UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 11
Spring 2023

1. **Generating Erdős–Rényi Random Graphs**

   Let G_1 and G_2 be independent Erdős–Rényi random graphs on n vertices with probabilities p_1 and p_2 respectively. Let G be $G_1 \cup G_2$, that is, the graph generated by combining the edges in G_1 and G_2.

   a. Is G an Erdős–Rényi random graph on n vertices with probability $p_1 + p_2$?
   b. Is G an Erdős–Rényi random graph?

   Solution:

   a. No; an edge appears in G if it appears in G_1 or in G_2, which occurs with probability $p_1 + p_2 - p_1 p_2$ by inclusion-exclusion.
   b. Yes; each edge appears with probability $p = p_1 + p_2 - p_1 p_2$ independently.

2. **Degree of Vertex in Random Graph**

Consider a random undirected graph on n vertices in which each of the $\binom{n}{2}$ possible edges is present with probability p, independent of all other edges.

   a. Fix a particular vertex of the graph, and let D be the random variable equal to its degree. Find the pmf and expected value of D.

   b. Suppose that p depends on n in a way such that np tends to a fixed constant λ as $n \to \infty$. For large n, how could we model the pmf of D without using n?

Solution:

   a. The degree of any vertex is the number of present edges among the $n-1$ possible adjacent edges, so $D \sim \text{Binomial}(n-1, p)$ and $\mathbb{E}(D) = (n-1)p$.

   b. By the Poisson limit theorem or the law of rare events, for large n, the distribution of D is approximately $\text{Poisson}(\lambda)$.

3. **Maximum Likelihood Estimation for the Binary Symmetric Channel**

You are testing a digital link that corresponds to a BSC with some error probability $\varepsilon \in [0, 0.5]$.

a. Suppose that you observe an input bit X and an output bit Y. Calculate the MLE of ε, which is the value of $\varepsilon \in [0, 0.5]$ maximizing the probability you observe the given data. Here, since the observation is (X, Y), we want to find

$$\varepsilon_{\text{MLE}} = \underset{\varepsilon \in [0,0.5]}{\operatorname{argmax}} \mathbb{P}_\varepsilon(X = x, Y = y).$$

b. You are now told that the inputs $X_1, \dots, X_n$ are i.i.d. Bernoulli(0.6) bits. Suppose that you observe n outputs $Y_1, \dots, Y_n$. Calculate the MLE of ϵ.

Solution:

a. Let $\mathbb{P}_\varepsilon$ denote the probability distribution of X and Y when the error probability of the BSC is ε. For $(x, y) \in \{0, 1\}^2$,

$$\varepsilon_{\text{MLE}} = \underset{\varepsilon \in [0,0.5]}{\operatorname{argmax}} \mathbb{P}_\varepsilon(X = x, Y = y) = \underset{\varepsilon \in [0,0.5]}{\operatorname{argmax}} \left(\varepsilon^{\mathbb{1}\{x \neq y\}}(1 - \varepsilon)^{\mathbb{1}\{x = y\}}\right).$$

If $x \neq y$, then $\varepsilon_{\text{MLE}} = 0.5$ maximizes the likelihood; otherwise, $\varepsilon_{\text{MLE}} = 0$.

b. Note that every use of the channel is independent. Given n outputs, ε_{MLE} maximizes

$$\begin{aligned}
\mathbb{P}_\varepsilon(Y_1 = y_1, \dots, Y_n = y_n) &= \prod_{i=1}^n \mathbb{P}_\varepsilon(Y_i = y_i) \\
&= \prod_{i=1}^n \left[(0.6(1 - \varepsilon) + 0.4\varepsilon)\, \mathbb{1}_{y_i = 1} + (0.4(1 - \varepsilon) + 0.6\varepsilon)\, \mathbb{1}_{y_i = 0}\right] \\
&= \prod_{i=1}^n (0.6 - 0.2\varepsilon)^{y_i}(0.4 + 0.2\varepsilon)^{1 - y_i} \\
&= (0.6 - 0.2\varepsilon)^{\sum_{i=1}^n y_i}(0.4 + 0.2\varepsilon)^{n - \sum_{i=1}^n y_i}.
\end{aligned}$$

We find that the likelihood only depends on $t = \sum_{i=1}^n y_i$. We can then find the maximizer as $\varepsilon_{\text{MLE}} = 3 - \frac{5t}{n}$ from differentiating the log-likelihood:

$$\frac{-0.2t}{0.6 - 0.2\varepsilon} + \frac{0.2(n - t)}{0.4 + 0.2\varepsilon} = \frac{t(\varepsilon + 2) + (n - t)(\varepsilon - 3)}{(\varepsilon - 3)(\varepsilon + 2)} = 0.$$

If ε_{MLE} falls outside of the interval $[0, 0.5]$, then we clip it to 0 or 0.5, whichever endpoint is closer to the maximizer.