Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to ask them on Ed Discussion (so that everyone can benefit from the answer) or stop at the office hours.

Problem 3.1 (2 points). *True or false? Give a justification if true and a counterexample if false. Note: "singular" is another way of saying "not invertible".*

(a) *A square matrix with a column of zeros cannot be invertible.*

True. Consider the matrix organized in columns $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ that takes the form $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_j = \vec{0} & \cdots & \vec{b}_n \end{bmatrix}$ where $j = 1, 2, \cdots, n$, then we can always find a vector in the input space $\vec{a} \in \mathbb{R}^n$ that takes the form $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_j \\ \vdots \\ 0 \end{bmatrix}$ where $x_j \neq 0$ and satisfies $\boldsymbol{A}\vec{a} = \vec{0}$. This shows that there are non-zero vectors in the $\mathcal{N}(\boldsymbol{A})$, which is logically equivalent to say $\mathcal{N}(\boldsymbol{A}) \neq \{0\}$, which implies that $\boldsymbol{A}$ is not invertible.

(b) *A square matrix whose all entries are non-zero is invertible.*

False. The counterexample would be $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ where $rk(\boldsymbol{A}) = 1 \geq 2$, which implies $\boldsymbol{A}$ is not invertible.

(c) *A square matrix where every row adds up to zero must be singular.*

True. Consider the matrix organized in columns $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ that takes the form $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{b}_n \end{bmatrix}$ with the property that $\sum_{i=1}^{n} \vec{a}_i = \vec{0}$. Rearranging the term we could get $\vec{a}_i = \sum_{j\neq i}(-1) \times \vec{a}_j \ \ \forall i = 1, 2, \cdots, n$, this is equivalent to say that $\vec{a}_i$ is linearly dependent with the other column vectors. Thus $Span(\{\vec{a}_j\}_{j\neq i}) = Span(\{\vec{a}_1.\cdots, \vec{a}_n\}), \forall i = 1, 2, \cdots, n$. Thus $rk(\boldsymbol{A}) = dim(Span(\{\vec{a}_1.\cdots, \vec{a}_n\})) = dim(Span(\{\vec{a}_j\}_{j\neq i})) \leq n-1, \forall i = 1, 2, \cdots, n$. By the theorem in the lecture we know that $\boldsymbol{A}$ is singular(not invertible).

(d) *Every matrix with zeros down the main diagonal is singular.*

False. Consider this matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The column of this matrix is clearly linearly independent and then spans the whole $\mathbb{R}^n$. Thus $rk(\boldsymbol{A}) = dim(Span(\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\})) = 2$, which by theorem in the lecture we know that $\boldsymbol{A}$ is nonsingular.

Problem 3.2 (2 points). *Let A be a square $n \times n$ matrix.*

(**a**) *Assume that A is invertible. Is A^{-1} invertible? If yes, what is its inverse?*

True. Since $\boldsymbol{A}$ is invertible, $\exists \boldsymbol{A}^{-1}$ s.t. $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$ and $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n$ by definition and by this definition we change the perspective to $\boldsymbol{A}^{-1}$, then $\exists \boldsymbol{A}$ s.t. $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n$ and $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$, which implies that $\boldsymbol{A}$ is invertible and the inverse of $\boldsymbol{A}^{-1}$ is just $\boldsymbol{A}$.

(**b**) *Assume that A is invertible. Show that $A^\top$ is invertible and that its inverse is $(A^{-1})^\top$: in other words, $(A^\top)^{-1} = (A^{-1})^\top$ (hint: calculate the entries of $A^\top(A^{-1})^\top$).*

We first verify the diagonal entries of $\boldsymbol{A}^\top(\boldsymbol{A}^{-1})^\top$. The derivations are as follows:

$$\begin{aligned}
\boldsymbol{A}^\top(\boldsymbol{A}^{-1})^\top_{ii} &= \sum_{k=1}^{n} \boldsymbol{A}^\top_{ik}(\boldsymbol{A}^{-1})^\top_{ki} && \text{(By definition)} \\
&= \sum_{k=1}^{n} \boldsymbol{A}_{ki}\boldsymbol{A}^{-1}_{ik} && \text{(By properties of transpose)} \\
&= \sum_{k=1}^{n} \boldsymbol{A}^{-1}_{ik}\boldsymbol{A}_{ki} && \text{(Rearrange the terms)} \\
&= 1 && \text{(By the property of matrix inverse)}
\end{aligned}$$

We then verify the non-diagonal entries of $\boldsymbol{A}^\top(\boldsymbol{A}^{-1})^\top_{ij}$ where $i \neq j$. The derivations are as follows:

$$\begin{aligned}
\boldsymbol{A}^\top(\boldsymbol{A}^{-1})^\top_{ij} &= \sum_{k=1}^{n} \boldsymbol{A}^\top_{ik}(\boldsymbol{A}^{-1})^\top_{kj} && \text{(By definition)} \\
&= \sum_{k=1}^{n} \boldsymbol{A}_{ki}\boldsymbol{A}^{-1}_{jk} && \text{(By properties of transpose)} \\
&= \sum_{k=1}^{n} \boldsymbol{A}^{-1}_{ik}\boldsymbol{A}_{kj} && \text{(Rearrange the terms)} \\
&= 0 && \text{(By the property of matrix inverse)}
\end{aligned}$$

Thus the result matrix is just an $n \times n$ diagonal matrix $\boldsymbol{I}_n$. The derivation of $(\boldsymbol{A}^{-1})^\top\boldsymbol{A}^\top = \boldsymbol{I}_n$ is the same, which implies that $(A^\top)^{-1} = (A^{-1})^\top$.

(c) *Assume that A is diagonal:*

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

What is the rank of A? When is A invertible? In such cases, what is its inverse? (hint: it is also diagonal).

The rank depends on the number of non-zero elements among $a_1, \cdots, a_n$. Mathematically, $rk(\boldsymbol{A}) = \sum_{i=1}^{n} \mathbf{1}\{a_i \neq 0\}$. When $a_i \neq 0, \quad \forall i = 1, 2, \cdots, n$, $\boldsymbol{A}$ is invertible. And its inverse is also a diagonal matrix,which is

$$A^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_n} \end{pmatrix}$$

, which makes $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n$ hold.

(d) *Assume that A is upper-triangular:*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{pmatrix}$$

When is A invertible?

Again, when $a_{i,i} \neq 0, \quad \forall i = 1, 2, \cdots, n$, $\boldsymbol{A}$ is invertible, otherwise, one can easily express that column j with $a_{j,j} = 0$ in terms of the linear combination of the columns before it(namely column $1, 2, \cdots, j-1$).

Problem 3.3 (2 points). (**a**) *Let $A, B \in \mathbb{R}^{n\times n}$. Suppose that AB is invertible. Show that A and B are both invertible. (In particular, if $AB = \mathrm{Id}$, then A is invertible). Express the inverse of AB in terms of the inverses of A and B.*

Since $\boldsymbol{AB}$ is invertible, we have that Rank$(\boldsymbol{AB}) = n$. Since we have Rank$(\boldsymbol{AB}) \leq$ Rank$(\boldsymbol{A})$ and Rank$(\boldsymbol{A}) \leq \min\{m, n\} = n$. We know that Rank$(\boldsymbol{A}) = n$. Asymmetric argument could show that Rank$(\boldsymbol{B}) = n$, which are equivalent to say that both $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible and we denote them by $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$. The expression of $(\boldsymbol{AB})^{-1}$ is $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ and we verify it. We have $(\boldsymbol{AB})\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} = \boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{I}_n$ and $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}(\boldsymbol{AB}) = \boldsymbol{B}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B} = \boldsymbol{I}_n$.

(**b**) *Let $A \in \mathbb{R}^{n\times m}$ with $n \neq m$, is it possible to find a matrix $B \in \mathbb{R}^{m\times n}$ such that $AB = \mathrm{Id}_n$ and $BA = \mathrm{Id}_m$?*

No, it is impossible. We prove it from the perspective of matrix rank and derive a contradiction. Suppose $\boldsymbol{AB} = \boldsymbol{I}_n$ holds, we have Rank$(\boldsymbol{AB}) = n \leq$ Rank$(\boldsymbol{A}) \leq \min\{m, n\}$, which implies Rank$(A) = n$ and $n < m$ since $n \neq m$. Now with this we have Rank$(\boldsymbol{BA}) \leq$ Rank$(\boldsymbol{A}) = n < m =$ Rank$(\boldsymbol{I}_m)$, which is a contradiction.
Now suppose $\boldsymbol{BA} = \boldsymbol{I}_m$ holds, we have Rank$(\boldsymbol{BA}) = m \leq$ Rank$(\boldsymbol{B}) \leq \min\{m, n\}$, which implies Rank$(B) = m$ and $m < n$ since $n \neq m$. Now with this we have Rank$(\boldsymbol{AB}) \leq$ Rank$(\boldsymbol{B}) = m < n =$ Rank$(\boldsymbol{I}_n)$, which is a contradiction.

Problem 3.4 (2 points). *Let A be a square $n \times n$ matrix. Define $A^k = \underbrace{A \times \cdots \times A}_{k \text{ times}}$, with $A^0 = \mathrm{Id}_n$.*

(a) *What can you say about the sequence $(\mathrm{rank}(A^k))_{k\in\mathbb{N}}$?*

Without further information, we know that $rk(\boldsymbol{A}^{m+1}) \leq rk(\boldsymbol{A}^m)$, $\forall m = 0, 1, 2, \cdots, k$. Since $rk(\boldsymbol{A}^0) = rk(\boldsymbol{I}_n) = n$, we know $rk(\boldsymbol{A}^k) \leq n$.

(b) *Assume that A is invertible. Show that for all $k \in \mathbb{N}$, A^k is invertible and its inverse is $(A^{-1})^k$ (in other words, $(A^k)^{-1} = (A^{-1})^k$, and we write this matrix A^{-k}).*

We can prove it by induction.
Proof. *Suppose our inductive hypothesis $P(k)$ is: $\boldsymbol{A}^k$ is invertible.*

(a) Base Step: When $k = 0$, $\boldsymbol{A} = \boldsymbol{I}_n$, which is indeed invertible, $P(0)$ is true.

(b) Inductive Step: Suppose for some $k = n \geq 0$, $P(n)$ is true, then $Rank(\boldsymbol{A}^n) = n$. We will have $Rank(\boldsymbol{A} \times \boldsymbol{A}^n) = Rank(\boldsymbol{A}^n) = n$ since $\boldsymbol{A}$ is invertible, which implies that $\boldsymbol{A}^{n+1}$ is invertible.

By mathematical induction, $\boldsymbol{A}^k$ is invertible $\forall k \in \mathbb{N}$. □

Now we move on to show its inverse, again we prove by induction:
Proof. *Suppose our inductive hypothesis is $P(k)$: $\boldsymbol{A}^k$'s inverse is $(\boldsymbol{A}^{-1})^k$*

(a) Base Step: When $k = 0$, $\boldsymbol{A}^0 = \boldsymbol{I}_n$, whose inverse is $I_n = (\boldsymbol{A}^0)^{-1}$, which implies that the inductive hypothesis $P(0)$ is true.

(b) Inductive Step: Suppose for some $n \geq 0$, $P(n)$ is true and that the inverse of $\boldsymbol{A}^n$ is $(\boldsymbol{A}^{-1})^n$. Then the inverse of $\boldsymbol{A}^{n+1}$ is the inverse of $\boldsymbol{A} \times \boldsymbol{A}^n$, which, as shown in problem 3.3, is $((\boldsymbol{A})^n)^{-1}\boldsymbol{A}^{-1} = (\boldsymbol{A}^{-1})^n\boldsymbol{A}^{-1} = (\boldsymbol{A}^{-1})^{n+1}$, which implies that $P(n+1)$ is true.

By mathematical induction, $P(k)$ is true $\forall k \in \mathbb{N}$. □

Problem 3.5 ($\star$)**.** *Let $A \in \mathbb{R}^{n\times n}$ given by*

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(**a**) *What is* $\operatorname{rank}(A)$*?*

$rk(\boldsymbol{A}) = n-1$. Since there are $n-1$ pivot elements.

(**b**) *Compute A^2. What is* $\operatorname{rank}(A^2)$*?*

We notice that it is a matrix that will zero out the first column for each right multiplication. From this we get that $\boldsymbol{A}^2 =$

$$A = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

and the $rank(\boldsymbol{A}^2) = n-2$.

(**c**) *Compute A^k for all $k \in \mathbb{N}$. What is* $\operatorname{rank}(A^k)$*?*

By intuition we would have $Rank(\boldsymbol{A}^k) = n-k$ Because each time we multiply the matrix $\boldsymbol{A}$ on the left, we will get a column of all zero and the rank will shrink by 1.