Discussion 9

1. Magic with constraints

In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn't hold in the other.

Let

$$f_0(x) \doteq \begin{cases} x^3 - 3x^2 + 4, & x \geq 0 \\ -x^3 - 3x^2 + 4, & x < 0 \end{cases} .$$

1) Consider the minimization problem

$$p^* = \min_{x \in \mathbb{R}} \quad f_0(x) \tag{1}$$

$$\text{s.t.} \ -1 \leq x, \quad x \leq 1. \tag{2}$$

(a) Show that $f_0(x)$ is differentiable everywhere and compute it's derivative.

(b) Show that $p^* = 2$ and the the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$ by examining the "critical" points, i.e., points where the gradient is zero, points on the boundaries, and $\pm\infty$.

(c) Show that the dual problem can be represented as

$$d^* = \max_{\lambda_1, \lambda_2 \geq 0} \quad g(\vec{\lambda}),$$

where

$$g(\vec{\lambda}) = \min\left\{ g_1(\vec{\lambda}), g_2(\vec{\lambda}) \right\},$$

with

$$g_1(\vec{\lambda}) = \min_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1)$$
$$g_2(\vec{\lambda}) = \min_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1).$$

(d) Next, show that

$$g_1(\vec{\lambda}) \leq -3\lambda_1 + \lambda_2$$
$$g_2(\vec{\lambda}) \leq \lambda_1 - 3\lambda_2.$$

Use this to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$.

(e) Show that $g(\vec{0}) = 0$ and conclude that $d^* = 0$.

(f) Does strong duality hold?

2) Now, consider a problem equivalent to the minimization in (1):

$$p^* = \min_{x \in \mathbb{R}} \quad f_0(x) \tag{3}$$

$$\text{s.t. } x^2 \leq 1. \tag{4}$$

Observe that $p^* = 2$ and the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$, since this problem is equivalent to the one in part 1).

(a) Show that the dual problem can be represented as

$$d^* = \max_{\lambda \geq 0} \quad g(\lambda),$$

where

$$g(\lambda) = \min(g_1(\lambda), g_2(\lambda)),$$

with

$$g_1(\lambda) = \min_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1)$$
$$g_2(\lambda) = \min_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1).$$

(b) Show that $g_1(\lambda) = g_2(\lambda) = \begin{cases} 4 - \lambda, & \lambda \geq 3 \\ -\frac{4}{27}(3 - \lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{cases}$

(c) Conclude that $d^* = 2$ and the optimal $\lambda = \frac{3}{2}$.

(d) Does strong duality hold?