Instructions:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a (*) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to ask them on Ed Discussion (so that everyone can benefit from the answer) or stop at the office hours.

Problem 4.1 (2.5 points)**.** *Given $x \in \mathbb{R}^n$, define*

$$N(x) := \max\{|x_1|, |x_2|, ..., |x_n|\}.$$

(a) Show that this defines a valid norm, i.e., verify that $N(\cdot)$ satisfies the three norm properties.

(b) Let $x = (10, 0)$ and $y = (9, 9)$. Using the norm $N(\cdot)$, which of x and y has bigger norm? Using the Euclidean norm, which of x and y has bigger norm?

(c) Show that $N(x) > 1$ implies the Euclidean norm of x is also greater than 1, i.e., $\|x\|_2 > 1$.

Problem 4.2 (2 points)**.** *Decide whether each of the following functions $N(\cdot)$ is a valid norm. If yes, you do not have to justify your answer. If no, then provide an explicit counterexample and state which norm property is violated.*

(a) $x \in \mathbb{R}^2$. $N(x) := x^\top Ax$, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$.

(b) $x \in \mathbb{R}^2$. $N(x) := x^\top Ax$, where $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$.

(c) $x \in \mathbb{R}^3$. $N(x) := 2|x_1| + \max\{|x_2|, |x_3|\}$.

(d) $x \in \mathbb{R}^n$. $N(x) := \left(\prod_{i=1}^n |x_i|\right)^{1/n} + \max_{i\in[n]}\{|x_i|\}$.

Problem 4.3 (1 point)**.** *Show that for any vector $x \in \mathbb{R}^n$,*

$$\left(\sum_{k=1}^n x_k\right)^2 \leq n \sum_{k=1}^n x_k^2.$$

Hint: use Cauchy-Schwarz.

Problem 4.4 (1.5 points)**.** *Consider the set V of random variables (on a probability space Ω) with a finite second moment (i.e., for $X \in V$, $\mathbb{E}[X^2] < \infty$), with the inner product $\langle X, Y\rangle = \mathbb{E}[XY]$. Prove the Cauchy-Schwarz inequality:*

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}.$$

Hint: given constants $a, b \geq 0$, evaluate the expression $\mathbb{E}[(aX - bY)^2]$. Then find appropriate values of a, b to prove the statement.

Problem 4.5 (3 points)**.** *Let S be a subspace of $\mathbb{R}^n$ with $x \mapsto P_S(x)$ the orthogonal projector of all $x \in \mathbb{R}^n$. We will use the Euclidean dot product as our inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. Show that for any $x \in \mathbb{R}^n$,*

(**a**) $\langle x, y \rangle = \langle P_S(x), y \rangle$, *for any vector y in S.*

(**b**) $x - P_S(x)$ *is orthogonal to S,*

(**c**) $\|P_S(x)\| \leq \|x\|$, *and the equality holds when $x \in S$.*

Hint: express y in terms of the orthonormal basis of S: $(v_1, v_2, ..., v_k)$.

Problem 4.6 ($\star$)**.** *Given $x \in \mathbb{R}^n$ and $p \in (1, \infty)$, define the ℓ_p-norm as*

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

Prove that the ℓ_p-norm satisfies the triangle inequality, that is, for any $x, y \in \mathbb{R}^n$, show that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

You may use the following inequality, known as Hölder's inequality, without proof: for any $x, y \in \mathbb{R}^n$, and constants $p, q \in \mathbb{R}_+$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\sum_{i=1}^n |x_i y_i| \leq \|x\|_p \|y\|_q.$$