UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 09
Spring 2023

1. **Product of Rolls of a Die**

   A fair die with labels 1 through 6 is rolled until the product of the last two rolls is 12. What is the expected number of rolls?

   Hint: You can model this process as a Markov chain with 3 states, choosing your states according to the outcome of last roll. For example, assign one state if its outcome was 1 or 5, which is useless if you want the product to be 12. If the outcome was 2, 3, 4 or 6, it's useful and can be assigned to another state. Assign a third state to the case when the product of the last two outcomes was 12.

   Solution: Taking the hint, we model this process as a Markov chain with 3 states, where the states correspond to the outcome of the last roll. Let s_1 be the state where the last outcome is 1 or 5; s_2 where the last outcome is 2, 3, 4, or 6; and s_3 where the product of the last two rolls is 12, so the transition probability matrix is

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 \end{bmatrix}.$$

   Let T_i be the expected number of rolls needed to get to state s_3 starting from state s_i for $i = 1, 2$. Then we have the first-step equations

$$T_1 = 1 + \frac{1}{3}T_1 + \frac{2}{3}T_2$$
$$T_2 = 1 + \frac{1}{3}T_1 + \frac{1}{2}T_2.$$

   Solving the equations, we get $T_1 = 10.5$ and $T_2 = 9$, so the expected number of rolls is

$$T = 1 + \frac{1}{3}T_1 + \frac{2}{3}T_2 = 10.5.$$

2. **Poisson Process Warmup**

Give an interpretation of the following fact in terms of a Poisson process with rate λ. If N is Geometric with parameter p and $(X_k)_{k\in\mathbb{N}}$ are i.i.d. Exponential(λ), then $X_1 + \cdots + X_N$ has an Exponential distribution with parameter λp.

Solution: Consider a Poisson process with rate λ, and split the process by keeping each arrival independently with probability p. In the original process, the interarrival times are i.i.d. Exponential(λ), and $X_1 + \cdots + X_N$ represents the amount of time until the first arrival we keep. By Poisson splitting, we know that the split process is a Poisson process with rate λp, so the time until its first arrival is an Exponential random variable with parameter λp.

3. **Customers in a Store**

Consider two independent Poisson processes with rates λ_1 and λ_2, which measure the number of customers arriving in store 1 and 2.

a. What is the probability that a customer arrives in store 1 before any arrives in store 2?

b. What is the probability that in the first hour, a total of exactly 6 customers arrive in the two stores?

c. Given that exactly 6 have arrived in total at the two stores, what is the probability that exactly 4 went to store 1?

Solution:

a. Consider the sum of the two processes, itself a Poisson process with rate $\lambda_1 + \lambda_2$ by Poisson merging. Each customer in this process is marked as 1 with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and 2 otherwise. The probability of the first customer going to store 1 is then the probability of marking the first customer as 1, which is

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Alternatively, the arrival times of the first customer of the two stores are respectively $X \sim \text{Exponential}(\lambda_1)$ and $Y \sim \text{Exponential}(\lambda_2)$. By the law of total probability,

$$\begin{aligned}
\mathbb{P}(X < Y) &= \int_0^\infty f_Y(y) \cdot \mathbb{P}(X < Y \mid Y = y)\, \mathrm{d}y \\
&= \int_0^\infty \lambda_2 e^{-\lambda_2 y} \cdot (1 - e^{-\lambda_1 y})\, \mathrm{d}y \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

b. Considering the merged process, this probability is

$$\mathbb{P}(\text{Poisson}((\lambda_1 + \lambda_2) \cdot 1) = 6) = \frac{(\lambda_1 + \lambda_2)^6}{6!} e^{-(\lambda_1+\lambda_2)}.$$

c. Conditioned on the total number of arrivals, the number of arrivals in each split process has Binomial distribution, so this probability is

$$\binom{6}{4} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^4 \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^2.$$