Week 9: Transportation and Minimum Cost Flow Problems – LP with special structure.

Transportation – Example: #7 from HW 2

#7) 3.4-11. The Medequip Company produces precision medical diagnostic equipment at two factories. Three medical centers have placed orders for this month's production output. The table below shows what the cost would be for shipping each unit from each factory to each of these customers. Also shown are the number of units that will be produced at each factory and the number of units ordered by each customer. A decision now needs to be made about the shipping plan for how many units to ship from each factory to each customer.

From \ To	Unit Shipping Cost Customer 1	Customer 2	Customer 3	Output
Factory 1	\$600	\$800	\$700	400 units
Factory 2	\$400	\$900	\$600	500 units
Order size	300 units	200 units	400 units	

As Linear Programming:

#7 (3.4-9) x_{ij} = number shipped from factory i to customer j
$i=1,2$ $j=1,2,3$ all ≥ 0

minimize cost (\$) $Z = 600x_{11} + 800x_{12} + 700x_{13} + 400x_{21} + 900x_{22} + 600x_{23}$

s.t.

enough sent to customers:

1: $x_{11} + x_{21} \geq 300$

2: $x_{12} + x_{22} \geq 200$

3: $x_{13} + x_{23} \geq 400$

limits for number that could be sent from factories:

1. $x_{11} + x_{12} + x_{13} \leq 400$

2. $x_{21} + x_{22} + x_{23} \leq 500$

Terminology

- Source ($i = 1..m$); Supply (s_i)
- Destination ($j = 1..n$); Demand (d_j)
- Transport cost (c_{ij})

Assumptions:

- Each source has a fixed supply available and each destination has a fixed demand required.
- The cost of transport is proportional to the amount transported (no volume discounts).

Consequences:

- Equality Constraints will be used, and there is only a feasible solution if the total supply equals the total demand. In cases where this is not the case we will add "dummy" sources or destinations.
- All solutions will be integer if supply and demand are integers.

Transportation Simplex Method

Traditional Simplex – BigM with (m+n) Artificials

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	a_1	a_2	a_3	a_4	a_5	rhs
$-Z$	←		c_{ij}			→	M	M	M	M	M	0
a_1	1	1	1				1					400
a_2				1	1	1		1				500
a_3	1			1					1			300
a_4		1			1					1		200
a_5			1			1					1	400

New variables:

u_i = multiple of source i constraint subtracted from row 0
v_j = multiple of destination j constraint subtracted from row 0

Fix the Z-row assuming M = 1:

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	a_1	a_2	a_3	a_4	a_5	rhs
$-Z$	←	$c_{ij} - u_i - v_j$				→	0	0	0	0	0	$-\sum s_i u_i - \sum d_j v_j$
a_1	1	1	1				1					400
a_2				1	1	1		1				500
a_3	1			1					1			300
a_4		1			1					1		200
a_5			1			1					1	400

Observations:

- All row operations will occur in the same way, depending on the entering and leaving basic variable.
- The numbers line up in such a way that the coefficients in the constraints will always end up as 1 or 0.
- We only need the artificial variables to provide an initial basic solution, so if we can find one without artificial variables we can eliminate them!
- If supply does not equal demand, we can add a dummy source or a dummy destination. (The Metro District Water Problem in Section 9.2 is an example)

NW corner method for finding an initial BFS:
Work from NW to SE inserting max possible in each cell.

	d_1	d_2	d_3	
S_1	600	800	700	400
S_2	400	900	600	500
	300	200	400	(= 900)

Continuing the method = the key ideas:

Optimality test: A BF solution is optimal if and only if $c_{ij} - u_i - v_j \geq 0$ for every (i, j) such that x_{ij} is nonbasic.[9]

Step 1: Find the Entering Basic Variable. Since $c_{ij} - u_i - v_j$ represents the rate at which the objective function will change as the nonbasic variable x_{ij} is increased, the entering basic variable must have a *negative* $c_{ij} - u_i - v_j$ value to decrease the total cost Z.

Step 2: Find the Leaving Basic Variable. Increasing the entering basic variable from zero sets off a *chain reaction* of compensating changes in other basic variables (allocations), in order to continue satisfying the supply and demand constraints. The first basic variable to be decreased to zero then becomes the leaving basic variable.

Step 3: Find the New BF Solution. The *new BF solution* is identified simply by adding the value of the leaving basic variable (before any change) to the allocation for each recipient cell and subtracting *this same amount* from the allocation for each donor cell.

	d_1	d_2	d_3	
S_1	600	800	700	400
S_2	400	900	600	500
	300	200	400	(= 900)

	d_1	d_2	d_3	
S_1	600	800	700	400
S_2	400	900	600	500
	300	200	400	(= 900)

	d_1	d_2	d_3	
S_1	600	800	700	400
S_2	400	900	600	500
	300	200	400	(= 900)

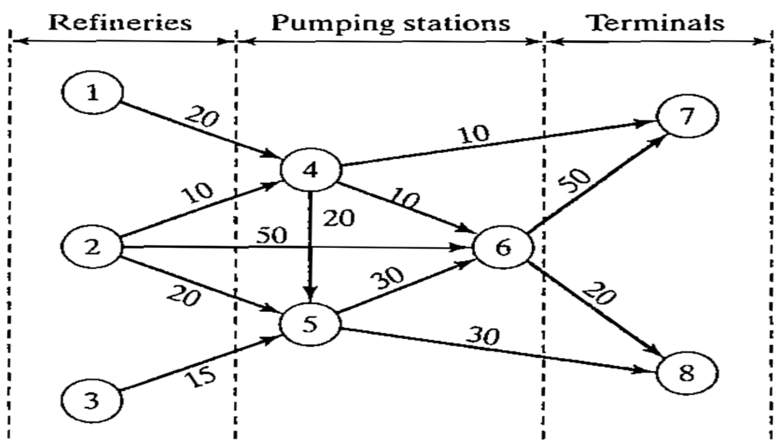
Maximal Flow and Minimum Cost Flow Problem – Networks

Terminology:

- Network: Set of points with lines connecting certain pairs of points. Usually the lines represent the ability to move (distance) or flow (quantity) between the points
- Node (aka Vertex): points or junctions on the network
  - Source (aka supply node): a node for which the flow out is greater than the flow in
  - Sink (aka demand node): a node for which the flow in is greater than the flow out
  - Transshipment node: a node whose flow in is equal to the flow out
- Arc (aka Edge or Branch): connections between vertices
  - Directed arc (aka Oriented): if flow can only happen in one direction; if all arcs are directed, the network is referred to as a directed network
  - Undirected arc: if flow can happen in either direction (often called links instead of arcs)
- Arc capacity: the maximum amount of flow through an arc
- Connected Network: a network in which every pair of nodes Is connected

Maximal Flow: A network with one or more source nodse and one or more sink nodes; maximum capacity of flow through each arc is known. The objective is to maximize the amount that can flow from the sources to the sinks.

Oil Pipeline Problem: Three refineries send gasoline to two distribution terminals through a pipeline network. Any demand that cannot be met through the network is acquired from other sources. The pipeline network is served by 3 pumping stations as shown in the figure. The product flows in the network in the direction shown by the arrows. The capacity of each pipe segment is shown on the arcs (in million bbl per day). Find the maximal flow to the terminals.



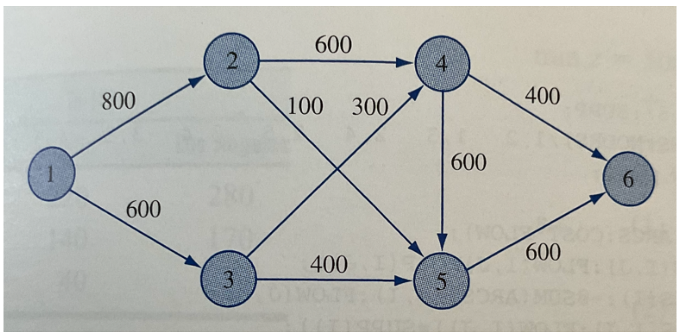
Variable: x_{ij} = flow from node i to node j. Linear Programming:

Minimum cost flow problem: A network with one or more supply nodes and one or more destination nodes; flow through each arc is in one direction with known maximum capacity, and the cost of the flow through each arc is proportional to the amount of the flow. The objective is to minimize the cost.

Traffic Problem: Each hour an average of 900 cars enter the highway network illustrated below at node 1 and seek to travel to node 6. The maximum number of cars that can travel on a given arc is provided along with the time it takes to travel through that arc. We want to determine a flow through the network that minimizes the total time required for all cars to travel from node one to node 6.

Travel Times for Traffic Example

Arc	Time (Minutes)
(1, 2)	10
(1, 3)	50
(2, 5)	70
(2, 4)	30
(5, 6)	30
(4, 5)	30
(4, 6)	60
(3, 5)	60
(3, 4)	10



Variable: x_{ij} = flow from node i to node j. Linear Programming: