Chapter 5: The Matrix Form

The Standard LP Problem in Matrix Form:

Maximize $c_1x_1 + c_2x_2 + ...c_nx_n$
s.t.
$a_{1,1}x_1 + a_{1,2}x_2 + ...a_{1,n}x_n \leq b_1$
$a_{2,1}x_1 + a_{2,2}x_2 + ...a_{2,n}x_n \leq b_2$
...
$a_{m,1}x_1 + a_{m,2}x_2 + ...a_{m,n}x_n \leq b_m$
$x_i \geq 0$

Maximize $\vec{c}^\top \vec{x}$
s.t.
$A\vec{x} \leq \vec{b}$
$\vec{x} \geq \vec{0}$
$A \in \mathbb{R}^{m \times n}, \vec{x} \in \mathbb{R}^n, \vec{b} \in \mathbb{R}^m$

Adding slack variables gives the format:

Maximize $c_1x_1 + c_2x_2 + ...c_nx_n$
s.t.
$a_{1,1}x_1 + a_{1,2}x_2 + ...a_{1,n}x_n + s_1 = b_1$
$a_{2,1}x_1 + a_{2,2}x_2 + ...a_{2,n}x_n + s_2 = b_2$
...
$a_{m,1}x_1 + a_{m,2}x_2 + ...a_{m,n}x_n + s_m = b_m$
$x_i \geq 0$

Maximize $\vec{c}^\top \vec{x}$
s.t.
$[A \quad I] \begin{bmatrix} \vec{x} \\ \vec{x}_s \end{bmatrix} = \vec{b}$
$\begin{bmatrix} \vec{x} \\ \vec{x}_s \end{bmatrix} \geq \vec{0}$
$A \in \mathbb{R}^{m \times n}; \vec{x} \in \mathbb{R}^n; \vec{x}_s, \vec{b} \in \mathbb{R}^m$, and I is the identity matrix of size m.

This means the constraints are a system of m equations in $m + n$ unknowns, so the simplex method essentially does the following:

1. Choose m variables in the vector $\begin{bmatrix} \vec{x} \\ \vec{x}_s \end{bmatrix}$ to be non-zero – the basic variables, and form the vector $\vec{x}_B$ of the basic variables. We start by selecting the slack variables for the first set of basic variables.
2. Pull the coefficients of the basic variables from the objective coefficient vector, to form the vector $\vec{c}_B$.
3. Pull the columns from $[A \quad I]$ that correspond to those variables and form a matrix B – the invertible "basis" matrix.
4. Solve the square invertible system $B\vec{x}_B = \vec{b}$, i.e., $\vec{x}_B = B^{-1}\vec{b}$. This gives values for the basic variables.
5. Calculate the objective value using only the basic variables (the other variables are 0!):
$$Z = \vec{c}_B^\top \vec{x}_B = \vec{c}_B B^{-1}\vec{b}$$
6. Check for optimality. If not optimal move to an adjacent corner by swapping out one basic variable for a non-basic variable, "moving" to an adjacent corner point through a change of basis.

The full tableau process in matrix form:

Initial

$$\boxed{\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}.}$$

What we know about the value of the right hand side at any intermediate tableau:

$$\begin{bmatrix} Z \\ \mathbf{x}_B \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

Had to happen to both sides of the initial tableau

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix},$$

Therefore, any subsequent tableau, including the final tableau:

$$\boxed{\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.}$$

Example: Ken and Barbie

Maximize $Z = 6b + 6.5k$

s.t.

$$12b + 14k \leq 100000$$
$$5b \leq 30000$$
$$4b + 4k \leq 35000$$
$$b, k, \geq 0$$

Ken and Barbie Tableaus:

	B	K	S1	S2	S3	RHS	
Z	-6	-6.5	0	0	0	0	RATIO
S1	12	14	1	0	0	100000	7142.85714
S2	5	0	0	1	0	30000	#DIV/0!
S3	4	4	0	0	1	35000	8750

	B	K	S1	S2	S3	RHS	
Z	-0.42857143	0	0.46428571	0	0	46428.5714	RATIO
K	0.857142857	1	0.07142857	0	0	7142.85714	8333.33333
S2	5	0	0	1	0	30000	6000
S3	0.571428571	0	-0.2857143	0	1	6428.57143	11250

	B	K	S1	S2	S3	RHS
Z	0	0	0.46428571	0.08571429	0	49000
K	0	1	0.07142857	-0.1714286	0	2000
B	1	0	0	0.2	0	6000
S3	0	0	-0.2857143	-0.1142857	1	3000

Knowing the basic set for the first new tableau, find the tableau:

x_B=

B = B^{-1} =

$c_B^T B^{-1}$ =

$c_B^T B^{-1} b$ =

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x_B=

B = B^{-1} =

$c_B^T B^{-1}$ =

$c_B^T B^{-1} b$ =

TABLE 5.8 Initial and later simplex tableaux in matrix form

Iteration	Basic Variable	Eq.	Coefficient of: Z	Original Variables	Slack Variables	Right Side
0	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 $\mathbf{0}$	$-\mathbf{c}$ $\mathbf{A}$	$\mathbf{0}$ $\mathbf{I}$	0 $\mathbf{b}$
Any	Z $\mathbf{x}_B$	(0) (1, 2, . . . , m)	1 $\mathbf{0}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$ $\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{c}_B\mathbf{B}^{-1}$ $\mathbf{B}^{-1}$	$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ $\mathbf{B}^{-1}\mathbf{b}$

Two important takeaways:

1) Fundamental Insight: Given the original problem (i.e., **A**, **b, c**), if we know the following two things at <u>optimum</u>, we can create the entire final tableau

$$\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1} = \text{coefficients of the } \textit{slack} \text{ variables in row 0}$$

$$\mathbf{S}^* = \mathbf{B}^{-1} = \text{coefficients of the } \textit{slack} \text{ variables in rows 1 to } m$$

For non-standard problems (BigM): The artificial variables are starting basic variables, so they will be included in the list of columns holding the inverse of **B** in the tableau.

2) Shadow prices and changes to **b:**

$$\mathbf{x}_B = \mathbf{S}^*\mathbf{b}$$
$$\mathbf{Z}^* = \mathbf{y}^*\mathbf{b},$$

Shadow prices for the non-standard problems:
≥: negative of surplus variable coefficient
=: artificial variable coefficient

Shadow prices examples:

$\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$ = coefficients of the *slack* variables in row 0

$\mathbf{S}^* = \mathbf{B}^{-1}$ = coefficients of the *slack* variables in rows 1 to m

$$\mathbf{x}_B = \mathbf{S}^*\mathbf{b}$$
$$Z^* = \mathbf{y}^*\mathbf{b},$$

Chapter 6: Duality Theory
Yet Another Way to Find a Solution: the "DUAL" problem…
Example: Barbie and Ken AGAIN

Maximize $Z = 6b + 6.5\,k$
s.t.

$$12\,b + 14\,k \leq 100000$$
$$5b \leq 30000$$
$$4b + 4k \leq 35000$$
$$b, k, \geq 0$$

PRIMAL	barbie	ken	plastic s1	nylon s2	cardboard s3	right hand side
Z	0	0	0.4642857	0.0857143	0	49000
ken	0	1	0.0714286	-0.171429	0	2000
barbie	1	0	0	0.2	0	6000
C s3	0	0	-0.285714	-0.114286	1	3000

DUAL	plastic	nylon	cardboard	barbie s1	ken s2	barbie a1	ken a2	rhs
negative Z	0	0	3000	6000	2000	994000	998000	-49000
nylon	0	1	0.11429	-0.20000	0.17143	0.20000	-0.17143	0.08571
plastic	1	0	0.28571	0	-0.07143	0	0.07143	0.46429

Standard problem in general:

Primal Problem	*Dual Problem*
Maximize $Z = \mathbf{cx}$, subject to $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.	Minimize $W = \mathbf{yb}$, subject to $\mathbf{yA} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

What about a non-standard problem???

Maximize $Z = 2x_1 + x_2$

s.t.

$$x_1 + x_2 = 2$$
$$2x_1 - x_2 \geq 3$$
$$x_1 - x_2 \leq 1$$
$$x_1 \geq 0, x_2 \geq 0$$

Make it standard first!!!

TABLE 6.14 Corresponding primal-dual forms

Label	Primal Problem (or Dual Problem)		Dual Problem (or Primal Problem)
	Maximize Z (or W)		Minimize W (or Z)
	Constraint i:		Variable y_i (or x_i):
Sensible	$\leq$ form	$\longleftrightarrow$	$y_i \geq 0$
Odd	$=$ form	$\longleftrightarrow$	Unconstrained
Bizarre	$\geq$ form	$\longleftrightarrow$	$y_i' \leq 0$
	Variable x_j (or y_j):		Constraint j:
Sensible	$x_j \geq 0$	$\longleftrightarrow$	$\geq$ form
Odd	Unconstrained	$\longleftrightarrow$	$=$ form
Bizarre	$x_j' \leq 0$	$\longleftrightarrow$	$\leq$ form

Another Example:

WHY DOES THIS WORK???

Weak duality property: If $\mathbf{x}$ is a feasible solution for the primal problem and $\mathbf{y}$ is a feasible solution for the dual problem, then

$$\mathbf{cx} \leq \mathbf{yb}.$$

Strong duality property: If $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{y}^*$ is an optimal solution for the dual problem, then

$$\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}.$$

Duality theorem: The following are the only possible relationships between the primal and dual problems.

1. If one problem has *feasible solutions* and a *bounded* objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
2. If one problem has *feasible solutions* and an *unbounded* objective function (and so *no optimal solution*), then the other problem has *no feasible solutions.*
3. If one problem has *no feasible solutions,* then the other problem has either *no feasible solutions* or an *unbounded* objective function.

Several reasons to care:

1) **Fewer constraints mean fewer row operations.**
2) **<= constraints mean less artificial variables.**
3) **Adding a variable in the primal is adding a constraint in the dual.**
4) **Changing the coefficient of a variable that is =0 at optimum.**

3) ADD skipper to Barbie and Ken: $6.25 profit, 9 oz plastic, 4 oz nylon and 4 oz cardboard.

New constraint:

4) change the coefficient of a variable in the constraints for 4.6-3:

Minimize $2x_1 + 3x_2 + x_3$
s.t

$$x_1 + 4x_2 + 2x_3 \geq 8$$
$$3x_1 + 2x_2 >= 6$$

Solution: x1 = 0.8, x2 = 1.8, x3 = 0, Z = 7

(1) Change the x3 constraint coefficients to: 2, -1 (from 2,0)
(2) Change the x1 constraint coefficients to: 2,5 (from 1,3)

Optimal tableau

	x1	x2	x3	s1	s2	a1	a2	rhs
z	0	0	0	0.5	0.5	99.5	99.5	-7
x2	0	1	0.6	-0.3	0.1	0.3	-0.1	1.8
x1	1	0	-0.4	0.2	-0.4	-0.2	0.4	0.8