CS1100: Test 3 Review

1 Proofs in first-order logic

The test will contain at least one medium-difficulty proof in first-order logic. Here is a sample of practice problems you should be able to do:

1. $$\frac{\forall x \forall y. P(x,y) \to Q(y,x) \lor R(x) \qquad \forall x. Q(a,x) \to R(x)}{\forall z. P(z,a) \to R(z)}$$

2. $$\frac{\exists v. P(a,v) \qquad \forall x \forall y. P(x,y) \land Q(x) \to R(y)}{Q(a) \to \exists z. R(z)}$$

3. $$\frac{\forall x. P(a,x) \to Q(x) \qquad \forall y. \exists z. P(y,z)}{\exists w. Q(w)}$$

4. $$\frac{\forall x. P(x,b) \lor Q(a,x) \to R(x) \qquad \forall y. R(y) \to Q(y,b)}{P(a,b) \to \exists z. Q(z,z)}$$

5. $$\frac{\forall x \exists y. P(a,x) \to Q(x,y) \qquad \forall x \forall y. Q(x,y) \to P(x,b)}{P(a,a) \to P(b,b)}$$

2 Induction and Recursion

2.1 Reading recursive definitions

Find $f(2), f(3), f(4)$, and $f(5)$ if $f(n)$ is defined recursively by $f(0) = f(1) = 1$ and, for $n = 1, 2, \ldots$:

1. $f(n+1) = f(n) - f(n-1)$
2. $f(n+1) = f(n)f(n-1)$
3. $f(n+1) = f(n)^2 + f(n-3)^3$
4. $f(n+1) = f(n)/f(n-1)$
5. $f(n+1) = f(n) - 2f(n-1) + 3n$

2.2 Formulating and proving explicit formulas

For each of the following problems, do the following:

- Compute $f(n)$ for $n \in \{1, 2, 3, 4, 5\}$.
- Guess an explicit formula for $f(n)$ — written only in terms of n, with no reference to $f(n), f(n-1)$, etc.
- Prove that your formula is correct for all n.

1. $f(0) = 1,\ f(n) = f(n-1) - 1$ for $n \geq 1$
2. $f(0) = 1,\ f(n+1) = -2f(n)$ for $n \geq 0$
3. $f(0) = 0,\ f(1) = 1,\ f(n+2) = 5f(n+1) - 6f(n)$ for $n \geq 0$.
   (*Hint.* Compare the first few values of $f(n)$ to the functions 2^n and 3^n.)
4. $f(1) = 1,\ f(2) = 4,\ f(n) = 4f(n-1) - 4f(n-2)$ for $n \geq 2$.
   (*Hint.* Compare $f(n)$ with 2^n.)

2.3 Computing Series

For each of the following examples, given a series of numbers $a_0 + a_1 + a_2 + \cdots$, do the following:

- Compute the sum of the first n numbers for $n \in \{0, 1, 2, 3, 4\}$.
- Guess an explicit formula for $\sum_{k=0}^{n} a_k$ for generic n.
- Prove that your formula is correct for all n.

$$\sum_{k=0}^{n} 2^k = 1 + 2 + 2^2 + \cdots + 2^n$$

$$\sum_{k=0}^{n} 2 \cdot 3^k = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^n$$

$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + 5 + \cdots + (2n-1)$$

$$\sum_{k=1}^{n} \frac{1}{k \cdot (k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$$

$$\sum_{k=0}^{n} k \cdot k! = 0 + 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$$

3 Sets and Cardinalities

Consider the following sets of numbers:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, 3, 4, \cdots\} = \{n \mid n \text{ is a non-negative integer}\} \\
\mathbb{N}_{<10} &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} = \{n \mid n \in \mathbb{N} \wedge n < 10\} \\
Odds &= 1, 3, 5, 7, \cdots = \{n \mid \exists k \in \mathbb{N}. n = 2k + 1\} \\
Squares &= 0, 1, 4, 9, \cdots = \{n \mid \exists k \in nat. n = k^2\} \\
Primes &= \{2, 3, 5, 7, \cdots\} = \{n \mid n \text{ is prime}\}
\end{aligned}$$

Now, define the sets

$$\begin{aligned}
A &= \mathbb{N}_{<10} \cap Odds \\
B &= \mathbb{N}_{<10} \cap Squares \\
C &= \mathbb{N}_{<10} \cap Primes
\end{aligned}$$

3.1 Basic operations on sets

Compute each of the following sets by explicitly listing its elements.

1. A, B, C
2. $A \cup B$, $A \cap B$, $A \cap C$, $B \cup C$
3. $A \times B$, $B \times C$, $A \sqcup B$, $A \sqcup C$
4. $A - B$, $B - A$, $B - C$, $C - A$
5. $\mathcal{P}(B), \mathcal{P}(A \cap B), \mathcal{P}(B \cap C), \mathcal{P}(\mathcal{P}(B \cap C))$
6. $(A \cap B)^{A \cap C}$

3.2 Cardinalities

Give $|X|$ for every set X you computed in the previous problem.

3.3 Rules of cardinalities

Express cardinality resulting from of operatons on sets in terms of cardinalities of sets themselves:

$$\begin{aligned}
|\mathbb{N}_{<10}| &= 10 \\
|\emptyset| &= 0 \\
|X \cup Y| &= |X| + |Y| \qquad \text{if } X \cap Y = \emptyset \\
|X \cup Y| &= \\
|X - Y| &= \\
|X \sqcap Y| &= \\
|X \times Y| &= \\
|\mathcal{P}(X)| &= \\
|Y^X| &=
\end{aligned}$$

4 Using the fundamental laws of counting

1. There are 3 spots left in a certain class section next semester, and 10 students that would like to register for it. How many possibilities for the final class roster are there, if all the seats are filled up?

2. How many license plates with 6 places can be made, if the first two must be letters, and the last four must be letters or numbers. (Upper and lower case are considered the same.)

3. The menu at a certain restaurant has 7 appetizers, 10 entrees, and 4 deserts.
   a) How many ways are there to order a non-desert item from the menu?
   b) How many ways are there to order two different items from the menu, deserts included?
   c) How many ways are there to order a three-course dinner, which includes one item from each category?
   d) A couple comes to a resaturant, and each wants to have a main course and either an appetizer or a desert. How many orders can they make?
   e) Another couple comes, with the same requirements as before, but now they want their main courses to be different. How many orders can they place? (*Hint.* Use the formula for $|X \cup Y|$.)

4. How many passwords can you make if:
   a) The length must be 7 characters, and the only allowed characters are uppercase letters, lowercase letters, and digits 0–9?
   b) Same requirements as above, but the passwords must begin with a letter?
   c) Same as above, but the length of the password can be 7 or 8?
   d) Same as above, but the password may also include some special characters, of which there are 7?

5. Somebody draws five cards from a full deck of 52. Three of the cards are hearts, and two are clubs. How many possibilities are there?