Cognitive Algorithms Assignment 1

Part 1 - Linear Algebra Recap

Due on Tuesday, May 2, 2017 10am via ISIS

Task 1 (8 points)

1. Compute the scalar product of the following vectors $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$.

   ☐ 3 ☐ 5 ☐ 7

2. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be two column vectors. Which of the following statements is always true?

   ☐ $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

   ☐ $\mathbf{v}^T \cdot \mathbf{w} = \mathbf{w}^T \cdot \mathbf{v}$

   ☐ $\mathbf{v} \cdot \mathbf{w}^T = \mathbf{w} \cdot \mathbf{v}^T$

3. The mapping $f : \mathbb{R}^2 \ni (x, y)^\top \mapsto (x + y, y - x)^\top \in \mathbb{R}^2$ is given by the following matrix:

   ☐ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

4. Which property does matrix multiplication *not* have?

   ☐ Associativity: $(AB)C = A(BC)$

   ☐ Commutativity: $AB = BA$

   ☐ Distributivity: $(A + B)C = AC + BC$

5. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ two column vectors with $A \cdot \mathbf{v} = \mathbf{w}$. Which of the following statements is always true?

   ☐ $A = \mathbf{w} \cdot \mathbf{v}^{-1}$ ☐ $\mathbf{v} = \mathbf{w} \cdot A^{-1}$ ☐ $\mathbf{v} = A^{-1} \cdot \mathbf{w}$

6. The rank of the matrix $\begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}$ is

   ☐ 1 ☐ 3 ☐ 4

7. For a square $n \times n$ matrix A holds

   ☐ $\operatorname{rank} A = n \Rightarrow A$ is invertible, but there are invertible A with $\operatorname{rank} A \neq n$

   ☐ A is invertible $\Rightarrow \operatorname{rank} A = n$, but there are A with $\operatorname{rank} A = n$, which are not invertible.

   ☐ $\operatorname{rank} A = n \Leftrightarrow A$ is invertible

8. Which of the following matrices is orthogonal:

☐ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Task 2 (3 points)

We consider two functions f and g which transform an input vector $\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$ into a scalar: $f(\mathbf{x}) = \mathbf{u}^\top \mathbf{x},\ \mathbf{u} = (u_1, \dots, u_d)^\top \in \mathbb{R}^d$ and $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x}$.

1. Compute the partial derivative of f and g with respect to one entry x_j $(j \in \{1, 2, \dots, d\})$
   (a) $\frac{\partial f(\mathbf{x})}{\partial x_j} =$

   (b) $\frac{\partial g(\mathbf{x})}{\partial x_j} =$

2. Compute the gradient $\nabla f(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_d}\right)^\top$ for f and g.
   (a) $\nabla f(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial f(\mathbf{x})}{\partial x_d}\right)^\top =$

   (b) $\nabla g(\mathbf{x}) = \left(\frac{\partial g(\mathbf{x})}{\partial x_1}, \dots, \frac{\partial g(\mathbf{x})}{\partial x_d}\right)^\top =$

Task 3 (4 points)

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ Eigenvectors of A corresponding to Eigenvalues $\lambda, \mu \in \mathbb{R}$, with $\lambda \neq \mu$. (Recall the definition of Eigenvectors: $A\mathbf{v} = \lambda \mathbf{v}$ and $A\mathbf{w} = \mu \mathbf{w}$)
Show: $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, i.e. $\mathbf{v}^\top \mathbf{w} = 0$.
Hint: $\lambda \mathbf{v}^\top \mathbf{w} = \dots$.