Homework 1

Self grades are due at 11 PM on February 2, 2023.

1. **Course Setup**

   Please complete the following steps to get access to all course resources.

   (a) Visit the course website at `http://www.eecs127.github.io/` and familiarize yourself with the syllabus.

   (b) Verify that you can access the class Ed site at `https://edstem.org/us/courses/35286/`.

   (c) Register for the class Gradescope site at `https://www.gradescope.com/courses/495974` using code **XV774D**.

   (d) When are self grades due for this homework? In general, when are self grades due? Where are the self-grade assignments?

   Solution: Self grades for this homework are due February 2, 2023 at 11 PM PST. In general, self grades are due one week following initial homework submission. Self-grade assignments are on Gradescope and are also accessible using the course website.

   (e) How many homework drops do you get? Are there exceptions?

   Solution: Your lowest homework grade will be dropped. There are no exceptions, and late work will not be accepted, though you may ask for extensions using the form on the course website.

2. What Prerequisites Have You Taken?

The prerequisites for this course are

- EECS 16A & 16B (Designing Information Devices and Systems I & II) **OR** MATH 54 (Linear Algebra & Differential Equations),
- CS 70 (Discrete Mathematics & Probability Theory), and
- MATH 53 (Multivariable Calculus).

Please list which of these courses you have taken. If you have taken equivalent courses at a separate institution, please list them here. If you are unsure of course material overlap, please refer to the EECS 16A, EECS 16B, and CS 70 websites (https://www.eecs16a.org/, https://www.eecs16b.org/, and http://www.sp22.eecs70.org/, respectively) and the MATH 53 textbook (Multivariable Calculus by James Stewart).

The course material this semester will rely on knowledge from these prerequisite courses. If you feel shaky on this material, please use the first week to reacquaint yourself with it.

3. Diagonalization and Singular Value Decomposition

Let matrix $A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(a) Compute the eigenvalues and associated eigenvectors of A.

Solution: Eigenvalues can be computed by first calculating A's characteristic polynomial:

$$\begin{aligned}
\det(sI - A) &= \det\left(\begin{bmatrix} s & -1 \\ -\frac{1}{2} & s - \frac{1}{2} \end{bmatrix}\right) && (1)\\
&= s\left(s - \frac{1}{2}\right) - (-1)\left(-\frac{1}{2}\right) && (2)\\
&= s^2 - \frac{1}{2}s - \frac{1}{2} && (3)\\
&= \left(s - \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{1}{2} && (4)\\
&= \left(s - \frac{1}{4}\right)^2 - \frac{9}{16} && (5)\\
&= \left(s - \frac{1}{4} - \frac{3}{4}\right)\left(s - \frac{1}{4} + \frac{3}{4}\right) \qquad a^2 - b^2 = (a-b)(a+b) && (6)\\
&= (s-1)\left(s + \frac{1}{2}\right). && (7)
\end{aligned}$$

The eigenvalues of A are thus $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$, the values of s at which $\det(sI - A) = 0$.

The eigenvectors associated with each eigenvalue λ can be calculated as values of $\vec{x} = \begin{bmatrix} x_a \\ x_b \end{bmatrix}$ for which $A\vec{x} = \lambda\vec{x}$, namely:

$$A\vec{x} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_a \\ x_b \end{bmatrix} = \begin{bmatrix} x_b \\ \frac{x_a + x_b}{2} \end{bmatrix} \tag{8}$$

$$A\vec{x}_1 = \vec{x}_1 \iff x_b = x_a \iff \vec{x}_1 = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \alpha_1 \neq 0 \in \mathbb{R}. \tag{9}$$

$$A\vec{x}_2 = -\frac{1}{2}\vec{x}_2 \iff x_b = -\frac{1}{2}x_a \iff \vec{x}_2 = \alpha_2 \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \ \alpha_2 \neq 0 \in \mathbb{R}. \tag{10}$$

Note that the expressions above are valid eigenvectors for any nonzero values of α_1 and α_2.

(b) Express A as $P\Lambda P^{-1}$, where Λ is a diagonal matrix and $PP^{-1} = I$. State P, Λ, and P^{-1} explicitly.

Solution: Combining the calculations in part (a), we have that

$$A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 \end{bmatrix} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{11}$$

For our calculations, we will use the eigenvalues and eigenvectors from part (a) with $\alpha_1 = \alpha_2 = 1$. (Your calculations may differ here; any nonzero values for α_1 and α_2 are permissible, and will result in scaled values of P and P^{-1}.) Filling in eigenvalue and eigenvector values, we have:

$$A\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \tag{12}$$

and rearranging,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1}. \tag{13}$$

Calculating the latter inverse explicitly, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1} = -\frac{2}{3} \begin{bmatrix} -\frac{1}{2} & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \text{ because } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{14}$$

so finally,

$$A = P \Lambda P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. \tag{15}$$

This is known as the *eigenvalue decomposition*, or *eigendecomposition*, of matrix A; for a more extensive description of this decomposition, see Calafiore & El Ghaoui section 3.5.

(c) Compute $\lim_{k \to \infty} A^k$.

Solution: Using the diagonalization of A from part (b), we have:

$$\begin{aligned}
A &= P \Lambda P^{-1} && (16) \\
A^k &= (P \Lambda P^{-1})^k && (17) \\
&= (P \Lambda P^{-1})(P \Lambda P^{-1}) \dots (P \Lambda P^{-1}) \qquad (k \text{ times}) && (18) \\
&= P \Lambda \underbrace{P^{-1} P}_{I} \Lambda P^{-1} \dots P \Lambda P^{-1} && (19) \\
&= P \Lambda^k P^{-1} && (20) \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}^k \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} && (21) \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & \left(-\frac{1}{2}\right)^k \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. && (22)
\end{aligned}$$

Finally, because $\lim_{k \to \infty} \left(-\frac{1}{2}\right)^k = 0$, we have

$$\lim_{k \to \infty} A^k = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}. \tag{23}$$

(d) Give the singular values σ_1 and σ_2 of A.

Solution: Each singular value σ_i of A can be calculated as $\sigma_i = \sqrt{\lambda_i(AA^\top)} = \sqrt{\lambda_i(A^\top A)}$. (This is because A's singular value decomposition, canonically written $A = U \Sigma V^\top$, can be multiplied by a transposed version to give $AA^\top = U \Sigma^2 U^\top$, where Σ^2 is a diagonal matrix containing the squared singular values of A and $UU^\top = I$. For a thorough treatment of SVD, see Calafiore & El Ghaoui chapter 5.)

To find A's singular values, we thus perform the same calculation used in part (a) to find each $\lambda_i(AA^\top) = \sigma_i^2$:

$$AA^\top = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \tag{24}$$

$$
\begin{aligned}
&= \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. && (25) \\
\det\left((sI - AA^\top)\right) &= \det\left(\begin{bmatrix} s-1 & -\frac{1}{2} \\ -\frac{1}{2} & s-\frac{1}{2} \end{bmatrix}\right) && (26) \\
&= (s-1)\left(s-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) && (27) \\
&= s^2 - s - \frac{1}{2}s + \frac{1}{2} - \frac{1}{4} && (28) \\
&= s^2 - \frac{3}{2}s + \frac{1}{4} && (29) \\
&= \left(s - \frac{3}{4}\right)^2 - \frac{9}{16} + \frac{1}{4} && (30) \\
&= \left(s - \frac{3}{4}\right)^2 - \frac{5}{16} && (31) \\
&= \left(s - \frac{3}{4} - \frac{\sqrt{5}}{4}\right)\left(s - \frac{3}{4} + \frac{\sqrt{5}}{4}\right) \qquad a^2 - b^2 = (a-b)(a+b) && (32) \\
&= \left(s - \frac{3+\sqrt{5}}{4}\right)\left(s - \frac{3-\sqrt{5}}{4}\right) && (33) \\
&= (s - \sigma_1^2)(s - \sigma_2^2). && (34)
\end{aligned}
$$

Thus, the singular values of A are $\sigma_1 = \frac{\sqrt{3+\sqrt{5}}}{2}$ and $\sigma_2 = \frac{\sqrt{3-\sqrt{5}}}{2}$.

4. Determinants

Consider a unit box $\mathcal{B}$ in $\mathbb{R}^2$ — i.e., the square with corners $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$, and $\begin{bmatrix}1\\1\end{bmatrix}$. Define $A(\mathcal{B})$ as the parallelogram generated by applying matrix A to every point in $\mathcal{B}$.

(a) For $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, calculate the location of each corner of $A(\mathcal{B})$.

Solution: Directly applying matrix A to each corner point, we have

$$A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$
$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}.$$
$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}.$$
$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\7\end{bmatrix}.$$

(b) Write the area of $A(\mathcal{B})$ as a function of $\det(A)$.

HINT: How are the basis vectors $\begin{bmatrix}0\\1\end{bmatrix}$ *and* $\begin{bmatrix}1\\0\end{bmatrix}$ *transformed by the matrix multiplication?*

Solution: The area of the parallelogram $A(\mathcal{B})$ can be computed geometrically from its corner points and is equal to $|\det(A)|$. A full explanation of this relation can be found in Calafiore & El Ghaoui section 3.3.2.

(c) Calculate the area of $A(\mathcal{B})$ for each of the following values of A.

i. $A = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$

ii. $A = \begin{bmatrix}2 & 1\\4 & 3\end{bmatrix}$

iii. $A = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$

iv. $A = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

Solution: We use the result from part (b) and use the determinant of A to calculate the area.

i. $\det(A) = \det\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right) = -2 \implies$ area$[A(\mathcal{B})] = |\det(A)| = 2$.

ii. $\det(A) = \det\left(\begin{bmatrix}2 & 1\\4 & 3\end{bmatrix}\right) = 2 \implies$ area$[A(\mathcal{B})] = |\det(A)| = 2$.

iii. $\det(A) = \det\left(\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\right) = 0 \implies$ area$[A(\mathcal{B})] = |\det(A)| = 0$ (Singular matrix!).

iv. $\det(A) = \det\left(\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\right) = 1 \implies$ area$[A(\mathcal{B})] = |\det(A)| = 1$ (Rotation matrix!).

5. Least Squares

The Michaelis-Menten model for enzyme kinetics relates the rate y of an enzymatic reaction to the concentration x of a substrate, as follows:

$$y = \frac{\beta_1 x}{\beta_2 + x}, \tag{35}$$

for constants $\beta_1, \beta_2 > 0$.

(a) Show that the model can be expressed as a linear relation between the values $1/y = y^{-1}$ and $1/x = x^{-1}$. Specifically, give an equation of the form $y^{-1} = w_1 + w_2 x^{-1}$, specifying the values of w_1 and w_2 in terms of β_1 and β_2.

Solution: Inverting each side of the equation, we have

$$\begin{aligned} y^{-1} &= \left(\frac{\beta_1 x}{\beta_2 + x}\right)^{-1} && (36)\\ &= \frac{\beta_2 + x}{\beta_1 x} && (37)\\ &= \frac{\beta_2}{\beta_1 x} + \frac{x}{\beta_1 x} && (38)\\ &= \frac{\beta_2}{\beta_1} x^{-1} + \frac{1}{\beta_1} && (39)\\ &= \frac{1}{\beta_1} + \frac{\beta_2}{\beta_1} x^{-1}. && (40)\\ & && (41) \end{aligned}$$

The above equation has exactly the desired form $y^{-1} = w_1 + w_2 x^{-1}$ for $w_1 = \frac{1}{\beta_1}$ and $w_2 = \frac{\beta_2}{\beta_1}$.

(b) In general, reaction parameters β_1 and β_2 (and, thus, w_1 and w_2) are not known a priori and must be fitted from data — for example, using least squares. Suppose you collect m measurements (x_i, y_i), $i = 1, \ldots, m$ over the course of a reaction. Formulate the least squares problem

$$\vec{w}^\star = \operatorname*{argmin}_{\vec{w}} \|X\vec{w} - \vec{y}\|_2^2, \tag{42}$$

where $\vec{w}^\star = \begin{bmatrix} w_1^\star & w_2^\star \end{bmatrix}^\top$, and you must specify $X \in \mathbb{R}^{m\times 2}$ and $\vec{y} \in \mathbb{R}^m$. Specifically, your solution should include explicit expressions for X and $\vec{y}$ as a function of (x_i, y_i) values and a final expression for $\vec{w}^\star$ in terms of X and $\vec{y}$, which should contain only matrix multiplications, transposes, and inverses.

Assume without loss of generality that $x_1 \neq x_2$.

Solution: To formulate the least squares problem as stated, X and $\vec{y}$ values should be set to

$$X = \begin{bmatrix} 1 & \cdots & 1 \\ x_1^{-1} & \cdots & x_m^{-1} \end{bmatrix}^\top, \quad \vec{y} = \begin{bmatrix} y_1^{-1} & \cdots & y_m^{-1} \end{bmatrix}^\top. \tag{43}$$

To solve this least squares problem, we note that the optimal residual vector $X\vec{w}^\star - \vec{y}$ must be orthogonal to $\mathcal{R}(X)$ by the orthogonality principle, and we have

$$X^\top (X\vec{w}^\star - \vec{y}) = 0. \tag{44}$$

Rearranging we get,

$$\vec{w}^\star = (X^\top X)^{-1} X^\top \vec{y}. \tag{45}$$

(c) Assume that we have used the above procedure to calculate values for $w_1^\star$ and $w_2^\star$, and we now want to estimate $\widehat{\vec{\beta}} = \begin{bmatrix} \widehat{\beta}_1 & \widehat{\beta}_2 \end{bmatrix}^\top$. Write an expression for $\widehat{\vec{\beta}}$ in terms of $w_1^\star$ and $w_2^\star$.

Solution: To calculate $\widehat{\vec{\beta}}$, we can simply reverse the calculations from part (a):

$$w_1 = \frac{1}{\beta_1} \implies \beta_1 = \frac{1}{w_1}, \tag{46}$$

$$w_2 = \frac{\beta_2}{\beta_1} \implies \beta_2 = \beta_1 w_2 = \frac{w_2}{w_1}. \tag{47}$$

Thus, $\widehat{\vec{\beta}} = \begin{bmatrix} \frac{1}{w_1^\star} & \frac{w_2^\star}{w_1^\star} \end{bmatrix}^\top$.

NOTE: This problem was taken (with some edits) from the textbook *Optimization Models* by Calafiore and El Ghaoui.

6. Vector Spaces and Rank

The *rank* of a $m \times n$ matrix A, $\mathrm{rk}(A)$, is the dimension of its range, also called span, and denoted $\mathcal{R}(A) := \{A\vec{x} \,:\, \vec{x} \in \mathbb{R}^n\}$.

(a) Assume that $A \in \mathbb{R}^{m\times n}$ takes the form $A = \vec{u}\vec{v}^\top$, with $\vec{u} \in \mathbb{R}^m$, $\vec{v} \in \mathbb{R}^n$, and $\vec{u}, \vec{v} \neq \vec{0}$. (Note that a matrix of this form is known as a dyad.) Find the rank of A.

HINT: Consider the quantity $A\vec{x}$ for arbitrary $\vec{x}$, i.e., what happens when you multiply any vector by matrix A.

Solution: For any $\vec{x} \in \mathbb{R}^n$, we have that $A\vec{x} = \vec{u}\vec{v}^\top\vec{x} = \vec{u}(\vec{v}^\top\vec{x}) = (\vec{v}^\top\vec{x})\vec{u}$. Note that $\vec{v}^\top\vec{x}$ is a scalar that can take on any value depending on choice of $\vec{x}$. Since the range of A is the subspace reachable through any choice of $\vec{x}$, $\mathcal{R}(A)$ is simply the 1-dimensional subspace spanned by $\vec{u}$ (i.e., the line pointing along $\vec{u}$). Since a single vector (namely, $\vec{u}$) spans $\mathcal{R}(A)$, the rank of A is 1.

(b) Show that for arbitrary $A, B \in \mathbb{R}^{m\times n}$,

$$\mathrm{rk}(A+B) \leq \mathrm{rk}(A) + \mathrm{rk}(B), \tag{48}$$

i.e., the rank of the sum of two matrices is less than or equal to the sum of their ranks.

HINT: First, show that $\mathcal{R}(A+B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$, meaning that any vector in the range of $A+B$ can be expressed as the sum of two vectors, each in the range of A and B, respectively. Remember that for any matrix A, $\mathcal{R}(A)$ is a subspace, and for any two subspaces S_1 and S_2, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.[1] (Note that the sum of vector spaces $S_1 + S_2$ is not equivalent to $S_1 \cup S_2$, but is defined as $S_1 + S_2 := \{\vec{s_1} + \vec{s_2} | \vec{s_1} \in S_1, \vec{s_2} \in S_2\}$.)

Solution: Given any vector $\vec{v} \in \mathcal{R}(A+B)$, there must by definition exist $\vec{x} \in \mathbb{R}^n$ such that $\vec{v} = (A+B)\vec{x}$. Thus, $\vec{v} = (A+B)\vec{x} = \underbrace{A\vec{x}}_{\in\mathcal{R}(A)} + \underbrace{B\vec{x}}_{\in\mathcal{R}(B)}$, so $\mathcal{R}(A+B) \subseteq \mathcal{R}(A) + \mathcal{R}(B)$, as hinted.

Computing the dimension of each side of the subset relationship, it follows that

$$\dim(\mathcal{R}(A+B)) \leq \dim(\mathcal{R}(A) + \mathcal{R}(B)). \tag{49}$$

Using the second part of the hint, we have that

$$\dim(\mathcal{R}(A) + \mathcal{R}(B)) \leq \dim(\mathcal{R}(A)) + \dim(\mathcal{R}(B)). \tag{50}$$

Combining the previous two equations,

$$\dim(\mathcal{R}(A+B)) \leq \dim(\mathcal{R}(A)) + \dim(\mathcal{R}(B)), \tag{51}$$

i.e., by definition,

$$\mathrm{rk}(A+B) \leq \mathrm{rk}(A) + \mathrm{rk}(B) \tag{52}$$

as desired.

(c) Consider an $m \times n$ matrix A that takes the form $A = UV^\top$, with $U \in \mathbb{R}^{m\times k}$, $V \in \mathbb{R}^{n\times k}$. Show that the rank of A is less or equal than k. *HINT: Use parts (a) and (b), and remember that this decomposition can*

[1]This fact can be proved by taking a basis of S_1 and extending it to a basis of S_2 (during which we can only add *at most* $\dim(S_2)$ basis vectors). This extended basis must now also be a basis of $S_1 + S_2$. Thus, $\dim(S_1 + S_2) \leq \dim(S_1) + \dim(S_2)$.

also be written as the dyadic expansion

$$A = UV^\top = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix} \begin{bmatrix} \vec{v}_1^\top \\ \vdots \\ \vec{v}_k^\top \end{bmatrix} = \sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top, \tag{53}$$

for $U = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_k \end{bmatrix}$ *and* $V = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{bmatrix}$.

Solution: Starting with the dyadic expansion above, iteratively pulling out terms from this summation, and using the result from (a) that the rank of a dyadic matrix is 1 (or 0, if any $\vec{v}_i = \vec{0}$), we know by the rank relation from (b) that

$$\operatorname{rk}(A) = \operatorname{rk}\left(\sum_{i=1}^{k} \vec{u}_i \vec{v}_i^\top\right) \leq \operatorname{rk}\left(\sum_{i=1}^{k-1} \vec{u}_i \vec{v}_i^\top\right) + \underbrace{\operatorname{rk}\left(\vec{u}_k \vec{v}_k^\top\right)}_{0 \text{ or } 1} \leq \dots \leq k, \tag{54}$$

as desired.

7. Homework Process

With whom did you work on this homework? List the names and SIDs of your group members.

NOTE: If you didn't work with anyone, you can put "none" as your answer.