Рассмотрим геометрический интеграл:

$$\prod_a^b f(x)^{dx} = \lim_{\Delta x \to 0} \prod_{i=1}^n f(x_i)^{\Delta x}$$

Данный интеграл имеет следующую связь с обыкновенным интегралом

$$\prod_a^b f(x)^{dx} = \lim_{\Delta x \to 0} e^{\sum\limits_{i=1}^n \ln f(x_i) \cdot \Delta x} = e^{\int\limits_a^b \ln f(x)\ dx}$$

Теперь рассмотрим следующий предел:

$$\lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x)}{f(x)} \right)^{\frac{1}{\Delta x}}$$

Назовём этот предел геометрической производной и обозначим f^*. Вычислив этот предел, можно получить связь с обыкновенной производной:

$$f^*(x) = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x)}{f(x)} \right)^{\frac{1}{\Delta x}} = e^{\ln \lim\limits_{\Delta x \to 0} \left(\frac{f(x + \Delta x)}{f(x)} \right)^{\frac{1}{\Delta x}}} =$$

$$= e^{\lim\limits_{\Delta x \to 0} \frac{\ln f(x + \Delta x) - \ln f(x)}{\Delta x}} = e^{\ln' f(x)} = e^{\frac{f'}{f}}$$

Частные производные

$$f^*_{x_i}(x_1,\ldots,x_i,\ldots,x_n) = \lim_{\Delta x\to 0}\left(\frac{f(x_1,\ldots,x_i+\Delta x,\ldots,x_n)}{f(x_1,\ldots,x_i,\ldots,x_n)}\right)^{\frac{1}{\Delta x}} =$$

$$= e^{\ln\lim\limits_{\Delta x\to 0}\left(\frac{f(x_1,\ldots,x_i+\Delta x,\ldots,x_n)}{f(x_1,\ldots,x_i,\ldots,x_n)}\right)^{\frac{1}{\Delta x}}} =$$

$$= e^{\lim\limits_{\Delta x\to 0}\frac{\ln f(x_1,\ldots,x_i+\Delta x,\ldots,x_n)-\ln f(x_1,\ldots,x_i,\ldots,x_n)}{\Delta x}} =$$

$$= e^{\ln'_{x_i}(f(x_1,\ldots,x_i,\ldots,x_n))} = e^{\frac{f'_{x_i}(x_1,\ldots,x_i,\ldots,x_n)}{f(x_1,\ldots,x_i,\ldots,x_n)}}, \quad i=\overline{1,\ldots,n}$$

Полный дифференциал

$$\Delta^* f = \frac{f(x+\Delta x,y+\Delta y)}{f(x,y)} = \frac{f(x+\Delta x,y+\Delta y)}{f(x,y)}\cdot\frac{f(x,y+\Delta y)}{f(x,y+\Delta y)} =$$

$$= \frac{f(x+\Delta x,y+\Delta y)}{f(x,y+\Delta y)}\cdot\frac{f(x,y+\Delta y)}{f(x,y)} = \left(\frac{f(x+\Delta x,y+\Delta y)}{f(x,y+\Delta y)}\right)^{\frac{\Delta x}{\Delta x}}\cdot\left(\frac{f(x,y+\Delta y)}{f(x,y)}\right)^{\frac{\Delta y}{\Delta y}}$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\Delta^* f = \lim_{(\Delta x,\Delta y)\to(0,0)}\left(\frac{f(x+\Delta x,y+\Delta y)}{f(x,y+\Delta y)}\right)^{\frac{\Delta x}{\Delta x}}\cdot\lim_{\Delta y\to 0}\left(\frac{f(x,y+\Delta y)}{f(x,y)}\right)^{\frac{\Delta y}{\Delta y}} =$$

$$= e^{\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\frac{[\ln f(x+\Delta x,y+\Delta y)-\ln f(x,y+\Delta y)]\Delta x}{\Delta x}}\cdot e^{\lim\limits_{\Delta y\to 0}\frac{[\ln f(x,y+\Delta y)-\ln f(x,y)]\Delta y}{\Delta y}} =$$

$$= e^{\lim\limits_{\Delta y\to 0}\frac{f'_x(x,y+\Delta y)}{f(x,y+\Delta y)}}\cdot e^{\frac{f'_y(x,y)}{f(x,y)}dy} = e^{\frac{f'_x(x,y)}{f(x,y)}dx}\cdot e^{\frac{f'_y(x,y)}{f(x,y)}dy} = f_x^{*dx}\cdot f_y^{*dy}$$

$$d^* f = f_x^{*dx}\cdot f_y^{*dy}$$

Геометрическая теорема Шварца. Пусть функция $f(x,y)$ имеет смешанные частные геометрические производные f^{**}_{xy} и f^{**}_{yx} в окрестности некоторой точки (x_0,y_0), причём они непрерывны в этой точке. Тогда в этой точке $f^{**}_{xy}=f^{**}_{yx}$.
Доказательство.

$$\Delta^{*^2} f = \frac{f(x_0+\Delta x, y_0+\Delta y)}{f(x_0+\Delta x, y_0)} \cdot \frac{f(x_0,y_0)}{f(x_0,y_0+\Delta y)}$$

$$\varphi(x) = \frac{f(x,y_0+\Delta y)}{f(x,y_0)}$$

Дважды юзаем $\ ^*Mean\ value\ theorem \left(f^*(c) = \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{b-a}} \right)$

$$\frac{\varphi(x_0+\Delta x)}{\varphi(x_0)} = \varphi^*(x_0+\theta_1\Delta x)^{\Delta x} = \left(\frac{f^*_x(x_0+\theta_1\Delta x, y_0+\Delta y)}{f^*_x(x_0+\theta_1\Delta x, y_0)}\right)^{\Delta x} =$$

$$= f^{**}_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)^{\Delta x\Delta y}$$

$$\psi(y) = \frac{f(x_0+\Delta x, y)}{f(x_0,y)}$$

$$\frac{\varphi(x_0+\Delta x)}{\varphi(x_0)} = \frac{\psi(y_0+\Delta y)}{\psi(y_0)}$$

$$\frac{\psi(y_0+\Delta y)}{\psi(y_0)} = \left(\frac{f^*_y(x_0+\Delta x, y_0+\theta'_2\Delta y)}{f^*_y(x_0, y+\theta'_2\Delta y)}\right)^{\Delta y} = f^{**}_{yx}(x_0+\theta'_1\Delta x, y_0+\theta'_2\Delta y)^{\Delta x\Delta y}$$

$$\frac{\varphi(x_0+\Delta x)}{\varphi(x_0)} = \frac{\psi(y_0+\Delta y)}{\psi(y_0)} \Rightarrow f^{**}_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)^{\Delta x\Delta y} = f^{**}_{yx}(x_0+\theta'_1\Delta x, y_0+\theta'_2\Delta y)^{\Delta x\Delta y}$$

$$f^{**}_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y) = f^{**}_{yx}(x_0+\theta'_1\Delta x, y_0+\theta'_2\Delta y)$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} f^{**}_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y) = \lim_{(\Delta x,\Delta y)\to(0,0)} f^{**}_{yx}(x_0+\theta'_1\Delta x, y_0+\theta'_2\Delta y)$$

$$f^{**}_{xy}(x_0,y_0) = f^{**}_{yx}(x_0,y_0)$$

Теперь поищем обратный оператор с f^*. Для этого попробуем выразить f. Обозначим $u = e^{\frac{f'}{f}}$. Тогда

$$f^*(x) = u(x)$$

$$\ln' f(x) = \ln u(x)$$

$$\ln f(x) = \int\limits_0^x \ln u(s)\ ds + \ln f(0)$$

$$f(x) = f(0) \cdot e^{\int\limits_0^x \ln u(s)\ ds}$$

Получили геометрический интеграл с переменным верхним пределом.

$$\prod_{\tilde{C}}^{x} f(s)^{ds} = e^{\int\limits_{\tilde{C}}^{x} \ln f(s)\ ds} = e^{F(x) - F(\tilde{C})} = \frac{e^{F(x)}}{e^{F(\tilde{C})}} = Ce^{F(x)}$$

$$\left(\prod_{\tilde{C}}^{x} f(s)^{ds} \right)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\int\limits_{\tilde{C}}^{x+\Delta x} \ln f(s)\ ds - \int\limits_{\tilde{C}}^{x} \ln f(s)\ ds}{\Delta x}} =$$

$$= e^{\lim\limits_{\Delta x \to 0} \frac{\int\limits_{\tilde{C}}^{x+\Delta x} \ln f(s)\ ds + \int\limits_{x}^{\tilde{C}} \ln f(s)\ ds}{\Delta x}} = e^{\lim\limits_{\Delta x \to 0} \frac{\int\limits_{x}^{x+\Delta x} \ln f(s)\ ds}{\Delta x}} = e^{\ln f(x)} = f(x)$$

Прикольные штуки для производных

$$a^* = \lim_{\Delta x \to 0} \left(\frac{a}{a}\right)^{\frac{1}{\Delta x}} = 1^\infty = 1$$

$$x^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x}} = e^{\frac{1}{x}}$$

$$(x^n)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln((x+\Delta x)^n) - \ln x^n}{\Delta x}} = e^{\frac{nx^{n-1}}{x^n}} = e^{\frac{n}{x}}$$

$$(a^x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{x+\Delta x} - \ln a^x}{\Delta x}} = e^{\ln a \lim\limits_{\Delta x \to 0} \frac{\Delta x}{\Delta x}} = a$$

$$(a^{x^n})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{(x+\Delta x)^n} - \ln a^{x^n}}{\Delta x}} = a^{nx^{n-1}}$$

$$\left(a^{e^x}\right)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{e^{x+\Delta x}} - \ln a^{e^x}}{\Delta x}} = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a \cdot \left(e^{x+\Delta x} - e^x\right)}{\Delta x}} = a^{e^x}$$

$$(\ln x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln(\ln(x+\Delta x)) - \ln(\ln x)}{\Delta x}} = e^{\frac{1}{x \ln x}}$$

$$(\sin x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln \sin(x+\Delta x) - \ln \sin x}{\Delta x}} = e^{\frac{\sin' x}{\sin x}} = e^{\cot x}$$

$$(\cos x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln \cos(x+\Delta x) - \ln \cos x}{\Delta x}} = e^{\frac{\cos' x}{\cos x}} = e^{-\tan x}$$

$$(\tan x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln \tan(x+\Delta x) - \ln \tan x}{\Delta x}} = e^{\frac{\tan' x}{\tan x}} = e^{\frac{1}{\cos x \cdot \sin x}} = e^{\frac{2}{\sin(2x)}}$$

$$(\cot x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln \cot(x+\Delta x) - \ln \cot x}{\Delta x}} = e^{\frac{\cot' x}{\cot x}} = e^{-\frac{1}{\cos x \cdot \sin x}} = e^{-\frac{2}{\sin(2x)}}$$

$$(\sinh x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln \sinh(x+\Delta x) - \ln \sinh x}{\Delta x}} = e^{\frac{\sinh' x}{\sinh x}} = e^{\coth x}$$

$$(\cosh x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln \cosh(x+\Delta x) - \ln \cosh x}{\Delta x}} = e^{\frac{\cosh' x}{\cosh x}} = e^{\tanh x}$$

$$(\tanh x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln \tanh(x+\Delta x) - \ln \tanh x}{\Delta x}} = e^{\frac{\tanh' x}{\tanh x}} = e^{\frac{1}{\cosh x \cdot \sinh x}} = e^{\frac{2}{\sinh(2x)}}$$

$$(\coth x)^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln \coth(x+\Delta x) - \ln \coth x}{\Delta x}} = e^{\frac{\coth' x}{\coth x}} = e^{-\frac{1}{\cosh x \cdot \sinh x}} = e^{-\frac{2}{\sinh(2x)}}$$

$$(a^{\sin x})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{\sin(x+\Delta x)} - \ln a^{\sin x}}{\Delta x}} = a^{\cos x}$$

$$(a^{\cos x})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{\cos(x+\Delta x)} - \ln a^{\cos x}}{\Delta x}} = a^{-\sin x}$$

$$(a^{\tan x})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{\tan(x+\Delta x)} - \ln a^{\tan x}}{\Delta x}} = a^{\frac{1}{\cos^2 x}}$$

$$(a^{\cot x})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{\cot(x+\Delta x)} - \ln a^{\cot x}}{\Delta x}} = a^{-\frac{1}{\sin^2 x}}$$

$$(a^{\sinh x})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{\sinh(x+\Delta x)} - \ln a^{\sinh x}}{\Delta x}} = a^{\cosh x}$$

$$(a^{\cosh x})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{\cosh(x+\Delta x)} - \ln a^{\cosh x}}{\Delta x}} = a^{\sinh x}$$

$$(a^{\tanh x})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{\tanh(x+\Delta x)} - \ln a^{\tanh x}}{\Delta x}} = a^{\frac{1}{\cosh^2 x}}$$

$$(a^{\coth x})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{\coth(x+\Delta x)} - \ln a^{\coth x}}{\Delta x}} = a^{-\frac{1}{\sinh^2 x}}$$

$$(a^{f(x)})^* = e^{\lim\limits_{\Delta x \to 0} \frac{\ln a^{f(x+\Delta x)} + \ln a^{f(x)}}{\Delta x}} = e^{\ln a \cdot \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} = a^{f'(x)}$$

Прикольные штуки для интегралов

$$\prod a^{dx} = Ca^x$$

$$\prod x^{dx} = Ce^{x(\ln x - 1)} = Ce^{-x}x^x$$

$$\prod e^{\frac{1}{x}^{dx}} = Cx$$

$$\prod a^{x^n dx} = Ca^{\frac{x^{n+1}}{n+1}}$$

$$\prod a^{e^x dx} = Ca^{e^x}$$

$$\prod \left(f(x)^{\ln f^*}\right)^{dx} = Ce^{\frac{\ln^2 f(x)}{2}}$$

Другие прикольные штуки для производных

$$\big(f(x)\cdot g(x)\big)^* = e^{\lim\limits_{\Delta x\to 0}\dfrac{\ln(f(x+\Delta x)g(x+\Delta x))-\ln(f(x)g(x))}{\Delta x}} =$$

$$= e^{\ln' f(x)+\ln' g(x)} = e^{\frac{f'}{f}+\frac{g'}{g}} = f^*(x)\cdot g^*(x)$$

$$\left(\frac{f(x)}{g(x)}\right)^* = e^{\lim\limits_{\Delta x\to 0}\dfrac{\ln\frac{f(x+\Delta x)}{g(x+\Delta x)}-\ln\frac{f(x)}{g(x)}}{\Delta x}} = e^{\ln' f(x)-\ln' g(x)} = e^{\frac{f'}{f}-\frac{g'}{g}} = \frac{f^*(x)}{g^*(x)}$$

$$\big(f(x)+g(x)\big)^* = e^{\lim\limits_{\Delta x\to 0}\dfrac{\ln(f(x+\Delta x)+g(x+\Delta x))-\ln(f(x)+g(x))}{\Delta x}} =$$

$$= e^{\frac{((f(x)+g(x))'}{f(x)+g(x)}} = e^{\frac{f'(x)+g'(x)}{f(x)+g(x)}} = \left(e^{f'(x)}\cdot e^{g'(x)}\right)^{\frac{1}{f(x)+g(x)}} =$$

$$= \left(e^{\frac{f(x)\cdot f'(x)}{f(x)}}\cdot e^{\frac{g(x)\cdot g'(x)}{g(x)}}\right)^{\frac{1}{f(x)+g(x)}} = \left(f^*(x)^{f(x)}\cdot g^*(x)^{g(x)}\right)^{\frac{1}{f(x)+g(x)}}$$

$$\big(f(g(x))\big)^* = e^{\lim\limits_{\Delta x\to 0}\dfrac{\ln\big(f(g(x+\Delta x))\big)-\ln\big(f(g(x))\big)}{\Delta x}} = e^{\frac{f'(g(x))\cdot g'(x)}{f(g(x))}} = f^*(g(x))^{g'(x)}$$

$$\left(f(x)^{g(x)}\right)^* = e^{\lim\limits_{\Delta x\to 0}\dfrac{\ln f^{g(x+\Delta x)}(x+\Delta x)-\ln f^{g(x)}(x)}{\Delta x}} = e^{\ln' f^{g(x)}(x)} =$$

$$= e^{\frac{(f^{g(x)}(x))'}{f^{g(x)}(x)}} = e^{\frac{f^{g(x)-1}(x)\cdot g(x)\cdot f'(x)+f^{g(x)}(x)\cdot g'(x)\cdot \ln f(x)}{f^{g(x)}(x)}} = e^{\frac{g(x)\cdot f'(x)}{f(x)}}\cdot e^{g'(x)\cdot\ln f(x)} =$$

$$= f^*(x)^{g(x)}\cdot\left(e^{\ln f(x)}\right)^{g'(x)} = f^*(x)^{g(x)}\cdot f(x)^{g'(x)}$$

Другие прикольные штуки для интегралов

Внесение под знак дифференциала

$$\prod f(x)^{g'(x)dx} = \prod f(x)^{d(g(x))}$$

Замена переменной

$$\prod f(x)^{dx} = e^{\int \ln f(x)\ dx} = e^{\int \frac{\ln f(u)}{u'}\ du} = e^{\int \ln f(u)^{\frac{1}{u'}}\ du} = \prod f(u)^{\frac{du}{u'}}$$

Интегрирование по частям для степеней

$$(f(x)^{g(x)})^* = f^*(x)^{g(x)} \cdot f(x)^{g'(x)}$$

$$f(x)^{g(x)} = \prod \left(f^*(x)^{g(x)}\right)^{dx} \cdot \prod \left(f(x)^{g'(x)}\right)^{dx}$$

$$\prod \left(f^*(x)^{g(x)}\right)^{dx} = \frac{f(x)^{g(x)}}{\prod \left(f(x)^{g'(x)}\right)^{dx}}$$

$$\prod \left(f(x)^{g'(x)}\right)^{dx} = \frac{f(x)^{g(x)}}{\prod \left(f^*(x)^{g(x)}\right)^{dx}}$$

Другое

$$\left(\prod f(x)^{dx}\right)' = \left(e^{\int \ln f(x)\ dx}\right)' = e^{\int \ln f(x)\ dx} \cdot \ln f(x) = \ln f(x) \cdot \prod f(x)^{dx}$$

$$\prod \left(a^{f(x)}\right)^{dx} = a^{\int f(x)\ dx}$$

$$\ln \prod f(x)^{dx} = \int \ln f(x)\ dx$$

Ряды

$$f(x) = \prod_{i=0}^{n} \left(f^{*(i)}(a)\right)^{\frac{(t-a)^i}{i!}} \cdot \left(f^{*(n+1)}(c)\right)^{\frac{(t-a)^{n+1}}{(n+1)!}}$$

$$f(x) = \prod_{i=0}^{\infty} \left(f^{*(i)}(a)\right)^{\frac{(t-a)^i}{i!}}$$

Дифуры

$$F(x, y, y^*) = 0$$

Допускающее разделение

$$y^* = f(x)$$

Линейное

Однородное

$$(y^*)^{f_1(x)} \cdot y^{f_0(x)} = 1$$

Общее решение

$$y_{\text{оо}} = C^{\prod\left(e^{-\frac{f_0(x)}{f_1(x)}}\right)^{dx}}$$

Неоднородное

$$y^* \cdot y^{\frac{f_0(x)}{f_1(x)}} = g(x)$$

Общее решение

$$y_{\text{он}} = \left(C \prod \left(g(x)^{\prod\left(e^{\frac{f_0(x)}{f_1(x)}}\right)^{dx}}\right)^{dx}\right)^{\prod\left(e^{-\frac{f_0(x)}{f_1(x)}}\right)^{dx}}$$

Метод подстановки

$$y^* \cdot y^{\frac{f_0(x)}{f_1(x)}} = g(x)$$

$$y = u(x)^{v(x)}$$

$$y^* = \left(u(x)^{v(x)}\right)^* = (u^*)^v \cdot u^{v'}$$

$$(u^*)^v \cdot u^{v'} \cdot u^{v\frac{f_0}{f_1}} = g(x)$$

$$(u^*)^v \cdot u^{v' + v\frac{f_0}{f_1}} = g(x)$$

$$v' + v\frac{f_0}{f_1} = 0$$

$$v = e^{-\displaystyle\int \frac{f_0}{f_1}\, dx} = \prod \left(e^{-\frac{f_0}{f_1}}\right)^{dx}$$

$$(u^*)^v = g(x)$$

$$u^* = g^{\frac{1}{v}}$$

$$u = C \prod \left(g^{v^{-1}}\right)^{dx} = C \prod \left(g^{\prod\left(e^{\frac{f_0}{f_1}}\right)^{dx}}\right)^{dx}$$

$$y = u^v = \left(C \prod \left(g^{\prod\left(e^{\frac{f_0}{f_1}}\right)^{dx}}\right)^{dx}\right)^{\prod\left(e^{-\frac{f_0}{f_1}}\right)^{dx}}$$

Приложение

К линейному уравнению

Однородное

$$(y^*)^{f_1(x)} \cdot y^{f_0(x)} = 1$$

$$\frac{y'}{y} + \frac{f_0(x)}{f_1(x)} \ln y = 0$$

Замена $u = \ln y$

$$u' + u\frac{f_0(x)}{f_1(x)} = 0$$

$$\int \frac{du}{u} = -\int \frac{f_0(x)}{f_1(x)}\, dx + C$$

$$\ln u = -\int \frac{f_0(x)}{f_1(x)}\, dx + C$$

$$u = Ce^{-\int \frac{f_0(x)}{f_1(x)}\, dx}$$

$$\ln y = Ce^{-\int \frac{f_0(x)}{f_1(x)}\, dx}$$

$$y = C^{e^{-\int \frac{f_0(x)}{f_1(x)}\, dx}}$$

$$y_{\text{оо}} = C^{\prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}}$$

Проверка

$$y^* = C^{-\frac{f_0(x)}{f_1(x)} \cdot \prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}}$$

$$C^{-f_0(x) \cdot \prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}} \cdot C^{f_0(x) \cdot \prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}} = 1$$

$$C^0 = 1$$

$$1 = 1$$

Неоднородное

$$y^* \cdot y^{\frac{f_0(x)}{f_1(x)}} = g(x)$$

$$\frac{y'}{y} + \frac{f_0(x)}{f_1(x)} \ln y = \ln g(x)$$

Замена $u = \ln y$

$$u' + u\frac{f_0(x)}{f_1(x)} = \ln g(x)$$

Решим соответствующее однородное

$$u' + u\frac{f_0(x)}{f_1(x)} = 0$$

$$\int \frac{du}{u} = -\int \frac{f_0(x)}{f_1(x)}\,dx + C$$

$$\ln u = -\int \frac{f_0(x)}{f_1(x)}\,dx + C$$

$$u_{\text{оо}} = Ce^{-\int \frac{f_0(x)}{f_1(x)}\,dx}$$

$$C = C(x)$$

$$u = C(x)e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}$$

$$u' = C'(x)e^{-\int \frac{f_0(x)}{f_1(x)}\,dx} - C(x)\frac{f_0}{f_1}e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}$$

$$C'(x)e^{-\int \frac{f_0(x)}{f_1(x)}\,dx} - C(x)\frac{f_0}{f_1}e^{-\int \frac{f_0(x)}{f_1(x)}\,dx} + C(x)\frac{f_0}{f_1}e^{-\int \frac{f_0(x)}{f_1(x)}\,dx} = \ln g(x)$$

$$C'(x)e^{-\int \frac{f_0(x)}{f_1(x)}\,dx} = \ln g(x)$$

$$C'(x) = e^{\int \frac{f_0(x)}{f_1(x)}\,dx} \cdot \ln g(x)$$

$$C(x) = \int e^{\int \frac{f_0(x)}{f_1(x)}\,dx} \cdot \ln g(x)\,dx + \tilde{C}$$

$$u = e^{-\int \frac{f_0(x)}{f_1(x)}\,dx} \left(\int e^{\int \frac{f_0(x)}{f_1(x)}\,dx} \cdot \ln g(x)\,dx + \tilde{C} \right)$$

$$\ln y = e^{-\int \frac{f_0(x)}{f_1(x)}\,dx} \int e^{\int \frac{f_0(x)}{f_1(x)}\,dx} \cdot \ln g(x)\,dx + \tilde{C} e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}$$

$$y = e^{\int e^{\int \frac{f_0(x)}{f_1(x)}\,dx} \cdot \ln g(x)\,dx \cdot e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}} \cdot \tilde{C}^{e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}}$$

$$y = \left(\tilde{C} e^{\int e^{\int \frac{f_0(x)}{f_1(x)}\,dx} \cdot \ln g(x)\,dx} \right)^{e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}}$$

$$y_{\text{он}} = \left(\tilde{C} \prod \left(g(x)^{\prod \left(e^{\frac{f_0(x)}{f_1(x)}} \right)^{dx}} \right)^{dx} \right)^{\prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}}$$

Проверка

$$y^* = \left(g(x)^{\prod \left(e^{\frac{f_0(x)}{f_1(x)}} \right)^{dx}} \right)^{\prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}} \cdot \left(\tilde{C} \prod \left(g(x)^{\prod \left(e^{\frac{f_0(x)}{f_1(x)}} \right)^{dx}} \right)^{dx} \right)^{-\frac{f_0(x)}{f_1(x)} \prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}}$$

$$\left(g(x)^{\prod \left(e^{\frac{f_0(x)}{f_1(x)}} \right)^{dx}} \right)^{\prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}} = g(x)$$

$$g(x)^{\prod \left(e^{\frac{f_0(x)}{f_1(x)}} \right)^{dx} \cdot \prod \left(e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}} = g(x)$$

$$g(x) = g(x)$$