Discussion 11

1. **The Duality of the ℓ_1 and ℓ_∞ Norms**

For this problem, we will prove the duality of the ℓ_1 and ℓ_∞ norms. Recall that the ℓ_1 and ℓ_∞ norms, denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively, are defined as follows for $\vec{x} \in \mathbb{R}^n$:

$$\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|\vec{x}\|_\infty = \max_{i\in[n]} |x_i|. \tag{1}$$

We will show that the ℓ_1 and ℓ_∞ norms are duals of each other; that is, we will show that:

$$\|\vec{x}\|_1 = \max_{\|\vec{y}\|_\infty=1} \vec{y}^\top\vec{x} \text{ and } \|\vec{x}\|_\infty = \max_{\|\vec{y}\|_1=1} \vec{y}^\top\vec{x}. \tag{2}$$

(a) To start, we will first prove the following inequality for all $\vec{x}, \vec{y} \in \mathbb{R}^n$:

$$\vec{x}^\top\vec{y} \le \|\vec{x}\|_1 \|\vec{y}\|_\infty. \tag{3}$$

Solution: Note that when $\vec{x} = 0$ or $\vec{y} = 0$, the left hand side of the inequality is 0 while the right hand side is non-negative. This proves the inequaliy when either $\vec{x}$ or $\vec{y}$ is 0. Now, assume that $\vec{x}, \vec{y} \neq 0$. We now show the inequality by the following series of inequalities:

$$\vec{x}^\top\vec{y} = \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} |x_i| |y_i| \le \sum_{i=1}^{n} |x_i| \|\vec{y}\|_\infty \le \|\vec{y}\|_\infty \sum_{i=1}^{n} |x_i| = \|\vec{x}\|_1 \|\vec{y}\|_\infty. \tag{4}$$

(b) Now, show that:

$$\max_{\|\vec{y}\|_\infty=1} \vec{y}^\top\vec{x} \ge \|\vec{x}\|_1 \tag{5}$$

and using (a), conclude that $\|\vec{x}\|_1 = \max_{\|\vec{y}\|_\infty=1} \vec{y}^\top\vec{x}$.

Solution: Let $\vec{y}$ be defined as $y_i = \text{sign}(x_i)$. Note that $\|\vec{y}\|_\infty = 1$. Therefore, we have:

$$\vec{y}^\top\vec{x} = \sum_{i=1}^{n} y_i x_i = \sum_{i=1}^{n} |x_i| = \|\vec{x}\|_1. \tag{6}$$

Therefore, we can conclude that $\max_{\|\vec{x}\|_\infty=1} \vec{y}^\top\vec{x} \ge \|\vec{x}\|_1$. Note that from (a), we we also get $\max_{\|\vec{x}\|_\infty=1} \vec{y}^\top\vec{x} \le \|\vec{x}\|_1$. From both of these inequalities, we have:

$$\|\vec{x}\|_1 = \max_{\|\vec{y}\|_\infty=1} \vec{y}^\top\vec{x}. \tag{7}$$

(c) Finally, show the following inequality:

$$\max_{\|\vec{y}\|_1=1} \vec{y}^\top \vec{x} \geq \|\vec{x}\|_\infty \tag{8}$$

and prove the second equality.

Solution: Let $i^* = \operatorname{argmax}_{i\in[n]} |x_i|$. Note that $|x_{i^*}| = \|\vec{x}\|_\infty$. Now, define $\vec{y}$ as:

$$y_i = \begin{cases} 0 & i \neq i^* \\ \operatorname{sign}(x_{i^*}) & \text{o.w} \end{cases}. \tag{9}$$

Notice that $\|\vec{y}\|_1 = 1$. With $\vec{y}$, we have:

$$\vec{y}^\top \vec{x} = |x_{i^*}| = \|\vec{x}\|_\infty. \tag{10}$$

From this, we get that $\max_{\|\vec{y}\|_1=1} \vec{y}^\top \vec{x} \geq \|\vec{x}\|_\infty$. From (a), we again obtain $\max_{\|\vec{y}\|_1=1} \vec{y}^\top \vec{x} \leq \|\vec{x}\|_\infty$. The previous two inequalities allow us to conclude:

$$\max_{\|\vec{y}\|_1=1} \vec{y}^\top \vec{x} = \|\vec{x}\|_\infty. \tag{11}$$

2. **Sphere Enclosure**

Let B_i, $i = 1, \ldots, m$, be m Euclidean balls in $\mathbb{R}^n$, with centers $\vec{x}_i$, and radii $\rho_i \geq 0$. We wish to find a ball B of minimum radius that contains all the B_i, $i = 1, \ldots, m$. Cast this problem as an SOCP.

Solution: Let $\vec{c} \in \mathbb{R}^n$ and $r \geq 0$ denote the center and radius of the enclosing ball B, respectively. We express the given balls B_i as

$$B_i = \{\vec{x} : \vec{x} = \vec{x}_i + \vec{\delta}_i, \ \left\|\vec{\delta}_i\right\|_2 \leq \rho_i\}, \quad i = 1, \ldots, m. \tag{12}$$

We have that $B_i \subseteq B$ if and only if

$$\max_{\vec{x}\in B_i} \|\vec{x} - \vec{c}\|_2 \leq r. \tag{13}$$

Note that

$$\max_{\vec{x}\in B_i} \|\vec{x} - \vec{c}\|_2 = \max_{\|\vec{\delta}_i\|_2 \leq \rho_i} \left\|\vec{x}_i - \vec{c} + \vec{\delta}_i\right\|_2 = \|\vec{x}_i - \vec{c}\|_2 + \rho_i. \tag{14}$$

The last step follows by choosing $\vec{\delta}_i$ in the direction of $\vec{x}_i - \vec{c}$. The problem is then cast as the following SOCP

$$\min_{\vec{c},r} \quad r \tag{15}$$

$$\text{s.t.} \quad \|\vec{x}_i - \vec{c}\|_2 + \rho_i \leq r, \qquad i = 1, \ldots, m. \tag{16}$$

3. **A review of standard problem formulations**

In this question, we review conceptually the standard forms of various problems and the assertions we can (and cannot!) make about each.

(a) **Linear programming (LP).**

(a) Write the most general form of a linear program (LP) and list its defining attributes.

Solution: A general LP can be written as

$$p^* = \min_{\vec{x} \in \mathbb{R}^n} \vec{c}^\top \vec{x} + d \tag{17}$$

$$\text{s.t. } A_{\text{eq}}\vec{x} = \vec{b}_{\text{eq}} \tag{18}$$

$$A\vec{x} \leq \vec{b}, \tag{19}$$

or equivalently,

$$p^* = \min_{\vec{x} \in \mathbb{R}^n} \vec{c}^\top \vec{x} + d \tag{20}$$

$$\text{s.t. } A_{\text{eq}}\vec{x} = \vec{b}_{\text{eq}} \tag{21}$$

$$\vec{x} \geq \vec{0}. \tag{22}$$

The first LP formulation is known as the **inequality form**; the second is known as the **conic form**. A full treatment of the equivalence of these forms and how to convert between them can be found in section 9.3 of Calafiore & El Ghaoui.

(b) Under what conditions is an LP convex?

Solution: An LP is **always convex**, as the objective function and all constraints are convex (affine) and all equality constraints are affine.

(b) **Quadratic programming (QP).**

(a) Write the most general form of a quadratic program (QP) and list its defining attributes.

Solution: A general QP can be written as

$$p^* = \min_{\vec{x} \in \mathbb{R}^n} \frac{1}{2}\vec{x}^\top H\vec{x} + \vec{c}^\top \vec{x} + d \tag{23}$$

$$\text{s.t. } A_{\text{eq}}\vec{x} = \vec{b}_{\text{eq}} \tag{24}$$

$$A\vec{x} \leq \vec{b}. \tag{25}$$

(b) Under what conditions is a QP convex?

Solution: A QP is convex if and only if $H \succeq 0$ (i.e., PSD).

(c) **Quadratically-constrained quadratic programming (QCQP).**

(a) Write the most general form of a quadratically-constrained quadratic program (QCQP) and list its defining attributes.

Solution: A general QCQP can be written as

$$p^* = \min_{\vec{x} \in \mathbb{R}^n} \vec{x}^\top H_0\vec{x} + 2\vec{c}^\top \vec{x} + d \tag{26}$$

$$\text{s.t. } \vec{x}^\top H_i\vec{x} + 2\vec{c}_i^\top \vec{x} + d_i \leq 0, \quad i = 1, \ldots, m \tag{27}$$

$$\vec{x}^\top H_j\vec{x} + 2\vec{c}_j^\top \vec{x} + d_j = 0, \quad j = 1, \ldots, q. \tag{28}$$

(b) Under what conditions is a QCQP convex?

Solution: A QCQP is convex if and only if all matrices H_0 and $H_i,\ i = 1, \ldots, m$ are PSD, and $H_j = 0$ for all $j = 1, \ldots, q$ (i.e., when the objective and all inequality constraints are convex quadratic, and all the equality constraints are actually affine.

(d) **Second-order cone programming (SOCP).**

(a) Write the most general form of a second-order cone program (SOCP) and list its defining attributes.

Solution: A general SOCP can be written as

$$p^* = \min_{\vec{x}\in\mathbb{R}^n} \vec{c}^\top \vec{x} \tag{29}$$
$$\text{s.t. } \|A_i\vec{x} + \vec{b}_i\|_2 \le \vec{c}_i^\top \vec{x} + d_i, \quad i = 1, \ldots, m, \tag{30}$$

or equivalently,

$$p^* = \min_{\vec{x}\in\mathbb{R}^n} \vec{c}^\top \vec{x} \tag{31}$$
$$\text{s.t. } (A_i\vec{x} + \vec{b}_i, \vec{c}_i^\top \vec{x} + d_i) \in \mathcal{K}_{m_i} \quad i = 1, \ldots, m, \tag{32}$$

where second-order cone (SOC) $\mathcal{K}_n \doteq \{(\vec{x}, t), \vec{x} \in \mathbb{R}^n, t \in \mathbb{R} \mid \|\vec{x}\|_2 \le t\}$. The first SOCP formulation is known as the **standard inequality form**; the second is known as the **conic standard form**.

(b) Under what conditions is an SOCP convex?

Solution: An SOCP is **always convex**, as the objective function is convex (linear) and the constraint stipulates that points lie within a convex set. A thorough discussion of the convexity of SOC constraints can be found in Calafiore & El Ghaoui chapter 10.

(e) **Relationships.** Recall that

$$LP \subset QP_{\text{convex}} \subset QCQP_{\text{convex}} \subset SOCP \subset \{\text{all convex programs}\}, \tag{33}$$

where LP denotes the set of all linear programs, QP_{convex} denotes the set of all convex quadratic programs, etc. Which of these problems can be solved most efficiently? Why are these categorizations useful?

Solution: In general, problems to the left side of the subset sequence above can be solved more efficiently than those on the right, for problems of comparable size: LPs are arguably easiest (optima are always achieved at a critical point, so we just need to check those analytically, e.g. using the simplex algorithm), while general convex problems must use numerical algorithms like gradient descent and Newton's method, whose efficiency depends on the particular geometric characteristics of the problem. Intermediate forms (convex QPs/QCQPs, SOCPs) fall somewhere in between in terms of difficulty: there's lots of research on how to solve them efficiently (one popular set of approaches is called "interior point methods"), and there are a number of off-the-shelf solvers available (e.g., CVX).

Though we don't get a chance to explore it much in this class, knowing these classes of problems is useful when encountering optimization problems in the wild — if you can write your problem in one of the forms above, you know what kinds of solutions and convergence guarantees you can expect, and often employ existing software to help you solve it. For further discussion of these problem classes — as well as an even more general class known as semidefinite programming (SDP), of which SOCPs are a subset — we encourage you to take EECS 227B and 227C.

