Homework 0

Due September 17 at 11 pm

Unless stated otherwise, justify any answers you give. You can work in groups, but each student must write their own solution based on their own understanding of the problem.

When uploading your homework to Gradescope you will have to select the relevant pages for each question. Please submit each problem on a separate page (i.e., 1a and 1b can be on the same page but 1 and 2 must be on different pages). We understand that this may be cumbersome but this is the best way for the grading team to grade your homework assignments and provide feedback in a timely manner. Failure to adhere to these guidelines may result in a loss of points. Note that it may take some time to select the pages for your submission. Please plan accordingly. We suggest uploading your assignment at least 30 minutes before the deadline so you will have ample time to select the correct pages for your submission. If you are using LaTeX, consider using the minted or listings packages for typesetting code.

This homework will help you review key mathematical concepts that we will need in the course.

1. (Sets) We will use set theory to define probability spaces. Are these statements true or false? Provide a proof if they are true (you can use Venn diagrams to gain intuition, but also write down a formal proof), or a counterexample if they are false.

   A partition of a set Ω is a collection of sets $S_1, \ldots, S_n$ such that $\Omega = \cup_i S_i$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

   (a) If $S_1, \ldots, S_n$ is a partition of Ω, then for any subset $A \subseteq \Omega$, $S_1 \cap A, \ldots, S_n \cap A$ is a partition of A.

   False, take A to be $\emptyset$, we have $\forall i S_i \cap A = \emptyset$. Thus $\cup_i (S_i \cap A) = \cup_i \emptyset = \emptyset \neq \Omega$.

(b) For any sets A and B, $A^c \cup B^c = (A \cup B)^c$.

False, and a counterexample would be $\Omega = \{1, 2, 3, 4, 5, 6\}, A = \{1, 2, 3\}, B = \{4, 5, 6\}, A^c \cup B^c = \Omega, (A \cup B)^c = \emptyset$.

(c) For any sets A, B, and C, $(A \cup B) \cap C = A \cup (B \cap C)$.

False, and a counterexample would be $A = \{1, 2, 3\}, B = \{2, 3, 4\}, C = \emptyset$, then $(A \cup B) \cap C = \emptyset$, $A \cup (B \cap C) = \{1, 2, 3\} = A$.

2. (Series) We will need series to compute probabilities and expectations related to discrete quantities.

   (a) Assuming $r \neq 1$, derive a simple expression for

$$S_n := \sum_{i=m}^{n} r^i \tag{1}$$

   as a function of r, m and n, and prove that it holds. Assume m and n are positive integers with $m \leq n$.

   By the formula of calculating geometric series, we have $S_n = \sum_{i=1}^{n} r^i - \sum_{i=1}^{m-1} r^i = \frac{r(1-r^n)}{1-r} - \frac{r(1-r^{m-1})}{1-r} = \frac{r^m - r^{n+1}}{1-r}$. We then prove that it holds by induction, and the inductive hypothesis is P(k): $\sum_{i=m}^{n} r^i = \frac{r^m - r^{n+1}}{1-r}$

   Proof. i. Base Step: When $n = 1$, we have $LHS = \sum_{i=1}^{1} r^i = r$ and $RHS = \frac{r-r^2}{1-r} = r$ so $P(1)$ holds.

   ii. Inductive Step: Assume when $n = k, k \geq 1$, $P(k)$ holds, we have $\sum_{i=m}^{k} r^i = \frac{r^m - r^{k+1}}{1-r}$. Then $\sum_{i=m}^{k+1} = \frac{r^m - r^{k+1}}{1-r} + r^{k+1} = \frac{r^m - r^{k+1} + r^{k+1} - r^{k+2}}{1-r} = \frac{r^m - r^{k+2}}{1-r}$, so $P(k+1)$ holds. Thus by induction our formula holds.

   $\square$

(b) Under what condition on r does the infinite series

$$\sum_{i=m}^{\infty} r^i = \lim_{n\to\infty} S_n \tag{2}$$

converge (where again m is a positive integer)?

We can decompose S_n as $\frac{r^m}{1-r} - \frac{r^{n+1}}{1-r}$. Since the limit $\lim_{n\to\infty} S_n = \frac{r^m}{1-r} - \lim_{n\to\infty} \frac{r^{n+1}}{1-r}$ and that $\lim_{n\to\infty} \frac{r^{n+1}}{1-r}$ exists when $|r| < 1$, so we arrive at the conclusion.

(c) Use induction to prove the identity

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \tag{3}$$

where n is a nonnegative integer greater than 1.

i. Base Case: $n = 1$, LHS=1, RHS=$\frac{1\times 2}{2} = 1$, LHS=RHS.

ii. Inductive Step: Let $P(k)$:$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ be true, then $P(k+1) : \sum_{i=1}^{k+1} \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$, which is true. Thus by induction we have $P(n), \forall n \geq 1$ is true.

(d) (Derivatives) We will use derivatives to define probability density functions. The derivative of a differentiable function f is defined as

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{4}$$

i. Briefly explain why the derivative of a function can be interpreted as an *instantaneous rate of change*.

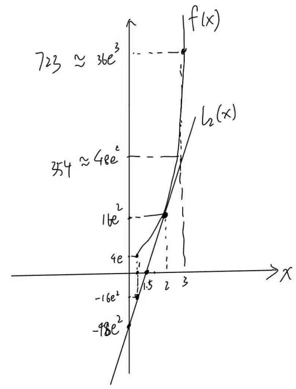
Denote f to be the amount of salt in the tank and is constantly changing, then within a very small time width Δt, the amount is increased by Δf. Then the $\frac{\Delta f}{\Delta t}$ is the change of amount of salt in Δt. When $\Delta t \to 0, \frac{\Delta f}{\Delta t} \to \frac{df}{dt}$, which means the instantaneous rate of change.

ii. Use the definition to derive the derivative of the function x^2.
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} 2x + h = 2x.$

iii. We would like to approximate a differentiable function f at y using a linear function $L_y(x) := ax + b$. We set a and b so that f and L_y have the same value and the same derivative at y (i.e., $L_y(y) = f(y)$ and $L'_y(y) = f'(y)$). Give an expression for $L_y(x)$ in terms of y, $f(y)$, and $f'(y)$.
$L_y(x) = f(y) + f'(y)(x - y)$ where $L'(y) = f'(y) + 0 = f'(y)$ and that $a = f'(y), b = f(y) - f'(y) \cdot y$.

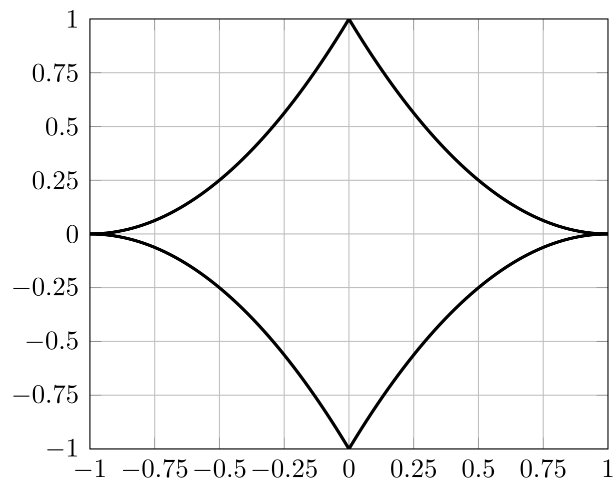
iv. Let $f(x) = 4x^2e^x$. Plot f and L_2 between 1 and 3.

$L_2(x) = f(2) + f'(2)(x - y) = 16e^2 + 32e^2 \cdot (x - 2)$.The graph is the following:



(e) (Integrals) We will use integrals to compute probabilities and expectations related to continuous quantities.

i. Express the area of the following shape in terms of an integral and solve it. Each of the four bounding curves are graphs of quadratic functions. As depicted, the bounding curve includes the points $(0,1)$, $(1,0)$, and $(1/2,1/4)$, and is symmetric about the x and y axes.
We first compute the expression for the four quadratic functions, which are given by $f_1(x) = (x+1)^2$, $f_2(x) = -(x+1)^2$, $f_3(x) = (x-1)^2$, $f_4(x) = -(x-1)^2$. We then calculate the double integral $4\int_0^1 \int_0^{(x-1)^2} 1 dy dx = \frac{4}{3}$ to get the bounded area since we can partition the area into four sub-areas with the same magnitude of area.



ii. Use change of variables to derive a closed-form expression for the function

$$f(t) := \int_0^t \frac{x}{1+x^2}\,\mathrm{d}x. \tag{5}$$

We let $1 + x^2 = u$, so by the properties of differential we have $du = 2xdx$. Then we have:

$$\begin{aligned} f(t) &= \int_0^t \frac{x}{1+x^2}dx \\ &= \int_1^{t^2+1} \frac{1}{2u}du \\ &= \frac{1}{2}ln|u|\Big|_1^{t^2+1} \\ &= \frac{1}{2}|t^2+1| \\ &= \frac{1}{2}(t^2+1) \end{aligned}$$