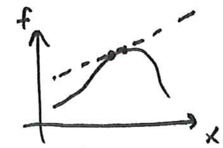
L02 : Sept 26, 2014 ME564, Fall 2014

Overview of Topics



① Review Calculus :
- Derivative
- Power Law
- Chain Rule

② Simple(st) ODE : $\dot{x} = \lambda x$

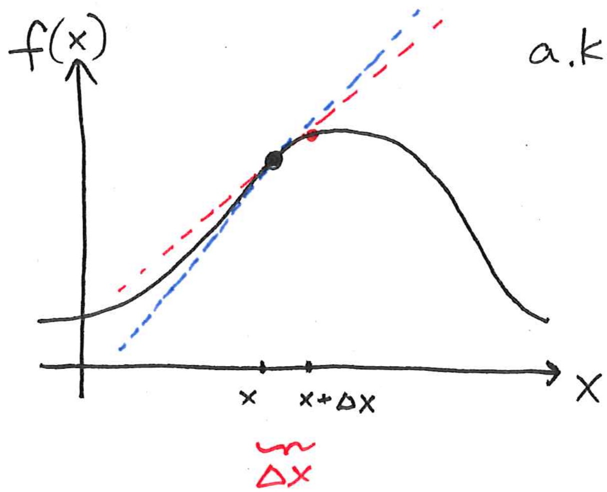
③ What is 'e' (Euler's number)

④ Solving $\dot{x} = \lambda x$ with Taylor Series

(Next Lecture?)

The Derivative : the rate of change of a function with respect to an independent variable...

a.k.a. slope of the tangent line!



$$\boxed{\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

Power Law : Try $f(x) = x^n$

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{1}{\Delta x}\left[(x+\Delta x)^n - x^n\right]$$

$$= \frac{1}{\Delta x}\left[x^n + n x^{n-1}\Delta x + \frac{n(n-1)}{2} x^{n-2}\Delta x^2 + \dots - x^n\right]$$

$$= \frac{1}{\Delta x}\left[n x^{n-1}\Delta x + \frac{n(n-1)}{2} x^{n-2}\Delta x^2 + \mathcal{O}(\Delta x^3)\right]$$

$$= n x^{n-1} + \underbrace{\mathcal{O}(\Delta x)}_{\to 0 \text{ when } \Delta x \to 0}$$

Power Law: $\frac{d}{dx} x^n = nx^{n-1}$

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

Ex: $\left.\begin{aligned} f(x) &= x^2 \\ g(x) &= x^3 \end{aligned}\right\}$ $f(g(x)) = (x^3)^2 = x^6$

Chain Rule: $\left.\begin{aligned} f'(g(x)) &= 2x^3 \\ g'(x) &= 3x^2 \end{aligned}\right\}$ $f'(g(x))g'(x) = 6x^5$

Power Law: $\frac{d}{dx} x^6 = 6x^5$.

Lets say that bunnies are

... procreating.

The bunny population size is x,

and the population grows at a rate

λ proportional to the population size:

$$\frac{dx}{dt} = \lambda x.$$

What is population as a function of time?

Method I:

$$\frac{dx}{dt} = \lambda x(t)$$

$$\Longrightarrow \quad \frac{dx}{x(t)} = \lambda dt$$

integration constant

$$\Longrightarrow \quad \int \frac{dx}{x(t)} = \int \lambda dt \Longrightarrow \ln(x(t)) = \lambda t + C$$

$$\Longrightarrow x(t) = e^{\lambda t + C}$$

$$= e^{\lambda t} K$$

What is K? $x(t=0) = e^0 \cdot K = K$ = initial population size!

$$\Longrightarrow \quad \boxed{x(t) = e^{\lambda t} x(0)}$$

Lets say I borrow money to buy a car, and the annual interest rate is 'r'.

- Compounded once at end of year:

$$\underbrace{x(1)}_{\text{loan amount on year 1}} = (1+r)\cdot \underbrace{x(0)}_{\text{initial loan amount}}$$

What is e?

- Compounded every month:

$$x(1) = \left(1 + r/12\right)^{12} \cdot x(0)$$

- Compounded every day:

$$x(1) = \left(1 + r/365\right)^{365} x(0)$$

- Compounded continuously:

$$x(1) = \lim_{n\to\infty} \left(1 + r/n\right)^{n} x(0)$$

$$= \underbrace{e}_{\text{e is Euler's number}}^{r} x(0)$$

Another way to phrase the interest problem is to say that the loan amount x is continuously increasing at a rate 'r', proportional to the current loan value x:

$$(*) \qquad \Delta x = r\, x(t)\, \Delta t$$

(divide by Δt
take limit $\Delta t \to 0$)

$$\boxed{\frac{dx}{dt} = r\, x(t), \quad x(0) = L}$$

$(*)$ is actually more general... think about bunnies that only reproduce 1 time a year...
2 times a year...
⋮

Example Radioactive decay

$$\dot{x} = -\lambda x \implies x(t) = e^{-\lambda t} x(0)$$

Plutonium has a half-life of $\approx$ 80 million years.

$$\frac{x(0)}{2} = e^{-\lambda \cdot 8\times 10^7} x(0)$$

$$\implies \lambda = \frac{-\ln(1/2)}{8\times 10^7} \approx$$

Polonium has a half life of $\approx$ 138 days.

Example Thermal Runaway

$$\dot{T} = \lambda T$$

... why doesn't $T \to \infty$?

Answer: $\dot{T} = \lambda T - \underbrace{T^3}_{\text{nonlinearity !!!}}$