

Discrete math: Homework 2

Proofs in Propositional Logic

Due date: Friday, September 13, in class

In each problem set below, prove the soundness of the given rules.

In every proof, state the assumptions, the goal to be proved, the rule which you are using in every step, and the place(s) earlier in the proof which gives the assumption(s) required by the rule.

EXAMPLE. *Show the soundness of the following rule.*

$$\frac{A \vee B \rightarrow C \wedge B \quad A \wedge B \rightarrow D}{A \rightarrow D}$$

Assumptions.

- $a_1 : A \vee B \rightarrow C \wedge B$
- $a_2 : A \wedge B \rightarrow D$

Goal: $A \rightarrow D$

- Proof.**
1. \lceil Assume A . Our new goal is to prove D .
 2. By \vee -intro with step 1, we derive $A \vee B$.
 3. By Modus Ponens with a_1 and step 2, we derive $C \wedge B$.
 4. By \wedge -elim with step 3, we derive B .
 5. By \wedge -intro with steps 1 and 4, we derive $A \wedge B$.
 6. By Modus Ponens with a_2 and step 5, we derive D . \rfloor
 7. By \rightarrow -intro with steps 1–6, we derive $A \rightarrow D$.

This was the goal, so the proof is complete.

In the following problems, *remember the notation*.

For example, $X \rightarrow Y \rightarrow Z$ stands for $X \rightarrow (Y \rightarrow Z)$.

If the rule has no formulas *above the line*, then this means the rule has *no assumptions*: all the facts you can derive must be generated from scratch (starting with Step 1).

Problem Set 1

1.
$$\frac{A \rightarrow A \rightarrow C \quad A}{C}$$
2.
$$\frac{A \wedge B \rightarrow C \quad D \wedge A \quad D \rightarrow B}{C \wedge D}$$
3.
$$\frac{A \vee B \rightarrow B \wedge C \quad A}{B}$$
4.
$$\frac{A \vee B \rightarrow B \wedge C \quad B \vee C \rightarrow D \quad A}{B \wedge D}$$

Problem Set 2

In the following problems, you will need to use the \rightarrow -intro rule.

1.
$$\frac{A \rightarrow B \rightarrow C \quad A \rightarrow B}{A \rightarrow C}$$
2.
$$\frac{A \vee B \rightarrow C \quad B \wedge C \rightarrow D}{A \wedge B \rightarrow D}$$
3.
$$\frac{A \vee B \rightarrow B \quad B \vee C \rightarrow C}{A \rightarrow C \vee D}$$
4.
$$\overline{(C \rightarrow B \rightarrow A) \rightarrow (C \rightarrow B) \rightarrow C \rightarrow A}$$

Problem Set 3

In the following problems, you will need to use the \vee -elim rule, and possibly the \perp -rules.

1.
$$\frac{A \rightarrow B \rightarrow C \quad A \rightarrow B \vee C}{A \rightarrow C}$$
2.
$$\frac{A \rightarrow B \quad A \wedge B \rightarrow C \vee D \quad B \wedge C \rightarrow D}{A \vee (B \wedge C) \rightarrow D}$$
3.
$$\frac{(A \rightarrow B) \rightarrow C \quad D \rightarrow \neg D \vee (A \rightarrow B)}{D \rightarrow C}$$
4.
$$\frac{A \vee B \rightarrow \neg B \quad B \vee C \rightarrow B \vee A}{C \rightarrow A}$$

Problem Set 4

In the following problems, you will need to use the Excluded Middle rule — or give a proof by contradiction.

1.
$$\frac{(\neg A \vee B) \rightarrow A}{A}$$
2.
$$\frac{X \vee Y \rightarrow Z}{\neg Z \rightarrow \neg X \wedge \neg Y}$$