- You have 1 hour and 30 minutes to work on this midterm exam.
- The exam has 2 pages with 8 problems.
- Please justify your answers, proving the statements you make. You are allowed to refer to results shown in lectures/recitations/homeworks as long as you state them precisely, meaning that you should say exactly which hypothesis are needed in the result you use.
- Partial answers will be graded. But not justified answers will not necessarily yield credit.
- Put your name and your netID on your paper. Number the pages on both sides with $1/x, 2/x$ etc... with x your total final number of pages.
- Once I announce that time is up, drop your pen immediately. Then you will be allowed to take out your phone to take a picture of all the pages of your exam before handing it to me. You will have to upload your pictures to Gradescope before the end of the day.

Problem 0.1 (5 points)**.** *Correct/Incorrect. No justification needed.*
Consider $A \in \mathbb{R}^{n\times m}$ with $n > m$ and $B \in \mathbb{R}^{k\times k}$.

(**a**) *A^2 is a square matrix.*

(**b**) *If $\lambda \in \mathbb{R}$ is an eigenvalue of B, there exists an infinite set of vectors $x \in \mathbb{R}^k$ such that $Bx = \lambda x$.*

(**c**) $\dim \ker A + \operatorname{rank}(A) = n$.

(**d**) *If B is a stochastic matrix, $B^\top$ has to also be a stochastic matrix.*

(**e**) $\operatorname{Im}(A)$ *admits an othonormal basis.*

Problem 0.2 (3 points)**.** *Are the following sets S subspaces or not? Justify carefully your positive/negative answers.*

(**a**) $S = \{(x, 0, y) \in \mathbb{R}^3 \mid x - y = 0\}$ *subspace of* $\mathbb{R}^3$*?*

(**b**) $S = \{A \in \mathbb{R}^{n\times n} \mid A = A^\top\}$ *subspace of* $\mathbb{R}^{n\times n}$*?*

(**c**) $S = \{x \in \mathbb{R} \mid x \leq 0\}$ *subspace of* $\mathbb{R}$*?*

Problem 0.3 (5 points)**.** *Consider the following matrices and vector*

$$M = \begin{pmatrix} 4 & 2 & b \\ 0 & 0 & 5 \\ 0 & a & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 3 \\ c \end{pmatrix} \text{ and } P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where a, b and c are real numbers.

(**a**) *Give the number of solutions $x \in \mathbb{R}^3$ to the linear system $Mx = y$ depending on the values of a, b and c.*

(**b**) *Same question for $MPx = y$.*

Problem 0.4 (6 points). *Recall that the trace operator is defined for a square matrix $M \in \mathbb{R}^{n\times n}$ as* $\mathrm{Tr}(M) = \sum_{i=1}^{n} M_{i,i}$. *For A and B in $\mathbb{R}^{n\times n}$, we define* $\langle A, B\rangle = \mathrm{Tr}(A^T B)$.

(**a**) *Show that the map* $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times n} & \to & \mathbb{R} \\ M & \mapsto & \mathrm{Tr}(M) \end{array} \right.$ *is a linear transformations on* $\mathbb{R}^{n\times n}$.

(**b**) *Show that* $\langle \cdot, \cdot \rangle$ *defines an inner product on* $\mathbb{R}^{n\times n}$.

(**c**) *Deduce that for any A and B symmetric matrices in* $\mathbb{R}^{n\times n}$, $\mathrm{Tr}(AB)^2 \leq \mathrm{Tr}(A^2)\mathrm{Tr}(B^2)$.

Problem 0.5 (8 points). *Consider $(v_1, \cdots, v_k)$ an orthonormal basis of a subspace S of $\mathbb{R}^n$, with $k \leq n$. Consider the matrix M of the orthogonal projection with respect to the Euclidian dot product on S, $M = VV^\top$ with*

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix}$$

(**a**) *Show that* $\mathrm{Im}(M) = \mathrm{Span}(v_1, \cdots, v_k)$. *What is the rank of M?*

(**b**) *Consider the case $n = 2$ and $k = 1$. Compute y defined as the orthogonal projection of $x = (3, 6)$ onto $S = \mathrm{Span}(v_1)$ with $v_1 = (1/\sqrt{2}, 1/\sqrt{2})$, and then compute z the orthogonal projection of y on S. Draw up a picture including v_1, x, y and z.*

(**c**) *For the setting of* (**b**), *give the dimension and an orthonormal basis of* $\ker(M)$ *and give the dimension and an orthonormal basis of* $\mathrm{Im}(M)$. *Justify as carefully as possible.*

(**d**) *Going back to the the general $\mathbb{R}^n$ case, show that $M = M^2$. Can you explain intuitively why?*

Problem 0.6 (3 points). *For a square symmetric matrix, we call eigen decomposition the collection of all its eigenvector-eigenvalue pairs. Let $A \in \mathbb{R}^{n\times n}$ be a square symmetric matrix.*

(**a**) *At which condition A^{-k} exists? Give the most precise condition you can and justify.*

(**b**) *Assuming it exists, give the eigen decomposition of A^{-k} as a function of the eigen decomposition of A for any integer $k > 0$.*

Problem 0.7 (Extra - 3 points). *A matrix $A \in \mathbb{R}^n$ is anti-symmetric if $A^\top = -A$. Give a basis of the subspace of $\mathbb{R}^{3\times 3}$ that contains all the anti-symmetric matrices of $\mathbb{R}^{3\times 3}$. No justification is needed.*

Problem 0.8 (Extra - 2 points). *Suppose that $A \in \mathbb{R}^{n\times n}$ preserves the Euclidian norm, that is for any $x \in \mathbb{R}^n$, $\|Ax\|_2 = \|x\|_2$. Show that A is an orthogonal matrix. [Hint: You could find the canonical basis vectors useful.]*