Recitation 1

1. (Correlation) If $a \neq 0$, $b \in \mathbb{R}$, $\tilde{a}$ is a random variable, compute the correlation coefficients $\rho_{\tilde{a},a\tilde{a}+b}$.

   Solution:

$$\begin{aligned} \rho_{\tilde{a},a\tilde{a}+b} &= \frac{\text{Cov}(\tilde{a}, a\tilde{a}+b)}{\sqrt{\text{Var}(\tilde{a})}\sqrt{\text{Var}(a\tilde{a}+b)}} && (1) \\ &= \frac{a\text{Var}(\tilde{a})}{|a|\,\text{Var}(\tilde{a})} && (2) \\ &= \frac{a}{|a|} && (3) \end{aligned}$$

2. (Uncorrelated term) Let $(\tilde{a}, \tilde{b})$ be bivariate normal with correlation ρ and $\sigma^2_{\tilde{a}} = \sigma^2_{\tilde{b}}$. Show that $\tilde{a}$ and $\tilde{b} - \rho\tilde{a}$ are independent.

   Solution: Since $(\tilde{a}, \tilde{b})$ is bivariate normal, the independence can be proved by:

$$\begin{aligned} & \text{E}\left[\tilde{a}(\tilde{b}-\rho\tilde{a})\right] - \text{E}\left[\tilde{a}\right]\text{E}\left[\tilde{b}-\rho\tilde{a})\right] && (4) \\ =& \text{E}\left[\tilde{a}\tilde{b}\right] - \rho\text{E}\left[\tilde{a}^2\right] - \text{E}\left[\tilde{a}\right]\text{E}\left[\tilde{b}\right] + \rho\text{E}\left[\tilde{a}\right]^2 && (5) \\ =& \text{Cov}\left(\tilde{a},\tilde{b}\right) - \rho\text{Var}\left(\tilde{a}\right) && (6) \\ =& \text{Cov}\left(\tilde{a},\tilde{b}\right) - \text{Cov}\left(\tilde{a},\tilde{b}\right)\frac{\sigma_{\tilde{a}}}{\sigma_{\tilde{b}}} && (7) \\ =& 0 && (8) \end{aligned}$$