Answer 1 (20 points). (**a**) *Yes. Proof: clearly S is a subset of $\mathbb{R}^3$, so we need only check that: (i) if $u, v \in S$ then $u = v = (0,0,0)$, so $u+v = (0,0,0) \in S$; (ii) if $u \in S$ and $\alpha \in \mathbb{R}$ then $\alpha u = (0,0,0) \in S$.*

(**b**) *No. Justification: $(0,0,0)$ is not in S.*

(**c**) *No. Justification: $x = (1,0,0) \in S$, however $-x = (-1,0,0) \notin S$.*

(**d**) *Yes. Proof: Clearly S is a subset of $\mathbb{R}^{n\times n}$, so we need only check that: (i) if $A, B \in S$ then $A + B = (-A^T)+(-B^T) = -(A+B)^T$, so $A+B \in S$; (ii) if $A \in S$ and $\alpha \in \mathbb{R}$ then $\alpha A = \alpha(-A^T) = -(\alpha A)^T$, so $\alpha A \in S$.*

Answer 2 (20 points). (**a**) *True. Proof:* $\mathrm{Ker}(A)$ *is a subspace, and in class we proved that any subspace admits an orthonormal basis. (Note: although incorrect, we also accepted "False" with a certain justification, see footnote[1].)*

(**b**) *False. Justification: A linear system can never have exactly two solutions, whereas there exist matrices with* $\mathrm{dimKer}A = 2$. *(For example the 2×2 zero matrix.)*

(**c**) *True. Proof: If λ is an eigenvalue of A, then there is some corresponding eigenvector $x \neq 0$, and therefore every scaling αx is an eigenvector too for every $\alpha \neq 0$.*

(**d**) *True. Proof: In lecture, we proved (i)* $\mathrm{rank}(XY) \leq \mathrm{rank}(X)$, *and (ii)* $\mathrm{rank}(XY) \leq \mathrm{rank}(Y)$ *for any matrices X, Y of compatible dimension. Thus* $\mathrm{rank}(ABC) \leq \mathrm{rank}(AB) \leq rank(B)$ *by applying (i) with $X = AB$, $Y = C$, and then applying (ii) with $X = A$ and $Y = B$.*

Answer 3 (20 points). (**a**) *True. If A is invertible, then A^{-1} exists, thus so does A^{-2}. Note that $A^2A^{-2} = AAA^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$, thus A^{-2} is the inverse of A^2.*

(**b**) *True. Proof: If A is invertible, then A^{-1} exists and $AA^{-1} = I$. This implies that A^{-1} is invertible with inverse A.*

(**c**) *True. If A and B are invertible, then A^{-1} and B^{-1} exist, and thus so does $B^{-1}A^{-1}$. Observe that $B^{-1}A^{-1}AB = B^{-1}IB = I$, thus $B^{-1}A^{-1}$ is the inverse of AB.*

(**d**) *False. Counterexample:* $A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. *This is not invertible since it has rank 1.*

Answer 4 (15 points). (**a**) *The rank of A is at least 1 since the first column is non-zero. The rank is 1 (else 2) if and only if the two columns are linearly dependent, i.e., if the columns are scalar multiples of each other, i.e., if $k = 6$. We conclude that A has rank 1 if $k = 6$, and otherwise has rank 2.*

(**b**) *In lecture, we proved that for a square matrix A, the linear system $Ax = b$ has exactly one solution if and only if A has full rank. Thus by part (a), $Ax = b$ has exactly one solution if and only if $k \neq 6$. (Note that this answer is independent of the value of the vector b.)*

Answer 5 (15 points). *$J = vv^T$ where $v \in \mathbb{R}^n$ is the vector with all entries 1. Thus by HW6, J has eigenvalue $\|v\|^2 = n$ with multiplicity 1, and eigenvalue 0 with multiplicity $n - 1$.*

Answer 6 (10 points). *From lecture, P_S can only have eigenvalues that are 0 and 1. Thus $Id_n - 2P_S$ can only have eigenvalues that are $1 - 2 * 0 = 1$ or $1 - 2 * 1 = -1$.*

Answer 7 (**Bonus: 3 points**). *For example, consider* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Answer 8 (**Bonus: 3 points**). *Apply the Cauchy-Schwarz inequality $\langle x, v\rangle^2 \leq \|x\|^2\|v\|^2$ where $v \in \mathbb{R}^n$ is the vector with all entries equal to 1.*

Answer 9 (**Bonus: 4 points**). *Since A is stochastic and has strictly positive entries, 1 is an eigenvalue of A. Thus $1 - 1 = 0$ is an eigenvalue of $A - Id$ by what was shown in lecture. Since 0 is an eigenvalue of $A - Id$, this matrix is not invertible.*

[1] Alternative answer: "False. Justification: For some matrices (e.g., the identity), $\mathrm{Ker}(A) = \{0\}$ which is a 0-dimensional subspace and thus doesn't admit an orthonormal basis." This is incorrect (since the empty set is technically an orthonormal basis of the set $\{0\}$), but we gave full marks for this since it demonstrates understanding of the main concepts.