Discussion 11

1. The Duality of the ℓ_1 and ℓ_∞ Norms

For this problem, we will prove the duality of the ℓ_1 and ℓ_∞ norms. Recall that the ℓ_1 and ℓ_∞ norms, denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively, are defined as follows for $\vec{x} \in \mathbb{R}^n$:

$$\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|\,, \quad \|\vec{x}\|_\infty = \max_{i\in[n]} |x_i|\,. \tag{1}$$

We will show that the ℓ_1 and ℓ_∞ norms are duals of each other; that is, we will show that:

$$\|\vec{x}\|_1 = \max_{\|\vec{y}\|_\infty=1} \vec{y}^\top \vec{x} \text{ and } \|\vec{x}\|_\infty = \max_{\|\vec{y}\|_1=1} \vec{y}^\top \vec{x}. \tag{2}$$

(a) To start, we will first prove the following inequality for all $\vec{x}, \vec{y} \in \mathbb{R}^n$:

$$\vec{x}^\top \vec{y} \leq \|\vec{x}\|_1 \|\vec{y}\|_\infty\,. \tag{3}$$

(b) Now, show that:

$$\max_{\|\vec{y}\|_\infty=1} \vec{y}^\top \vec{x} \geq \|\vec{x}\|_1 \tag{4}$$

and using (a), conclude that $\|\vec{x}\|_1 = \max_{\|\vec{y}\|_\infty=1} \vec{y}^\top \vec{x}$.

(c) Finally, show the following inequality:

$$\max_{\|\vec{y}\|_1=1} \vec{y}^\top \vec{x} \geq \|\vec{x}\|_\infty \tag{5}$$

and prove the second equality.

2. Sphere Enclosure

Let B_i, $i = 1, \ldots, m$, be m Euclidean balls in $\mathbb{R}^n$, with centers $\vec{x}_i$, and radii $\rho_i \geq 0$. We wish to find a ball B of minimum radius that contains all the B_i, $i = 1, \ldots, m$. Cast this problem as an SOCP.

3. A review of standard problem formulations

In this question, we review conceptually the standard forms of various problems and the assertions we can (and cannot!) make about each.

(a) **Linear programming (LP).**

(a) Write the most general form of a linear program (LP) and list its defining attributes.

(b) Under what conditions is an LP convex?

(b) **Quadratic programming (QP).**

(a) Write the most general form of a quadratic program (QP) and list its defining attributes.

(b) Under what conditions is a QP convex?

(c) **Quadratically-constrained quadratic programming (QCQP).**

(a) Write the most general form of a quadratically-constrained quadratic program (QCQP) and list its defining attributes.

(b) Under what conditions is a QCQP convex?

(d) **Second-order cone programming (SOCP).**

(a) Write the most general form of a second-order cone program (SOCP) and list its defining attributes.

(b) Under what conditions is an SOCP convex?

(e) **Relationships.** Recall that

$$LP \subset QP_{\text{convex}} \subset QCQP_{\text{convex}} \subset SOCP \subset \{\text{all convex programs}\}, \tag{6}$$

where LP denotes the set of all linear programs, QP_{convex} denotes the set of all convex quadratic programs, etc. Which of these problems can be solved most efficiently? Why are these categorizations useful?

