UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 02
Spring 2023

1. **Minimum of Geometrics**

   Suppose that you are flipping two coins at the same time. The coins are independent of each other, and have probability of heads p and q respectively. Starting at time step 1, at each time step, you flip both coins, and stop if at least one shows heads. What is the expected number of time steps before you stop (including the last flip)? Use this to prove that the minimum of two Geometric random variables is itself Geometric.

   Solution: Let Z be the number of time steps, i.e. number of flips, before you stop. If we let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ be the number of flips before each individual coin shows heads, then $Z = \min(X, Y)$. We find that

   $$\mathbb{P}(Z > k) = \mathbb{P}(X > k, Y > k) = \mathbb{P}(X > k) \cdot \mathbb{P}(Y > k) = (1-p)^k \cdot (1-q)^k$$

   by independence. We note that for a Geometric random variable like X, we have $\mathbb{P}(X > k) = (1-p)^k$, so

   $$\mathbb{P}(Z > k) = \big((1-p)(1-q)\big)^k$$

   shows that Z is a Geometric random variable with parameter $1-(1-p)(1-q)$. The expected number of time steps is now simply $\mathbb{E}(Z) = \frac{1}{1-(1-p)(1-q)}$.

2. **Expected Sorting Distance**

Let $(a_1, \ldots, a_n)$ be a random permutation of $\{1, \ldots, n\}$, so that it is equally likely to be any possible permutation. When sorting the list $(a_1, \ldots, a_n)$, the element a_i must move a distance of $|a_i - i|$ places from its current position to reach the position in the sorted order. Find the expected total distance that the elements will have to be moved,

$$\mathbb{E}\left(\sum_{i=1}^{n} |a_i - i|\right)$$

Note: To simplify your answer, you can use the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: By the linearity of expectation, we have that

$$\mathbb{E}\left(\sum_{i=1}^{n} |a_i - i|\right) = \sum_{i=1}^{n} \mathbb{E}(|a_i - i|).$$

Because all of the permuations are equally likely, a_i is equally likely to be any number from 1 to n. Thus

$$\begin{aligned}
\mathbb{E}(|a_i - i|) &= \sum_{k=1}^{n} \frac{1}{n} |k - i| \\
&= \frac{1}{n} \sum_{k=1}^{n-i} k + \frac{1}{n} \sum_{k=1}^{i-1} k \\
&= \frac{(n-i)(n-i+1) + (i-1)i}{2n}.
\end{aligned}$$

Putting it all together, and using the closed-form formula for $\sum_{k=1}^{n} k^2$, we obtain

$$\mathbb{E}\left(\sum_{i=1}^{n} |a_i - i|\right) = \frac{n^2 - 1}{3}.$$

3. **Lightbulbs**

Consider an $n \times n$ array of switches. Each row i of switches corresponds to a single lightbulb L_i, so that L_i lights up if at least i switches in row i are flipped ON. All of the switches start in the OFF position, and each is flipped ON with probability p, independently of all others. What is the expected number of lightbulbs that will be lit up? Express your answer in closed form without any summations.

Solution:

Method 1: Linearity of Expectation

Let X_i be the number of switches flipped ON in row i and let Y_i be the indicator random variable that the ith lightbulb lights up, for $1 \le i \le n$.

Consider the following alternative setup, where only the switches in the first row are flipped ON independently with probability p, and the switch status of the other rows all duplicate that of the first row. In this setup, the number of lightbulbs that lights up is equal to the number of switches flipped ON in the first row – if the first row has x switches ON, then the first x lightbulbs will light up. Let X_1' be the number of switches flipped ON in the first row and let Y_i' be the indicator random variable that the ith lightbulb lights up, for $1 \le i \le n$.

We see that X_i and X_1' have the same distribution for all i, and thus Y_i and Y_i' have the same distribution for all i. Then,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = \sum_{i=1}^{n} \mathbb{E}[Y_i'] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i'\right] = \mathbb{E}[X_1'] = np.$$

This gives a strong insight into what linearity of expectation is really saying. You might have already seen the powerfulness of LOE in that it allows us to break dependencies between the random variables of interest when finding their expectations. Here, we go the other direction. We start from independent random variables, but cleverly engineer a counterpart setup where these random variables are dependent in a desired way that is easy to calculate the expectation of. Then, LOE tells us that the expectation of the desired random variable is the same in both cases.

Method 2: Tail Sum

The number of switches ON in each row $i = 1, \ldots, n$ is a random variable $X_i \sim \text{Binomial}(n, p)$. We are interested in the following expectation, which is

$$\begin{aligned} &\mathbb{E}(\mathbb{1}_{X_1 \ge 1} + \mathbb{1}_{X_2 \ge 2} + \cdots + \mathbb{1}_{X_n \ge n}) \\ = &\sum_{i=1}^{n} \mathbb{E}(\mathbb{1}_{X_i \ge i}) = \sum_{i=1}^{n} \mathbb{P}(X_i \ge i) = \sum_{i=1}^{n} \mathbb{P}(X \ge i) = \sum_{i=1}^{\infty} \mathbb{P}(X \ge i) \end{aligned}$$

by linearity. X is any random variable with distribution $\text{Binomial}(n, p)$. Then, by the tail-sum formula, this is just $\mathbb{E}(X) = np$.

4. **Compact Arrays**

Consider an array of $n \geq 1$ entries, where each entry is chosen uniformly randomly from $\{0, \dots, 9\}$. We want to make the array more compact by moving all the zeros to the end of the array. For example, if we take the array

$$\begin{bmatrix} 6 & 4 & 0 & 0 & 5 & 3 & 0 & 5 & 1 & 3 \end{bmatrix}$$

and make it compact, we now have

$$\begin{bmatrix} 6 & 4 & 5 & 3 & 5 & 1 & 3 & 0 & 0 & 0 \end{bmatrix}$$

Let i be a fixed positive integer in $\{1, \dots, n\}$. Suppose that the ith entry of the array is nonzero. (The array is indexed starting from 1.) Let X_i be the random variable equal to the index that the ith entry has been moved to after making the array compact. Calculate $\mathbb{E}(X_i)$ and $\text{var}(X_i)$.

Solution: Let Y_j, $j = 1, \dots, i-1$, be the indicator that the jth entry of the original array is 0. Then the ith entry is moved backwards $\sum_{j=1}^{i-1} Y_j$ positions, so

$$\mathbb{E}(X_i) = i - \sum_{j=1}^{i-1} \mathbb{E}(Y_j) = i - \frac{i-1}{10} = \frac{9i+1}{10}.$$

The variance is also straightforward to compute by the independence of the indicators Y_j. We note that $\text{var}(Y_j) = \frac{1}{10} \cdot \frac{9}{10} = \frac{9}{100}$, so

$$\text{var}(X_i) = \text{var}\left(i - \sum_{j=1}^{i-1} Y_j\right) = \sum_{j=1}^{i-1} \text{var}(Y_j) = \frac{9(i-1)}{100}.$$

5. **Poissson Properties**

a. Suppose X and Y are independent Poisson random variables with means λ and μ respectively. Prove that $X+Y$ has the Poisson distribution with mean $\lambda+\mu$. **Note**: It is *not* enough to use linearity of expectation to say that $X+Y$ has mean $\lambda+\mu$. You are asked to prove that the *distribution* of $X+Y$ is Poisson.

b. Given X and Y as above, what is the distribution of X conditioned on $X+Y=z$, $z \in \mathbb{N}$?

Solution:

a. The distribution of the sum of two independent random variables is the *convolution* of their individual distributions. For $z \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}(X+Y=z) &= \sum_{x=0}^{z} \mathbb{P}(X=x) \cdot \mathbb{P}(Y=z-x) \\
&= \sum_{x=0}^{z} \frac{\lambda^x}{x!} e^{-\lambda} \cdot \frac{\mu^{z-x}}{(z-x)!} e^{-\mu} \\
&= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \lambda^x \mu^{z-x} \\
&= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \mu^{z-x} \\
&= \frac{e^{-(\lambda+\mu)}}{z!} (\lambda+\mu)^z,
\end{aligned}$$

which shows that $X+Y \sim \text{Poisson}(\lambda+\mu)$.

b.

$$\begin{aligned}
\mathbb{P}(X=x \mid X+Y=z) &= \frac{\mathbb{P}(X=x, X+Y=z)}{\mathbb{P}(X+Y=z)} \\
&= \mathbb{P}(X=x) \cdot \mathbb{P}(Y=z-x) \Big/ \left(\frac{(\lambda+\mu)^z}{z!} e^{-(\lambda+\mu)} \right) \\
&= \frac{\lambda^x}{x!} e^{-\lambda} \cdot \frac{\mu^{z-x}}{(z-x)!} e^{-\mu} \Big/ \left(\frac{(\lambda+\mu)^z}{z!} e^{-(\lambda+\mu)} \right) \\
&= \binom{z}{x} \left(\frac{\lambda}{\lambda+\mu} \right)^x \left(\frac{\mu}{\lambda+\mu} \right)^{z-x},
\end{aligned}$$

which shows that $X \mid X+Y=z \sim \text{Binomial}(z, \frac{\lambda}{\lambda+\mu})$.

Remark: These properties will be used extensively when we discuss the Poisson process.

6. **Even Rolls**

Suppose you roll a fair six-sided die until you get a 6. What is the expected number of rolls given that all the numbers you see are even?

Hint: Let N be the number of rolls, and let A be the event that all rolls are even. Try applying the conditional expectation formula for $\mathbb{E}(N \mid A)$.

Solution: Let N be the number of rolls with this procedure and A be the event that all numbers seen are even. We can find $\mathbb{E}(N \mid A)$ as

$$\begin{aligned}\mathbb{E}(N \mid A) &= \sum_{n=1}^{\infty} n\, \mathbb{P}(N = n \mid A) \\ &= \sum_{n=1}^{\infty} n \frac{\mathbb{P}(A \mid N = n) \cdot \mathbb{P}(N = n)}{\mathbb{P}(A)} \\ &= \sum_{n=1}^{\infty} n \frac{\left(\frac{2}{5}\right)^{n-1} \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}}{\frac{1}{4}} \\ &= \sum_{n=1}^{\infty} n \frac{2}{3}\left(\frac{1}{3}\right)^{n-1} \\ &= \frac{3}{2}.\end{aligned}$$

As a side note, $\mathbb{P}(A) = \frac{1}{4}$ comes the fact that A is equivalent to the event that the first number in $\{1, 3, 5, 6\}$ to appear is a 6, which has probability $\frac{1}{4}$ by symmetry.

Alternatively, consider counting the number of die rolls until we get a 1, 3, 5, or 6. Let M be the number of rolls for this new problem, and B be the value of the last roll. We want to find $\mathbb{E}(M \mid B = 6)$, but notice that $\mathbb{E}(M \mid B) = \mathbb{E}(\mathbb{E}(M \mid B))$ by symmetry between the possible values of the last number. Thus, our answer is

$$\mathbb{E}(M \mid B = 6) = \mathbb{E}(\mathbb{E}(M \mid B)) = \mathbb{E}(M) = \frac{3}{2},$$

since $M \sim \text{Geometric}\left(\frac{2}{3}\right)$.