Incompressible Kernel - Computation Strategy

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1 θ - scheme

Under the hypothesis of 2D incompressible flow, the Navier-Stokes equations can be written as following :

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{1}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right] \tag{2}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right] \tag{3}$$

One can derive a Poisson equation for the pressure by combining these equation ($\frac{\partial}{\partial x} \times (2) + \frac{\partial}{\partial y} \times (3)$) and then eliminating most of the terms using the mass equation :

$$\Delta P = -\rho\left[\frac{\partial^2 U^2}{\partial x^2} + 2\frac{\partial^2 UV}{\partial x \partial y} + \frac{\partial^2 V^2}{\partial y2}\right] \tag{4}$$

For stability and flexibility purposes, the θ-scheme was chosen to perform the temporal integration of this differential system. In the general case of a variable X, it can be written as following :

$$\frac{\partial X}{\partial t} = F(t, X) \quad \Longrightarrow \quad \frac{X^{n+1} - X^n}{\Delta t} = (1-\theta)F(n, X^n) + \theta F(n+1, X^{n+1}) \tag{5}$$

One can notice that the momentum equations (2) and (3) can perfectly be discretized in time with this scheme. Here is what it looks like :

$$\left[U - \theta\Delta t\left(\nu\Delta U + \frac{1}{\rho}\frac{\partial P}{\partial x} - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right)\right]_{n+1} = \left[U + (1-\theta)\Delta t\left(\nu\Delta U - \frac{1}{\rho}\frac{\partial P}{\partial x} - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right)\right]_n \tag{6}$$

$$\left[V - \theta\Delta t\left(\nu\Delta V + \frac{1}{\rho}\frac{\partial P}{\partial y} - U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y}\right)\right]_{n+1} = \left[V + (1-\theta)\Delta t\left(\nu\Delta V - \frac{1}{\rho}\frac{\partial P}{\partial y} - U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y}\right)\right]_n \tag{7}$$

On the other hand, the pressure equation (4) is not an equation defining the temporal evolution of P, but rather an equation defining the value of P as a function of the velocity. Moreover, P can't be eliminated from the equations above. In order to provide a scheme consistent with the philosophy of the original θ-scheme, the pressure term will be interpreted as a derivative-type variable that will be computed between the steps of the main variables, with a θ-like scheme. The resulting set of discretized equation is the following :

$$\left[U - \theta\Delta t\left(\nu\Delta U - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right)\right]_{n+1} + \left[\Delta t\frac{1}{\rho}\frac{\partial P}{\partial x}\right]_{n+\theta} = \left[U + (1-\theta)\Delta t\left(\nu\Delta U - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right)\right]_n \tag{8}$$

$$\left[V - \theta\Delta t\left(\nu\Delta V - U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y}\right)\right]_{n+1} + \left[\Delta t\frac{1}{\rho}\frac{\partial P}{\partial y}\right]_{n+\theta} = \left[V + (1-\theta)\Delta t\left(\nu\Delta V - U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y}\right)\right]_n \tag{9}$$

$$[\Delta P]_{n+\theta} + \theta\rho\left[\frac{\partial^2 U^2}{\partial x^2} + 2\frac{\partial^2 UV}{\partial x\partial y} + \frac{\partial^2 V^2}{\partial y2}\right]_{n+1} = -(1-\theta)\rho\left[\frac{\partial^2 U^2}{\partial x^2} + 2\frac{\partial^2 UV}{\partial x\partial y} + \frac{\partial^2 V^2}{\partial y2}\right]_n \tag{10}$$

2 Linearization

Since the chosen scheme is not explicit (unless if $\theta = 0$, where it becomes a simple Forward Euler scheme) it has to be put as a matricial system to be solved. However, with the current set of equations (8 - 9 - 10), the advection terms and the whole pressure equation are non-linear, and must all those unknown at the step n must be linearized.

To do so, one can write the following Taylor developments :

$$U^{n+1} \approx \tilde{U} := U^n + \Delta t\left.\frac{\partial U}{\partial t}\right|_n \tag{11}$$

$$V^{n+1} \approx \tilde{V} := V^n + \Delta t\left.\frac{\partial V}{\partial t}\right|_n \tag{12}$$

NB : $\left.\frac{\partial U}{\partial t}\right|_n = \left(\nu\Delta U - \frac{1}{\rho}\frac{\partial P}{\partial x} - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right)_n$ and can therefore be computed explicitely. The result also applies to $\left.\frac{\partial U}{\partial t}\right|_n$.

By replacing some of the U^{n+1} and V^{n+1} in the non linear terms by these approximations, the system can completely be linearized :

$$U^{n+1} - \theta\Delta t\left(\nu\Delta U^{n+1} - \tilde{U}\left.\frac{\partial U}{\partial x}\right|_{n+1} - \tilde{V}\left.\frac{\partial U}{\partial y}\right|_{n+1}\right) + \Delta t\frac{1}{\rho}\left.\frac{\partial P}{\partial x}\right|_{n+\theta} = \left[U + (1-\theta)\Delta t\left(\nu\Delta U - U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y}\right)\right]_n \tag{13}$$

$$V^{n+1} - \theta\Delta t\left.\left(\nu\Delta V^{n+1} - \tilde{U}\left.\frac{\partial V}{\partial x}\right|_{n+1} - \tilde{V}\frac{\partial V}{\partial y}\right)\right|_{n+1} + \Delta t\frac{1}{\rho}\left.\frac{\partial P}{\partial y}\right|_{n+\theta} = \left[V + (1-\theta)\Delta t\left(\nu\Delta V - U\frac{\partial V}{\partial x} - V\frac{\partial V}{\partial y}\right)\right]_n \tag{14}$$

$$\Delta P^{n+\theta} + \theta\rho\left[\left.\frac{\partial^2 \tilde{U}U}{\partial x^2}\right|_{n+1} + \left.\frac{\partial^2 \tilde{U}V}{\partial x\partial y}\right|_{n+1} + \left.\frac{\partial^2 U\tilde{V}}{\partial x\partial y}\right|_{n+1} + \left.\frac{\partial^2 \tilde{V}V}{\partial y^2}\right|_{n+1}\right] = -(1-\theta)\rho\left[\frac{\partial^2 U^2}{\partial x^2} + 2\frac{\partial^2 UV}{\partial x\partial y} + \frac{\partial^2 V^2}{\partial y2}\right]_n \tag{15}$$

3 Matricial system

The system of equations above can now be put as $AX = B$ where X is a vector containing all the unknown variables at $n+1$ and $n+\theta$, B a vector representing the second member of the system and depending only on known variables (ie variables at n), and A a square matrix. Let's assume $X = [U|V|P]^T$ with the variables ordered such as $U_{i,j} = X_{i+n_x j}$ and so on. After spatial discretization of the field operators, the matrix A can be shown to be equal to :

$$A = \begin{bmatrix}
\alpha & \beta_1 & & & \gamma_1 & & & & & & & & & & & \delta_1^u & & & & & \\
\beta_2 & \alpha & \beta_1 & & & \gamma_1 & & & & & & & & & \delta_2^u & & \delta_1^u & & & & \\
 & \beta_2 & \alpha & \beta_1 & & & \gamma_1 & & & & & & & & & \delta_2^u & & \delta_1^u & & & \\
 & & \beta_2 & \alpha & \beta_1 & & & & & & & & & & & & \delta_2^u & & \delta_1^u & & \\
\gamma_2 & & & \beta_2 & \alpha & \beta_1 & & & & & & & & & & & & \delta_2^u & & \delta_1^u & \\
 & \gamma_2 & & & \beta_2 & \alpha & \beta_1 & & & & & & & & & & & & \delta_2^u & & \delta_1^u \\
 & & \gamma_2 & & & \beta_2 & \alpha & & & & & & & & & & & & & \delta_2^u & \\
 & & & & & & & \alpha & \beta_1 & & & \gamma_1 & & & & & & & \delta_1^v & & \\
 & & & & & & & \beta_2 & \alpha & \beta_1 & & & \gamma_1 & & & & & & & \delta_1^v & \\
 & & & & & & & & \beta_2 & \alpha & \beta_1 & & & \gamma_1 & & & & & & & \delta_1^v \\
 & & & & & & & & & \beta_2 & \alpha & \beta_1 & & & & & & & & & \\
 & & & & & & & \gamma_2 & & & \beta_2 & \alpha & \beta_1 & & \delta_2^v & & & & & & \\
 & & & & & & & & \gamma_2 & & & \beta_2 & \alpha & \beta_1 & & \delta_2^v & & & & & \\
 & & & & & & & & & \gamma_2 & & & \beta_2 & \alpha & & & \delta_2^v & & & & \\
A^u & B_1^u & & D_3 & & D_1 & & A^v & & & E_3 & C_1^v & E_1 & & A & B_1 & & & C_1 & & \\
B_2^u & A^u & B_1^u & & D_3 & & D_1 & & A^v & & & E_3 & C_1^v & E_1 & B_2 & A & B_1 & & & C_1 & \\
 & B_2^u & A^u & B_1^u & & D_3 & & & & A^v & & & E_3 & C_1^v & & B_2 & A & B_1 & & & C_1 \\
D_2 & & B_2^u & A^u & B_1^u & & D_3 & E_2 & & & A^v & & & E_3 & & & B_2 & A & B_1 & & \\
 & D_2 & & B_2^u & A^u & B_1^u & & C_2^v & E_2 & & & A^v & & & C_2 & & & B_2 & A & B_1 & \\
D_4 & & D_2 & & B_2^u & A^u & B_1^u & E_4 & C_2^v & E_2 & & & A^v & & & C_2 & & & B_2 & A & B_1 \\
 & D_4 & & D_2 & & B_2^u & A^u & & E_4 & C_2^v & E_2 & & & A^v & & & C_2 & & & B_2 & A
\end{bmatrix}$$

All the non-zero coefficients of the matrix A are defined below.

NB : If a coefficient seems to have to point to a value out of bounds, it must not be implemented in the matrix. Instead, one should use the boundary conditions to adjust the vector B.

$$\nu_x^* = \nu \frac{\Delta t}{\Delta x^2}$$

$$\nu_y^* = \nu \frac{\Delta t}{\Delta y^2}$$

$$\tilde{U}_{i,j} = U_{i,j}^n + \Delta t \left. \frac{\partial U_{i,j}}{\partial t} \right|_n \approx U_{i,j}^{n+1}$$

$$\tilde{V}_{i,j} = V_{i,j}^n + \Delta t \left. \frac{\partial V_{i,j}}{\partial t} \right|_n \approx V_{i,j}^{n+1}$$

$$\tilde{U}^*_{i,j} = \frac{\Delta t}{\Delta x} \tilde{U}_{i,j}$$

$$\tilde{V}^*_{i,j} = \frac{\Delta t}{\Delta y} \tilde{V}_{i,j}$$

$$\alpha = 1 + 2\theta\ (\nu_x^* + \nu_y^*)$$

$$\beta_1 = -\theta\ (\nu_x^* - \frac{1}{2}\tilde{U}^*_{i,j})$$

$$\beta_2 = -\theta\ (\nu_x^* + \frac{1}{2}\tilde{U}^*_{i,j})$$

$$\gamma_1 = -\theta\ (\nu_y^* - \frac{1}{2}\tilde{V}^*_{i,j})$$

$$\gamma_2 = -\theta\ (\nu_y^* + \frac{1}{2}\tilde{V}^*_{i,j})$$

$$\delta_1^u = \frac{\Delta t}{2\rho\Delta x}$$

$$\delta_2^u = -\frac{\Delta t}{2\rho\Delta x}$$

$$\delta_1^v = \frac{\Delta t}{2\rho\Delta y}$$

$$\delta_2^v = -\frac{\Delta t}{2\rho\Delta y}$$

$$A = -2(1 + \frac{\Delta x^2}{\Delta y^2})$$

$$B_1 = B_2 = 1$$

$$C_1 = C_2 = \frac{\Delta x^2}{\Delta y^2}$$

$$A^u = -2\rho\theta\tilde{U}_{i,j}$$

$$B_1^u = \rho\theta\tilde{U}_{i+1,j}$$

$$B_2^u = \rho\theta\tilde{U}_{i-1,j}$$

$$A^v = -2\rho\theta\frac{\Delta x^2}{\Delta y^2}\tilde{V}_{i,j}$$

$$C_1^v = \rho\theta\frac{\Delta x^2}{\Delta y^2}\tilde{V}_{i,j+1}$$

$$C_2^v = \rho\theta\frac{\Delta x^2}{\Delta y^2}\tilde{V}_{i,j-1}$$

$$D_1 = \frac{1}{4}\frac{\Delta x}{\Delta y}\rho\theta\tilde{V}_{i+1,j+1}$$

$$D_2 = -\frac{1}{4}\frac{\Delta x}{\Delta y}\rho\theta\tilde{V}_{i+1,j-1}$$

$$D_3 = -\frac{1}{4}\frac{\Delta x}{\Delta y}\rho\theta\tilde{V}_{i-1,j+1}$$

$$D_4 = \frac{1}{4}\frac{\Delta x}{\Delta y}\rho\theta\tilde{V}_{i-1,j-1}$$

$$E_1 = \frac{1}{4}\frac{\Delta x}{\Delta y}\rho\theta\tilde{U}_{i+1,j+1}$$

$$E_2 = -\frac{1}{4}\frac{\Delta x}{\Delta y}\rho\theta\tilde{U}_{i+1,j-1}$$

$$E_3 = -\frac{1}{4}\frac{\Delta x}{\Delta y}\rho\theta\tilde{U}_{i-1,j+1}$$

$$E_4 = \frac{1}{4}\frac{\Delta x}{\Delta y}\rho\theta\tilde{U}_{i-1,j-1}$$

4 Solving algorithm

In order to process this huge matricial system, an explicit resolution with a LU decomposition is very unefficient. Instead, an iterative resolution with the Gauss-Seidel algorithm is much more useful.

This algorithm relies on the LDU decomposition of the matrix $A = L + D + U$, where L is the lower part, D the diagonal and U the upper part. The algorithm then consists only in repeating the same update of the vector X until the values converge :

$$(L + D)X^{n+1} = B - UX^n \tag{16}$$

If updated "from first to last", the computation of the values of X^{n+1} is fully explicit :

$$X_i^{n+1} = \frac{1}{D_{ii}}\left(B_i - \sum_{k=i+1}^{N} U_{ik}X_k^n - \sum_{k=1}^{i-1} L_{ik}X_k^{n+1}\right) \tag{17}$$