UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Discussion 06
Spring 2023

1. **Gambling Game**

   Let's play a game. You stake a positive initial amount of money w_0. Then you toss a fair coin. If it is heads, you earn an amount equal to three times your stake, so you quadruple your wealth. If it comes up tails, you lose everything. There is one requirement, though: you are not allowed to quit, and you have to keep playing by staking all your available wealth, over and over again. Let W_n be a random variable equal to your wealth after n plays.

   a. Find $\mathbb{E}(W_n)$ and show that $\lim_{n\to\infty} \mathbb{E}(W_n) = \infty$.
   b. Since $\lim_{n\to\infty} \mathbb{E}(W_n) = \infty$, this game sounds like a good deal. But wait a moment! To where does the sequence of random variables $\{W_n\}_{n\geq 0}$ converge almost surely?

Solution:

a. By the law of total expectation, or considering $\mathbb{E}(W_n \mid W_{n-1} = w_{n-1})$,

$$\mathbb{E}(W_n) = \mathbb{E}(\mathbb{E}(W_n \mid W_{n-1})) = \mathbb{E}\left(\frac{1}{2} \cdot 4W_{n-1} + \frac{1}{2} \cdot 0\right) = 2\,\mathbb{E}(W_{n-1}).$$

Unfolding the recursion, with the base case being given by $\mathbb{E}(W_0) = w_0$, yields $\mathbb{E}(W_n) = 2^n w_0$. Thus $\lim_{n\to\infty} \mathbb{E}(W_n) = \infty$.

b. The probability that all of the first n coin flips will come up heads is 2^{-n}, so the probability that the coin flips will keep on coming up heads forever is zero. This means that, with probability one, at some point you will lose your entire wealth:

$$\mathbb{P}\left(\lim_{n\to\infty} W_n = 0\right) = 1.$$

2. **More Almost Sure Convergence**

a. Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

b. Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

$$\mathbb{P}\big(X_n = (y + n^{-1})^{-1} \mid Y = y\big) = 1.$$

Does $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

c. Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \ldots, 2^{k+1} - 1\}$, and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

d. Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}(X_n) \to \mathbb{E}(X)$ as $n \to \infty$?

Solution:

a. Yes. If a sequence oscillates between two distinct values infinitely often, then it does not converge. Here, we have a sequence that oscillates infinitely often, with probability 1, which means that the sequence in fact **diverges** with probability 1.

The above may have been very cumbersome to read, which is why we often abbreviate "with probability 1" with "a.s." Then the above reads "$(X_n)_{n\in\mathbb{N}}$ oscillates between two values infinitely often a.s., so $(X_n)_{n\in\mathbb{N}}$ does not converge a.s."

b. Yes. Observe that when $Y = y \neq 0$, $(X_n)_{n\in\mathbb{N}}$ will converge to y^{-1}, but when $Y = 0$, $(X_n)_{n\in\mathbb{N}}$ does not converge. However, $\mathbb{P}(Y = 0) = 0$, since Y is a continuous random variable. In other words,

$$\begin{aligned}
\mathbb{P}(X_n \text{ does not converge as } n \to \infty) &= \mathbb{P}(Y = 0) = 0 \\
\mathbb{P}(X_n \text{ converges as } n \to \infty) &= \mathbb{P}(Y \neq 0) = 1,
\end{aligned}$$

so $(X_n)_{n\in\mathbb{N}}$ converges a.s.

c. No. The sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between 0 and successively higher powers of two infinitely often a.s., so it does not converge a.s.

d. Yes. Fix $\varepsilon > 0$. For $n \in \mathbb{Z}^+$, one has

$$\mathbb{P}(|X_n| > \varepsilon) = \frac{1}{2^k},$$

where $k = \lfloor \log_2 n \rfloor$. As $n \to \infty$, the above probability goes to 0, so $X_n \to 0$ in probability. Intuitively, $(X_n)_{n\in\mathbb{N}}$ has infinitely many oscillations, so it cannot converge a.s. However, the probability of each oscillation shrinks to 0, so $(X_n)_{n\in\mathbb{N}}$ converges in probability.

The expectations do not converge. For all n, one has $\mathbb{E}(X_n) = 1$, so it is not the case that $\mathbb{E}(X_n) \to 0$ as $n \to \infty$. Hence, convergence in probability is not sufficient to imply that the expectations converge. In fact, almost sure convergence is not sufficient either.

3. **Sum of Rolls**

You roll a fair 6-sided die 100 times, and you call the sum of the values of all your rolls X. Use the Central Limit Theorem to approximate the probability that $X > 400$. You may use a calculator and Gaussian lookup table.

Solution: The value of an individual roll, distributed as Uniform([6]), has mean 3.5 and variance $\frac{6^2-1}{12} = \frac{35}{12}$. Call $\sigma = \sqrt{\frac{35}{12}}$. Then, by the Central Limit Theorem, $(X - 350)/(10\sigma)$ is approximately $\mathcal{N}(0, 1)$ distributed, i.e. X is approximately $\mathcal{N}(350, 100\sigma^2)$. Therefore

$$\begin{aligned}\mathbb{P}(X > 400) &\approx \mathbb{P}(\mathcal{N}(350, 100\sigma^2) > 400)\\ &= \mathbb{P}(\mathcal{N}(0, 1) > \tfrac{400-350}{10\sigma})\\ &\approx 1 - \Phi(2.93)\\ &= 1 - 0.9983 = 0.0017.\end{aligned}$$

4. **Entropy Warmup**

Suppose that the random variable X takes values in $\{\text{lecture}, \text{midterm}, \text{pop quiz}\}$. Every day you go to class, you observe a random value of X determined according to the distribution p_X, for instance $p_X(\text{lecture}) = 0.85$, $p_X(\text{midterm}) = 0.1$, and $p_X(\text{pop quiz}) = 0.05$. The *surprise*

$$S(x) = \log_2 \frac{1}{p_X(x)}$$

describes how "interesting" it is to see a particular $X = x$.

a. For the probabilities above, calculate $S(\text{lecture})$, $S(\text{midterm})$, and $S(\text{pop quiz})$.

b. Calculate the surprises for $p_X(\text{lecture}) = \frac{1}{3}$, $p_X(\text{midterm}) = \frac{1}{3}$, and $p_X(\text{pop quiz}) = \frac{1}{3}$. Given that $\log_2 \frac{1}{0.85} \approx 0.234$, $\log_2 \frac{1}{1/3} \approx 1.58$, $\log_2 \frac{1}{0.1} \approx 3.32$, and $\log_2 \frac{1}{0.05} \approx 4.32$, do the relative magnitudes of the values in parts (a) and (b) make sense intuitively?

c. The entropy of X is its *expected surprise*. Formally,

$$H(X) = \mathbb{E}(S(X)) = \sum_x p_X(x) \log_2 \frac{1}{p_X(x)}.$$

We will follow the convention that if $p_X(x) = 0$ for some value x, then $p_X(x) \log_2 \frac{1}{p_X(x)} = 0$. Calculate the entropy for the original distribution $(0.85, 0.1, 0.05)$, the uniform distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the deterministic distribution $(1, 0, 0)$. Do these entropy values make sense?

Solution:

a.

$$S(\text{lecture}) = \log_2 \frac{1}{0.85} \approx 0.234$$
$$S(\text{midterm}) = \log_2 \frac{1}{0.1} \approx 3.32$$
$$S(\text{pop quiz}) = \log_2 \frac{1}{0.05} \approx 4.32.$$

b.

$$S(\text{lecture}) = S(\text{midterm}) = S(\text{pop quiz}) = \log_2 \frac{1}{1/3} \approx 1.58.$$

c. For $(0.85, 0.1, 0.05)$,

$$\begin{aligned} H(X) &= 0.85 \cdot S(\text{lecture}) + 0.10 \cdot S(\text{midterm}) + 0.05 \cdot S(\text{pop quiz}) \\ &\approx (0.85)(0.234) + (0.1)(3.32) + (0.05)(4.32) = 0.747. \end{aligned}$$

For $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$,

$$\begin{aligned} H(X) &= \frac{1}{3} S(\text{lecture}) + \frac{1}{3} S(\text{midterm}) + \frac{1}{3} S(\text{pop quiz}) \\ &\approx \frac{1}{3}(1.58) + \frac{1}{3}(1.58) + \frac{1}{3}(1.58) = 1.58. \end{aligned}$$

For $(1, 0, 0)$,

$$H(X) = 1 \cdot S(\text{lecture}) + 0 \cdot S(\text{midterm}) + 0 \cdot S(\text{pop quiz}) = 0.$$

The entropy of a deterministic random variable is 0: the outcome should never be a surprise to us. The uniform distribution has the highest entropy of $\log_2 3$, as it contains the most randomness as to which value we will see. The other distribution lies somewhere in the middle.