Induction Step

We use $P(j)$, $j \leq k$ to show $P(k+1)$: $f_{k+1} \leq \left(\frac{5}{3}\right)^{k+1}$. Consider the left-hand side of $P(k+1)$:

$$\begin{aligned}
f_{k+1} = f_k + f_{k-1} &\leq \left(\frac{5}{3}\right)^k + \left(\frac{5}{3}\right)^{k-1} \\
&= \left(\frac{5}{3}\right)^{k-1}\left(\frac{5}{3}+1\right) \\
&= \left(\frac{5}{3}\right)^{k-1}\left(\frac{8}{3}\right) \\
&< \left(\frac{5}{3}\right)^{k-1}\left(\frac{5}{3}\right)^2 \\
&= \left(\frac{5}{3}\right)^{k+1}, \quad \text{the right-hand side of } P(k+1).
\end{aligned}$$

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3.5 Exercises

In Exercises 1 through 6, give the first four terms and identify the given recurrence relation as linear homogeneous or not. If the relation is a linear homogeneous relation, give its degree.

1. $a_n = 2.5a_{n-1}$, $a_1 = 4$
2. $b_n = -3b_{n-1} - 2b_{n-2}$, $b_1 = -2$, $b_2 = 4$
3. $c_n = 2^n c_{n-1}$, $c_1 = 3$
4. $d_n = nd_{n-1}$, $d_1 = 2$
5. $e_n = 5e_{n-1} + 3$, $e_1 = 1$
6. $g_n = \sqrt{g_{n-1} + g_{n-2}}$, $g_1 = 1$, $g_2 = 3$
7. Let $A = \{0, 1\}$. Give a recurrence relation for the number of strings of length n in A^* that do not contain 01.
8. Let $A = \{0, 1\}$. Give a recurrence relation for the number of strings of length n in A^* that do not contain 111.
9. On the first of each month Mr. Martinez deposits \$100 in a savings account that pays 6% compounded monthly. Assuming that no withdrawals are made, give a recurrence relation for the total amount of money in the account at the end of n months.
10. An annuity of \$10,000 earns 8% compounded monthly. Each month \$250 is withdrawn from the annuity. Write a recurrence relation for the monthly balance at the end of n months.
11. A game is played by moving a marker ahead either 2 or 3 steps on a linear path. Let c_n be the number of different ways a path of length n can be covered. Give a recurrence relation for c_n.

In Exercises 12 through 17, use the technique of backtracking to find an explicit formula for the sequence defined by the recurrence relation and initial condition(s).

12. $a_n = 2.5a_{n-1}$, $a_1 = 4$

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13. $b_n = 5b_{n-1} + 3$, $b_1 = 3$
14. $c_n = c_{n-1} + n$, $c_1 = 4$
15. $d_n = -1.1d_{n-1}$, $d_1 = 5$
16. $e_n = e_{n-1} - 2$, $e_1 = 0$
17. $g_n = ng_{n-1}$, $g_1 = 6$

In Exercises 18 through 23, solve each of the recurrence relations.

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18. $a_n = 4a_{n-1} + 5a_{n-2}$, $a_1 = 2$, $a_2 = 6$
19. $b_n = -3b_{n-1} - 2b_{n-2}$, $b_1 = -2$, $b_2 = 4$
20. $c_n = -6c_{n-1} - 9c_{n-2}$, $c_1 = 2.5$, $c_2 = 4.7$
21. $d_n = 4d_{n-1} - 4d_{n-2}$, $d_1 = 1$, $d_2 = 7$
22. $e_n = 2e_{n-2}$, $e_1 = \sqrt{2}$, $e_2 = 6$
23. $g_n = 2g_{n-1} - 2g_{n-2}$, $g_1 = 1$, $g_2 = 4$
24. Develop a general explicit formula for a nonhomogeneous recurrence relation of the form $a_n = ra_{n-1} + s$, where r and s are constants.
25. Test the results of Exercise 24 on Exercises 13 and 16.
26. Let r_n be the number of regions created by n lines in the plane, where each pair of lines has exactly one point of intersection.
    (a) Give a recurrence relation for r_n.
    (b) Solve the recurrence relation of part (a).
27. Let a_n be the number of ways a set with n elements can be written as the union of two disjoint subsets.
    (a) Give a recurrence relation for a_n.
    (b) Solve the recurrence relation of part (a).
28. Prove Theorem 1(b). (*Hint*: Find the condition on r_1 and r_2 that guarantees that there is one solution s.)