# Examples of proofs about sets and functions

## Example 1

Let $A$ and $B$ be sets. Prove that $A \cup B = B \cup A$.

PROOF.

We need to show equality between two sets: $A \cup B$ and $B \cup A$.

To prove that two sets are equal, we need to prove that each is a subset of the other. We proceed to do this in turn.

$\underline{A \cup B \subseteq B \cup A}$

1. Let $x \in A \cup B$. We will show that $x \in B \cup A$.
2. By step 1, $x \in A$ or $x \in B$. We consider two cases of this disjunction.
3. **Case 1.** Suppose $x \in A$. Then $x \in B \vee x \in A$.

   Therefore, $x \in B \cup A$.

   **Case 2.** Suppose $x \in B$. Then $x \in B \vee x \in A$.

   Therefore, $x \in B \cup A$.
4. In either case, we obtain $x \in B \cup A$.
5. Steps 1–4 show that, if $x \in A \cup B$, then $x \in B \cup A$. Therefore $A \cup B \subseteq B \cup A$.

$\underline{B \cup A \subseteq A \cup B}$ This proof is completely symmetric to the first one, by interchanging the roles of $A$ and $B$.

Having established that $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$, we conclude that $A \cup B = B \cup A$.

## Example 2

Let $A$ and $B$ be sets. Prove that $A \subseteq B \leftrightarrow A \cup B = B$.

PROOF. To prove a logical equivalence, we show that each side implies the other. That is, we need to establish two claims.

$\underline{A \subseteq B \to A \cup B = B}$ 1. This goal is an implication, so we begin by assuming $A \subseteq B$, and proceed to show that $A \cup B = B$.

To that end, we must show that $A \cup B \subseteq B$ as well as $B \subseteq A \cup B$.

CLAIM. $\underline{A \cup B \subseteq B}$

2. Assume $x \in A \cup B$. We will show that $x \in B$.
3. From $x \in A \cup B$, we deduce that $x \in A \vee x \in B$.
4. This is a disjunction, so we proceed to prove by cases.

**Case 1.** Suppose $x \in A$.

By assumption in step 1, this implies that $x \in B$.

This proves the goal.

**Case 2.** Suppose $x \in B$. The goal is immediate.

5. In either case, we conclude $x \in B$. Since $x \in A \cup B$ was arbitrary, we conclude that $A \cup B \subseteq B$.

CLAIM. $\underline{B \subseteq A \cup B}$

6. This goal is immediate: If $x \in B$, then certainly $x \in A \vee x \in B$. Thus $x \in A \cup B$.

We have proved that $A \cup B \subseteq B$ and $B \subseteq A \cup B$.

Therefore, $A \cup B = B$.

$\underline{A \cup B = B \to A \subseteq B}$ 1. This goal is an implication, so we begin by assuming that $A \cup B = B$. We then seek to prove that $A \subseteq B$.

2. Let $x \in A$. We seek to prove that $x \in B$.
3. Since $x \in A$, we also have $x \in A \vee x \in B$ ($\vee$-introduction).

   Thus, $x \in A \cup B$.
4. By assumption in step 1, $A \cup B = B$, so the previous step entails that $x \in B$.
5. Since $x \in A$ was arbitrary, steps 2–4 show that $A \subseteq B$.

This completes the proof.

## Example 3

Let $f : A \to B$ be a function.

For $X \subseteq B$, the *inverse image of $X$ under $f$*, denoted $f^{-1}[X]$, is the subset of $A$ defined as follows:

$$f^{-1}[X] = \{a \in A \mid f(a) \in X\}$$

Prove that $f^{-1}[X \cap Y] = f^{-1}[X] \cap f^{-1}[Y]$.

PROOF. Let $X, Y \subseteq B$.

We are asked to show equality between the two sets $f^{-1}[X \cap Y]$ and $f^{-1}[X] \cap f^{-1}[Y]$; we do this by showing that each is a subset of the other.

$\underline{f^{-1}[X \cap Y] \subseteq f^{-1}[X] \cap f^{-1}[Y]}$.

1. Let $a \in f^{-1}[X \cap Y]$. We will show $a \in f^{-1}[X] \cap f^{-1}[Y]$.
2. By definition of inverse image, $a \in f^{-1}[X \cap Y]$ implies $f(a) \in X \cap Y$.
3. By step 2, $f(a) \in X$ and $f(a) \in Y$.
4. By definition of inverse image, $a \in f^{-1}[X]$ and $a \in f^{-1}[Y]$.
5. That is, $a \in f^{-1}[X] \cap f^{-1}[Y]$.

$\underline{f^{-1}[X] \cap f^{-1}[Y] \subseteq f^{-1}[X \cap Y]}$

1. Let $a \in f^{-1}[X] \cap f^{-1}[Y]$. We will show that $a \in f[X \cap Y]$.
2. By step 1, $f(a) \in X$ and $f(a) \in Y$.
3. That is, $f(a) \in X \cap Y$.
4. By definition of inverse image, $a \in f^{-1}[X \cap Y]$.

This completes the proof.