Nondensity of orbital shadowing in C^1-topology.

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Problem

Is a set of dynamical systems having some shadowing property generic or nondense?

What is shadowing?

Theory of shadowing studies relations between approximate and exact trajectories of dynamical systems on unbounded time intervals. Shadowing means that any sufficiently precise approximate trajectory (a pseudotrajectory) is close to some exact trajectory.

POTP

Let f be a homeomorphism of a metric space (M, dist).

Definition 1. *A sequence $\{x_k\}_{k\in\mathbb{Z}}$ of points of M is a d-pseudotrajectory of f iff*

$$\text{dist}(x_{k+1}, f(x_k)) < d, \quad \forall k \in \mathbb{Z}.$$

Definition 2. *A homeomorphism f has pseudoorbit tracing property or standard shadowing property ($f \in$ POTP) iff for any $\epsilon > 0$ there exists a $d > 0$ such that for any d-pseudotrajectory $\xi = \{x_k\}_{k\in\mathbb{Z}}$ there exists a point $p \in M$ such that*

$$\text{dist}(x_k, f^k(p)) < \epsilon.$$

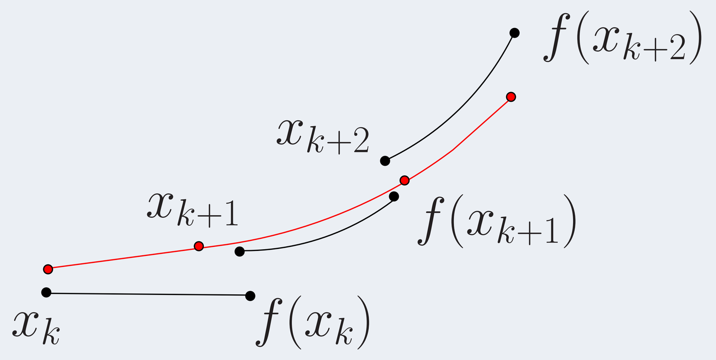

Figure : Example when POTP holds: black denotes a pseudotrajectory, red denotes an exact trajectory.

Any structurally stable diffeomorphism has POTP (**Robinson, Sawada, Morimoto, 1977–80**).

OSP and WSP

By $N(\epsilon, A)$ denote the ϵ-neighborhood of $A \subset M$. By $O(p, f) = \{f^k(p)\}_{k\in\mathbb{Z}}$ denote the trajectory of p.

Definition 3. *A homeomorphism f has orbital shadowing property ($f \in$ OSP) iff for any $\epsilon > 0$ there exists a $d > 0$ such that for any d-pseudotrajectory $\xi = \{x_k\}_{k\in\mathbb{Z}}$ there exists a point $p \in M$ such that*

$$\xi \subset N(\epsilon, O(p, f)) \quad \text{and} \quad O(p, f) \subset N(\epsilon, \xi).$$

Example of $f \in$ OSP\POTP: *irrational rotation of the circle.*

Definition 4. *A homeomorphism f has weak shadowing property ($f \in$ WSP) iff for any $\epsilon > 0$ there exists a $d > 0$ such that for any d-pseudotrajectory $\xi = \{x_k\}_{k\in\mathbb{Z}}$ there exists a point $p \in M$ such that*

$$\xi \subset N(\epsilon, O(p, f)).$$

Example of $f \in$ WSP\OSP: *the Plamenevskaya map* (an Ω-stable diffeomorphism constructed in 1999).

Spaces of dynamical systems

Let M be a closed smooth Riemannian manifold. By $H(M)$ denote the space of all homeomorphisms of M with C^0-metric. By Diff(M) denote the space of all C^1-diffeomorphisms of M with C^1-metric.

Definition 5. *A set is called generic if it is a Baire second cathegory set (i.e. contains a countable union of open and dense sets) in the corresponding space.*

Previous results

	POTP	OSP	WSP
H(M)	generic Pilyugin, Plamenevskaya 1999	generic Pilyugin, Plamenevskaya 1999	generic Pilyugin, Plamenevskaya 1999
Diff(M)	nondense Bonatti, Diaz, Turcat 2000	?	generic Crovisier 2006

Figure : Previous results on the topic.

Main result: C^1-nondensity of OSP

Theorem (Osipov,2010). *There exists a domain $W \subset$ Diff$(S^2 \times S^1)$ such that any $f \in W$ does not have OSP.*

It is possible to construct similar domains in Diff(M) for any n-dimensional manifold M with $n \geq 3$.

General strategy of the proof

In essense, we use a strategy of Ilyashenko and Gorodetski developed for construction of domains in Diff(M) with specific properties.

Step 1. Prove that there exists a *sufficiently good* neighborhood $\mathcal{U}$ in the space of skew products of a certain type such that any skew product from $\mathcal{U}$ does not have OSP.

Step 2. For transfer from skew products to diffeomorphisms apply the result of Gorodetski that can be informally formulated as follows:

Theorem (Gorodetski, 2001). *Let $\mathcal{U}$ be a sufficiently good neighborhood in the space of skew products of a certain type. Then there exists a C^1-domain $\mathcal{V} \subset$ Diff(M) such that any $f \in \mathcal{V}$ has a partially hyperbolic locally maximal invariant set Δ, and $F|_\Delta$ is topologically conjugate with a skew product from $\mathcal{U}$.*

Skew products

Note that skew products can be interpreted as random dynamical systems.

Let Σ^2 be a space of biinfinite sequences of 0 and 1.

Let $\sigma : \Sigma^2 \mapsto \Sigma^2$ be a Bernoulli shift given by

$$(\sigma(\omega))_k = \omega_{k+1}, \quad \forall \omega \in \Sigma^2, k \in \mathbb{Z}.$$

Definition 6. *Let $\{f_\omega\}_{\omega\in\Sigma^2}$ be a family of diffeomorphisms of S^1. A map $f : \Sigma^2 \times S^1 \mapsto \Sigma^2 \times S^1$ is called a mild skew product iff*

$$f(\omega, \phi) = (\sigma(\omega), f_\omega(\phi)).$$

Let g_0 be a rotation of S^1 by angle b, g_1 be a Morse-Smale diffeomorphism with two hyperbolic fixed points p_1 and p_2, and $Df(p_2) = a < 1$, $Df(p_1) = 1/a$ (cf. the figure).

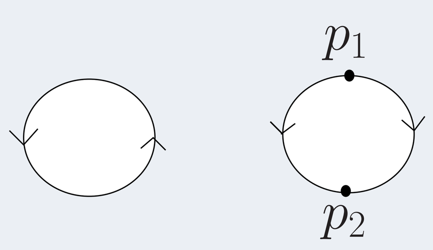

Figure : Diffeomorphisms g_0 and g_1.

Let a be close to 1, b be close to 0, and δ be a sufficiently small number; then $\mathcal{U}$ is a set of all mild skew products $\{f_\omega\}_{\omega\in\Sigma^2}$ such that

- f_ω is δ-close to g_0 in Diff(S^1) if $\omega_0 = 0$;
- f_ω is δ-close to g_1 in Diff(S^1) if $\omega_0 = 1$.

Case 1

Let F be a diffeomorphism from $\mathcal{V} \subset$ Diff$(S^2 \times S^1)$ which is assigned to some mild skew product G. Assume that there exist two hyperbolic periodic points q_1 and q_2 lying on different fibres such that $\dim W^u(q_1) = \dim W^s(q_2) = 1$ and

$$W^u(q_1) \cap W^s(q_2) \neq \emptyset.$$

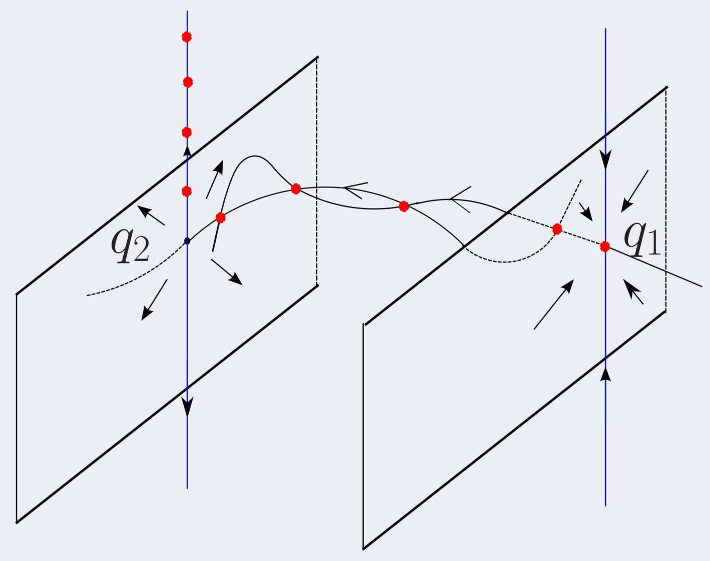

Figure : F in Case 1: red denotes a pseudotrajectory, blue denotes fibres.

Using expansivity of the Bernoulli shift, we prove that the constructed pseudotrajectory (cf. the figure) cannot be orbitally close to any exact trajectory.

Note that in order to derive contradiction it is important to prove that the intermediate part of the pseudotrajectory is far from the points q_1 and q_2.

Case 2

Let F be a diffeomorphism from $\mathcal{V} \subset$ Diff$(S^2 \times S^1)$ which is assigned to some mild skew product G.

Any such F has the following property:

hyperbolic periodic points with different indices are dense in the partially hyperbolic invariant set Δ.

Since Case 1 does not hold, for any two hyperbolic periodic points q_1 and q_2 lying on different fibres such that $\dim W^u(q_1) = \dim W^s(q_2) = 1$ the following holds:

$$W^u(q_1) \cap W^s(q_2) = \emptyset.$$

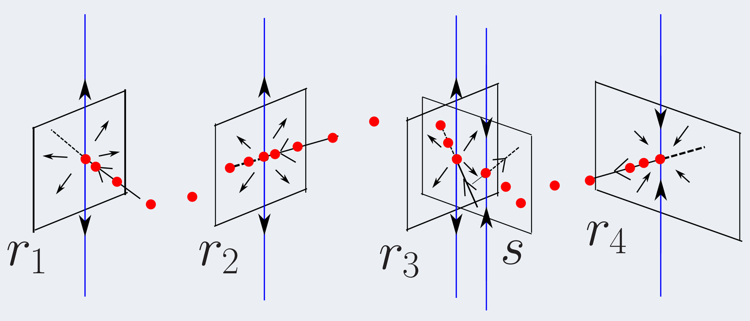

Figure : F in Case 2: red denotes a pseudotrajectory, blue denotes fibres.

Using density of hyperbolic periodic points with different indices, we contruct a pseudotrajectory joining r_1 and r_4 with $\dim W^s(r_1) = \dim W^u(r_4) = 1$ (cf. the figure). Assume that this pseudotrajectory is orbitally close to an exact trajectory of some point p.

Using methods from theory of skew products and symbolic dynamics, we prove that

$$p \in W^u(r_4) \cap W^s(r_1)$$

which contradicts to our conditions.

Note that in order to derive contradiction it is important to prove that the intermediate part of the pseudotrajectory is far from the points r_1 and r_4. In fact it is the key technical difficulty.

Possible directions for further research.

1. **Conjecture:** OSP is C^1-generic in Diff(M) if M is two-dimensional.
2. One-sided shadowing properties.
3. Inverse shadowing properties.
4. Limit shadowing.