UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Homework 09
Spring 2023

1. **System Shocks**

   For a positive integer n, let $X_1, \dots, X_n$ be independent Exponentially distributed random variables, each with mean 1. Let $\gamma > 0$. A system experiences shocks at times $k = 1, \dots, n$, and the size of the shock at time k is X_k.

   a. Suppose that the system fails if any shock exceeds the value γ. What is the probability of system failure?

   b. Suppose instead that the effect of the shocks is cumulative, i.e. the system fails when the total amount of shock received exceeds γ. What is the probability of system failure?

   Solution:

   a. The system fails if $\max\{X_1, \dots, X_n\} > \gamma$, so

   $$\begin{aligned}\mathbb{P}(\max\{X_1, \dots, X_n\} > \gamma) &= 1 - \mathbb{P}(\max\{X_1, \dots, X_n\} \le \gamma) \\ &= 1 - \prod_{k=1}^{n} \mathbb{P}(X_k \le \gamma) = 1 - (1 - e^{-\gamma})^n.\end{aligned}$$

   b. $\mathbb{P}(X_1 + \cdots + X_n > \gamma) = \mathbb{P}(N_\gamma < n)$, where $(N_t)_{t \ge 0}$ is a Poisson process with rate 1, so

   $$\mathbb{P}(X_1 + \cdots + X_n > \gamma) = \sum_{k=0}^{n-1} \frac{\gamma^k}{k!} e^{-\gamma}.$$

2. **Poisson Process Arrival Times**

   Consider a Poisson process $(N_t)_{t \geq 0}$ with rate 1. Let T_k be the time of the kth arrival, $k \geq 1$.

   a. Find $\mathbb{E}(T_3 \mid N_1 = 2)$.
   b. Given $T_3 = s$, where $s > 0$, find the joint distribution of T_1 and T_2.
   c. Find $\mathbb{E}(T_2 \mid T_3 = s)$.

   Solution:

   a. By the memoryless property, $\mathbb{E}(T_3 \mid N_1 = 2) = 1 + \mathbb{E}(T_1) = 2$.
   b. The distribution of the sum of k i.i.d. Exponential random variables is Erlang:

   $$f_{T_k}(s) = \frac{s^{k-1} e^{-s}}{(k-1)!} \mathbb{1}_{s \geq 0} \, .$$

   Then, by Bayes' rule and the memorylessness of Exponential distributions,

   $$\begin{aligned} f_{T_1,T_2|T_3}(s_1, s_2 \mid s) &= \frac{f_{T_1,T_2,T_3}(s_1, s_2, s)}{f_{T_3}(s)} \\ &= \frac{e^{-s_1} e^{-(s_2 - s_1)} e^{-(s - s_2)}}{s^2 e^{-s}/2!} \mathbb{1}_{\{0 \leq s_1 \leq s_2 \leq s\}} \\ &= \frac{2}{s^2} \mathbb{1}_{\{0 \leq s_1 \leq s_2 \leq s\}} \, . \end{aligned}$$

   In other words, T_1 and T_2 are uniformly distributed on the feasible region $\{0 \leq s_1 \leq s_2 \leq s\}$. In particular, the joint distribution is precisely that of the order statistics of 2 i.i.d. Uniform$([0, s])$ random variables.

   c. By part b, T_2 is the maximum of 2 Uniform$([0, s])$ random variables. Thus, for $0 \leq x \leq s$,

   $$\begin{aligned} F_{T_2|T_3}(x \mid s) &= \mathbb{P}(T_2 \leq x \mid T_3 = s) = \left(\frac{x}{s}\right)^2 \\ f_{T_2|T_3}(x \mid s) &= \frac{2x}{s^2} \\ \mathbb{E}(T_2 \mid T_3 = s) &= \int_0^s \frac{2x^2}{s^2} \, dx = \frac{2s}{3}. \end{aligned}$$

3. **Random Telegraph Wave**

   Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ, let X_0 be a Bernoulli random variable independent of $(N_t)_{t\geq 0}$, and define $X_t = X_0(-1)^{N_t}$.

   a. Does the process $(X_t)_{t\geq 0}$ have independent increments?
   b. Calculate $\mathbb{P}(X_t = 1)$ if $\mathbb{P}(X_0 = 1) = p$.
   c. Assume that $p = \frac{1}{2}$. Calculate $\mathbb{E}(X_{t+s}X_s)$ for $s, t \geq 0$.

   Solution:

   a. No, the process does not have independent increments. If it did, for any $0 < t_0 < t_1 < t_2$, we should have $X_{t_2} - X_{t_1}$ is independent of $X_{t_1} - X_{t_0}$. However, suppose $X_0 = 1$ and $X_{t_1} - X_{t_0} = 2$. This means that from t_0 to t_1, X_t increases from -1 to 1. Then it is impossible to have $X_{t_2} - X_{t_1} = 2$, since $X_t \in \{-1, 1\}$ for all $t > 0$ when $X_0 = 1$.
   b. Considering the parity of N_t, we observe that

   $$\mathbb{P}(X_t = 1) = \mathbb{P}(X_0 = 1 \text{ and } N_t \text{ is even}) = p \cdot \mathbb{P}(N_t \text{ is even}).$$

   As $N_t \sim \text{Poisson}(\lambda t)$, the probability that N_t is even is

   $$\begin{aligned}\sum_{k \text{ even}} \frac{(\lambda t)^k}{k!} e^{-\lambda t} &= \frac{e^{-\lambda t}}{2} \left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} + \sum_{k=0}^{\infty} \frac{(-\lambda t)^k}{k!} \right) \\ &= \frac{e^{-\lambda t}}{2} (e^{\lambda t} + e^{-\lambda t}) = \frac{1 + e^{-2\lambda t}}{2}.\end{aligned}$$

   Note that $(\lambda t)^k = \frac{1}{2}((\lambda t)^k + (-\lambda t)^k)$ for all k, even or odd. Thus $\mathbb{P}(X_t = 1) = \frac{p(1+e^{-2\lambda t})}{2}$.
   c. If $X_0 = 0$, then $X_{t+s}X_s = 0$ for all $s, t \geq 0$. For $X_0 = 1$, we observe that

   $$\mathbb{P}(X_{t+s}X_s = 1) = \mathbb{P}(N_{t+s} - N_s \text{ is even}) = \mathbb{P}(N_t \text{ is even}) = \frac{1 + e^{-2\lambda t}}{2}$$

   and $\mathbb{P}(X_{t+s}X_s = -1) = \frac{1-e^{-2\lambda t}}{2}$. Therefore, we get

   $$\begin{aligned}\mathbb{E}(X_{t+s}X_s) &= \mathbb{P}(X_0 = 1) \cdot \mathbb{E}(X_{t+s}X_s \mid X_0 = 1) \\ &= \frac{1}{2} \left[\frac{1 + e^{-2\lambda t}}{2} \cdot 1 + \frac{1 + e^{-2\lambda t}}{2} \cdot -1 \right] = \frac{1}{2} e^{-2\lambda t}.\end{aligned}$$

4. **Frogs**

Three frogs are playing near a pond. When they are in the sun, they get too hot and jump in the lake at rate 1. When they are in the lake, they get too cold and jump onto land at rate 2. The rates here refer to those of the Exponential distribution. Let X_t be the number of frogs in the sun at time $t \geq 0$.

a. Find the stationary distribution of $(X_t)_{t\geq 0}$.

b. Find the answer to part a again, this time using the observation that the three frogs are independent two-state Markov chains.

Solution:

a. Let the states $S = \{0, 1, 2, 3\}$ be the number of frogs in the sun. The Markov chain has $\lambda_0 = 6$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\mu_3 = 3$, $\mu_2 = 2$, and $\mu_1 = 1$, where λ_i and μ_i are the rates of jumping forwards and backwards respectively from state i. Using detailed balance, we compute the stationary distribution to be

$$\pi = \frac{1}{27}\begin{bmatrix} 1 & 6 & 12 & 8 \end{bmatrix}.$$

b. The individual frogs follow independent Markov chains, each with stationary distribution

$$\pi = \frac{1}{3}\begin{bmatrix} 1 & 2 \end{bmatrix}.$$

The stationary probability of being in state $i \in S$ is therefore

$$\mathbb{P}(X_t = i) = \binom{3}{i}\left(\frac{1}{3}\right)^{3-i}\left(\frac{2}{3}\right)^{i}.$$

5. **Lazy Server**

Customers arrive at a queue at the times of a Poisson process with rate λ. The queue is in a service facility with infinite capacity, in which there is an infinitely powerful but lazy server who visits the facility at the times of a Poisson process with rate μ. These two processes are independent. When the server visits the facility, it instantaneously serves all the customers in the queue, then immediately leaves. In other words, at any time, the only customers waiting in the queue are those who arrived after the server's most recent visit.

a. Model the queue length as a CTMC, and find its stationary distribution.
b. Supposing that the CTMC is at stationarity, find the mean number of customers waiting in the queue at any given time.

Solution:

a. We can model the queue length as a continuous-time Markov chain on the state space $S = \mathbb{N}$. The rate at at which a customer arrives is λ, and the rate at which the server arrives is μ, so the rates are $q(i, i+1) = \lambda$ for $i \in \mathbb{N}$ and $q(i, 0) = \mu$ for $i \in \mathbb{Z}^+$. Now, the balance equation for state $i \in \mathbb{Z}^+$ reads $\lambda \cdot \pi(i-1) = (\lambda + \mu) \cdot \pi(i)$, a recurrence relation whose base case we can find by

$$\sum_{i \in \mathbb{N}} \pi(i) = \sum_{i \in \mathbb{N}} \left(\frac{\lambda}{\lambda + \mu}\right)^i \pi(0) = \frac{1}{1 - \frac{\lambda}{\lambda+\mu}} \pi(0) = \frac{\lambda + \mu}{\mu} \pi(0) = 1.$$

With $\pi(0) = \frac{\mu}{\lambda+\mu}$, the stationary distribution is given by

$$\pi(i) = \frac{\mu}{\lambda + \mu} \left(\frac{\lambda}{\lambda + \mu}\right)^i.$$

b. If X is a random variable with $\mathbb{P}(X = i) = \pi(i)$ for all $i \in S$, then we see that

$$X + 1 \sim \text{Geometric}\left(\frac{\mu}{\lambda + \mu}\right).$$

Thus $\mathbb{E}(X) = \frac{\lambda+\mu}{\mu} - 1 = \frac{\lambda}{\mu}$. One possible interpretation of this fact is that $\frac{1}{\mu}$ is the mean amount of time a customer spends in the system.

6. $M/M/2$ **Queue**

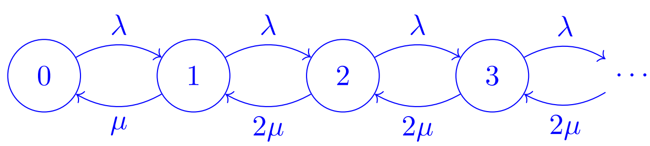
A queue has Poisson arrivals with rate λ and two servers with i.i.d. Exponential(μ) service times. The two servers work in parallel: when there are at least two customers in the queue, two are being served; when there is only one customer, only one server is active. Let X_t be the number of customers either in the queue or in service at time t.

a. Argue that the process $(X_t)_{t\geq 0}$ is a Markov process, and draw its state transition diagram.

b. Find the range of values of μ for which the Markov chain is positive recurrent. For this range of values, calculate the stationary distribution of the Markov chain.

Solution:

a. The queue length is a MC: customer arrivals are independent of the current number of customers in the queue, and departures only depend on the current number of customers being served. Also, even when $k = 1$ or 2 customers are being served, the completion of their services are independent of one another. Finally, when $k = 2$, even if one of the customers has been completely served, the other customer has the same service time distribution as before, because the Exponential distribution is memoryless.



b. It suffices to solve the detailed balance equations

$$\pi(1) = \frac{\lambda}{\mu}\pi(0)$$

$$\pi(i+1) = \frac{\lambda}{2\mu}\pi(i), \qquad i \in \mathbb{Z}^+.$$

Iterating these recurrences yields the following expression for the stationary distribution. We can find the base case $\pi(0)$ as the stationary distribution must normalize:

$$\sum_{i=0}^{\infty} \pi(i) = \pi(0) + \pi(0) \cdot \frac{\lambda}{\mu} \sum_{i=1}^{\infty} \left(\frac{\lambda}{2\mu}\right)^{i-1} = 1.$$

This series converges iff $\lambda < 2\mu$, in which case the Markov chain is positive recurrent. For μ in this range, we find the stationary distribution

$$\pi(0) = \frac{2\mu - \lambda}{2\mu + \lambda}$$

$$\pi(i) = \frac{2\mu - \lambda}{2\mu + \lambda}\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{2\mu}\right)^{i-1}, \qquad i \in \mathbb{Z}^+.$$