UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 10
Spring 2023

1. **Expected Squared Arrival Times**

   Let $(N_t)_{t\geq 0}$ be a Poisson process with arrival times $(T_n)_{n\geq 1}$. Find $\mathbb{E}(T_1^2 + T_2^2 + T_3^2 \mid N_1 = 3)$.

   Solution: Conditioned on $\{N_1 = 3\}$, the arrival times T_1, T_2, T_3 are jointly distributed as the order statistics of 3 i.i.d. Uniform($[0, 1]$) random variables. If $U_1, U_2, U_3 \sim_{\text{i.i.d.}}$ Uniform($[0, 1]$),

   $$\mathbb{E}(T_1^2 + T_2^2 + T_3^2 \mid N_1 = 3) = \mathbb{E}(U_1^2 + U_2^2 + U_3^2) = 3 \cdot \frac{1}{3} = 1.$$

   Remark: We can also find the result by successive integration with proper bounds. Although T_1, T_2, T_3 are ordered random variables, we will not need to use order in this problem as we are interested in the sum of expectations.

2. **CTMC Introduction**

Consider the continuous-time Markov chain defined on the state space $\{1,2,3,4\}$ which has transition rate matrix

$$Q = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

a. Find the stationary distribution π of this chain.

b. Find the stationary distribution μ of the jump chain, the DTMC which only keeps track of the jumps. Formally, if $(X_t)_{t\geq 0}$ transitions at times $T_1, T_2, \ldots$, then its jump chain is $(Y_n)_{n=1}^{\infty}$, where $Y_n \coloneqq X_{T_n}$.

c. Suppose the chain starts in state 1. What is the expected amount of time until it changes state for the first time?

d. From state 1, what is the expected amount of time until the chain is in state 4?

Solution:

a. Solving $\pi Q = 0$ with $\sum_i \pi(i) = 1$, we find that the stationary distribution is

$$\pi = \begin{bmatrix} \frac{3}{38} & \frac{7}{38} & \frac{9}{38} & \frac{1}{2} \end{bmatrix}.$$

b. Recall that μ is given by

$$\mu(i) = \frac{q(i)\pi(i)}{\sum_{j=1}^{4} q(j)\pi(j)}.$$

Using part a, we have

$$\mu = \begin{bmatrix} \frac{9}{85} & \frac{21}{85} & \frac{36}{85} & \frac{19}{85} \end{bmatrix}.$$

c. The *holding time*, the time the chain remains in state 1 before jumping, has an Exponential distribution with rate 3, so the expected amount of time it stays in state 1 is $\frac{1}{3}$.

d. We can compute the expected hitting times using first-step equations. Let $\beta(i)$ be the mean time needed to reach state 4 from i, so that $\beta(4) = 0$. For $i = 1, 2, 3$, we have

$$\beta(i) = \frac{1}{q(i)} + \sum_{j \neq i} p(i,j)\beta(j) = \frac{1}{q(i)} + \sum_{j \neq i} \frac{q(i,j)}{q(i)}\beta(j).$$

If we delete row and column 4 of the rate matrix Q and consider the submatrix corresponding to the remaining rows and columns, we get

$$Q' = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -3 & 2 \\ 1 & 2 & -4 \end{bmatrix}.$$

Let $\beta' = \begin{bmatrix} \beta(1) & \beta(2) & \beta(3) \end{bmatrix}^{\mathsf{T}}$. Then we can rewrite the first-step equations as

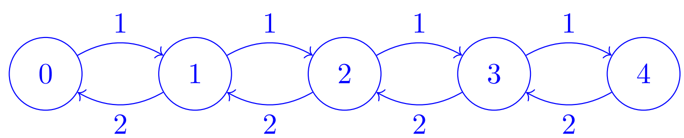
$$Q'\beta' = -\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}},$$

which yields the solution $\beta' = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$. Thus the expected time until the chain is in state 4, starting from state 1, is 1.

3. **Taxi Queue**

Empty taxis pass by a street corner according to a Poisson process of rate 2 per minute, and passengers arrive at the street corner according to a Poisson process of rate 1 per minute. Taxis always pick up a passenger if one is waiting; passengers wait for a taxi only if there are less than four people waiting, otherwise leaving and never returning. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

Solution: Consider the continuous-time Markov chain with states $\{0, 1, 2, 3, 4\}$ that describe the number of people waiting. For $i < 4$, the transitions from i to $i + 1$ have rate 1, and the transitions from $i + 1$ to i have rate 2:



The balance equations are then $\pi(i) = \frac{1}{2}\pi(i-1) = \frac{1}{2^i}\pi(0)$ for $i \geq 1$. By $\sum_{i=0}^{4} \pi(i) = 1$, we find that $\pi(0) = \frac{16}{31}$. Since the expected waiting time for a new taxi is 0.5, the expected waiting time of John given that he joins the queue is

$$\mathbb{E}(T) = \frac{\pi(0) \times 0.5 + \pi(1) \times 1 + \pi(2) \times 1.5 + \pi(3) \times 2}{\pi(0) + \pi(1) + \pi(2) + \pi(3)} = \frac{13}{15},$$

where the denominator reflects the fact that we are conditioning on the event that $n < 4$.