Discussion 10

1. An optimization problem

Consider the primal optimization problem,

$$p^\star = \min_{x \in \mathbb{R}^2} \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{1}$$

$$\text{s.t.} \quad x_1 \geq 0 \tag{2}$$

$$x_1 + x_2 \geq 2. \tag{3}$$

First we solve the primal problem directly.

(a) Sketch the feasible region and argue that $x^\star = (1, 1)$ and $p^\star = 1$.

(b) The *critical points* of an optimization problem are points where the gradient is 0 or undefined, and also points which are on the boundary of the constraint set.

Compute the value of the objective function at its critical points and find $p^\star$ and $x^\star$.

(c) Next we solve the problem with the help of the dual. First, find the Lagrangian $\mathcal{L}(x, \lambda)$.

(d) Formulate the dual problem.

(e) Solve the dual problem to find $d^\star$ and $\lambda^\star$.

(f) Does strong duality hold?

(g) Find $p^\star$ and $x^\star$.

(h) Finally we use KKT conditions to find $x^\star$, $\lambda^\star$. First, Write down the KKT conditions and find $\tilde{x}$ and $\tilde{\lambda}$ that satisfy it.

(i) Argue why the optimal primal and dual solutions are given by $x^\star = \tilde{x}$ and $\lambda^\star = \tilde{\lambda}$.

2. Complementary Slackness

Consider the problem:

$$p^\star = \min_{x \in \mathbb{R}} \quad x^2 \tag{4}$$

$$\text{s.t.} \quad x \geq 1,\; x \leq 2. \tag{5}$$

(a) Does Slater's condition hold? Is the problem convex? Does strong duality hold?

(b) Find the Lagrangian $\mathcal{L}(x, \lambda_1, \lambda_2)$.

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(c) Find the dual function $g(\lambda_1, \lambda_2)$ so that the dual problem is given by,

$$d^\star = \max_{\lambda_1, \lambda_2 \in \mathbb{R}_+} g(\lambda_1, \lambda_2). \tag{6}$$

(d) Solve the dual problem in (6) for $d^\star$.

(e) Solve for $x^\star, \lambda_1^\star, \lambda_2^\star$ that satisfy KKT conditions.

(f) Can you spot a connection between the values of $\lambda_1^\star, \lambda_2^\star$ in relation to whether the corresponding inequality constraints are strict or not at the optimal $x^\star$?

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