EECS 127/227AT | Optimization Models in Engineering | UC Berkeley | Spring 2023

Discussion 9

1. Magic with constraints

In this question, we will represent a problem in two different ways and show that strong duality holds in one case but doesn't hold in the other.

Let

$$f_0(x) \doteq \begin{cases} x^3 - 3x^2 + 4, & x \geq 0 \\ -x^3 - 3x^2 + 4, & x < 0 \end{cases}.$$

1) Consider the minimization problem

$$p^* = \min_{x \in \mathbb{R}} \; f_0(x) \tag{1}$$

$$\text{s.t.} \; -1 \leq x, \;\; x \leq 1. \tag{2}$$

(a) Show that $f_0(x)$ is differentiable everywhere and compute it's derivative.

Solution: By differentiability of polynomials, $f_0(x)$ is differentiable everywhere except possibly at $x = 0$. We show that $f_0(x)$ is in fact differentiable everywhere by taking the right and left derivatives at $x = 0$ and showing that they are equivalent.

We can calculate these right and left derivatives as follows. For $h > 0$, the right derivative at $x = 0$ is given by

$$\begin{aligned} &\lim_{h \to 0} \frac{f_0(0+h) - f_0(0)}{h} \\ &= \lim_{h \to 0} \frac{h^3 - 3h^2 + 4 - 4}{h} \\ &= 0. \end{aligned}$$

Similarly, for $h > 0$, the left derivative at $x = 0$ is given by

$$\begin{aligned} &\lim_{h \to 0} \frac{f_0(0) - f_0(0-h)}{h} \\ &= \lim_{h \to 0} \frac{4 - h^3 + 3h^2 - 4}{h} \\ &= 0. \end{aligned}$$

Thus, f_0 is differentiable everywhere. The derivative is

$$\frac{df_0(x)}{dx} = \begin{cases} 3x^2 - 6x, & x \geq 0 \\ -3x^2 - 6x, & x < 0 \end{cases}.$$

(b) Show that $p^* = 2$ and the the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$ by examining the "critical" points, i.e., points where the gradient is zero, points on the boundaries, and $\pm\infty$.

Solution: Since, $f_0(x)$ is differentiable everywhere, the minimum must be achieved at a boundary point or when the derivative is 0. From the previous part, we showed that $f_0'(0) = 0$ so, $x = 0$ is one of the critical points.

Next, we calculate the remaining critical points at which the derivative is zero:

$$\frac{d}{dx} f_0(x) = \begin{cases} 3x^2 - 6x, & x \geq 0 \\ -3x^2 - 6x, & x < 0 \end{cases} = 0 \Rightarrow x \in \{0, \pm 2\}$$

We now have a list of all critical points to test: $x \in \{0, \pm 2\}$ (where the derivative is 0), $x = \pm 1$ (constraint boundaries), and $x = \pm\infty$. The only critical points that fall within our constraints are $x \in \{0, \pm 1\}$, so we examine the function at these 3 points:

$$\begin{aligned} f_0(1) &= f_0(-1) = 2 \\ f_0(0) &= 4. \end{aligned}$$

Thus, $p^* = 2$ and $\mathcal{X}^* = \{-1, 1\}$.

(c) Show that the dual problem can be represented as

$$d^* = \max_{\lambda_1, \lambda_2 \geq 0} \; g(\vec{\lambda}),$$

where

$$g(\vec{\lambda}) = \min\left\{g_1(\vec{\lambda}), g_2(\vec{\lambda})\right\},$$

with

$$\begin{aligned} g_1(\vec{\lambda}) &= \min_{x \geq 0} x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1) \\ g_2(\vec{\lambda}) &= \min_{x < 0} -x^3 - 3x^2 + 4 - \lambda_1(x+1) + \lambda_2(x-1). \end{aligned}$$

Solution:

The Lagrangian is given by

$$\mathcal{L}(x, \vec{\lambda}) = f_0(x) + \lambda_1(-x-1) + \lambda_2(x-1).$$

The dual function $g(\vec{\lambda})$ is then given by

$$\begin{aligned} g(\vec{\lambda}) &= \min_{x} \mathcal{L}(x, \vec{\lambda}) \\ &= \min\left\{\min_{x \geq 0} \mathcal{L}(x, \vec{\lambda}),\ \min_{x<0} \mathcal{L}(x, \vec{\lambda})\right\} \\ &= \min\left\{g_1(\vec{\lambda}), g_2(\vec{\lambda})\right\} \end{aligned}$$

for the given $g_1(\vec{\lambda})$ and $g_2(\vec{\lambda})$, as desired.

(d) Next, show that

$$\begin{aligned} g_1(\vec{\lambda}) &\leq -3\lambda_1 + \lambda_2 \\ g_2(\vec{\lambda}) &\leq \lambda_1 - 3\lambda_2. \end{aligned}$$

Use this to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$.

Solution: Because $g_1(\vec{\lambda})$ is the minimum over all $x \geq 0$ of $\mathcal{L}(x, \vec{\lambda})$, it is less than or equal to any instantiation of $\mathcal{L}(x, \vec{\lambda})$ at a particular value of $x \geq 0$. Thus, for instantiation $x = 2$, we can write

$$\begin{aligned} g_1(\vec{\lambda}) &= \min_{x \geq 0} \mathcal{L}(x, \vec{\lambda}) \\ &\leq \mathcal{L}(2, \vec{\lambda}) \\ &= \lambda_1 - 3\lambda_2 \end{aligned}$$

as desired. Analogously, we can instantiate $g_2(\vec{\lambda})$ at $x = -2$ and write

$$\begin{aligned} g_2(\vec{\lambda}) &= \min_{x<0} \mathcal{L}(x, \vec{\lambda}) \\ &\leq \mathcal{L}(-2, \vec{\lambda}) \\ &= \lambda_1 - 3\lambda_2, \end{aligned}$$

giving us the two desired inequalities.

We now use these inequalities to show that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$. Since $g(\vec{\lambda})$ is the minimization of $g_1(\vec{\lambda})$ and $g_2(\vec{\lambda})$, we can use the upper bounds we just established to write

$$\begin{aligned} g(\vec{\lambda}) &= \min \left\{ g_1(\vec{\lambda}), g_2(\vec{\lambda}) \right\} \\ &\leq \min \left\{ -3\lambda_1 + \lambda_2, \lambda_1 - 3\lambda_2 \right\} \\ &\leq 0. \end{aligned}$$

The last inequality follows from a subtle relationship between the two expressions over which we are minimizing. First, note that it is sufficient to show that either $-3\lambda_1 + \lambda_2$ or $\lambda_1 - 3\lambda_2$ must be negative, since $g(\vec{\lambda})$ is determined by the minimum of the two values. Consider the case in which $-3\lambda_1 + \lambda_2 \geq 0$, i.e., $\lambda_2 \geq 3\lambda_1$; this implies that the second expression $\lambda_1 - 3\lambda_2 \leq 0$, so $g(\vec{\lambda}) \leq 0$ holds. Alternatively, if $\lambda_1 - 3\lambda_2 > 0$, i.e., $\lambda_1 > 3\lambda_2$, then the first expression $-3\lambda_1 + \lambda_2 < 0$, so $g(\vec{\lambda}) \leq 0$ holds. Thus, as these cases are exhaustive, $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$ as desired.

(e) Show that $g(\vec{0}) = 0$ and conclude that $d^* = 0$.

Solution: In part (d), we proved that $g(\vec{\lambda}) \leq 0$ for all $\lambda_1, \lambda_2 \geq 0$. Since d^* is the maximum over all feasible values of $g(\vec{\lambda})$, it is sufficient to show that there exists a $\vec{\lambda}$ for which this upper bound is attained.

Toward this objective, consider g at $\vec{\lambda} = 0$:

$$\begin{aligned} g(\vec{0}) &= \min \left\{ g_1(\vec{0}), g_2(\vec{0}) \right\} \\ &= \min \left\{ \min_{x \geq 0} x^3 - 3x^2 + 4,\ \min_{x<0} -x^3 - 3x^2 + 4 \right\} \\ &= \min \{0, 0\} \\ &= 0. \end{aligned}$$

Note that the third equality can be shown by examining the critical points of each objective function, which are the same as those of the unconstrained primal function in part (b); this minimum is achieved at $x = \pm 2$.

We can now conclude that the maximum possible value of the dual (i.e., zero) is attained for $\vec{\lambda} = \vec{0}$, and thus $d^* = 0$ as desired.

(f) Does strong duality hold?

Solution: Since $d^* = 0 < 2 = p^*$, strong duality does not hold. This is not surprising, since the objective function $f_0(x)$ is non-convex.

2) Now, consider a problem equivalent to the minimization in (1):

$$p^* = \min_{x \in \mathbb{R}} \quad f_0(x) \tag{3}$$

$$\text{s.t. } x^2 \leq 1. \tag{4}$$

Observe that $p^* = 2$ and the set of optimizers $x \in \mathcal{X}^*$ is $\mathcal{X}^* = \{-1, 1\}$, since this problem is equivalent to the one in part 1).

(a) Show that the dual problem can be represented as

$$d^* = \max_{\lambda \geq 0} \; g(\lambda),$$

where

$$g(\lambda) = \min(g_1(\lambda), g_2(\lambda)),$$

with

$$\begin{aligned} g_1(\lambda) &= \min_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \\ g_2(\lambda) &= \min_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1). \end{aligned}$$

Solution: This solution is identical in strategy to that in part (c) The Lagrangian is given by

$$\mathcal{L}(x, \lambda) = f_0(x) + \lambda(x^2 - 1).$$

The dual function $g(\lambda)$ is then given by

$$\begin{aligned} g(\lambda) &= \min_x \mathcal{L}(x, \lambda) \\ &= \min \left\{ \min_{x \geq 0} \mathcal{L}(x, \lambda), \min_{x < 0} \mathcal{L}(x, \lambda) \right\} \\ &= \min \{g_1(\lambda), g_2(\lambda)\} \end{aligned}$$

for the given $g_1(\lambda)$ and $g_2(\lambda)$, as desired.

(b) Show that $g_1(\lambda) = g_2(\lambda) = \begin{cases} 4 - \lambda, & \lambda \geq 3 \\ -\frac{4}{27}(3 - \lambda)^3 + 4 - \lambda, & 0 \leq \lambda < 3. \end{cases}$

Solution: We first show that $g_2(\vec{\lambda}) = g_1(\vec{\lambda})$:

$$\begin{aligned} g_2(\lambda) &= \min_{x < 0} -x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \\ &= \min_{-x > 0} (-x)^3 - 3(-x)^2 + 4 + \lambda((-x)^2 - 1) \\ &= \min_{x \geq 0} x^3 - 3x^2 + 4 + \lambda(x^2 - 1) \\ &= g_1(\lambda). \end{aligned}$$

The last equality follows from a change in the variable over which we compute the minimum ($-x$ to x), which does not affect the value of the minimum. Note also that we have added the point $x = 0$ as a feasible point by amending our constraint from $-x > 0$ to $-x \geq 0$; this does not affect the value of the minimum either, since we do not require it to be attained as we do when minimizing.

Next, let us compute $g_1(\lambda)$ directly. Setting the derivative of g_1's objective function with respect to x to zero, we have

$$3x^2 - 2(3 - \lambda)x = 0 \implies x = 0 \text{ or } x = \frac{2}{3}(3 - \lambda).$$

We now consider all critical points of g_1's objective function: $x \in \{0, \frac{2}{3}(3 - \lambda)\}$ (where the derivative is 0) and $x \in \{0, \infty\}$ (boundary points).

First, suppose $\lambda \geq 3$. In this case, $x = \frac{2}{3}(3-\lambda)$ is no longer in the range $x \geq 0$, so we need only check boundary points $x = 0$ and $x = \infty$. As $x \to \infty$, the function value also approaches infinity, so the minimum is attained at $x = 0$, and thus $g_1(\lambda) = 4 - \lambda$.

Next, assume $0 \leq \lambda < 3$. In this case, we must check the function value at $x = 0$, $x = \frac{2}{3}(3-\lambda)$, and $x = \infty$ to determine where the minimum is attained. As previously established, the function approaches infinity as $x \to \infty$, so we need only compare the values $4 - \lambda$ (at $x = 0$) and $-\frac{4}{27}(3-\lambda)^3 + 4 - \lambda$ at $x = \frac{2}{3}(3-\lambda)$. Since $3 - \lambda > 0$, we know that $-\frac{4}{27}(3-\lambda)^3$ is always negative, and thus the minimum is $-\frac{4}{27}(3-\lambda)^3 + 4 - \lambda$.

Combining the two cases above yields the desired expression for $g_1(\lambda) = g_2(\lambda)$.

(c) Conclude that $d^* = 2$ and the optimal $\lambda = \frac{3}{2}$.

Solution: Since $g_1(\lambda) = g_2(\lambda)$, we have $g(\lambda) = g_1(\lambda) = g_2(\lambda)$. We examine each range of possible λ values in turn to determine the maximum.

For $\lambda \geq 3$, the maximum value of $g(\lambda) = 1$ is achieved at $\lambda = 3$.

For $0 \leq \lambda < 3$, the maximum of $g(\lambda)$ is computed as follows. First, we set the derivative of $g(\lambda)$ with respect to λ to 0:

$$\frac{12}{27}(3-\lambda)^2 - 1 = 0 \implies (3-\lambda)^2 = \frac{9}{4} \implies \lambda = \frac{3}{2} \text{ or } \lambda = \frac{9}{2}.$$

Since the expression is valid only for $0 \leq \lambda < 3$, we examine values at $\lambda \in \{0, 3\}$ (boundary points) and at the computed $\lambda = \frac{3}{2}$. We observe that the maximum is achieved at $\lambda = \frac{3}{2}$ with $g(\frac{3}{2}) = 2$.

Finally, we note that the overall maximum occurs in the second case, at $\lambda = \frac{3}{2}$, and thus $d^* = 2$ as desired.

(d) Does strong duality hold?

Solution: In this case, $p^* = 2 = d^*$, so strong duality holds.