UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 13
Spring 2023

1. **Balls in Bins Estimation**

   You throw n balls into m bins, where $n \geq 1$ and $m \geq 2$. Each ball lands in each bin with the same probability, independently of all other events. Let X and Y be the number of balls in bin 1 and 2 respectively.

   a. What is $\mathbb{E}(Y \mid X)$?

   b. Define $\mathbb{Q}(Y \mid X)$ to be the best quadratic function in X that minimizes mean squared error when used to estimate Y. Without doing any mathematical work, what are $\mathbb{L}(Y \mid X)$ and $\mathbb{Q}(Y \mid X)$? Justify your answer.

   c. Your friend from UCLA who hasn't learned about the Hilbert space of random variables isn't convinced by your explanation. Use the formula

   $$\mathbb{L}(Y \mid X) = \mathbb{E}(Y) + \frac{\text{cov}(X, Y)}{\text{var}(X)}(X - \mathbb{E}(X))$$

   to calculate the LLSE and verify your claim.

2. **Basic Properties of Jointly Gaussian Random Variables**

   Let $(X_1, \dots, X_n)$ be a collection of jointly Gaussian random variables with mean vector μ and covariance matrix Σ. Their joint density is given by, for $x \in \mathbb{R}^n$,

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left\{-\frac{1}{2}(x-\mu)^\mathsf{T}\Sigma^{-1}(x-\mu)\right\}.$$

   a. Show that $X_1, \dots, X_n$ are independent if and only if they are pairwise uncorrelated.

   b. Show that any linear combination of $X_1, \dots, X_n$ will also be a Gaussian random variable. *Hint*: Consider using moment-generating functions.

3. **Gaussian Random Vector MMSE**

Consider the Gaussian random vector

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right),$$

and define the sign of Y to be the random variable

$$W = \begin{cases} 1 & \text{if } Y > 0 \\ 0 & \text{if } Y = 0 \\ -1 & \text{if } Y < 0 \end{cases}.$$

a. Find $\mathbb{E}(WX \mid Y)$.

b. Is the LLSE $\mathbb{L}(WX \mid Y)$ the same as the MMSE you found in part a?

c. Are WX and Y jointly Gaussian?

4. **MMSE for Jointly Gaussian**

Find $\mathbb{E}(X \mid Y, Z)$ for the jointly Gaussian random vector

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \sim \mathcal{N}\left(\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 2 \end{bmatrix} \right).$$

5. **Even-Times Kalman Filter**

Consider a random process $(X_n)_{n\in\mathbb{N}}$ with state space model

$$\begin{aligned} X_{n+1} &= aX_n + V_n, \quad V_n \sim_{\text{i.i.d.}} \mathcal{N}(0, \sigma_V^2) \\ Y_n &= X_n + W_n, \quad W_n \sim_{\text{i.i.d.}} \mathcal{N}(0, \sigma_W^2) \end{aligned}$$

where $(V_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are independent. We can only observe the process at even times, i.e. we observe the random variables $Y_0, Y_2, Y_4, \ldots$.

a. Derive a recurrence relation for the estimator $\hat{X}_{2n|2n} := \mathbb{L}(X_{2n} \mid Y_0, Y_2, \ldots, Y_{2n})$ in terms of $\hat{X}_{2n-2|2n-2}$.

b. Derive a recurrence relation for $\hat{X}_{2n+1|2n}$ in terms of $\hat{X}_{2n|2n}$.

6. **Kalman Filter with Correlated Noise**

   Consider the state space model

$$
\begin{aligned}
X_n &= aX_{n-1} + V_n \\
Y_n &= X_n + V_n,
\end{aligned}
$$

   with $X_0 = 0$ and $(V_n)_{n \geq 0} \sim_{\text{i.i.d.}} \mathcal{N}(0, 1)$. Note that the observation noise is the same as the process noise V_n, not independent of it, so this is different from the usual Kalman filter model. Derive recursive update equations for $\hat{X}_{n|n} := \mathbb{L}(X_n \mid Y_0, \dots, Y_n)$.

   Hint: You may use the fact that the equations will be of the form

$$
\begin{aligned}
\hat{X}_{n|n} &= a\hat{X}_{n-1|n-1} + K_n \tilde{Y}_n \\
\tilde{Y}_n &= Y_n - a\hat{X}_{n-1|n-1},
\end{aligned}
$$

   where you should find the Kalman gain and the estimator covariance recurrence relation

$$
\begin{aligned}
K_n &= ? \\
\sigma^2_{n|n-1} &= a^2 \sigma^2_{n-1|n-1} + 1 \\
\sigma^2_{n|n} &= ?(\sigma^2_{n|n-1}).
\end{aligned}
$$