UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 03
Spring 2023

1. **Uncorrelatedness and Independence**

   a. Show that if $X_1, \ldots, X_n$ are pairwise uncorrelated, then

   $$\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{var}(X_i).$$

   b. Find an example where a pair of random variables are uncorrelated but not independent.

Solution:

a. By linearity of expectation, pairwise uncorrelatedness of $X_1, \ldots, X_n$ implies uncorrelatedness of $X_1 + \cdots + X_k$ and X_{k+1} for $k = 1, 2, \ldots, n-1$ (you should verify this yourself). Then, since $\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y)$ for uncorrelated X and Y, we have

$$\begin{aligned} &\operatorname{var}(X_1 + \cdots + X_n) \\ &= \operatorname{var}(X_1 + \cdots + X_{n-1}) + \operatorname{var}(X_n) \\ &= \operatorname{var}(X_1 + \cdots + X_{n-2}) + \operatorname{var}(X_{n-1}) + \operatorname{var}(X_n) \\ &\quad \vdots \\ &= \operatorname{var}(X_1) + \cdots + \operatorname{var}(X_n). \end{aligned}$$

b. Consider $X \sim \text{Uniform}\{-1, 0, 1\}$, $Z \sim \text{Uniform}\{-1, 1\}$, independent of each other (Z is called a *Rademacher* random variable). Let $Y = XZ$. Then,

$$\begin{aligned} \operatorname{cov}(X, Y) &= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] \\ &= \mathbb{E}[X^2 Z] - 0 \cdot 0 \\ &= \mathbb{E}[X^2]\,\mathbb{E}[Z] \\ &= \tfrac{2}{3} \cdot 0 \\ &= 0. \end{aligned}$$

However, X and Y are not independent since

$$\mathbb{P}(X = 0, Y = 0) = \mathbb{P}(X = 0) = \tfrac{1}{3},$$

$$\mathbb{P}(X = 0)\,\mathbb{P}(Y = 0) = \tfrac{1}{3} \cdot \tfrac{1}{3} \neq \mathbb{P}(X = 0, Y = 0).$$

2. **Sampling Without Replacement**

   Suppose you have N items, G of which are good and B of which are bad. $B + G = N$ are all positive integers. You start to draw items without replacement, and suppose that the first good item appears on draw X. Compute the mean and variance of X.

   Solution: The expectation is computed with a clever trick: let X_i be the indicator that the ith bad item appears before the first good item, $i = 1, \ldots, B$. Then $X = 1 + \sum_{i=1}^{B} X_i$, and by the linearity of expectation,

$$\mathbb{E}(X) = 1 + B\,\mathbb{E}(X_1) = 1 + \frac{B}{G+1} = \frac{N+1}{G+1}.$$

   Now we observe that $\operatorname{var} X = \operatorname{var}(X-1)$, which we can find using the same indicators:

$$\begin{aligned}
\operatorname{var}(X-1) &= \mathbb{E}((X-1)^2) - \mathbb{E}(X-1)^2 \\
&= \mathbb{E}\left(\sum_{i=1}^{B} X_i^2 + \sum_{i \neq j} X_i X_j\right) - \mathbb{E}(X-1)^2 \\
&= B\,\mathbb{E}(X_1^2) + B(B-1)\,\mathbb{E}(X_1X_2) - (B\,\mathbb{E}(X_1))^2 \\
&= \frac{B}{G+1} + \frac{2B(B-1)}{(G+1)(G+2)} - \left(\frac{B}{G+1}\right)^2.
\end{aligned}$$

   Optionally, with a little algebra, we can simplify the result:

$$\begin{aligned}
\operatorname{var}(X) &= \frac{B(G+1)(G+2) + 2B(B-1)(G+1) - B^2(G+2)}{(G+1)^2(G+2)} \\
&= \frac{BG(N+1)}{(G+1)^2(G+2)}.
\end{aligned}$$

3. **Galton–Watson Branching Process**

Consider a population of N individuals for some positive integer N. Let ξ be a random variable taking values in $\mathbb{N}$ with $\mathbb{E}(\xi) = \mu$ and $\operatorname{var}(\xi) = \sigma^2$. At the end of each year, each individual, independently of all other individuals and generations, leaves behind a number of offspring which has the same distribution as ξ. For each $n \in \mathbb{N}$, let X_n denote the size of the population at the end of the nth year.

a. Compute $\mathbb{E}(X_n)$.

b. Compute $\operatorname{var}(X_n|X_{n-1})$. Then, write $\operatorname{var}(X_n)$ in terms of $\operatorname{var}(X_{n-1})$.

Solution:

a. We first note that $X_0 = N$, so $\mathbb{E}(X_0) = N$ and $\operatorname{var}(X_0) = 0$. Then, conditioned on the number of people in the previous year X_{n-1}, we have

$$\begin{aligned}\mathbb{E}(X_n) = \mathbb{E}(\mathbb{E}(X_n \mid X_{n-1})) &= \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{X_{n-1}} \xi_i \mid X_{n-1}\right)\right) \\ &= \mathbb{E}\left(X_{n-1}\,\mathbb{E}(\xi)\right) \\ &= \mu\,\mathbb{E}(X_{n-1}).\end{aligned}$$

By recursion, we find that $\mathbb{E}(X_n) = \mu^n N$.

b. As we computed above, $\mathbb{E}(X_n \mid X_{n-1}) = \mu X_{n-1}$. The conditional variance is $\operatorname{var}(X_n \mid X_{n-1}) = \sigma^2 X_{n-1}$. Then, we have

$$\operatorname{var} X_n = \mathbb{E}[\sigma^2 X_{n-1}] + \operatorname{var}(\mu X_{n-1}) = \sigma^2 \mu^{n-1} N + \mu^2 \operatorname{var} X_{n-1}.$$

First, suppose that $\mu = 1$. Then, the recurrence simplifies to $\operatorname{var} X_n = \sigma^2 N + \operatorname{var} X_{n-1}$, which means that the variance increases linearly:

$$\operatorname{var}(X_n) = \sigma^2 N n.$$

For $\mu \neq 1$, the solution to the recurrence is obtained by finding a pattern after a few iterations:

$$\begin{aligned}\operatorname{var}(X_n) &= \sigma^2 \mu^{n-1} N + \mu^2 \operatorname{var} X_{n-1} = \sigma^2 \mu^{n-1} N + \sigma^2 \mu^n N + \mu^4 \operatorname{var} X_{n-2} \\ &= \cdots = \sigma^2 \mu^{n-1} N \sum_{k=0}^{n-1} \mu^k = \sigma^2 \mu^{n-1} N \frac{1-\mu^n}{1-\mu}\end{aligned}$$

We have used the formula for a finite geometric series.