Lab 2: Higher-Order Functions, Lambda Expressions

lab02.zip (lab02.zip)

Due by 11:59pm on Wednesday, September 7.

Starter Files

Download lab02.zip (lab02.zip). Inside the archive, you will find starter files for the questions in this lab, along with a copy of the Ok (ok) autograder.

Topics

Consult this section if you need a refresher on the material for this lab. It's okay to skip directly to the questions and refer back here should you get stuck.

Higher-Order Functions

Higher-Order Functions

Variables are names bound to values, which can be primitives like `3` or `'Hello World'`, but they can also be functions. And since functions can take arguments of any value, other functions can be passed in as arguments. This is the basis for higher-order functions.

A higher order function is a function that manipulates other functions by taking in functions as arguments, returning a function, or both. We will introduce the basics of higher order functions in this lab and will be exploring many applications of higher order functions in our next lab.

Functions as arguments

In Python, function objects are values that can be passed around. We know that one way to create functions is by using a `def` statement:

```
def square(x):
    return x * x
```

The above statement created a function object with the intrinsic name `square` as well as binded it to the name `square` in the current environment. Now let's try passing it as an argument.

First, let's write a function that takes in another function as an argument:

```
def scale(f, x, k):
    """ Returns the result of f(x) scaled by k. """
    return k * f(x)
```

We can now call `scale` on `square` and some other arguments:

```
>>> scale(square, 3, 2) # Double square(3)
18
>>> scale(square, 2, 5) # 5 times 2 squared
20
```

Note that in the body of the call to `scale`, the function object with the intrinsic name `square` is bound to the parameter `f`. Then, we call `square` in the body of `scale` by calling `f(x)`.

As we saw in the above section on `lambda` expressions, we can also pass `lambda` expressions into call expressions!

```
>>> scale(lambda x: x + 10, 5, 2)
30
```

In the frame for this call expression, the name `f` is bound to the function created by the `lambda` expression `lambda x: x + 10`.

Functions that return functions

Because functions are values, they are valid as return values! Here's an example:

```
def multiply_by(m):
    def multiply(n):
        return n * m
    return multiply
```

In this particular case, we defined the function `multiply` within the body of `multiply_by` and then returned it. Let's see it in action:

```
>>> multiply_by(3)
<function multiply_by.<locals>.multiply at ...>
>>> multiply(4)
Traceback (most recent call last):
File "<stdin>", line 1, in <module>
NameError: name 'multiply' is not defined
```

A call to `multiply_by` returns a function, as expected. However, calling `multiply` errors, even though that's the name we gave the inner function. This is because the name `multiply` only exists within the frame where we evaluate the body of `multiply_by`.

So how do we actually use the inner function? Here are two ways:

```
>>> times_three = multiply_by(3) # Assign the result of the call expression to a name
>>> times_three(5) # Call the inner function with its new name
15
>>> multiply_by(3)(10) # Chain together two call expressions
30
```

The point is, because `multiply_by` returns a function, you can use its return value just like you would use any other function.

Lambda Expressions

Lambda Expressions

Lambda expressions are expressions that evaluate to functions by specifying two things: the parameters and a return expression.

```
lambda <parameters>: <return expression>
```

While both `lambda` expressions and `def` statements create function objects, there are some notable differences. `lambda` expressions work like other expressions; much like a mathematical expression just evaluates to a number and does not alter the current environment, a `lambda` expression evaluates to a function without changing the current environment. Let's take a closer look.

	lambda	**def**
Type	*Expression* that evaluates to a value	*Statement* that alters the environment
Result of execution	Creates an anonymous lambda function with no intrinsic name.	Creates a function with an intrinsic name and binds it to that name in the current environment.
Effect on the environment	Evaluating a `lambda` expression does *not* create or modify any variables.	Executing a `def` statement both creates a new function object *and* binds it to a name in the current environment.
Usage	A `lambda` expression can be used anywhere that expects an expression, such as in an assignment statement or as the operator or operand to a call expression.	After executing a `def` statement, the created function is bound to a name. You should use this name to refer to the function anywhere that expects an expression.

Example	`# A lambda expression by itself does not alter`
	`# the environment`
	`lambda x: x * x`
	
	`# We can assign lambda functions to a name`
	`# with an assignment statement`
	`square = lambda x: x * x`
	`square(3)`
	
	`# Lambda expressions can be used as an operator`
	`# or operand`
	`negate = lambda f, x: -f(x)`
	`negate(lambda x: x * x, 3)`
		`def square(x):`
		`  return x * x`
		
		`# A function created by a def statement`
		`# can be referred to by its intrinsic name`
		`square(3)`

YouTube link (https://youtu.be/vCeNq_P3akI?list=PLx38hZJ5RLZcUPWZ1-3HYsRPgZ8OCrvqz)

Currying

Currying

We can transform multiple-argument functions into a chain of single-argument, higher order functions. For example, we can write a function `f(x, y)` as a different function `g(x)(y)`. This is known as **currying**.

For example, to convert the function `add(x, y)` into its curried form:

```
def curry_add(x):
    def add2(y):
        return x + y
    return add2
```

Calling `curry_add(1)` returns a new function which only performs the addition once the returned function is called with the second addend.

```
>>> add_one = curry_add(1)
>>> add_one(2)
3
>>> add_one(4)
5
```

> Refer to the textbook (http://composingprograms.com/pages/16-higher-order-functions.html#currying) for more details about currying.

Environment Diagrams

Environment Diagrams

Environment diagrams are one of the best learning tools for understanding `lambda` expressions and higher order functions because you're able to keep track of all the different names, function objects, and arguments to functions. We highly recommend drawing environment diagrams or using Python tutor (https://tutor.cs61a.org) if you get stuck doing the WWPD problems below. For examples of what environment diagrams should look like, try running some code in Python tutor. Here are the rules:

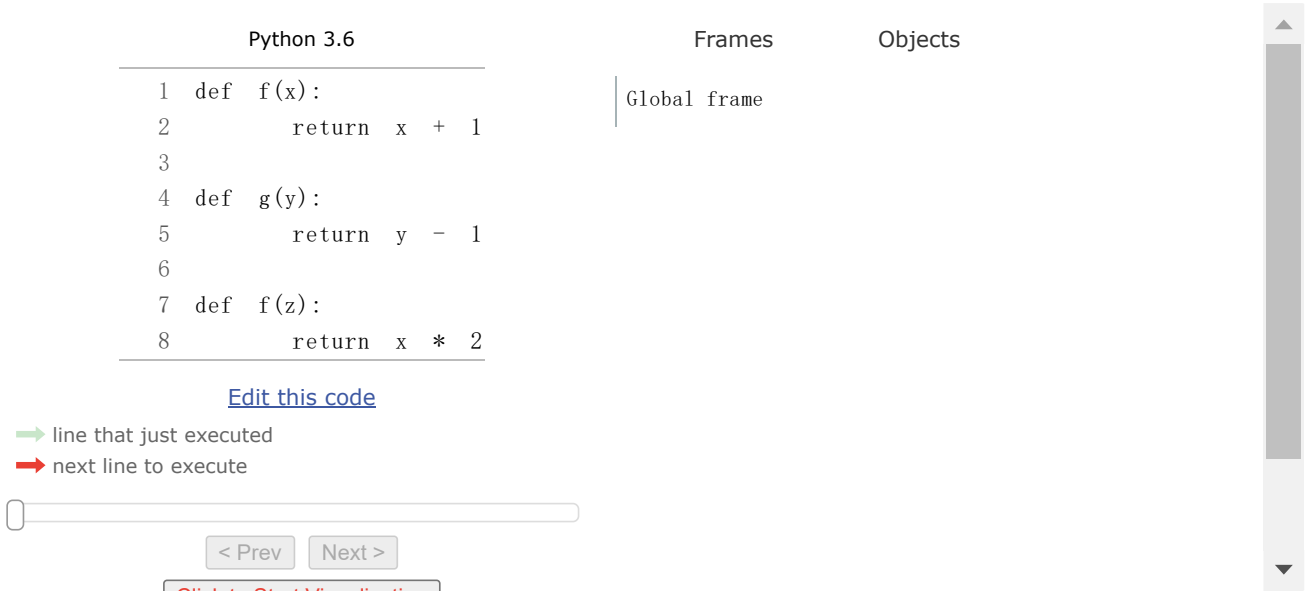
Assignment Statements

1. Evaluate the expression on the right hand side of the `=` sign.
2. If the name found on the left hand side of the `=` doesn't already exist in the current frame, write it in. If it does, erase the current binding. Bind the *value* obtained in step 1 to this name.

If there is more than one name/expression in the statement, evaluate all the expressions first from left to right before making any bindings.

def Statements

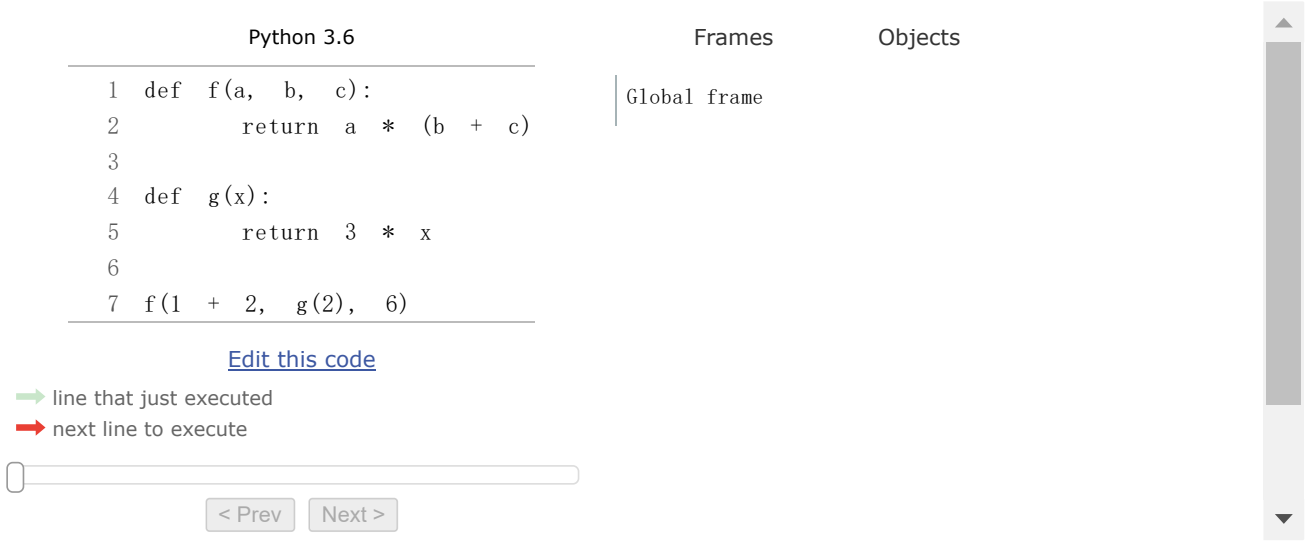
1. Draw the function object with its intrinsic name, formal parameters, and parent frame. A function's parent frame is the frame in which the function was defined.
2. If the intrinsic name of the function doesn't already exist in the current frame, write it in. If it does, erase the current binding. Bind the newly created function object to this name.



Call expressions

> Note: you do not have to go through this process for a built-in Python function like `max` or `print`.

1. Evaluate the operator, whose value should be a function.
2. Evaluate the operands left to right.
3. Open a new frame. Label it with the sequential frame number, the intrinsic name of the function, and its parent.
4. Bind the formal parameters of the function to the arguments whose values you found in step 2.
5. Execute the body of the function in the new environment.



Lambdas

> *Note:* As we saw in the `lambda` expression section above, `lambda` functions have no intrinsic name. When drawing `lambda` functions in environment diagrams, they are labeled with the name `lambda` or with the lowercase Greek letter λ. This can get confusing when there are multiple lambda functions in an environment diagram, so you can distinguish them by numbering them or by writing the line number on which they were defined.

1. Draw the lambda function object and label it with λ, its formal parameters, and its parent frame. A function's parent frame is the frame in which the function was defined.

This is the only step. We are including this section to emphasize the fact that the difference between `lambda` expressions and `def` statements is that `lambda` expressions *do not* create any new bindings in the environment.

YouTube link (https://youtu.be/IPec2A7j2bY?list=PLx38hZJ5RLZcUPWZ1-3HYsRPgZ8OCrvqz)

Required Questions

Getting Started Videos

What Would Python Display?

Important: For all WWPD questions, type `Function` if you believe the answer is `<function...>`, `Error` if it errors, and `Nothing` if nothing is displayed.

Q1: WWPD: Lambda the Free

Use Ok to test your knowledge with the following "What Would Python Display?" questions:

```
python3 ok -q lambda -u
```

As a reminder, the following two lines of code will not display anything in the Python interpreter when executed:

```
>>> x = None
>>> x
```

```
>>> lambda x: x  # A lambda expression with one parameter x
------

>>> a = lambda x: x  # Assigning the lambda function to the name a
>>> a(5)
------

>>> (lambda: 3)()  # Using a lambda expression as an operator in a call exp.
------

>>> b = lambda x: lambda: x  # Lambdas can return other lambdas!
>>> c = b(88)
>>> c
------

>>> c()
------

>>> d = lambda f: f(4)  # They can have functions as arguments as well.
>>> def square(x):
...     return x * x
>>> d(square)
------
```

```
>>> z = 3
>>> e = lambda x: lambda y: lambda: x + y + z
>>> e(0)(1)()
------

>>> f = lambda z: x + z
>>> f(3)
------
```

```
>>> x = None # remember to review the rules of WWPD given above!
>>> x
------
```

```
>>> higher_order_lambda = lambda f: lambda x: f(x)
>>> g = lambda x: x * x
>>> higher_order_lambda(2)(g)  # Which argument belongs to which function call?
------

>>> higher_order_lambda(g)(2)
------

>>> call_thrice = lambda f: lambda x: f(f(f(x)))
>>> call_thrice(lambda y: y + 1)(0)
------

>>> print_lambda = lambda z: print(z)  # When is the return expression of a lambda expression executed?
>>> print_lambda
------

>>> one_thousand = print_lambda(1000)
------

>>> one_thousand # What did the call to print_lambda return?
------
```

Q2: WWPD: Higher Order Functions

Use Ok to test your knowledge with the following "What Would Python Display?" questions:

```
python3 ok -q hof-wwpd -u
```

```
>>> def even(f):
...     def odd(x):
...         if x < 0:
...             return f(-x)
...         return f(x)
...     return odd
>>> steven = lambda x: x
>>> stewart = even(steven)
>>> stewart
------

>>> stewart(61)
------

>>> stewart(-4)
------
```

```
>>> def cake():
...     print('beets')
...     def pie():
...         print('sweets')
...         return 'cake'
...     return pie
>>> chocolate = cake()
------

>>> chocolate
------

>>> chocolate()
------

>>> more_chocolate, more_cake = chocolate(), cake
------

>>> more_chocolate
------

>>> def snake(x, y):
...     if cake == more_cake:
...         return chocolate
...     else:
...         return x + y
>>> snake(10, 20)
------

>>> snake(10, 20)()
------

>>> cake = 'cake'
>>> snake(10, 20)
------
```

Coding Practice

Q3: Lambdas and Currying

Write a function `lambda_curry2` that will curry any two argument function using lambdas.

Your solution to this problem should only be one line. You can try first writing a solution without the restriction, and then rewriting it into one line after.

> **If the syntax check isn't passing:** Make sure you've removed the line containing `"***YOUR CODE HERE***"` so that it doesn't get treated as part of the function for the syntax check.

```
def lambda_curry2(func):
    """
    Returns a Curried version of a two-argument function FUNC.
    >>> from operator import add, mul, mod
    >>> curried_add = lambda_curry2(add)
    >>> add_three = curried_add(3)
    >>> add_three(5)
    8
    >>> curried_mul = lambda_curry2(mul)
    >>> mul_5 = curried_mul(5)
    >>> mul_5(42)
    210
    >>> lambda_curry2(mod)(123)(10)
    3
    """
    "*** YOUR CODE HERE ***"
    return ______
```

Use Ok to test your code:

```
python3 ok -q lambda_curry2
```

Q4: Count van Count

Consider the following implementations of `count_factors` and `count_primes`:

```
def count_factors(n):
    """Return the number of positive factors that n has.
    >>> count_factors(6)
    4   # 1, 2, 3, 6
    >>> count_factors(4)
    3   # 1, 2, 4
    """
    i = 1
    count = 0
    while i <= n:
        if n % i == 0:
            count += 1
        i += 1
    return count

def count_primes(n):
    """Return the number of prime numbers up to and including n.
    >>> count_primes(6)
    3   # 2, 3, 5
    >>> count_primes(13)
    6   # 2, 3, 5, 7, 11, 13
    """
    i = 1
    count = 0
    while i <= n:
        if is_prime(i):
            count += 1
        i += 1
    return count

def is_prime(n):
    return count_factors(n) == 2 # only factors are 1 and n
```

The implementations look quite similar! Generalize this logic by writing a function `count_cond`, which takes in a two-argument predicate function `condition(n, i)`. `count_cond` returns a one-argument function that takes in `n`, which counts all the numbers from 1 to `n` that satisfy `condition` when called.

> **Note:** When we say `condition` is a predicate function, we mean that it is a function that will return `True` or `False` based on some specified condition in its body.

```
def count_cond(condition):
    """Returns a function with one parameter N that counts all the numbers from
    1 to N that satisfy the two-argument predicate function Condition, where
    the first argument for Condition is N and the second argument is the
    number from 1 to N.

    >>> count_factors = count_cond(lambda n, i: n % i == 0)
    >>> count_factors(2)   # 1, 2
    2
    >>> count_factors(4)   # 1, 2, 4
    3
    >>> count_factors(12)  # 1, 2, 3, 4, 6, 12
    6

    >>> is_prime = lambda n, i: count_factors(i) == 2
    >>> count_primes = count_cond(is_prime)
    >>> count_primes(2)    # 2
    1
    >>> count_primes(3)    # 2, 3
    2
    >>> count_primes(4)    # 2, 3
    2
    >>> count_primes(5)    # 2, 3, 5
    3
    >>> count_primes(20)   # 2, 3, 5, 7, 11, 13, 17, 19
    8
    """
    "*** YOUR CODE HERE ***"
```

Use Ok to test your code:

```
python3 ok -q count_cond
```

Submit

Make sure to submit this assignment by running:

```
python3 ok --submit
```

Environment Diagram Practice

There is no Ok submission for this component.

However, we still encourage you to do this problem on paper to develop familiarity with Environment Diagrams, which might appear in an alternate form on the exam. To check your work, you can try putting the code into PythonTutor.

Q5: HOF Diagram Practice

Draw the environment diagram that results from executing the code below.

```
n = 7

def f(x):
    n = 8
    return x + 1

def g(x):
    n = 9
    def h():
        return x + 1
    return h

def f(f, x):
    return f(x + n)

f = f(g, n)
g = (lambda y: y())(f)
```

Optional Questions

Q6: Composite Identity Function

Write a function that takes in two single-argument functions, `f` and `g`, and returns another **function** that has a single parameter `x`. The returned function should return `True` if `f(g(x))` is equal to `g(f(x))`. You can assume the output of `g(x)` is a valid input for `f` and vice versa. Try to use the `composer` function defined below for more HOF practice.

```
def composer(f, g):
    """Return the composition function which given x, computes f(g(x)).

    >>> add_one = lambda x: x + 1        # adds one to x
    >>> square = lambda x: x**2
    >>> a1 = composer(square, add_one)   # (x + 1)^2
    >>> a1(4)
    25
    >>> mul_three = lambda x: x * 3      # multiplies 3 to x
    >>> a2 = composer(mul_three, a1)    # ((x + 1)^2) * 3
    >>> a2(4)
    75
    >>> a2(5)
    108
    """
    return lambda x: f(g(x))

def composite_identity(f, g):
    """
    Return a function with one parameter x that returns True if f(g(x)) is
    equal to g(f(x)). You can assume the result of g(x) is a valid input for f
    and vice versa.

    >>> add_one = lambda x: x + 1        # adds one to x
    >>> square = lambda x: x**2
    >>> b1 = composite_identity(square, add_one)
    >>> b1(0)                            # (0 + 1)^2 == 0^2 + 1
    True
    >>> b1(4)                            # (4 + 1)^2 != 4^2 + 1
    False
    """
    "*** YOUR CODE HERE ***"
```

Use Ok to test your code:

```
python3 ok -q composite_identity
```

Q7: I Heard You Liked Functions...

Define a function `cycle` that takes in three functions `f1`, `f2`, `f3`, as arguments. `cycle` will return another function that should take in an integer argument `n` and return another function. That final function should take in an argument `x` and cycle through applying `f1`, `f2`, and `f3` to `x`, depending on what `n` was. Here's what the final function should do to `x` for a few values of `n`:

- `n = 0`, return `x`
- `n = 1`, apply `f1` to `x`, or return `f1(x)`
- `n = 2`, apply `f1` to `x` and then `f2` to the result of that, or return `f2(f1(x))`
- `n = 3`, apply `f1` to `x`, `f2` to the result of applying `f1`, and then `f3` to the result of applying `f2`, or `f3(f2(f1(x)))`
- `n = 4`, start the cycle again applying `f1`, then `f2`, then `f3`, then `f1` again, or `f1(f3(f2(f1(x))))`
- And so forth.

Hint: most of the work goes inside the most nested function.

```
def cycle(f1, f2, f3):
    """Returns a function that is itself a higher-order function.

    >>> def add1(x):
    ...     return x + 1
    >>> def times2(x):
    ...     return x * 2
    >>> def add3(x):
    ...     return x + 3
    >>> my_cycle = cycle(add1, times2, add3)
    >>> identity = my_cycle(0)
    >>> identity(5)
    5
    >>> add_one_then_double = my_cycle(2)
    >>> add_one_then_double(1)
    4
    >>> do_all_functions = my_cycle(3)
    >>> do_all_functions(2)
    9
    >>> do_more_than_a_cycle = my_cycle(4)
    >>> do_more_than_a_cycle(2)
    10
    >>> do_two_cycles = my_cycle(6)
    >>> do_two_cycles(1)
    19
    """
    "*** YOUR CODE HERE ***"
```

Use Ok to test your code:

```
python3 ok -q cycle
```