Induction exercises

If $c_1, c_2, c_3, \ldots$ is a sequence of numbers, then

$$\sum_{k=1}^{n} c_k = c_1 + \cdots + c_n$$

In particular,

$$\sum_{k=1}^{n+1} c_k = \left(\sum_{k=1}^{n} c_k\right) + c_{n+1}$$

The *factorial function* is defined by

$$n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n$$

More explicitly, this function can be defined by recursion as follows:

$$\begin{aligned} 0! &= 1 \\ (n+1)! &= n! \cdot (n+1) \end{aligned}$$

Using the above equations, it is possible to compute $n!$ for any number n.

Prove by induction the following statements. (You may wish to first "test" the formula for the first few values of n.)

$$\forall n \geq 0. \quad \sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{1}$$

$$\forall n \geq 0. \quad \sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2} \tag{2}$$

$$\forall n \geq 0. \quad \sum_{k=0}^{n} (2k+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3} \tag{3}$$

$$\forall n \geq 1. \quad \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 \tag{4}$$

$$\forall n \geq 1. \quad \sum_{k=1}^{n} k \cdot (k+1) = \frac{n(n+1)(n+2)}{3} \tag{5}$$

$$\forall n \geq 4. \quad n^2 \leq n! \tag{6}$$

$$\forall n \geq 1. \quad n! \leq n^n \tag{7}$$

$$\forall n \geq 4. \quad n^2 - 7n + 12 \geq 0 \tag{8}$$

$$\forall n \geq 1. \quad n^2 + n \text{ is even} \tag{9}$$

$$\forall n \geq 1. \quad n^3 + 2n \text{ is a multiple of } 3 \tag{10}$$