Homework 1

This homework is due at 11 PM on January 26, 2023.

Submission Format: Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Course Setup**

   Please complete the following steps to get access to all course resources.

   (a) Visit the course website at http://www.eecs127.github.io/ and familiarize yourself with the syllabus.

   (b) Verify that you can access the class Ed site at https://edstem.org/us/courses/35286/.

   (c) Register for the class Gradescope site at https://www.gradescope.com/courses/495974 using code **XV774D**.

   (d) When are self grades due for this homework? In general, when are self grades due? Where are the self-grade assignments?

   (e) How many homework drops do you get? Are there exceptions?

2. What Prerequisites Have You Taken?

The prerequisites for this course are

- EECS 16A & 16B (Designing Information Devices and Systems I & II) **OR** MATH 54 (Linear Algebra & Differential Equations),
- CS 70 (Discrete Mathematics & Probability Theory), and
- MATH 53 (Multivariable Calculus).

Please list which of these courses you have taken. If you have taken equivalent courses at a separate institution, please list them here. If you are unsure of course material overlap, please refer to the EECS 16A, EECS 16B, and CS 70 websites (`https://www.eecs16a.org/`, `https://www.eecs16b.org/`, and `http://www.sp22.eecs70.org/`, respectively) and the MATH 53 textbook (Multivariable Calculus by James Stewart).

The course material this semester will rely on knowledge from these prerequisite courses. If you feel shaky on this material, please use the first week to reacquaint yourself with it.

3. Diagonalization and Singular Value Decomposition

Let matrix $A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(a) Compute the eigenvalues and associated eigenvectors of A.

(b) Express A as $P\Lambda P^{-1}$, where Λ is a diagonal matrix and $PP^{-1} = I$. State P, Λ, and P^{-1} explicitly.

(c) Compute $\lim_{k\to\infty} A^k$.

(d) Give the singular values σ_1 and σ_2 of A.

4. Determinants

Consider a unit box $\mathcal{B}$ in $\mathbb{R}^2$ — i.e., the square with corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Define $A(\mathcal{B})$ as the parallelogram generated by applying matrix A to every point in $\mathcal{B}$.

(a) For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, calculate the location of each corner of $A(\mathcal{B})$.

(b) Write the area of $A(\mathcal{B})$ as a function of $\det(A)$.

HINT: How are the basis vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *transformed by the matrix multiplication?*

(c) Calculate the area of $A(\mathcal{B})$ for each of the following values of A.

i. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

ii. $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

iii. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

iv. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

5. Least Squares

The Michaelis-Menten model for enzyme kinetics relates the rate y of an enzymatic reaction to the concentration x of a substrate, as follows:

$$y = \frac{\beta_1 x}{\beta_2 + x}, \tag{1}$$

for constants $\beta_1, \beta_2 > 0$.

(a) Show that the model can be expressed as a linear relation between the values $1/y = y^{-1}$ and $1/x = x^{-1}$. Specifically, give an equation of the form $y^{-1} = w_1 + w_2 x^{-1}$, specifying the values of w_1 and w_2 in terms of β_1 and β_2.

(b) In general, reaction parameters β_1 and β_2 (and, thus, w_1 and w_2) are not known a priori and must be fitted from data — for example, using least squares. Suppose you collect m measurements (x_i, y_i), $i = 1, \ldots, m$ over the course of a reaction. Formulate the least squares problem

$$\vec{w}^\star = \underset{\vec{w}}{\operatorname{argmin}} \|X\vec{w} - \vec{y}\|_2^2, \tag{2}$$

where $\vec{w}^\star = \begin{bmatrix} w_1^\star & w_2^\star \end{bmatrix}^\top$, and you must specify $X \in \mathbb{R}^{m \times 2}$ and $\vec{y} \in \mathbb{R}^m$. Specifically, your solution should include explicit expressions for X and $\vec{y}$ as a function of (x_i, y_i) values and a final expression for $\vec{w}^\star$ in terms of X and $\vec{y}$, which should contain only matrix multiplications, transposes, and inverses.

Assume without loss of generality that $x_1 \neq x_2$.

(c) Assume that we have used the above procedure to calculate values for $w_1^\star$ and $w_2^\star$, and we now want to estimate $\widehat{\vec{\beta}} = \begin{bmatrix} \widehat{\beta}_1 & \widehat{\beta}_2 \end{bmatrix}^\top$. Write an expression for $\widehat{\vec{\beta}}$ in terms of $w_1^\star$ and $w_2^\star$.

NOTE: This problem was taken (with some edits) from the textbook *Optimization Models* by Calafiore and El Ghaoui.

6. Vector Spaces and Rank

The *rank* of a $m \times n$ matrix A, $\operatorname{rk}(A)$, is the dimension of its range, also called span, and denoted $\mathcal{R}(A) := \{A\vec{x} \,:\, \vec{x} \in \mathbb{R}^n\}$.

(a) Assume that $A \in \mathbb{R}^{m \times n}$ takes the form $A = \vec{u}\vec{v}^\top$, with $\vec{u} \in \mathbb{R}^m$, $\vec{v} \in \mathbb{R}^n$, and $\vec{u}, \vec{v} \neq \vec{0}$. (Note that a matrix of this form is known as a dyad.) Find the rank of A.

HINT: Consider the quantity $A\vec{x}$ for arbitrary $\vec{x}$, i.e., what happens when you multiply any vector by matrix A.

(b) Show that for arbitrary $A, B \in \mathbb{R}^{m \times n}$,

$$\operatorname{rk}(A + B) \leq \operatorname{rk}(A) + \operatorname{rk}(B), \tag{3}$$

i.e., the rank of the sum of two matrices is less than or equal to the sum of their ranks.

HINT: First, show that $\mathcal{R}(A + B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$, meaning that any vector in the range of $A + B$ can be expressed as the sum of two vectors, each in the range of A and B, respectively. Remember that for any matrix A, $\mathcal{R}(A)$ is a subspace, and for any two subspaces S_1 and S_2, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.[1] (Note that the sum of vector spaces $S_1 + S_2$ is not equivalent to $S_1 \cup S_2$, but is defined as $S_1 + S_2 := \{\vec{s_1} + \vec{s_2} | \vec{s_1} \in S_1, \vec{s_2} \in S_2\}$.)

(c) Consider an $m \times n$ matrix A that takes the form $A = UV^\top$, with $U \in \mathbb{R}^{m \times k}$, $V \in \mathbb{R}^{n \times k}$. Show that the rank of A is less or equal than k. *HINT: Use parts (a) and (b), and remember that this decomposition can also be written as the dyadic expansion*

$$A = UV^\top = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_k^\top \end{bmatrix} = \sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top, \tag{4}$$

for $U = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix}$ and $V = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}$.

[1] This fact can be proved by taking a basis of S_1 and extending it to a basis of S_2 (during which we can only add *at most* $\dim(S_2)$ basis vectors). This extended basis must now also be a basis of $S_1 + S_2$. Thus, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.

7. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.

