UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 05

Spring 2023

1. **Midterm**

   Solve again the midterm problems which you got incorrect. Please demonstrate understanding of the questions without simply copying the solutions.

2. **Bernoulli Convergence**

   Consider an independent sequence of random variables $X_n \sim \text{Bernoulli}(\frac{1}{n})$.

   a. Show that X_n converges to 0 in probability.

   b. Argue that
   $$\mathbb{P}\left(\left\{\lim_{n\to\infty} X_n = 0\right\}\right) = \mathbb{P}\left(\bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}\right).$$

   c. Using part b, show that X_n does **not** converge almost surely to 0.
   Hint: Consider applying the union bound and the independence of the X_n.

3. **The CLT Implies the WLLN**

a. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. Show that if X_n converges in distribution to a constant c, then X_n converges in probability to c.

b. Now let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with mean μ and finite variance σ^2. Show that the CLT implies the WLLN: that is,

$$\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n}(X_i-\mu) \xrightarrow{\mathsf{d}} Z \sim \mathcal{N}(0,1) \implies \frac{1}{n}\sum_{i=1}^{n}X_i \xrightarrow{\mathbb{P}} \mu,$$

where $\xrightarrow{\mathsf{d}}$ is short for "converges in distribution" and $\xrightarrow{\mathbb{P}}$ for "converges in probability."

4. **CLT Cannot Be Upgraded**

   a. Show that if X_n converges to X in probability and Y_n to Y in probability, then $aX_n + Y_n$ converges to $aX + Y$ in probability.

   b. Show that the CLT cannot be upgraded to convergence in probability or almost surely. That is, if $X_1, X_2 \dots$ are i.i.d. with mean 0 and variance 1, prove that it cannot be the case that

$$Z_n \coloneqq \frac{X_1 + \cdots + X_n}{\sqrt{n}} \to Z, \quad \text{where } Z \sim \mathcal{N}(0,1), \text{ almost surely or in probability.}$$

   Hint: From part a, the sequence of random variables $\sqrt{2}Z_{2n} - Z_n$ converges in probability to $(\sqrt{2}-1)Z$. Does this contradict the fact that Z_n converges to Z in probability?

5. **Finite Exit Time**

   Consider the random walk $S_n = \sum_{i=1}^n X_i$, where the X_i are i.i.d. with mean zero and variance 1. (Note that the X_i do not have to be discrete.) Show that almost surely the random walk will leave the interval $[-a, a]$ in finite time.

   Hint: Let T be the first time that the random walk leaves the interval $[-a, a]$, and show that $\lim_{n\to\infty} \mathbb{P}(T > n) = 0$.

6. **Coupon Collector Convergence**

   In the coupon collector's problem, there are n different types of coupons, and you are trying to collect them all. Each time you purchase an item, you receive one of the n coupons uniformly at random. Let T_n denote the number of purchases it takes to collect all n coupons. Prove that $T_n/(n \ln n) \to 1$ in probability as $n \to \infty$.

   Hint: Consider using Chebyshev's inequality to show that for every $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{T_n - nH_n}{n \ln n}\right| \geq \varepsilon\right) \to 0.$$

   H_n denotes the nth harmonic sum $\sum_{i=1}^{n} \frac{1}{i}$. You may use the fact that $H_n \sim \ln n$.