Unbiased Risk Estimates for Singular Value Thresholding and Spectral Estimators

Motivation: High noise levels in cardiac Magnetic Resonance Imaging (MRI) data impede the ability of doctors to deliver accurate patient evaluations. A powerful strategy for denoising lies in a procedure called singular value thresholding

$$\mathsf{SVT}_\lambda(X) = \operatorname*{argmin}_{Z \in \mathbb{R}^{m \times n}} \frac{1}{2} \|Z - X\|_F^2 + \lambda \|Z\|_*$$

which (effectively) finds a low rank matrix that best approximates observed MRI data. The rank of the solution is determined by the parameter λ, and ideally one would choose λ to minimize the risk

$$\mathbb{E}_\theta \left[\|\mathsf{SVT}_\lambda(X) - \theta\|_F^2 \right].$$

Since the risk is unknown, this paper proposes an unbiased estimate of the risk which can be used to select λ.

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Model: $X_{ij} \overset{\text{ind}}{\sim} \mathcal{N}(\theta_{ij}, \tau^2)$ for unknown $\theta \in \mathbb{R}^{m \times n}$, known τ

Estimand: $g(\theta) = \mathbb{E}_\theta \left[\|\mathsf{SVT}_\lambda(X) - \theta\|_F^2 \right]$ for a fixed $\lambda \in \mathbb{R}$

Decision Space: $\mathcal{D} = \mathbb{R}$ **Loss:** $L(\theta, d) = (g(\theta) - d)^2$

Initial estimator: Let $\sigma_i(X) = i$-th largest singular value. Then,

$$\delta(X) = -mn\tau^2 + \sum_{i=1}^{\min(m,n)} \min(\lambda^2, \sigma_i^2) + 2\tau^2 \mathsf{div}(\mathsf{SVT}_\lambda(X))$$

where the expression $\mathsf{div}(\mathsf{SVT}_\lambda(Y))$ is given by
$\sum_{i=1}^{\min(m,n)} \left[\mathbb{I}\left(\sigma_i > \lambda\right) + |m - n|(1 - \frac{\lambda}{\sigma_i})_+ \right] + 2 \sum_{i \neq j, i,j=1}^{\min(m,n)} \frac{\sigma_i (\sigma_i - \lambda)_+}{\sigma_i^2 - \sigma_j^2}$.

- UMVUE / UMRUE, Not Bayes, Inadmissible

Alternative estimator: $\delta'(X) = (\delta(X))_+$

- Dominates δ

Preference: If my only goal were to estimate $g(\theta)$, then I would prefer the dominating δ' over δ. However, I have not investigated whether this will also lead to better selection of λ for SVT.