UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 05
Spring 2023

1. **Order Statistics Practice**

For the random variables $X_1, \dots, X_5 \sim_{\text{i.i.d.}} \text{Uniform}([0,1])$, let $X_{(i)}$ be the ith order statistic, i.e. the ith smallest value of $\{X_1, \dots, X_5\}$. Recall that the pdf of $X_{(i)}$ is

$$f_{X_{(i)}}(x) = n\binom{n-1}{i-1} f(x)F(x)^{i-1}(1-F(x))^{n-i},$$

where f and F are the pdf and cdf of X_i.

a. What is the pdf of $X_{(i)}$ for $i = 1, \dots, 5$?
b. What is $\mathbb{E}(X_{(i)})$?
c. Find the expected value of the range of $\{X_1, \dots, X_5\}$, that is, the difference between the lowest and highest values.

Solution:

a. Plugging $n = 5$ into the formula, we have

$$f_{X_{(i)}}(x) = 5\binom{4}{i-1} x^{i-1}(1-x)^{5-i}.$$

b. Imagine picking points uniformly randomly on a circle with respect to a fixed reference point, then cutting the circle at that point afterwards to get an extra segment. Since the location of the reference point doesn't actually affect the pdf, as the distribution is uniform, we can also imagine uniformly selecting the reference point. Thus, this is equivalent to picking $n+1$ points uniformly at random on a circle. By symmetry, the expected length between any two segments is $\frac{1}{n+1}$, and $\mathbb{E}(X_{(i)}) = \frac{i}{n+1}$.
c. The range can be represented as $X_{(5)} - X_{(1)}$. By the previous part,

$$\mathbb{E}(X_{(5)} - X_{(1)}) = \frac{5}{6} - \frac{1}{6} = \frac{2}{3}.$$

2. **Exponential Bounds**

Let $X \sim \text{Exponential}(\lambda)$. For $x > \lambda^{-1}$, find bounds on $\mathbb{P}(X \geq x)$ using Markov's inequality, Chebyshev's inequality, and the Chernoff bound.

Solution: Since $\mathbb{E}(X) = \lambda^{-1}$, Markov's inequality gives

$$\mathbb{P}(X \geq x) \leq \frac{\mathbb{E}(X)}{x} = \frac{1}{\lambda x},$$

and from $\text{var}(X) = \lambda^{-2}$, Chebyshev's inequality gives

$$\begin{aligned}\mathbb{P}(X \geq x) = \mathbb{P}(X - \lambda^{-1} \geq x - \lambda^{-1}) &\leq \mathbb{P}(|X - \lambda^{-1}| \geq x - \lambda^{-1}) \\ &\leq \frac{\text{var}(X)}{(x - \lambda^{-1})^2} = \frac{1}{(\lambda x - 1)^2}.\end{aligned}$$

By the Chernoff bound, for any $s > 0$,

$$\mathbb{P}(X \geq x) = \mathbb{P}\left(\exp(sX) \geq \exp(sx)\right) \leq \frac{M_X(s)}{\exp(sx)} = \frac{\lambda}{(\lambda - s)\exp(sx)}.$$

We wish to optimize this bound over $s > 0$; we note that it suffices to maximize the denominator $(\lambda - s)\exp(sx)$. Differentiating,

$$-\exp(sx) + x(\lambda - s)\exp(sx) = 0,$$

so $1 = x(\lambda - s)$, that is, $s = \lambda - x^{-1}$. Thus

$$\begin{aligned}\mathbb{P}(X \geq x) &\leq \frac{\lambda}{(\lambda - (\lambda - x^{-1}))\exp((\lambda - x^{-1})x)} = \frac{\lambda}{x^{-1}\exp(\lambda x - 1)} \\ &= \lambda x \exp\left(-(\lambda x - 1)\right).\end{aligned}$$

Observe that the Chernoff bound is the only one which decreases exponentially with x, which is the true behavior: $\mathbb{P}(X \geq x) = \exp(-\lambda x)$.

3. **Convergence in Probability**

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables distributed uniformly in $[-1,1]$. Show that the following sequences $(Y_n)_{n\in\mathbb{N}}$ converge in probability to some limit.

a. $Y_n = \prod_{i=1}^n X_i$.

b. $Y_n = \max\{X_1,\dots,X_n\}$.

c. $Y_n = (X_1^2 + \cdots + X_n^2)/n$.

Solution:

a. By the independence of the random variables,

$$\begin{aligned}\mathbb{E}(Y_n) &= \mathbb{E}(X_1)\cdots\mathbb{E}(X_n) = 0\\ \operatorname{var}(Y_n) &= \mathbb{E}(Y_n^2) = (\operatorname{var}(X_1))^n = \left(\frac{1}{3}\right)^n.\end{aligned}$$

Since $\operatorname{var}(Y_n) \to 0$ as $n \to \infty$, by Chebyshev's inequality, the sequence converges to its mean 0 in probability.

b. Consider $\varepsilon \in (0,1]$. We see that

$$\begin{aligned}\mathbb{P}(|Y_n - 1| \geq \varepsilon) &= \mathbb{P}(\max\{X_1,\dots,X_n\} \leq 1-\varepsilon)\\ &= \mathbb{P}(X_1 \leq 1-\varepsilon,\dots,X_n \leq 1-\varepsilon)\\ &= \mathbb{P}(X_1 \leq 1-\varepsilon)^n\\ &= \left(1 - \frac{\varepsilon}{2}\right)^n,\end{aligned}$$

so $\mathbb{P}(|Y_n - 1| \geq \varepsilon) \to 0$ as $n \to \infty$, and we are done.

c. We can find the expectation, then bound the variance:

$$\begin{aligned}\mathbb{E}(Y_n) &= \frac{1}{n}\cdot n\,\mathbb{E}(X_1^2) = \frac{1}{3},\\ \operatorname{var}(Y_n) &= \frac{1}{n}\operatorname{var}(X_1^2) \leq \frac{1}{n} \to 0 \qquad \text{as } n \to \infty,\end{aligned}$$

since $X_1^2 \leq 1$. Hence, we see that $Y_n \to \frac{1}{3}$ in probability as $n \to \infty$.