UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 05
Spring 2023

1. **Midterm**

   Solve again the midterm problems which you got incorrect. Please demonstrate understanding of the questions without simply copying the solutions.

   Solution: See midterm solutions.

2. **Bernoulli Convergence**

Consider an independent sequence of random variables $X_n \sim \text{Bernoulli}(\frac{1}{n})$.

a. Show that X_n converges to 0 in probability.

b. Argue that

$$\mathbb{P}\left(\left\{\lim_{n\to\infty} X_n = 0\right\}\right) = \mathbb{P}\left(\bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}\right).$$

c. Using part b, show that X_n does **not** converge almost surely to 0.
Hint: Consider applying the union bound and the independence of the X_n.

Solution:

a. We want to show that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - 0| > \varepsilon) = 0.$$

Because each X_n can only be 0 or 1, if $\varepsilon \geq 1$, then $\mathbb{P}(|X_n - 0| > \varepsilon) = 0$, so the limit is also zero. If $0 < \varepsilon < 1$, then

$$\mathbb{P}(|X_n - 0| > \varepsilon) = \mathbb{P}(X_n = 1) = \frac{1}{n} \to 0.$$

b. Since each X_n can only take on the values 0 or 1, the limit of X_n is 0 iff the sequence $X_1, X_2, \ldots$ is eventually always 0. In other words, $\{\lim_{n\to\infty} X_n = 0\}$ occurs if and only if there exists an N such that for all $n \geq N$, $X_n = 0$. Thus

$$\left\{\lim_{n\to\infty} X_n = 0\right\} = \bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}.$$

c. Applying the union bound to the equality in part b,

$$\begin{aligned}\mathbb{P}\left(\lim_{n\to\infty} X_n = 0\right) &\leq \sum_{N=1}^{\infty} \mathbb{P}(X_n = 0 \text{ for all } n \geq N) \\ &= \sum_{N=1}^{\infty} \mathbb{P}\left(\bigcap_{n=N}^{\infty} \{X_n = 0\}\right)\end{aligned}$$

Because the X_n are independent, this equals

$$\begin{aligned}&= \sum_{N=1}^{\infty} \prod_{n=N}^{\infty} \mathbb{P}(X_n = 0) \\ &= \sum_{N=1}^{\infty} \frac{N-1}{N} \cdot \frac{N}{N+1} \cdot \frac{N+1}{N+2} \cdots\end{aligned}$$

By telescoping, this infinite product is zero for any value of N, so we have

$$= \sum_{N=1}^{\infty} 0 = 0.$$

Since this probability is not 1, X_n does not converge almost surely to 0. In fact, since this probability is 0, X_n *almost surely does not converge* to 0. A related result is Kolmogorov's 0–1 law, which states that a sequence of independent random variables either converges or does not converge with probability 1.

3. **The CLT Implies the WLLN**

a. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. Show that if X_n converges in distribution to a constant c, then X_n converges in probability to c.

b. Now let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with mean μ and finite variance σ^2. Show that the CLT implies the WLLN: that is,

$$\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i-\mu)\xrightarrow{\mathsf{d}} Z\sim\mathcal{N}(0,1) \implies \frac{1}{n}\sum_{i=1}^{n}X_i\xrightarrow{\mathbb{P}}\mu,$$

where $\xrightarrow{\mathsf{d}}$ is short for "converges in distribution" and $\xrightarrow{\mathbb{P}}$ for "converges in probability."

Solution:

a. Since X_n converges in distribution to c, we know that for all $\varepsilon>0$,

$$\lim_{n\to\infty}F_{X_n}(c-\varepsilon)=F_c(c-\varepsilon)=0$$
$$\lim_{n\to\infty}F_{X_n}(c+\tfrac{\varepsilon}{2})=F_c(c+\tfrac{\varepsilon}{2})=1.$$

Using these limits, we have convergence in probability:

$$\begin{aligned}\lim_{n\to\infty}\mathbb{P}(|X_n-c|\geq\varepsilon)&=\lim_{n\to\infty}\mathbb{P}(X_n\leq c-\varepsilon)+\lim_{n\to\infty}\mathbb{P}(X_n\geq c+\varepsilon)\\&\leq\lim_{n\to\infty}\mathbb{P}(X_n\leq c-\varepsilon)+\lim_{n\to\infty}\mathbb{P}(X_n> c+\tfrac{\varepsilon}{2})\\&=\lim_{n\to\infty}F_{X_n}(c-\varepsilon)+\lim_{n\to\infty}1-F_{X_n}(c+\tfrac{\varepsilon}{2})\\&=0+1-1=0.\end{aligned}$$

(The reason we take $c+\frac{\varepsilon}{2}$ instead of $c+\varepsilon$ is because $1-F_{X_n}(x)=\mathbb{P}(X_n>x)$, but we have $\mathbb{P}(X_n\geq c+\varepsilon)$, which is not a strict inequality.)

b. From the CLT, we know that

$$Z_n:=\frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\right)\text{ converges to }Z\sim\mathcal{N}(0,1)\text{ in distribution.}$$

Additionally, $a_n:=\frac{\sigma}{\sqrt{n}}\to 0$. Then $Y_n:=a_nZ_n=\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\to 0$ in distribution. By part a, since $c=0$ is a constant, Y_n also converges to 0 in probability. In other words,

$$\frac{1}{n}\sum_{i=1}^{n}X_i\to\mu\text{ in probability,}$$

which is precisely the Weak Law of Large Numbers.

Note. The claim that "if $Z_n\to Z$ in distribution and $a_n\to 0$ as constants, then $a_nZ_n\to 0$ in distribution" requires proof, which we present below.

For $x<0$ and any $N\geq 1$, we know that $\frac{x}{a_n}\leq -N$ eventually, so

$$\lim_{n\to\infty}\mathbb{P}(a_nZ_n\leq x)=\lim_{n\to\infty}\mathbb{P}(Z_n\leq\tfrac{x}{a_n})\leq\lim_{n\to\infty}\mathbb{P}(Z_n\leq -N)=\mathbb{P}(Z\leq -N).$$

The left-hand side does not depend on N, so taking the limit as $N \to \infty$ of both sides, by continuity from above, we find that

$$\lim_{n\to\infty} \mathbb{P}(a_n Z_n \leq x) \leq \mathbb{P}(Z = -\infty) = 0.$$

Similarly, for $x > 0$, we know that $\frac{x}{a_n} \geq N$ eventually for any N, so

$$\lim_{n\to\infty} \mathbb{P}(a_n Z_n > x) \leq \lim_{n\to\infty} \mathbb{P}(Z_n > N) = \mathbb{P}(Z > N),$$

and taking the limit as $N \to \infty$, we find that $\lim_{n\to\infty} \mathbb{P}(a_n Z_n > x) \leq 0$, or $\lim_{n\to\infty} \mathbb{P}(a_n Z_n \leq x) = 1$. In other words, we have shown that

$$\lim_{n\to\infty} \mathbb{P}(a_n Z_n \leq x) = \mathbb{1}\{0 \leq x\},$$

i.e. $a_n Z_n$ converges to 0 in distribution. This is a specific case of a more general result called *Slutsky's theorem.*

4. **CLT Cannot Be Upgraded**

   a. Show that if X_n converges to X in probability and Y_n to Y in probability, then $aX_n + Y_n$ converges to $aX + Y$ in probability.

   b. Show that the CLT cannot be upgraded to convergence in probability or almost surely. That is, if $X_1, X_2, \ldots$ are i.i.d. with mean 0 and variance 1, prove that it cannot be the case that

   $$Z_n \coloneqq \frac{X_1 + \cdots + X_n}{\sqrt{n}} \to Z, \quad \text{where } Z \sim \mathcal{N}(0,1), \text{ almost surely or in probability.}$$

   Hint: From part a, the sequence of random variables $\sqrt{2}Z_{2n} - Z_n$ converges in probability to $(\sqrt{2}-1)Z$. Does this contradict the fact that Z_n converges to Z in probability?

Solution:

a. Let $\varepsilon > 0$. By the union bound,

$$\begin{aligned}\mathbb{P}(|(aX + Y) - (aX_n + Y_n)| > \varepsilon) &\le \mathbb{P}(|a(X - X_n)| > \varepsilon/2 \text{ or } |Y - Y_n| > \varepsilon/2) \\ &\le \mathbb{P}(|X - X_n| > \varepsilon/(2|a|)) + \mathbb{P}(|Y - Y_n| > \varepsilon/2),\end{aligned}$$

which we know converges to 0. Hence $aX_n + Y_n$ converges to $aX + Y$ in probability.

b. We observe that

$$\sqrt{2}Z_{2n} - Z_n = \frac{X_{n+1} + X_{n+2} + \cdots + X_{2n}}{\sqrt{n}},$$

is equal in distribution to Z_n, and hence must converge in distribution to Z. However, convergence in probability implies convergence in distribution, so $\sqrt{2}Z_{2n} - Z_n$ must also converge to $(\sqrt{2}-1)Z$ in distribution. As $Z \neq (\sqrt{2}-1)Z$, this is a contradiction.

Lastly, as Z_n cannot converge in probability, it cannot converge almost surely either, since almost sure convergence is a stronger form of convergence.

5. **Finite Exit Time**

Consider the random walk $S_n = \sum_{i=1}^n X_i$, where the X_i are i.i.d. with mean zero and variance 1. (Note that the X_i do not have to be discrete.) Show that almost surely the random walk will leave the interval $[-a, a]$ in finite time.

Hint: Let T be the first time that the random walk leaves the interval $[-a, a]$, and show that $\lim_{n\to\infty} \mathbb{P}(T > n) = 0$.

Solution: We note that $T > n$ only if $|S_n| \le a$, so

$$\mathbb{P}(T > n) \le \mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}}\right| \le \frac{a}{\sqrt{n}}\right).$$

Fix a particular value of N. By the Central Limit Theorem, $S_n/\sqrt{n}$ converges to $\mathcal{N}(0, 1)$ in distribution, and hence for all $n \ge N$,

$$\mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}}\right| \le \frac{a}{\sqrt{n}}\right) \le \mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}}\right| \le \frac{a}{\sqrt{N}}\right) \to \mathbb{P}\left(|\mathcal{N}(0,1)| \le \frac{a}{\sqrt{N}}\right).$$

In other words, for any value of N,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}}\right| \le \frac{a}{\sqrt{n}}\right) \le \mathbb{P}\left(|\mathcal{N}(0,1)| \le \frac{a}{\sqrt{N}}\right).$$

By continuity from above, the right-hand side converges to $\mathbb{P}(|\mathcal{N}(0, 1)| \le 0) = 0$ as $N \to \infty$. Since $\mathbb{P}(T = \infty) = \lim_{n\to\infty} \mathbb{P}(T > n) = 0$, we conclude that $T < \infty$ almost surely.

Remark: We could not directly conclude that $\mathbb{P}(|S_n/\sqrt{n}| \le a/\sqrt{n}) \to \mathbb{P}(|\mathcal{N}(0, 1)| \le 0)$ because $a/\sqrt{n}$ is varying with n. Hence, we showed that $\lim_{m\to\infty} \lim_{n\to\infty} p_{n,m} \to 0$ for the 2d "grid" of values $p_{n,m} = \mathbb{P}(|S_n/\sqrt{n}| \le a/\sqrt{m})$, which implies that the "diagonal" sequence $p_{n,n} \to 0$, as desired. This is a common style of argument for working with sequences like $\mathbb{P}(X_n \le x_n)$.

6. **Coupon Collector Convergence**

In the coupon collector's problem, there are n different types of coupons, and you are trying to collect them all. Each time you purchase an item, you receive one of the n coupons uniformly at random. Let T_n denote the number of purchases it takes to collect all n coupons. Prove that $T_n/(n \ln n) \to 1$ in probability as $n \to \infty$.

Hint: Consider using Chebyshev's inequality to show that for every $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{T_n - nH_n}{n \ln n}\right| \geq \varepsilon\right) \to 0.$$

H_n denotes the nth harmonic sum $\sum_{i=1}^n \frac{1}{i}$. You may use the fact that $H_n \sim \ln n$.

Solution: We can write T_n as an independent sum $\sum_{i=1}^n X_i$, where $X_i \sim \text{Geometric}(\frac{n-i+1}{n})$ is the number of purchases required to collect the ith new coupon. Noting that the variance of a Geometric(p) random variable is $\frac{1-p}{p^2} \leq \frac{1}{p^2}$, we have

$$\mathbb{E}(T_n) = \sum_{i=1}^n \frac{n}{n-i+1} = n \sum_{i=1}^n \frac{1}{i} = nH_n$$

$$\text{var}(T_n) = \sum_{i=1}^n \text{var}(X_i) \leq \sum_{i=1}^n \left(\frac{n}{n-i+1}\right)^2 = \sum_{i=1}^n \left(\frac{n}{i}\right)^2 \leq n^2 \sum_{i=1}^\infty \frac{1}{i^2}.$$

Here, it suffices to note that the summation converges. Notably,

$$\text{var}\left(\frac{T_n - nH_n}{n \ln n}\right) \leq \frac{\cancel{n^2}}{\cancel{n^2}(\ln n)^2} \sum_{i=1}^\infty \frac{1}{i^2} \to 0$$

as $n \to \infty$, and now Chebyshev's inequality gives

$$\mathbb{P}\left(\left|\frac{T_n - nH_n}{n \ln n}\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \text{var}\left(\frac{T_n - nH_n}{n \ln n}\right) \to 0$$

as $n \to \infty$ for every $\varepsilon > 0$. Hence, $(T_n - nH_n)/(n \ln n) \to 0$ in probability as $n \to \infty$. To conclude, we note that $H_n \sim \ln n$ asymptotically, so $T_n/(n \ln n) \to 1$ in probability as $n \to \infty$.

Remark: From previous analysis, we know that $\mathbb{E}(T_n)$ is close to $n \ln n$, so we have shown a result similar in spirit to a "weak law for the coupon collector problem": as $n \to \infty$, T_n is "close" to its expected value. However, since we are not dealing with i.i.d. random variables, we cannot use the version of the WLLN proved in lecture to deal with this problem.