Linear Independence:
Suppose that V is a vector space and that $x_1, x_2, \ldots, x_k$ belong to V.

- $\{x_1, x_2, \ldots, x_k\}$ are linearly independent if

$$r_1x_1 + r_2x_2 + \cdots + r_kx_k = 0$$

only for $r_1 = r_2 = \cdots = r_k = 0$.

- The vectors $x_1, x_2, \ldots, x_k$ are linearly dependent if they are not linearly independent; that is, if there exist scalars $r_1, r_2, \ldots, r_k$ which are not all zero such that

$$r_1x_1 + r_2x_2 + \cdots + r_kx_k = 0$$

- A basis of V is a set of linearly independent vectors which span V.

This lecture: basis and dimension §4.4

Question Why is this useful?
Example Is $\{\cos x, \sin x, 1\}$ is linearly independent?
If $s\cos x + t\sin x + r \cdot 1 = 0$ then

$$\begin{aligned}
x = 0 : \qquad & s \cdot 0 + t \cdot 1 + r \cdot 1 && = 0 \\
x = \tfrac{\pi}{2} : \qquad & s \cdot 1 + t \cdot 0 + r \cdot 1 && = 0 \\
x = \tfrac{\pi}{4} : \qquad & s \cdot \tfrac{1}{\sqrt{2}} + t \cdot \tfrac{1}{\sqrt{2}} + r \cdot 1 && = 0
\end{aligned}$$

Therefore, $\{\cos x, \sin x, 1\}$ is linearly independent.

The order of the logic is very important here:

For any particular value $x = a$ of x we can find $r, s, t \in \mathbb{R}$ such that

$$r \cdot 1 + s \cos a + t \sin a = 0.$$

The point is that we have to find $r, s, t \in \mathbb{R}$ such that

$$r \cdot 1 + s \cos x + t \sin x = 0 \quad \text{for all} \quad x \in \mathbb{R}.$$

If we pick 'good' test values of x then we can show that we must have $r = s = t = 0$.

Basis of a Vector Space: We now combine spanning sets and linear independence.

Definition Suppose that V is a vector space.
A basis of V is a set of vectors $\{x_1, x_2, \dots, x_k\}$ in V such that

- $V = \text{Span}(x_1, x_2, \dots, x_k)$ and
- $\{x_1, x_2, \dots, x_k\}$ is linearly independent.

Examples

- $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right.$ is a basis of $\mathbb{R}^2$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.
- $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^m$.
- $\{1, x, x^2\}$ is a basis of $\mathbb{P}_2$.
- $\{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathbb{P}_n$.
- Typically, if W is a vector subspace of V then our challenge is to find a basis for W.

Another basis of $\mathbb{R}^3$ From the last slide, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$. There are many other bases of $\mathbb{R}^3$.

Example Show that $X = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is another basis of $\mathbb{R}^3$

We need to check two things:

- $\mathbb{R}^3 = \text{Span}(X)$.
- X is linearly independent.

$\underline{\mathbb{R}^3 = \text{Span}(X):}$ Suppose that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$.

Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{Span}(X)$ if and only if we can find $r, s, t \in \mathbb{R}$ such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 2 & 1 & 0 & y \\ 3 & 1 & 1 & z \end{array}\right]$$

We apply Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 2 & 1 & 0 & y \\ 3 & 1 & 1 & z \end{array}\right] \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & -1 & 0 & y-2x \\ 0 & -2 & 1 & z-3x \end{array}\right]$$

$$\xrightarrow{R_2=-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 0 & 2x-y \\ 0 & -2 & 1 & z-3x \end{array}\right]$$

$$\xrightarrow[R_3=R_3+2R_2]{R_1=R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & y-x \\ 0 & 1 & 0 & 2x-y \\ 0 & 0 & 1 & x-2y+z \end{array}\right]$$

Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (y-x)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (2x-y)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (x-2y+z)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, $\mathrm{Span}(X) = \mathbb{R}^3$.

We also need to check that X is linearly independent.

Taking $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ above,
we see that $0 = 0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the only linear combination of X giving the zero vector.

Hence, X is linearly independent.

Therefore, X is a basis of $\mathbb{R}^3$.

The independence theorem

Suppose that $x_1, x_2, \ldots, x_d$ is a basis of V and let $v \in V$. Then v can be expressed as a linear combination of $\{x_1, x_2, \ldots, x_d\}$ in exactly one way.

Proof

Suppose that $r_1 x_1 + r_2 x_2 + \cdots + r_d x_d = v$
$= s_1 x_1 + s_2 x_2 + \cdots + s_d x_d$,
for some $r_1, r_2, \ldots, r_d, s_1, s_2, \ldots, s_d \in \mathbb{R}$.

So $0 = v - v = (r_1 x_1 + r_2 x_2 + \cdots + r_d x_d)$
$- (s_1 x_1 + s_2 x_2 + \cdots + s_d x_d)$
$= (r_1 - s_1)x_1 + (r_2 - s_2)x_2 + \cdots + (r_d - s_d)x_d$.

But, $x_1, x_2, \ldots, x_d$ are linearly independent so this means that
$r_1 - s_1 = 0, r_2 - s_2 = 0, \ldots, r_d - s_d = 0.$

That is, $r_1 = s_1, r_2 = s_2, \ldots, r_d = s_d.$

Hence, we can write v as a linear combination of $x_1, x_2, \ldots, x_d$ in a unique way as claimed!

How big can a basis be? Suppose that we could find a basis $\{w, x, y, z\}$ of $\mathbb{R}^3$ with four elements.

Write $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$.

Let $a, b, c, d \in \mathbb{R}$ be scalars such that
$aw + bx + cy + dz = 0.$

That is, $a \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + b \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + c \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + d \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

To solve this we use Gaussian elimination:

$$\left[\begin{array}{cccc|c} w_1 & x_1 & y_1 & z_1 & 0 \\ w_2 & x_2 & y_2 & z_2 & 0 \\ w_3 & x_3 & y_3 & z_3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & * & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 1 & * & 0 \end{array}\right]$$

(at best)

We must have at least one free variable. So there is no way that $\{w, x, y, z\}$ can be linearly independent.

The dependence theorem

Suppose that $\{x_1, x_2, \ldots, x_d\}$ is basis of V.
Then every linearly independent subset of V has at most d elements.

Proof

Let $y_1, y_2, \ldots, y_n$ are vectors in V, where $n > d$.

We have to show the vectors $y_1, y_2, \ldots, y_n$ are linearly dependent. That is, we have to show that we can find scalars $r_1, r_2, \ldots, r_n$ which are not all zero and
$r_1 y_1 + r_2 y_2 + \cdots + r_n y_n = 0$.

As $\{x_1, x_2, \ldots, x_d\}$ is basis of V we can certainly write:

$$\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d \\
y_3 &= a_{31}x_1 + a_{32}x_2 + \cdots + a_{3d}x_d \\
&\ \ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d
\end{aligned}$$

Hence,
$$\begin{aligned}
& r_1\big(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d\big) \\
&+ r_2\big(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d\big) \\
&\qquad \vdots \qquad\qquad\qquad \vdots \\
&+ r_n\big(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d\big) = 0.
\end{aligned}$$

Rearranging the last equation we have:

$$\begin{aligned} & \left(r_1 a_{11} + r_2 a_{21} + \cdots + r_n a_{n1}\right) x_1 \\ + \quad & \left(r_1 a_{12} + r_2 a_{22} + \cdots + r_n a_{n2}\right) x_2 \\ & \qquad \vdots \qquad\qquad\qquad \vdots \\ + \quad & \left(r_1 a_{1d} + r_2 a_{2d} + \cdots + r_n a_{nd}\right) x_d \\ & = 0. \end{aligned}$$

However, $x_1, x_2, \ldots, x_d$ are linearly independent, so:

$$\begin{aligned} r_1 a_{11} + r_2 a_{21} + \cdots + r_n a_{n1} &= 0 \\ r_1 a_{12} + r_2 a_{22} + \cdots + r_n a_{n2} &= 0 \\ \vdots \qquad\qquad\qquad \vdots & \\ r_1 a_{1d} + r_2 a_{2d} + \cdots + r_n a_{nd} &= 0 \end{aligned}$$

This is a system of d equations in the n unknowns $r_1, r_2, \ldots, r_n$.

As $n > d$ there are infinitely many solutions.
In particular, we must have a non–zero solution to

$$r_1 y_1 + r_2 y_2 + \cdots + r_n y_n = 0.$$

So, $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent, as claimed.

Basis Theorem 2

Suppose that $\{x_1, x_2, \ldots, x_d\}$ is a basis of V and that $\{y_1, y_2, \ldots, y_n\}$ is a linearly independent subset of V.

By the last result we must have $n \leq d$.

The dimension theorem
Every basis of V has the same size.

That is, if $\{x_1, x_2, \ldots, x_d\}$ and $\{y_1, y_2, \ldots, y_n\}$ are two bases of V then $n = d$.

Proof
As $\{x_1, x_2, \ldots, x_d\}$ is a basis of V and $\{y_1, y_2, \ldots, y_n\}$ is linearly independent we have $n \leq d$.

Similarly, as $\{y_1, y_2, \ldots, y_n\}$ is a basis of V and $\{x_1, x_2, \ldots, x_d\}$ is linearly independent we have $d \leq n$.

Hence, $n \leq d \leq n$. So $n = d$!

Definition
Suppose that V is a vector space with basis $\{x_1, x_2, \ldots, x_d\}$. Then the dimension of V is $\dim V = d$.

Dimensions of common vector spaces

Examples

- $\dim \mathbb{R}^2 = 2$ since $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right.$ is a basis of $\mathbb{R}^2$.
- $\dim \mathbb{R}^3 = 3$ since $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.
- $\dim \mathbb{R}^m = m$ since $\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^m$.
- $\dim \mathbb{P}_0 = 1$ since $\{1\}$ is a basis of $\mathbb{P}_0$.
- $\dim \mathbb{P}_1 = 2$ since $\{1, x\}$ is a basis of $\mathbb{P}_1$.
- $\dim \mathbb{P}_2 = 3$ since $\{1, x, x^2\}$ is a basis of $\mathbb{P}_2$.
- $\dim \mathbb{P}_n = n + 1$ since $\{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathbb{P}_n$.
- $\dim \mathbb{P} = \infty$
- $\dim \mathbb{F} = \infty$

Example

Let $a(x) = 1$, $b(x) = x - 1$ and $c(x) = (x-1)^2$.
Is $\{a(x), b(x), c(x)\}$ a basis of $\mathbb{P}_2$?

Let $p(x) = u + vx + wx^2$ be an arbitrary element of $\mathbb{P}_2$.

Then $p(x) \in \text{Span}\big(a(x), b(x), c(x)\big)$ if and only if

$$u + vx + wx^2 = ra(x) + sb(x) + tc(x),$$

for some $r, s, t \in \mathbb{R}$.

That is, $u + vx + wx^2 = r + s(x-1) + t(x^2 - 2x + 1)$.

Equating coefficients we require:

$$\begin{aligned} x^0: \quad & r - s + t = u \\ x^1: \quad & s - 2t = v \\ x^2: \quad & t = w \end{aligned}$$

Hence, $p(x) = (u+v+w)a(x) + (v+2w)b(x) + wc(x)$.

Check: $u + vx + wx^2$
$= (u+v+w)\cdot 1 + (v+2w)(x-1) + w(x^2 - 2x + 1)$.

Therefore, $\text{Span}(a(x), b(x), c(x)) = \mathbb{P}_2$.

Question Does this mean that $\{a(x), b(x), c(x)\}$ must be linearly independent?