UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

**Homework 08**
Spring 2023

1. **Ant**

   An ant is walking on the nonnegative integers. At each step, the ant moves forward one step with probability $p \in (0, 1)$, or slides back down to 0 with probability $1-p$. What is the average time it takes for the ant to get to $n$, where $n$ is a positive integer, starting from state 0?

2. **Basketball II**

   Captain America and Superman are playing an untimed basketball game in which the two players score points according to independent Poisson processes with rates $\lambda_C$ and $\lambda_S$ respectively. The game is over when one player has scored $k$ more points than the other.

   a. Suppose $\lambda_C = \lambda_S$, and suppose Captain America has a head start of $m < k$ points. Find the probability that Captain America wins.
   *Hint*: if $\alpha_i = \frac{1}{2}\alpha_{i-1} + \frac{1}{2}\alpha_{i+1}$, then $\alpha_{i+1} - \alpha_i = \alpha_i - \alpha_{i-1}$.

   b. Keeping the assumptions, find the expected time $\mathbb{E}(T)$ it will take for the game to end.
   *Hint*: consider the telescoping sum $\beta_j = \beta_0 + (\beta_1 - \beta_0) + \cdots + (\beta_j - \beta_{j-1})$.

3. **Checking Reversibility**

a. *Cut property.* A **cut** of a graph is a partition of its states $S$ into two disjoint subsets $T$, $S \setminus T$. Show that for an irreducible Markov chain at stationarity, flow-in equals flow-out holds across any cut of the Markov chain. That is, for any time $n$,

$$\mathbb{P}(X_n \in T,\, X_{n+1} \in S \setminus T) = \mathbb{P}(X_n \in S \setminus T,\, X_{n+1} \in T).$$

b. *Sufficient condition for reversibility.* We can convert the transition diagram of any chain into an undirected graph by removing any self-loops and making all edges undirected. For an irreducible chain whose resulting graph is a **tree**, show that if it has a stationary distribution, then it must also satisfy detailed balance.

(In particular, this shows that positive recurrent birth-death chains are reversible, even on infinite state spaces.)

4. **Metropolis–Hastings**

We will prove some properties of the *Metropolis–Hastings* algorithm, an example of Markov Chain Monte Carlo (MCMC) sampling that you will see more of in lab. The goal of MH is to draw samples from a distribution $p(x)$; the algorithm assumes that

- We can compute $p(x)$ up to a normalizing constant $C$ via $f(x)$, and
- We have a proposal distribution $g(x, \cdot)$.

The steps in making a transition are:

i. Propose the next state $y$ according to the distribution $g(x, \cdot)$.

ii. Accept the proposal with probability

$$A(x, y) = \min\left\{1, \frac{f(y)}{f(x)} \frac{g(y, x)}{g(x, y)}\right\}.$$

iii. If the proposal is accepted, move the chain to $y$; otherwise, stay at $x$.

*Remark.* The normalizing factor $C = 1/\sum_{x \in \mathcal{X}} f(x)$ is sometimes called the *partition function*, and it can be difficult to compute for large sets $\mathcal{X}$, even if $f(x)$ is efficient to compute.

In the following, we will verify that the Metropolis–Hastings chain has stationary distribution $p$, and in fact approaches stationarity after running for some time, at which point we can draw samples from $p$ by sampling from the chain.

a. The key to why Metropolis–Hastings works is the **detailed balance equations**. Suppose we have a finite irreducible Markov chain on a state space $\mathcal{X}$ with transition probability matrix $P$. Show that if there exists a distribution $\pi$ on $\mathcal{X}$ satisfying detailed balance,

$$\pi(x)P(x, y) = \pi(y)P(y, x) \quad \text{for all } x, y \in \mathcal{X},$$

then $\pi P = \pi$ is a stationary distribution of the chain.

b. Returning to the Metropolis–Hastings chain, find $P(x, y)$. For simplicity, assume $x \neq y$.

c. Show that the target distribution $p(x)$ satisfies the detailed balance equations for $P(x, y)$, and conclude that $p(x)$ is the stationary distribution of the chain.

d. If the chain is aperiodic, then it will converge to the stationary distribution. If not, we can force the chain to be aperiodic by considering the **lazy chain**: on each transition, the chain decides not to move with probability $\frac{1}{2}$, independently of the propose-accept step. Explain why the lazy chain is aperiodic, and explain why the stationary distribution is the same as before.

5. **Poisson Process Practice**

   Let $(N_t)_{t\geq 0}$ be a Poisson process with rate $\lambda$. Let $T_k$, $k \in \mathbb{N}$ denote the time of the $k$th arrival. Given $0 \leq s < t$, we write $N(s,t) := N(t) - N(s)$. Compute the following:

   a. $\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0)$.
   b. $\mathbb{E}(N(1,3) \mid N(1,2) = 3)$.
   c. $\mathbb{E}(T_2 \mid N(2) = 1)$.

6. **Bus Arrivals at Cory Hall**

   Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate $\lambda$. Students arrive at the stop according to an independent Poisson process of rate $\mu$. Every time the bus arrives, all students waiting get on.

   a. Given that the interarrival time between bus $i-1$ and bus $i$ is $x$, find the distribution for the number of students entering the $i$th bus. Here, $x$ is a given number, not a random quantity.

   b. Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.