Simple mask in the ϑ plane

We are interested in seeing how applying a mask to some data affects the recovered power spectrum. First, let us look at the theoretical power spectrum for a simple mask. We start by considering a very simple mask, where we only allow data through in a central region.

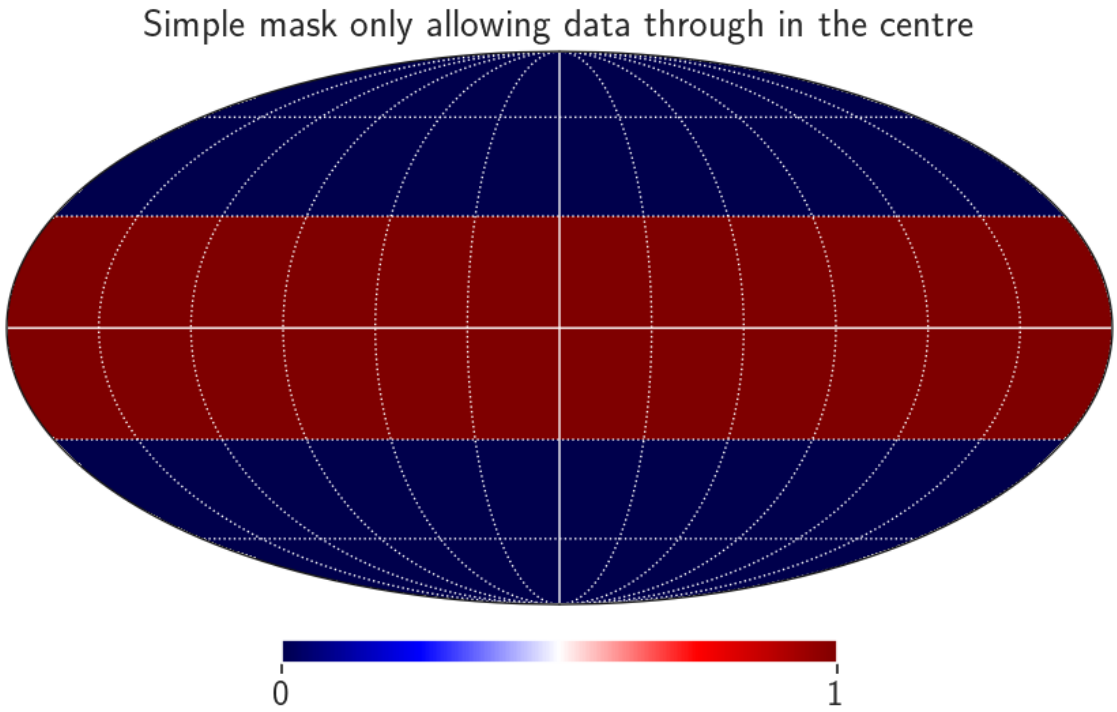

Figure 1: A simple mask in the ϑ plane.

Here, we are only allowing data through a region between $\vartheta = \pi/3$ and $\vartheta = 2\pi/3$, however these angles are arbitrary and can be changed to anything. The important thing to note is that our mask is independent of ϕ, as we are simply making two cuts in the ϑ plane.

Now that we have a rough mask, we can try to find what its power spectrum is. To do so, we first need to compute what its $a_{\ell m}$ values are. To do so, we first recall that for a spherical harmonic expansion of a general field $f(\vartheta, \phi)$ its $a_{\ell m}$ values are given by

$$a_{\ell m} = \int \mathrm{d}\Omega \, Y^*_{\ell m}(\vartheta, \phi) \, f(\vartheta, \phi). \tag{1}$$

Here, our field f is simply the values of our mask. In this case, we find that

$$f(\vartheta, \phi) = f(\vartheta), \tag{2}$$

as we have previously identified that it is independent of ϕ.

Now that we have split the mask function, we can also decompose the spherical harmonics $Y_{\ell m}$ using

$$Y_{\ell m}(\vartheta, \phi) = \sqrt{\frac{(2\ell + 1)}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} \, P_{\ell m}(\cos\vartheta) \, e^{im\phi}, \tag{3}$$

where $P_{\ell m}(x)$ are the associated Legendre polynomials. Using this, we find we can split the $a_{\ell m}$ integral into

$$a_{\ell m} = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \int_0^\pi \mathrm{d}\vartheta \, \sin\vartheta \, f(\vartheta) P_{\ell m}(\cos\theta) \int_0^{2\pi} \mathrm{d}\phi \, e^{-im\phi}. \tag{4}$$

Here, we note that the ϕ integral is zero when m is non-zero, and equal to 2π only when $m = 0$, ie.

$$\int_0^{2\pi} \mathrm{d}\phi \; e^{-im\phi} = 2\pi\delta_m. \tag{5}$$

Using this result, we find our $a_{\ell m}$ values as

$$a_{\ell m} = 2\pi\delta_m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \int_0^\pi \mathrm{d}\vartheta \, \sin\vartheta \, f(\vartheta) P_{\ell m}(\cos\theta). \tag{6}$$

Using this, we can simplify the prefactor and integrand, to give

$$a_{\ell m} = \delta_m \sqrt{\pi(2\ell+1)} \int_0^\pi \mathrm{d}\vartheta \, \sin\vartheta \, f(\vartheta) P_\ell(\cos\theta), \tag{7}$$

where now $P_\ell(x)$ is the regular Legendre polynomials. If we now use the fact that our mask is only non-zero within a certain range of ϑ values, which we will now generalise to a range $[A, B]$ where it is one within this boundary, we find the ϑ integral becomes

$$a_{\ell m} = \delta_m \sqrt{\pi(2\ell+1)} \int_A^B \mathrm{d}\vartheta \, \sin\vartheta \, P_\ell(\cos\theta). \tag{8}$$

This ϑ integral is now a known integral of Legendre polynomials, which is

$$\int_A^B \mathrm{d}\vartheta \, \sin\vartheta \, P_\ell(\cos\theta) = \frac{-P_{\ell-1}(\cos A) + P_{\ell+1}(\cos A) + P_{\ell-1}(\cos B) - P_{\ell+1}(\cos B)}{2\ell+1}. \tag{9}$$

Hence, the $a_{\ell m}$ values for our mask become

$$a_{\ell m} = \delta_m \sqrt{\frac{\pi}{2\ell+1}} \left[-P_{\ell-1}(\cos A) + P_{\ell+1}(\cos A) + P_{\ell-1}(\cos B) - P_{\ell+1}(\cos B)\right]. \tag{10}$$

Now that we have the $a_{\ell m}$ values, we can transform these into C_ℓ values using

$$C_\ell = \frac{1}{2\ell+1} \sum_m |a_{\ell m}|^2. \tag{11}$$

The Kronecker-δ in the $a_{\ell m}$'s enforcing $m = 0$ makes this sum trivial, and so we find the C_ℓ values of our mask to be

$$C_\ell = \frac{\pi}{(2\ell+1)^2} \left[-P_{\ell-1}(\cos A) + P_{\ell+1}(\cos A) + P_{\ell-1}(\cos B) - P_{\ell+1}(\cos B)\right]^2. \tag{12}$$

We can now compare our simple analytic formula with the C_ℓ values that were numerically derived from the mask in Figure 1 using **HealPy**, which gave us the following plot

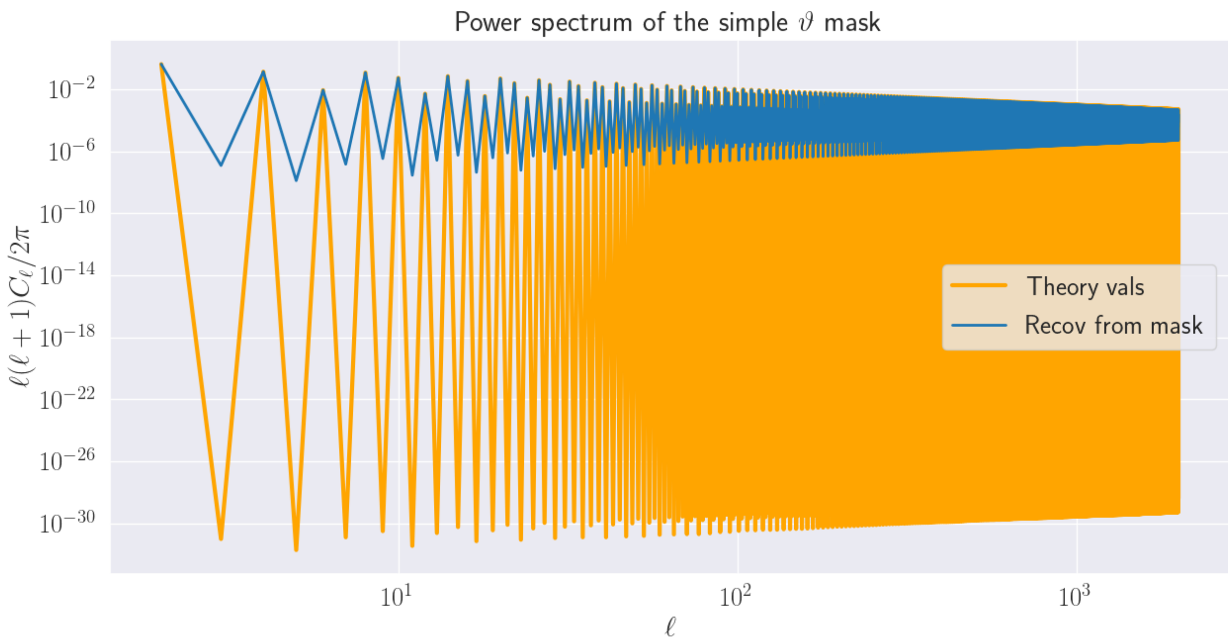

Figure 2: Comparison of the theoretical and numerical C_ℓ values for our simple mask in the ϑ plane.

Here, we see that for odd values of ℓ, the theoretical values are *extremely* small, of the order $\mathcal{O}(10^{-30})$, which indicates that these are actually zero, and are just non-zero due to finite numerical precision. This makes sense as the parity of the Legendre polynomials gives that

$$P_\ell(-x) = (-1)^\ell P_\ell(x), \tag{13}$$

and so as our mask is symmetric around $\vartheta = \pi/2$, we expect the C_ℓ values to be zero for odd ℓ. Therefore, if we only include even ℓ values in the plot, we get a much cleaner plot, giving

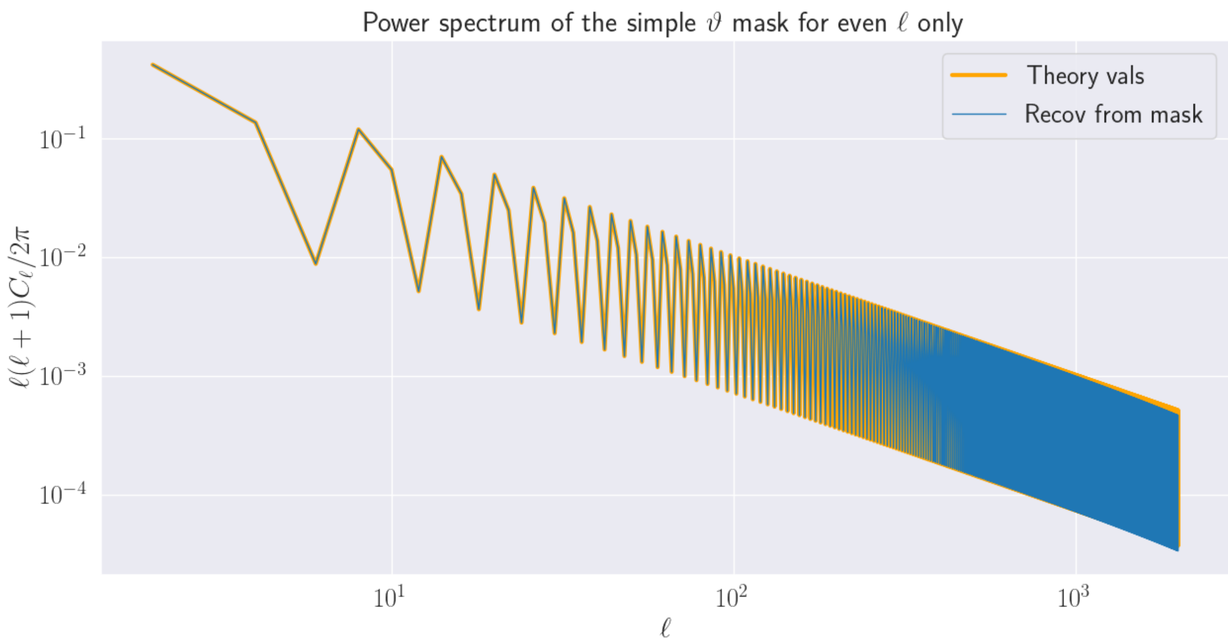

Figure 3: Comparison of the theoretical and numerical C_ℓ values for even ℓ only, for our simple mask in the ϑ plane.

Here, we see very good agreement between the numerically derived values and our analytic results. We can quantify the agreement between the two sets of values by taking their ratio, giving

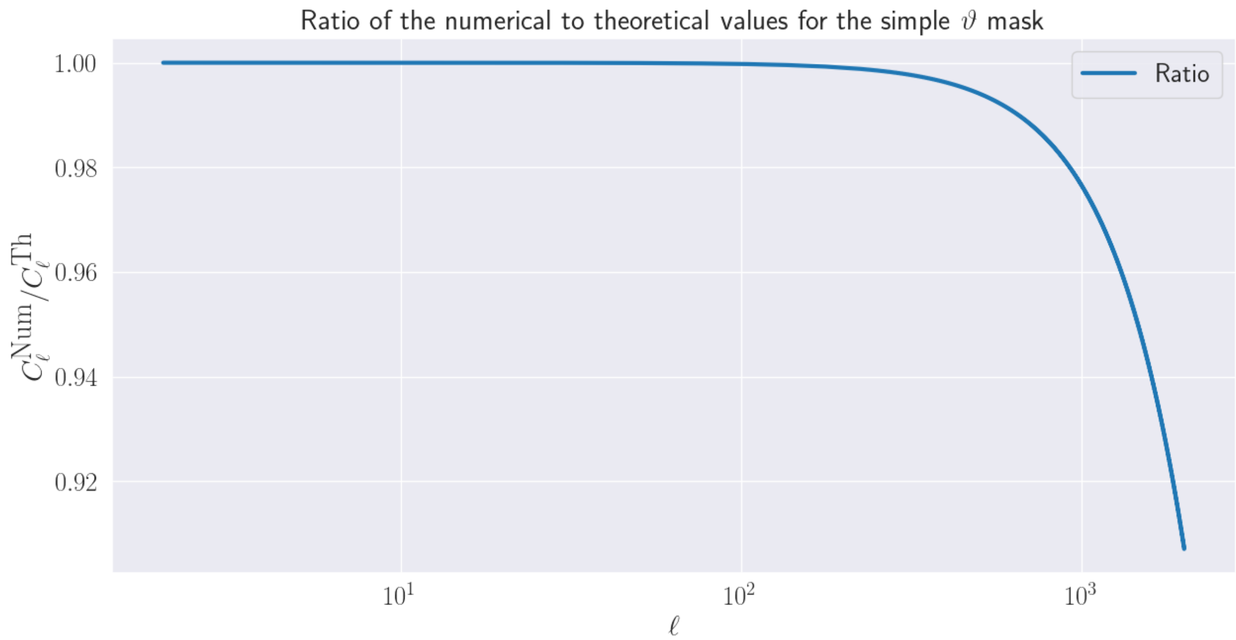

Figure 4: Ratio of the numerical to theoretical C_ℓ values for even ℓ only, for our simple mask in the ϑ plane.

Here, we see very good precision at low-ℓ, only loosing precision at ℓ's greater than one thousand, or so. Do note that our "theoretical" values still involve numerically evaluating Legendre polynomials, and so are still susceptible to numerical noise at high ℓ values.

Simple mask in the ϕ plane

Above, we looked at masks which only vary in the ϑ plane, which limits us in extending our analysis to general masks. Here, we wish to repeat the same analysis (deriving analytic expressions and comparing to numerical results) but for a mask that only varies in the ϕ plane. For example, let us consider the following mask

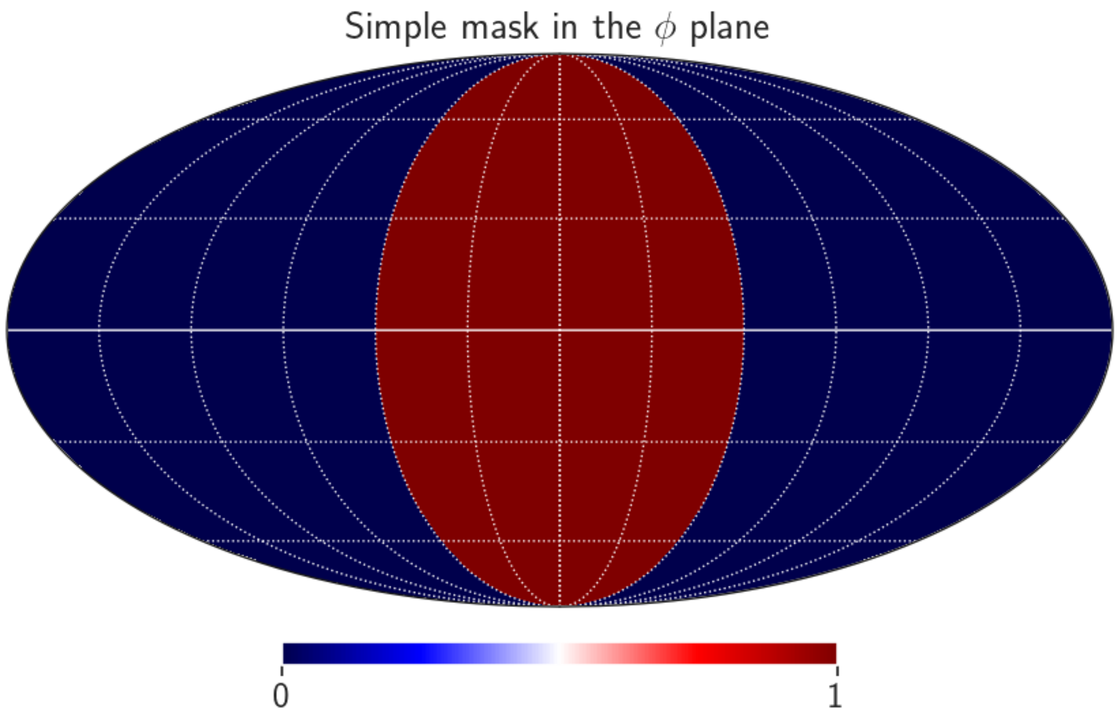

Figure 5: A simple mask in the ϕ plane.

Here, we only let data through in a specific region, from $\phi = A$ to $\phi = B$

Now, our mask is only a function of ϕ, and so when comparing with Equation 1, we find that our mask function can be expressed as

$$f(\vartheta, \phi) = f(\phi). \tag{14}$$

Using this, we can jump straight into writing down the $a_{\ell m}$ values that such a mask would predict, which is

$$a_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} \int_0^{\pi} \mathrm{d}\vartheta \, \sin\vartheta P_{\ell m}(\cos\vartheta) \int_0^{2\pi} \mathrm{d}\phi \; f(\phi)\, e^{-im\phi} \tag{15}$$

Let us first consider the ϕ integral

$$I_\phi \equiv \int_0^{2\pi} \mathrm{d}\phi \; f(\phi)\, e^{-im\phi} \tag{16}$$

As we are assuming that $f(\phi)$ is only non-zero between $\phi = A$ and $\phi = B$, where it is unity, we can re-write this integral as

$$I_\phi \equiv \int_A^{B} \mathrm{d}\phi \; e^{-im\phi} \tag{17}$$

We can evaluate this integral, to find

$$I_\phi = \frac{i}{m}\left[e^{-imB} - e^{-imA}\right] \tag{18}$$

Rewriting the complex exponentials in terms of trigonometric functions gives us

$$I_\phi = \frac{i}{m}\left[\cos(mB) - i\sin(mB) - \cos(mA) + i\sin(mA)\right], \tag{19}$$

where we have used $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$.

Now, we are free to rewrite the A and B angular positions in terms of a new parameter a, which is defined through

$$A = \pi - a \tag{20a}$$
$$B = \pi + a \tag{20b}$$

Using this parametrisation, we find that our mask is centred on $\phi = \pi$, and has an angular width of $2a$. Using these expressions for A and B, we find the ϕ integral becomes

$$I_\phi = -\frac{i}{m}\left[\cos(m\pi - ma) - \cos(m\pi + ma) + i\sin(m\pi + ma) - i\sin(m\pi - ma)\right], \tag{21}$$

where we have re-ordered terms to hint at the use of some trigonometric two-angle formulas. If we apply the use of them here, then we find that we can write this as

$$I_\phi = -\frac{i}{m}\left[2\sin(m\pi)\sin(ma) + 2i\cos(m\pi)\sin(ma)\right]. \tag{22}$$

We note that $\sin(m\pi) = 0$ for all m, and so this simplifies down to

$$I_\phi = \frac{2}{m}\cos(m\pi)\sin(ma). \tag{23}$$

This is now a nice and simple formula for the ϕ integral expressed as a function of the mask's width. Importantly, we note that this is non-zero for $m \neq 0$, which is *not* what we found for the ϑ mask.

We now switch our focus to the ϑ integral, which is

$$I_\vartheta \equiv \int_0^\pi \mathrm{d}\vartheta\, \sin\vartheta\, P_{\ell m}(\cos\vartheta). \tag{24}$$

Previously, we had a simplifying condition that $m = 0$ for an $a_{\ell m}$ to be non-zero, however we have no such condition here. Hence, we must do this for general ℓ and m.

First, let us define a new variable x, given by

$$x = \cos\vartheta. \tag{25}$$

Hence,

$$\mathrm{d}x = -\sin\vartheta\, \mathrm{d}\vartheta$$

This gives our integral as

$$I_\vartheta = \int_{-1}^{+1} \mathrm{d}x\, P_{\ell m}(x). \tag{26}$$

Fortunately, this is a known integral which can be expressed in terms of the Γ function as

$$I_\vartheta = \frac{\left[(-1)^\ell + (-1)^m\right] 2^{m-2}\, m\, \Gamma\left(\frac{\ell}{2}\right)\, \Gamma\left(\frac{\ell+m+1}{2}\right)}{\Gamma\left(\frac{\ell+3}{2}\right)\left(\frac{\ell-m}{2}\right)!}, \tag{27}$$

which is valid for all ℓ and m values that we are considering.

Now that we have both the ϕ and ϑ integrals known in terms of simple functions, we can evaluate the power spectrum coefficients using Equation 11, to find

$$C_\ell = \frac{1}{2\ell + 1}\sum_m \frac{2\ell + 1}{4\pi}\, \frac{(\ell - m)!}{(\ell + m)!} \times I_\phi^2 \times I_\vartheta^2. \tag{28}$$

Substituting our ϕ and ϑ integrals gives us

$$C_\ell = \sum_m \frac{1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \frac{4}{m^2} \cos^2(m\pi) \sin^2(ma) \frac{\left[2+2(-1)^{\ell+m}\right] 2^{2m-4} m^2 \, \Gamma^2\left(\frac{\ell}{2}\right) \Gamma^2\left(\frac{\ell+m+1}{2}\right)}{\Gamma^2\left(\frac{\ell+3}{2}\right) \left[\left(\frac{\ell-m}{2}\right)!\right]^2}. \tag{29}$$

Here, we note that $\cos^2(m\pi)$ is always unity, and further simplifications occur, thus giving us a final result of

$$C_\ell = \frac{1}{\pi} \sum_m \frac{(\ell-m)!}{(\ell+m)!} \sin^2(ma) \frac{\left[2+2(-1)^{\ell+m}\right] 2^{2m-4} \, \Gamma^2\left(\frac{\ell}{2}\right) \Gamma^2\left(\frac{\ell+m+1}{2}\right)}{\Gamma^2\left(\frac{\ell+3}{2}\right) \left[\left(\frac{\ell-m}{2}\right)!\right]^2}. \tag{30}$$

This now a simple expression given in terms of analytic functions which we can implement numerically to compare the analytic C_ℓ values with the numerical values obtained from the mask given in Figure 5. Here, we note that when $m = 0$, $|a_{\ell m}|^2 = 0$, which is in stark contrast to our ϑ map in which only the $m = 0$ modes contributed.

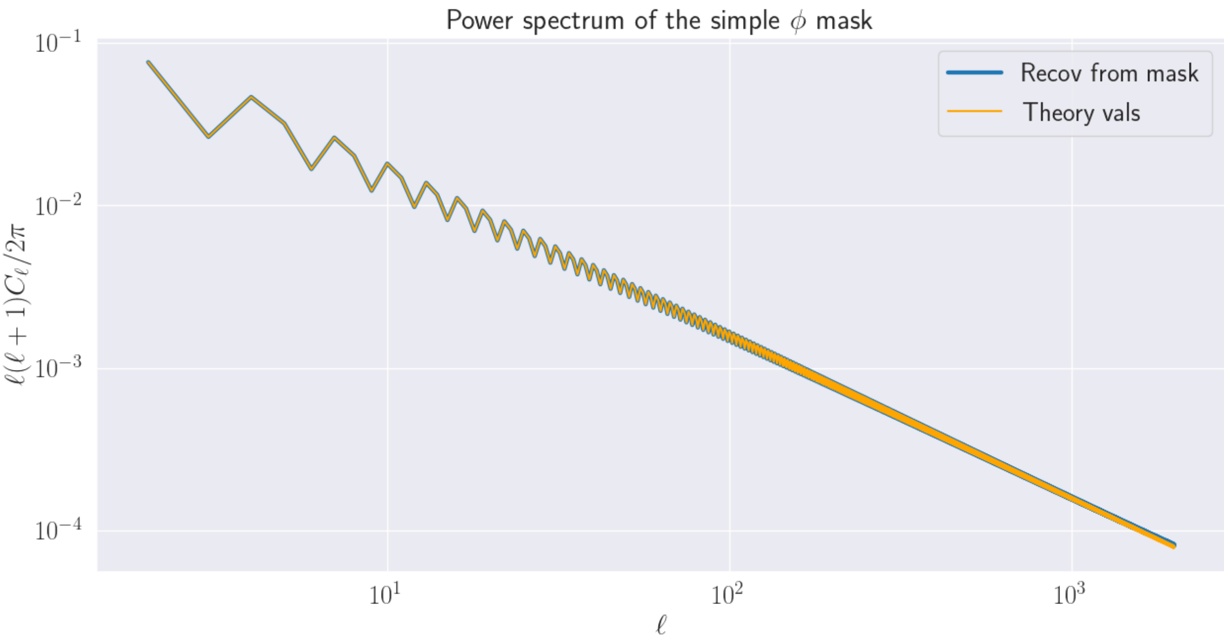

Figure 6: Comparison of the theoretical and numerical C_ℓ values for our simple mask in the ϕ plane.

Again, we see very good agreement between the theoretically derived and numerical results for our ϕ mask. Again, we can quantify this agreement by taking the ratio of the two values, to give

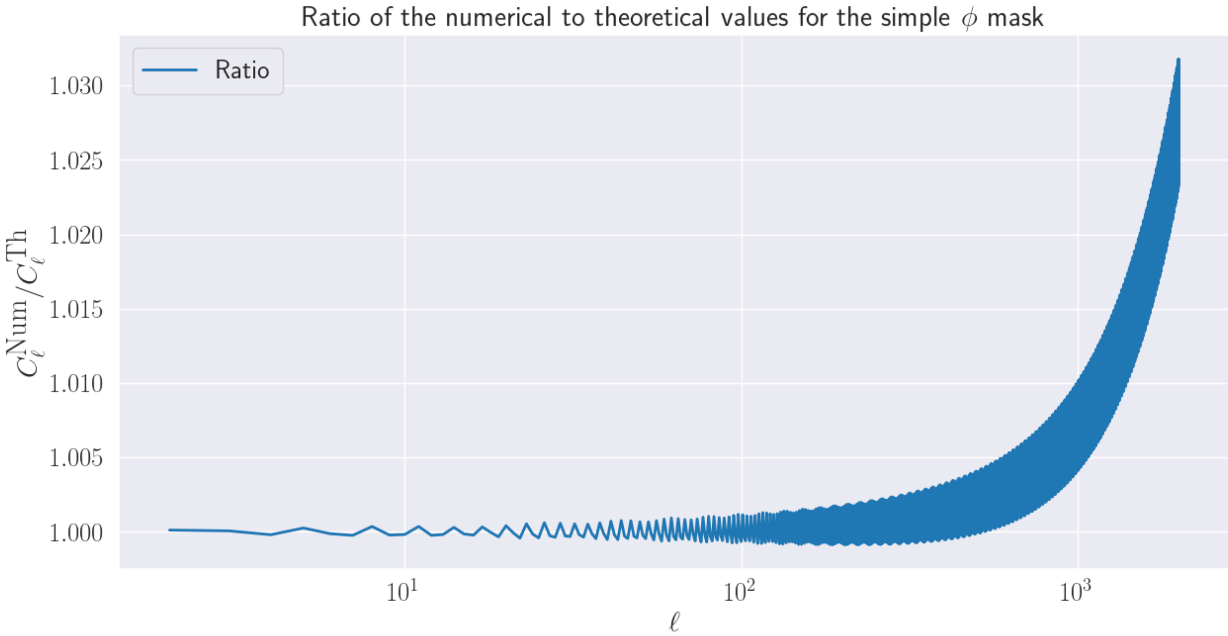

Figure 7: Ratio of the numerical to theoretical C_ℓ values for our simple mask in the ϕ plane.

Here, we see the same trend as before: very good agreement at low ℓ, with the difference gradually growing at high ℓ. Do note though that at large ℓ, there will be growing numerical inaccuracies in evaluating the theoretical values, especially as when $\ell \sim 2000$, we have to evaluate $(2000!)^2 \sim \mathcal{O}(10^{11\,000})$, and so numerical precision becomes a limiting factor.

Extension to a separable mask

Above, we have dealt with the cases where we can express our mask as either a function of ϑ or ϕ exclusively. However, general masks will not have this easy separability, and be a function of both variables. However, if the mask can be separated into a product of two function, i.e.,

$$f(\vartheta, \phi) = g(\vartheta)\, h(\phi),$$

then the above methods can be combined to predict the C_ℓ values for such a mask.

Legendre polynomials in the limit of large ℓ

In Equation 12, the C_ℓ values for our simple mask in the ϑ direction was found to be

$$C_\ell = \frac{\pi}{(2\ell+1)^2}\left[-P_{\ell-1}(\cos A) + P_{\ell+1}(\cos A) + P_{\ell-1}(\cos B) - P_{\ell+1}(\cos B)\right]^2 . \tag{31}$$

When we recovered the C_ℓ values for our masks that have both galactic and ecliptic cuts (and the Euclid-prototype mask), we found that the $\ell(\ell+1)C_\ell$ values generally had a relationship of $\sim 1/\ell$. Since this is an interesting relationship, we would like to see if this can be derived analytically starting from Equation 31.

Since we are only interested in the large-ℓ behaviour of the C_ℓ values, we can use the symtopic limit of the Legendre polynomials of

$$P_\ell(\cos\vartheta) \simeq \frac{2}{\sqrt{2\pi\ell\sin\vartheta}}\cos\left[\left(\ell+\frac{1}{2}\right) - \frac{\pi}{4}\right], \tag{32}$$

up to order $\mathcal{O}(\ell^{-3/2})$ corrections. This can now be applied to Equation 31, where we note that if our mask is symmetric around $\vartheta = \pi/2$, then $\sin A = \sin B$, we can write this as

$$\begin{aligned} C_\ell = \frac{\pi}{(2\ell+1)^2}\Bigg[& \frac{2}{\sqrt{\sqrt{3}\pi(\ell-1)}}\left(\cos\left[\left\{\ell-\frac{1}{2}\right\}B-\frac{\pi}{4}\right] - \cos\left[\left\{\ell-\frac{1}{2}\right\}A-\frac{\pi}{4}\right]\right) \\ & + \frac{2}{\sqrt{\sqrt{3}\pi(\ell+1)}}\left(\cos\left[\left\{\ell+\frac{3}{2}\right\}A-\frac{\pi}{4}\right] - \cos\left[\left\{\ell+\frac{3}{2}\right\}B-\frac{\pi}{4}\right]\right)\Bigg]^2 \end{aligned} \tag{33}$$

The cosine terms will now cause some oscillations in the C_ℓ values that depend on the ℓ values, however as they are bounded between $[-2, +2]$, we can simply label their values as α and β, as they will only contribute to $\mathcal{O}(1)$ corrections, and no impact the ℓ scaling of the C_ℓ values. Therefore, this gives us

$$C_\ell = \frac{\pi}{(2\ell+1)^2}\left[\frac{2}{\sqrt{\sqrt{3}\pi(\ell-1)}}\alpha + \frac{2}{\sqrt{\sqrt{3}\pi(\ell+1)}}\beta\right]^2 \tag{34}$$

If we now take $\ell \gg 1$, we find the C_ℓ values as approximately

$$C_\ell \simeq \frac{\pi}{4\ell^2}\left[\frac{2}{\sqrt{\sqrt{3}\pi\ell}}(\alpha+\beta)\right]^2, \tag{35}$$

and so we find

$$C_\ell \simeq \frac{1}{\sqrt{3}\ell^3}(\alpha+\beta)^2, \tag{36}$$

which is the desired scaling ratio, as then $\ell(\ell+1)C_\ell$ will scale like $1/\ell$, just as we found numerically.