Econometrics - Homework 2

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2.2 Show that the norm of $\frac{x}{\|x\|}$ is 1.

$$\|x\| = \sqrt{x^T x} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \text{constant}$$

Thus

$$\|\frac{x}{\|x\|}\| = \frac{1}{\sqrt{x_1^2+x_2^2+\ldots+x_n^n}}\|x\|$$

Since the denominator $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^n}$ is a constant, we can take it out of the norm. We already know that $\|x\|$ is also equal to $\sqrt{x_1^2 + x_2^2 + \ldots + x_n^n}$, it is easy to see that the result is 1.

Compute the norm of the sum and of the difference of x and y normalized

The norm of the sum is

$$\sqrt{[\frac{x}{\|x\|} + \frac{y}{\|y\|}]^T[\frac{x}{\|x\|} + \frac{y}{\|y\|}]} = \sqrt{\frac{x^T x}{\|x\|^2} + \frac{2x^T y}{\|x\|\|y\|} + \frac{y^T y}{\|y\|^2}} = \sqrt{1 + \frac{2x^T y}{\|x\|\|y\|} + 1} = \sqrt{2(1 + \frac{x^T y}{\|x\|\|y\|})}$$

To take the square root the expression must be non-negative, thus $\frac{x^T y}{\|x\|\|y\|} \geq -1$.

The norm of the difference is given by

$$\sqrt{[\frac{x}{\|x\|} - \frac{y}{\|y\|}]^T[\frac{x}{\|x\|} - \frac{y}{\|y\|}]} = \sqrt{2(1 - \frac{x^T y}{\|x\|\|y\|})}$$

Here again $\frac{x^T y}{\|x\|\|y\|} \leq 1$ for the equation to be non-negative.

Prove the Cauchy-Schwartz inequality and show that this inequality becomes an equality when x and y are parallel.

By taking the last two equations, it is straightforward to obtain the Cauchy-Schwartz inequality $|x^T y| \leq \|x\|\|y\|$.
If x and y are parallel we can write $y = \alpha x$ and $\|y\| = \alpha\|x\|$, where α is a constant. Thus

$$x^T y = x^T \alpha x = \alpha\|x\|^2 = \alpha\|x\|\|x\| = \|y\|\|x\|$$

Therefore it is an equality when the vectors are parallel.

2.10 Show that $P = X(W^T X)^{-1} X^T$ is idempotent but not symmetric.

It is easy to show that P is idempotent since

$$PP = X(W^T X)^{-1} W^T X(W^T X)^{-1} W^T = X(W^T X)^{-1} W^T = P$$

We can see that the transpose of P is given by

$$P^T = W(X^T W)^{-1} X^T$$

To prove that they are not equal let's show that they span different subspace. Let y be a n-vector, such that

$$Py = X(W^T X)^{-1} W^T y = Xa$$

where $a = (W^T X)^{-1} W^T y$ is a k-vector. Thus Py is on $S(X)$.

$$P^T y = W(X^T W)^{-1} X^T y = Wb$$

where $b = (X^T W)^{-1} X^T y$. Thus P^T is on $S(W)$. Since $S(W) \neq S(X), P \neq P^T$ and P is not symmetric.

Characterize the spaces that P and $I - P$ project on to, and show that they are not orthogonal.

Let $P_w = W(W^T W)^{-1} W^T$ and $P_x = X(X^T X)^{-1} X^T$. We can see that $(I - P)$ is $\perp$ to $S(W)$ since
$P_w(I - P) = P_w - W(W^T W)^{-1} W^T X(W^T X)^{-1} W^T = 0$

To prove that P and $(I - P)$ are not orthogonal we need to show that their product is not equal to 0. Since P is on to $S(X)$ and $(I - P)$ is on to $S^{\perp}(W)$ we can rewrite them as P_x and M_w respectively. Thus

$$P_x M_w = P_x(I - P_w) = P_x - P_x P_w$$

where $P_x P_w = X(X^T X)^{-1} X^T W(W^T W)^{-1} W^T \neq P_x$. Therefore $P_x M_w \neq 0$ and $S(X)$ and $S^{\perp}(W)$ are not $\perp$.

2.13 Consider the two regressions

$$y = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u, \text{ and}$$
$$y = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + u,$$

where $z_1 = x_1 - 2x_2$, $z_2 = x_2 + 4x_3$, and $z3 = 2x_1 - 3x_2 + 5x_3$. Let $X = [x_1 x_2 x_3]$ and $Z = [z_1 z_2 z_3]$. Show that the columns of Z can be expressed as linear combinations of the columns of X, that is, that $Z = XA$, for some 3 $\times$ 3 matrix A. Find the elements of this matrix A.

Since $Z = XA$ we can easily show that

$$A = \begin{bmatrix} 1\ 0\ 2 \\ -2\ 1\ -3 \\ 0\ 4\ 5 \end{bmatrix}$$

For A to be invertible we must be able to express X as a linear combination of Z, such that $X = ZA^{-1}$

$$A^{-1} = \begin{bmatrix} 17\ 8\ -2 \\ 10\ 5\ -1 \\ -8\ -4\ 1 \end{bmatrix}$$

Show that the two regressions give the same fitted values and residuals.

$$P_z = Z(Z^T Z)^{-1} Z^T = XA(X^T A^T XA)^{-1}(XA)^T =$$
$$XAA^{-1}(X^T X)^{-1}(A^T)^{-1} A^T X^T = X(X^T X)^{-1} X^T = P_x$$

Since $P_z = P_x$, $M_z = M_x$ since they are define by the identity minus their respective fitted value. Therefore they have the same fitted value and residuals.

Precisely how is the OLS estimate β_1 related to the OLS estimates α_i?

Since $Z\alpha = y$, $X\beta = y$ and $Z = XA$ we can replace Z in the first equation and get $XA\alpha = y$. We can see that $A\alpha$ must be equal to β for the equality to hold.

$$\beta_1 = \alpha_1 + 2\alpha_3$$

Precisely how is α_1 related to β_i?

We can rewrite the previous equation as $\alpha = A^{-1}\beta$. Therefore

$$\alpha_1 = 17\beta_1 + 8\beta_2 - 2\beta_3$$

2.23 For the period 1950:4 to 1996:4, run the regression $\Delta r_t = \beta_1 + \beta_2 \pi_{t-1} + \beta_3 \Delta y_{t-1} + \beta_4 \Delta r_{t-1} + \beta_5 \Delta r_{t-2} + u_t$

Call: lm(formula = dr   pi1 + dGDP1 + dr1 + dr2, data = tbills)

Residuals: Min 1Q Median 3Q Max

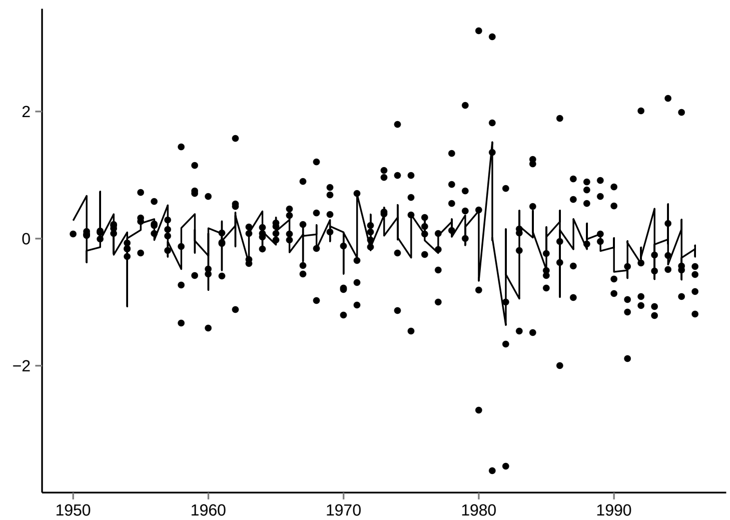

Figure 1: Actual and fitted values

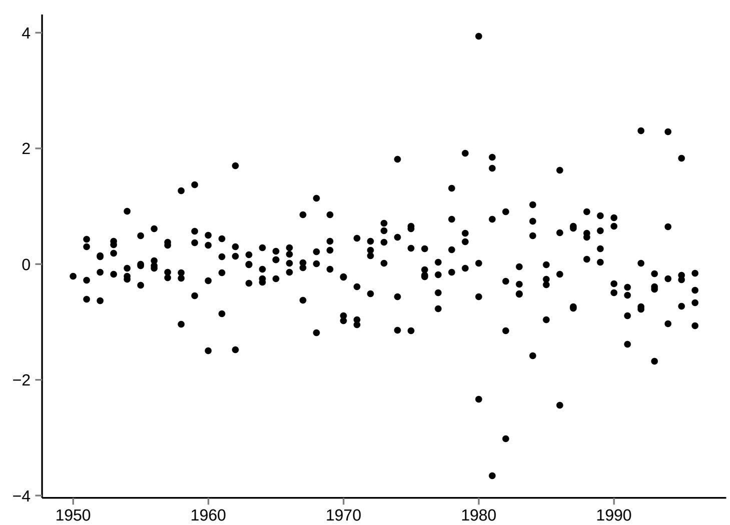

Figure 2: Residuals

Then regress the residuals on the fitted values and on a constant.

Both coefficient are 0, which is easy to see since the residuals are centered at 0 (for the constant) and the coefficient is also 0 because the fitted values lie on $S(X)$ and the residuals on $S^{\perp}(X)$.

```
##
## Call:
## lm(formula = tbills$res ~ tbills$fit)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -3.660 -0.389 -0.010  0.428  3.935
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -2.84e-18   6.50e-02       0        1
## tbills$fit   2.12e-16   1.88e-01       0        1
##
## Residual standard error: 0.884 on 183 degrees of freedom
## Multiple R-squared:  1.07e-32,Adjusted R-squared:  -0.00546
## F-statistic: 1.96e-30 on 1 and 183 DF,  p-value: 1
```

Now regress the fitted values on the residuals and on a constant.

We can see that the coefficient of the residuals is 0 since they lie on $S^{\perp}(X)$ and the intercept is the average of Δr .

```
##
## Call:
## lm(formula = tbills$fit ~ tbills$res)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.3741 -0.1802  0.0132  0.2263  1.5055
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.0134     0.0256    0.52      0.6
## tbills$res    0.0000     0.0291    0.00      1.0
##
## Residual standard error: 0.348 on 183 degrees of freedom
## Multiple R-squared:  1.98e-32,Adjusted R-squared:  -0.00546
## F-statistic: 3.62e-30 on 1 and 183 DF,  p-value: 1
```

2.24 For the same sample period, regress Δr_t on a constant, Δy_{t-1}, Δr_{t-1}, and Δr_{t-2}. Save the residuals from this regression, and call them e_t.

```
##
## Call:
## lm(formula = dr ~ dGDP1 + dr1 + dr2, data = tbills)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -3.579 -0.417 -0.031  0.442  4.046
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -0.1580     0.0852   -1.85   0.0654
## dGDP1        17.4756     5.6416    3.10   0.0023
## dr1           0.2437     0.0736    3.31   0.0011
## dr2          -0.1472     0.0720   -2.05   0.0423
##
## Residual standard error: 0.89 on 181 degrees of freedom
## Multiple R-squared:  0.131,Adjusted R-squared:  0.117
## F-statistic: 9.12 on 3 and 181 DF,  p-value: 1.18e-05
```

Regress π_{t-1} on a constant, Δy_{t-1} , Δr_{t-1} , and Δr_{t-2} . Save the residuals and call them v_t

```
##
## Call:
## lm(formula = pi1 ~ dGDP1 + dr1 + dr2, data = tbills)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -6.528 -2.262 -0.591  1.800 10.366
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)    4.602      0.317   14.54   <2e-16
## dGDP1        -56.346     20.953   -2.69   0.0078
## dr1            0.391      0.273    1.43   0.1547
## dr2            0.427      0.267    1.60   0.1122
##
## Residual standard error: 3.31 on 181 degrees of freedom
## Multiple R-squared:  0.0605,Adjusted R-squared:  0.0449
## F-statistic: 3.89 on 3 and 181 DF,  p-value: 0.0101
```

Now regress e_t on v_t.

We can see that the coefficient is the same than the one for inflation (π) and the residuals are the same than the one in the first regression.

```
##
## Call:
## lm(formula = e ~ v + 0)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -3.660 -0.389 -0.010  0.428  3.935
##
## Coefficients:
##   Estimate Std. Error t value Pr(>|t|)
## v   0.0161     0.0198    0.81     0.42
##
## Residual standard error: 0.881 on 184 degrees of freedom
## Multiple R-squared:  0.00356,Adjusted R-squared:  -0.00186
## F-statistic: 0.657 on 1 and 184 DF,  p-value: 0.419
```