Compressing images with Discrete Cosine Basis

In [1]:

```
%matplotlib inline
import numpy as np
import scipy.fftpack
import scipy.misc
import matplotlib.pyplot as plt
plt.gray()
```

```
<Figure size 640x480 with 0 Axes>
```

In [2]:

```
# Two auxiliary functions that we will use. You do not need to read them (but make sure to run tl

def dct(n):
    return scipy.fftpack.dct(np.eye(n), norm='ortho')

def plot_vector(v, color='k'):
    plt.plot(v,linestyle='', marker='o',color=color)
```

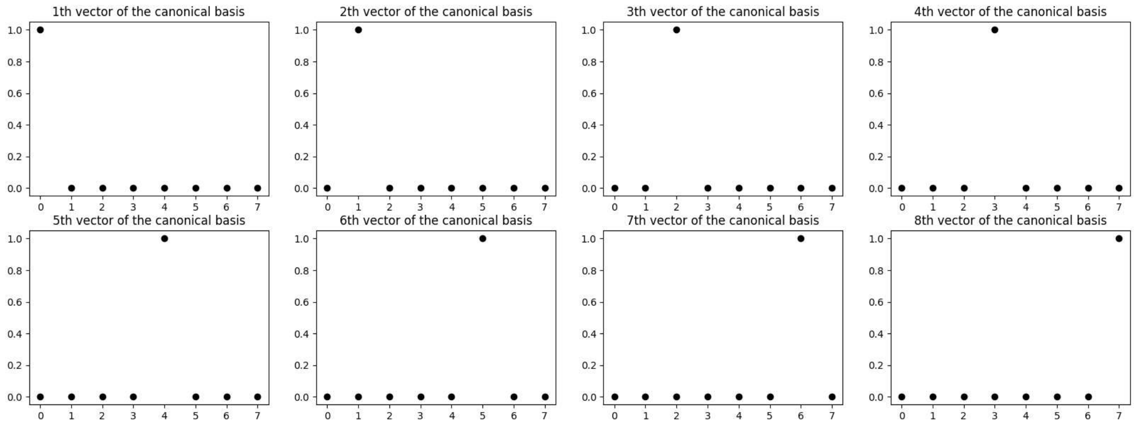
5.3.1 The canonical basis

The vectors of the canonical basis are the columns of the identity matrix in dimension n. We plot their coordinates below for $n = 8$.

In [3]:

```
identity = np.identity(8)
print(identity)

plt.figure(figsize=(20,7))
for i in range(8):
    plt.subplot(2,4,i+1)
    plt.title(f"{i+1}th vector of the canonical basis")
    plot_vector(identity[:,i])
```

```
[[1. 0. 0. 0. 0. 0. 0. 0.]
 [0. 1. 0. 0. 0. 0. 0. 0.]
 [0. 0. 1. 0. 0. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0. 0. 0.]
 [0. 0. 0. 0. 1. 0. 0. 0.]
 [0. 0. 0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 0. 0. 1.]]
```



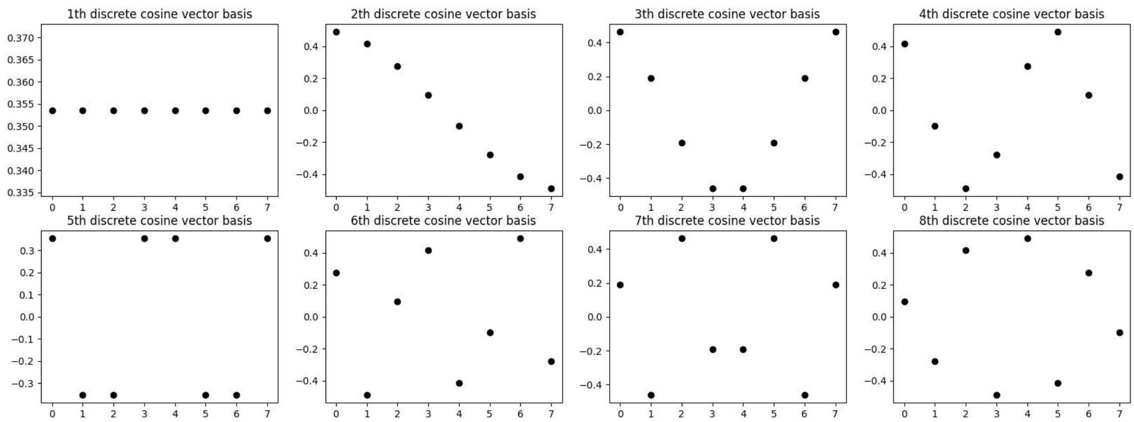
5.3.2 Discrete Cosine basis

The discrete cosine basis is another basis of $\mathbb{R}^n$. The function `dct(n)` outputs a square matrix of dimension n whose columns are the vectors of the discrete cosine basis.

In [4]:

```
# Discrete Cosine Transform matrix in dimension n = 8
D8 = dct(8)
print(np.round(D8,3))

plt.figure(figsize=(20,7))

for i in range(8):
    plt.subplot(2,4,i+1)
    plt.title(f"{i+1}th discrete cosine vector basis")
    plot_vector(D8[:,i])
```

```
[[ 0.354  0.49   0.462  0.416  0.354  0.278  0.191  0.098]
 [ 0.354  0.416  0.191 -0.098 -0.354 -0.49  -0.462 -0.278]
 [ 0.354  0.278 -0.191 -0.49  -0.354  0.098  0.462  0.416]
 [ 0.354  0.098 -0.462 -0.278  0.354  0.416 -0.191 -0.49 ]
 [ 0.354 -0.098 -0.462  0.278  0.354 -0.416 -0.191  0.49 ]
 [ 0.354 -0.278 -0.191  0.49  -0.354 -0.098  0.462 -0.416]
 [ 0.354 -0.416  0.191  0.098 -0.354  0.49  -0.462  0.278]
 [ 0.354 -0.49   0.462 -0.416  0.354 -0.278  0.191 -0.098]]
```



5.3 (a) Check numerically (in one line of code) that the columns of `D8` are an orthonormal basis of $\mathbb{R}^8$ (ie verify that the discrete cosine basis is an orthonormal basis).

In [13]:

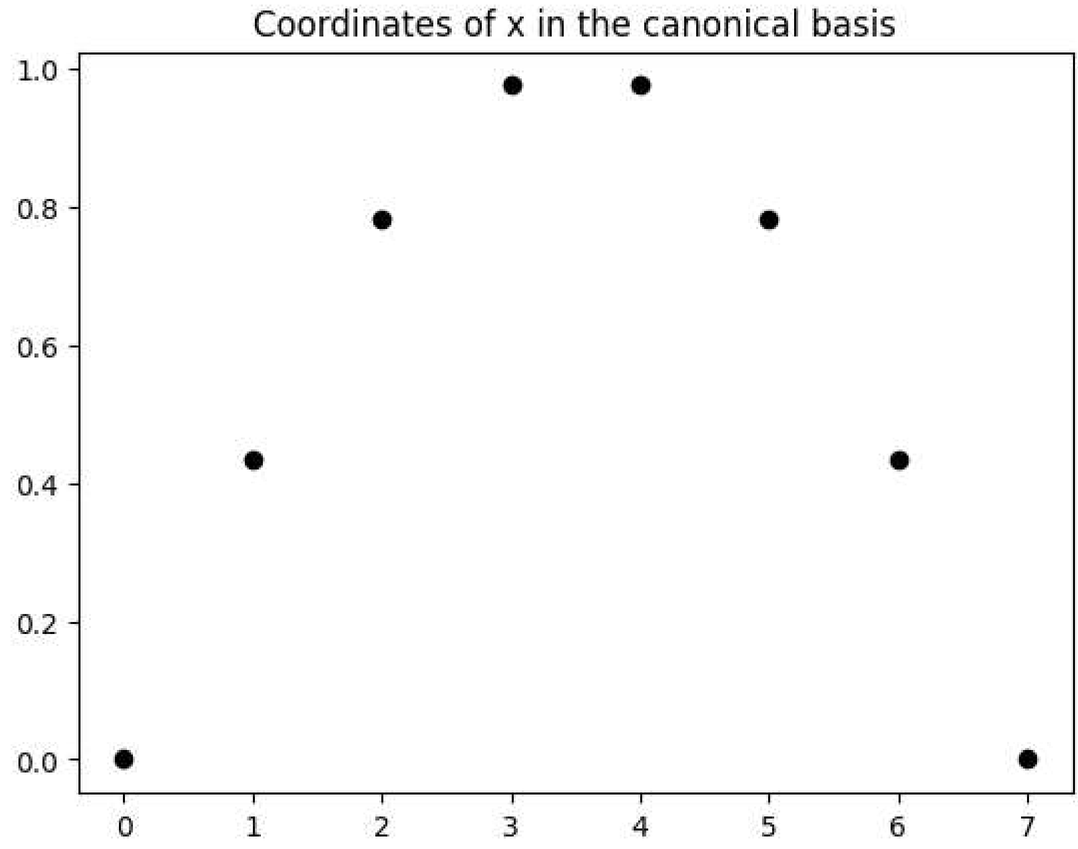
```
# Your answer here
D8.T @ D8 == np.eye(8)
```

Out[13]:

```
array([[False, False,  True,  True,  True,  True,  True,  True],
       [False, False, False, False,  True, False,  True,  True],
       [ True, False, False,  True,  True,  True,  True, False],
       [ True, False,  True, False,  True, False,  True, False],
       [ True,  True,  True,  True, False,  True, False, False],
       [ True, False,  True, False,  True, False, False, False],
       [ True,  True,  True,  True, False, False,  True, False],
       [ True,  True, False, False, False, False, False, False]])
```

In [14]:

```
# Let consider the following vector x
x = np.sin(np.linspace(0,np.pi,8))
plt.title('Coordinates of x in the canonical basis')
plot_vector(x)
```

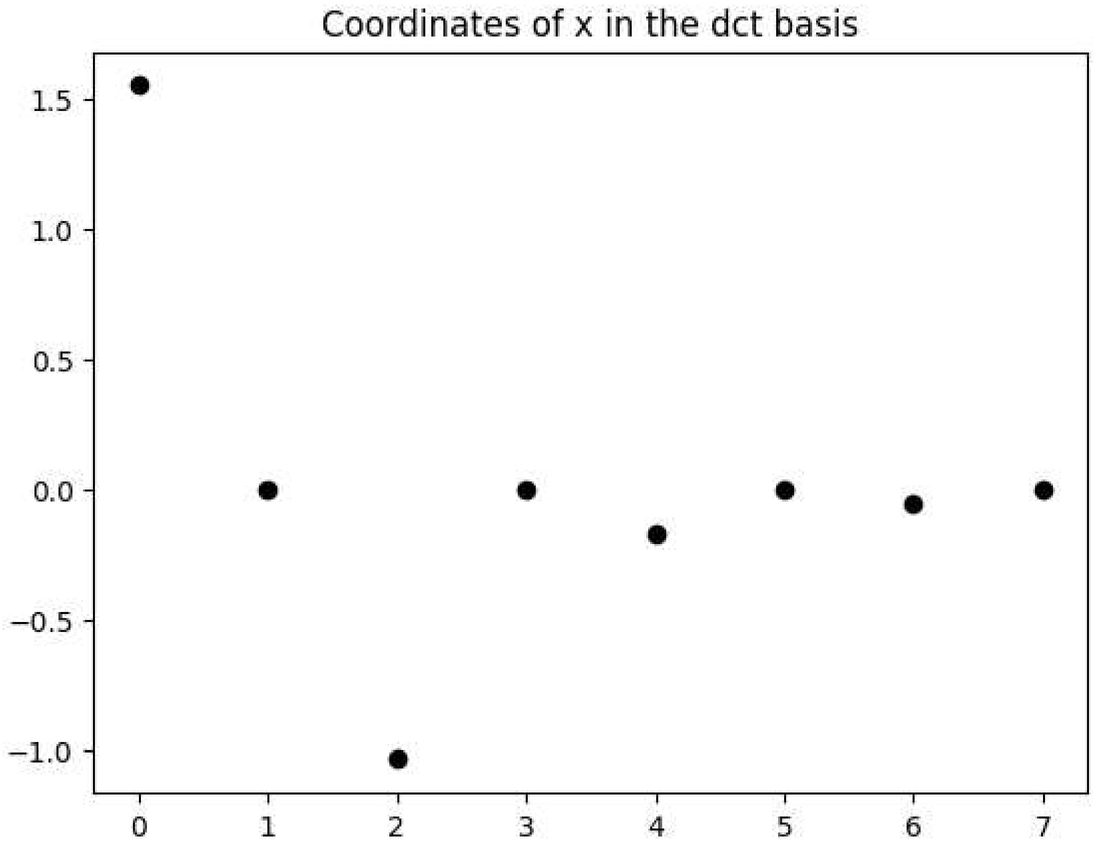


5.3 (b) Compute the vector $v \in \mathbb{R}^8$ of DCT coefficients of x. (1 line of code!), and plot them.

How can we obtain back x from v ? (1 line of code!).

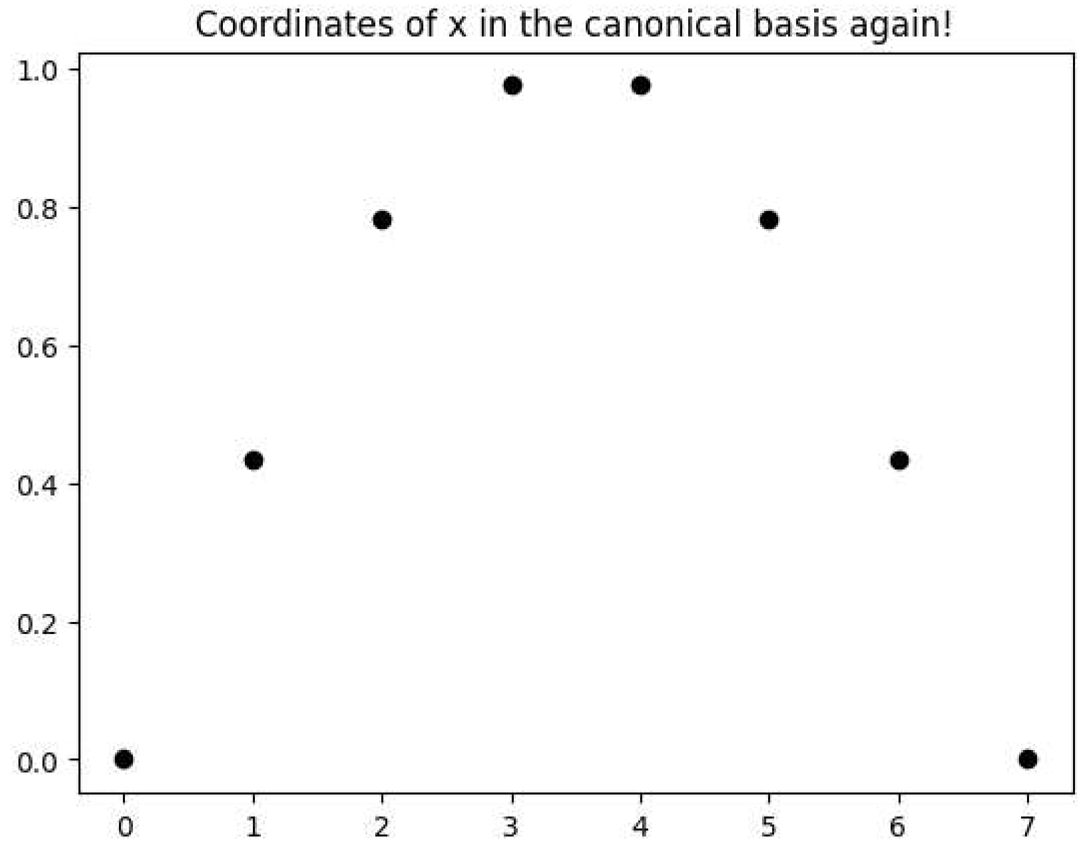
In [17]:

```
# Write your answer here
x_dct = D8.T @ x

plt.title('Coordinates of x in the dct basis')
plot_vector(x_dct)
```



In [18]:

```
x = D8 @ x_dct

plt.title('Coordinates of x in the canonical basis again!')
plot_vector(x)
```



5.3.3 Image compression

In this section, we will use DCT modes to compress images. Let's use one of the template images of python.

In [24]:

```
image = scipy.misc.face(gray=True)
h,w = image.shape
print(f'Height: {h}, Width: {w}')

plt.imshow(image)
```

```
Height: 768, Width: 1024

C:\Users\alexm\AppData\Local\Temp\ipykernel_20444\3842338847.py:1: DeprecationWar
ning: scipy.misc.face has been deprecated in SciPy v1.10.0; and will be completel
y removed in SciPy v1.12.0. Dataset methods have moved into the scipy.datasets mo
dule. Use scipy.datasets.face instead.
  image = scipy.misc.face(gray=True)
```

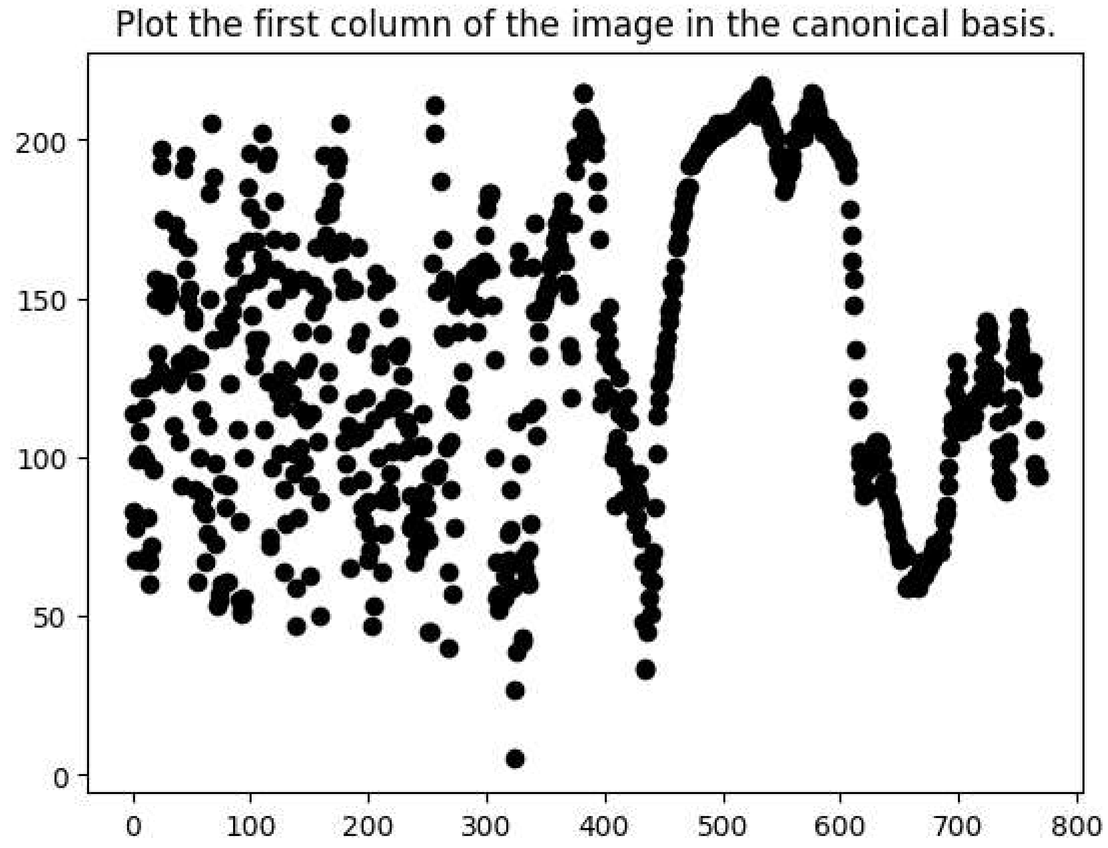
Out[24]:

```
<matplotlib.image.AxesImage at 0x1c298b90460>
```

5.3 (c) We will see each column of pixels as a vector in $\mathbb{R}^{768}$, and compute their coordinates in the DCT basis of $\mathbb{R}^{768}$. Plot the entries of x , the first column of our image.

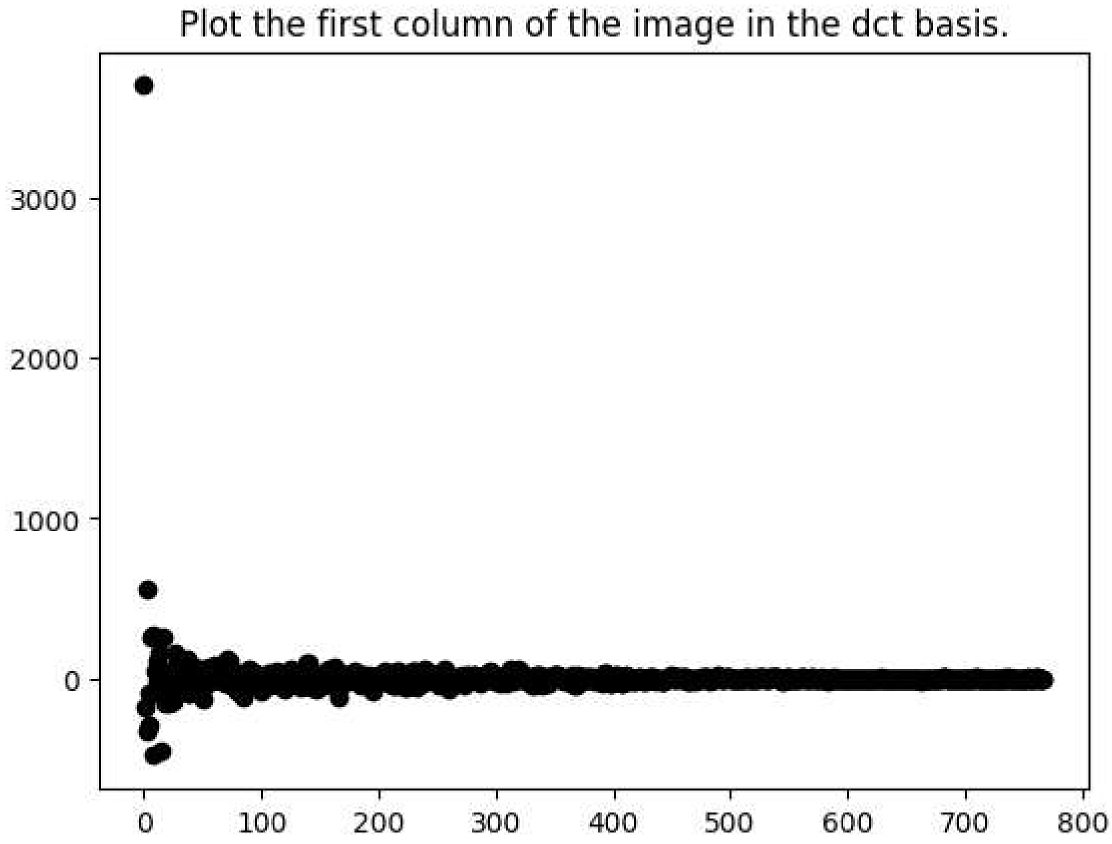
In [44]:

```
# Your answer here
plt.title('Plot the first column of the image in the canonical basis.')
plot_vector(image[:,0])
```



In [45]:

```
D768 = dct(768)
plt.title('Plot the first column of the image in the dct basis.')
plot_vector(D768.T @ image[:,0])
```



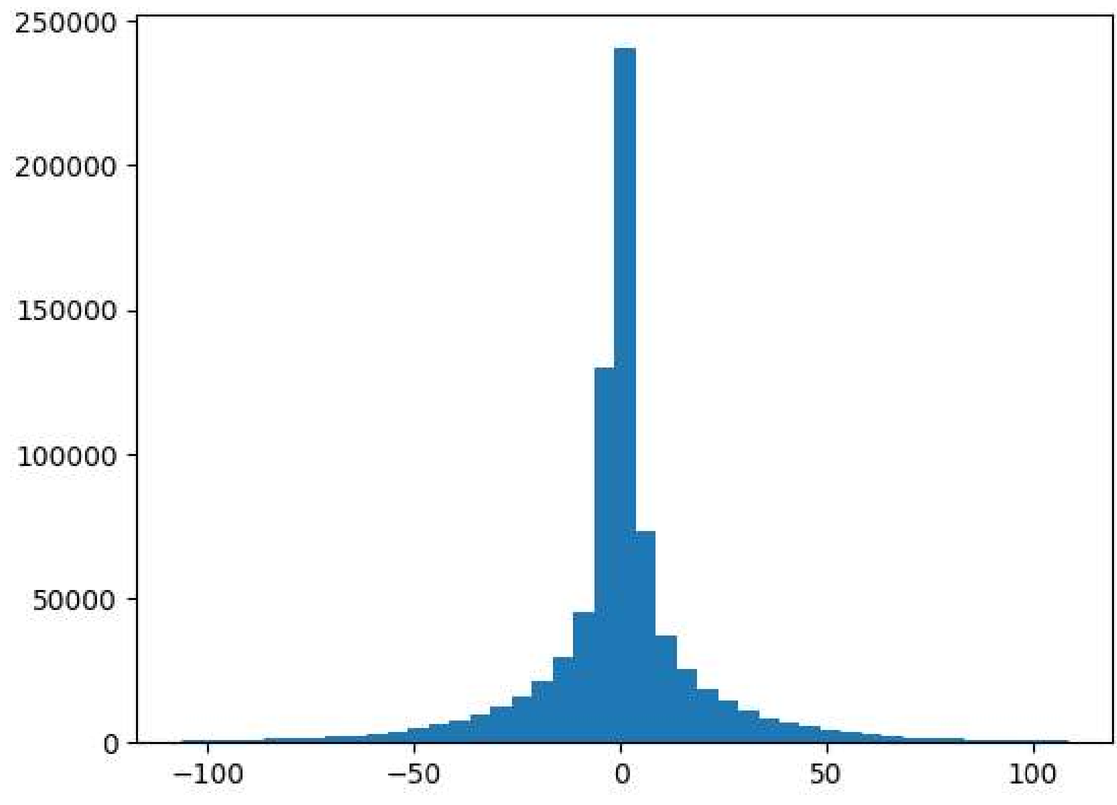
5.3 (d) Compute the 768 x 1024 matrix `dct_coeffs` whose columns are the DCT coefficients of the columns of `image`. Plot a histogram of their magnitudes using `plt.hist`.

In [65]:

```
# Your answer here
dct_coeffs = D768.T @ image
dct_coeffs_flat = dct_coeffs.flatten()
data_min = np.min(dct_coeffs_flat) / 10
data_max = np.max(dct_coeffs_flat) / 40
w = 5
bin_edges = np.arange(start=data_min, stop=data_max + w, step=w)
plt.hist(dct_coeffs.flatten(),bins=bin_edges)
```

Out[65]:

```
(array([   689.,    844.,    890.,    973.,   1243.,   1452.,   1717.,
          2170.,   2495.,   3000.,   3809.,   4762.,   6095.,   7556.,
          9500.,  12470.,  16127.,  21415.,  29855.,  45333., 130007.,
        240372.,  73255.,  36979.,  25785.,  18809.,  14493.,  11133.,
          8702.,   6732.,   5336.,   4294.,   3511.,   2785.,   2268.,
          1817.,   1582.,   1333.,   1111.,   1041.,    817.,    709.,
           648.]),
 array([-106.44312388, -101.44312388,  -96.44312388,  -91.44312388,
         -86.44312388,  -81.44312388,  -76.44312388,  -71.44312388,
         -66.44312388,  -61.44312388,  -56.44312388,  -51.44312388,
         -46.44312388,  -41.44312388,  -36.44312388,  -31.44312388,
         -26.44312388,  -21.44312388,  -16.44312388,  -11.44312388,
          -6.44312388,   -1.44312388,    3.55687612,    8.55687612,
          13.55687612,   18.55687612,   23.55687612,   28.55687612,
          33.55687612,   38.55687612,   43.55687612,   48.55687612,
          53.55687612,   58.55687612,   63.55687612,   68.55687612,
          73.55687612,   78.55687612,   83.55687612,   88.55687612,
          93.55687612,   98.55687612,  103.55687612,  108.55687612]),
 <BarContainer object of 43 artists>)
```



Since a large fraction of the DCT coefficients seems to be negligible, we see that the vector x can be well approximated by a linear combination of a small number of discrete cosines vectors.

Hence, we can "compress" the image by only storing the few DCT coefficients with the largest magnitude.

For instance, to reduce the size by 98%, we store only the top 2% largest (in absolute value) coefficients of `wavelet_coeffs` .

5.3 (e) Compute a matrix `thres_coeffs` which is the matrix `dct_coeffs` where about 98% smallest entries have been put to 0.

In [69]:

```
# Your answer here
# Flatten and sort the matrix(in the ascending order)
sorted_coeffs = np.sort(np.abs(dct_coeffs).flatten())

# Input the threshold
threshold_index = int(0.98 * len(sorted_coeffs))
threshold_value = sorted_coeffs[threshold_index]

# Set values below the threshold to 0
thres_coeffs = np.where(np.abs(dct_coeffs) < threshold_value, 0, dct_coeffs)

thres_coeffs
```

Out[69]:

```
array([[3697.60371463, 3687.50008492, 3683.24212669, ..., 2767.52851536,
         2777.70431385, 2782.39528479],
       [-170.00988   , -147.53068464,    0.        , ..., -294.3850494 ,
        -295.07337131, -294.44949706],
       [-320.49471312, -322.25987969, -322.86057333, ...,  155.39217275,
         159.72835212,  162.24937077],
       ...,
       [   0.        ,    0.        ,    0.        , ...,    0.        ,
           0.        ,    0.        ],
       [   0.        ,    0.        ,    0.        , ...,    0.        ,
           0.        ,    0.        ],
       [   0.        ,    0.        ,    0.        , ...,    0.        ,
           0.        ,    0.        ]])
```

5.3 (f) Compute and plot the `compressed_image` corresponding to `thres_coeffs` .

In [74]:

```
# Your answer here
plt.imshow(D768 @ thres_coeffs)
```

Out[74]:

<matplotlib.image.AxesImage at 0x1c2a4c584c0>