MTH 354 10-12-17

Recurrence Relation of degree 2

Def. A recurrence relation is a linear homoge-us relation of degree k (order k) if it is of the form $a_n = r_1 a_{n-1} + r_2 a_{n-2} + \dots + r_k a_{n-k}$ with the r_i's constant.

Note: We note that on the right-hand side, the terms are each built in the same (homogeneous) way, as a multiple of one of the k (degree) previous terms.

Def. For a linear homoge recurrence relation of degree k, $a_n = r_1 a_{n-1} + r_2 a_{n-2} + \dots + r_k a_{n-k}$, we call the associated poly of degree k

$x^k = r_1 x^{k-1} + r_2 x^{k-2} + \dots + r_k$, its characteristic equation. The roots of the characteristic equation play a key role in the explicit formula for the sequence defined by the RR & the initial conditions.

Note: While the problem can be solved in general, we only consider the case of degree 2.

For that case, the RR would be $a_n = r_1 a_{n-1} + r_2 a_{n-2}$.

Hence the characteristic equation is $x^2 = r_1 x + r_2$.

It is common to write this as $x^2 - r_1 x - r_2 = 0$.

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Use Gersting's book

To solve RR of the form $S(n) = c_1 S(n-1) + c_2 S(n-2)$ subject to intial conditions $S(1)$ and $S(2)$

1. Solve the characteristic equation

$$t^2 - c_1 t - c_2 = 0$$

2. If the chara. equation has distinct roots r_1 and r_2, the solution is

$$S(n) = p r_1^{n-1} + q r_2^{n-1}$$

where

$$p + q = S(1)$$
$$p r_1 + q r_2 = S(2)$$

3. If the charac. equation has a repeated root r, the solution is

$$S(n) = p r^{n-1} + q (n-1) r^{n-1}$$

where

$$p = S(1)$$
$$pr + qr = S(2).$$

Ex: Solve $a_n = 4a_{n-1} + 5a_{n-2}$ $a_1 = 2$, $a_2 = 6$

Sol: The charac equation is $t^2 - 4t - 5 = 0$

$$(t-5)(t+1) = 0$$

There are 2 distinct roots $r_1 = 5$, $r_2 = -1$.

Therefore $a_n = p(5)^{n-1} + q(-1)^{n-1}$

$$\left.\begin{array}{l} p + q = 2 \\ 5p + (-1)q = 6 \end{array}\right\} \Rightarrow \begin{array}{l} p + q = 2 \\ 5p - q = 6 \end{array}$$

$$+ \quad 6p = 8 \Rightarrow p = \frac{8}{6} = \boxed{\frac{4}{3} = p}$$

$$\frac{4}{3} + q = 2$$
$$q = 2 - \frac{4}{3} = \boxed{\frac{2}{3} = q}$$

Therefore, the RR $a_n = 4a_{n-1} + 5a_{n-2}$ with initial conditions $a_1 = 2$, and $a_2 = 6$ has the closed form $a_n = \frac{4}{3}(5)^{n-1} + \frac{2}{3}(-1)^{n-1}$

RR	closed form
$a_1 = 2$	$a_1 = 2$
$a_2 = 6$	$a_2 = 6$
$a_3 = 4a_2 + 5a_1 = 4(6) + 5(2) = 34$	$a_3 = 34$
$a_4 = 4a_3 + 5a_2 = 4(34) + 5(6) = 166$	$a_4 = 166$

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show $a_n = 4a_{n-1} + 5a_{n-2}$ and $a_n = \left(\frac{4}{3}\right)(5)^{n-1} + \left(\frac{2}{3}\right)(-1)^{n-1}$ produce the same sequence of values.

Pf (induction)

B.C. ✓

I.C Assume $a_r = \left(\frac{4}{3}\right)(5)^{r-1} + \left(\frac{2}{3}\right)(-1)^{r-1}$ for all $1 \leq r \leq k$ for some $k \geq 1$. We need to show $a_{k+1} = \left(\frac{4}{3}\right)(5)^{k} + \left(\frac{2}{3}\right)(-1)^{k}$

We note that

$$a_{k+1} \overset{RR}{=} 4a_k + 5a_{k-1} \overset{I.H}{=} 4\underbrace{\left(\frac{4}{3}(5)^{k-1} + \frac{2}{3}(-1)^{k-1}\right)}_{a_k} + 5\underbrace{\left(\frac{4}{3}(5)^{k-2} + \frac{2}{3}(-1)^{k-2}\right)}_{a_{k-1}}$$

$$= \left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\underbrace{5^{k}}_{5\cdot 5^{k-1}} + \left(\frac{2}{3}\right)(-4)(-1)(-1)^{k-1} + \left(\frac{4}{3}\right)\left(\frac{25}{5}\right)(5)^{k-2} + \left(\frac{2}{3}\right)(5)(-1)^{k-1}$$

$$= \left(\frac{4}{3}\right)\left(\frac{4}{5}\right)5^{k} + \left(\frac{2}{3}\right)(-4)(-1)^{k} + \left(\frac{4}{3}\right)\left(\frac{1}{5}\right)(5)^{k} + \left(\frac{2}{3}\right)(5)(-1)^{k}$$

$$= \left(\frac{4}{3}\right)(5)^{k}\underbrace{\left(\frac{4}{5} + \frac{1}{5}\right)}_{=1} + \left(\frac{2}{3}\right)(-1)^{k}\underbrace{(-4+5)}_{=1}$$

$$a_{k+1} = \left(\frac{4}{3}\right)5^{k} + \left(\frac{2}{3}\right)(-1)^{k}.$$

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One more example for recurrence relation of degree 2.
Solve the following RR.

$a_n = 2a_{n-1} - a_{n-2}$, $a_1 = 1.5$, $a_2 = 3$

Sol: We note that the charac equation is

$t^2 - 2t + 1 = 0$ $(t^2 = +2t - 1)$

$(t-1)^2 = 0$

$t = 1$ is the only real root. So $r = 1$.

$a_n = P(1)^{n-1} + Q(n-1)(1)^{n-1}$

$P = \boxed{a_1 = 1.5}$ $\boxed{P = 1.5}$

$P(1) + Q(1) = a_2 = 3 \Rightarrow \boxed{Q = 1.5}$

$a_n = 1.5 + 1.5(n-1) = \cancel{1.5} + 1.5n - \cancel{1.5} = 1.5n$

$a_n = 1.5n$

RR	Closed form
$a_1 = 1.5$	$a_1 = 1.5$
$a_2 = 3$	$a_2 = 3$
$a_3 = 2a_2 - a_1 = 6 - 1.5 = 4.5$	$a_3 = 1.5(3) = 4.5$
$a_4 = 2a_3 - a_2 = 9 - 3 = 6$	$a_4 = (1.5)(4) = 6$