EECS 127/227AT Optimization Models in Engineering UC Berkeley Spring 2023

Discussion 8

1. Simple Constrained Optimization Problem with Duality

Consider the optimization problem

$$\begin{aligned} \min_{x_1, x_2 \in \mathbb{R}} \quad & f(x_1, x_2) && (1) \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 1 && (2) \\ & x_1 + 3x_2 \geq 1 && (3) \\ & x_1 \geq 0, && (4) \\ & x_2 \geq 0 && (5) \end{aligned}$$

(a) Express the Lagragian of the problem $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

(b) Show that $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

(c) Express the dual function of the problem, and show that it is concave.

(d) Assume f is convex. Show that $\mathcal{L}$ is convex in (x_1, x_2).

(e) Denoting $\mathcal{X} = \{(x_1, x_2) \mid 2x_1 + x_2 \geq 1,\ x_1 + 3x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\}$, show that

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \quad (6)$$

(f) Conclude that $\min_{(x_1,x_2)\in\mathcal{X}} \max_{\lambda_1\geq 0,\lambda_2\geq 0,\lambda_3\geq 0,\lambda_4\geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1,x_2)\in\mathcal{X}} f(x_1, x_2)$.

(g) Assuming f is convex, formulate the first order condition on $\mathcal{L}$ as a function of ∇f and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 to solve:

$$\min_{x_1,x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{7}$$

2. Lagrangian Dual of a QP

Consider the general form of a convex quadratic program, with $Q \succ 0$:

$$\begin{aligned} \min_{\vec{x}} \quad & \frac{1}{2}\vec{x}^\top Q\vec{x} && (8) \\ \text{s.t.} \quad & A\vec{x} \leq \vec{b} && (9) \end{aligned}$$

(a) Write the Lagrangian function $\mathcal{L}(\vec{x}, \vec{\lambda})$.

(b) Write the Lagrangian dual function, $g(\vec{\lambda})$.

(c) Show that the Lagrangian dual problem is convex by writing it in standard QP form. Is the Lagrangian dual problem convex in general?