A Different Approach in the Shrinkage Estimation for the Rayleigh Model

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Abstract

In the present paper, the shrinkage estimators have been developed based on the Upper record values under the Bayesian Minimax Risk Criteria. The properties of the proposed shrinkage estimators have been studied in terms of the relative efficiency with respect to improved estimator.

Keywords

Bayes Estimator; Linex Loss Function; Record Values; Reliability Function; Minimax Estimation

Introduction

The Rayleigh distribution is a suitable model for life testing experiments and clinical studies. A significant amount of work has been done related to Rayleigh model. The origin and other aspects of this distribution are found in Siddiqui (1962) and Hirano (1986). Polovko (1968), Dyer & Whisenand (1973) demonstrated the importance of this distribution in electro vacuum devices and communication engineering respectively. Its many applications have also been discussed by Ariyawansa and Templeton (1984). Hendi et al. (2007) presented a Bayesian analysis for record values from the Rayleigh model. Recently, Prakash and Prasad (2010) have present a Bayes Prediction interval for the Rayleigh model.

The objective of the present paper is to study the properties of the Shrinkage estimator based on the Upper record values under the Bayesian Minimax Risk Criteria.

The Considered Model and Prior Distributions

The probability density function and distribution function of the Rayleigh model with the parameter θ are given as

$$f\left(x\,;\theta\right)=\frac{2\,x}{\theta}\exp\left(-\frac{x^2}{\theta}\right);\, x\geq 0,\theta>0 \qquad (2.1)$$

and

$$F\left(x\,;\theta\right)=1-\exp\left(-\frac{x^2}{\theta}\right). \qquad (2.2)$$

Here θ is known as the shape parameter. The Reliability function $\Psi(t)$ and Failure rate $\rho(t)$ for any specific mission time $t>0$ are obtained respectively as

$$\Psi(t)=\exp\left(-\frac{t^2}{\theta}\right);\theta>0 \qquad (2.3)$$

and

$$\rho(t)=\frac{2\,t}{\theta};\theta>0. \qquad (2.4)$$

Here, we are not going in to the debate or to justify the questions of the proper choice of the prior distribution. For situations where life tester has no prior information about the parameter, they may use the quasi-density prior. Here we consider a quasi prior of the following form:

$$\pi(\theta)\propto\frac{1}{\theta^d}\ ;d>0,\theta>0. \qquad (2.5)$$

If $d=1$ we get a non-informative prior (Jeffrey's (1961))

$$\pi_1(\theta)\propto\frac{1}{\theta}$$

and if $d=3$ we get the asymptotically invariant prior, proposed by Hartigan (1964)

$$\pi_2(\theta)\propto\frac{1}{\theta^3}.$$

Record Values and the Estimation

Record values found in many conditions of daily life as well as in numerous statistical applications. The record values, can be viewed as order statistic from a sample whose size is determined by the values and the order of occurrence of the observations. In a little over thirty years, a large number of publications devoted to records have appeared.

This is possibly due to the fact that we encounter this notion frequently in daily life, especially in singling out record values from a set of other ones and in registering and recalling record values.

Motivated by extreme weather conditions, Chandler (1952) introduced the study of record values and documented many of the basic properties of records. Feller (1966) gave some examples of record values with respect to gambling problems. Resnick (1973) discussed the asymptotic theory of records. The theory of record values and its distributional properties have been extensively studied by Ahsanullah (1995), Arnold et al. (1998).

Let $X_1, X_2, \ldots$ be the sequence of independent and identically distributed random variables following the model (2.1). The observation X_j is an upper record value if it is larger than all preceding observations, i.e., if $X_j > X_i$, for all $i < j$. The indices at which these upper record values occur are called upper record times $\{U(m),\ m \geq 0\}$, where $U(0) = 1$ and $U(m) = \min\{j : j > U(m-1),\ X_j > X_{U(m-1)}\}$.

Let $X = X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ be the n upper record values from model (2.1). The likelihood function is thus obtained as

$$L(x \mid \theta) = f\left(x_{U(n)}, \theta\right) \prod_{i=1}^{n-1} \rho\left(x_{U(i)}\right) \tag{3.1}$$

It follows from (2.1), (2.4) and (3.1) that

$$L(x \mid \theta) = \left(\prod_{i=1}^{n} 2\, x_{U(i)}\right) \theta^{-n} \exp\left(-\frac{S}{\theta}\right); \quad \theta > 0, S = x_{U(n)}^2 . \tag{3.2}$$

The Maximum likelihood estimator (MLE) and unbiased estimator for the parameter θ based on the upper records values are given respectively as

$$\hat{\theta}_{ML} = \frac{S}{n} \text{ and } \hat{\theta}_U = S.$$

Bayes Estimation

Under the Bayes theorem, the posterior distribution of the parameter θ is obtained as

$$g(\theta \mid x) = \frac{L(x \mid \theta) \cdot \pi(\theta)}{\int_\theta L(x \mid \theta) \cdot \pi(\theta)\, d\theta} = \frac{S^{n+d-1}}{\Gamma(n+d-1)\, \theta^{n+d}} \exp\left(-\frac{S}{\theta}\right); \theta > 0, d > 0.$$

The Bayes estimator for the parameter θ under the SELF (squared error loss function) is simply the posterior mean and given as

$$\hat{\theta}_{BS} = \omega_1 S\ ; \quad \omega_1 = \frac{1}{n+d-2}. \tag{4.1}$$

Based on the Bayesian context, it is standard practice for most statisticians to consider the squared error loss function as it does not lead to extensive numerical computation. The symmetric nature of this function confers equal importance to overestimation as well as underestimation. If the SELF is taken as a measure of inaccuracy then the resulting risk is often too sensitive to the assumptions about the behaviour of the tail of the probability distribution. In addition, in some estimation problems overestimation is more serious than underestimation, or vice-versa (Parsian & Kirmani (2002)).

To deal with such cases, a useful and flexible class of asymmetric loss function (LINEX loss function (LLF)) was introduced by Varian (1975). The re-parameterized version of LLF (Singh et al. (2007)) for any parameter θ is given as

$$L(\Delta) = e^{a\Delta} - a\Delta - 1; \quad a \neq 0 \text{ and } \Delta = \left(\hat{\theta} - \theta\right)/\theta. \tag{4.2}$$

The sign and magnitude of $'a'$ represents the direction and degree of asymmetry respectively. The positive (negative) value of $'a'$ is used when overestimation is more (less) serious than underestimation. $L(\Delta)$ is approximately square error and almost symmetric if $|a|$ near to zero.

Bayes estimator of the parameter θ under the LLF is denoted by $\hat{\theta}_{BL}$ and obtained by simplifying the equation

$$E_p\left[\frac{1}{\theta} \exp\left(a\, \frac{\hat{\theta}_{BL}}{\theta}\right)\right] = e^a\, E_p\left(\frac{1}{\theta}\right).$$

Here the suffix p indicates the expectation taken under the posterior. Hence, the Bayes estimator under the LLF is

$$\hat{\theta}_{BL} = \omega_2 S \ ; \ \ \omega_2 = \frac{1}{a}\left(1 - \exp\left(-\frac{a}{n+d}\right)\right). \qquad (4.3)$$

The expressions of the risk for the Bayes estimators are summarized in the following table under different loss

Estimator	Risk Under SELF
$\hat{\theta}_{BS}$	$R\left(\hat{\theta}_{BS}\right) = \theta^2\left(\omega_1^2 + (\omega_1 - 1)^2\right)$
$\hat{\theta}_{BL}$	$R\left(\hat{\theta}_{BL}\right) = \theta^2\left(\omega_2^2 + (\omega_2 - 1)^2\right)$
Estimator	Risk Under LLF
$\hat{\theta}_{BS}$	$R\left(\hat{\theta}_{BS}\right) = \frac{e^{-a}}{1 - a\,\omega_1} - a\,\omega_1 + a - 1$
$\hat{\theta}_{BL}$	$R\left(\hat{\theta}_{BL}\right) = \frac{e^{-a}}{1 - a\,\omega_2} - a\,\omega_2 + a - 1$

The Minimax Bayes Risk Estimator

The basic principle of Minimax Bayes risk estimation is to minimize the loss. Following Hodge and Lehmann (1950):

Let $\omega = \left\{F_\theta : \theta \in \Theta\right\}$ be a family of distribution functions and C be a class of estimators of the parameter θ. Suppose that $c^* \in C$ is a Bayes estimator against a prior distribution $\pi(\theta)$ on the parameter space Θ. Then the Bayes estimator c^* is said to be the Minimax estimator if the risk function of the estimator c^* is independent on Θ.

It is clear from the above table that the risks of the Bayes estimators under the SELF are not independent with the parameter θ. Hence, the Bayes estimators $\hat{\theta}_{BS}$ and $\hat{\theta}_{BL}$ are not the Minimax estimators. Therefore, under the SELF risk criterion the Minimax estimators do not exist. However, the risk of Bayes estimators $\hat{\theta}_{BS}$ and $\hat{\theta}_{BL}$ are independent of the parameter θ under the LLF risk criterion. Hence, both the estimators $\hat{\theta}_{BS}$ and $\hat{\theta}_{BL}$ are Minimax estimators under the LLF loss criterion.

The following statistical problem (Minimax Estimation) is equivalent to two person zero sum game between the Statistician (Player–II) and Nature (Player–I). Here the pure strategies of Nature are the different values of θ in the interval $(0,\infty)$ and the mixed strategies of Nature are the prior densities of θ in the interval $(0,\infty)$. The pure strategies of Statistician are all possible decision functions in the interval $(0,\infty)$.

The expected value of the loss function is the risk function and it is the gain of the Player–I. Further, the Bayes risk of the Bayes estimator $\hat{\theta}_B$ is defined as

$$R\left(\eta, \hat{\theta}_B\right) = E_\theta\, R\left(\hat{\theta}_B\right)$$

Here, the expectation has been taken under the prior density of the parameter θ. If the loss function is continuous in both the estimator $\hat{\theta}_B$ and the parameter θ, and convex in $\hat{\theta}_B$ for each value of θ then their exist a measures η^* and $\hat{\theta}_B^*$ for all θ and $\hat{\theta}_B$ so that, the following relation holds:

$$R\left(\eta, \hat{\theta}_B^*\right) \le R\left(\eta^*, \hat{\theta}_B^*\right) \le R\left(\eta^*, \hat{\theta}_B\right).$$

The number $R\left(\eta^*, \hat{\theta}_B^*\right)$ is known as the value of the game, and η^* and $\hat{\theta}_B^*$ are the corresponding optimum strategies of the Player I and II. In statistical terms η^* is the least favourable prior density of θ and the estimator $\hat{\theta}_B^*$ is the Minimax estimator. In fact, the value of the game is the loss of the Player–II. Hence, the optimum strategy of Player–II and the value of game are given for present case as

Optimum Strategy	Corresponding Loss	Value of Game
$\hat{\theta}_{BS} = \omega_1 S$	LLF	$\frac{e^{-a}}{1 - a\,\omega_1} - a\omega_1 + a - 1$
$\hat{\theta}_{BL} = \omega_2 S$	LLF	$\frac{e^{-a}}{1 - a\,\omega_2} - a\omega_2 + a - 1$

The Shrinkage Estimation

Thompson (1968) suggested a procedure, which makes use of the prior information of the parameter θ in form of a guessed value θ_0 by shrinking the usual estimator towards the guess value of the parameter with the help of a shrinkage factor (see Prakash & Singh (2009)).

The shrinkage procedure has been applied in numerous problems, including mean survival time in epidemiological studies (Harries & Shakarki, 1979), forecasting of the money supply (Tso, 1990), estimating mortality rates (Marshall, 1991), and improved estimation in sample surveys (Wooff, 1985). Following Thompson (1968), the shrinkage estimators based on the upper record values under the Bayesian Minimax criteria are defined as

$$\bar{\theta}_1 = \omega_1 \hat{\theta}_{BS} + (1-\omega_1)\theta_0 \ ; 0 \le \omega_1 \le 1. \tag{6.1}$$

and

$$\bar{\theta}_2 = \omega_2 \hat{\theta}_{BL} + (1-\omega_2)\theta_0 \ ; 0 \le \omega_2 \le 1. \tag{6.2}$$

It is observed later that for the considered set of parametric set of values, both ω_1 and ω_2 lie between zero and one. Hence, they are chosen as the shrinkage factors. The risk of the shrinkage estimators $\bar{\theta}_i \ ; \ i=1,2$ under the LLF is given as

$$R\left(\bar{\theta}_i\right) = \frac{\exp\left((1-\omega_i)\, a\,\delta - a\right)}{1 - a\,\omega_i^2} - a\,\omega_i^2 - a\,\delta\,(1-\omega_i) + a - 1 \ ; \ i = 1, 2.$$

Further, an unbiased estimator $\hat{\theta}_U$ for the parameter θ is also available. Therefore, a class of the unbiased estimator for the parameter θ is constructed as

$$T = k\,\hat{\theta}_U \ ; \ k \in R^+. \tag{6.3}$$

The value of k which minimizes the risk of the estimator T under the LLF is obtained as

$$k_{min} = \frac{1}{a}\left(1 - e^{-a/2}\right).$$

Hence, the improved class of estimator among the class (6.3) is

$$\hat{T} = k_{min}\,\hat{\theta}_U \,, \tag{6.4}$$

with the risk under the LLF is given as

$$R\left(\hat{T}\right) = 2\left(e^{-a/2} - 1\right) + a\,.$$

***Remark*:**

Here we take only the LLF risk criterion because the Minimax estimator have been obtained only for the LLF loss criterion.

Numerical Study

The relative efficiency for the Bayes shrinkage Minimax estimator $\bar{\theta}_1$ and $\bar{\theta}_2$ with respect to improved estimator $\hat{T}$ are defined as

$$RE\left(\bar{\theta}_i, \hat{T}\right) = \frac{R\left(\hat{T}\right)}{R\left(\bar{\theta}_i\right)} \ ; i = 1, 2.$$

The expression of the relative efficiencies are the functions of a, d, δ and n. For the selected set of values for $a = 0.50, 1.00, 2.00$; $d = 0.50, 1, 2, 5, 10, 50$; $\delta = 0.50\,(0.25)\,1.50$ and $n = 05\,(05)\,20$; the relative efficiencies have been calculated and presented here in the Tables 1-4.

It is observed that both the Bayes Shrinkage Minimax estimators based on upper records value are uniformly performs better than the estimator $\hat{T}$ for all the considered set of parametric values. For both shrinkage estimators, the relative efficiency increases as sample size n increases for $\delta \le 1.00$ and decreases otherwise.

Similar trend has also been observed for $'d'$. The relative efficiency attains maximum when $\delta = 1.00$ (except for $d \le 2.00$). Further, it is also noted that the relative efficiency decreases in the interval $\delta \ge 0.75$ as the $'a'$ increases.

Conclusions

In present paper, we obtained the Shrinkage estimator based on the Bayes Minimax risk estimation criteria under the upper record values for the Rayleigh model. It is observed that on the basis of the relative efficiency, the proposed both Shrinkage Bayes Minimax estimators performs uniformly better than the improved estimator in a wide range of δ which is defined here as the ratio between the true value and predicted (prior point) value of the unknown parameter θ under the LLF criterion. Based on the magnitude of the relative efficiency, $\bar{\theta}_2$ (Shrinkage Bayes Minimax estimator under LLF) is recommended for the interval $\delta \le 1.00$ and the estimator $\bar{\theta}_1$ otherwise.

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TABLE 1: RELATIVE EFFICIENCIES BETWEEN THE ESTIMATOR $\overline{\theta}_1$ AND T^* FOR $n = 5$

a	$\delta \downarrow d \rightarrow$	0.50	1.00	2.00	5.00	10.00	25.00	50.00
0.50	0.50	1.6240	1.6381	1.6649	1.7298	1.7983	1.8866	1.9347
	0.75	3.9547	4.0918	4.3412	4.9284	5.5622	6.4351	6.9445
	1.00	21.952	25.627	32.561	59.666	123.42	452.90	1462.8
	1.25	92.965	71.050	46.648	23.669	15.063	10.109	8.5535
	1.50	5.9795	5.3324	4.4666	3.2825	2.6100	2.0936	1.8991
1.00	0.50	1.6452	1.6585	1.6836	1.7442	1.8083	1.8917	1.9374
	0.75	3.9387	4.0642	4.2925	4.8317	5.4164	6.2270	6.7031
	1.00	21.389	25.556	32.522	58.519	118.88	427.82	1368.8
	1.25	73.138	57.062	38.834	20.612	13.334	8.9896	7.5999
	1.50	4.9450	4.4462	3.7637	2.7978	2.2304	1.7845	1.6143
2.00	0.50	1.6828	1.6946	1.7166	1.7694	1.8259	1.9006	1.9423
	0.75	3.9083	4.0136	4.2058	4.6627	5.1636	5.8680	6.2873
	1.00	21.325	25.291	32.478	56.662	111.20	385.24	1209.9
	1.25	46.626	37.947	27.515	15.803	10.534	7.1620	6.0432
	1.50	3.4167	3.1154	2.6857	2.0370	1.6304	1.2966	1.1659

TABLE 1: RELATIVE EFFICIENCIES BETWEEN THE ESTIMATOR $\overline{\theta}_2$ AND T^* FOR $n = 5$

a	$\delta \downarrow d \rightarrow$	0.50	1.00	2.00	5.00	10.00	25.00	50.00
0.50	0.50	1.5747	1.5796	1.6029	1.6832	1.7720	1.8786	1.9321
	0.75	3.2185	3.3998	3.7308	4.5080	5.3159	6.3527	6.9163
	1.00	9.9191	12.218	17.437	38.465	91.858	390.24	1348.3
	1.25	141.12	115.48	62.146	37.033	17.551	10.424	8.6254
	1.50	14.659	11.332	7.4561	4.0389	2.8273	2.1307	1.9085
1.00	0.50	1.5930	1.5966	1.6180	1.6950	1.7805	1.8832	1.9347
	0.75	3.1748	3.3452	3.6577	4.3942	5.1598	6.1409	6.6736
	1.00	9.5300	11.693	16.588	36.236	85.958	362.93	1250.7
	1.25	127.84	104.41	56.589	33.589	15.794	9.3043	7.6719
	1.50	12.952	10.054	6.5974	3.5270	2.4417	1.8208	1.6236
2.00	0.50	1.6248	1.6262	1.6446	1.7156	1.7954	1.8912	1.9392
	0.75	3.0977	3.2496	3.5305	4.1971	4.8902	5.7759	6.2556
	1.00	8.8571	10.792	15.136	32.450	75.976	316.92	1086.4
	1.25	105.77	85.874	47.090	27.879	12.898	7.4707	6.1142
	1.50	10.144	7.9578	5.1979	2.7024	1.8262	1.3306	1.1746

TABLE 2: RELATIVE EFFICIENCIES BETWEEN THE ESTIMATOR $\overline{\theta}_1$ AND T^* FOR $n = 10$

a	$\delta \downarrow d \rightarrow$	0.50	1.00	2.00	5.00	10.00	25.00	50.00
0.50	0.50	1.7385	1.7467	1.7618	1.7983	1.8397	1.9012	1.9398
	0.75	5.0077	5.0827	5.2213	5.5622	5.9620	6.5862	7.0004
	1.00	65.005	70.574	82.403	123.42	210.21	608.81	1734.0
	1.25	22.106	20.798	18.741	15.063	12.268	9.5835	8.4154
	1.50	3.1749	3.0808	2.9237	2.6100	2.3351	2.0300	1.8808
1.00	0.50	1.7523	1.7600	1.7741	1.8083	1.8473	1.9055	1.9423
	0.75	4.9047	4.9738	5.1015	5.4164	5.7869	6.3680	6.7554
	1.00	63.597	68.888	80.109	118.88	200.55	573.41	1621.0
	1.25	19.313	18.219	16.483	13.334	10.898	8.5215	7.4758
	1.50	2.7081	2.6291	2.4969	2.2304	1.9941	1.7290	1.5983
2.00	0.50	1.7766	1.7833	1.7957	1.8259	1.8606	1.9131	1.9467
	0.75	4.7249	4.7839	4.8931	5.1636	5.4841	5.9917	6.3336
	1.00	61.287	66.097	76.269	111.20	184.15	513.38	1430.0
	1.25	14.894	14.118	12.867	10.534	8.6662	6.7875	5.9424
	1.50	1.9740	1.9182	1.8239	1.6304	1.4552	1.2542	1.1535

TABLE 2: RELATIVE EFFICIENCIES BETWEEN THE ESTIMATOR $\overline{\theta}_2$ AND T^* FOR $n = 10$

a	$\delta \downarrow d \rightarrow$	0.50	1.00	2.00	5.00	10.00	25.00	50.00
0.50	0.50	1.6947	1.7055	1.7252	1.7720	1.8233	1.8951	1.9376
	0.75	4.6114	4.7086	4.8867	5.3159	5.8021	6.5230	6.9763
	1.00	42.769	47.303	57.061	91.858	168.28	535.78	1609.1
	1.25	32.655	29.308	24.577	17.551	13.245	9.7957	8.4743
	1.50	3.8170	3.6322	3.3426	2.8273	2.4355	2.0560	1.8886
1.00	0.50	1.7060	1.7164	1.7353	1.7805	1.8299	1.8991	1.9400
	0.75	4.4922	4.5844	4.7531	5.1598	5.6201	6.3019	6.7302
	1.00	40.251	44.478	53.571	85.958	156.98	497.87	1492.2
	1.25	29.588	26.529	22.207	15.794	11.870	8.7340	7.5349
	1.50	3.3279	3.1622	2.9027	2.4417	2.0921	1.7544	1.6059
2.00	0.50	1.7258	1.7355	1.7532	1.7954	1.8415	1.9060	1.9441
	0.75	4.2859	4.3694	4.5222	4.8902	5.3061	5.9210	6.3066
	1.00	35.976	39.685	47.654	75.976	137.90	434.03	1295.5
	1.25	24.507	21.929	18.289	12.898	9.6113	6.9962	6.0006
	1.50	2.5410	2.4069	2.1973	1.8262	1.5465	1.2780	1.1606

TABLE 3: RELATIVE EFFICIENCIES BETWEEN THE ESTIMATOR $\overline{\theta}_1$ AND T^* FOR $n = 15$

a	$\delta\downarrow d\rightarrow$	0.50	1.00	2.00	5.00	10.00	25.00	50.00
0.50	0.50	1.8033	1.8081	1.8171	1.8397	1.8672	1.9124	1.9442
	0.75	5.6102	5.6561	5.7422	5.9620	6.2360	6.7048	7.0483
	1.00	131.06	138.93	155.37	210.21	320.03	787.76	2028.1
	1.25	14.659	14.293	13.655	12.268	10.909	9.2123	8.3008
	1.50	2.5725	2.5379	2.4762	2.3351	2.1865	1.9839	1.8655
1.00	0.50	1.8130	1.8176	1.8260	1.8473	1.8732	1.9162	1.9465
	0.75	5.4608	5.5033	5.5830	5.7869	6.0415	6.4787	6.8004
	1.00	126.09	133.51	148.99	200.55	303.53	740.30	1894.4
	1.25	12.984	12.666	12.112	10.898	9.6992	8.1900	7.3728
	1.50	2.1983	2.1686	2.1156	1.9941	1.8654	1.6887	1.5848
2.00	0.50	1.8301	1.8341	1.8416	1.8606	1.8839	1.9228	1.9506
	0.75	5.2019	5.2386	5.3074	5.4841	5.7058	6.0892	6.3735
	1.00	117.67	124.32	138.18	184.15	275.50	659.92	1668.5
	1.25	10.269	10.028	9.6036	8.6662	7.7257	6.5208	5.8585
	1.50	1.6068	1.5850	1.5458	1.4552	1.3581	1.2232	1.1430

TABLE 3: RELATIVE EFFICIENCIES BETWEEN THE ESTIMATOR $\overline{\theta}_2$ AND T^* FOR $n = 15$

a	$\delta\downarrow d\rightarrow$	0.50	1.00	2.00	5.00	10.00	25.00	50.00
0.50	0.50	1.7784	1.7844	1.7956	1.8233	1.8560	1.9077	1.9423
	0.75	5.3753	5.4318	5.5372	5.8021	6.1242	6.6547	7.0276
	1.00	98.464	105.30	119.66	168.28	267.74	704.36	1892.9
	1.25	16.878	16.284	15.282	13.245	11.422	9.3649	8.3499
	1.50	2.7707	2.7194	2.6301	2.4355	2.2440	2.0030	1.8721
1.00	0.50	1.7866	1.7925	1.8033	1.8299	1.8614	1.9112	1.9445
	0.75	5.2161	5.2696	5.3694	5.6201	5.9248	6.4264	6.7787
	1.00	92.101	98.458	111.81	156.98	249.30	654.12	1754.9
	1.25	15.180	14.638	13.725	11.870	10.212	8.3429	7.4221
	1.50	2.3911	2.3454	2.2656	2.0921	1.9216	1.7074	1.5913
2.00	0.50	1.8011	1.8065	1.8166	1.8415	1.8709	1.9173	1.9483
	0.75	4.9411	4.9895	5.0796	5.3061	5.5811	6.0332	6.3502
	1.00	81.340	86.889	98.537	137.90	218.21	569.53	1522.9
	1.25	12.383	11.928	11.164	9.6113	8.2266	6.6712	5.9071
	1.50	1.7857	1.7490	1.6852	1.5465	1.4106	1.2408	1.1491

TABLE 4: RELATIVE EFFICIENCIES BETWEEN THE ESTIMATOR $\overline{\theta}_1$ AND T^* FOR $n = 20$

a	$\delta \downarrow d \rightarrow$	0.50	1.00	2.00	5.00	10.00	25.00	50.00
0.50	0.50	1.8429	1.8460	1.8519	1.8672	1.8866	1.9214	1.9480
	0.75	5.9939	6.0247	6.0831	6.2360	6.4351	6.8003	7.0900
	1.00	220.15	230.33	251.37	320.03	452.90	989.75	2345.3
	1.25	12.091	11.925	11.624	10.909	10.109	8.9363	8.2041
	1.50	2.3164	2.2987	2.2661	2.1865	2.0936	1.9488	1.8525
1.00	0.50	1.8504	1.8533	1.8588	1.8732	1.8917	1.9247	1.9501
	0.75	5.8165	5.8451	5.8993	6.0415	6.2270	6.5680	6.8395
	1.00	209.89	219.44	239.19	303.53	427.82	928.51	2189.1
	1.25	10.742	10.597	10.332	9.6992	8.9896	7.9431	7.2859
	1.50	1.9779	1.9626	1.9345	1.8654	1.7845	1.6579	1.5734
2.00	0.50	1.8634	1.8660	1.8710	1.8839	1.9006	1.9306	1.9539
	0.75	5.5098	5.5347	5.5818	5.7058	5.8680	6.1679	6.4081
	1.00	192.46	200.95	218.48	275.50	385.24	824.85	1925.4
	1.25	8.5449	8.4312	8.2237	7.7257	7.1620	6.3213	5.7877
	1.50	1.4430	1.4315	1.4103	1.3581	1.2966	1.1996	1.1342

TABLE 4: RELATIVE EFFICIENCIES BETWEEN THE ESTIMATOR $\overline{\theta}_2$ AND T^* FOR $n = 20$

a	$\delta \downarrow d \rightarrow$	0.50	1.00	2.00	5.00	10.00	25.00	50.00
0.50	0.50	1.8272	1.8310	1.8380	1.8560	1.8786	1.9176	1.9463
	0.75	5.8401	5.8766	5.9455	6.1242	6.3527	6.7597	7.0720
	1.00	177.19	186.33	205.30	267.74	390.24	895.98	2199.7
	1.25	12.998	12.770	12.362	11.422	10.424	9.0512	8.2457
	1.50	2.4106	2.3873	2.3450	2.2440	2.1307	1.9635	1.8581
1.00	0.50	1.8337	1.8373	1.8441	1.8614	1.8832	1.9207	1.9484
	0.75	5.6561	5.6906	5.7558	5.9248	6.1409	6.5256	6.8206
	1.00	165.25	173.74	191.35	249.30	362.93	831.67	2039.0
	1.25	11.646	11.438	11.066	10.212	9.3043	8.0582	7.3276
	1.50	2.0699	2.0492	2.0115	1.9216	1.8208	1.6723	1.5789
2.00	0.50	1.8450	1.8484	1.8547	1.8709	1.8912	1.9261	1.9519
	0.75	5.3386	5.3698	5.4287	5.5811	5.7759	6.1225	6.3879
	1.00	145.10	152.49	167.82	218.21	316.92	723.43	1768.8
	1.25	9.4235	9.2499	8.9396	8.2266	7.4707	6.4348	5.8289
	1.50	1.5288	1.5122	1.4822	1.4106	1.3306	1.2131	1.1393