UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 07
Spring 2023

1. **Mutual Information for Markov Chain**

   In the proof of Homework 06 Q4, we stated without proof the fact that $H(X \mid Y) \leq H(X \mid \hat{X})$, where $\hat{X} = g(Y)$. Here, we will explore why this inequality is true. We define the *conditional mutual information* between random variables X and Y given Z to be

   $$I(X;Y \mid Z) \coloneqq \sum_{x,y,z} p(x,y,z) \log \frac{p(x,y \mid z)}{p(x \mid z)\, p(y \mid z)}.$$

   a. Let $(X_n)_{n \in \mathbb{N}}$ be a Markov chain. Show that $I(X_{n-1}; X_{n+1} \mid X_n) = 0$ for any $n \geq 1$.

   b. Give an interpretation of part a.

   c. Show that $I(X;Y \mid Z) = H(X \mid Z) - H(X \mid Y, Z)$. Returning to the setting of Homework 06 Q4, conclude that $H(X \mid Y) \leq H(X \mid \hat{X})$.
   Hint: Show that $I(X; \hat{X} \mid Y) = 0$ using part a.

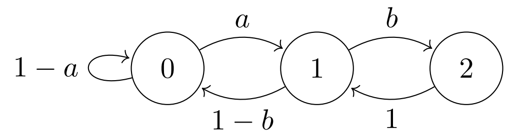
2. **Umbrellas**

A professor has n umbrellas, $n \geq 1$. Every morning, she commutes from her home to her office, and every night she commutes from her office back home. On every commute, if it is raining outside and there is at least one umbrella at her starting location, she takes an umbrella with her; otherwise, she does not take any umbrellas.

Assume that on each commute, it rains with probability $p \in (0, 1)$, independently of all other times. Give the state space and transition probabilities for the Markov chain corresponding to the number of umbrellas the professor has at her current location.

3. **Three-State Chain**

Consider the following Markov chain, where $0 < a, b < 1$.



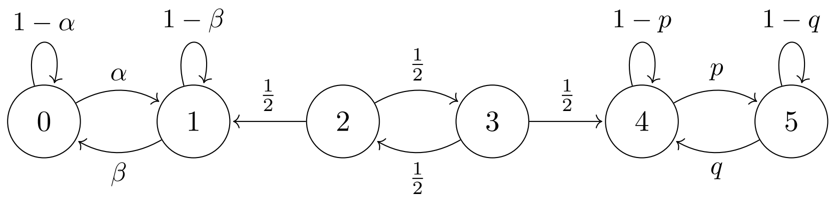
a. Calculate $\mathbb{P}(X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 1 \mid X_0 = 0)$.

b. Show that the Markov chain is irreducible and aperiodic.

c. Find the invariant or stationary distribution.

4. **Reducible Markov Chain**

Consider the following Markov chain, where $\alpha, \beta, p, q \in (0, 1)$.



a. Find all the recurrent and transient classes.

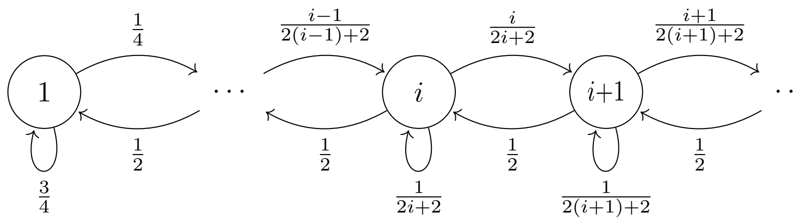
b. Given that we start in state 2, what is the probability we reach state 0 before state 5?

c. What are all of the possible stationary distributions of this chain?
Hint: Consider the recurrent classes.

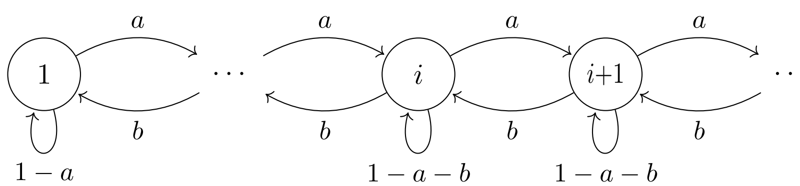
d. Suppose we start with initial distribution $\pi_0 := \begin{bmatrix} 0 & 0 & \gamma & 1-\gamma & 0 & 0 \end{bmatrix}$ for some $\gamma \in [0, 1]$. Does the distribution of the chain converge, and if so, to what?

5. **Markov Chains with Countably Infinite State Space**

a. Show that the Markov chain with state space $\mathbb{Z}^+$ and the following transition diagram is not positive recurrent. Also find the expected time it takes to return to state i starting from i for any $i \in \mathbb{Z}^+$.



b. Let $0 < a < b < a + b \leq 1$. Consider now the Markov chain with state space $\mathbb{Z}^+$ and the following transition diagram:



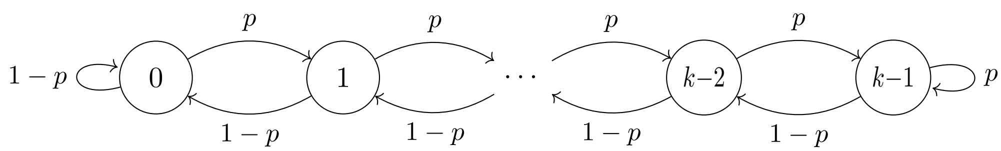
Show that a stationary distribution of this Markov chain is given by

$$\pi(i) = \left(\frac{a}{b}\right)^{i-1}\left(1 - \frac{a}{b}\right).$$

Also compute the expected time it takes to return to state i starting from i.

6. **Finite Random Walk**

Let $0 < p < 1$, and consider the following finite *random walk* with bias p on $\mathcal{X} = \{0, \ldots, k-1\}$, also known as the finite *birth-death chain*.



a. Find the stationary distribution π.

   Hint: Write $q = 1 - p$ and define $r := \frac{p}{q}$. Be careful when $r = 1$.

b. Find the limit of $\pi(0)$ and $\pi(k-1)$ as $k \to \infty$.