CS1100: Homework 3

Due Friday, October 12, in class

Problem Set 1 Prove the soundness of each of the following rules using first-order logic.

1. $\dfrac{\exists x.P(a) \to Q(x)}{P(a) \to \exists x.Q(x)}$

2. $\dfrac{\forall x.P(x,x) \to Q(x) \qquad \exists x.\forall y.P(x,y)}{\exists z.Q(z)}$

3. $\dfrac{\forall x.\forall y.P(x,y) \to P(y,x) \qquad \forall x.P(a,x) \to P(x,b)}{\forall z.P(z,a) \to P(b,z)}$

EXAMPLE. $\dfrac{\forall x.\forall y.P(x,y) \to Q(a,y) \qquad \forall x.\exists y.R(x,a) \to P(y,x)}{\forall z.R(z,a) \to \exists y.Q(y,z)}$

PROOF.

Assumption a1: $\forall x.\forall y.P(x,y) \to Q(a,y)$

Assumption a2: $\forall x.\exists y.R(x,a) \to P(y,x)$

Goal: $\forall z.R(z,a) \to \exists y.Q(y,z)$

1. Let z be given. We will show $\boxed{R(z,a) \to \exists y.Q(y,z)}$.
2. Assume $R(z,a)$. We will show $\boxed{\exists y.Q(y,z)}$.
3. By $\forall$-elim on a2, with $x = z$, we get $\exists y.R(z,a) \to P(y,z)$.
4. By $\exists$-elim on step 3, let b be such that $R(z,a) \to P(b,z)$.
5. By $\forall$-elim on a1, with $x = b$ and $y = z$, we get $P(b,z) \to Q(a,z)$.
6. By MP with steps 4 and 2, we get $P(b,z)$.
7. By MP with steps 5 and 6, we get $Q(a,z)$.
8. By $\exists$-intro with step 7, with $y = a$, we get $\exists y.Q(y,z)$.
9. By $\to$-intro, steps 2–8, we get $R(z,a) \to \exists y.Q(y,z)$.
10. By $\forall$-intro, steps 1–9, we get $\forall z.R(z,a) \to \exists y.Q(y,z)$.

This completes the proof.

Problem Set 2 A relation $R(x,y)$ on a set A is called *dense* if it satisfies

$$\forall x \forall y.R(x,y) \to \exists z.R(x,z) \land R(z,y)$$

For each relation below, determine whether the relation is reflexive, symmetric, antisymmetric, transitive, dense, a partial order, or an equivalence relation.

Indicate each property above that the relation satisfies.

1. Let $\{0,1\}^*$ be the set of binary strings.

   For $s, t \in \{0,1\}^*$, $R(s,t)$ is true if and only if s is an *initial segment*, or a *prefix*, of t.

   For example, $R(01, 0101101)$ is true, but $R(011, 0101101)$ is false.

2. Let $A = \mathbb{R} \times \mathbb{R}$ be the set of points in the Cartesian coordinate plane.

   Given two elements of A, $p = (x, y)$ and $p' = (x', y')$, $R(p, p')$ is true if the distance from p to the origin is no greater than the distance from p' to the origin. In other words,

$$P((x,y),(x',y')) \iff \sqrt{x^2+y^2} \leq \sqrt{(x')^2+(y')^2}$$

3. Let $A = \mathbb{Q}$, the set of rational numbers. (Fractions.)

   For $r, r' \in A$, $P(r, r')$ is true if $r < r'$.

4. Let A be the set of all people in the world.

   $R(x, y)$ is true if x and y have a common parent.

Problem Set 3

1. Let A, B be sets. Prove that

$$A = B \leftrightarrow A \cup B = A \cap B$$

2. Let A, B, C be sets. Prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

3. Let $R(x, y)$ be a relation on a set A which satisfies

$$\forall x \forall y \forall z . R(x,y) \wedge R(z,y) \rightarrow R(x,z)$$

   Prove that if R is reflexive, then R is an equivalence relation.

4. Let $f : A \to B$ be a function. For $X, Y \subseteq A$, prove that the f-image $f(X \cup Y)$ of $X \cup Y$ is equal to the union of f-images $f(X) \cup f(Y)$.