UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 04
Spring 2023

1. **Drawing Batteries I**

   You have an endless box of used batteries. The number of hours remaining in a battery is i.i.d. Uniform($[0,1]$).

   a. Suppose you draw n batteries, and the ith battery you draw has X_i hours remaining. What is $\mathbb{P}(X_1 \leq X_2 \leq \cdots \leq X_n)$?

   b. Now suppose you draw batteries until you have enough batteries to last one hour in total. Let N be the number of batteries you draw. What is $\mathbb{P}(N > 2)$? $\mathbb{P}(N > 3)$?

Solution:

a. Note that each ordering of the random variables is equally likely, and there are $n!$ such orderings. Thus we have

$$\mathbb{P}(X_1 \leq X_2 \leq \cdots \leq X_n) = \frac{1}{n!}$$

b. Note that by the definition of N,

$$\mathbb{P}(N > 2) = \mathbb{P}(X_1 + X_2 \leq 1), \quad \mathbb{P}(N > 3) = \mathbb{P}(X_1 + X_2 + X_3 \leq 1).$$

Let us first find $\mathbb{P}(X_1 + X_2 \leq 1)$. We observe that (X_1, X_2) is uniformly distributed on the unit square in the (X_1, X_2) plane, and the inequality $X_1 + X_2 \leq 1$ is represented by the triangular region with vertices $(0,0)$, $(0,1)$, and $(1,0)$. Thus $\mathbb{P}(X_1 + X_2 \leq 1) = \frac{1}{2}$. We now tackle $\mathbb{P}(X_1 + X_2 + X_3 \leq 1)$. By the same construction, (X_1, X_2, X_3) is uniform on the unit cube in the first orthant, and the region of interest is the triangular pyramid with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$. By a bit of geometry,

$$\mathbb{P}(X_1 + X_2 + X_3 \leq 1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

2. **Graphical Density**

   Figure 1 shows the joint density $f_{X,Y}$ of the random variables X and Y.

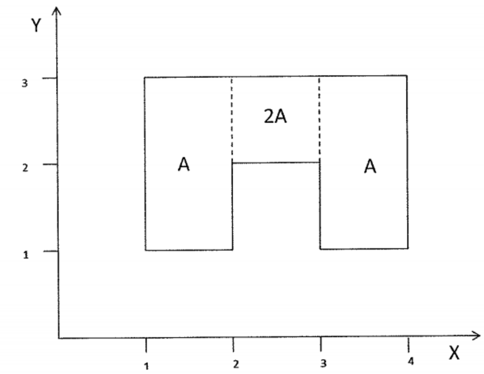

Figure 1: Joint density of X and Y.

   a. Find A and sketch f_X, f_Y, and $f_{X|X+Y\leq 3}$.
   b. Find $\mathbb{E}(X \mid Y = y)$ for $1 \leq y \leq 3$ and $\mathbb{E}(Y \mid X = x)$ for $1 \leq x \leq 4$.
   c. Find $\operatorname{cov}(X, Y)$.

Solution:

   a. The integral of the density over the entire shown region should be 1, i.e.

$$1 = \int_1^3 \int_1^2 A \,\mathrm{d}x \,\mathrm{d}y + \int_2^3 \int_2^3 2A \,\mathrm{d}x \,\mathrm{d}y + \int_1^3 \int_3^4 A \,\mathrm{d}x \,\mathrm{d}y$$
$$= 2A + 2A + 2A = 6A.$$

So, $A = \frac{1}{6}$. Now we find the densities as follows. X is clearly uniform over each of the intervals $[1, 2]$, $[2, 3]$, and $[3, 4]$. The probability of X being in any one of these intervals is $2A = \frac{1}{3}$, which means that

$$f_X(x) = \frac{1}{3}\,\mathbb{1}\{1 \leq x \leq 4\}.$$

Y is uniform in each of the intervals $[1, 2]$ and $[2, 3]$. The probability of Y falling in the first interval is $\frac{1}{3}$, and the second interval $\frac{2}{3}$, so

$$f_Y(y) = \frac{1}{3}\,\mathbb{1}\{1 \leq y \leq 2\} + \frac{2}{3}\,\mathbb{1}\{2 < y \leq 3\}.$$

Finally, given that $X + Y \leq 3$, (X, Y) is uniformly distributed over the triangle with vertices $(1, 1)$, $(1, 2)$, and $(2, 1)$. Thus

$$f_{X|X+Y\leq 3}(x) = \int_1^{3-x} 2 \,\mathrm{d}y = 2(2 - x)\,\mathbb{1}\{1 \leq x \leq 2\}.$$

Sketching the densities is then a straightforward matter of plotting the functions.

b. Given that Y takes on any value of $y \in [1,3]$, we see that X has a conditional distribution symmetric about the line $x = 2.5$. Thus

$$\mathbb{E}(X \mid Y = y) = 2.5 \quad \text{for all } y,\ 1 \leq y \leq 3.$$

To calculate $\mathbb{E}(Y \mid X = x)$, we will have to consider two cases:

- When $2 \leq x \leq 3$, then $\mathbb{E}(Y \mid X = x) = 2.5$.
- When $1 \leq x < 2$ or $3 < x \leq 4$, then $\mathbb{E}(Y \mid X = x) = 2$.

In both cases, we found the conditional expectation of Y using graphical symmetry.

c. As $\mathbb{E}(X \mid Y = y) = \mathbb{E}(X)$, we have that

$$\begin{aligned}
\mathbb{E}(XY) &= \int_1^3 \mathbb{E}(XY \mid Y = y) \cdot f_Y(y)\, \mathrm{d}y = \int_1^3 \mathbb{E}(X) y \cdot f_Y(y)\, \mathrm{d}y \\
&= \mathbb{E}(X) \cdot \mathbb{E}(Y).
\end{aligned}$$

Thus $\operatorname{cov}(X, Y) = 0$.

3. **Revisiting Proofs Using Transforms**

   a. Calculate the MGF of $X \sim \text{Poisson}(\lambda)$.

   b. Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ be independent. Calculate the MGF of $X+Y$, and use this to show that $X+Y \sim \text{Poisson}(\lambda+\mu)$.

   c. Repeat parts a and b above, this time for the standard normal distribution.

Solution:

a. The MGF of X is

$$\mathbb{E}(e^{sX}) = \sum_{x=0}^{\infty} e^{sx} \frac{\lambda^x}{x!} e^{-\lambda} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^s)^x}{x!} = e^{\lambda(e^s-1)},$$

which converges for all $s \in \mathbb{R}$.

b. The MGF of $X+Y$ is, by the independence of X and Y,

$$\mathbb{E}(e^{s(X+Y)}) = \mathbb{E}(e^{sX}) \cdot \mathbb{E}(e^{sY}) = e^{\lambda(e^s-1)} \cdot e^{\mu(e^s-1)} = e^{(\lambda+\mu)(e^s-1)},$$

which we recognize as the MGF of a $\text{Poisson}(\lambda+\mu)$ random variable.

Remark: In general, arguing that the MGF uniquely determines the probability distribution requires a few assumptions on the MGF itself, but we will not worry about these issues in this course.

c. The MGF of $X \sim \mathcal{N}(0,1)$ is, by completing the square,

$$\begin{aligned} M_X(s) &= \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2-2sx)/2} \,\mathrm{d}x \\ &= e^{s^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-s)^2/2} \,\mathrm{d}x \\ &= e^{s^2/2}. \end{aligned}$$

If $Y \sim \mathcal{N}(0,1)$ is independent of X, then the MGF of $X+Y$ is

$$M_{X+Y}(s) = \mathbb{E}(e^{sX}) \cdot \mathbb{E}(e^{sY}) = e^{s^2/2} \cdot e^{s^2/2} = e^{s^2},$$

which we recognize as the MGF of a $\mathcal{N}(0,2)$ distribution.