UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 04
Spring 2023

1. **Moment-Generating Functions Practice**

   The **moment-generating function** (mgf) of a random variable X is the function

$$M_X(s) = \mathbb{E}(e^{sX}) = \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{(sX)^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{s^k}{k!}\,\mathbb{E}(X^k).$$

   In this class, we will not worry about technical details about the convergence of Taylor series, so we will say that the mgf is equal to any of the expressions above.

   The mgf gets its name because it is the function *generating* the moments $\mathbb{E}(X^p)$, $p \geq 1$, of X. More specifically, by evaluating the pth derivative of the mgf at $s = 0$, we have a method to explicitly find the pth moment of X from its mgf:

$$\left[\frac{\mathrm{d}^p}{\mathrm{d}s^p} M_X(s)\right]_{s=0} = \left[\sum_{k=p}^{\infty} \frac{s^{k-p}}{p!}\,\mathbb{E}(X^k)\right]_{s=0} = \mathbb{E}(X^p) + \sum_{k=p+1}^{\infty} 0 = \mathbb{E}(X^p).$$

   Consider a random variable Z with moment-generating function

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8} \qquad \text{for } |s| < 2.$$

   Calculate the following quantities:

   a. The numerical value of the parameter a.
   b. $\mathbb{E}(Z)$.
   c. $\mathrm{var}(Z)$.

2. **Transforms and Independence**

In this problem, we will make use of multivariate moment generating functions, defined for a random vector $X = (X_1, \ldots, X_n)$ as

$$M_X(t) = \mathbb{E}(e^{t \cdot X}) = \mathbb{E}(e^{\sum_{i=1}^n t_i X_i})$$

for $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. You may assume that MGFs are unique: if $M_X(t) = M_Y(t)$ for all t, then $X \sim Y$.

Consider the random vector $X = (X_1, \ldots, X_n)$. Show that $X_1, \ldots, X_n$ are independent if and only if for all $t \in \mathbb{R}^n$,

$$M_X(t) = \prod_{i=1}^n M_{X_i}(t_i).$$

3. **Coupon Collector Bounds**

Recall the coupon collector's problem, in which there are n different types of coupons. Every box contains a single coupon, and we let the random variable X be the number of boxes bought until one of every type of coupon is obtained. The expected value of X is nH_n, where $H_n := \sum_{i=1}^{n} \frac{1}{i}$ is the *harmonic number* of order n, which satisfies the inequality

$$\ln n \le H_n \le \ln n + 1.$$

a. Use Markov's inequality in order to show that

$$\mathbb{P}(X > 2nH_n) \le \frac{1}{2}.$$

b. Use Chebyshev's inequality in order to show that

$$\mathbb{P}(X > 2nH_n) \le \frac{\pi^2}{6(\ln n)^2}.$$

Note: You can use Euler's solution to the Basel problem, the identity $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$.

c. Define appropriate events and use the union bound in order to show that

$$\mathbb{P}(X > 2nH_n) \le \frac{1}{n}.$$

Note: $a_n = (1 - \frac{1}{n})^n$ is a strictly increasing sequence with limit e^{-1}.

4. **Confidence Interval Comparisons**

In order to estimate the probability p of heads of a coin flip, you flip a coin $n \geq 1$ times and count the number of heads S_n. You use the estimator $\hat{p} = \frac{S_n}{n}$, and you choose the sample size n to have the guarantee

$$\mathbb{P}(|\hat{p} - p| \geq \varepsilon) \leq \delta.$$

Using Chebyshev's inequality, determine n for the following parameters. You should not have p in your final answer.

a. Compare the value of n when $\varepsilon = 0.05, \delta = 0.1$ to the value of n when $\varepsilon = 0.1, \delta = 0.1$.

b. Compare the value of n when $\varepsilon = 0.1, \delta = 0.05$ to the value of n when $\varepsilon = 0.1, \delta = 0.1$.

5. **Convergence in L^p**

   Let $p \geq 1$. A sequence of random variables $(X_n)_{n\geq 1}$ is said to **converge in** L^p (norm) to a random variable X if

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0.$$

   Prove that if $X_n \to X$ in L^p, then $X_n \to X$ in probability.

6. **Breaking a Stick**

   I break a stick n times, $n \geq 1$, in the following manner: the ith time I break the stick, I keep a fraction $X_i \sim \text{Uniform}((0,1])$ of the remaining stick. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. Let $P_n = \prod_{i=1}^{n} X_i$ be the fraction of the original stick that I end up with at time n.

   a. Show that $P_n^{1/n}$ converges almost surely, and find its limit.

   b. Compute $\mathbb{E}(P_n)^{1/n}$.

   c. Now compute $\mathbb{E}(P_n^{1/n})$. Do you find the same answer as in part b? Is the limit of $\mathbb{E}(P_n^{1/n})$ equal to the limit you found in part a?