UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 01
Spring 2023

1. **Coin Flipping and Symmetry**

   Alice and Bob have $2n + 1$ fair coins, $n \geq 1$. Bob tosses $n + 1$ coins, while Alice tosses the remaining n coins. A fair coin lands on heads with probability $\frac{1}{2}$; assume that coin tosses are independent.

   a. Formulate this scenario in terms of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Describe explicitly what the outcomes are and what the probability measure of any event is defined to be.
   b. Show that the probability that after all coins have been tossed, Bob will have gotten more heads than Alice is $\frac{1}{2}$.
   Hint: Consider the event $A = \{\text{more heads in the first } n+1 \text{ tosses than the last } n \text{ tosses}\}$.

2. **Expanding the NBA**

The NBA is looking to expand to another city. In order to decide which city will receive a new team, the commissioner interviews potential owners from each of the N potential cities, $N \geq 1$, one at a time. The cities are interviewed in a uniformly random order.

a. Formulate this scenario in terms of a probability space. Describe explicitly what the outcomes are and what the probability measure of any event equals.

Unfortunately, the owners would like to know immediately after the interview whether their city will receive the team or not. The commissioner decides to use the following strategy: she will interview the first m owners and reject all of them, $m \in \{1, \ldots, N\}$. After the mth owner is interviewed, she will pick the first city that is better than all previous cities. Assume that the commissioner has an objective scoring method, and each city receives a unique score.

b. What is the probability that the best city is selected?

   You should arrive at an exact answer for the probability in terms of a summation. *Hint*: Consider the events $B_i = \{\text{the } i\text{th city is the best}\}$ and $A = \{\text{the best city is chosen}\}$.

c. Approximate your answer using $\sum_{i=1}^{n} i^{-1} \approx \ln n$, and find the optimal value of m that maximizes the probability that the best city is selected.

   You may also use the fact that $\ln(n-1) \approx \ln n$.

3. **Passengers on a Plane**

   There are n passengers in a plane with n assigned seats, but after boarding, the passengers take the seats randomly. n is a positive integer. Assuming all seating arrangements are equally likely, what is the probability that no passenger is in their assigned seat? Also find the limit of this probability as $n \to \infty$.

   Hint: Use the principle of inclusion-exclusion and the power series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

4. **Upperclassmen**

You meet two students in the library. Assume that each student is an upperclassman or underclassman with equal probability, and each student takes EECS 126 with probability $\frac{1}{10}$, independent of each other and independent of their class standing. What is the probability that both students are upperclassmen, given at least one of them is an upperclassman currently taking EECS 126?

5. **Conditional Probability Space**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and fix an event $B \in \mathcal{F}$ with probability $\mathbb{P}(B) > 0$.

a. Show that $\mathcal{F}|_B \coloneqq \{A \cap B : A \in \mathcal{F}\}$ is a σ-algebra on the sample space B (*not* Ω).

b. Consider the function $\mathbb{P}(\cdot \mid B) \colon \mathcal{F}|_B \to [0, 1]$, which takes as input $A \in \mathcal{F}|_B$ and outputs its conditional probability $\mathbb{P}(A \mid B)$. Show that $(B, \mathcal{F}|_B, \mathbb{P}(\cdot \mid B))$ is a probability space satisfying Kolmogorov's axioms.

6. **Independence and Pairwise Independence**

A collection of events $\{A_i\}_{i \in I}$ is said to be *pairwise independent* if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j)$ for all distinct indices $i \neq j$.

You flip a fair coin 99 times, where the result of each flip is independent of all other flips. For $i = 1, \ldots, 99$, let A_i be the event that the ith flip comes up heads. Let B be the event that in total, an *odd* number of heads are seen. Show that the events $A_1, \ldots, A_{99}, B$ are pairwise independent but *not* independent.