Recitation 2

1. Consider two random variables $\tilde{x}$ and $\tilde{y}$ with the joint PMF given in the following table:

	$\tilde{y}=0$	$\tilde{y}=1$
$\tilde{x}=0$	$1/5$	$2/5$
$\tilde{x}=1$	$2/5$	0

   (a) Find the linear MMSE estimator of $\tilde{x}$ given $\tilde{y}$.

   Solution: Using the table we find out,

$$p_{\tilde{x}}(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}, \tag{1}$$
$$p_{\tilde{x}}(1) = \frac{2}{5} + 0 = \frac{2}{5}, \tag{2}$$
$$p_{\tilde{y}}(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}, \tag{3}$$
$$p_{\tilde{y}}(1) = \frac{2}{5} + 0 = \frac{2}{5}. \tag{4}$$

   Therefore, we have

$$E[\tilde{x}] = E[\tilde{y}] = \frac{2}{5}, \tag{5}$$
$$\mathrm{Var}(\tilde{x}) = \mathrm{Var}(\tilde{y}) = \frac{2}{5}\cdot\frac{3}{5} = \frac{6}{25}. \tag{6}$$
$$E[\tilde{x}\tilde{y}] = \sum x_i y_j p_{\tilde{x},\tilde{y}}(x,y) = 0$$
$$\mathrm{Cov}(\tilde{x},\tilde{y}) = E[\tilde{x}\tilde{y}] - E[\tilde{x}]E[\tilde{y}] \tag{7}$$
$$= -\frac{4}{25}. \tag{8}$$

   Therefore, the linear MMSE estimator of $\tilde{x}$ given $\tilde{y}$ is

$$\begin{aligned}\ell_{\mathrm{MMSE}}(\tilde{y}) &= \frac{\mathrm{Cov}(\tilde{x},\tilde{y})}{\mathrm{Var}(\tilde{y})}(\tilde{y} - E[\tilde{y}]) + E[\tilde{x}] \\ &= \frac{-4/25}{6/25}\left(\tilde{y} - \frac{2}{5}\right) + \frac{2}{5} \\ &= -\frac{2}{3}\tilde{y} + \frac{2}{3}.\end{aligned}$$

   (b) Find the MMSE estimator of $\tilde{x}$ given $\tilde{y}$.

   Solution:

$$\begin{aligned}p_{\tilde{x}|\tilde{y}}(0 \mid 0) &= \frac{p_{\tilde{x},\tilde{y}}(0,0)}{p_{\tilde{y}}(0)} \\ &= \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}.\end{aligned}$$

Thus,

$$p_{\tilde{x}|\tilde{y}}(1 \mid 0) = 1 - \frac{1}{3} = \frac{2}{3}.$$

and

$$\tilde{x} \mid \tilde{y} = 0 \sim \text{ Bernoulli } \left(\frac{2}{3}\right).$$

Similarly, we find

$$\begin{aligned} p_{\tilde{x}|\tilde{y}}(0 \mid 1) &= 1, \\ p_{\tilde{x}|\tilde{y}}(1 \mid 1) &= 0. \end{aligned}$$

Thus, the MMSE estimator of X given Y is

$$\ell_{\mathrm{MMSE}}(\tilde{y}) = E[\tilde{x} \mid \tilde{y}]$$