UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 02
Spring 2023

1. **Minimum of Geometrics**

   Suppose that you are flipping two coins at the same time. The coins are independent of each other, and have probability of heads p and q respectively. Starting at time step 1, at each time step, you flip both coins, and stop if at least one shows heads. What is the expected number of time steps before you stop (including the last flip)? Use this to prove that the minimum of two Geometric random variables is itself Geometric.

2. **Expected Sorting Distance**

Let $(a_1, \ldots, a_n)$ be a random permutation of $\{1, \ldots, n\}$, so that it is equally likely to be any possible permutation. When sorting the list $(a_1, \ldots, a_n)$, the element a_i must move a distance of $|a_i - i|$ places from its current position to reach the position in the sorted order. Find the expected total distance that the elements will have to be moved,

$$\mathbb{E}\left(\sum_{i=1}^{n} |a_i - i|\right)$$

Note: To simplify your answer, you can use the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. **Lightbulbs**

   Consider an $n \times n$ array of switches. Each row i of switches corresponds to a single lightbulb L_i, so that L_i lights up if at least i switches in row i are flipped ON. All of the switches start in the OFF position, and each is flipped ON with probability p, independently of all others. What is the expected number of lightbulbs that will be lit up? Express your answer in closed form without any summations.

4. **Compact Arrays**

Consider an array of $n \geq 1$ entries, where each entry is chosen uniformly randomly from $\{0, \ldots, 9\}$. We want to make the array more compact by moving all the zeros to the end of the array. For example, if we take the array

$$\begin{bmatrix} 6 & 4 & 0 & 0 & 5 & 3 & 0 & 5 & 1 & 3 \end{bmatrix}$$

and make it compact, we now have

$$\begin{bmatrix} 6 & 4 & 5 & 3 & 5 & 1 & 3 & 0 & 0 & 0 \end{bmatrix}$$

Let i be a fixed positive integer in $\{1, \ldots, n\}$. Suppose that the ith entry of the array is nonzero. (The array is indexed starting from 1.) Let X_i be the random variable equal to the index that the ith entry has been moved to after making the array compact. Calculate $\mathbb{E}(X_i)$ and $\text{var}(X_i)$.

5. **Poisson Properties**

a. Suppose X and Y are independent Poisson random variables with means λ and μ respectively. Prove that $X + Y$ has the Poisson distribution with mean $\lambda + \mu$. **Note**: It is *not* enough to use linearity of expectation to say that $X + Y$ has mean $\lambda + \mu$. You are asked to prove that the *distribution* of $X + Y$ is Poisson.

b. Given X and Y as above, what is the distribution of X conditioned on $X + Y = z$, $z \in \mathbb{N}$?

6. **Even Rolls**

Suppose you roll a fair six-sided die until you get a 6. What is the expected number of rolls given that all the numbers you see are even?

Hint: Let N be the number of rolls, and let A be the event that all rolls are even. Try applying the conditional expectation formula for $\mathbb{E}(N \mid A)$.