UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 12
Spring 2023

1. **Exponential MLE, MAP, and Hypothesis Testing**

   Let X be Exponentially distributed with rate 1. Given X, the random variable Y is Exponentially distributed with parameter X.

   a. Find $\text{MLE}(X \mid Y)$.

   b. Find $\text{MAP}(X \mid Y)$.

   c. Let $c > 1$. Suppose that
      - The null hypothesis is $X = 1$: $Y \sim \text{Exponential}(1)$, and
      - The alternative hypothesis is $X = c$: $Y \sim \text{Exponential}(c)$.

      Find the decision rule $\hat{X}$ (a function of Y) that maximizes $\mathbb{P}(\hat{X} = 1 \mid X = 1)$ subject to $\mathbb{P}(\hat{X} = 1 \mid X = c) \leq 5\%$.

Solution:

a. The derivative of the log-likelihood function of Y given X is

$$\frac{\partial}{\partial x} \ln f_{Y|X}(y \mid x) = \frac{\partial}{\partial x} \ln(xe^{-xy}) = \frac{\partial}{\partial x}(\ln x - xy) = \frac{1}{x} - y,$$

which equals zero when $x = 1/y$. Thus $\text{MLE}(X \mid Y) = 1/Y$.

b. The posterior distribution of X is

$$f_{X|Y}(x \mid y) \propto f_{Y|X}(y \mid x) \cdot f_X(x) = xe^{-x(y+1)},$$

so we can maximize $\ln x - x(y+1)$ over x. Its derivative $1/x - 1 - y$ equals zero when $1/x = 1 + y$, and thus $\text{MAP}(X \mid Y) = 1/(1 + Y)$.

c. The likelihood ratio is

$$L(y) = \frac{f_{Y|X}(y \mid c)}{f_{Y|X}(y \mid 1)} = \frac{ce^{-cy}}{e^{-y}} = ce^{-(c-1)y},$$

which is decreasing in y. Then the decision rule will be of the form $\hat{X} = \mathbb{1}_{Y>t}$, where the threshold is determined by

$$\mathbb{P}(\hat{X} = 1 \mid X = c) = \mathbb{P}(Y > t \mid X = c) = e^{-ct} = 0.05$$

to be $t = (\ln 20)/c$.

2. **Hypothesis Testing for Bernoulli Random Variables**

Suppose that

- The null hypothesis is $X = 0$: $Y \sim \text{Bernoulli}(\frac{1}{4})$, and
- The alternative hypothesis is $X = 1$: $Y \sim \text{Bernoulli}(\frac{3}{4})$.

Using the Neyman–Pearson formulation of hypothesis testing, find the optimal randomized decision rule $\hat{X}$ with respect to the criterion

$$\begin{aligned} \min \quad & \mathbb{P}(\hat{X} = 0 \mid X = 1) \\ \text{s.t.} \quad & \mathbb{P}(\hat{X} = 1 \mid X = 0) \le \beta, \end{aligned}$$

where $\beta \in [0, 1]$ is a given upper bound on the probability of false alarm (PFA).

(Note that the Neyman–Pearson decision rule may change depending on the value of β. In particular, consider the two separate cases of $\beta \le \frac{1}{4}$ and $\beta > \frac{1}{4}$.)

Solution: The likelihood ratio is the discrete function

$$L(y) = \frac{f_{Y|X}(y \mid 1)}{f_{Y|X}(y \mid 0)} = \begin{cases} 3 & \text{if } y = 1 \\ \frac{1}{3} & \text{if } y = 0. \end{cases}$$

By Neyman–Pearson, the optimal decision rule with randomization is given by

- If $\mathbb{P}(Y = 1 \mid X = 0) = \frac{1}{4} \ge \beta$, then

$$\hat{X} = \begin{cases} 0 & \text{if } Y = 0 \\ \text{Bernoulli}(\gamma) & \text{with } \gamma = \beta / \frac{1}{4} \text{ if } Y = 1. \end{cases}$$

- Otherwise, the threshold is $Y = 0$, and

$$\hat{X} = \begin{cases} \text{Bernoulli}(\gamma) & \text{with } \gamma = \frac{4}{3}\beta - \frac{1}{3} \text{ if } Y = 0 \\ 1 & \text{if } Y = 1. \end{cases}$$

  The value of γ above is chosen to make

$$\mathsf{PFA} = \mathbb{P}(Y = 1 \mid X = 0) + \gamma \cdot \mathbb{P}(Y = 0 \mid X = 0) = \frac{1}{4} + \frac{3}{4}\gamma = \beta.$$

3. **Hypothesis Testing for Uniform Random Variables**

Suppose that

- The null hypothesis is $X = 0$: $Y \sim \text{Uniform}([-1, 1])$, and
- The alternative hypothesis is $X = 1$: $Y \sim \text{Uniform}([0, 2])$.

Using the Neyman–Pearson formulation of hypothesis testing, find the optimal randomized decision rule $\hat{X}$ with respect to the criterion

$$\begin{aligned} \min \ & \mathbb{P}(\hat{X} = 0 \mid X = 1) \\ \text{s.t. } & \mathbb{P}(\hat{X} = 1 \mid X = 0) \leq \beta, \end{aligned}$$

where $\beta \in [0, \frac{1}{2}]$ is a given upper bound on the probability of false alarm (PFA).

Solution: The likelihood ratio is the discrete function

$$L(y) = \frac{f_{Y|X}(y \mid 1)}{f_{Y|X}(y \mid 0)} = \frac{\mathbb{1}_{\{0 \leq y \leq 2\}}}{\mathbb{1}_{\{-1 \leq y \leq 1\}}} = \begin{cases} 0 & \text{if } y < 0 \\ 1 & \text{if } y \in [0, 1] \\ \infty & \text{if } y > 1. \end{cases}$$

Thus, $\hat{X} = 0$ if $Y < 0$ and $\hat{X} = 1$ if $Y > 1$. Otherwise, when $Y \in [0, 1]$, we need to introduce randomization to ensure that

$$\mathsf{PFA} = \mathbb{P}(\hat{X} = 1 \mid X = 0) = \gamma \cdot \mathbb{P}(Y \in [0, 1] \mid X = 0) = \beta,$$

so we set $\hat{X} = 1$ with probability $\gamma = 2\beta$.