Discussion 6

1. Convexity of Functions

Definition. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is convex if $\mathrm{dom}(f)$ is a convex set and if for all $\vec{x}, \vec{y} \in \mathrm{dom}(f)$ and $\theta \in [0, 1]$, we have,

$$f(\theta\vec{x} + (1-\theta)\vec{y}) \le \theta f(\vec{x}) + (1-\theta) f(\vec{y}). \tag{1}$$

The function f is strictly convex if the inequality is strict.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave if $\mathrm{dom}(f)$ is a convex set and if for all $\vec{x}, \vec{y} \in \mathrm{dom}(f)$ and θ with $0 \le \theta \le 1$, we have,

$$f(\theta\vec{x} + (1-\theta)\vec{y}) \ge \theta f(\vec{x}) + (1-\theta) f(\vec{y}). \tag{2}$$

The function f is strictly concave if the inequality is strict.

Property. A function f is concave if and only if $-f$ is convex. An affine function is both convex and concave.

Property: Jensen's inequality. The inequality in Equation (1) is known as **Jensen's Inequality**. This can be extended to convex combinations of more than one point. If f is convex, and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathrm{dom}(f)$, and $\theta_1, \theta_2, \ldots, \theta_k \ge 0$ with $\sum_{i=1}^k \theta_i = 1$ then,

$$f(\theta_1\vec{x}_1 + \theta_2\vec{x}_2 + \cdots + \theta_k\vec{x}_k) \le \theta_1 f(\vec{x}_1) + \theta_2 f(\vec{x}_2) + \cdots + \theta_k f(\vec{x}_k). \tag{3}$$

Property: first order condition. Suppose f is differentiable. Then f is convex if and only if $\mathrm{dom}(f)$ is convex and

$$f(\vec{y}) \ge f(\vec{x}) + \nabla f(\vec{x})^\top (\vec{y} - \vec{x}), \tag{4}$$

for all $\vec{x}, \vec{y} \in \mathrm{dom}(f)$.

Property: Second order condition. Suppose f is twice differentiable. Then f is convex if and only if, $\mathrm{dom}(f)$ is convex and the Hessian of f, $\nabla^2 f(\vec{x})$, is positive semi-definite for all $\vec{x} \in \mathrm{dom}(f)$.

(a) **Restriction to a line**.

Show that a function f is convex if and only if for all $\vec{x} \in \mathrm{dom}(f)$ and all $\vec{v}$, the function $g : \mathrm{dom}(g) \to \mathbb{R}$ given by $g(t) = f(\vec{x} + t\vec{v})$ is convex for $\mathrm{dom}(g) = \{t \in \mathbb{R} \mid \vec{x} + t\vec{v} \in \mathrm{dom}(f)\}$.

Solution: In the first direction: assume f is convex and consider $\vec{x} \in \mathrm{dom}(f)$, $\vec{v}$ and the function $g : \mathrm{dom}(g) \to \mathbb{R}$ given by $g(t) = f(\vec{x} + t\vec{v})$ where $\mathrm{dom}(g) = \{t \in \mathbb{R} \mid \vec{x} + t\vec{v} \in \mathrm{dom}(f)\}$.

Because f is convex, $\mathrm{dom}(f)$ is convex, therefore $\mathrm{dom}(g)$ is also convex. For $t_1, t_2 \in \mathrm{dom}(g)$ and $\lambda \in [0, 1]$:

$$\begin{aligned}
g(\lambda t_1 + (1-\lambda)t_2) &= f(\vec{x} + (\lambda t_1 + (1-\lambda)t_2)\vec{v}) && (5)\\
&= f(\lambda(\vec{x} + t_1\vec{v}) + (1-\lambda)(\vec{x} + t_2\vec{v})) && (6)\\
&\le \lambda f(\vec{x} + t_1\vec{v}) + (1-\lambda) f(\vec{x} + t_2\vec{v}) && (7)\\
&= \lambda g(t_1) + (1-\lambda) g(t_2) && (8)
\end{aligned}$$

Therefore g is convex.

In the other direction: Consider $\vec{x}_1, \vec{x}_2 \in \text{dom}(f)$ and $\lambda \in [0,1]$. Define $g : t \to f(\vec{x}_2 + t(\vec{x}_1 - \vec{x}_2))$. g is convex and $0 \in \text{dom}(g)$ and $1 \in \text{dom}(g)$, so $[0,1] \in \text{dom}(g)$. Therefore $\lambda\vec{x}_1 + (1-\lambda)\vec{x}_2 \in \text{dom}(f)$ and $\text{dom}(f)$ is convex.

Because g is convex:

$$g(\lambda 1 + (1-\lambda)0) = g(\lambda) \leq \lambda g(1) + (1-\lambda)g(0) \tag{9}$$
$$f(\vec{x}_2 + \lambda(\vec{x}_1 - \vec{x}_2)) \leq \lambda f(\vec{x}_2 + 1(\vec{x}_1 - \vec{x}_2)) + (1-\lambda) f(\vec{x}_2 + 0(\vec{x}_1 - \vec{x}_2)) \tag{10}$$
$$f(\lambda\vec{x}_2 + (1-\lambda)\vec{x}_2) \leq \lambda f(\vec{x}_1) + (1-\lambda) f(\vec{x}_2) \tag{11}$$

Therefore f is convex.

(b) **Non-negative weighted sum**.

Show that the non-negative weighted sum of convex functions is convex: i.e. if $f_1, \ldots, f_n$ are n convex functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $w_1, \ldots, w_n \in \mathbb{R}_+$ are n positive scalars, then the function:

$$f = \sum_{i=1}^{n} w_i f_i \tag{12}$$

is convex. To make the question easier, you can assume that the functions $f_1, \ldots, f_n$ are twice-differentiable.

Solution: Check convexity by using the second order condition. First, the weighted sum of twice-differentiable function is also twice-differentiable:

$$\nabla^2 f = \nabla^2 \left(\sum_{i=1}^{n} w_i f_i \right) \tag{13}$$
$$= \sum_{i=1}^{n} w_i \nabla^2 f_i \qquad \text{(linearity of } \nabla^2\text{)} \tag{14}$$

Next we check that $\nabla^2 f$ is PSD.

$$\forall \vec{y}, \forall \vec{x} \quad \vec{y}^\top (\nabla^2 f(\vec{x}))\vec{y} = \vec{y}^\top (\sum_{i=1}^{n} w_i \nabla^2 f_i(\vec{x}))\vec{y} \tag{15}$$
$$= \sum_{i=1}^{n} w_i \vec{y}^\top (\nabla^2 f_i(\vec{x}))\vec{y} \tag{16}$$
$$\geq 0 \qquad (\vec{y}^\top (\nabla^2 f_i(\vec{x}))\vec{y} \geq 0, \text{ because } f_i \text{ is convex}) \tag{17}$$

So $\forall \vec{x},\ \nabla^2 f(\vec{x})$ is PSD, so f is convex.

(c) **Point-wise maximum**.

Show that if f_1 and f_2 are convex functions then their pointwise maximum f, defined by

$$f(\vec{x}) = \max(f_1(\vec{x}), f_2(\vec{x})), \tag{18}$$

with $\text{dom}(f) = \text{dom}(f_1) \cap \text{dom}(f_2)$, is also convex.

Solution: Because f_1 and f_2 are convex, then $\text{dom}(f_1)$ and $\text{dom}(f_2)$ are convex sets. Because convexity of sets is preserved under intersection, $\text{dom}(f) = \text{dom}(f_1) \cap \text{dom}(f_2)$ is also convex.

$$\text{epi}(f) = \{(\vec{x}, t) \mid \vec{x} \in \text{dom}(f), f(\vec{x}) \leq t\} \tag{19}$$

$$= \{(\vec{x}, t) \mid \vec{x} \in \operatorname{dom}(f), \max(f_1(\vec{x}), f_2(\vec{x})) \le t\} \tag{20}$$
$$= \{(\vec{x}, t) \mid \vec{x} \in \operatorname{dom}(f_1) \cap \operatorname{dom}(f_2), f_1(\vec{x}) \le t \text{ and } f_2(\vec{x}) \le t\} \tag{21}$$
$$= \{(\vec{x}, t) \mid \vec{x} \in \operatorname{dom}(f_1), f_1(\vec{x}) \le t\} \cap \{(\vec{x}, t) \mid \vec{x} \in \operatorname{dom}(f_2), f_2(\vec{x}) \le t\} \tag{22}$$
$$= \operatorname{epi}(f_1) \cap \operatorname{epi}(f_2) \tag{23}$$

Because f_1 and f_2 are convex, then $\operatorname{epi}(f_1)$ and $\operatorname{epi}(f_2)$ are convex. Because convexity of sets is preserved under intersection, $\operatorname{epi}(f)$ is convex. Because of the equivalence between the convexity of functions and the convexity of their epigraphs, f is convex.

2. **Convexity of Constraint Sets**

Let $f_1, \ldots, f_m, h_1, \ldots, h_p \colon \mathbb{R}^n \to \mathbb{R}$ be functions. Let $S \subseteq \mathbb{R}^n$ be defined as

$$S \doteq \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} f_i(\vec{x}) \le 0 & \forall i = 1, \ldots, m \\ h_j(\vec{x}) = 0 & \forall j = 1, \ldots, p \end{array} \right\}. \tag{24}$$

Show that if $f_1, \ldots, f_m$ are convex functions, and $h_1, \ldots, h_p$ are affine functions, then S is a convex set.

Solution: Let $\vec{x}, \vec{y} \in S$ and let $\theta \in [0, 1]$. Then for any $i = 1, \ldots, m$, we have

$$\begin{aligned} f_i(\theta\vec{x} + (1-\theta)\vec{y}) &\le \theta \underbrace{f_i(\vec{x})}_{\le 0} + (1-\theta) \underbrace{f_i(\vec{y})}_{\le 0} \\ &\le 0. \end{aligned}$$

And for any $j = 1, \ldots, p$, we have

$$\begin{aligned} h_j(\theta\vec{x} + (1-\theta)\vec{y}) &= \theta \underbrace{h_j(\vec{x})}_{=0} + (1-\theta) \underbrace{h(\vec{y})}_{=0} \\ &= 0. \end{aligned}$$

Thus $\theta\vec{x} + (1-\theta)\vec{y} \in S$. Thus S is convex.

3. **Disproving Convexity: Finding Counter-Examples**

Though we spend a lot of time in this course learning how to prove convexity of sets and functions, in practical scenarios we may not have a mathematical representation of a set/function and so it is not possible to prove convexity. Instead, we may be able to represent this set/function in terms of a query $Q(\vec{x})$ that returns some information about the element $\vec{x}$ in relation to the set/function. For example, instead representing the set $S = \{\vec{x} \mid \text{some condition on } \vec{x}\}$ we only have $Q(\vec{x})$ which returns whether or not $\vec{x} \in S$. In these cases we can **disprove** convexity by showing that one or more of the properties of convex sets/functions are violated by finding counterexamples. In this problem we will see how we can disprove convexity for sets/functions given limited information that can be accessed via certain types of queries.

(a) **Disproving convexity of set S (Proving non-convexity of set S).**

Assume that we know that the set lies within some $\mathcal{D}$. Define the query:

- $Q(\vec{x})$: For $\vec{x} \in \mathcal{D}$, returns `true` if $\vec{x} \in S$ and `false` if $\vec{x} \notin S$.

Solution: Choose $\vec{x}$ and $\vec{y}$ randomly in $\mathcal{D}$ and if both lie in S then check if $(\vec{x} + \vec{y})/2$ lies in S. We can choose any point on line segment joining $\vec{x}, \vec{y} \in S$ instead of the mid-point.

(b) **Disproving convexity of function f (Proving non-convexity of function f).**

Assume that we know $\mathrm{dom}(f)$, denoted as $\mathcal{D}$ and that $\mathcal{D}$ is convex.

i. Define the query:

- $G(\vec{x})$: For $\vec{x} \in \mathcal{D}$, returns function value $f(\vec{x})$.

How can you use G to check/disprove convexity of f?

Solution: Get $G(\vec{x}), G(\vec{y})$ for $\vec{x}, \vec{y} \in \mathcal{D}$ and then check if $G(\frac{\vec{x}+\vec{y}}{2}) \leq \frac{G(\vec{x})+G(\vec{y})}{2}$. Can also check for other points on line segment joining $\vec{x}$ and $\vec{y}$.

ii. Define the query:

- $H(\vec{x})$: For $\vec{x} \in \mathcal{D}$, returns $f(\vec{x})$ and $\nabla f(\vec{x})$. (Here we assume that f is differentiable).

How can you use H to check/disprove convexity of f?

Solution: Check first order condition $f(\vec{y}) \geq f(\vec{x}) + \nabla f(\vec{x})^\top(\vec{y} - \vec{x})$.

4. Properties of Convex Functions

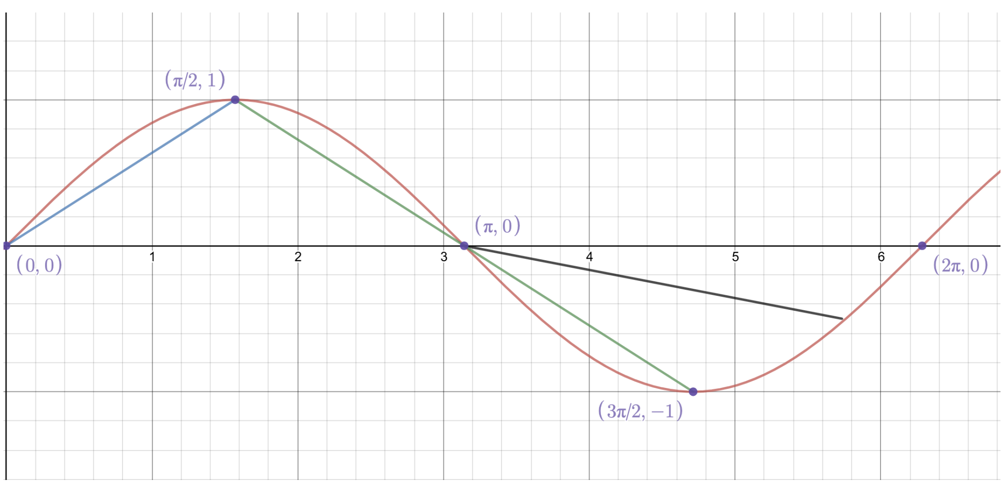
In this exercise, we examine convexity and what it represents graphically.

(a) In what region between $[0, 2\pi]$ is $\sin(x)$ a convex function? In what region between $[0, 2\pi]$ is $\sin(x)$ a concave function? Give a region between $[0, 2\pi]$ where $\sin(x)$ is neither convex nor concave.

Solution: The function $\sin(x)$ is convex (in fact, strictly convex) between $[\pi, 2\pi]$; similarly, it is concave (in fact, strictly concave) between $[0, \pi]$. It is non-convex and non-concave for any interval between $[0, 2\pi]$ that is not a subset of the two aforementioned intervals. Note that our interval could even be disjoint!

(b) Plot $\sin(x)$ between $[0, 2\pi]$. For each of the 3 intervals defined above in part (a), draw a chord to illustrate graphically on what regions the function is convex, concave, and neither convex nor concave.

Solution:



In the region $[0, \pi]$, the function is concave and all chords (e.g., the *blue* chord above) lie below the function. In the region $[\pi, 2\pi]$, the function is convex and all chords (e.g., the *black* chord above) lie above the function. When considering the full region $[0, 2\pi]$, or any region that is not a subset of the two regions above, chords (like the example *green* chord above) do not lie strictly above or strictly below the function.

(c) Show that for all $x \in [0, \frac{\pi}{2}]$,

$$\frac{2}{\pi}x \leq \sin x \leq x. \tag{25}$$

Solution: From part (a), we know that $\sin(x)$ is concave on $[0, \frac{\pi}{2}]$, and thus every value lies below every tangent and above every chord that can be defined in the region.

In the region $[0, \frac{\pi}{2}]$, $\sin(x)$ can therefore be upper bounded by its tangent at 0 (the identity function $f(x) = x$) and lower bounded by the chord between $(0, \sin(0))$ and $(\pi/2, \sin(\pi/2))$ (the linear function $\frac{2}{\pi}x$).

Note that we could establish different upper and lower bounds as well; all values of $\sin(x)$ lie below any tangent line of the function, and values within the span of a chord lie above that chord.

