# 2012 Math Subject Test Problems

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1.A 12-feet-length ladder is leaning against the wall, find the maximum value of the area bounded by the wall, the ground and the ladder.

2.An object is tossing upward with an initial speed of $64feet/s$. Suppose the gravitational acceleration is $32feet/s^2$. Find the smallest $t$, such that the object reach the height of $96feet$ at time $t$.

3.There are three red balls and two black balls in box A, one red ball and two black balls in box B. Taking 2 balls without replacement from both boxes respectively. Consider the probability that there are exactly three red balls in the four chosen balls.

4.The area of triangle ABC is 144. Denote the midpoint of BC as P, the midpoint of the segment AP Q and R as the midpoint of AC. Calculate the area of triangle PQR.

5.Calculate $\log \sqrt{e^3}$.

6.We define a graph without loops as a graph that no edge links the same vertex. If a vertex has an odd valency, we call it an odd vertex, denote the number of odd vertex as O. Analogously, we have E. Which of the following statement must be true?

A. O can be even or odd, and E can also be even or odd.

B. O can be even or odd, and E must be odd.

C. O must be even, and E must be odd.

D. O must be even, and E might be odd or even.

E. O must be odd, and E might be odd or even.

7.The radius of two circles that are externally tangent to each other is r, and s. Suppose $r > s$, and two outer tangent of the circles intersects at point P. Denote the center of the smaller circle as O. Find OP.

8.The distance between the origin and the line $x + 2y - 5 = 0$.

9.Let S be the set of all maps from $X = \{1, 2, 3, 4\}$ to itself. T is the power set of S, i.e., the set of all subsets of S. Determine the cardinality of T.

10.Choose two distinct numbers $a$ and $b$ randomly out of the set of integers:$\{1, 2, \ldots, 53\}$. How many ways are possible that $a + b$ is even.

11.$f(x) = \frac{x}{x-1}$,$g\left(f(x)\right) = 2 + x$. Calculate $g(-1)$.

12.Let S be the space of all $3 * k$ matrices, T be the space of all column vectors consists of seven components. If S is isomorphic to a subspace of T. What are possible values of $k$?

13. Determine the dimension of the solution space of the following equations:

$$\begin{cases} x_1 + 2x_2 + x_5 & = 0 \\ x_3 - x_5 & = 0 \\ x_4 + 2x_5 & = 0 \end{cases}$$

14. Suppose A be a real square matrix with order $n$. There is at least one nontrivial solution for the homogeneous system: $Ax = 0$. For any real column vector $b$ with n components, the equation $Ax = b$ has

A. no solutions;

B. unique solution;

C. infinitely many solutions;

D. finitely many solutions.

15. If T is an orthogonal matrix of a rotation on $x - y$ plane. If the first row of T is $(0.6, 0.8)$, the second row would be:

A. $(0.8, -0.6)$;

B. $(0.8, 0.6)$;

C. $(-0.8, 0.6)$;

D. $(-0.8, -0.6)$;

E. not unique.

16. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transform. Given $L(1, 2) = (1, 0)$ and $L(2, 3) = (0, 1)$. Calculate $L(3, 1)$.

17. Find $b$ such that two eigenvalues of the matrix

$$A = \begin{pmatrix} b & 2 \\ -2 & -2 \end{pmatrix}$$

are same.

18. M is a 5*5 matrix, apply the following operations onto M respectively:

(1)double the elements in the first row;

(2)subtract three times of second row from the third row(without changing the second row);

(3)substitute the fourth row with the fifth.

If the $|M|$ is known, determine $|K|$.

19.Suppose $f(x)$ be a strictly increasing function, let $f(x) = f(2x) + g(x)$. Denote sgn the sign function, i.e.

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Which of the following is correct?

A. sgn g(x) is the same as sgn x;

B. sgn g(x) is the opposite of sgn x;

C. sgn g(x) is the same as sgn f(x);

D. sgn g(x) is the opposite of sgn f(x);

E. sgn g(x) is a constant.

20. Let $f(x)$ be a continuous differentiable function. Given $f(3) = 1, f'(3) = -3$, calculate $\frac{d}{\mathrm{dx}}(\frac{f(x)}{x^2})$ at $x = 3$.

21. $f(x) = \int_0^{x^2}(x^2 - t^2)\mathrm{dt}$.Calculate $f'(x)$.

22. $\int_{-1}^{1}(|x| - 2x)\mathrm{dx}$.

23. $l$ is the tangent line of $y = \frac{1}{2}x^2 - 2x + 3$ at point $(a, b)$, and $l$ is perpendicular to $y = -2x$. Find $a$.

24. Of all the followings, which one is different from $\int_0^\pi \sqrt{1+\cos^2 t}\mathrm{d}t$, numerically?

A. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+\cos^2 t}\mathrm{d}t$;

B. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+\sin^2 t}\mathrm{d}t$;

C. $\int_0^\pi \sqrt{1+\sin^2 t}\mathrm{d}t$;

D. $2\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2(2t)}\mathrm{d}t$;

E. $2\int_0^\pi \sqrt{1+\cos^2(2t)}\mathrm{d}t$.

25.Suppose $f(x)$ be a continuous function on the interval $(a,b)$, then:

A. there exists $x \in (a,b)$ such that $f(x) \leqslant f(t) \forall t \in (a,b)$;

B. there exists $M \in \mathbb{R}$ such that $|f(x)| \leqslant M \forall x \in (a,b)$;

C. if there exists $u, v \in (a,b)$ with $f(u) * f(v) < 0$, then there also exists $w \in (a,b)$ and $f(w) = 0$;

D. $f(x)$ is differentiable on the interval $(a,b)$;

26. Suppose sequence $a_n$ and $b_n$ both have limits and are finite, which of the following is sufficient for $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$?

$A.$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$

$B.$ $\lim_{n\to\infty} \frac{a_n^2}{b_n^2} = 1$;

$C.$ $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$;

$D.$ $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = 1$;

$E.$ $\left|(1-\frac{1}{n})a_n\right| \leqslant |b_n| \leqslant |a_n|$.

27. Compare the value of $10!, 10^9, 9^{10}$.

28. Determine $\prod_{k=1}^{\infty} 2^{\frac{1}{2^k}}$.

29. The sequence $\{x_n\}$ is defined iteratively as $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, if the limit of the sequence exists, what is it?

30. How many roots does the equation $(x^x)^x = x^{x^x} (x > 0)$ have?

31. Consider the implicit function $y = F(x)$ satisfying the parameter form $x = 2e^t - e^{-t}, y = 2e^t + e^{-t}$.Find the minimum of $y$.

32. Calculate $\int_0^1 f(x)\mathrm{dx}$, while the function is defined as:

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ n, & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n}] \end{cases}$$

33. Select the convergence region for the improper integral:$\int_1^\infty \frac{x^\alpha}{1+x^\alpha}\mathrm{dx}$.

34. Which of the following conclusion can be proved by the Mean Value Theorem?

$$|\sin x - \sin y| \leqslant |x - y|;$$

$$|\tan x - \tan y| \geqslant |x - y|, 0 \leq x, y \leq \frac{\pi}{2};$$

$$|2^x - 2^y| \geqslant |x - y|;$$

35. Suppose $f(x) = x + \sin x$, which of the following is correct?

A. $f(x)$ is continuous on $\mathbb{R}$, but is not uniformly continuous;

B. $f(x)$ is uniformly continuous on $\mathbb{R}$, but does not have bounded variation;

C. $f(x)$ has bounded variation on $\mathbb{R}$, but is not uniformly continuous;

D. $f(x)$ has bounded variation on $\mathbb{R}$, and is uniformly continuous;

E. $f(x)$ does not have bounded variation on bounded region in $\mathbb{R}$, and is uniformly continuous.

36. $\lim_{x\to 1} \frac{(x-\frac{1}{x})^3}{(x-1)^3}$.

37. Evaluate the angle between two curves at their intersection: $y - x^2 = 1$ and $x^2 + y^2$=1.

38. Which of the following is the normal vector of the surface $x^2+3y^2-z=0$ at point (1,1,4)?

39. Let $a_n \in A$, and $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$, then:
$\exists a \in A$, such that $a = \lim a_n$;
$\exists b \in \mathbb{R}$, such that $b = \lim a_n$.
$a_n$ is bounded.

40. $f(x), g(x)$ are defined on $[-1,1]$,$f'(0), g'(0)$ exist, $f(0) = g(0)$, and $f(x) \geqslant g(x)$ holds for an open interval containing 0. Then,
I. $f(x)$ and $g(x)$ have the same tangent line at $(0,0)$;
II. $f'(0) \geqslant g'(0)$;
III. $g''(0)$ exists.

41. $C : x^2 + y^2 = 4$, evaluate $\oint_C (y + e^x)\mathrm{dx} + (x - e^{-y})\mathrm{dy}$.

42. $f(x,y)$ is differentiable on $\mathbb{R}^2$ , $\frac{\partial f}{\partial x} = xy, \frac{\partial f}{\partial y} = x + y$,what is $f(x,y)$?

43. $f$ is continuous differentiable function, which of the following is sufficient for $\lim_{n\to\infty} \int_{-n}^{x} f'(s)\mathrm{ds} \to f(x)$?
A.
$$\lim_{x\to-\infty} f(x) = 0;$$
B.
$$\lim_{x\to-\infty} f'(x) = 0;$$

44. Evaluate the volume of the solid determined by: $x, y, z \in [0,1], z \geqslant xy$.

45. How many roots does the polynomial equation $x^5 - 5x + 1 = 0$ have?

46. Consider $\Omega : \{(x,y,z)|x^2 + y^2 \leqslant 1, 0 \leqslant z \leqslant 2\}$ and the transform

$T : (x, y, z) \to (x, y + \tan \alpha z, z)$, where $\alpha \in (0, \pi)$. What is the volume of $T(\Omega)$?

47. What is the residue of $x^{15} - 1$ divided by $x^2 - 1$?

48. Which of the following is correct?
I. $n^3 \equiv n(\mod 3)$.
II. $n^3 \equiv n(\mod 6)$.
II. $n^6 \equiv 1(\mod 7)$.

49.$x, y \in \mathbb{Z}$, find the smallest positive integer that can be written as $70x + 42y$.

50.b=0;
input(n)
while(n>1)
n=n/2,
b=b+1
output(b)
Find the estimation of $b$.

51. An Abel group $G$ is generated by $x$ and $y$, with $|x| = 16, |y| = 24$,$x^2 = y^3$. What is the order of G?

52. $p$ is a prime, G is a cyclic group with order $p^2$. How many subgroups does $G$ have?

53. Let $\phi$ be a homomorphism from $(X, \oplus)$ to $(Y, *)$. Try to find the necessary condition of $\oplus$ is $*$ is a known operator?

54. In $\mathbb{Z}_p$, what is the number of roots of $x^2 + 1 = \bar{0}$? (p is a prime).

A. 0;

B. $\frac{p}{2}$;

C. 2;

D. it depends;

E. 4.

55. Let $f$ be a homomorphism from group $G$ to group $H$, $K$ be the kernel of $f$,

I. $K$ is a normal subgroup in $G$;

II. $f(G)$ is a normal subgroup in $H$;

III. $G/K$ is isomorphic to a subgroup in H.

56.$S$ is the power set of $\mathbb{Z}$. Define two binary operations: $\oplus$(the symmetric difference set which means A $\oplus$ B=(A$\bigcup$B)-(A$\bigcap$B) and $*$(the intersection of two sets A*B=A$\bigcap$B), which form a ring,

1.$*$ has a unit.

2. If $A * B$ is an empty set， $A$ or $B$ is empty.

3. Every element's inverse in $S$ is itself.

57. Suppose $X, Y$ be random variables satisfying $Y = \frac{X+4}{5}$, and the standard deviation of $X$ is 20, what is the standard deviation of $Y$?

58. Find $b$ such that $f(x)$ is a probability density function

$$f(x) = \begin{cases} \frac{1}{x(\ln bx)^2} & \text{for } x \geqslant 2 \\ 0 & \text{for } x < 2 \end{cases}$$

59. Scratch the graph of $y' + y^3 = 0$.

60. $y'' + by' + cy = 0$, $b, c > 0$. Then:

A. $\lim_{t\to-\infty} y(t) = 0$;

B. $\lim_{t\to\infty} y(t) = 0$;

C. $y(t)$ is periodic;

D. $\lim_{t\to-\infty} y(t)$ does not exist.

61. What is $\cup_{i=1}^{\infty} \cap_{j\geq i}^{\infty} A_i$?

A. All elements that are in finitely many $A_i$;

B. All elements that are in infinitely many $A_i$;

C. All elements that are in all but infinite numbers of $A_i$.

62.What is the type of the graph of $|z-i|+|z+i|=4$?

A. ellipse(non-circular);

B. Circle;

C. Parabola;

D. Hyperbola;

E. Two lines.

63. Calculate the integral of $\frac{1}{z(z-1)}$ along the curve $C$:

64. Suppose $A$ be a connected subset of topology space $X$, which of the following must be connected:

I. the interior of A;

II. the closure of A;

III. the complement of A.

65. Suppose $X$ and $Y$ be metric space,$A \subseteq X$, $f$ be a continuous map from $A$ to $Y$, then:

I. $A$ is compact, then $f(A)$ is compact;

II. If $A$ is bounded, then $f(A)$ is uniformly continuous;

III. Given $f$ is bijection,$A$ is bounded, $f(A)$ is compact, then $f^{-1}$ must be continuous.