UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

**Discussion 06**
Spring 2023

1. **Gambling Game**

   Let's play a game. You stake a positive initial amount of money $w_0$. Then you toss a fair coin. If it is heads, you earn an amount equal to three times your stake, so you quadruple your wealth. If it comes up tails, you lose everything. There is one requirement, though: you are not allowed to quit, and you have to keep playing by staking all your available wealth, over and over again. Let $W_n$ be a random variable equal to your wealth after $n$ plays.

   a. Find $\mathbb{E}(W_n)$ and show that $\lim_{n\to\infty} \mathbb{E}(W_n) = \infty$.

   b. Since $\lim_{n\to\infty} \mathbb{E}(W_n) = \infty$, this game sounds like a good deal. But wait a moment! To where does the sequence of random variables $\{W_n\}_{n\geq 0}$ converge almost surely?

2. **More Almost Sure Convergence**

   a. Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

   b. Suppose that $Y$ is uniform on $[-1, 1]$, and $X_n$ has distribution

   $$\mathbb{P}\big(X_n = (y + n^{-1})^{-1} \mid Y = y\big) = 1.$$

   Does $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

   c. Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each $X_n$ to 0. Then, for each $k \in \mathbb{N}$, pick $j$ uniformly randomly in $\{2^k, \dots, 2^{k+1} - 1\}$, and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

   d. Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some $X$? If so, is it true that $\mathbb{E}(X_n) \to \mathbb{E}(X)$ as $n \to \infty$?

3. **Sum of Rolls**

   You roll a fair 6-sided die 100 times, and you call the sum of the values of all your rolls $X$. Use the Central Limit Theorem to approximate the probability that $X > 400$. You may use a calculator and Gaussian lookup table.

4. **Entropy Warmup**

Suppose that the random variable $X$ takes values in $\{\text{lecture}, \text{midterm}, \text{pop quiz}\}$. Every day you go to class, you observe a random value of $X$ determined according to the distribution $p_X$, for instance $p_X(\text{lecture}) = 0.85$, $p_X(\text{midterm}) = 0.1$, and $p_X(\text{pop quiz}) = 0.05$. The *surprise*

$$S(x) = \log_2 \frac{1}{p_X(x)}$$

describes how "interesting" it is to see a particular $X = x$.

a. For the probabilities above, calculate $S(\text{lecture})$, $S(\text{midterm})$, and $S(\text{pop quiz})$.

b. Calculate the surprises for $p_X(\text{lecture}) = \frac{1}{3}$, $p_X(\text{midterm}) = \frac{1}{3}$, and $p_X(\text{pop quiz}) = \frac{1}{3}$. Given that $\log_2 \frac{1}{0.85} \approx 0.234$, $\log_2 \frac{1}{1/3} \approx 1.58$, $\log_2 \frac{1}{0.1} \approx 3.32$, and $\log_2 \frac{1}{0.05} \approx 4.32$, do the relative magnitudes of the values in parts (a) and (b) make sense intuitively?

c. The entropy of $X$ is its *expected surprise*. Formally,

$$H(X) = \mathbb{E}(S(X)) = \sum_x p_X(x) \log_2 \frac{1}{p_X(x)}.$$

We will follow the convention that if $p_X(x) = 0$ for some value $x$, then $p_X(x) \log_2 \frac{1}{p_X(x)} = 0$. Calculate the entropy for the original distribution $(0.85, 0.1, 0.05)$, the uniform distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the deterministic distribution $(1, 0, 0)$. Do these entropy values make sense?