Dynamic Data Structures and Algorithms.

ENGF0002: Design and Professional Skills

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Partly based on slides from George Danezis

Term 1, 2018

Sorting.

Binary search, and a number of other efficient algorithms, rely on sorted sequences.

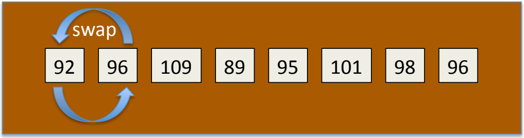
We next study:

- A sorting algorithm, **bubble sort**.
- A better sorting algorithm, **merge sort**.
- The **time complexity** of sorting.
- How to show sorting is **correct**.
- How to write **generic** sorting and searching algorithms.

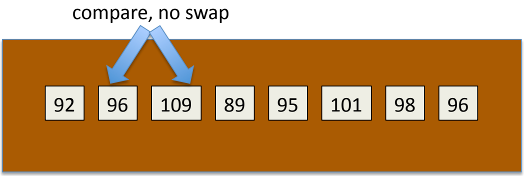
Bubble Sort

96	92	109	89	95	101	98	96

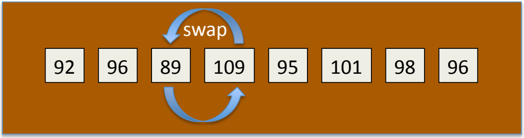
Bubble Sort



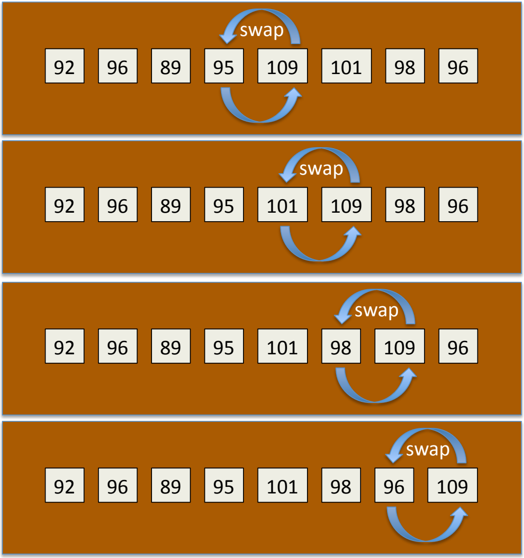
Bubble Sort



Bubble Sort



Bubble Sort


swap
92
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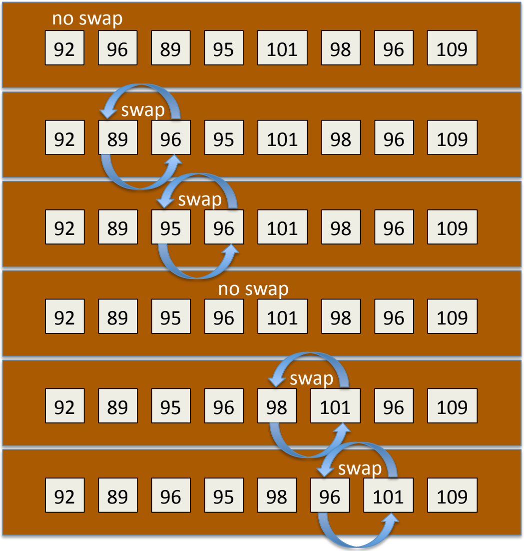
Bubble Sort


no swap
92 96 89 95 101 98 96 109
swap
92 89 96 95 101 98 96 109
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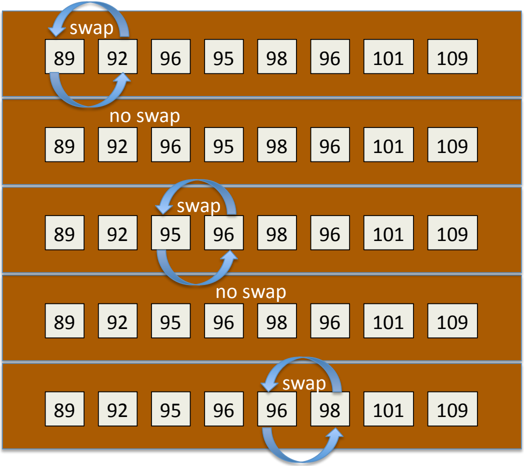
Bubble Sort



swap
89 92 96 95 98 96 101 109
no swap
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89 92 95 96 98 96 101 109
no swap
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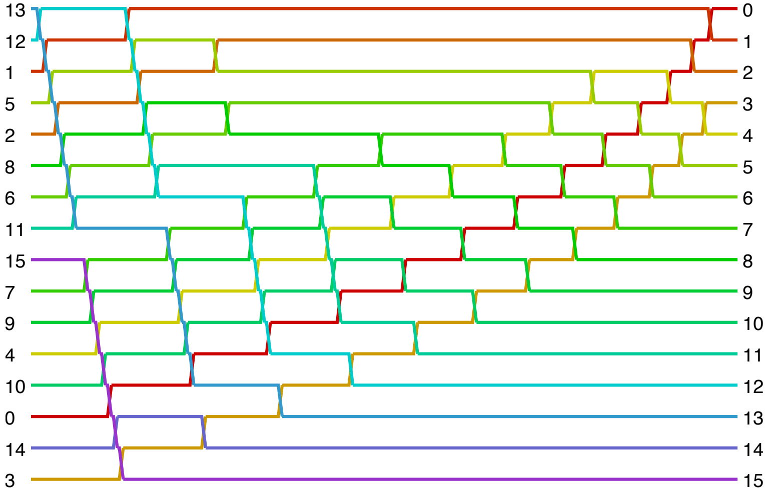
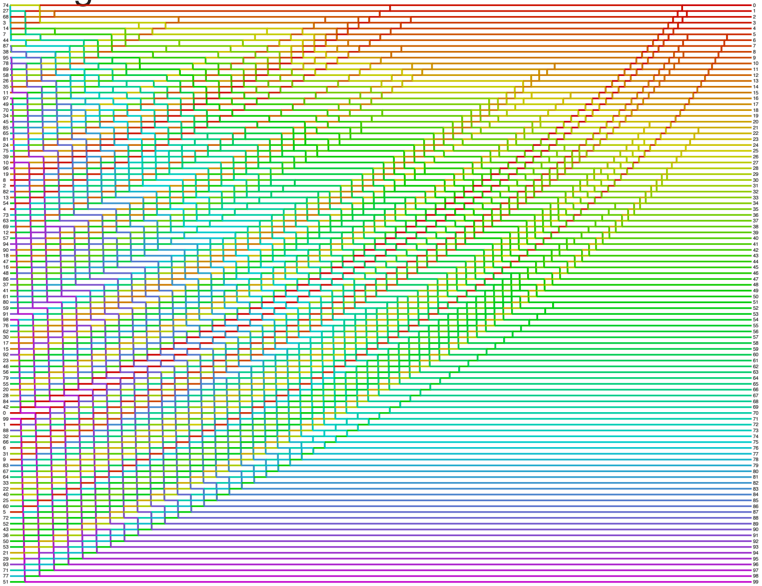
swap
89 92 95 96 96 98 101 109

Visualizing Bubble Sort



Visualizing Bubble Sort



Complexity of Bubble Sort

What is the computational complexity of Bubblesort with n items?

- Each pass through the data involves $n - 1$ comparisons.
- Each pass moves one more item to the end of the list.
  - After 1 pass, the largest item is at the end.
  - After 2 passes, the largest two items are at the end and sorted.
  - After k passes, the largest k items are at the end and sorted.
- After n passes, the whole list is sorted.

Requires $n(n - 1)$ comparisons.

$n(n - 1) = n^2 - n = \mathcal{O}(n^2)$

Optimization of Bubblesort

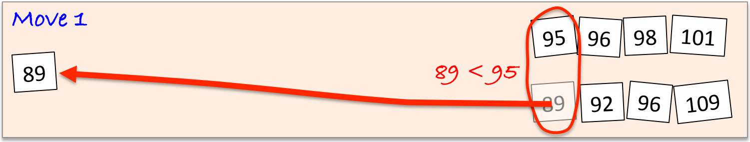
Observe we don't need to compare all n items in each pass.

- In pass k, the last $k - 1$ items are already sorted, so we can stop the pass early.
- Cuts the total number of comparisons in half.

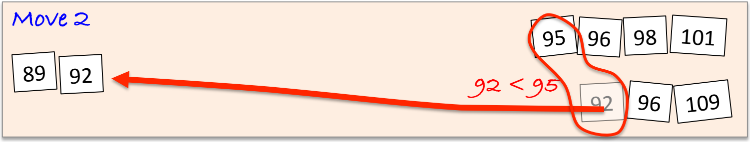
Does this change the complexity?

Merging lists

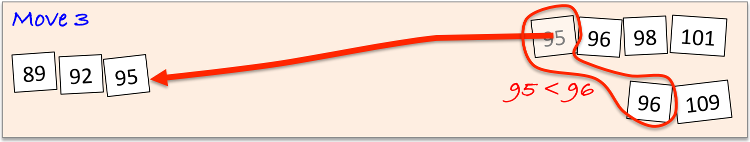
Merging lists



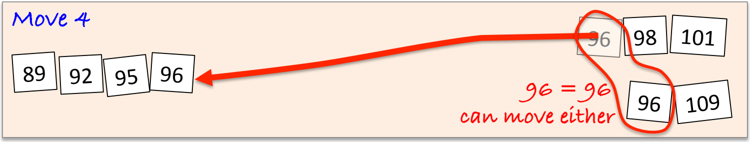
Merging lists



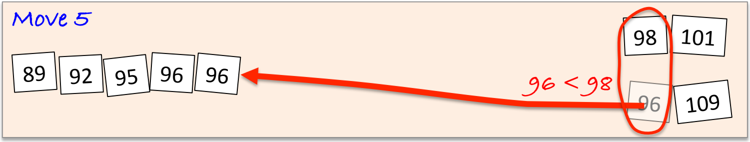
Merging lists



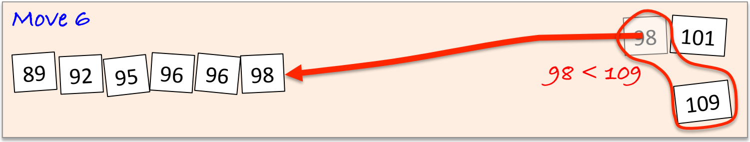
Merging lists



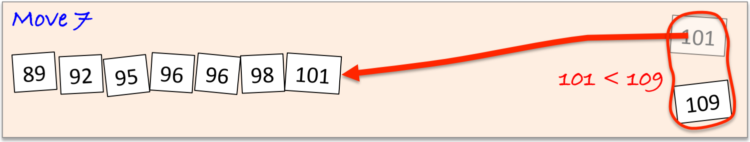
Merging lists



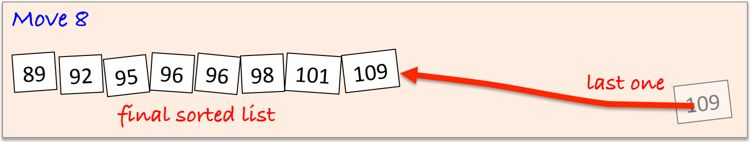
Merging lists



Merging lists



Merging lists



Observation

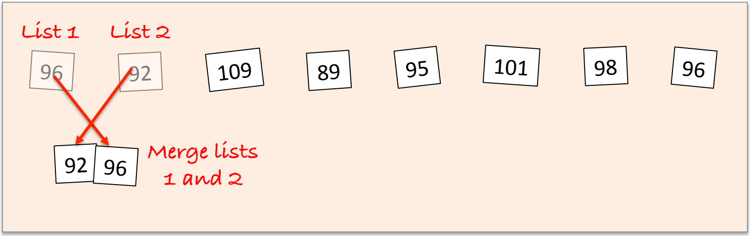
Merging two **already sorted** lists containing *n* items only requires *n* comparisons and moves.

Little lists

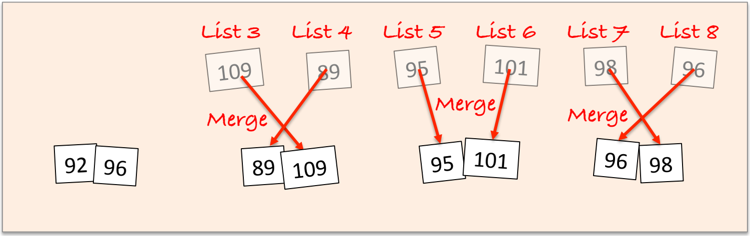
Observation

Lists of length 1 are **already sorted!**.

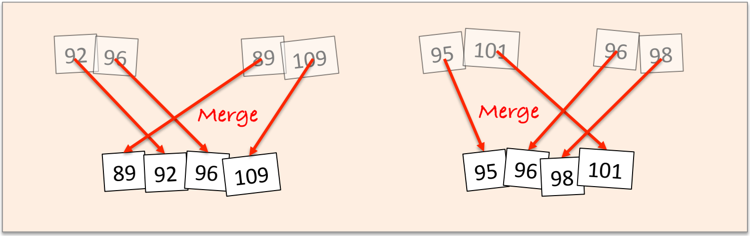
Merging lists (round 1)



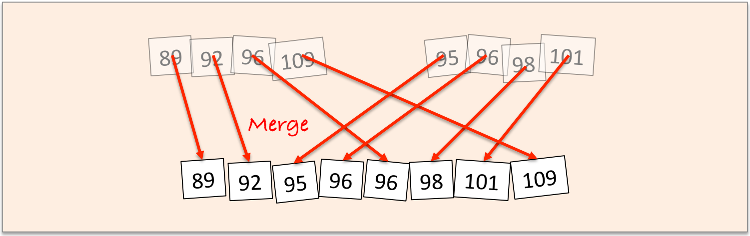
Merging lists (round 1 continued)



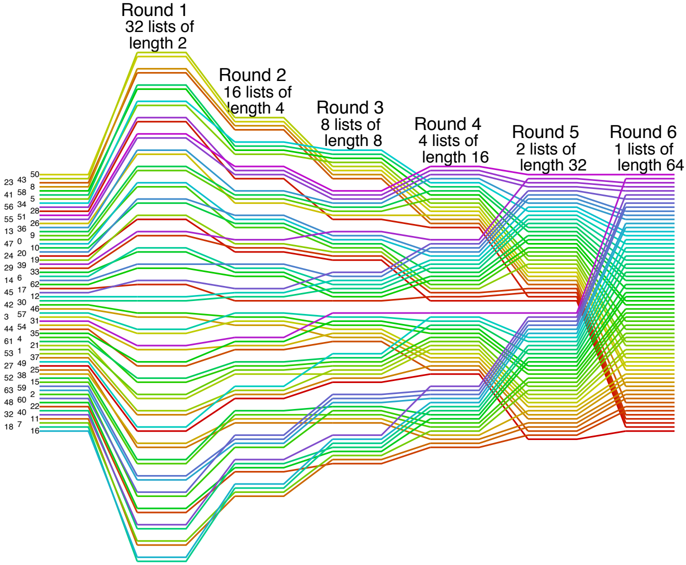
Merging lists (round 2)



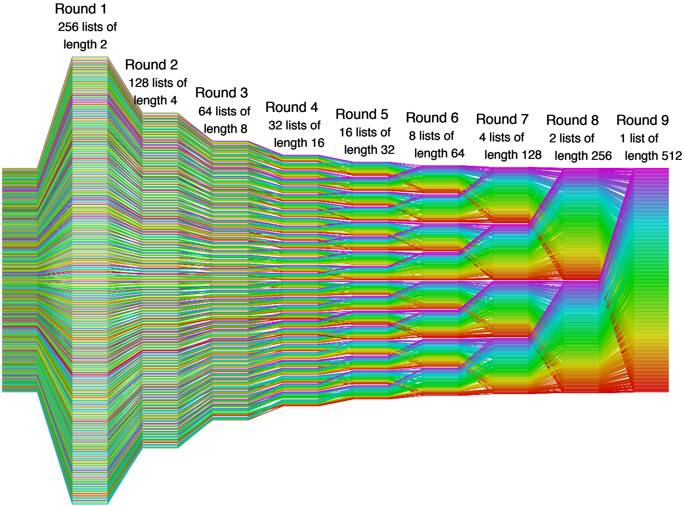
Merging lists (round 3)



Visualizing merging lists: 64 items



Visualizing merging lists: 64 items



Merge sort

Mergesort was proposed by John von Neumann in 1945.

Essentially two algorithms:

Algorithm 1. Merge two sorted lists into one sorted list:

- While both lists are not empty, consider the first unprocessed item from both.
- Add to the front of the new list the smaller or available one, and remove it from its list. Repeat.
- Append remaining elements of one of the lists.

Merge sort

Algorithm 2. Merge Sort.

- If the list has length 1, we're done.
- Otherwise, divide into two sub-lists of equal length.
- Merge Sort both lists separately.
- Merge the two sorted lists using Algorithm 1.

Tests for sorting.

```python
def test_simple_sort():
    assert mergesort([2, 3, 1, 4, 0]) == [0, 1, 2, 3, 4]

def test_simple_sort10(): # Test edge cases, for 1 or no elements.
    assert mergesort([190]) == [190]
    assert mergesort([]) == []

def test_simple_longlist():
    import random
    N = 1000
    L = list(range(N))
    random.shuffle(L) # Shuffle at random in-place
    L2 = mergesort(L)
    assert L2 == list(range(N))
```

Merge two sorted lists (algorithm 1).

```python
def mergelists(lst1, lst2):
    ''' Merge two sorted lists into a sorted list.'''
    merged_lst = []
    len1 = len(lst1)
    len2 = len(lst2)
    index1 = 0
    index2 = 0
    # while either list has any items left
    while index1 < len1 or index2 < len2:
        if (index2 == len2    #either we've used all of lst2
            or (index1 != len1     #or both lists have elements left
                and lst1[index1] <= lst2[index2])):  #and lst1 is smaller
            merged_lst.append(lst1[index1])
            index1 += 1
        else:
            merged_lst.append(lst2[index2])
            index2 += 1
    return merged_lst
```

A recursive implementation of mergesort (algorithm 2).

```
def mergesort(lst):
    if len(lst) <= 1:
        return lst.copy()
    pivot = len(lst) // 2
    lst1 = mergesort(lst[:pivot])
    lst2 = mergesort(lst[pivot:])
    return mergelists(lst1, lst2)
```

- Python **slicing**: `items[start:end]` returns the list from index `start` up to, but not including, index `end`. If omitted from zero to the length of the list.
- Use `items.copy()` to get a new **copy** of the list (otherwise we'll just modify the original!).

How to argue mergesort is correct.

Structural induction (verbal reasoning):

- Argue that the merge operation is correct: given two sorted lists it produces a sorted list.
- Base case of mergesort: a single element list is sorted.
- Inductive case of mergesort: assume mergesort for a previous step works, prove that the result is sorted. Follows from the properties of merge.

Note that the recursive structure of the program supports the argument of correctness through induction.

What is the cost of mergesort.

The time complexity of merge (algorithm 1) is $\mathcal{O}(n)$, where n is the sum of the lengths of both lists.

Mergesort (algorithm 2) can only divide an array of n elements $\mathcal{O}(\log n)$ times. At each level of subdivision the overall costs of the merge operations will be $\mathcal{O}(n)$ (total from algorithm 1).

Therefore the overall time complexity of mergesort is of the **order** $\mathcal{O}(n \log n)$.

This is **optimal** for comparison based algorithms (but note that **constant factors may vary**.) Space complexity may also vary.

More sequences and generic programming

Strings and byte sequences.

Besides **lists**, Python supports a number of other sequence types:

- **tuples**: like a list, but content is immutable once assigned.
- **strings** (`str`): represent sequences of unicode characters.
- **bytes** (`bytes`): represent a sequences of raw bytes (ie. values from 0 to 255).
- Strings are transformed into and from sequences of bytes through an **encoding** such as `utf8` or `ASCII`.

```
>>> name = "École"          # A unicode string
>>> name.encode('utf-8')  # A byte sequence
b'\xc3\x89cole'
```

Internationalization (i18n).

Old encodings such as ASCII only map latin characters to bytes. Others cannot exist!

Software and services need to be **accessible to all people in their native languages**. Always use string representations that can represent all languages (and emoji!) such as **unicode**, and encodings into bytes such as **utf-8** to represent those strings as bytes.

- Don't assume English is the only possible language.
- Plan for internationalization: do not hard code english strings into software, but rather load them from a file that can be localized.
- Dates, currency, number representations also vary per locale.

Operations on string.

Strings support a number of operations:

```
>>> s = "All quiet on the western front"
>>> s[10]       # Indexing
'o'
>>> s[4:9]      # Slicing
'quiet'
>>> s[4:9] + s[-6:]   # Concatenation & neg indexing
'quiet front'
>>> s.split(' ')      # Split to str list
['All', 'quiet', 'on', 'the', 'western', 'front']
>>> "hello" < "world" # Comparison
True
```

For full reference see:

`https://docs.python.org/3.7/library/string.html`

String literals & formatting.

- String constants in programs can be encosed in single or double quotes.
- You can represent special characters in strings using the '`\`' symbol, such as new line as `\n` and tab as `\t`.
- The `format` method substitutes into the string:

```
>>> "My name is {name}".format(name = "Alice")
'My name is Alice'
>>> "Dec: {num:d} | Hex: {num:x} | Oct: {num:o}".format(num=14)
'Dec: 14 | Hex: e | Oct: 16'
>>> template = "Float: {num:.2e} Fixed: {num:.2f} Perc: {num:.2%}"
>>> template.format(num=0.12345)
'Float: 1.23e-01 Fixed: 0.12 Perc: 12.35%'
```

Generic programming: how to sort a list of strings.

Do you need to **re-implement mergesort** for lists of strings? Remember the **abstraction principle**.

However our implementation of mergesort seems to just work:

```
def test_str_sort():
    from mergesort import mergesort
    items = ["BB", "A", "C", "DAAX", "BA"]
    sorted_items = ["A", "BA", "BB", "C", "DAAX"]
    assert mergesort(items) == sorted_items
```

Generic programming ensures that algorithms are independent of and agnostic about the exact types they are processing.

Why generic programming works.

Merge (algorithm 1) only uses ‘<=’ on items within the list to sort:

```
    # while either list has any items left
    while index1 < len1 or index2 < len2:
        if (index2 == len2    #either we've used all of lst2
            or (index1 != len1     #or both lists have elements left
                and lst1[index1] <= lst2[index2])):  #and lst1 is smaller
            merged_lst.append(lst1[index1])
            index1 += 1
        else:
            merged_lst.append(lst2[index2])
            index2 += 1
```

If the operation greater-or-equal (compare) is supported correctly by the type of object in the list, mergesort will return the correct result.

Sorting strings by length, rather than lexicographically.

Strategy: allow the programmer to specify a function that performs a comparison. Parametrize mergesort by the comparison function.

```
def mergesort(lst, compare):
    if len(lst) <= 1:
        return lst.copy()
    pivot = len(lst) // 2
    lst1 = mergesort(lst[:pivot], compare)
    lst2 = mergesort(lst[pivot:], compare)
    return mergelists(lst1, lst2, compare)
```

```python
def mergelists(lst1, lst2, compare):
    ''' Merge two sorted lists into a sorted list.'''
    merged_lst = []
    len1 = len(lst1)
    len2 = len(lst2)
    index1 = 0
    index2 = 0
    # while either list has any items left
    while index1 < len1 or index2 < len2:
        if (index2 == len2    #either we've used all of lst2
            or (index1 != len1     #or both lists have elements left
                and compare(lst1[index1], lst2[index2]))):
            merged_lst.append(lst1[index1])
            index1 += 1
        else:
            merged_lst.append(lst2[index2])
            index2 += 1
```

Example of different comparison functions for strings.

```python
def test_order():
    items = ["A", "BB", "CCC", "DD", "E"]

    def compare_lex(l1, l2):
        return l1 <= l2
    assert mergesort(items, compare_lex) == items

    def compare_len(l1, l2):
        return len(l1) <= len(l2)
    assert mergesort(items, compare_len) \
```

Functions are just variables.

Functions are first-class objects.

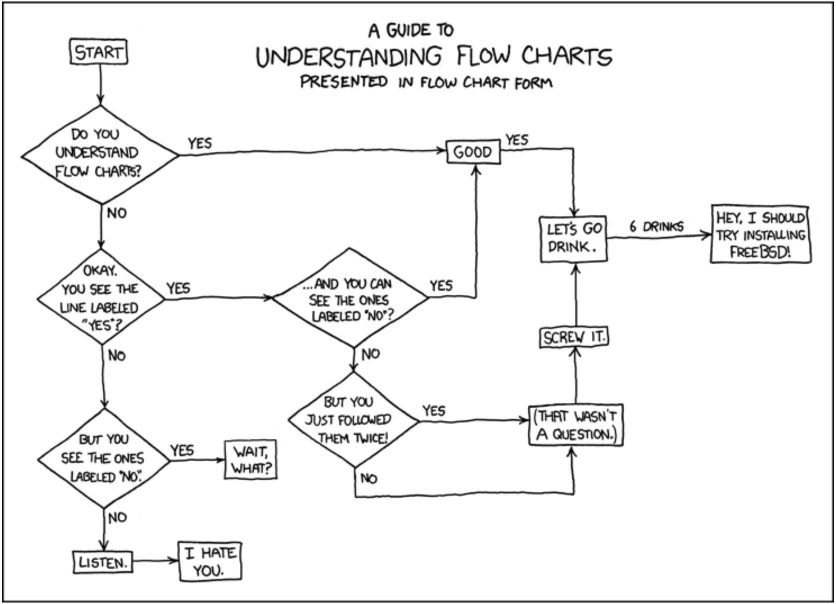
In modern programming languages functions are first-class objects (of type `function`). They are assigned to names, and they can be passed as arguments to function calls; stored in data structures; etc.

The shorthand `lambda` notation can be used to define functions within expressions.

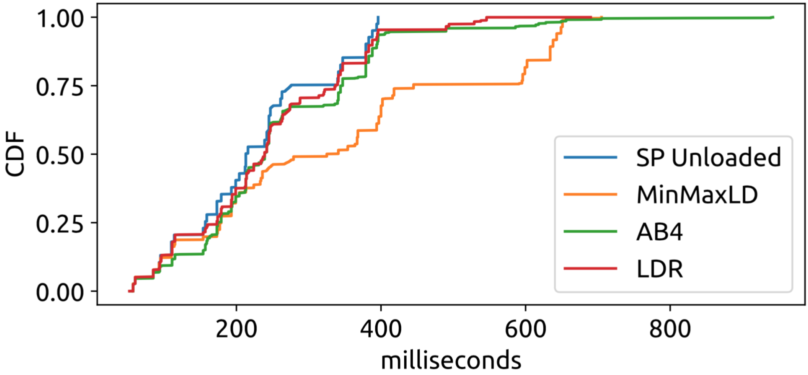
```
def test_order_lambda():
    items = ["A", "BB", "CCC", "DD", "E"]

    assert mergesort(items, lambda x,y: x <= y) == items
    assert mergesort(items, lambda x,y: len(x) <= len(y)) \
        == ["A", "E", "BB", "DD", "CCC"]
```

Visualization Diversion


A GUIDE TO
UNDERSTANDING FLOW CHARTS
PRESENTED IN FLOW CHART FORM
START
DO YOU UNDERSTAND FLOW CHARTS?
YES
GOOD
YES
NO
LET'S GO DRINK.
6 DRINKS
HEY, I SHOULD TRY INSTALLING FREEBSD!
OKAY. YOU SEE THE LINE LABELED "YES"?
YES
...AND YOU CAN SEE THE ONES LABELED "NO"?
YES
NO
NO
SCREW IT.
BUT YOU JUST FOLLOWED THEM TWICE!
YES
(THAT WASN'T A QUESTION.)
NO
BUT YOU SEE THE ONES LABELED "NO".
YES
WAIT, WHAT?
NO
LISTEN.
I HATE YOU.

Cumulative Distribution Functions (CDFs)



y value shows the cumulative fraction of data values less than the x value