Discussion 8

1. Simple Constrained Optimization Problem with Duality

Consider the optimization problem

$$
\begin{aligned}
\min_{x_1, x_2 \in \mathbb{R}} \quad & f(x_1, x_2) && (1)\\
\text{s.t.} \quad & 2x_1 + x_2 \geq 1 && (2)\\
& x_1 + 3x_2 \geq 1 && (3)\\
& x_1 \geq 0, && (4)\\
& x_2 \geq 0 && (5)
\end{aligned}
$$

(a) Express the Lagragian of the problem $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Solution:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2 \quad (6)$$

(b) Show that $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Solution: $-\mathcal{L}$ is convex in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ as a affine function of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. So $\mathcal{L}$ is concave in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

(c) Express the dual function of the problem, and show that it is concave.

Solution: $g(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

We can show that by showing that $-g$ is convex.

$$
\begin{aligned}
-g(\vec{\lambda}) &= -\min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) && (7)\\
&= \max_{x_1, x_2} -\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) && (8)
\end{aligned}
$$

When (x_1, x_2) is fixed, the function $-\mathcal{L}$ is linear in $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, therefore convex.

Because the max of convex functions is convex, $-g$ is convex. Therefore g is concave.

(d) Assume f is convex. Show that $\mathcal{L}$ is convex in (x_1, x_2).

Solution: $\mathcal{L}$ is convex in (x_1, x_2) because it is the sum of convex functions.

(e) Denoting $\mathcal{X} = \{(x_1, x_2) \mid 2x_1 + x_2 \geq 1,\ x_1 + 3x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\}$, show that

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \quad (9)$$

Solution: Let's just do it for λ_4:

$$\max_{\lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \max_{\lambda_4 \geq 0} (f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 - \lambda_4 x_2) \tag{10}$$

$$= f(x_1, x_2) + \lambda_1(-2x_1 - x_2 + 1) + \lambda_2(1 - x_1 - 3x_2) - \lambda_3 x_1 + \max_{\lambda_4 \geq 0} -\lambda_4 x_2 \tag{11}$$

$$\max_{\lambda_4 \geq 0} -\lambda_4 x_2 = \begin{cases} 0 & \text{if } x_2 \geq 0 \\ +\infty & \text{otherwise} \end{cases} \tag{12}$$

One can show the same results for λ_1, λ_2 and λ_3, resulting in:

$$\max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{13}$$

(f) Conclude that $\min_{(x_1,x_2) \in \mathcal{X}} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{(x_1,x_2) \in \mathcal{X}} f(x_1, x_2)$.

Solution:

$$\min_{x_1, x_2} \max_{\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \min_{x_1, x_2} \begin{cases} f(x_1, x_2) & \text{if } (x_1, x_2) \in \mathcal{X} \\ +\infty & \text{otherwise} \end{cases} \tag{14}$$

$$= \min_{(x_1, x_2) \in \mathcal{X}} f(x_1, x_2) \tag{15}$$

(g) Assuming f is convex, formulate the first order condition on $\mathcal{L}$ as a function of ∇f and $\lambda_1, \lambda_2, \lambda_3$ and λ_4 to solve:

$$\min_{x_1, x_2} \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \tag{16}$$

Solution:

$$\nabla_{x_1, x_2} \mathcal{L}(x_1^\star, x_2^\star, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = 0 \tag{17}$$

$$= \nabla_{x_1, x_2} f(x_1^\star, x_2^\star) + \begin{bmatrix} -2\lambda_1 - \lambda_2 - \lambda_3 \\ -\lambda_1 - 3\lambda_2 - \lambda_4 \end{bmatrix} \tag{18}$$

2. Lagrangian Dual of a QP

Consider the general form of a convex quadratic program, with $Q \succ 0$:

$$\min_{\vec{x}} \quad \frac{1}{2} \vec{x}^\top Q \vec{x} \tag{19}$$

$$\text{s.t.} \quad A\vec{x} \leq \vec{b} \tag{20}$$

(a) Write the Lagrangian function $\mathcal{L}(\vec{x}, \vec{\lambda})$.

Solution:

$$\mathcal{L}(\vec{x}, \vec{\lambda}) = \frac{1}{2} \vec{x}^\top Q \vec{x} + \vec{\lambda}^\top (A\vec{x} - \vec{b}) \tag{21}$$

(b) Write the Lagrangian dual function, $g(\vec{\lambda})$.

Solution:

$$g(\vec{\lambda}) = \inf_{\vec{x}} \ \mathcal{L}(\vec{x}, \vec{\lambda}) \tag{22}$$

We can find this infimum by setting $\nabla_{\vec{x}} \mathcal{L}(\vec{x}^*, \vec{\lambda}) = 0$:

$$Q\vec{x}^* + A^\top \vec{\lambda} = 0 \implies \vec{x}^* = -Q^{-1} A^\top \vec{\lambda} \tag{23}$$

Substituting, we get

$$\begin{aligned} g(\vec{\lambda}) &= \mathcal{L}(\vec{x}^*, \vec{\lambda}) && (24)\\ &= \frac{1}{2}\vec{\lambda}^\top A Q^{-\top} A^\top \vec{\lambda} - \vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b} && (25)\\ &= -\frac{1}{2}\vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b} && (26) \end{aligned}$$

(c) Show that the Lagrangian dual problem is convex by writing it in standard QP form. Is the Lagrangian dual problem convex in general?

Solution: The Lagrangian dual problem writes

$$\max_{\vec{\lambda} \geq 0} \ g(\vec{\lambda}) = \max_{\vec{\lambda} \geq 0} \ -\frac{1}{2}\vec{\lambda}^\top A Q^{-1} A^\top \vec{\lambda} - \vec{\lambda}^\top \vec{b}, \tag{27}$$

the maximization of a concave function of $\vec{\lambda}$ over the convex region given by the non-negative orthant $\vec{\lambda} \geq 0$. The dual problem is therefore convex.

While in this problem, the primal problem was convex, it turns out that the Lagrangian dual problem is a convex problem even when the primal is not. To see this, examine its general form:

$$\max_{\vec{\lambda} \geq 0} \min_{\vec{x}} \mathcal{L}(\vec{x}, \vec{\lambda}) = \max_{\vec{\lambda} \geq 0} \min_{\vec{x}} \left[f_0(\vec{x}) + \sum_{i=1}^{n} \lambda_i f_i(\vec{x}) \right] \tag{28}$$

This represents the pointwise minimum of affine functions of $\vec{\lambda}$, which we know to be concave. The resulting maximization problem of a concave objective in $\vec{\lambda}$ over the convex region $\vec{\lambda} \geq 0$ is then a convex optimization problem!