Utility functions.

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The paradox.

The game.

Toss a coin. In case of **heads** get 2 dollars, in case of **tails** toss again. Each time in case of **heads** the reward is payed (and the game is over), in case of **tails** the reward doubles.

- How much will you pay to participate in this game?
- $$ER = \sum_x R(x)p(x) =$$ $$= 2 \cdot 1/2 + 4 \cdot 1/4 + \ldots + 2^n \cdot (1/2^n) + \ldots = +\infty$$

Why Saint Petersburg?

Bernoulli, Commentaries of the Imperial Academy of Science of **Saint Petersburg**, 1738.

Possible solutions to the paradox.

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- How much will you pay to participate in this game?

$$ER = \sum_x R(x)p(x) = 1/2{\cdot}2 + 1/4{\cdot}4 + \ldots + (1/2^n){\cdot}2^n + \ldots = +\infty$$

- **Solution 3.** Events with small probability do not exist. **Answer** 9 dollars for probability 0.001.
- **Solution 2.** Infinite capitals do not exist. **Answer** 20.9 dollars if reward of more than 1000000 dollars seems impossible.

Solution with utility functions.

The game.

Toss a coin. In case of **heads** get 2 dollars, in case of **tails** toss again. Each time in case of **heads** the reward is payed (and the game is over), in case of **tails** the reward doubles.

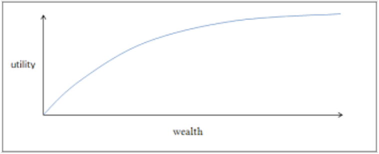
- The more money you have, the less is change of **utility** of 1 dollar for you!
- Maximize not mathematical expectation of **reward**, but mathematical expectation of **utility of reward**.
- **Solution 1.** Not enough information. The answer should depend on your **wealth**!

Choosing utility function.

- **Utility function** $U(v)$, where v is the wealth.
- utility of 2 dollars is more than utility of 1 dollar:

$$U(x) < U(y), \qquad \text{for } x < y.$$

- **Desirable:** The more money you have, the less is change in utility when wealth changes by 1 dollar!



- **Choice**: $U(v) = \ln(v+1)$.

Solution with utility functions.

The game.

Toss a coin. In case of **heads** get 2 dollars, in case of **tails** toss again. Each time in case of **heads** the reward is payed (and the game is over), in case of **tails** the reward doubles.

- **Utility function** $U(v) = \ln(v+1)$.
- Utility of doing nothing is $U(w) = \ln(w+1)$, where w is initial wealth.
- Calculate mathematical expectation of $U(w+R-y)$ (y is a price of a ticket). If $EU(w+R-y) \geq \ln(w+1)$, then it is reasonable to play a game.
- Find a ticket price $y = y(w)$ such that $EU(w+R-y) = \ln(w+1)$.

$$EU(w+R-y) = \sum_{n\geq 1} \frac{\ln(2^n + w - y + 1)}{2^n} < \infty.$$

Three solutions to paradox.

The game.

Toss a coin. In case of **heads** get 2 dollars, in case of **tails** toss again. Each time in case of **heads** the reward is payed (and the game is over), in case of **tails** the reward doubles.

Solutions.

- **Solution 1** (utility functions).
  Answer: it depends on wealth w, if initial wealth $w = 50000$ dollars, ticket price $y = 16.5$ dollars.
- **Solution 2** (infinite capitals do not exist).
  Answer: $y = 20.9$ dollars if reward of more than 1000000 dollars is impossible.
- **Solution 3** (events with small probability do not exist).
  Answer: $y = 9$ dollars if events with probability less than 0.001 just do not occur.

Other types of utility functions

Attitude towards risk is something like

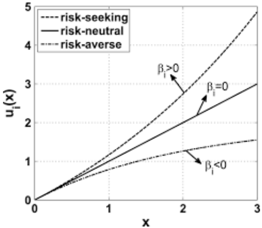
$$\frac{dU}{dv} \qquad \text{or} \qquad U(v+1) - U(v).$$

Risk-averse:

- Bernoulli: $U(v) = \ln(v)$ (we used $U(v) = \ln(v+1)$)
- $U(v) = \frac{e^{v\beta}-1}{\beta}$, where $\beta < 0$ is a constant.

Risk-neutral: $U(v) = v$ (we optimize standard mathematical expectation).

Risk-seeking: $U(v) = \frac{e^{v\beta}-1}{\beta}$, where $\beta > 0$ is a constant.



Kelly strategy (1956).

The game.

A series of games on tossing the coin. If your guess is correct and your bet is s dollars, you receive s more dollars. You know that the coin is biased (0.6 is probability of getting heads). Your capital is S dollars, what fraction of the capital should you bet?

Mathematical expectation approach.

Let R be a reward in 1 game on s dollars

$$E(S+R) = 0.6 \cdot (S+s) + 0.4 \cdot (S-s) = S + 0.2s.$$

The more s is, the better it is! You should invest all your money. Not a very good idea.

Kelly strategy (1956).

The game.

A series of games on tossing the coin. If your guess is correct and your bet is s dollars, you receive s more dollars. You know that the coin is biased (0.6 is probability of getting heads). Your capital is S dollars, what fraction of the capital should you bet?

Utility function approach. Let R be a reward in 1 game on s dollars, and let $U(v) = \ln v$ be the utility function.

$$E \ln(S + R) = 0.6 \cdot \ln(S + s) + 0.4 \cdot \ln(S - s).$$

The derivative is equal to:

$$0.6/(S + s) - 0.4/(S - s).$$

It is equal to 0, when $s = 0.2S$. Sounds reasonable!
In his paper Kelly gave economic reasoning reasoning to maximize logarithm of wealth.

Kelly strategy for more general game (1956).

The game.

A series of games on tossing the coin. If your guess is correct and your bet is s dollars, you receive ks more dollars. You know the probability of getting heads $p > 0.5$. Your capital is S dollars, what fraction of the capital should you bet?

Solution with utility functions

Similarly one can prove that we should bet:

$$\frac{edge}{odds} S = \frac{kp - q}{k} S$$

In his paper Kelly gave economic reasoning reasoning to maximize logarithm of wealth.

Applications of utility functions.

- first application was in **blackjack**, Edward O. Thorp, Beat the Dealer, 1966.
- portfolio optimization in **finance** (now it is one of the standard methods)
- for bet sizing: in **poker** (e.g. Texas holdem), in **betting**.
- used in **insurance** (the next slide)

Applications in insurance.

- To find the **risk premium**.
  X — losses, w — initial capital, P is the premium, $U(x)$ is the utility function. Find P such that
  $$EU(w-X) = U(w-P).$$
  We must have seen it somewhere...
- In **risk measures**: For example, **utility based shortfall** (J. Maes, Utility based risk measures).
- To construct portfolio (by Kelly method), instead of Efficient frontier.
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$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Conclusion

Main messages.

1. It may be a good idea to maximize mathematical expectation of an utility function (the most standard is the logarithm).
2. Often behavior of different people is modeled by different utility functions, since they have different attitudes towards risk.

Thank you very much for your attention!