Instructions

- Write your name and netID on each answer booklet that you use.
- The exam stops at the end of the lecture time.
- The exam has 2 pages with 6 problems (100 points total) and 3 bonuses (10 points total).
- Justify answers. Answers without justification will not necessarily receive credit. You may refer to results from lectures/labs/homeworks, so long as you clearly state what result you are using.
- This exam is open notes. You may use notes and books that you bring, but you may not use electronic devices of any kind (e.g., phones, laptops, iPads, calculators, etc.)
- You may not talk to any other students during the exam.

Problem 1 (20 points). *Are the following subspaces? Justify.*

(**a**) *Is* $S = \{(0, 0, 0)\}$ *a subspace of* $\mathbb{R}^3$*?*

(**b**) *Is* $S = \{(x, 10 + x, 10 - x) : x \in \mathbb{R}\}$ *a subspace of* $\mathbb{R}^3$*?*

(**c**) *Is* $S = \{(x, 0, 0) : x \geq 0\}$ *a subspace of* $\mathbb{R}^3$*?*

(**d**) *Is* $S = \{A \in \mathbb{R}^{n\times n} : A = -A^T\}$ *a subspace of* $\mathbb{R}^{n\times n}$*?*

Problem 2 (20 points). *True or false? Prove if true; give a counterexample if false.*

(**a**) *If* A *is a matrix, then* $\mathrm{Ker}(A)$ *admits an orthonormal basis.*

(**b**) *If* $\mathrm{dimKer}A = 2$*, then the linear system* $Ax = 0$ *has exactly two solutions* x*.*

(**c**) *If* λ *is an eigenvalue of* A*, then there are infinitely many vectors* x *satisfying* $Ax = \lambda x$*.*

(**d**) *If* $A, B, C \in \mathbb{R}^{n\times n}$*, then* $\mathrm{rank}(ABC) \leq \mathrm{rank}(B)$*.*

Problem 3 (20 points). *True or false? If false, give a counterexample; if true, prove and state the inverse. Do not forget to state the inverse in each case that is invertible. Below, let* $A, B \in \mathbb{R}^{n\times n}$*.*

(**a**) *If* A *is invertible, then* A^2 *is invertible.*

(**b**) *If* A *is invertible, then* A^{-1} *is invertible.*

(**c**) *If* A *and* B *are invertible, then* AB *is invertible.*

(**d**) *If* A *is stochastic, then* A *is invertible.*

Problem 4 (15 points). *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix}$$

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(**a**) *Determine the rank of A for all values of k.*

(**b**) *Let $b = (2022, 2022) \in \mathbb{R}^2$. For what values of k does the linear system $Ax = b$ have exactly one solution?*

Problem 5 (15 points)**.** *Let $J \in \mathbb{R}^{n\times n}$ be the matrix which has 1 in each entry. Compute all the eigenvalues of J and their multiplicities. (Hint: can you write $J = vv^T$ for some vector v?)*

Problem 6 (10 points)**.** *Let P_S be the matrix for the orthogonal projection onto a subspace $S \subset \mathbb{R}^n$. Let $M = Id_n - 2P_S$. What are all the possible values for eigenvalues of M? Justify.*

Bonus 1 (**Bonus: 3 points**)**.** *Provide a matrix $A \in \mathbb{R}^{2\times 2}$ for which $A^2 = 0$ yet $A \neq 0$.*

Bonus 2 (**Bonus: 3 points**)**.** *Prove that $\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} x_k^2$ for any $x_1, \cdots, x_n \in \mathbb{R}$.*

Bonus 3 (**Bonus: 4 points**)**.** *Suppose $A \in \mathbb{R}^{n\times n}$ is a stochastic matrix and all its entries are strictly positive. Prove that $A - Id_n$ is not invertible.*