UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 14
Spring 2023

1. **MMSE for Jointly Gaussian Random Variables**

   Provide justification for each of the following steps to prove that the LLSE $g(X) \coloneqq \mathbb{L}(Y \mid X)$ is equal to the MMSE estimator for jointly Gaussian random variables X and Y.

$$
\begin{aligned}
& \mathbb{E}((Y - g(X)) \cdot X) = 0 && (1)\\
\implies\ & \operatorname{cov}(Y - g(X), X) = 0 && (2)\\
\implies\ & Y - g(X) \text{ is independent of } X && (3)\\
\implies\ & \mathbb{E}((Y - g(X)) \cdot f(X)) = 0 \ \forall f && (4)\\
\implies\ & g(X) = \mathbb{E}(Y \mid X). && (5)
\end{aligned}
$$

2. **Joint Gaussian Probability**

   Let $X \sim \mathcal{N}(1, 1)$ and $Y \sim \mathcal{N}(0, 1)$ be jointly Gaussian with covariance ρ. What is $\mathbb{P}(X > Y)$?

3. **Joint Gaussians As Linear Transformations of IID Gaussians**

Let $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^\mathsf{T}$ be a jointly Gaussian random vector with mean $\begin{bmatrix} 0 & 0 \end{bmatrix}^\mathsf{T}$ and covariance

$$\Sigma_X = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \end{bmatrix} := AA^\mathsf{T}.$$

a. Express X_1, X_2 as linear combinations of i.i.d. standard Gaussian random variables.

b. Find a matrix B such that the components of BX are i.i.d. $\mathcal{N}(0,1)$. You do not have to simplify your answer.