Problem 2.1 (2 points). *Which of the following are linear transformations? Justify.*

(a) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (-2x+y, x+3y) \end{array} \right.$

It is a linear transformation. And we prove it by definition:

Proof.

(a) $\forall v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2$, *we have* $T(v+w) = T((v_1+w_1, v_2+w_2)) = (-2(v_1+w_1)+(v_2+w_2), (v_1+w_2)+3(v_2+w_2)) = (-2v_1+v_2+(-2w_1+w_2), v_1+3v_2+w_2+3w_2) = (-2v_1+v_2, v_1+3v_2) + (-2w_1+w_2, w_1+3w_2) = T(v)+T(w)$

(b) $\forall v = (v_1, v_2) \in \mathbb{R}^2, \alpha \in \mathbb{R}$, $L(v) = L(\alpha(v_1, v_2)) = L((\alpha v_1, \alpha v_2)) = (-2\alpha v_1 + \alpha v_2, \alpha v_1 + 3\alpha v_2) = \alpha(-2v_1+v_2, v_1+3v_2) = \alpha L(v)$.

□

(b) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x+y+1 \end{array} \right.$

It is not a linear transformation. The counterexample would be easy to find. For example, we choose $v = (1,2), w = (3,4)$, *then* $T(v+w) = T((4,6)) = 4+6+1 = 11$ *whereas* $T(v)+T(w) = (1+2+1)+(3+4+1) = 12$, *which shows that* $T(v+w) \neq T(v)+T(w)$.

(c) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times m} & \to & \mathbb{R}^n \\ A & \mapsto & Ax \end{array} \right.$ *where* $x \in \mathbb{R}^m$ *is a fixed vector.*

It is a linear transformation. And we prove it by definition:

Proof.

(a) $\forall A, B \in \mathbb{R}^{m\times n}$, *we have* $T(A+B) = (A+B)x = Ax + Bx = T(A) + T(B)$

(b) $\forall \alpha \in \mathbb{R}, A \in \mathbb{R}^{m\times n}$, *we have* $T(\alpha A) = (\alpha A)x = \alpha Ax = \alpha T(A)$.

□

(d) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times n} & \to & \mathbb{R}^{n\times n} \\ A & \mapsto & \begin{cases} A^{-1} & \text{if } A \text{ is invertible} \\ 0 & \text{otherwise} \end{cases} \end{array} \right.$

It is not a linear transformation. The counterexample could be if we choose $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ *where* A^{-1} *and* B^{-1} *both exists but* $(A+B)^{-1}$ *doesn't exist and thus doesn't equal to* $A^{-1}+B^{-1}$.

Problem 2.2 (3 points). *Let $L\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear map.*

(a) *Show that* $\mathrm{Ker}(L)$ *is a subspace of* $\mathbb{R}^m$.

Proof.

(a) Since $L(0) = 0$, $0 \in Ker(L)$.

(b) $\forall v, w \in Ker(L)$, we have $L(v) = 0$ and $L(w) = 0$, then we have $L(v+w) = L(v+w) = 0$, which means $v + w \in Ker(L)$.

(c) $\forall v \in Ker(L), \alpha \in \mathbb{R}$, we have $L(v) = 0$ and $L(\alpha v) = \alpha L(v) = 0m$ which implies $\alpha v \in Ker(L)$.

□

(b) *Show that* $\mathrm{Im}(L)$ *is a subspace of* $\mathbb{R}^n$.

Proof.

(a) Since $L(0) = 0$, thus $0 \in \mathbb{R}^n$ has a pre-image $0 \in \mathbb{R}^m$, and this implies $0 \in Im(L)$.

(b) $\forall v, w \in Im(L) = \mathbb{R}^n$, we have $L(v_0) = v$ and $L(w_0) = w$ where $v_0, w_0 \in \mathbb{R}^m$, then we have $L(v_0 + w_0) = L(v_0) + L(w_0) = v + w$, which means $v + w$ has a pre-image and thus $v + w \in Im(L)$.

(c) $\forall v \in Im(L), \alpha \in \mathbb{R}$, we have $L(v_0) = v, v_0 \in \mathbb{R}^m$ and $L(\alpha v_0) = \alpha L(v_0) = \alpha v$, which implies αv has a pre-image αv_0 and $\alpha v \in Im(L)$.

□

(c) *For each subspace below, find a linear map L such that it is its kernel or its image. You do not have to prove that L is linear.*

(i) $E_1 = \{(x, y) \in \mathbb{R}^2 \,|\, 3x - y = 0\}$.

i. We can construct a linear map $L :\left|\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & 3x - y \end{array}\right.$ where E_1 is its kernel.

ii. We can construct a linear map $L :\left|\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x + 2y, 3x + 6y) \end{array}\right.$ where E_1 is its image.

(ii) $E_2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x = 0 \text{ and } y - z = 0\}$.

i. We can construct a linear map $L :\left|\begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R} \\ (x,y,z) & \mapsto & x + y - z \end{array}\right.$ where E_2 is its kernel.

ii. We can construct a linear map $L :\left|\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}^3 \\ x & \mapsto & (0, x, x) \end{array}\right.$ where E_1 is its image.

(iii) $E_3 = \{(4a + b - 3c, 5a - 2b) \,|\, (a, b, c) \in \mathbb{R}^3\}$.

We first rearrange the E_3 into $Span\{\begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix}\} = Span\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix}\}$

i. We can construct a linear map $L :\left|\begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & 0 \end{array}\right.$ where E_3 is its kernel.

ii. We can construct a linear map $L :\left|\begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^2 \\ (x,y,z) & \mapsto & (4x + y - 3z, 5x - 2y) \end{array}\right.$ where E_3 is its image.

Problem 2.3 (3 points). *Let*

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & -3 \\ 3 & 1 & 2 \end{pmatrix}$$

.

(a) *Let $(a, b, c) \in \mathbb{R}^3$. Using Gaussian elimination, find all the $(x, y, z) \in \mathbb{R}^3$ such that $A(x, y, z) = (a, b, c)$. You will have to separate cases according to the values of (a, b, c) (e.g., if you find the equation $a = 0$, then there are no solutions when $a \neq 0$).*

We perform gaussian elimination on augmented matrix $\begin{bmatrix} A | (a, b, c) \end{bmatrix}$ and could get

$$\begin{bmatrix} 1 & 0 & 1 & 5a - 2b \\ 0 & 1 & -1 & b - 2a \\ 0 & 0 & 0 & 5b + c - 13a \end{bmatrix}$$

(a) When $5b + c - 13a \neq 0$, we have no solution.

(b) When $5b+c-13a = 0$, we have infinite solutions $(x, y, z) = \{ \begin{bmatrix} 5a - 2b \\ b - 2a \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} | \in \mathbb{R}\}$

(b) *Using the first question, give $\mathrm{Im}(A)$ as a set of vectors satisfying a linear equation.*

Since the pivot elements appear at the first two columns, we can pick these two pivot columns as our basis for the image of A, and we have $Im(A) = Span(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}) = \begin{bmatrix} x + 2y \\ 2x + 5y \\ 3x + y \end{bmatrix}, \forall x, y \in \mathbb{R}$.

(c) *Using the first question, give a basis of $\mathrm{Ker}(A)$.*

The $Ker(A) = Span(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix})$, which is given by the format of solution in part (a) where $dim(Ker(A)) = 1$.

Problem 2.4 (2 points). *Let B and P be the following matrices in $\mathbb{R}^{3\times 3}$:*

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix}$$

with arbitrary entries for B.

(a) *Compute the matrix product BP. Why is P called a permutation matrix?*

$BP = \begin{pmatrix} B_{1,2} & B_{1,1} & B_{1,3} \\ B_{2,2} & B_{2,1} & B_{2,3} \\ B_{3,2} & B_{3,1} & B_{3,3} \end{pmatrix}$, *since the first and second column of B is swapped to form a new permutation of the columns of B, P is called a permutation matrix.*

(b) *Compute PB. What can you notice?*

$BP = \begin{pmatrix} B_{2,1} & B_{2,2} & B_{2,3} \\ B_{1,1} & B_{1,2} & B_{1,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix}$, *since the first and second row of B is swapped to form a new permutation of the rows of B.*

The observation is that, if the permutation matrix appears at the left matrix, we are permuting the rows, otherwise we are permuting the columns.

Problem 2.5 ($\star$). **(a)** *Show that the set $V = \{L \mid L\colon \mathbb{R}^m \to \mathbb{R}^n \text{ linear}\}$ of all linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a vector space.*

Proof. *By definition of linear maps, we first prove additivity:*

(a) $\forall S, T \in V, v, w \in \mathbb{R}^m$, we have $(S+T)(v+w) = S(v+w)+T(v+w) = S(v)+S(w)+T(v)+T(w) = (S+T)v+(S+T)w$

(b) $\forall S, T \in V, v \in \mathbb{R}^m, \alpha \in \mathbb{R}$, we have $(S+T)(\alpha v) = S(\alpha v)+T(\alpha v) = \alpha S(v)+\alpha T(v) = (\alpha(S+T))(v)$

, thus $S+T$ is linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $S+T \in V$.
We next prove scalar multiplication:

(a) $\forall S \in V, \alpha \in \mathbb{R}, v, w \in \mathbb{R}^m$, we have $(\alpha S)(v+w) = \alpha S(v+w) = \alpha(S(v)+S(w)) = (\alpha S)(v)+(\alpha S)(w)$

(b) $\forall S \in V, \alpha \in \mathbb{R}, v \in \mathbb{R}^m, \beta \in \mathbb{R}$, we have $(\alpha S)(\beta v) = \alpha S(\beta v) = \alpha\beta S(v) = \beta\alpha S(v) = \beta(\alpha S)(v)$

, thus αS is linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $\alpha S \in V$.
Finally we know that zero map always exists, so V is a vector space. $\square$

(b) *Give an example of a linear map from V to $\mathbb{R}^n$.*

One example should be $\mathcal{T} : \left| \begin{array}{ccc} \mathbb{V} & \to & \mathbb{R}^n \\ S & \mapsto & \begin{bmatrix} \sum_{j=1}^m \widetilde{S}_{1j} \\ \sum_{j=1}^m \widetilde{S}_{2j} \\ \vdots \\ \sum_{j=1}^m \widetilde{S}_{nj} \end{bmatrix} \end{array} \right.$ where $\widetilde{S}_{ij}$ is the ij-entry of the representation matrix of the linear map S.

(c) *What is the dimension of V? Hint: what would be the equivalent for linear maps of the proof we did in class for matrices?*

The dimension of V is $n \times m$, because the input is a matrix(representation matrix of a linear map).