Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to ask them on Brightspace (so that everyone can benefit from the answer) or stop at the office hours.

Problem 1.1 (3 points). *Are the following sets subspaces of $\mathbb{R}^2$? Draw a picture and justify your answer using the definition of a subspace. If yes, indicate the dimension of the subspace and a basis, add the basis vectors on your drawing.*

(a) $E_1 = \{(x, y) \in \mathbb{R}^2 \,|\, 3x - y = 0\}$.

(b) $E_2 = \{(x, y) \in \mathbb{R}^2 \,|\, 3x - y = -1\}$.

(c) $E_3 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 0\}$.

(d) $E_4 = \{(x, y) \in \mathbb{R}^2 \,|\, x + y \geq 0\}$.

Problem 1.2 (2 points). *Are the following sets subspaces of $\mathbb{R}^3$? Justify your answer using the definition of a subspace. If yes, indicate the dimension of the subspace and a basis.*

(a) $E_5 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x = 0 \text{ and } y - z = 0\}$.

(b) $E_6 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x = 0 \text{ or } y - z = 0\}$.

Problem 1.3 (2 points). *Let us define the vectors $e_1, \dots, e_n \in \mathbb{R}^n$ by*

$$\begin{aligned} e_1 &= (1, 0, 0, \dots, 0) \\ e_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \dots, 1). \end{aligned}$$

(a) *Verify that the family $(e_1, \dots, e_n)$ is a basis of $\mathbb{R}^n$. This basis is called the “canonical basis” of $\mathbb{R}^n$. What is the dimension of $\mathbb{R}^n$?*

(b) *Give an example of hyperplane and an example of a line of $\mathbb{R}^n$ using spans of subsets of $(e_1, \dots, e_n)$.*

Problem 1.4 (3 points). **(a)** *Consider $v_1, \dots, v_p \in \mathbb{R}^n$. Prove that* $\mathrm{Span}(v_1, \dots, v_p)$ *is the smallest subspace which contains $v_1, \dots, v_p$ (i.e., it is a subspace which contains $v_1, \dots, v_p$, and it is contained in any other such subspace).*

(**b**) *Let V, W be two subspaces of $\mathbb{R}^n$. Show that $V \cap W$ is a subspace of $\mathbb{R}^n$.*

(**c**) *Let V, W be two subspaces of $\mathbb{R}^n$. Show that $V \cup W$ may not be a subspace of $\mathbb{R}^n$ with a counter-example.*

Problem 1.5 ($\star$)**.** *Let V be a vector space of dimension n and let $x_1, \dots, x_n \in V$. Show that:*

(**a**) *If $x_1, \dots, x_n$ are linearly independent, then $(x_1, \dots, x_n)$ is a basis of V.*

(**b**) *If $\mathrm{Span}(x_1, \dots, x_n) = V$, then $(x_1, \dots, x_n)$ is a basis of V.*