Discussion 1

1. Eigenvalues

Let $A \in \mathbb{R}^{n\times n}$ have the eigendecomposition $P\Lambda P^{-1}$ where $\Lambda \in \mathbb{R}^{n\times n}$ is a diagonal matrix with entries consisting of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $P \in \mathbb{R}^{n\times n}$ is an invertible matrix. Note that this is equivalent to stating that A is diagonalizable via the transformation,

$$P^{-1}AP = \Lambda. \tag{1}$$

(a) Show that $A^m = P\Lambda^m P^{-1}$, for integer $m \geq 1$.

(b) Show that determinant of A is the product of its eigenvalues, i.e.

$$\det(A) = \prod_{i=1}^{n} \lambda_i. \tag{2}$$

HINT: We have the identity $\det(XY) = \det(X)\det(Y)$.

2. Invertibility of $A^\top A$

In this problem, we show that if the matrix $A \in \mathbb{R}^{m \times n}$ has a full column rank, then the matrix $A^\top A$ is invertible.

(a) Show that if a vector $\vec{x}$ is in the null space of A then $\vec{x}$ is in the null space of $A^\top A$.

(b) Conversely, show that if $\vec{x}$ is in the null space of $A^\top A$ then $\vec{x}$ is in the null space of A.

(c) Given that matrix A has a full column rank, what can you say about its null space? What does this imply about the null space and invertibility of the matrix $A^\top A$?

3. Least Squares and Gram-Schmidt

Consider the least squares problem

$$\vec{x}^\star = \underset{\vec{x}\in\mathbb{R}^n}{\operatorname{argmin}} \left\| A\vec{x} - \vec{b} \right\|_2^2 \tag{3}$$

where $A \in \mathbb{R}^{m\times n}, \vec{b} \in \mathbb{R}^m$ and assume A is full column rank. One way to solve this least-squares problem is to use Gram-Schmidt Orthonormalization (GSO). Using GSO, the matrix A can be written as,

$$A = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \tag{4}$$

where Q is an orthonormal matrix and R is an upper-triangular matrix. The columns of Q_1 form an orthonormal basis for the range space $\mathcal{R}(A)$ and columns of Q_2 form an orthonormal basis for the range space $\mathcal{R}(A)^\perp$. Moreover, R_1 is upper triangular and invertible.

(a) Show that the squared norm of the residual is given by

$$\|\vec{r}\|_2^2 := \left\| \vec{b} - A\vec{x} \right\|_2^2 = \left\| Q_1^\top \vec{b} - R_1\vec{x} \right\|_2^2 + \left\| Q_2^\top \vec{b} \right\|_2^2. \tag{5}$$

(b) Find $\vec{x}^\star$ such that the squared norm of the residual in Equation (5) is minimized. Your expression for $\vec{x}^\star$ should only use some or all of the following terms: Q_1, Q_2, R_1, $\vec{b}$.

(c) Check if the expression for $\vec{x}^\star$ obtained in the previous part is equivalent to the one obtained by the formula, $\vec{x}^\star = (A^\top A)^{-1}A^\top\vec{b}$.

