Homework 0

This homework is *optional* and consists of review problems. It will be run like a regular homework, but will not be graded, so you may also use it to familiarize yourself with the submission and self-grading system.

Self grades are due at 11 PM on January 26, 2023.

1. **Subspaces and Dimensions**

   Consider the set $\mathcal{S}$ of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$x_1 + 2x_2 + 3x_3 = 0, \quad 3x_1 + 2x_2 + x_3 = 0. \tag{1}$$

   (a) Find a 2×3 matrix A for which $\mathcal{S}$ is exactly the null space of A.

   Solution: Recall the definition of the null space of a matrix A as the set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$. The equations

$$x_1 + 2x_2 + 3x_3 = 0 \tag{2}$$
$$3x_1 + 2x_2 + x_3 = 0 \tag{3}$$

   can be written in matrix-vector form as

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

   The set of $\vec{x} = \begin{bmatrix} x_1, x_2, x_3 \end{bmatrix}^\top$ which satisfy this equation form the null space of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$. This is the matrix we are looking for.

   (b) Determine the dimension of $\mathcal{S}$ and find a basis for it.

   Solution: Recall the definitions of a basis and the dimension of a subspace, which are related. A basis for a space is a set of linearly independent vectors that span the space. The dimension of this space is then the number of vectors in the basis.

   To find the dimension, we solve the equation and find that any solution to the equations is of the form $x_1 = x_3$, $x_2 = -2x_3$, where x_3 is free. Thus, the solutions are of the form $\begin{bmatrix} 1, -2, 1 \end{bmatrix}^\top u$ for $u \in \mathbb{R}$, and so $\mathcal{S} = \operatorname{span}\left(\begin{bmatrix} 1, -2, 1 \end{bmatrix}^\top\right)$. Hence, the dimension of $\mathcal{S}$ is 1, and a basis for $\mathcal{S}$ is the vector $\begin{bmatrix} 1, -2, 1 \end{bmatrix}^\top$.

2. Orthogonality

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ be two linearly independent unit-norm vectors; that is, $\|\vec{x}\|_2 = \|\vec{y}\|_2 = 1$.

(a) Show that the vectors $\vec{u} = \vec{x} - \vec{y}$ and $\vec{v} = \vec{x} + \vec{y}$ are orthogonal.

Solution: Orthogonal means dot product is 0. When x, y are both unit-norm, we have

$$(\vec{x} - \vec{y})^\top (\vec{x} + \vec{y}) = \vec{x}^\top \vec{x} - \vec{y}^\top \vec{y} - \vec{y}^\top \vec{x} + \vec{x}^\top \vec{y} = \vec{x}^\top \vec{x} - \vec{y}^\top \vec{y} = 0, \tag{5}$$

(b) Find an orthonormal basis for $\operatorname{span}(\vec{x}, \vec{y})$, the subspace spanned by $\vec{x}$ and $\vec{y}$.

Solution: Since $\vec{x}$ and $\vec{y}$ are linearly independent, we have $\vec{x} \neq \vec{y}$ and $\vec{x} \neq -\vec{y}$ (since they are both unit norm). Thus $\vec{u}$ and $\vec{v}$ are nonzero.

Motivated by the first part that asks us to show $\vec{u}$ and $\vec{v}$ are orthogonal, we first see that $\vec{u}, \vec{v}$ form an orthogonal basis for $\operatorname{span}(\vec{u}, \vec{v})$.

The subspace spanned by $\vec{x}, \vec{y}$ is $S_1 = \operatorname{span}(\vec{x}, \vec{y})$, We want to check if $S_2 = \operatorname{span}(\vec{u}, \vec{v})$ is the same set as S_1.

If this is true then $\vec{u}$ and $\vec{v}$ are orthogonal basis vectors for $\operatorname{span}(\vec{x}, \vec{y})$.

First we show $S_1 \subseteq S_2$ by checking that $\vec{z} \in S_1 \implies \vec{z} \in S_2$.

We can express any vector $\vec{z} \in \operatorname{span}(\vec{x}, \vec{y})$ as $\vec{z} = \lambda\vec{x} + \mu\vec{y}$, for some $\lambda, \mu \in \mathbb{R}$. We have $\vec{z} = \alpha\vec{u} + \beta\vec{v}$, where

$$\alpha = \frac{\lambda - \mu}{2}, \quad \beta = \frac{\lambda + \mu}{2}. \tag{6}$$

Hence $\vec{z} \in \operatorname{span}(\vec{u}, \vec{v})$. The converse is also true for similar reasons.

We can find orthonormal basis vectors by dividing each orthogonal basis vector by its norm. The desired orthonormal basis is thus given by $((\vec{x} - \vec{y}) / \|\vec{x} - \vec{y}\|_2, (\vec{x} + \vec{y}) / \|\vec{x} + \vec{y}\|_2)$.

We could have also gotten that $\operatorname{span}(\vec{x}, \vec{y}) = \operatorname{span}(\vec{u}, \vec{v})$ in a slightly faster way by noting that

$$\vec{u} = \vec{x} - \vec{y}, \quad \vec{v} = \vec{x} + \vec{y}, \quad \vec{x} = \frac{\vec{u} + \vec{v}}{2}, \quad \vec{y} = \frac{\vec{v} - \vec{u}}{2} \tag{7}$$

so linear combinations of $\vec{u}$ and $\vec{v}$ are linear combinations of $\vec{x}$ and $\vec{y}$, and vice versa. Thus the spans are the same, e.g., $\operatorname{span}(\vec{x}, \vec{y}) = \operatorname{span}(\vec{u}, \vec{v})$. And the solution proceeds from there in the same way.

3. **Homework Process**

   With whom did you work on this homework? List the names and SIDs of your group members.

   NOTE: If you didn't work with anyone, you can put "none" as your answer.

