MATH5605 Assignment 2

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Question 1

Assume that T is a positive, bounded operator over a complex Hilbert space $\mathcal{H}$, that is:

$$T \in B(\mathcal{H}, \mathcal{H}), \qquad \langle Tx, x \rangle \geq 0, \qquad \forall x \in H$$

Now as T is bounded, there exists its adjoint T^* such that:

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \qquad \forall x, y \in \mathcal{H}$$

$$\therefore \langle x, T^*x \rangle = \overline{\langle T^*x, x \rangle} = \langle T^*x, x \rangle = \langle x, Tx \rangle \implies \langle Tx - T^*x, x \rangle = 0, \forall x \in \mathcal{H}$$

With the second equality holding as $\langle Tx, y \rangle$ real, from by positivity.

Now we fix some $y \in \mathcal{H}$ and consider $x + y \in \mathcal{H}$,

$$\langle T(x+y) - T^*(x+y), x+y \rangle = \langle Tx - T^*x, y \rangle + \langle Ty - T^*y, x \rangle$$
$$\implies \langle Tx - T^*x, y \rangle = -\langle Ty - T^*y, x \rangle, \forall x \in \mathcal{H}$$

Which follows as $\langle Tz - T^*z, z \rangle = 0, \forall z \in \mathcal{H}$ and linearity properties of $\langle . \rangle$.

We now consider instead $x + iy \in H$ and obtain a similarly that:

$$\langle T(x+iy) - T^*(x+iy), x+iy \rangle = i\langle Ty - T^*y, x \rangle - i\langle Tx - T^*x, y \rangle$$
$$\implies \langle Tx - T^*x, y \rangle = \langle Ty - T^*y, x \rangle, \forall x \in \mathcal{H}$$

By anti-linearity in the second variable. Combining the results we arrive at:

$$\langle Ty - T^*y, x \rangle = 0, \forall x \in \mathcal{H} \implies Ty = T^*y$$

As can be seen by taking $x = Ty - T^*y$, thus as y is arbitrary we end up with the result that $T = T^*$, hence T is self-adjoint.

As a small aside, it is worth noting that if we assume that our field is real, the result fails. We can see this easily with the following example:

Let $\mathcal{H} = \mathbb{R}^2$ with usual innerproduct, and consider the linear map L with matrix representation

$$L = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

This linear operator is clearly bounded and positive ($\langle Lx, x \rangle = (x_1 + x_2)^2$, $x = (x_1, x_2)$), however its clear that $L \neq L^*$.

Question 2 1)

Let $T = T_A$ be a Toeplitz operator on ℓ^2, where the matrix A has entries drawn from the following two-sided complex sequence $\{a_m\}_{m\in\mathbb{Z}}$,

$$A = \{a_{i-j}\}_{i,j=1}^{\infty}$$

Suppose that

$$\sum_{m\in\mathbb{Z}} |a_m| < +\infty$$

For clarity we let

$$\|T\|_B \quad = \quad \sup_{x\in\{x\in H:\|x\|\leq 1\}} \|Tx\|$$

where $T \in L(\ell^2)$, the space of linear operators on ℓ^2 and $\|.\|$ is the ℓ^2 norm.

First we define the following operators on ℓ^2,

$$A_m = \begin{cases} a_m S^m & m \geq 0 \\ a_m (S^*)^m & m < 0 \end{cases}$$

Where we have S and S^* the left and right shift operators respectively, we note that for $\mathbf{x} = \{x_k\}_{k=1}^{\infty} \in \ell^2$

$$S(\mathbf{x}) = (0, x_1, x_2, x_3, ...), \qquad S^*(\mathbf{x}) = (x_2, x_3, x_4, ...) \implies \|S\|_B = 1 = \|S^*\|_B$$

$$\therefore \qquad S^m(\mathbf{x}) = (\underbrace{0, ..., 0}_{\text{m}}, x_1, x_2, ...), \qquad (S^*)^m(\mathbf{x}) = (x_{m+1}, x_{m+2}, x_{m+3}, ...)$$

$$\implies \|S^m\|_B = 1 = \|(S^*)^m\|_B \implies \|A_m\|_B = |a_m| \qquad \forall m \in \mathbb{Z}$$

Using the above and the fact that $B(\ell^2)$ forms a Banach space with respect to $\|.\|_B$, we observe that:

$$\|\sum_{m\in\mathbb{Z}} A_m\|_B \leq \sum_{m\in\mathbb{Z}} \|A_m\|_B = \sum_{m\in\mathbb{Z}} |a_m| < +\infty$$

Which implies that $\sum_{m\in\mathbb{Z}} A_m \in B(\ell^2)$ as it is complete. We now note that:

$$(T_A(\mathbf{x}))_j = \sum_{k=1}^{\infty} a_{j-k}x_k = \sum_{k=1}^{\infty} (A_{j-k}(\mathbf{x}))_j = \sum_{m\in\mathbb{Z}} (A_m(\mathbf{x}))_j = \big(\sum_{m\in\mathbb{Z}} A_m(\mathbf{x})\big)_j$$

With the third equality holding as $m \geq j \implies (A_m(\mathbf{x}))_j = 0$,

$$\therefore T_A = \sum_{m\in\mathbb{Z}} A_m(\mathbf{x}) \implies \|T_A\|_B = \Big\|\sum_{m\in\mathbb{Z}} A_m\Big\|_B \leq \sum_{m\in\mathbb{Z}} |a_m| < +\infty$$

Question 2 3)

Let us proceed by way of contradiction. Assume that $\|T_A\|_{HS} < +\infty$ and $\exists a_n \in \{a_m\}_{m\in\mathbb{Z}} \quad s.t. \quad a_n \neq 0$, then by definition:

$$\|T_A\|_{HS}^2 = \sum_{i=0}^{\infty} \|T_A(e_i)\|^2 = \sum_{i=1}^{\infty} \big(\sum_{m=1-i}^{\infty} |a_m|^2 \big) \geq \sum_{i=n+1}^{\infty} |a_n|^2$$

Which is clearly unbounded, hence we must have that the two sided sequence is the zero sequence and so $T_A = 0$, so the only Hilbert-Schmidt Toeplitz operator is the zero operator.

Question 3 1)

Let H be a Hilbert space and $T \in B(\mathcal{H})$, we note the following:

$$Im(T) = \{Tx : x \in \mathcal{H}\}$$

Is a vector subspace of $\mathcal{H}$, which can easily be seen as:

$$\alpha \in \mathbb{C}, \;\; x, y \in Im(T) \implies \exists x', y' \in \mathcal{H} : T(x') = x, T(y') = y$$

$$\therefore T(x' + \alpha y') = x + \alpha y \implies x + \alpha y \in Im(T)$$

$$Im(T) \leq \mathcal{H}$$

Now as $T \in B(\mathcal{H}) \implies T^*$ exists and is bounded, then we have that:

$$x \in (Im(T))^{\perp} \iff \langle x, Ty \rangle = 0 \;\; \forall y \in \mathcal{H} \tag{1}$$

$$\langle x, Ty \rangle = 0 \;\; \forall y \in \mathcal{H} \iff \langle T^* x, y \rangle = 0 \;\; \forall y \in \mathcal{H} \tag{2}$$

$$\langle T^* x, y \rangle = 0 \;\; \forall y \in \mathcal{H} \iff T^* x = 0 \tag{3}$$

$$T^* x = 0 \iff x \in Ker(T^*) \tag{4}$$

$$\therefore (Im(T))^{\perp} = Ker(T^*)$$

Question 3 2)

This is a simple corollary of the previous proof.

As $Im(T) \leq \mathcal{H}$, then we have that:

$$(Im(T))^{\perp\perp} = \overline{\text{Span}(Im(T))} = \overline{Im(T)} \implies (Im(T^*))^{\perp\perp} = \overline{Im(T^*)}$$

Thus using the result from Question 3) 1) but considering T^* rather than T:

$$(Ker(T))^{\perp} = ((Im(T^*))^{\perp\perp} = \overline{Im(T^*)}$$

Question 2 2)

The answer to the question of sharpness is unfortunately no, a proof of this is as follows and is due directly to the following Lemma:

Lemma: [**?**]

The Toeplitz matrix corresponding to a two-sided sequence $\{a_m\}_{m\in\mathbb{Z}}$ of complex numbers is the matrix of a bounded operator on ℓ^2 if and only if there exists a function $f \in L^\infty(\mathbb{T}, \mu)$ such that $a_n = \hat{f}(n)$ for every integer n.

Here $\mathbb{T}$ is the unit circle of the complex plane and μ is the arc-length measure normalised by 2π. L^∞ is the space of all essentially bounded functions, that is functions that are bounded μ almost everywhere, further the fourier coefficients $\hat{f}(n)$ are with respect to the basis $\{z^n\}_{n\in\mathbb{Z}} \quad L^2(\mathbb{T}, \mu)$.

Now consider the monomial $f = z \in L^\infty(\mathbb{T}, \mu)$, we note that this has Fourier coefficients:

$$\hat{f}(n) = \begin{cases} \frac{4}{n-4} & n \quad \text{odd} \\ 2\pi i & n = 0 \\ 0 & \text{else} \end{cases}$$

Thus by our lemma we note that the matrix of the Toeplitz operator with entries taken from the series above is a bounded operator on ℓ^2, however the double sided sum is most definitely not absolutely convergent, as it is bounded below by the harmonic sum. Hence we have that the condition is not sharp.

Pardon the wordiness here Denis, I was not entirely comfortable going through and proving something that seems a bit far from what we've done in the lectures thus far, even if it isn't technically difficult it still felt a bit off. I included this just so I'd have something for this section. The lemma was draw from a bit of trawling through the web after I failed to prove that a similar operator was bounded.

I'd also like to thank Roberto for pointing out to me that I should probably mention that the real case fails in the first question, and Ed for this eternal patients and help with question 2 2) in particular.

References

[1] Arlen Brown and Paul Halmos, Algebraic properties of Toeplitz Operators. J. Reigne Angew. Math. 213 (1963/1964), 89–102