UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 10
Spring 2023

1. **Midterm**

   Solve again the midterm problems which you got incorrect, resorting to the solutions if necessary. Make sure that you understand the concepts tested without simply copying the solutions, as you may find similar questions on the final exam.

2. **Connected Random Graph**

   We start with the empty graph on n vertices. Iteratively, we add an undirected edge, chosen uniformly at random from the edges that are not yet present in the graph, until the graph is connected.

   Hint: Recall the coupon collector's problem.

   a. Suppose that there are currently k connected components in the graph. Let X_k be the number of edges we need to add until there are $k-1$ connected components. Show that $\mathbb{E}(X_k) \leq \frac{n-1}{k-1}$.
   b. Let X be the total number of edges in the final connected graph. Show that $\mathbb{E}(X) \leq Cn \log n$ for some constant C.

3. **Isolated Vertices**

   Consider an Erdős–Rényi random graph $\mathcal{G}(n, p(n))$, where n is the number of vertices and $p(n)$ is the probability that any specific edge appears in the graph. Let X_n be the number of isolated vertices in $\mathcal{G}(n, p(n))$.

   a. Show that $\mathbb{E}(X_n) \to \exp(-c)$ as $n \to \infty$ when $p(n) = \frac{(\ln n)+c}{n}$ for some constant c.
   b. Conclude that $\mathbb{E}(X_n) \to \infty$ when $p(n) \ll \frac{\ln n}{n}$.
   c. Conclude that $\mathbb{E}(X_n) \to 0$, and $X_n \to 0$ in probability, when $p(n) \gg \frac{\ln n}{n}$.

   The asymptotic notation $f(n) \ll g(n)$ means that $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$.

   Hint: From Taylor series expansion, $\ln(1 + x) \approx x$ when x is small.

4. **Random Bipartite Graph**

Consider a random bipartite graph with K left nodes and M right nodes. Each of the $K \cdot M$ possible edges of this graph is present independently with probability p.

a. Find the distribution of the degree of a particular right node.

b. Now fix a left node u and right node v. Conditioned on the event that the edge (u, v) is present, find the distribution of the degree of v. Is what you find the same as in part a?

c. Call a right node with degree one a *singleton*. What is the expected number of singletons in a random bipartite graph?

d. Find the expected number of left nodes connected to at least one singleton.

5. **Random Graph with Partition**

   Let $G = (V, E)$ be a graph on n vertices, where the vertex set V is deterministically partitioned into two disjoint subsets V_1 and V_2. You do not know the sizes of V_1 and V_2, but assume they both contain at least $n/100$ vertices.

   The edges of G are independently randomly generated as follows. If u and v belong to different subsets, then the edge $\{u, v\}$ appears with probability 1. Otherwise, the edge $\{u, v\}$ appears with probability $p < 1$, where p does not depend on n.

   a. Is G an Erdős–Rényi random graph?

   b. Suppose you do not know which vertices are in V_1 or V_2. You try to determine this by finding a partition $V = \hat{V}_1 \sqcup \hat{V}_2$ such that $\hat{V}_1, \hat{V}_2$ are nonempty subsets and $\{u, v\} \in E$ for all $u \in \hat{V}_1, v \in \hat{V}_2$. If there are multiple candidates, you choose one arbitrarily.

      Your choice is considered correct if $\hat{V}_1 = V_1$ or $\hat{V}_2 = V_2$ (since both identify the same partition of V). What is the limit as $n \to \infty$ of the probability that the procedure above recovers the correct partition?

      Hint. Consider the complementary graph G', in which an edge appears if and only if it does not appear in the original graph G.