, \varepsilon_{x_i})$. Hence, $d(f_t)_x$ is injective. So, f_t is an immersion at x, for all $x \in X$, $t \in [0, \varepsilon)$. Thus, f_t is an immersion for all $t \in [0, \varepsilon)$.

(b) The same idea applies to submersions. It is because a surjective *linear* map is still surjective if we change it a little bit.

(c) Suppose $Z \subseteq Y$ is a closed submanifold and $f \pitchfork Z$. First, consider $Z = \{\text{point}\} = \{y_0\}$. In the case that X is compact, we need to show that there exists $\varepsilon > 0$ such that $f_t \pitchfork \{y_0\}$ when $0 \le t < \varepsilon$. Then, $M = f^{-1}(y_0)$ is a submanifold of X since $\{y_0\}$ is a regular value of f. Consider $U = f^{-1}(V)$ where V is given in 15.8. Notice U is open and contains $M = f^{-1}(y_0)$, and $f|_{\overline{U}} : \overline{U} \to V \subseteq Y$ is a submersion. Observe that for t small enough, $(f_t)^{-1}(y_0) \subseteq U$. Also, $\overline{U} \subseteq X$ is compact (here you can achieve this by choosing V small enough). Now, applying (b) to $f_t|_{\overline{U}}$, then we obtain that $f_t|_{\overline{U}}$ is a submersion for t small enough. In particular, $\{y_0\}$ is a regular value of $f_t|_{\overline{U}}$. Since $(f_t)^{-1}(y_0) \subseteq U$, this means that $\{y_0\}$ is a regular value of f_t.

Let $Z \subseteq Y$ be a closed submanifold. In the case that X is compact and Z is closed, we need to show that $\pitchfork Z$ is stable. The idea is similar to before, where we use a projection map to reduce to the case when Z is a single point.

□

Lemma 15.8 (Homework)**.** *If X is compact, then there exists $V \ni y_0$ and V is open in Y such that every $y \in V$ is a regular value of f.*

Lecture 16

October 19

16.1 Stability

(P) is **stable** if $f : X \to Y$ satisfies (P) and for all homotopies $F = \{f_t\}$, where $f_0 = f$, there exists $\varepsilon > 0$ such that for all $t < \varepsilon$, f_t satisfies (P).

Theorem: Given that X is compact, then the following properties are stable:

(a) immersion;

(b) submersion;

(c) $\pitchfork Z$, where Z is a closed submanifold of Y;

(d) embedding.

Proof of 15.4, Continued. (c) In the case $Z = \{y\}$, $f \pitchfork \{y\}$, i.e., y is a regular value of f. We need to show that for all $t < \varepsilon$, $f_t \pitchfork \{y\}$. Choose a closed neighborhood V of y small enough so that every point in V is regular, so if $U = f^{-1}(V)$, then f is a submersion in U.

Now consider the case when Z is a general closed submanifold of Y. Start with $f \pitchfork Z$. $M = f^{-1}(Z)$ is closed. Since X is compact, M is also compact. Consider $Z' = f(X) \cap Z$, which is compact. For each $z_0 \in Z'$, let U_{z_0} be a neighborhood of z_0 and choose local coordinates near z_0 in Y, $(x_1, \ldots, x_n)$, such that Z' in these local coordinates is given by $\{x_1 = \cdots = x_k = 0\}$. Here, $n = \dim Y$, $k = \operatorname{codim} Z$. Consider $\pi_{z_0} : U_{z_0} \to \mathbb{R}^n$ given by $\pi_{z_0}(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$. Then, $f \pitchfork (Z' \cap U_{z_0})$ iff $\pi_{z_0} \circ f \pitchfork \{0\}$. ($f \pitchfork Z$ means $\operatorname{image} df_z + T_z Z = T_z Y$, but applying $d\pi_{z_0}$ we obtain $d\pi_{z_0}(\operatorname{image} df_z) + d\pi_{z_0}(T_z Z) = d\pi_{z_0}(T_z Y)$, but $d\pi_{z_0}(T_z Z) = 0$ and $d\pi_{z_0} T_z Y = T_{\pi(z)}\mathbb{R}^n$, so by the Chain Rule 10.2, $d\pi_{z_0}(\operatorname{image} df_z) = \operatorname{image} d(\pi_{z_0} \circ f)_z = T_{\pi(z)}\mathbb{R}^n$.) Hence, for all z_0, we get a U_{z_0}, π_{z_0}. Since Z' is compact and $\{U_{z_0}\}_{z_0 \in Z'}$ is an open cover, there is a finite subcover:

$$\begin{array}{cccc}
U_{z_1} & U_{z_2} & \cdots & U_{z_N} \\
\downarrow{\scriptstyle \pi_{z_1}} & \downarrow{\scriptstyle \pi_{z_2}} & & \downarrow{\scriptstyle \pi_{z_N}} \\
\mathbb{R}^n & \mathbb{R}^n & \cdots & \mathbb{R}^n
\end{array}$$

Then, $f \pitchfork Z$ is equivalent to for all $i = 1, \ldots, N$, $\pi_{z_i} \circ f \pitchfork \{0\}$. Let $U = \bigcup_{i=1}^N U_{z_i}$. Let $V = f^{-1}(U)$. For t small enough, $f_t^{-1}(Z) \subseteq V$. Hence, when t is small enough, $f_t \pitchfork Z$ is equivalent to $\pi_{z_i} \circ f_t \pitchfork \{0\}$ for all $i = 1, \ldots, N$. From the case $Z = \{\text{point}\}$ we proved earlier, there exists $\varepsilon > 0$ such that when $t < \varepsilon$, $\pi_{z_i} \circ f_t \pitchfork \{0\}$ for $i = 1, \ldots, N$, and therefore $t < \varepsilon \implies f_t \pitchfork Z$.

(d) Recall that an embedding is a proper one-to-one immersion. Immersions are stable. Proper is automatic since X is compact. One needs to show that there exists $\varepsilon > 0$ such that for $t < \varepsilon$, f_t is one-to-one. Suppose not. Then, there exist $t_J \xrightarrow{J\to\infty} 0$ together with $\{u_J\}, \{v_J\} \subseteq X$, where $u_J \neq v_J$, such that $f_{t_J}(u_J) = f_{t_J}(v_J)$. X is compact, so it satisfies: every sequence in X has a convergent subsequence. Since $\{u_J\} \subseteq X$, there exists a subsequence of u_J, u_{n_J}, which converges to $u \in X$. Look at $\{v_{n_J}\} \subseteq X$. There exists a subsequence of v_{n_J}, $v_{n'_J}$, converging to $v \in X$. Since $f_{t_{n'_J}}(u_{n'_J}) = f_{t_{n'_J}}(v_{n'_J})$, as $J \to \infty$, the sequences converge to $f(u)$ and $f(v)$ respectively, and hence $u = v$ since f is one-to-one. Consider $H : X \times I \to Y \times I$ given by $H(x,t) = (f_t(x), t)$. One checks that $dH_{(u,0)}$ is injective (HW). Hence, by the Local Immersion Theorem 11.4, H is injective near $(u, 0)$. This is a contradiction since $(u_{n'_J}, t_{n'_J})$ and $(v_{n'_J}, t_{n'_J})$ are near $(u, 0)$ when n'_J is large, but $(f_{t_{n'_J}}(u_{n'_J}), t_{n'_J}) = (f_{t_{n'_J}}(v_{n'_J}), t_{n'_J})$ so $H(u_{n'_J}, t_{n'_J}) = H(v_{n'_J}, t_{n'_J})$. Therefore, f_t has to be an embedding when $t < \varepsilon$.

□

If $f : X \to Y$ is an embedding, then equivalently (since f is injective), $X \sim f(X) \overset{\text{submanifold}}{\subseteq} Y$. If f_t is a homotopy, then when $t < \varepsilon$, if $X \pitchfork Z$, then $X_t = f_t(X)$ is a submanifold and $X_t \pitchfork Z$.

If X is compact, $Z \subseteq Y$ is a closed submanifold, $f : X \to Y$, then $f \pitchfork Z$ is stable.

16.2 Critical Values

Suppose that originally, $f \not\pitchfork Z$. Is there a small perturbation of f such that $\tilde{f} \pitchfork Z$?

Given $f : X \to Y$, recall that $\{y\}$ is a regular *value* if df_x is surjective at every $x \in f^{-1}(y)$. In the case where $f^{-1}(y) = \varnothing$, we still call y a regular value. If y is not a regular value, we call y a **critical value**, i.e., there exists $x \in f^{-1}(y)$ such that df_x is not surjective. A point x at which df_x is not surjective is called a **critical point**. Given $f : X \to Y$, how big can {critical points of f} be? If $f : X \to Y$ is constant, then {critical points of f} $= X$.

Theorem 16.1 (Sard's Theorem)**.** *Given $f : X \to Y$ smooth, the set of critical values of f has measure 0 in Y.*

Lecture 17

October 24

17.1 Sard's Theorem

Sard's Theorem: Given a smooth map $f : X \to Y$, the set of critical values of f in Y has measure 0 in Y.

Example 17.1. Let $X = S^1$, $Y = \mathbb{R}$, and $f : X \to Y$ be given by drawing a closed loop in $\mathbb{R}^2$ and projecting down onto the horizontal axis. The critical values are the points at which the closed loop has a vertical tangent line.

Corollary 17.2. *If $\dim X < \dim Y$, then the set of critical values is $f(X)$. Sard's Theorem 16.1 implies $f(X)$ has measure 0.*

17.1.1 Measure-0 Subsets in $\mathbb{R}^n$

For a box $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, we can define volume $B = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$.

Definition 17.3. A subset $A \subseteq \mathbb{R}^n$ has **measure** 0 if for all $\varepsilon > 0$, there exists a countable collection of boxes $\{S_i\}_{i=1}^{\infty}$ such that $A \subseteq \bigcup_{i=1}^{\infty} S_i$ and $\sum_{i=1}^{\infty} \operatorname{volume} S_i < \varepsilon$.

Example 17.4. If we consider $\mathbb{R}^m \subset \mathbb{R}^n$ for $m < n$, we can identify $\mathbb{R}^m = \{(x_1, \ldots, x_m, 0, \ldots, 0)\}$, then $\mathbb{R}^m$ has measure 0. First, cover $\mathbb{R}^m$ by $\{B_i\}_{i=1}^{\infty}$, where the $B_i \subseteq \mathbb{R}^m$ are boxes. For each i, choose ε_i small enough so that $B_i \times [-\varepsilon_i, \varepsilon_i]^{n-m} \subset \mathbb{R}^n$ is a box with

$$\operatorname{volume}(B_i \times [-\varepsilon_i, \varepsilon_i]) = (\operatorname{volume} B_i)(2\varepsilon_i)^{n-m} < \frac{\varepsilon}{2^i}.$$

Now, $\sum_{i=1}^{\infty} \operatorname{volume}(B_i \times [-\varepsilon_i, \varepsilon_i]) < \varepsilon$.

Example 17.5. $\mathbb{Q} \subset \mathbb{R}$. Enumerate $\{q_i\}_{i=1}^{\infty}$. Choose ε_i small,

$$\varepsilon_i < \frac{\varepsilon}{2^i}.$$

Take $B_i = (q_i - \varepsilon_i, q_i + \varepsilon_i)$. Then, $\bigcup_{i=1}^{\infty} B_i \supset \mathbb{Q}$ and $\sum_{i=1}^{\infty} \operatorname{volume} B_i < \varepsilon$.

Proposition 17.6 (Properties). *(a) If $\{A_i\}_{i=1}^{\infty}$ is a countable collection of measure-0 subsets, then $\bigcup_{i=1}^{\infty} A_i$ has measure 0.*

(b) *If $A \subset \mathbb{R}^n$ is such that $A \cap K$ has measure 0 for all K compact, then A has measure 0.*

(c) *$A \subset \mathbb{R}^n$ has measure 0 iff for all $x \in \mathbb{R}^n$, there exists U_x open such that $U_x \cap A$ has measure 0.*

Proof. (a) Given $\varepsilon > 0$, for each i, find a countable collection $\{B_m^{(i)}\}_{m\in\mathbb{N}}$ such that $\bigcup_{m\in\mathbb{N}} B_m^{(i)} \supseteq A_i$ and

$$\sum_{m\in\mathbb{N}} \text{volume}\, B_m^{(i)} < \frac{\varepsilon}{2^i}.$$

Now, $\mathcal{B} = \bigcup_{i=1}^{\infty}\{B_m^{(i)}\}_{m\in\mathbb{N}}$ is a countable collection,

$$\sum_{B\in\mathcal{B}} \operatorname{vol} B < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon,$$

and $\bigcup_{B\in\mathcal{B}} B \supseteq \bigcup_{i=1}^{\infty} A_i$. Hence, $\bigcup_{i=1}^{\infty} A_i$ has measure 0.

(b) $\mathbb{R}^n = \bigcup_{m\in\mathbb{N}} \overline{B_m(0)}$, where $K_m = \overline{B_m(0)}$ is compact. Then, $A_m = A \cap K_m$ has measure 0, but $A = \bigcup_{m\in\mathbb{N}} A_m$. By (a), A has measure 0.

(c) We need to show that $A \cap K$ has measure 0 for all K compact. For each such $x \in K$, let U_x be such that $U_x \cap A$ has measure 0. Then, $\{U_x\}_{x\in K}$ is an open cover of K. By compactness of K, pick $U_{x_1}, \ldots, U_{x_m}$ such that $\bigcup_{i=1}^{m} U_{x_i} \supseteq K$. Since $U_{x_i} \cap A$ has measure 0, $\bigcup_{i=1}^{m}(U_{x_1} \cap A)$ has measure 0. Hence, $K \cap A$ has measure 0. □

Definition 17.7. $A \subseteq X$, where $\dim X = n$, has measure 0 if for all parameterizations $\phi : U \subseteq \mathbb{R}^n \to X$, $\phi^{-1}(A \cap \phi(U))$ has measure 0.

Theorem 17.8. *If $f : \mathbb{R}^n \xrightarrow{smooth} \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ has measure 0, then $f(A)$ has measure 0.*

Proof. Given $a \in A$ and $U \subseteq \mathbb{R}^n$ open containing a, we will show that $f(U) \cap f(A)$ has measure 0. In $\overline{U}$, df is bounded. Say $\|df_x\| < M$ when $x \in \overline{U}$. Hence, there exists $c > 0$ such that if $x, y \in \overline{U}$, $\|f(x) - f(y)\| < c\|x - y\|$. Since $U \cap A$ has measure 0, given $\varepsilon > 0$, there exist boxes $\{B_m\}_{m\in\mathbb{N}} \supseteq U \cap A$, $B_m \subseteq U$, such that $\sum_{m\in\mathbb{N}} \text{volume}\, B_m < \varepsilon$. The $f(B_m)$ may not be boxes. However, by the inequality, we can put $f(B_m)$ inside some box $\tilde{B}_m$ and volume $\tilde{B}_m < 2c^n$ volume B_m. Then, $\bigcup_{m\in\mathbb{N}} \tilde{B}_m \supseteq f(U) \cap f(A)$ and

$$\begin{aligned}\sum_{m\in\mathbb{N}} \text{volume}\, \tilde{B}_m &< 2c^n \sum_{m\in\mathbb{N}} \text{volume}\, B_m \\ &< 2c^n \varepsilon.\end{aligned}$$

Thus, $f(A) \cap f(U)$ has measure 0. Hence, $f(A)$ has measure 0. □

Corollary 17.9. *If $f : \mathbb{R}^m \to \mathbb{R}^n$, $m < n$, then $f(\mathbb{R}^m)$ has measure 0.*

Proof. Consider the projection $\pi : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ given by $f \circ \pi : \mathbb{R}^n \to \mathbb{R}^n$. If we take $A = \mathbb{R}^m \subset \mathbb{R}^n$, then $g|_A = f$. By 17.8, $g(A) = f(\mathbb{R}^m)$ has measure 0 in $\mathbb{R}^n$. □

This is exactly a local mini-Sard's Theorem.

Theorem 17.10 (Mini-Sard's Theorem)**.** *If $f : X \to Y$, where $\dim X = m$, $\dim Y = n$, is smooth ($m < n$), then $f(X)$ has measure 0.*

Proof. Using countably many parameterizations of X and Y, we automatically get the global Mini-Sard's Theorem from the local one. □

Corollary 17.11. *If you have $\{f_i\}_{i \in \mathbb{N}}$, $f_i : X \to Y$ is smooth, $\dim X = m$, $\dim Y = n$ ($m < n$), then $\bigcup_{i \in \mathbb{N}} \operatorname{image} f_i$ has measure 0. In particular, for any $y \in Y$ and any $\delta > 0$, there exists $y_0 \in B_\delta(y) \cap Y$ such that $y_0 \notin \operatorname{image} f_i$ for all $i \in \mathbb{N}$.*

Exercise. If $A \subset \mathbb{R}^n$ has measure 0, then $\mathbb{R}^n \setminus A$ is dense in A.

17.1.2 Applications (Whitney's Embedding Theorem)

Theorem 17.12. *Given X, $\dim X = n$, there exists an embedding of $X \to \mathbb{R}^{2n+1}$.*

Actually, it is true that $X \xrightarrow{\text{embedded}} \mathbb{R}^{2n}$.

TX is the tangent space of X. If $X \subseteq \mathbb{R}^N$, then $TX = \{(x, v) \in \mathbb{R}^N \times \mathbb{R}^N \mid v \in T_x X\} \subseteq \mathbb{R}^N \times \mathbb{R}^N$.

Proposition 17.13. *TX is a manifold of dimension $2n$.*

Proof. It is good to think about $T\mathbb{R}$. □

Lecture 18

October 26

18.1 Application of Mini-Sard's Theorem

18.1.1 Whitney's Embedding Theorem

Sard's Theorem: If $f : X \to Y$ is smooth, then the set of critical values of f has measure 0.

Special case (Mini-Sard's Theorem): If $\dim X < \dim Y$, then the critcal values of f, $\operatorname{image} f$, has measure 0.

Theorem (Whitney's Embedding Theorem): If X is a manifold of dimension n, then there exists an embedding $X \to \mathbb{R}^{2n+1}$.

Recall that if $X \subseteq \mathbb{R}^n$, $TX = \{(x, v) \mid x \in X, v \in T_x X\} \subseteq \mathbb{R}^N \times \mathbb{R}^N$.

Example 18.1. If $X = \mathbb{R}$, then $TX = T\mathbb{R} \subseteq \mathbb{R} \times \mathbb{R}$. In fact, $TX = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Check that $TX \subseteq \mathbb{R}^N \times \mathbb{R}^N$ is a manifold of dimension $2n$.

Example 18.2. If $X = \mathbb{R}$ and we map $\mathbb{R} \to \mathbb{R}^3$ by $t \mapsto (t^2, t^3, t)$, where H is the xy-plane. The projection of the curve onto H is the function $x = y^{2/3}$ which has a cusp at 0.

Proof of Whitney's Embedding Theorem 17.12. Let $X \subseteq \mathbb{R}^N$. We will reduce

$$\mathbb{R}^N \to \mathbb{R}^{N-1} \to \cdots \to \mathbb{R}^{2n+1}.$$

Assume X is compact (so that we do not worry about properness of our map). Given $\mathbf{n}$, we can determine a hyperplane $H_{\mathbf{n}}$ going through 0. $\pi_H : \mathbb{R}^N \to H_{\mathbf{n}}$ is the projection. The issue is that $\pi_H : X \to H_{\mathbf{n}}$ may fail to be injective and immersive. The idea is to choose $\mathbf{n}$ correctly. Say $x_1, x_2 \in X$ and suppose $\pi_H(x_1) = \pi_H(x_2)$. Then, $x_1 - x_2 \parallel \mathbf{n}$, so there exists $t \in \mathbb{R}$ such that $x_1 - x_2 = t\mathbf{n}$. Define $F : X \times X \times I \to \mathbb{R}^N$ (where $N > 2n+1$), and $X \times X \times I$ has dimension $2n+1$, given by $F(x_1, x_2, t) = t(x_1 - x_2)$ (for $X \subseteq \mathbb{R}^N$). Then, $\operatorname{image} F$ has measure 0. This means that there exists $v \neq 0$ such that $v \notin \operatorname{image} F$, so there do not exist t, x_1, x_2 such that $t(x_1 - x_2) = v$, i.e., there do not exist x_1, x_2 such that $x_1 - x_2 \parallel v$. Choose $\mathbf{n} = v$, so π_H is injective. For an immersion, consider $G : TX \to \mathbb{R}^N$ given by $G(x, v) = v$. Then, π_H an immersion at x if and only if for all $v \in T_x X$, $v \nparallel \mathbf{n}$. We know that $N > 2n+1 > 2n$, $\dim TX = 2n$, and $\dim \mathbb{R}^N = N$, so $\operatorname{image} G$ has measure 0 in $\mathbb{R}^N$. Hence, there exists $w \in \mathbb{R}^N$ such that $w \notin \operatorname{image} G$. Choose $\mathbf{n} = w$ so π_H is an immersion. To solve for both issues, choose non-zero $\mathbf{n} \notin \operatorname{image} F \cup \operatorname{image} G$ since the latter set has measure 0. Hence, π_H is both injective and an immersion. Since X is compact, $\pi_H : X \to H$ is proper. Thus, π_H is an embedding. Repeat the argument until $N \rightsquigarrow 2n+1$.

In the case when X is non-compact, there is a diffeomorphism $\mathbb{R}^{2n+1} \xrightarrow{\phi} B_1(0)$. Since we may assume $X \subseteq \mathbb{R}^{2n+1}$, then using ϕ we may assume $X \subseteq B_1(0)$. Then, $X \subseteq B_1(0)$ may fail to be proper because all of the coordinate functions are not proper functions. Thus, $X \subseteq B_1(0) \subseteq \mathbb{R}^{2n+1}$, so we may consider

$$X \subseteq \underbrace{\mathbb{R}^{2n+1} \times \mathbb{R}}_{(x,\rho(x))}.$$

Now, notice that if you tilt $H_{\mathbf{n}}$ a bit and use that hyperplane to project, then the projection is proper. $\square$

Lemma 18.3. *For any manifold X, there is a proper function $\rho : X \to \mathbb{R}$.*

Example 18.4. Suppose $X \overset{\text{embedded}}{\subseteq} \mathbb{R}^N$, then you can take $\rho(x) = \|x\|^2$ which is proper.

Theorem 18.5 (Stronger Version)**.** *There exists an embedding $X \to \mathbb{R}^{2n}$.*

Recall that the immersion argument works as before, but the previous injectivity argument may fail. A lot of work is required to prove this (Whitney trick).

18.1.2 Another Application

Recall that S^n is the n-dimensional sphere.

Theorem 18.6. *Any smooth map $S^k \to S^n$ $(k < n)$ is homotopic to a constant map.*

Example 18.7. For $k = 1$, S^n is **simply connected**, i.e., any map $S^1 \to S^n$ is homotopic to a constant map.

Proof of 18.6. Recall the stereographic projection $\pi : S^n \setminus \{\star\} \xrightarrow{\text{diffeomorphism}} \mathbb{R}^n$. Since $k < n$, $f(S^k)$ has measure 0. So, there is a point $\star \in S^n$ that is not in $f(S^k)$. Hence, $\pi \circ f : S^k \to \mathbb{R}^n$ is homotopic to a constant map since $\mathbb{R}^n$ is contractible. Now, we use the inverse of π to pull back the homotopy $\pi \circ f \sim$ constant map to get $f \sim$ constant map in $S^n \setminus \{\star\} \subseteq S^n$. $\square$

18.1.3 Sard's Theorem

To prove Sard's Theorem 16.1 for $f : X \to Y$, $\dim Y \leq \dim X$, then write the set of critical points as $\bigcup_{k=1}^{\infty} C_k$, where C_k is the set of points at which all higher derivatives up to kth order of f vanish. Then, one must show that $f(C_1 \setminus C_2)$, $f(C_2 \setminus C_3)$, $f(C_3 \setminus C_4)$, ... all have measure 0, but for large enough k, $f(C_k)$ has measure 0.

Lecture 19

October 31

19.1 Proof of Sard's Theorem

The following are applications of Mini-Sard's Theorem 17.10.

Whitney's Embedding Theorem: If $\dim X = k$, there exists an embedding $X \to \mathbb{R}^{2k+1}$.

Whitney's Immersion Theorem: If $\dim X = k$, there exists an immersion $X \to \mathbb{R}^{2k}$.

Stronger version. There exists an embedding $X \to \mathbb{R}^{2k}$. This requires more work (Whitney trick).

Theorem (Sard's Theorem): If $f : X \to Y$ is smooth, then the critical values of f has measure 0 in Y.

Proof of 16.1. By Mini-Sard's Theorem 17.10, if $\dim X < \dim Y$, the critical values of f, $\operatorname{image} f$, has measure 0. If S is the set of critical points of f, then $S = \bigcup_{k \geq 0} C_k$, where

$$\begin{aligned}
C_0 &= \{x \in X \mid df_x \text{ is not surjective but not } 0\}, \\
C_1 &= \{x \in X \mid df_x = 0 \text{ but some 2nd-order derivative} \neq 0\}, \\
&\vdots \\
C_k &= \left\{ x \in X \;\middle|\; \begin{array}{c} \text{all derivatives up to } k\text{th order vanish} \\ \text{but some } (k+1)\text{th order derivative does not vanish} \end{array} \right\}
\end{aligned}$$

The idea is to show that $f(C_i)$ has measure 0. If we can show this, then the critical values of f, $\bigcup_{k \geq 0} f(C_k)$, has measure 0. For k sufficiently large, $f(C_k)$ has measure 0. For small k, you have to do something else. Pick y in $f(C_k)$. We must show that there exists an open set U containing y such that $U \cap f(C_k)$ has measure 0. Look at $f^{-1}(U) \subseteq X$, which is open in X. Technically, one needs to show there exists C, a cube in X containing $x \in C_k$ such that $f(C) \cap f(C_k)$ has measure 0. If $x \in C_k$, f locally can be written as a Taylor series.

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \cdots + \frac{1}{k!}f^{(k)}(x)h^k + \frac{1}{(k+1)!}f^{(k+1)}(x)h^{k+1} + \cdots$$

Hence, in C, we have the following inequality: There exists $M > 0$ such that for all $x \in C_k \cap C$ and $x + h \in C$, then $|f(x+h) - f(x)| < M|h|^{k+1}$. Say the cube C has size δ. Divide the cube into r^n small cubes of equal size δ/r ($n = \dim X$). For each small cube C' that intersects C_k, by the inequality, we have $f(C')$ is inside some cube of size $(\delta/r)^{k+1}$, which has volume $(\delta/r)^{(k+1)\dim Y}$. Also, there are only

r^n cubes, so $f(C_k) \cap f(C)$ is roughly at most

$$r^n \left(\frac{\delta}{r}\right)^{(k+1)\dim Y} = \frac{\delta^{(k+1)\dim Y}}{r^{(k+1)\dim Y - n}}.$$

As $r \to \infty$ and $(k+1)\dim Y - n > 0$, then $f(C_k) \cap f(C)$ has measure 0. $\square$

19.2 Manifolds with Boundaries

Definition 19.1. $X \subseteq \mathbb{R}^N$ is a k**-dimensional manifold with boundary** if for every point $x \in X$, there exists an open set U in X such that U is diffeomorphic to an open set in $\mathbb{H}^k$, where

$$\mathbb{H}^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k \geq 0\}.$$

Example 19.2. $\mathbb{H}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$. If U is an open set in $\mathbb{R}^2$ which intersects the horizontal axis, then $U \cap \mathbb{H}^2$ is not open in $\mathbb{R}^2$, but it is open in $\mathbb{H}^2$.

Topology. In general, if $X \subseteq \mathbb{R}^n$, then U is open in X by definition if there exists $\tilde{U}$, open in $\mathbb{R}^N$, such that $U = \tilde{U} \cap X$.

Lemma 19.3. *$\mathbb{H}^k$ and $\mathbb{R}^k$ are not diffeomorphic.*

Proof. Suppose there exists a diffeomorphism ϕ from the open disk U in $\mathbb{R}^k$ to $U \cap \mathbb{H}^k$. Let $x \in U$ such that $\phi(x)$ is in $U \cap \mathbb{H}^k$. Think of ϕ as a map from the open set U in $\mathbb{R}^k$ to $U \cap \mathbb{H}^k \subset \mathbb{R}^k$. Notice that $d\phi_x$ is an isomorphism, so by the Inverse Function Theorem 10.3, $\phi(U)$ is an open set in $\mathbb{R}^k$ (contradiction). $\square$

From 19.3, we can distinguish points on the boundary of X as follows.

$$\partial X := \{\text{those points that belong to the image of } \partial\mathbb{H}^k \text{ under some local parameterization}\}.$$

Note that

$$\begin{aligned}
\partial\mathbb{H}^k &= \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k = 0\} = \mathbb{R}^{k-1}, \\
\mathbb{H}^k &= \{x_k \geq 0\},
\end{aligned}$$

where x_k is the last coordinate.

Corollary 19.4. *∂X is a manifold without boundary of dimension $k-1$.*

Suppose X has $\partial X \neq \varnothing$, Y has $\partial Y = \varnothing$. Then, $X \times Y$ has boundary $\partial(X \times Y) = \partial X \times Y$.

If $X = Y = [0,1]$, then $X \times Y$ is not a manifold with boundary because there is no way to parameterize the corner. How do we construct a manifold with boundary?

Lemma 19.5. *Let $\pi : X \to \mathbb{R}$, where X is a manifold without boundary. Say $0 \in \mathbb{R}$ is a regular value of π. Then, $Y = \{x \in X \mid \pi(x) \geq 0\} = \pi^{-1}([0,\infty))$ is a manifold with boundary $\partial Y = \pi^{-1}(0)$.*

Example 19.6. If $\pi : \mathbb{R}^n \to \mathbb{R}$ is given by $(x_1, \ldots, x_n) \mapsto x_1^2 + \cdots + x_n^2 - 1$, then $\{0\}$ is a regular value of π, so $Y = \pi^{-1}([0,\infty))$ is the complement of the open unit ball.

Proof of 19.5. Let $x \in \pi^{-1}(0)$. Since 0 is a regular value of π, π is a submersion at x. The Local Submersion Theorem 12.3 implies that there exists $U \ni x$ such that:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \mathbb{R} \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\psi} \\ U & \longrightarrow & \mathbb{R} \end{array}$$

where the map $U \to \mathbb{R}$ is the projection of the last coordinate $(x_1, \ldots, x_k) \mapsto x_k$. Hence, in U, $\pi^{-1}([0, \infty)) \cap U$ is exactly $\{(x_1, \ldots, x_k) \in U \mid x_k \geq 0\}$. Hence, $U \cap Y$ is diffeomorphic to

$$\{(x_1, \ldots, x_k) \in U \mid x_k \geq 0\},$$

so Y is a manifold with boundary and the boundary is exactly $\pi^{-1}(0)$. $\square$

Lecture 20

November 2

20.1 Manifolds with Boundaries

The boundary of X is

$$\partial X = \bigcup_{\phi \text{ parameterization of } X} \phi(\partial \mathbb{H}^k).$$

Then, $\operatorname{int} X = X - \partial X$ is the **interior of** X.

Theorem: Let $Y \xrightarrow{\pi} \mathbb{R}$ be a smooth map ($\partial Y = \varnothing$) and $\{0\}$ be a regular value of π. Then,

$$X = \pi^{-1}\big([0,\infty)\big) \subseteq Y$$

is a manifold with boundary $\partial X = \pi^{-1}(0)$.

If we have a map $f : X \to Y$, where $\partial X \neq \varnothing$, $\partial Y = \varnothing$, how do we define df_x? If $x \in \operatorname{int} X$, then you can define df_x as usual. If $x \in \partial X$, recall that if $f : \mathbb{H}^k \to \mathbb{R}^n$ is smooth, then around each point $x \in \mathbb{H}^k$, we can extend f to a smooth map $\tilde{f}$ defined on the open set $U_x \subseteq \mathbb{R}^k$. Define $df_x := d\tilde{f}_x$. Note that if $\tilde{f} : U_x \to \mathbb{R}^n$, then $d\tilde{f}_x : T_x U_x \to T_{\tilde{f}(x)}\mathbb{R}^n$, where $T_x U_x = T_x \mathbb{H}^k = T_x \mathbb{R}^k$.

The **restriction of** f **on** ∂X is denoted by $\partial f : \partial X \to Y$, where ∂X is a manifold without boundary.

Notice that if X has dimension m, then $T_x X$ has dimension m, and $T_x \partial X$ has dimension $m-1$ (for $x \in \partial X$).

20.1.1 Transversality

Suppose $f : X \to Y$, where $\partial X \neq \varnothing$, $\partial Y = \varnothing$. Let $Z \subseteq Y$ be a submanifold with $\partial Z = \varnothing$.

Before, if $X = S^2$, $Y = \mathbb{R}^3$, and Z is the (x,y)-plane, then $f \pitchfork Z$ since $df_x T_x X + T_{f(x)} Z = T_{f(x)}\mathbb{R}^3$ (f is the inclusion). However, if we let X be the upper half of the sphere, then slightly tilting X changes the intersection so the intersection is not transversal. On the other hand, if f tilts X slightly, then $\partial f : \partial X \to Y$ and $\partial f \pitchfork Z$ since $\operatorname{image} d(\partial f)_x + T_{\partial f(x)} Z = T_{\partial f(x)}\mathbb{R}^3$.

Definition 20.1. If $f : X \to Y$, then $f \pitchfork Z$ if $f \pitchfork Z$ and $\partial f \pitchfork Z$.

Theorem 20.2. *If f is transversal to Z, then $M = f^{-1}(Z)$ is a manifold with boundary*

$$\partial M = (\partial f)^{-1}(Z).$$

Proof. If $x \in M$ and $x \in \operatorname{int} M$, then the Pre-Image Theorem 12.6 we proved earlier can be applied here. Now, say $x \in M$ and $x \in \partial X$. Since f is transversal to Z, in local coordinates, $df_0 : T_0\mathbb{H}^k \to T_0\mathbb{R}^n$ satisfies

$$\operatorname{image} df_0 + T_0 Z = T_0\mathbb{R}^n \tag{20.1}$$

and $d(\partial f)_0 : T_0\partial\mathbb{H}^k \to T_0\mathbb{R}^n$ satisfies

$$\operatorname{image} d(\partial f)_0 + T_0 Z = T_0\mathbb{R}^n. \tag{20.2}$$

Notice that $T_0\partial\mathbb{H}^k \subseteq T_0\mathbb{H}^k$, and $df_0|_{T_0\partial\mathbb{H}^k} = d(\partial f)_0$. Hence, if $d(\partial f)_0$ satisfies the transversality condition, then df_0 automatically satisfies the transversality condition. Say that $\tilde{f}$ is an extension of f near 0, i.e., $\tilde{f} : U_0 \to \mathbb{R}^n$ where U_0 is open in $\mathbb{R}^k$, and $\tilde{f}|_{\mathbb{H}^k \cap U_0} = f$. Since $d\tilde{f}_0 = df_0$ (by definition), $d\tilde{f}_0$ satisfies $\operatorname{image} d\tilde{f}_0 + T_0 Z = T_0\mathbb{R}^n$. Applying the Pre-Image Theorem 12.6 for $\tilde{f}$, we get $S = \tilde{f}^{-1}(Z)$ is a manifold. We claim that $S \cap \mathbb{H}^k = f^{-1}(Z)$ is a manifold with boundary $S \cap \partial\mathbb{H}^k = (\partial f)^{-1}(Z)$. Consider the map $S \xrightarrow{\pi} \mathbb{R}$ given by $\pi(x) = x_k$, where x_k is the last coordinate of x. Notice that $S \cap \mathbb{H}^k = \pi^{-1}([0,\infty))$. By 19.5, if we can show 0 is a regular value of π, then we are done. Suppose 0 is not a regular value, i.e., there exists $x \in \pi^{-1}(0)$ such that $d\pi_x : T_x S \to T_0\mathbb{R} = \mathbb{R}$ is not surjective, i.e., $d\pi_x = 0$. Hence, $T_x S \subseteq \ker d\pi_x = T_x\partial\mathbb{H}^k$. Since $S = \tilde{f}^{-1}(Z)$,

$$\begin{aligned} T_x S &= d\tilde{f}_x^{-1}(T_{\tilde{f}(x)} Z) \\ &= df_x^{-1}(T_{f(x)} Z). \end{aligned}$$

By Condition (20.1), $\operatorname{codim}_X T_x S = \operatorname{codim}_Y Z$. By Condition (20.2) plus the fact that $T_x S \subseteq T_x\partial\mathbb{H}^k$, then $T_x S = d(\partial f)_x^{-1}(T_{f(x)} Z)$, so $\operatorname{codim}_{\partial X} T_x S = \operatorname{codim}_Y Z$. This is a contradiction, since if $\dim X = n$, then we obtain

$$\begin{aligned} \operatorname{codim}_X T_x S &= n - \dim T_x S, \\ \operatorname{codim}_{\partial X} T_x S &= (n-1) - \dim T_x S. \end{aligned}$$

So, this implies that $f^{-1}(Z)$ is a manifold with boundary $\partial(f^{-1}(Z)) = (\partial f)^{-1}(Z)$. □

Thus, f is transversal to Z if and only if for all $x \in f^{-1}(Z)$,

1. if $x \in \operatorname{int} X$, $\operatorname{image} df_x + T_{f(x)} Z = T_{f(x)} Y$,
2. if $x \in \partial X$, $\operatorname{image} d(\partial f)_x + T_{f(x)} Z = T_{f(x)} Y$.

Theorem 20.3 (Sard's Theorem)**.** *If $f : X \to Y$ ($\partial X \neq \varnothing$, $\partial Y = \varnothing$), the set of critical values has measure 0 in Y.*

Proof. If y is a critical value, then there exists $x \in f^{-1}(y)$ such that either 1 or 2 fails, i.e., y is a critical value of either $f : \operatorname{int} X \to Y$ or $\partial f : \partial X \to Y$. Then, Sard's Theorem 16.1 applied to these maps implies that the set of critical values of f has measure 0. □

Lecture 21

November 7

21.1 Review

21.1.1 Measure Zero

$A \subset \mathbb{R}^k$ has measure 0 if for all $\varepsilon > 0$, there exists $\{B_i\}_{i\in\mathbb{N}}$, boxes, such that

- $A \subseteq \bigcup_{i\in\mathbb{N}} B_i$;
- $\sum_{i\in\mathbb{N}} \operatorname{vol} B_i < \varepsilon$.

If $A \subset \mathbb{R}$ is such that A has countably many elements, $A = \{q_1, q_2, \dots\}$, then choose boxes $B_i \ni q_i$ so that B_i has volume $\varepsilon/2^i$. Then, $\bigcup_{i=1}^{\infty} B_i \supset A$ and $\sum_{i=1}^{\infty} \operatorname{vol} B_i = \varepsilon$.

As another example, consider $\mathbb{R} \subset \mathbb{R}^2$. One can write $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [n, n+1]$. So,

$$\mathbb{R} \subseteq \bigcup_{n\in\mathbb{Z}} \underbrace{[n, n+1] \times \left[-\frac{\varepsilon}{2^{|n|}}, \frac{\varepsilon}{2^{|n|}}\right]}_{B_n}$$

and $\sum_{n\in\mathbb{Z}} \operatorname{vol} B_n = 3\varepsilon$.

If $\{A_i\}_{i=1}^{\infty}$ is such that each A_i has measure 0, then $\bigcup_{i=1}^{\infty} A_i$ has measure 0.

$A \subset \mathbb{R}^n$ has measure 0 if and only if $A \cap B_r(0)$ has measure 0 for all $r > 0$, since $A = \bigcup_{n=1}^{\infty} (A \cap B_n(0))$.

Theorem: A has measure 0 if there exists $x \in A$ and open $U \ni x$ such that $U \cap A$ has measure 0.

Sard's Theorem: The critical values of f in Y has measure 0.

Note that $\mathbb{R} \subset \mathbb{R}^2$ has measure 0, but $\mathbb{R} \subseteq \mathbb{R}$ does not have measure 0.

21.1.2 Critical Values & Whitney's Immersion Theorem

Definition: Let $f : X \to Y$. y_0 is a critical value if there exists $x_0 \in f^{-1}(y_0)$ such that df_{x_0} is *not* surjective.

If X has dimension n, then TX has dimension $2n$. Let $f : X \to \mathbb{R}^N$ be the embedding of X in $\mathbb{R}^N$ and

$$G : TX \to T\mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N \xrightarrow{\text{second coordinate projection}} \mathbb{R}^N,$$

i.e., $G(x, v) = df_x(v)$ for $v \in T_xX$. If $N > 2n$, then $\operatorname{image} G$ has measure 0. $\pi : X \to H$ fails to be an immersion at x if T_xX contains $\mathbf{n}$. Using Mini-Sard's Theorem 17.10 and $2n < N$, there exists $\mathbf{n} \neq 0$ that satisfies $\mathbf{n} \notin T_x X$ for all $x \in X$.

Consider the map $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by $t \mapsto (t^2, t^3, t)$. If $\pi : X \to \mathbb{R}^2$ is the projection onto the plane, then π fails to be an immersion at $t = 0$.

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_{t=0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$T_{(0,0,0)}X = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Then, $d\pi_{(0,0,0)} : T_{(0,0,0)}X \to T_{(0,0)}\mathbb{R}^2$ is 0, i.e., not injective.

From the homework: If $f : X \to Y$, where X is compact, then the set of regular values is open in Y, so the set of critical values cannot be dense in Y.

$df_{x_0} : T_{x_0}X \to T_{f(x_0)}Y$ is an isomorphism if it is an isomorphism as a linear map. If $L : V_1 \to V_2$ is linear, where V_1, V_2 are vector spaces, then L is an isomorphism if $\dim V_1 = \dim V_2$, $\ker L = \{0\}$, and L is onto.

21.1.3 Transversality

If $f : X \to Y$ and $\pi : Y \to \mathbb{R}^k$ where $k = \operatorname{codim}_Y Z$, then $f \pitchfork Z$ is equivalent to $\pi \circ f \pitchfork \{0\}$ for all such π, which is equivalent to the statement that 0 is a regular value of $\pi \circ f$. In this case, $f^{-1}(Z) = (\pi \circ f)^{-1}(0)$ has codimension in X equal to k, i.e., $\operatorname{codim}_X f^{-1}(Z) = k$.

If $f : X \to Y$ is an embedding, then $f(X)$ is a submanifold in Y, $f(X) \cong X$. Then, $f \pitchfork Z$ is equivalent to the submanifold $f(X) \pitchfork Z$, so by the Pre-Image Theorem 12.6, $f^{-1}(Z) \subseteq Z$ is a submanifold in X. Also, $\operatorname{codim}_X(f(X) \cap Z) = \operatorname{codim}_Y Z$, so $\dim X - \dim(f(X) \cap Z) = \dim Y - \dim Z$. Then,

$$\dim\big(f(X) \cap Z\big) = -\dim Y + \dim Z + \dim X.$$

If $Z_1 \subseteq Y$, $Z_2 \subseteq Y$, then $Z_1 \pitchfork Z_2$ at y means $T_y Z_1 + T_y Z_2 = T_y Y$, and if $\dim Z_1 + \dim Z_2 < \dim Y$, then this condition can never hold, so transversality in this case is equivalent to $Z_1 \cap Z_2 = \varnothing$.

Lecture 22

November 14

22.1 Recap

We have studied manifolds with boundary. Both $\operatorname{int} X$ and ∂X are manifolds without boundary of dimension $\dim X$ and $\dim X - 1$ respectively.

Definition: If $f : X \to Y$ is a smooth map, where $\partial X \neq \varnothing$, $\partial Y = \varnothing$, and Z is a submanifold of Y with $\partial Z = \varnothing$, then $f \pitchfork Z$ if:

- $\forall x \in f^{-1}(Z) \cap \operatorname{int} Z,\ \operatorname{image} df_x + T_{f(x)}Z = T_{f(x)}Y;$
- $\forall x \in f^{-1}(Z) \cap \partial X,\ \operatorname{image} d(\partial f)_x + T_{f(x)}Z = T_{f(x)}Y.$

Theorem: If $f : X \to Y$, $Z \subseteq Y$ is a submanifold, $\partial X \neq \varnothing$, $\partial Y = \partial Z = \varnothing$, and $f \pitchfork Z$, then $f^{-1}(Z)$ is a manifold with boundary. The boundary is $f^{-1}(Z) \cap \partial X = (\partial f)^{-1}(Z)$.

Theorem (Sard's Theorem): If $f : X \to Y$ is smooth, then the set of critical values of f has measure 0.

22.2 Classification of Compact 1-Manifolds with or without Boundary

Every connected component of a compact 1-manifold is diffeomorphic to the closed unit interval or to S^1.

Corollary 22.1. *The number of boundary points of X ($\dim X = 1$) is even.*

Theorem 22.2. *Suppose X is a compact smooth manifold ($\dim X = n$) with $\partial X \neq \varnothing$. Then, there does not exist a map $g : X \to \partial X$ such that $g|_{\partial X} = \operatorname{id}$ (i.e., $\partial g : \partial X \to \partial X$ is $\partial g = \operatorname{id}$).*

Proof. Assume that $g : X \to \partial X$ exists. Sard's Theorem 20.3 implies that if we choose $z \in \partial X$ such that z is a regular value of g, then $g^{-1}(\{z\})$ is a manifold with boundary $(\partial g)^{-1}(\{z\}) = \{z\}$. Since $\operatorname{codim}_{\partial X}\{z\} = \dim \partial X$, we have $\operatorname{codim}_X g^{-1}(\{z\}) = \dim \partial X$. Hence, $g^{-1}(\{z\})$ is a 1-dimensional manifold with boundary $\{z\}$. Since $g^{-1}(\{z\})$ is closed in X and X is compact, we obtain $g^{-1}(\{z\})$ is compact. Since $g^{-1}(\{z\})$ only has 1 boundary point, this is a contradiction. □

Corollary 22.3 (Brouwer's Fixed Point Theorem)**.** *Suppose $f : B \to B$ is smooth, where B^n is a compact n-dimensional ball. Then f must have a fixed point.*

Proof. Assume f has no fixed point. We will construct a smooth map $g : B \to \partial B$ such that $g|_{\partial B} = \mathrm{id}$ as follows. Since $f(x) \neq x$, construct the ray starting at $f(x)$ and connecting to x, and define $g(x)$ as the unique point in the intersection of the ray and ∂B. Then, g is smooth. Also, when $x \in \partial B$, $g(x) = x$. So, $g : B \to \partial B$ and $g|_{\partial B} = \mathrm{id}$ (contradiction). $\square$

22.3 Genericity of $\pitchfork$

Suppose $f : X \to Y$, with $Z \subseteq Y$ a submanifold, and $f \not\pitchfork Z$. Let $F : X \times S \to Y$ be a deformation, where S contains a deformation parameter (S is a finite-dimensional manifold). We will try to say that when s is small, $f_s \pitchfork Z$. We need $F \pitchfork Z$. Q: For what $s \in S$ is it true that $f_s \pitchfork Z$? Consider the map $\pi : W = f^{-1}(Z) \to S$. We will see that if s is a regular value of π, then $f_s \pitchfork Z$.

Next time, we will discuss the Transversality Homotopy Theorem.

Lecture 23

November 16

23.1 Genericity of $\pitchfork$

Recall that if $\partial X \neq \varnothing$, $f \pitchfork Z$ means $f|_{\text{int } X} \pitchfork Z$ and $\partial f \pitchfork Z$.

Theorem 23.1 (Transversality Homotopy Theorem)**.** *Given $f : X \to Y$ smooth (X can have boundary), $\partial Y = \varnothing$, and $Z \subseteq Y$ is a submanifold, $\partial Z = \varnothing$, then there exists $g : X \to Y$ that is homotopic to f such that $g \pitchfork Z$.*

Lemma 23.2 (Transversality Theorem)**.** *Consider $F : X \times S \to Y$, where S is a manifold with no boundary and we have a general Y with no boundary. Think of $F = \{f_s\}_{s \in S}$ where $f_s(x) = F(x, s)$. Assume $F \pitchfork Z$. Define $W = F^{-1}(Z)$, a submanifold in $X \times S$, and consider the projection $\pi : W \to S$. If s is a regular value of π, then $f_s \pitchfork Z$.*

Proof. Suppose s_0 is a regular value of $\pi : W \to S$, i.e., for all $(x, s_0) \in W$, $d\pi_{(x,s_0)} : T_{(x,s_0)}W \to T_{s_0}S$ is surjective. Notice that

$$\underbrace{W \cap (X \times \{s_0\})}_{\pi^{-1}(\{s_0\})} = f_{s_0}^{-1}(Z).$$

To show that $f_{s_0} \pitchfork Z$, we need to show that for all $x_0 \in f_{s_0}^{-1}(Z)$, $d(f_{s_0})_{x_0}(T_{x_0}X) + T_{f_{s_0}(x_0)}Z = T_{f_{s_0}(x_0)}Y$. Given $v \in T_{f_{s_0}(x_0)}Y$, we want to solve $d(f_{s_0})_{x_0}(u) + z = v$, where $u \in T_{x_0}X$, $z \in T_{f_{s_0}(x_0)}Z$. Since $F \pitchfork Z$, $dF_{(x_0,s_0)}(T_{(x_0,s_0)}(X \times S)) + T_{F(x_0,s_0)}Z = T_{F(x_0,s_0)}Y$. Hence,

$$\begin{aligned} v &= dF_{(x_0,s_0)}(u_{x_0}, u_{s_0}) + z \\ &= dF_{(x_0,s_0)}(u_{x_0}, 0) + dF_{(x_0,s_0)}(0, u_{s_0}) + z \\ &= d(f_{s_0})_{x_0}(u_{x_0}) + dF_{(x_0,s_0)}(0, u_{s_0}) + z. \end{aligned}$$

Since s_0 is a regular value of π, we can find a vector $k \in T_{(x_0,s_0)}W$ such that $u_{s_0} = d\pi_{(x_0,s_0)}(k)$. Define $k_{x_0} = k - (0, u_{s_0})$. So,

$$\begin{aligned} dF_{(x_0,s_0)}(0, u_{s_0}) &= dF_{(x_0,s_0)}(0, k - k_{x_0}) \\ &= dF_{(x_0,s_0)}(k) - \underbrace{dF_{(x_0,s_0)}(k_{x_0})}_{d(f_{s_0})_{x_0}(k_{x_0})} \\ &= dF_{(x_0,s_0)}(k) - d(f_{s_0})_{x_0}(k_{x_0}) \end{aligned}$$

since k_{x_0} has no S-component, and $dF_{(x_0,s_0)}(k) \in T_{f_{s_0}(x_0)}Z$ since $k \in T_{(x_0,s_0)}(F^{-1}(Z))$. Thus, the equation $d(f_{s_0})_{x_0}(u) + z = v$ is solved by

$$\begin{aligned} v &= d(f_{s_0})_{x_0}(u_{x_0}) + z + dF_{(x_0,s_0)}(k) - d(f_{s_0})_{x_0}(k_{x_0}) \\ &= \underbrace{d(f_{s_0})_{x_0}(u_{x_0} - k_{x_0})}_{\in \text{image } df_{s_0}} + \underbrace{z + dF_{(x_0,s_0)}(k)}_{\in T_{f_{s_0}(x_0)} Z}. \end{aligned}$$

Hence, $f_{s_0} \pitchfork Z$. □

We now apply 23.2 to prove 23.1. One should think of S as the parameter of deformation.

Proof of 23.1. Case 1. $Y = \mathbb{R}^n$. Take

$$F : X \times \underbrace{\mathbb{R}^n}_{S} \to Y = \mathbb{R}^n,$$

given by $F(x,s) = f(x) + s$. Then, $f_s(x) = f(x) + s$ is the translation of f along s. Also, $F \pitchfork Z$ since $dF_{(x,s)}(0,v) = v$ for all $v \in T_{F(x,s)}Y$. So, 23.2 implies that if s_0 is a regular value of $\pi : W \to S = \mathbb{R}^n$, then $f_{s_0} \pitchfork Z$. Sard's Theorem 20.3 implies that such an s_0 exists (almost all points in S are regular values of π). Define $g = f_{s_0}$ and g is homotopic to f by $\{f_{ts_0}\}_{t\in[0,1]}$.

Case 2. $Y \subset \mathbb{R}^n$ is compact. Define

$$F : X \times \underbrace{B_\varepsilon(0)}_{S} \to Y$$

by $F(x,s) = \pi(f(x) + s)$, where ε is chosen small enough so that π is defined. Since $F \pitchfork Z$ (since F is the composition of $(x,s) \mapsto f(x) + s$ and π, both of which are submersions), then by 23.2 there exists $s \in S$ such that $f_s \pitchfork Z$. In the case when Y is not compact, define $F : X \times B_1(0) \to Y$ by $F(x,s) = \pi(f(x) + \varepsilon(f(x))s)$. □

Theorem 23.3 (ε-Neighborhood Theorem)**.** *There exists Y_ε, an open set containing Y, such that for each $p \in Y_\varepsilon$, there is a unique point $y_0 \in Y$ such that $d(p, y_0) = \min_{y\in Y} d(p,y)$, and also $\pi : Y_\varepsilon \to Y$ given by $\pi(p) = y_0$ is a submersion.*

Example 23.4. If $Y = S^1 \subset \mathbb{R}^2$, then take Y_ε to be an annulus containing S^1 and $p - y_0 \perp T_{y_0}Y$.

Theorem 23.5 (Extension Theorem)**.** *Let $f : X \to Y$, $Z \subseteq Y$ be a closed submanifold, and let $C \subseteq X$ be a closed set where $f \pitchfork Z$ in C. Then, there exists $g : X \to Y$ that is homotopic to f such that $g \pitchfork Z$ and also $g = f$ in some open set containing C.*

Proof. Some important parts are left for the homework. □

If $\dim X + \dim Z < \dim Y$, then we can pull X and Z apart from each other by a small perturbation (since $X \pitchfork Z$ implies $X \cap Z = \varnothing$).

Next Time: $\dim X + \dim Z = \dim Y$. In this case, we can perturb them so that $X \pitchfork Z$. We can count the intersections in $X \cap Z$, which is a collection of points.

Lecture 24

November 28

24.1 Mod 2 Intersection Theory

Let $f : X \to Y$, where $\partial X = \partial Y = \varnothing$. Let $Z \subseteq Y$ be a submanifold. Assume $\boxed{\dim X + \dim Z = \dim Y}$. If $f \pitchfork Z$, then $f^{-1}(Z) \subseteq X$ is a submanifold of dimension 0. Also, if X is compact, $f^{-1}(Z)$ is a finite collection of points $\{x_1, \ldots, x_n\}$.

Definition 24.1. $I_2(f, Z) = n \bmod 2$.

In the case when $f \not\pitchfork Z$, the Transversality Homotopy Theorem 23.1 gives another map $\tilde{f} : X \to Y$ that is $\tilde{f} \overset{\text{homotopic}}{\sim} f$, $\tilde{f} \pitchfork Z$.

Definition 24.2. $I_2(f, Z) = I_2(\tilde{f}, Z)$.

Check: The definition is independent of the choice of $\tilde{f}$. Suppose $\tilde{f}_1 \overset{F}{\sim} \tilde{f}_2$, $\tilde{f}_1 \pitchfork Z$, $\tilde{f}_2 \pitchfork Z$. Using the Extension Theorem 23.5, make $\tilde{F} \pitchfork Z$ such that $\tilde{F} \sim F$, and $\tilde{F} = F$ near $\partial(X \times I)$. $\tilde{F}^{-1}(Z)$ is a compact 1-dimensional manifold with boundary $(\partial \tilde{F})^{-1}(Z) = \tilde{f}_1^{-1}(Z) \cup \tilde{f}_2^{-1}(Z)$, which has to have an even number of points. Thus, $\#\tilde{f}_1^{-1} \equiv \#\tilde{f}_2^{-1}(Z) \pmod 2$.

Now, assume $\dim X = \dim Y$, $Z = \{y_0\} \in Y$. Then, $\deg_2 f = I_2(f, Z)$ is the number of points in $\tilde{f}^{-1}(\{y_0\})$, modulo 2.

Observation: If y_0, y_1 are regular values of f, then $I_2(f, \{y_0\}) = I_2(f, \{y_1\})$, assuming Y is path-connected.

Theorem 24.3. *Let $X = \partial W$ where W is compact. Assume $f : X \to Y$ ($\dim X = \dim Y$) can be extended smoothly to W. Then,* $\deg_2 f = 0$.

Proof. Take a regular value y_0 of f, which exists by Sard's Theorem 20.3. Assume that g is the extension of f to W, that is, $g : W \to Y$ and $g|_{\partial W = X} = f$. By the Extension Theorem 23.5, let $\tilde{g} \overset{\text{homotopic}}{\sim} g$, $\tilde{g} \pitchfork \{y_0\}$, and $\tilde{g} = f$ in ∂W. Here, $g^{-1}(\{y_0\})$ is a compact 1-dimensional manifold with boundary $f^{-1}(\{y_0\})$. So, $\deg_2 f = \#f^{-1}(\{y_0\}) \bmod 2 = 0$. $\square$

Theorem 24.4 (Half Fundamental Theorem of Algebra)**.** *Every polynomial of odd degree with $\mathbb{C}$-coefficients has at least one root.*

Proof. Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ (n is odd). Choose X (with sufficiently large radius) such that the map $f = p/|p| : X \to S^1$ is homotopic to $\tilde{f}(z) = z^n/|z^n|$. Take

$$f_t(z) = \frac{(1-t)p(z) + tz^n}{|(1-t)p(z) + tz^n|}.$$

Thus,

$$\begin{aligned}(1-t)p(z) + tz^n &= (1-t)(z^n + a_{n-1}z^{n-1} + \cdots + a_0) + tz^n \\ &= z^n + (1-t)a_{n-1}z^{n-1} + \cdots + (1-t)a_0.\end{aligned}$$

When the radius of X is large, there is no $t \in [0,1]$ and $z \in X$ such that $(1-t)p(z) + tz^n = 0$. So, $\{f_t : X \to S^1\}_{t\in[0,1]}$ is a homotopy between f and $\tilde{f}$, and so $\deg_2 f = \deg_2 \tilde{f} = 1$. However, W does not have a root of p, then $f : X \to S^1$ can be extended to $g : W \to S^1$ by $g = p/|p|$. So, $\deg_2 f = 0$ (contradiction), and therefore, W must contain a zero of p. □

If we want to upgrade the invariant from $\mathbb{Z}_2$ to $\mathbb{Z}$ (algebra), we need extra structure on the spaces (topology). The extra structure is the orientation.

Lecture 25

November 30

25.1 Orientation

If $f : X \to Y$, $Z \subseteq Y$, $\dim X + \dim Y = \dim Z$, then $I_2(f, Z)$ is the number of points in $f^{-1}(Z)$. Here, f may be perturbed to be $\pitchfork Z$. The homotopy F is such that $F^{-1}(Z)$ is a compact 1-dimensional manifold ($F \pitchfork Z$) with boundary.

$$\begin{aligned} \partial F^{-1}(Z) &= (\partial F)^{-1}(Z) \\ &= f_1^{-1}(Z) \cup f_2^{-1}(Z), \end{aligned}$$

and since there are an even number of points in $\partial F^{-1}(Z)$, then $\# f_1^{-1}(Z) = \# f_2^{-1}(Z) \pmod 2$.

For a compact 1-dimensional manifold with boundary, the number of boundary points with *orientation* is 0 so

$$\begin{aligned} \underbrace{\partial F^{-1}(Z)}_{0} &= (\partial F)^{-1}(Z) \\ &= -f_1^{-1}(Z) + f_2^{-1}(Z). \end{aligned}$$

Given a vector space V with dimension n, an **orientation** of V is an ordered basis $\{v_1, \dots, v_n\}$ up to equivalence. $\{v_1, \dots, v_n\} \sim \{w_1, \dots, w_n\}$ if there exists A, an isomorphism $A : V \to V$, with $Av_i = w_i$ for all $i = 1, \dots, n$, such that $\det A > 0$.

Example 25.1. Let $V = \mathbb{R}^2$. (e_1, e_2) and (e_2, e_1) are two different basis representations, but if $Ae_1 = e_2$, $Ae_2 = e_1$, then

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so $\det A = -1 < 0$, and $(e_1, e_2) \not\sim (e_2, e_1)$.

Example 25.2. Consider (e_1, e_2) and $(e_2, -e_1)$. Then, $Ae_1 = e_2$, $Ae_2 = -e_1$, so

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $\det A > 0$. Thus, $(e_1, e_2) \sim (e_2, -e_1)$.

Observation: There are always two choices of equivalence classes.

If X is a manifold, $x \in X$, then T_xX is a vector space. An **orientation** of X is a *smooth* choice of orientations on each T_xX ($x \in X$).

Definition 25.3. If you can find a smooth choice for X, then X is called **orientable**.

Observation: If X is orientable, there are exactly 2 choices of orientation.

The Möbius strip is not orientable, as no such orientation exists. The Klein bottle is also not orientable.

If $\partial X \neq \varnothing$, if X has an orientation, it induces an orientation on ∂X. There is only one orientation on ∂X such that, when added to the outward normal vector, is consistent with the orientation of X.

Suppose X, Y, and Z are all oriented. Let $f : X \to Y$, $f \pitchfork Z$. $f^{-1}(Z)$ is a submanifold of X. What is the orientation of $f^{-1}(Z)$? $f^{-1}(Z)$ has an induced orientation coming from the orientations of X, Y, and Z.

In the case when X has a boundary, it is important to understand the orientation of $\partial f^{-1}(Z)$ and the orientation of $(\partial f)^{-1}(Z)$. (This is the **coherence issue**.) If the orientation is coming from being the boundary of the one-dimensional $f^{-1}(Z)$, then $\partial f^{-1}(Z) = (-1)^{\operatorname{codim}_Y Z}(\partial f)^{-1}(Z)$.

Definition 25.4. $I(f, Z)$ is the signed number of points in $f^{-1}(Z)$ where $f \pitchfork Z$.

$f^{-1}(Z)$ is a collection of points and each point has an orientation ($+$ or $-$) coming from the orientation defined above.

Definition 25.5. If $f : X \to Y$ and $\dim X = \dim Y$, $Z = \{y_0\}$, then $\deg f = I(f, Z = \{y_0\})$.

Theorem 25.6 (Fundamental Theorem of Algebra)**.** *Every polynomial of degree $n > 0$ has a root.*

Proof. Assume no root of p exists. Then, for $f = p/|p|$, $\deg f = 0$, but f is homotopic to the map $z \mapsto z^n/|z^n|$, and thus $\deg f = n \neq 0$. Hence, there exists a root inside X (where X is a big circle). ☐

25.2 Further Topics in Differential Topology

We have seen positive results and negative results. For example, there does not exist $g : W \to \partial W$ such that $g|_{\partial W} = \text{id}$. *Invariants* are useful for classification or for checking whether a certain construction is potentially doable.

For a manifold X, we can consider the zero section $X \subseteq TX$, where $(x, 0) \in \{(x, v) : v \in T_xX\}$. Then, $I(X, X)$ is the self-intersection of X, and it is an invariant of the manifold. If v is a vector field on X, what is $X_v \cap X$? For example, $I(S^2, S^2) = 2$, and for a torus, $I(S_g, S_g) = 2 - 2g$, and this is the **Euler characteristic** of X. In the case when the dimension is 2, $2 - 2g$ characterizes all orientable surfaces.

In **algebraic topology**, there are more invariants such as π_1, π_k, and H_k. These invariants are groups (richer structure than a number).

Example 25.7. Suppose that X has the same invariants as the sphere, i.e., $H_\cdot(X) \sim H_\cdot(S^n)$ and $\pi_1(X) = 0$. Is X diffeomorphic to the sphere?

Result: For $n \geq 5$, X is homeomorphic to the sphere. For $n = 5, 6$, then X is diffeomorphic to S^n. For $n = 7$, there are exotic spheres. For $n = 3$, we have Poincaré's Theorem (proved by Perelman). The case $n = 4$ is open.

For $n \geq 5$, the tools are mostly algebraic topology. For $n = 2, 3$, there is geometry. For $n = 4$, there is both geometry and physics.