Optimization and Computational Linear Algebra
Homework 1 - Fall 2023

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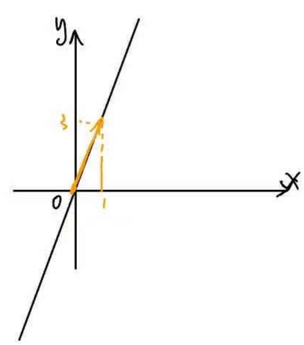
1 Problem 1.1

Are the following sets subspaces of $\mathbb{R}^2$? Draw a picture and justify your answer using the definition of a subspace. If yes, indicate the dimension of the subspace and a basis, add the basis vectors on your drawing.

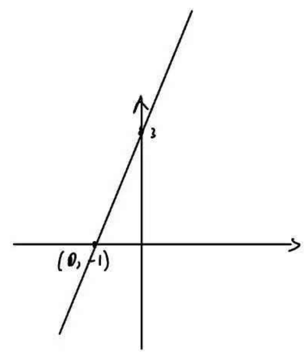
- (a) $E_1 = \{(x, y) \in \mathbb{R}^2 \mid 3x - y = 0\}$.
  We first show that it is a subspace.

  1. $\vec{0} = (0, 0) \in E_1$. Zero vector is in E_1.
  2. $\forall \vec{v} = (a_1, a_2), \vec{w} = (b_1, b_2) \in E_1$, we have $3a_1 = a_2, 3b_1 = b_2$. Then $\vec{v} + \vec{w} = (a_1 + b_1, a_2 + b_2) = (a_1 + b_1, 3a_1 + 3b_1) \in E_1$. Thus E_1 is closed under addition.
  3. $\forall \vec{v} = (a_1, a_2) \in E_1, \alpha \in \mathbb{R}$, we have $3a_1 = a_2$. Then $\alpha\vec{v} = (\alpha a_1, \alpha a_2) \in E_1$. Thus E_1 is closed under scalar multiplication.

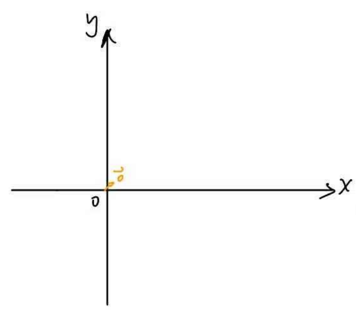
  Then the basis for it: $\forall \vec{u} = (x, y) \in E_1$, we have $\vec{u} = (x, 3x) = x(1, 3), x \in \mathbb{R}$. So E_1 is a line passing through the origin towards the $(1, 3)$. The basis is then $\{\begin{bmatrix}1\\3\end{bmatrix}\}$ and the dimension $dim(E_1) = 1$.
  Finally the drawing(basis shown in orange):



- (b) $E_2 = \{(x, y) \in \mathbb{R}^2 \mid 3x - y = -1\}$.
  It is not a subspace since $\vec{0} \notin E_2$ but it is a translation of subspace(affine subspace). $\forall \vec{u} = (x, y) \in E_2$, we have $\vec{u} = (x, 3x - 1) = x(1, 3) + (0, -1), x \in \mathbb{R}$. So E_2 is a line passing through the point $(0, -1)$ along the direction $(1, 3)$.
  The drawing is:



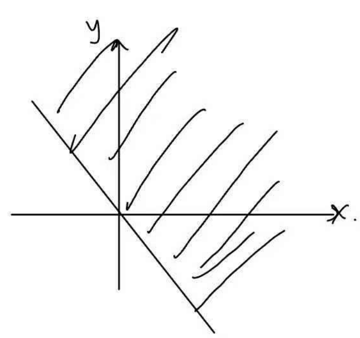
- (c) $E_3 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0\}$.
  It is a subspace and the only element in E_3 is $\vec{0}$ and the dimension is $dim(E_3) = 0$.
  The drawing is(basis shown in orange):



- (d) $E_4 = \{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0\}$.
  It is a halfspace and thus not a subspace.
  We verify it as follows:

  1. $\vec{0} = (0, 0) \in E_4$. Zero vector is in E_4.
  2. $\forall \vec{v} = (a_1, a_2), \vec{w} = (b_1, b_2) \in E_4$, we have $a_1 + a_2 \geq 0, b_1 + b_2 \geq 0$. Then $\vec{v} + \vec{w} = (a_1 + b_1, a_2 + b_2) \in E_4$. Thus E_4 is closed under addition.
  3. $\forall \vec{v} = (a_1, a_2) \in E_4, \alpha \in \mathbb{R}$, we have $a_1 + a_2 \geq 0$ but $\alpha(a_1 + a_2)$ is not necessarily greater or equal to zero if we choose $\alpha < 0$. Then $\alpha\vec{v} = (\alpha \cdot a_1, \alpha \cdot a_2) \notin E_4$. So E_4 is not closed under scalar multiplication.

  . Thus E_4 is not a subspace.
  The drawing is:



2 Problem 1.2

Are the following sets subspaces of $\mathbb{R}^3$? Justify your answer using the definition of a subspace. If yes, indicate the dimension of the subspace and a basis.

1. (a) $E_5 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ and } y - z = 0\}$.
   Yes, and we will prove it by definition:

   - 1. Zero Vector: $\vec{0} \in E_5$ in that $\vec{0} = (0,0,0) = (x,y,z)$ where $x = 0, y = 0, z = 0$ and $y = z$.
   - 2. Addition: $\forall \vec{x} = (a_1, a_2, a_3), \vec{y} = (b_1, b_2, b_3) \in E_5$, we have $a_1 = b_1 = 0, a_2 = a_3, b_2 = b_3$ and that $a_1 + a_2 = 0, a_2 + b_2 = a_3 + b_3$. Then $\vec{x} + \vec{y} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \in E_5$. So E_5 is closed under addition.
   - 3. Scalar Multiplication: $\forall \vec{x} = (a_1, a_2, a_3)$ and $\alpha \in \mathbb{R}$, we have $a_1 = 0, a_2 = a_3$. So $\alpha \cdot a_1 = 0, \alpha \cdot a_2 = \alpha \cdot a_3$. This means $\alpha\vec{x} \in E_5$. Thus, $E_5$is closed under scalar multiplication.

   Bring all these pieces together, E_5 is a vector space.
   For $\forall v = (x, y, z) \in E_5$, we have $v = (0, y, y) = y(0, 1, 1)$, thus the dimension of E_5 is 1 and one valid basis is $(0, 1, 1)$.

2. (b) $E_6 = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } y - z = 0\}$.
   No. And we will show that it doesn't satisfy the closure under addition.
   $\forall \vec{x} = (a_1, a_2, a_3) \in E_6$, we have $a_1 = 0$ or $a_2 = a_3$. So if we pick $\vec{u} = (0, 1, 2)$ and $\vec{v} = (2, 3, 3)$, we have $\vec{u} + \vec{v} = (2, 4, 5)$ where $\vec{u} + \vec{v} \notin E_6$. So E_6 is not a subspace.

3 Problem 1.3

Let us define the vectors $e_1, \ldots, e_n \in \mathbb{R}^n$ by

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \ldots, 1). \end{aligned}$$

1. (a) Verify that the family $(e_1, \ldots, e_n)$ is a basis of $\mathbb{R}^n$. This basis is called the "canonical basis" of $\mathbb{R}^n$. What is the dimension of $\mathbb{R}^n$?

   We examine the following systems of linear equations: $a_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and we solve it to get $a_1 = 0, a_2 = 0, \cdots, a_n = 0$, which implies that $\{\vec{e}_1, \cdots, \vec{e}_n\}$ are linearly independent.
   To show $\{\vec{e}_1, \cdots, \vec{e}_n\}$ spans $\mathbb{R}^n$, we notice that $\forall \vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$, we have $\vec{v} = \sum_{i=1}^{n} a_i \vec{e}_i \in Span(\{\vec{e}_1, \cdots, \vec{e}_n\})$, which ends our proof of basis.
   Finally by definition, we have $dim(\mathbb{R}^n) = n$ since it has a length-n basis.

2. (b) Give an example of hyperplane and an example of a line of $\mathbb{R}^n$ using spans of subsets of $(e_1, \ldots, e_n)$

   (a) An example of $n-1$-dimensional hyperplane can be defined as the $\mathcal{H} = \{\vec{x} \in \mathbb{R}^n | \vec{e}_1^\top \vec{x} = 0\}$ since $Span(\{\vec{e}_1\}) \oplus \mathcal{H} = \mathbb{R}^n$, where $dim(Span(\{\vec{e}_1\})) + dim(\mathcal{H}) = dim(\mathbb{R}^n) = n$ and $Span(\{\vec{e}_1\}) = 1$ and $dim(\mathcal{H}) = n - 1$.

   (b) An example of a line of $\mathbb{R}^n$ could be formed by any of the basis vector $\vec{e}_i$. For example, $S = \{\vec{x} \in \mathbb{R}^n | \vec{x} = \alpha \vec{e}_i, \alpha \in \mathbb{R}^n\}$.

4 Problem 1.4

1. (a) Consider $v_1, \ldots, v_p \in \mathbb{R}^n$. Prove that Span $(v_1, \ldots, v_p)$ is the smallest subspace which contains $v_1, \ldots, v_p$ (i.e., it is a subspace which contains $v_1, \ldots, v_p$, and it is contained in any other such subspace).

   - We first prove that $Span(\vec{v}_1, \cdots, \vec{v}_n)$ contains $\vec{v}_1, \cdots, \vec{v}_n$.

     Proof. Since $\vec{v_i} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 1 \cdot \vec{v_i} + \cdots + 0 \cdot \vec{v}_n \forall i = 1, 2, \cdots, n$, which is a linear combination of $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$. Then we know that $\forall \vec{v}_i, \vec{v_i} \in Span(\vec{v}_1, \cdots, \vec{v}_n)$ by definition of span. □

   - We then prove that it is a subspace.

     Proof.
     - Zero Vector: Taking all the coefficients before $\vec{v}_i, i = 1, 2, \cdots n$ to be zero we know that $\vec{0} \in Span(\vec{v}_1, \cdots, \vec{v}_n)$
     - Addition: $\forall \vec{v}, \vec{w} \in Span(\vec{v}_1, \cdots, \vec{v}_n)$, by definition we have $\vec{v} = \sum_{i=1}^n \alpha_i \vec{v}_i$ and $\vec{w} = \sum_{i=1}^n \beta_i \vec{v}_i$. Then $\vec{v} + \vec{w} = \sum_{i=1}^n (\alpha_i + \beta_i) \vec{v}_i \in Span(\vec{v}_1, \cdots, \vec{v}_n)$.
     - Scalar Multiplication: $\forall \vec{v} \in Span(\vec{v}_1, \cdots, \vec{v}_n), t \in \mathbb{R}$, by definition we have $\vec{v} = \sum_{i=1}^n \alpha_i \vec{v}_i$. Then $\alpha \vec{v} = \sum_{i=1}^n (t\alpha_i) \vec{v}_i \in Span(\vec{v}_1, \cdots, \vec{v}_n)$.

     Thus $Span(\vec{v}_1, \cdots, \vec{v}_n)$ is a subspace. □

   - Finally, we prove that it is the smallest such subspaces.

     Proof. We know that any subspaces containing $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ must contain the linear combination of them(which is just $Span(\vec{v}_1, \cdots, \vec{v}_n)$) since any subspaces are closed under addition and scalar multiplication(in short, linear combination). In other words, any subspaces containing $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ must contain $Span(\vec{v}_1, \cdots, \vec{v}_n)$. Thus, $Span(\vec{v}_1, \cdots, \vec{v}_n)$ is the smallest such subspaces. □

2. (b) Let V, W be two subspaces of $\mathbb{R}^n$. Show that $V \cap W$ is a subspace of $\mathbb{R}^n$.

   We will prove by definition:

   - Zero Vector: Since V and W are both subspaces, we know $\vec{0} \in V$ and $\vec{0} \in W$, thus $\vec{0} \in V \cap W$ by definition.
   - Addition: $\forall \vec{v}, \vec{w} \in V \cap W$, we have $\vec{v}, \vec{w} \in V$ and $\vec{v}, \vec{w} \in W$. Since V, W are both subspaces, we have $\vec{v} + \vec{w} \in V$, and $\vec{v} + \vec{w} \in W$, which implies $\vec{v} + \vec{w} \in V \cap W$ by definition. Thus, $V \cap W$ is closed under addition.
   - Scalar Multiplication: $\forall \vec{x} \in V \cap W, \alpha \in \mathbb{R}$, we have $\vec{x} \in V$ and $\vec{x} \in W$. Since V, W are both subspaces, we have $\alpha \vec{x} \in V$, and $\alpha \vec{x} \in W$. Thus, $\alpha \vec{x} \in V \cap W$. So $V \cap W$ is closed under scalar multipication.

   To sum up, $V \cap W$ is a subspace.

3. (c) Let V, W be two subspaces of $\mathbb{R}^n$. Show that $V \cup W$ may not be a subspace of $\mathbb{R}^n$ with a counter-example.

   We will prove by counter-example. Suppose that $V = \{(x, y) \in \mathbb{R}^2 | x = 3y\}, W = \{(x, y) \in \mathbb{R}^2 | x = y\}$. Then if we choose $\vec{v} = (1, 3) \in V$ and $\vec{w} = (1, 1) \in W$, then surely $\vec{v} \in V \cup W$and $\vec{w} \in V \cup W$Properties of set. Then $\vec{v} + \vec{w} = (2, 4) \notin V$ and $\vec{v} + \vec{w} \notin W$, which is equivalent to say that $\vec{v} + \vec{w} \notin (V \cup W)$.

5 Problem 1.5

Let V be a vector space of dimension n and let $x_1, \ldots, x_n \in V$. Show that:

1. (a) If $x_1, \ldots, x_n$ are linearly independent, then $(x_1, \ldots, x_n)$ is a basis of V.

   We will first prove a lemma: If $(x_1, \cdots, x_n) \in V$ are linearly independent. Then if $\vec{x} \notin Span(v_1, \cdots, v_n)$, we will have that $(v_1, \cdots, v_n, x)$ are linearly independent.

   Proof. We prove by contradiction. If $(v_1, \cdots, v_k, x)$are linearly dependent, there exists $(\beta_1, \cdots, \beta_{k+1}) \neq \vec{0}$ such that $\beta_1 v_1 + \beta_2 v_2 + \cdots \beta_k v_k + \beta_{k+1} x = 0$ holds.
   If $\beta_{k+1} \neq 0$, $x = \frac{\beta_1}{\beta_{k+1}} v_1 + \frac{\beta_2}{\beta_{k+1}} v_2 + \cdots + \frac{\beta_k}{\beta_{k+1}} v_k \in Span(v_1, \cdots, v_k)$, which contradicts the assumption that $x \notin Span(v_1, \cdots, v_n)$.
   If $\exists \beta_i \neq 0, i = 1, 2, \cdots, k$ and $\beta_{k+1} = 0$, $v_i = \frac{\beta_1}{\beta_i} v_1 + \cdots + \frac{\beta_{i-1}}{\beta_i} v_{i-1} + \cdots + \frac{\beta_{i+1}}{\beta_i} v_{i+1} + \cdots + \frac{\beta_k}{\beta_i} v_k$, which contradicts the fact that v_i's are linearly independent. To sum up$(v_1, \cdots, v_k, x)$ are linearly independent. □

   Then we prove our main theorem:

   Proof. We prove by contradiction, assume that $(x_1, \cdots, x_n)$ is not a basis for V. Then by our criterion of basis, since $(x_1, \cdots, x_n)$ are linearly independent, we must have that $Span(x_1, \cdots, x_n) \neq V$. This means $\vec{v} \in V, \quad s.t. \quad \vec{v} \notin Span(x_1, \cdots, x_n)$. Then by our lemma, $(x_1, \cdots, x_n, v)$ are linearly independent. But we know from lecture that any linear independent list of vectors of V with dimension n should have length $k \leq n$, but $(x_1, \cdots, x_n, v)$ has length $n + 1$, which is a contradiction. □

2. (b) If $\operatorname{Span}(x_1, \ldots, x_n) = V$, then $(x_1, \ldots, x_n)$ is a basis of V.

   We will first prove a lemma: Let $x_1, \ldots, x_k \in \mathbb{R}^n$. If $x_1 \in \operatorname{Span}(x_2, \ldots, x_k)$, then $\operatorname{Span}(x_1, \ldots, x_k) = \operatorname{Span}(x_2, \ldots, x_k)$.

   Proof. We prove by definition of set equivalence.
   1. $\forall x \in Span(x_2, \cdots, x_k)$, we have $x = \alpha_2 x_2 + \cdots + \alpha_k x_k = 0 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \in Span(x_1, \cdots, x_k)$ So $Span(x_2, \cdots, x_k) \subseteq Span(x_1, \cdots, x_k)$.
   2. $\forall x \in Span(x_1, \cdots, x_k)$, we have $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$. Since $x_1 \in Span(x_2, \cdots, {}_k)$, we have $x_1 = \beta_2 x_2 + \cdots + \beta_n x_n$, fitting into the equation above we have $x = (\alpha_2 + \alpha_1 \beta_2) x_2 + \cdots + (\alpha_k + \alpha_1 \beta_k) x_k \in Span(x_2, \cdots, x_k) \in Span(x_2, \cdots, x_k)$, thus $Span(x_1, \cdots, x_k) \subseteq Span(x_2, \cdots, x_k)$.
   To sum up, we have $Span(x_1, \cdots, x_k) = Span(x_2, \cdots, x_k)$. □

   Then we prove our main theorem:

   Proof. We prove by contradiction. Assume that $(x_1, \cdots, x_n)$ is not a basis of V, then by definition of a basis, $(x_1, \cdots, x_n)$ must be linearly dependent. By symmetry, we can assume that $x_i \in Span((x_j)_{j \neq i})$. Then by our lemma we have $Span(x_1, \cdots, x_n) = Span((x_j)_{j \neq i}) = V$. However we know from lecture that any spanning list of V with dimension n should have length of at least n, but $Span((x_j)_{j \neq i})$ has length of just $n - 1$, which is a contradiction. □