Probability

TOLD TO US:

SET Ω (finite or countable)

$$P(\omega) > 0, \sum_{\omega} P(\omega) = 1$$

$$A \subseteq \Omega, P(A) = \sum_{\omega \in A} P(\omega)$$

EX. Birthday Problem: "How many people do we need to have chance 50% that 2 or more have same birthday date?" Say have n People and C categories. What is Ω?

$$(\omega_1, \omega_2, \cdots, \omega_n), \omega_i \in \{1, 2, \cdots, C\}$$

What is $P(\omega)$? Try $P(\omega) = \frac{1}{C^n}$

What is A?

$$A = \{\omega : \omega_i \neq \omega_j, \forall i, j\}$$

$$P(A) = \sum_{\omega \in A} P(\omega) = \frac{1}{C^n}|A|$$

where $|A| = C(C-1)(C-2)\dots(C-n+1)$

Answer 1:

$$P(A) = (1 - 1/C)(1 - 2/C)\dots(1 - (n-1)/C)$$

Answer 2(HUMANS) Use $\log(1-x)\tilde{-}X$

$$P(A) = \exp \sum_{i=1}^{n} log(1 - i/C) \approx \exp - \sum_{i=1}^{n} i/C = \exp - \frac{(n2)}{C}$$

Now set

$$e^{-(n2)}2 = \frac{1}{2} \to n = 1.2\sqrt{C} \approx 23$$

Answer 3 $\log(1-x) = -x + O(x^2)$ $-x - x^2 \leq \log(1-x) \leq -x, 0 \leq x \leq \frac{1}{2}$

Theorem: if n, C tend to ∞ so that

$$\frac{n^3}{C^2} \to 0, \frac{(n2)}{C} \to E$$

$$P(A) \approx e^{-E}$$

Problem \$10 How many people: Do we need to have even odds to have triple match

More examples: Put N points down at random in $[0,1]^2$, put ε Ball around each, what's chance cover?
We put probabilities on $\tau[0,1]$
Manifolds

Half Way House:

$$\Omega = (0,1]$$

Work with intervals $(a_n, b_n]$, $A = \cup_{i=1}^n I_n, I_n$ are disjoint intervals.

Model for fair coin tossing:

$$\omega = \omega_1, \cdots$$

$$\omega = \sum \frac{d_n(\omega)}{2^n}$$

$$A = \{\omega : d_1(\omega) = 1\}$$

has $P(A) = \frac{1}{2}$, similarly,

$$P(A_1 = E_1, \cdots, A_n = E_n) = \frac{1}{2^n}$$

Theorem (Bernoulli Weak Law of Large Numbers)

$$\forall \varepsilon > 0, P\{|\frac{d_1 + \cdots + d_n}{n} - \frac{1}{2}| > \varepsilon\} \to 0$$

Proof: Define $\Omega_n(\omega) = 2 \times d_n(\omega) - 1$ Same to prove

$$\forall \varepsilon P\{|\frac{1}{n}\sum_{i=1}^{n} \Omega_i| > 2\varepsilon\} \to 0$$

Note $\int_0^1 \Omega_i(\omega) \mathrm{d}\omega = 0$

$$\int_0^1 \Omega_i(\omega)\Omega_j(\omega) = \delta_{ij}$$

So $\int_0^1 (\sum \Omega_n) \mathrm{d}\omega = 0$, $\int_0^1 (\sum \Omega_n)^2 \mathrm{d}\omega = n$
$P\{|\frac{1}{n}\sum \Omega_i| > 2\varepsilon\} \leq \frac{1}{4\varepsilon^2} var(\frac{1}{n}\sum_{i=1}^n \Omega_i) = \frac{1}{4\varepsilon^2 n}$
This is Markov's inequality.

Lemma: if $f : (0,1) \to R$,

$$f(\omega) \geq 0$$

Then $\forall a > 0$

$$P\{\omega : f(\omega) \geq a\} \leq \frac{\int_0^1 f(\omega)\mathrm{d}\omega}{a}$$

PF.

$$\int_0^1 f(\omega)\mathrm{d}\omega \geq \int_A f(\omega) \geq aP(A)$$

We want strong law: Borel's Strong Law of Large Numbers theorem:

$$\lim \frac{1}{n}\sum_{i=1}^{n} \Omega_n(\omega) = 0$$

Problems: not tame
$w = .1111, w = 0110000111110$, limit doesn't exist

Def. $A \subseteq \Omega$ to be negligable
If $\forall \in A$ can be converted by countably many intervals of total length $< \varepsilon$.
eq. $x \in \Omega$ negligable, rationals in $(0, 1]$ negligable.

Theorem (Borel SLLN). Except for a set $N \subseteq \Omega$ negligible, $\forall \omega \in N^c$,

$$\lim \frac{1}{n}\sum_{i=1}^{n} \Omega_i(\omega) = 0$$

Let $A = N^c$,

$$A = \cap_{h=1}^{\infty} \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} \{\omega : |\frac{1}{n}\sum_{i=1}^{n} \Omega_i(\omega)| \leq \frac{1}{h}\}$$