CS1100: Final Exam Review

1 Proof in Propositional Logic (20 pts)

The exam will have a question asking you to prove the soundness of an inference rule that contains propositional formulas.

There are several types of rules that are involved in this type of problem, and you should make sure you are comfortable with all of them.

Basic Proofs: Modus Ponens (AKA $\rightarrow$-elimination), $\vee$-intro, $\wedge$-intro and $\wedge$-elim.

These are the easiest types of steps — you only need to name the rule correctly, and point to the facts where the premises of the rules are established.

Hypothetical Proofs: Proofs involving $\rightarrow$-introduction.

These are used when you have to prove a goal of the form $X \rightarrow Y$.

Such proofs will typically look like:

⌈1. Assume X. New goal: Y.
2.
⋮
n. By ⋯ , get Y. ⌋
n+1. By $\rightarrow$-introduction, steps 1–n, get $X \rightarrow Y$.

Proofs by Cases: Proofs involving $\vee$-elimination.

These are used when you have to use an assumption or a previously derived fact of the form $X \vee Y$.

In order to use this fact in order to prove a goal, Z, you would use the pattern

- From $X \vee Y$, we get two cases:

  Case 1. Assume X. (Goal remains Z.)
  ⋮
  By ⋯ , get Z.

  Case 2. Assume Y. (Goal remains Z.)
  ⋮
  By ⋯ , get Z.

- By $\vee$-elimination, in either case we obtain Z.

Proofs involving negation and absurdity: Whenever two mutually contradictory facts X and $\neg X$ are derived, you can generate the contradictory proposition $\bot$ using $\bot$-introduction.

If you are ever able to derive $\bot$, you can get any goal using the *ex falso* rule, also known as $\bot$-elimination.

Alternatively, these steps can be chained together, allowing to obtain any goal whenever both X and $\neg X$ have been derived (or available among the assumptions). In this case, you can end the whole (sub)proof by saying: "By the facts derived in steps n and m, we get a contradiction."

Practice problems

Prove soundness of the following rules:

1. $\dfrac{A \vee B \to C \wedge D \qquad B \vee C \to A \wedge B}{A \to B}$
2. $\dfrac{A \vee B \to B \wedge C \qquad A \wedge B \to C \wedge D}{A \to D}$
3. $\dfrac{A \vee B \to C \wedge D \qquad B \wedge C \to A \vee \neg D}{B \to A}$
4. $\dfrac{A \to B \to C \qquad A \to B \vee C}{A \to C}$
5. $\dfrac{A \to B \to C \qquad B \to C \to D \qquad B \vee D}{A \vee C \to D}$

2 Proof in first-order logic (20 pts)

3 Proof in first-order logic (20 pts)

The exam will have *two* proofs in first-order logic.

In addition to all of the rules above, these add two new levels of complexity.

First of all, since the formulas are now made of predicates rather than mere propositions, one can substitute constants or other terms in place of variables.

Secondly, there are two new formula formers: the universal and the existential quantifiers. The "for all" $\forall$ and the "exists" $\exists$. Each of these has a corresponding introduction rule and elimination rule.

The **first** question will involve the rules $\forall$-elim and $\exists$-intro. These are the rules where YOU get to choose what to substitute for the quantified variable to make the formula true.

The **second** question will involve the rules $\forall$-intro and $\exists$-elim. These are the rules where you do NOT get to choose what substitution makes the formula true. However, you can choose how to *name* the object substituted for the variable that makes it true.

The following example from homework 3 illustrates all four rules in action. Notice how each quantifier rule falls into one of the two types above.

Example.

Prove the soundness of the following rule:

$$\frac{\forall x.\forall y.P(x,y) \to Q(a,y) \qquad \forall x.\exists y.R(x,a) \to P(y,x)}{\forall z.R(z,a) \to \exists y.Q(y,z)}$$

Assumption a1: $\forall x.\forall y.P(x,y) \to Q(a,y)$

Assumption a2: $\forall x.\exists y.R(x,a) \to P(y,x)$

Goal: $\forall z.R(z,a) \to \exists y.Q(y,z)$

PROOF.

1. Let z be given. We will show $\boxed{R(z,a) \to \exists y.Q(y,z)}$.
2. Assume $R(z,a)$. We will show $\boxed{\exists y.Q(y,z)}$.
3. By $\forall$-elim on a2, with $x = z$, we get $\exists y.R(z,a) \to P(y,z)$.
4. By $\exists$-elim on step 3, let b be such that $R(z,a) \to P(b,z)$.
5. By $\forall$-elim on a1, with $x = b$ and $y = z$, we get $P(b,z) \to Q(a,z)$.
6. By MP with steps 4 and 2, we get $P(b,z)$.
7. By MP with steps 5 and 6, we get $Q(a,z)$.
8. By $\exists$-intro with step 7, with $y = a$, we get $\exists y.Q(y,z)$.
9. By $\to$-intro, steps 2–8, we get $R(z,a) \to \exists y.Q(y,z)$.
10. By $\forall$-intro, steps 1–9, we get $\forall z.R(z,a) \to \exists y.Q(y,z)$.

This completes the proof.

Practice problems: type 1

1. $\dfrac{\forall x.P(a,x) \to Q(x,b) \qquad \forall y.Q(a,y) \to P(a,y)}{P(a,a) \to Q(b,b)}$

2. $\dfrac{\forall x.P(x,x) \lor Q(x,x) \qquad \forall y.P(a,y) \to R(y) \qquad \forall z.Q(z,a) \to R(z)}{\exists v.R(v)}$

3. $\dfrac{\forall x.P(x) \to Q(a,x) \lor Q(b,x) \qquad \forall y.Q(y,a) \to P(y) \lor C \qquad \forall x \forall y.Q(x,y) \to Q(y,x) \to \bot}{P(a) \to C}$

4. $\dfrac{\forall x.P(x) \to Q(a,x) \lor Q(b,x) \qquad \forall y.Q(y,a) \to P(y) \qquad \forall x \forall y.Q(x,y) \land Q(y,x) \to R(x)}{P(a) \to \exists z.Q(z,z) \lor R(z)}$

Practice problems: type 2

1. $\dfrac{\exists x.Z \to P(x)}{Z \to \exists y.P(y)}$

2. $\dfrac{\exists x.\forall y.Q(x,y)}{\forall y.\exists x.Q(x,y)}$

3. $\dfrac{\forall x.P(x) \lor Q(a,x) \to R(x) \qquad \forall y.R(y) \to Q(a,b) \lor \neg P(y)}{\forall z.P(z) \to R(b)}$

4. $\dfrac{\exists x.P(a,x) \land Q(x) \qquad \forall x.Q(x) \to P(x,a) \qquad \forall x.\forall y.P(x,y) \land P(y,x) \to R(x)}{R(a)}$

5. $\dfrac{\forall x.\forall y.P(x,y) \land P(y,x) \to Q(x,y) \qquad \exists v.\forall w.Q(v,w) \to P(w,w)}{(\forall x.Q(x,a)) \to Q(a,a)}$

4 Reading first-order formulas (20 pts)

The final exam will also test your ability to understand what first-order formulas actually *mean*. Emphasis will be placed on interpreting the same formulas in different contexts. (That is, when the domain of interpretation is different.)

You should be familiar with the following standard sets of numbers

$$
\begin{array}{rccl}
\textbf{Natural numbers:} & \mathbb{N} & = & \{0,1,2,3,\dots\} \\
\textbf{Integers:} & \mathbb{Z} & = & \{0,1,-1,2,-2,\dots\} \\
\textbf{Rationals:} & \mathbb{Q} & = & \left\{0,1,\frac{1}{2},\frac{1}{3},-\frac{4}{3},10,-0.7,\frac{19}{199},\dots\right\} = \left\{\frac{m}{n} \;\middle|\; m,n \in \mathbb{Z}, n \neq 0\right\} \\
\textbf{Real numbers:} & \mathbb{R} & = & \left\{0,1,\frac{2}{3},\sqrt{2},\pi,e,\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2},\dots\right\} = (-\infty,\infty)
\end{array}
$$

Furthermore, if $\square \in \{\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{R}\}$, then $\square^+ = \square \cap (0,\infty)$. This introduces the notation for

$$
\begin{array}{rccccl}
\textbf{Positive Integers:} & \mathbb{N}^+ & = & \mathbb{Z}^+ & = & \{1,2,3,\dots\} \\
\textbf{Positive Rationals:} & \mathbb{Q}^+ & = & \mathbb{Q} \cap (0,\infty) & = & \left\{1,\frac{1}{2},\frac{1}{3},10,\frac{19}{199},\dots\right\} \\
\textbf{Positive Reals:} & \mathbb{R}^+ & = & (0,\infty) & = & \left\{1,\frac{2}{3},\sqrt{2},\pi,e,\frac{1+\sqrt{5}}{2},\dots\right\}
\end{array}
$$

Practice problems

For each formula F below, state whether it is true or false in the given domain.

Give your answers in the following table, filling out one whole row at a time.

F	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Z}^+$	$\mathbb{Q}$	$\mathbb{Q}^+$	$\mathbb{R}$
$\forall x. x \geq 0$						
$\forall x. x + x > 0$						
$\forall x. x \times x > 0$						
$\forall x. x \times x \geq 0$						
$\forall x \exists y. x < y$						
$\forall x \exists y. x > y$						
$\forall x \exists y. x \times y = x$						
$\forall x \exists y. x \times y = y$						
$\forall x \exists y. x + y = 0$						
$\forall x \exists y. x \times y = 1$						
$\exists x \forall y. x \leq y$						
$\exists x \forall y. x + y = x$						
$\exists x \forall y. x + y = y$						
$\exists x \forall y. x \times y = x$						
$\exists x \forall y. x \times y = y$						
$\exists x \forall y. y + (x \times y) = y$						
$\forall x \forall y. x < y \to (\exists z. x < z \land z < y)$						
$\forall x \forall y. (x \times x = y \times y) \to x = y$						
$\forall x \forall y \forall z. x < y \to (x + z \leq y + z)$						
$\forall x \forall y \forall z. x < y \to (x \times z \leq y \times z)$						

5 Identifying properties of relations (20 pts)

This problem will have the same flavor as the previous one, but will additionally test your knowledge of the terminology about binary relations on a set.

Let A be a set, and $R \subseteq A \times A$.

R has property on the left if the corresponding formula on the right is true.

Reflexive:	$\forall x. R(x, x)$
Symmetric:	$\forall x \forall y. R(x, y) \to R(y, x)$
Transitive:	$\forall x \forall y \forall z. R(x, y) \land R(y, z) \to R(x, z)$
Antisymmetric:	$\forall x \forall y. R(x, y) \land R(y, x) \to x = y$
Equivalence:	Reflexive, Symmetric, and Transitive
Partial order:	Reflexive, Antisymmetric, and Transitive

Practice problems

For each relation below, determine whether the relation is reflexive, symmetric, antisymmetric, transitive, dense, a partial order, or an equivalence relation.

1. Let D be the set of words in the English language. Say that words v and w are related if w has at least as many letters as v (repetitions included). That is, $R(v, w)$ is true if w is at least as long as v.

2. Let D be the set of all triangles in the Cartesian plane. Say that two triangles are related if they are congruent. Equivalently, $R(\Delta_1, \Delta_2)$ is true if the lengths of sides of triangle Δ_1, when listed in increasing length, are the same as those of triangle Δ_2.

3. Let D be the set of natural numbers. Say that two numbers are related if they share a prime factor. For example, $R(2, 4)$ is true, but $R(10, 21)$ is false.

4. Let D be the set of students at Appstate. $R(x, y)$ is true if x and y are connected on Facebook.

5. Let D be the set of all monotonically increasing functions $f : \mathbb{R}^+ \to \mathbb{R}^+$. Say that $R(f, g)$ is true if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = +\infty$$

6. Let $D = \mathbb{N}$, the non-negative integers. $R(x, y)$ is true if x divides y. More precisely, for $m, n \in D$,

$$R(m, n) \iff \exists k \in D.\ mk = n$$

Part II

6 Truth tables (20 pts)

The exam will test your knowledge of the truth table method.

You should understand the meaning of and be comfortable with the following concepts:

- A propositional formula is *satisfiable* if there is a truth assignment that makes the formula true.

  For example, $X \vee Y$ is satisfiable, because the truth assignment where $X=\top$ and $Y=\bot$ makes it true. (There may be other assignments that make it true, but the requirement is that there be *at least one*.)

- A propositional formula is a *contradiction* if it is not satisfiable. In other words, every truth assignment makes the formula false.

  For example, $X \wedge \neg X$ is unsatisfiable, because no truth assignment can make it true. Thus, $X \wedge \neg X$ is a contradiction.

- A propositional formula is a *tautology* if every truth assignment makes the formula true.

  For example, $X \vee \neg X$ is a tautology, because, no matter whether X is $\top$ or $\bot$, the result is always true.

  An important observation: A formula is a tautology if and only if its negation is a contradiction — and vice versa. (Check this!)

- Two propositional formulas are *logically equivalent*, or just "equal", if one is true if and only if the other one is true.

  In practice, this means that two formulas are equal if the truth assignments that make them true are the same for both formulas.

  Another way to say this is that formulas A and B are logically equivalent if and only if formula $A \leftrightarrow B$ is a tautology.

- An inference rule is *sound* if, whenever all the assumptions are true, then also the conclusion is true.

  That is, the rule is sound if every truth assignment that makes all the assumptions true, also makes the conclusion true.

  Another way to say this is that the rule

  $$\frac{X_1 \quad X_2 \quad \cdots \quad X_n}{Z}$$

  is sound if and only if the formula

  $$X_1 \to X_2 \to \cdots \to X_n \to Z$$

  is a tautology.

Practice problems

1. Determine whether each of the following formulas is satisfiable, a tautology, or a contradiction.
   - $(Y \vee \neg X) \wedge X$
   - $X \vee (X \to Y)$
   - $X \vee Y \iff X$
   - $X \vee Y \to Y$
   - $(X \wedge Y) \vee (\neg X \wedge Z) \vee (\neg Y \wedge \neg Z)$
2. Determine whether the following equations are valid:
   - $X \wedge (Y \to Z) = (X \wedge \neg Y) \vee (X \wedge Z)$
   - $X \wedge Y \to Z = X \to Y \to Z$
   - $X \vee Y \to Z = (X \to Z) \wedge (Y \to Z)$
   - $X \to Y \to X = \top$
   - $(X \to Y) \wedge (Y \to Z) = (X \wedge Y) \vee (\neg X \wedge \neg Y)$

7 Sets and Functions (20 pts)

You will have a problem that will ask you to perform various operations on sets and list the elements of the resulting sets.

You should also know the concepts of domain, codomain, and range of a function, and be able to determine whether the given function is one-to-one, onto, or a bijection.

Practice problems on sets

Let A, B, and C denote the following sets:

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$B = \{n^2 \mid n \in \mathbb{N}\}$$
$$C = \{n \mid n \text{ is prime}\}$$

(Notice that $B = \{0, 1, 4, 9, \dots\}$, and $C = \{2, 3, 5, 7, 11, \dots\}$.)

Compute the following sets by listing all of their elements explicitly:

1. $A \cap B$
2. $A \cap C$
3. $(A \cap B) \cup (A \cap C)$
4. $(A \cap B) \times (A \cap C)$
5. $(A \cap B) \sqcup (A \cap C)$
6. $\mathcal{P}(A \cap B)$

Practice problems on functions

For each of the following functions, give the domain, codomain, and range, and determine whether the function is one-to-one, onto, or bijection.

If you are not sure about the domain, provide the most general one you can think of.

1. $f(x) = \sqrt{x}$
2. $f(x) = 10x$
3. $f(x) = x^2 - x$
4. $f(x) = |x|$, the absolute value of x (modulus)
5. $f(x) = \sqrt{|x|}$
6. $f(x) = x^x$
7. $f(x) = x + \frac{1}{x}$
8. $f(x) =$ number of prime factors of x
9. $f(x) = \begin{cases} 0 & x \text{ is even} \\ 1 & x \text{ is odd} \end{cases}$
10. $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

8 Induction, Recursion, and Summations (30 pts)

The final exam will include *two* questions about proofs by induction.

The first question will be about proving an explicit formula for a recursive function.

The second question will be about proving a summation formula by induction.

Practice problems on recursive functions

1. Let $f(0) = 3$, $f(n+1) = 2f(n)$. Prove that

$$\forall n \geq 0. f(n) = 3 \cdot 2^n$$

2. Let $f(0) = -3$, $f(n+1) = f(n) + 5$. Prove that

$$\forall n \geq 0. f(n) = 5n - 3$$

3. Let $f(0) = 1$, $f(n+1) = 2f(n) - n$. Prove that

$$\forall n \geq 0. f(n) = n + 1$$

4. Let $f(0) = 1$, $f(n+1) = f(n) + 2n$. Prove that
$$\forall n \geq 0. f(n) = n^2 - n + 1$$

5. Let $f(0) = 1$, and $f(n+1) = -f(n)$. Prove that
$$\forall n \geq 0. f(n) = (-1)^n$$

6. Let $f(0) = 0$, and $f(n+1) = f(n) + (-1)^n(2n+1)$. Prove that
$$\forall n \geq 0. f(n) = (-1)^{n+1} \cdot n$$

Practice problems on summations

$$\forall n \geq 1. \qquad \sum_{k=1}^{n} (2n-1) = n^2 \tag{1}$$

$$\forall n \geq 0. \qquad \sum_{k=0}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{2}$$

$$\forall n \geq 0. \qquad \sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2} \tag{3}$$

$$\forall n \geq 1. \qquad \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 \tag{4}$$

$$\forall n \geq 1. \qquad \sum_{k=1}^{n} k \cdot (k+1) = \frac{n(n+1)(n+2)}{3} \tag{5}$$

9 Counting and finite combinatorics (20 pts)

The exam will include questions about:

- Cardinality of a finite set.
- The sum rule and the product rule of counting.
- Combinations, Permutations — when to use each.

Practice problems

1. Let A be a set with cardinality 3. Let B be a set with $|B| = 4$. Suppose that there are two elements which are both in A and in B. What's $|A \cup B|$?

2. Every customer in the Lost Province restaurant and bar is either eating, having a drink, or both.

   Right now, in Lost Province there are 16 customers that are eating. 20 are having a drink. 12 are doing both. How many customers are there in total?

3. 5 people arrive to the El Tacorriendo lunch truck at the exact same time. How many ways are there for them to queue up in sequence to order tacos?

4. El Tacorriendo has 5 types of tacos, and 7 types of burritos on the food menu.

   a) How many ways are there to order a single item from the menu?

   b) How many ways are there to order one taco and one burrito?

   c) How many ways are there to order 2 different types of tacos?

   d) How many ways are there to order 2 different tacos and 3 different burritos?

5. In Norway, Social Security Numbers are made of 11 digits, where the first six digits are the person's birthdate, and the last 5 digits are assigned by availability. What is the maximum number of people this system can sustain?

   To keep computation simple, you may assume that every month has exactly 30 days, and that there are no leap years.

6. In the original TCP/IP internet protocol, every user of the internet was assigned a unique "IP" address, which was a sequence of 4 8-bit numbers.

   a) How many 8-bit numbers are there? (That is, how many numbers are there that can be made of eight binary digits?)

   b) How many possible addresses were there in the original protocol?

   c) Would the protocol be enough to give every person on the planet their own internet connection?

7. One round of Powerball consists of drawing in sequence 5 numbers from the set $\{1, 2, \ldots, 69\}$. How many total drawings are there?

10 Base conversions (10 pts)

The exam will include a problem asking you to convert numbers between decimal, binary, and hexadecimal representations.

Practice problems

1. Convert the following numbers from decimal to binary and hexadecimal:

$$10, 20, 30, 100, 200, 1000, 1025$$

2. Convert the following numbers from binary to hexadecimal and to decimal:

$$1101, 11, 111101, 1010101, 10000000, 10000100001$$

3. Convert the following numbers from hexadecimal to binary and to decimal:

$$9, B, 9B, B9, FF, 10000, ABC$$