UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 09
Spring 2023

1. **System Shocks**

   For a positive integer n, let $X_1, \ldots, X_n$ be independent Exponentially distributed random variables, each with mean 1. Let $\gamma > 0$. A system experiences shocks at times $k = 1, \ldots, n$, and the size of the shock at time k is X_k.

   a. Suppose that the system fails if any shock exceeds the value γ. What is the probability of system failure?

   b. Suppose instead that the effect of the shocks is cumulative, i.e. the system fails when the total amount of shock received exceeds γ. What is the probability of system failure?

2. **Poisson Process Arrival Times**

   Consider a Poisson process $(N_t)_{t\geq 0}$ with rate 1. Let T_k be the time of the kth arrival, $k \geq 1$.

   a. Find $\mathbb{E}(T_3 \mid N_1 = 2)$.
   b. Given $T_3 = s$, where $s > 0$, find the joint distribution of T_1 and T_2.
   c. Find $\mathbb{E}(T_2 \mid T_3 = s)$.

3. **Random Telegraph Wave**

Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ, let X_0 be a Bernoulli random variable independent of $(N_t)_{t\geq 0}$, and define $X_t = X_0(-1)^{N_t}$.

a. Does the process $(X_t)_{t\geq 0}$ have independent increments?

b. Calculate $\mathbb{P}(X_t = 1)$ if $\mathbb{P}(X_0 = 1) = p$.

c. Assume that $p = \frac{1}{2}$. Calculate $\mathbb{E}(X_{t+s}X_s)$ for $s, t \geq 0$.

4. **Frogs**

Three frogs are playing near a pond. When they are in the sun, they get too hot and jump in the lake at rate 1. When they are in the lake, they get too cold and jump onto land at rate 2. The rates here refer to those of the Exponential distribution. Let X_t be the number of frogs in the sun at time $t \geq 0$.

a. Find the stationary distribution of $(X_t)_{t \geq 0}$.

b. Find the answer to part a again, this time using the observation that the three frogs are independent two-state Markov chains.

5. **Lazy Server**

Customers arrive at a queue at the times of a Poisson process with rate λ. The queue is in a service facility with infinite capacity, in which there is an infinitely powerful but lazy server who visits the facility at the times of a Poisson process with rate μ. These two processes are independent. When the server visits the facility, it instantaneously serves all the customers in the queue, then immediately leaves. In other words, at any time, the only customers waiting in the queue are those who arrived after the server's most recent visit.

a. Model the queue length as a CTMC, and find its stationary distribution.

b. Supposing that the CTMC is at stationarity, find the mean number of customers waiting in the queue at any given time.

6. $M/M/2$ **Queue**

A queue has Poisson arrivals with rate λ and two servers with i.i.d. Exponential(μ) service times. The two servers work in parallel: when there are at least two customers in the queue, two are being served; when there is only one customer, only one server is active. Let X_t be the number of customers either in the queue or in service at time t.

a. Argue that the process $(X_t)_{t\geq 0}$ is a Markov process, and draw its state transition diagram.

b. Find the range of values of μ for which the Markov chain is positive recurrent. For this range of values, calculate the stationary distribution of the Markov chain.