Lipschitz Periodic Shadowing

Aleksei V. Osipov[1]
Joined research with Sergei Yu. Pilyugin[1] and Sergei B. Tikhomirov[2].

[1]Saint-Petersbourg State University

[2]National Taiwan University

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Shadowing and Lipschitz Shadowing

- $f : M \mapsto M$, $f \in C^1$, $M \in C^\infty$, dist
- $\xi = \{x_n\}$ is a d-pseudotrajectory, if

$$\mathrm{dist}(x_{n+1}, f(x_n)) < d.$$

- Shadowing property (POTP)
  $\forall \epsilon > 0\ \exists d > 0$ such that $\forall d$-pseudotrajectory ξ there exists an exact trajectory $\{p_n\}$ such that

$$\mathrm{dist}(x_n, p_n) < \epsilon.$$

- Lipschitz shadowing property (LipSP)
  $\exists L, d_0 > 0$ such that $\forall d < d_0$ and d-pseudotrajectory ξ there exists an exact trajectory $\{p_n\}$ such that

$$\mathrm{dist}(x_n, p_n) < Ld.$$

Periodic shadowing and structural stability

- Periodic shadowing (PerSh)
  $\forall \epsilon > 0\ \exists d > 0$ such that $\forall$ periodic d-pseudotrajectory ξ there exists a periodic exact trajectory $\{p_n\}$ such that
  $$\operatorname{dist}(x_n, p_n) < \epsilon.$$
- Lipschitz periodic shadowing (LipPerSh)
  PerSh with $\epsilon = Ld$
- S — the set of structurally stable diffeomorphisms
  There exists a neighborhood U of f in the C^1-topology such that $\forall g \in U$ the diffeomorphisms f and g are topologically conjugate.
- ΩS — the set of Ω-stable diffeomorphisms
  There exists a neighborhood U of f in the C^1-topology such that $\forall g \in U$ the diffeomorphisms f and g are topologically conjugate on their nonwandering sets.

Known Facts and Results

- $\mathrm{Int}^1(\mathrm{POTP}) = S$ (Sakai, 1994)
- $\mathrm{POTP} \neq S$
- $\mathrm{PerSh} \neq \Omega S$
- $\mathrm{LipSP} = S$ (Pilyugin, Tikhomirov, 2009)
- variational shadowing is equivalent to structural stability (Pilyugin, 2009)

Main results

Theorems (Osipov, Pilyugin, Tikhomirov, 2009)

- $\mathrm{Int}^1(\mathrm{PerSh}) = \Omega S$
- $\mathrm{LipPerSh} = \Omega S$

General Scheme of the Proof

- $\Omega S \subset \mathrm{LipPerSh}$
- $\mathrm{Int}^1(\mathrm{PerSh}) \subset \Omega S$
- $f \in \mathrm{LipPerSh} \Rightarrow f \in \Omega S$
    - Step 1. hyperbolicity of periodic points
    - Step 2. uniform hyperbolicity of periodic points
    - Step 3. f has the Axiom A
    - Step 4. f satisfies the no-cycle condition

Proof of $\Omega S \subset \mathrm{LipPerSh}$

- Spectral decomposition theorem:
  $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_m$, Ω_j is hyperbolic and has a dense semi-trajectory
- ξ is a periodic d-pseudotrajectory, $\xi \subset U(\Omega_j)$ for some j
- Shadowing lemma:
  if Λ is hyperbolic then f has LipSh and is expansive in some $U(\Lambda)$

Proof of $\mathrm{Int}^1(\mathrm{PerSh}) \subset \Omega S$

- HP — set of diffeomorphisms f such that every periodic point of f is hyperbolic
  Lemma (Aoki, 1992, Hayashi, 1992). $\mathrm{Int}^1(\mathrm{HP}) = \Omega S$
- It is enough to prove that $\mathrm{Int}^1(\mathrm{PerSh}) \subset \mathrm{HP}$
- h is a C^1-small pertubation of f that is linear in $U(p)$, p is a nonhyperbolic periodic point for h

Proof of LipPerSh $\subset \Omega S$, Steps 1 and 2

- $f, f^{-1} \in$ LipPerSh with $L > 1$
- Lemma: Every periodic point is hyperbolic
- Key lemma: Set of all periodic points of f has all properties of a standart hyperbolic set except compactness.

$$|Df^j(p)v_s| \le C\lambda^j|v_s|, \quad |Df^{-j}(p)v_u| \le C\lambda^j|v_u|,$$

where $j \ge 0$, $v_s \in S(p)$, $v_u \in U(p)$

  - p is an m-periodic point, let $v_0 = v_u \in U(p)$, $v_{i+1} = Df^i(p)v_i$,

$$\lambda_i = |v_{i+1}|/|v_i|, \quad a_0 = \tau, \quad a_{i+1} = \lambda_i a_i - 1,$$

  where τ is chosen such that $a_m = 0$

  - $w_i = a_i v_i/|v_i|$ for $0 \le i \le m-1$, $\{w_i\}$ is an $m(n+1)$-periodic
  - $|Df^i(p)v_u| = \lambda_0 \cdot \ldots \lambda_{i-1} > \frac{1}{16L}\left(1 + \frac{1}{8L}\right)^i |v_u|, \quad 0 \le i \le m-1.$

Proof of LipPerSh $\subset \Omega S$, Steps 3 and 4

- Lemma: f satisfies the Axiom A
    - P_l — the set of periodic points of index l
    - $\mathrm{Cl} P_l$ is a hyperbolic set.
    - density of periodic points in $\Omega(f)$
- Lemma: f has no cycles
    - any cycle is approximated by periodic pseudotrajectories
    - any cycle is approximated by periodic exact trajectories

Conclusion

Theorems (Osipov, Pilyugin, Tikhomirov, 2009)

- $\mathrm{Int}^1(\mathrm{PerSh}) = \Omega S$
- $\mathrm{LipPerSh} = \Omega S$

Thank you very much for your attention!