Chapter 6: Duality Theory (replaces 6 pages from Week 6 note outline)

Standard Problem Conversion

Primal Problem	*Dual Problem*
Maximize $Z = \mathbf{cx}$, subject to $\mathbf{Ax} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$.	Minimize $W = \mathbf{yb}$, subject to $\mathbf{yA} \ge \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$.

Example: Barbie and Ken AGAIN

Primal

Maximize $Z = 6b + 6.5\,k$

s.t

$12\,b + 14\,k \le 100000$

$5b \le 30000$

$4b + 4k \le 35000$

$b, k \ge 0$

Dual

Minimize $W = 100000\,p + 30000\,n + 35000\,c$

s.t

$12\,p + 5\,n + 4\,c \ge 6$

$14\,p + 0\,n + 4\,c \ge 6.5$

$p, n, c \ge 0$

Optimal Tableaus

PRIMAL	barbie	ken	plastic s1	nylon s2	cardboard s3	right hand side
Z	0	0	0.4642857	0.0857143	0	49000
ken	0	1	0.0714286	-0.171429	0	2000
barbie	1	0	0	0.2	0	6000
cardboard s3	0	0	-0.285714	-0.114286	1	3000

DUAL	plastic	nylon	cardboard	barbie s1	ken s2	*barbie a1*	*ken a2*	rhs
W = negative Z	0	0	3000	6000	2000	994000	998000	-49000
nylon	0	1	0.11429	-0.20000	0.17143	0.20000	-0.17143	0.08571
plastic	1	0	0.28571	0	-0.07143	0	0.07143	0.46429

Note that cardboard is not basic and so cardboard = 0

TABLE 6.14 Corresponding primal-dual forms

Label	Primal Problem (or Dual Problem)		Dual Problem (or Primal Problem)
	Maximize Z (or W)		Minimize W (or Z)
	Constraint i:		Variable y_i (or x_i):
Sensible	$\leq$ form	$\longleftrightarrow$	$y_i \geq 0$
Odd	$=$ form	$\longleftrightarrow$	Unconstrained
Bizarre	$\geq$ form	$\longleftrightarrow$	$y_i' \leq 0$
	Variable x_j (or y_j):		Constraint j:
Sensible	$x_j \geq 0$	$\longleftrightarrow$	$\geq$ form
Odd	Unconstrained	$\longleftrightarrow$	$=$ form
Bizarre	$x_j' \leq 0$	$\longleftrightarrow$	$\leq$ form

One More Example:

<u>Original Problem</u>

Maximize $Z = 2x_1 + x_2$

s.t.

$$x_1 + x_2 = 2$$
$$2x_1 - x_2 \geq 3$$
$$x_1 - x_2 \leq 1$$
$$x_1 x_2 \geq 0$$

<u>"Standardized" Problem</u>

Maximize $Z = 2x_1 + x_2$

s.t.

$$x_1 + x_2 \leq 2$$
$$-x_1 - x_2 \leq -2$$
$$-2x_1 + x_2 \leq -3$$
$$x_1 - x_2 \leq 1$$
$$x_1 x_2 \geq 0$$

<u>From Table:</u>

Minimize $W = 2\,y_1 + 3\,y_2 + y_3$

s.t

$$y_1 + 2\,y_2 + y_3 \geq 2$$
$$y_1 - y_2 - y_3 \geq 1$$
$$y_1\ URS, y_2 \leq 0, y_3 \geq 0$$

<u>From "standardized" problem:</u>

Minimize $W = 2\,y_1 - 2\,y_1^* - 3\,y_2 + y_3$

s.t

$$y_1 - y_1^* - 2\,y_2 + y_3 \geq 2$$
$$y_1 - y_1^* + y_2 - y_3 \geq 1$$
$$y_1, y_1^*, y_2, y_3 \geq 0$$

WHY DOES THIS WORK???

Weak duality property: If **x** is a feasible solution for the primal problem and **y** is a feasible solution for the dual problem, then

$$\mathbf{cx} \leq \mathbf{yb}.$$

Strong duality property: If **x*** is an optimal solution for the primal problem and **y*** is an optimal solution for the dual problem, then

$$\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}.$$

Complementary solutions property: At each iteration, the simplex method simultaneously identifies a CPF solution **x** for the primal problem and a **complementary solution y** for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{cx} = \mathbf{yb}.$$

If **x** is *not optimal* for the primal problem, then **y** is *not feasible* for the dual problem.

Duality theorem: The following are the only possible relationships between the primal and dual problems.

1. If one problem has *feasible solutions* and a *bounded* objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
2. If one problem has *feasible solutions* and an *unbounded* objective function (and so *no optimal solution*), then the other problem has *no feasible solutions.*
3. If one problem has *no feasible solutions,* then the other problem has either *no feasible solutions* or an *unbounded* objective function.

Several reasons to care:

1) Fewer constraints mean fewer row operations, so sometimes solving the dual is faster by hand
2) ≤ constraints mean less artificial variables, so more ≥ constraints may mean the dual is faster by hand.
3) Changing the constraint coefficient(s) of a variable changes ONE constraint in the dual.
4) Adding a variable in the primal is adding a constraint in the dual.

Example for 4) ADD skipper to Barbie and Ken: $6.25 profit, 9 oz plastic, 4 oz nylon and 4 oz cardboard.

In the Primal – new variable:

Maximize $Z = 6b + 6.5\,k$

s.t

$$
\begin{aligned}
12\,b + 14\,k &\leq 100000 \\
5b &\leq 30000 \\
4b + 4k &\leq 35000 \\
b, k &\geq 0
\end{aligned}
$$

In the Dual – new constraint:

Minimize $W = 100000\,p + 30000\,n + 35000\,c$

s.t

$$
\begin{aligned}
12\,p + 5\,n + 4\,c &\geq 6 \\
14\,p + 0\,n + 4\,c &\geq 6.5
\end{aligned}
$$

$p, n, c \geq 0$

Changes to the A and c coefficients for non-basic and basic variables…

Homework Problem 5 parts b and c:

D,I **7.2-3.** Consider the following problem.

Maximize $Z = 2x_1 + 7x_2 - 3x_3$,

subject to

$$x_1 + 3x_2 + 4x_3 \leq 30$$
$$x_1 + 4x_2 - x_3 \leq 10$$

and

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$

By letting x_4 and x_5 be the slack variables for the respective constraints, the simplex method yields the following final set of equations:

$$\begin{array}{ll} (0) & Z + x_2 + x_3 + 2x_5 = 20 \\ (1) & -x_2 + 5x_3 + x_4 - x_5 = 20 \\ (2) & x_1 + 4x_2 - x_3 + x_5 = 10. \end{array}$$

Final tableau

	x1	x2	x3	x4	x5	rhs
z	0	1	1	0	2	20
x4	0	-1	5	1	-1	20
x1	1	4	-1	0	1	10

(b) Change the coefficients of x_3 to

$$\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}.$$

$$c_b^\top B^{-1}A - c^\top = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 5 \\ 1 & 4 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 7 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$$

$$B^{-1}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 4 & -2 \end{bmatrix}$$

(c) Change the coefficients of x_1 to

$$\begin{bmatrix} c_1 \\ a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

$$c_b^\top B^{-1}A - c^\top = \begin{bmatrix} 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 5 \\ 2 & 4 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 7 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 9 & -1 \end{bmatrix}$$

$$B^{-1}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 4 & -1 \end{bmatrix}$$