

Рассмотрим геометрический интеграл:

$$\prod_a^b f(x)^{dx} = \lim_{\Delta x \rightarrow 0} \prod_{i=1}^n f(x_i)^{\Delta x}$$

Данный интеграл имеет следующую связь с обыкновенным интегралом

$$\prod_a^b f(x)^{dx} = \lim_{\Delta x \rightarrow 0} e^{\sum_{i=1}^n \ln f(x_i) \cdot \Delta x} = e^{\int_a^b \ln f(x) dx}$$

Теперь рассмотрим следующий предел:

$$\lim_{\Delta x \rightarrow 0} \left( \frac{f(x + \Delta x)}{f(x)} \right)^{\frac{1}{\Delta x}}$$

Назовём этот предел геометрической производной и обозначим  $f^*$ . Вычислив этот предел, можно получить связь с обыкновенной производной:

$$\begin{aligned} f^*(x) &= \lim_{\Delta x \rightarrow 0} \left( \frac{f(x + \Delta x)}{f(x)} \right)^{\frac{1}{\Delta x}} = e^{\ln \lim_{\Delta x \rightarrow 0} \left( \frac{f(x + \Delta x)}{f(x)} \right)^{\frac{1}{\Delta x}}} = \\ &= e^{\lim_{\Delta x \rightarrow 0} \frac{\ln f(x + \Delta x) - \ln f(x)}{\Delta x}} = e^{\ln' f(x)} = e^{\frac{f'}{f}} \end{aligned}$$

Частные производные

$$\begin{aligned}
 f_{x_i}^*(x_1, \dots, x_i, \dots, x_n) &= \lim_{\Delta x \rightarrow 0} \left( \frac{f(x_1, \dots, x_i + \Delta x, \dots, x_n)}{f(x_1, \dots, x_i, \dots, x_n)} \right)^{\frac{1}{\Delta x}} = \\
 &= e^{\lim_{\Delta x \rightarrow 0} \left( \frac{f(x_1, \dots, x_i + \Delta x, \dots, x_n)}{f(x_1, \dots, x_i, \dots, x_n)} \right)^{\frac{1}{\Delta x}}} = \\
 &= e^{\lim_{\Delta x \rightarrow 0} \frac{\ln f(x_1, \dots, x_i + \Delta x, \dots, x_n) - \ln f(x_1, \dots, x_i, \dots, x_n)}{\Delta x}} = \\
 &= e^{\ln'_{x_i} (f(x_1, \dots, x_i, \dots, x_n))} = e^{\frac{f'_{x_i}(x_1, \dots, x_i, \dots, x_n)}{f(x_1, \dots, x_i, \dots, x_n)}}, \quad i = \overline{1, \dots, n}
 \end{aligned}$$

Полный дифференциал

$$\begin{aligned}
 \Delta^* f &= \frac{f(x + \Delta x, y + \Delta y)}{f(x, y)} = \frac{f(x + \Delta x, y + \Delta y)}{f(x, y)} \cdot \frac{f(x, y + \Delta y)}{f(x, y + \Delta y)} = \\
 &= \frac{f(x + \Delta x, y + \Delta y)}{f(x, y + \Delta y)} \cdot \frac{f(x, y + \Delta y)}{f(x, y)} = \left( \frac{f(x + \Delta x, y + \Delta y)}{f(x, y + \Delta y)} \right)^{\frac{\Delta x}{\Delta x}} \cdot \left( \frac{f(x, y + \Delta y)}{f(x, y)} \right)^{\frac{\Delta y}{\Delta y}} = \\
 \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \Delta^* f &= \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \left( \frac{f(x + \Delta x, y + \Delta y)}{f(x, y + \Delta y)} \right)^{\frac{\Delta x}{\Delta x}} \cdot \lim_{\Delta y \rightarrow 0} \left( \frac{f(x, y + \Delta y)}{f(x, y)} \right)^{\frac{\Delta y}{\Delta y}} = \\
 &= e^{\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{[\ln f(x + \Delta x, y + \Delta y) - \ln f(x, y + \Delta y)] \Delta x}{\Delta x}} \cdot e^{\lim_{\Delta y \rightarrow 0} \frac{[\ln f(x, y + \Delta y) - \ln f(x, y)] \Delta y}{\Delta y}} = \\
 &= e^{\lim_{\Delta y \rightarrow 0} \frac{f'_x(x, y + \Delta y)}{f(x, y + \Delta y)}} \cdot e^{\frac{f'_y(x, y)}{f(x, y)} dy} = e^{\frac{f'_x(x, y)}{f(x, y)} dx} \cdot e^{\frac{f'_y(x, y)}{f(x, y)} dy} = f_x^{*dx} \cdot f_y^{*dy}
 \end{aligned}$$

$$d^* f = f_x^{*dx} \cdot f_y^{*dy}$$

Теперь найдем обратный оператор с  $f^*$ . Для этого попробуем выразить  $f$ . Обозначим  $u = e^{\frac{f'}{f}}$ . Тогда

$$f^*(x) = u(x)$$

$$\ln' f(x) = \ln u(x)$$

$$\ln f(x) = \int_0^x \ln u(s) \, ds + \ln f(0)$$

$$f(x) = f(0) \cdot e^{\int_0^x \ln u(s) \, ds}$$

Получили геометрический интеграл с переменным верхним пределом.

$$\prod_{\tilde{C}}^x f(s)^{ds} = e^{\int_{\tilde{C}}^x \ln f(s) \, ds} = e^{F(x) - F(\tilde{C})} = \frac{e^{F(x)}}{e^{F(\tilde{C})}} = C e^{F(x)}$$

$$\begin{aligned} \left( \prod_{\tilde{C}}^x f(s)^{ds} \right)^* &= e^{\lim_{\Delta x \rightarrow 0} \frac{\int_{\tilde{C}}^{x+\Delta x} \ln f(s) \, ds - \int_{\tilde{C}}^x \ln f(s) \, ds}{\Delta x}} = \\ &= e^{\lim_{\Delta x \rightarrow 0} \frac{\int_{\tilde{C}}^{x+\Delta x} \ln f(s) \, ds + \int_x^{\tilde{C}} \ln f(s) \, ds}{\Delta x}} = e^{\lim_{\Delta x \rightarrow 0} \frac{\int_x^{x+\Delta x} \ln f(s) \, ds}{\Delta x}} = e^{\ln f(x)} = f(x) \end{aligned}$$

## Прикольные штуки для производных

$$a^* = \lim_{\Delta x \rightarrow 0} \left( \frac{a}{a} \right)^{\frac{1}{\Delta x}} = 1^\infty = 1$$

$$x^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x}} = e^{\frac{1}{x}}$$

$$(x^n)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln((x + \Delta x)^n) - \ln x^n}{\Delta x}} = e^{\frac{nx^{n-1}}{x^n}} = e^{\frac{n}{x}}$$

$$(a^x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln a^{x+\Delta x} - \ln a^x}{\Delta x}} = e^{\ln a \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x}} = a$$

$$(a^{x^n})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln a^{(x+\Delta x)^n} - \ln a^{x^n}}{\Delta x}} = a^{nx^{n-1}}$$

$$(a^{e^x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln a^{e^{x+\Delta x}} - \ln a^{e^x}}{\Delta x}} = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln a \cdot (e^{x+\Delta x} - e^x)}{\Delta x}} = a^{e^x}$$

$$(\ln x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln(\ln(x + \Delta x)) - \ln(\ln x)}{\Delta x}} = e^{\frac{1}{x \ln x}}$$

$$(\sin x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \sin(x + \Delta x) - \ln \sin x}{\Delta x}} = e^{\frac{\sin' x}{\sin x}} = e^{\cot x}$$

$$(\cos x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \cos(x + \Delta x) - \ln \cos x}{\Delta x}} = e^{\frac{\cos' x}{\cos x}} = e^{-\tan x}$$

$$(\tan x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \tan(x + \Delta x) - \ln \tan x}{\Delta x}} = e^{\frac{\tan' x}{\tan x}} = e^{\frac{1}{\cos x \cdot \sin x}} = e^{\frac{2}{\sin(2x)}}$$

$$(\cot x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \cot(x + \Delta x) - \ln \cot x}{\Delta x}} = e^{\frac{\cot' x}{\cot x}} = e^{-\frac{1}{\cos x \cdot \sin x}} = e^{-\frac{2}{\sin(2x)}}$$

$$(\sinh x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \sinh(x + \Delta x) - \ln \sinh x}{\Delta x}} = e^{\frac{\sinh' x}{\sinh x}} = e^{\coth x}$$

$$(\cosh x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \cosh(x + \Delta x) - \ln \cosh x}{\Delta x}} = e^{\frac{\cosh' x}{\cosh x}} = e^{\tanh x}$$

$$(\tanh x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \tanh(x + \Delta x) - \ln \tanh x}{\Delta x}} = e^{\frac{\tanh' x}{\tanh x}} = e^{\frac{1}{\cosh x \cdot \sinh x}} = e^{\frac{2}{\sinh(2x)}}$$

$$(\coth x)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \coth(x + \Delta x) - \ln \coth x}{\Delta x}} = e^{\frac{\coth' x}{\coth x}} = e^{-\frac{1}{\cosh x \cdot \sinh x}} = e^{-\frac{2}{\sinh(2x)}}$$

$$(e^{\sin x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln e^{\sin x + \Delta x} - \ln e^{\sin x}}{\Delta x}} = e^{\cos x}$$

$$(e^{\cos x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln e^{\cos x + \Delta x} - \ln e^{\cos x}}{\Delta x}} = e^{-\sin x}$$

$$(e^{\tan x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln e^{\tan x + \Delta x} - \ln e^{\tan x}}{\Delta x}} = e^{\frac{1}{\cos^2 x}}$$

$$(e^{\cot x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln e^{\cot x + \Delta x} - \ln e^{\cot x}}{\Delta x}} = e^{-\frac{1}{\sin^2 x}}$$

$$(e^{\sinh x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln e^{\sinh x + \Delta x} - \ln e^{\sinh x}}{\Delta x}} = e^{\cosh x}$$

$$(e^{\cosh x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln e^{\cosh x + \Delta x} - \ln e^{\cosh x}}{\Delta x}} = e^{\sinh x}$$

$$(e^{\tanh x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln e^{\tanh x + \Delta x} - \ln e^{\tanh x}}{\Delta x}} = e^{\frac{1}{\cosh^2 x}}$$

$$(e^{\coth x})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln e^{\coth x + \Delta x} - \ln e^{\coth x}}{\Delta x}} = e^{-\frac{1}{\sinh^2 x}}$$

## Прикольные штуки для интегралов

$$\prod a^{dx} = Ca^x$$

$$\prod x^{dx} = Ce^{x(\ln x - 1)} = Ce^{-x} x^x$$

$$\prod e^{\frac{1}{x} dx} = Cx$$

$$\prod e^{x^{n+1} dx} = Ce^{\frac{x^{n+1}}{n+1}}$$

$$\prod a^{e^x dx} = Ca^{e^x}$$

$$\prod (f(x)^{\ln f^*})^{dx} = Ce^{\frac{\ln^2 f(x)}{2}}$$

## Другие прикольные штуки для производных

$$(f(x) \cdot g(x))^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln(f(x + \Delta x)g(x + \Delta x)) - \ln(f(x)g(x))}{\Delta x}} =$$

$$= e^{\ln' f(x) + \ln' g(x)} = e^{\frac{f'}{f} + \frac{g'}{g}} = f^*(x) \cdot g^*(x)$$

$$\left(\frac{f(x)}{g(x)}\right)^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln \frac{f(x+\Delta x)}{g(x+\Delta x)} - \ln \frac{f(x)}{g(x)}}{\Delta x}} = e^{\ln' f(x) - \ln' g(x)} = e^{\frac{f'}{f} - \frac{g'}{g}} = \frac{f^*(x)}{g^*(x)}$$

$$(f(x)+g(x))^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln(f(x + \Delta x) + g(x + \Delta x)) - \ln(f(x) + g(x))}{\Delta x}} =$$

$$= e^{\frac{((f(x)+g(x))')}{f(x)+g(x)}} = e^{\frac{f'(x)+g'(x)}{f(x)+g(x)}} = \left(e^{f'(x)} \cdot e^{g'(x)}\right)^{\frac{1}{f(x)+g(x)}} =$$

$$= \left(e^{\frac{f(x) \cdot f'(x)}{f(x)}} \cdot e^{\frac{g(x) \cdot g'(x)}{g(x)}}\right)^{\frac{1}{f(x)+g(x)}} = \left(f^*(x)^{f(x)} \cdot g^*(x)^{g(x)}\right)^{\frac{1}{f(x)+g(x)}}$$

$$(f(g(x)))^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln(f(g(x + \Delta x))) - \ln(f(g(x)))}{\Delta x}} = e^{\frac{f'(g(x)) \cdot g'(x)}{f(g(x))}} = f^*(g(x))^{g'(x)}$$

$$(f(x)^{g(x)})^* = e^{\lim_{\Delta x \rightarrow 0} \frac{\ln f^{g(x+\Delta x)}(x + \Delta x) - \ln f^{g(x)}(x)}{\Delta x}} = e^{\ln' f^{g(x)}(x)} =$$

$$= e^{\frac{(f^{g(x)}(x))'}{f^{g(x)}(x)}} = e^{\frac{f^{g(x)-1}(x) \cdot g(x) \cdot f'(x) + f^{g(x)}(x) \cdot g'(x) \cdot \ln f(x)}{f^{g(x)}(x)}} = e^{\frac{g(x) \cdot f'(x)}{f(x)}} \cdot e^{g'(x) \cdot \ln f(x)} =$$

$$= f^*(x)^{g(x)} \cdot (e^{\ln f(x)})^{g'(x)} = f^*(x)^{g(x)} \cdot f(x)^{g'(x)}$$



## Другие прикольные штуки для интегралов

### Внесение под знак дифференциала

$$\prod f(x)^{g'(x)dx} = \prod f(x)^{d(g(x))}$$

### Замена переменной

$$\prod f(x)^{dx} = e^{\int \ln f(x) dx} = e^{\int \frac{\ln f(u)}{u'} du} = e^{\int \ln f(u)^{\frac{1}{u'}} du} = \prod f(u)^{\frac{du}{u'}}$$

### Интегрирование по частям для степеней

$$(f(x)^{g(x)})^* = f^*(x)^{g(x)} \cdot f(x)^{g'(x)}$$

$$f(x)^{g(x)} = \prod \left( f^*(x)^{g(x)} \right)^{dx} \cdot \prod \left( f(x)^{g'(x)} \right)^{dx}$$

$$\prod \left( f^*(x)^{g(x)} \right)^{dx} = \frac{f(x)^{g(x)}}{\prod \left( f(x)^{g'(x)} \right)^{dx}}$$

$$\prod \left( f(x)^{g'(x)} \right)^{dx} = \frac{f(x)^{g(x)}}{\prod \left( f^*(x)^{g(x)} \right)^{dx}}$$

### Другое

$$\left( \prod f(x)^{dx} \right)' = \left( e^{\int \ln f(x) dx} \right)' = e^{\int \ln f(x) dx} \cdot \ln f(x) = \ln f(x) \cdot \prod f(x)^{dx}$$

$$\prod \left( e^{f(x)} \right)^{dx} = e^{\int f(x) dx}$$

$$\ln \prod f(x)^{dx} = \int \ln f(x) dx$$

## Ряды

$$f(x) = \prod_{i=0}^n \left( f^{*(i)}(a) \right)^{\frac{(t-a)^i}{i!}} \cdot \left( f^{*(n+1)}(c) \right)^{\frac{(t-a)^{n+1}}{(n+1)!}}$$

$$f(x) = \prod_{i=0}^{\infty} \left( f^{*(i)}(a) \right)^{\frac{(t-a)^i}{i!}}$$

## Дифуры

$$F(x, y, y^*) = 0$$

## Допускающее разделение

$$y^* = f(x)$$

## Линейное

### Однородное

$$(y^*)^{f_1(x)} \cdot y^{f_0(x)} = 1$$

### Решение

$$y_{\text{оо}} = C \prod \left( e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}$$

### Неоднородное

$$(y^*)^{f_1(x)} \cdot y^{f_0(x)} = g(x)$$

### Решение

$$y_{\text{он}} = C \prod \left( e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx} \cdot \prod \left( g(x) \prod \left( e^{\frac{f_0}{f_1}} \right)^{dx} \right)^{dx}$$

# Приложение

## К линейному уравнению

### Однородное

$$(y^*)^{f_1(x)} \cdot y^{f_0(x)} = 1$$

$$\frac{y'}{y} + \frac{f_0(x)}{f_1(x)} \ln y = 0$$

Замена  $u = \ln y$

$$u' + u \frac{f_0(x)}{f_1(x)} = 0$$

$$\int \frac{du}{u} = - \int \frac{f_0(x)}{f_1(x)} dx + C$$

$$\ln u = - \int \frac{f_0(x)}{f_1(x)} dx + C$$

$$u = C e^{- \int \frac{f_0(x)}{f_1(x)} dx}$$

$$\ln y = C e^{- \int \frac{f_0(x)}{f_1(x)} dx}$$

$$y = C e^{- \int \frac{f_0(x)}{f_1(x)} dx}$$

$$y_{\text{oo}} = C \Pi \left( e^{- \frac{f_0(x)}{f_1(x)}} \right)^{dx}$$

*Проверка*

$$y^* = C^{-\frac{f_0(x)}{f_1(x)} \cdot \Pi \left( e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}}$$

$$C^{-f_0(x) \cdot \Pi \left( e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}} \cdot C^{f_0(x) \cdot \Pi \left( e^{-\frac{f_0(x)}{f_1(x)}} \right)^{dx}} = 1$$

$$C^0 = 1$$

## Неоднородное

$$(y^*) \cdot y^{\frac{f_0(x)}{f_1(x)}} = g(x)$$

$$\frac{y'}{y} + \frac{f_0(x)}{f_1(x)} \ln y = \ln g(x)$$

Замена  $u = \ln y$

$$u' + u \frac{f_0(x)}{f_1(x)} = \ln g(x)$$

Решим соответствующее однородное

$$u' + u \frac{f_0(x)}{f_1(x)} = 0$$

$$\int \frac{du}{u} = - \int \frac{f_0(x)}{f_1(x)} dx + C$$

$$\ln u = - \int \frac{f_0(x)}{f_1(x)} dx + C$$

$$u_{\text{оо}} = C e^{- \int \frac{f_0(x)}{f_1(x)} dx}$$

$$C = C(x)$$

$$u = C(x) e^{- \int \frac{f_0(x)}{f_1(x)} dx}$$

$$u' = C'(x) e^{- \int \frac{f_0(x)}{f_1(x)} dx} - C(x) \frac{f_0}{f_1} e^{- \int \frac{f_0(x)}{f_1(x)} dx}$$

$$C'(x) e^{- \int \frac{f_0(x)}{f_1(x)} dx} - C(x) \frac{f_0}{f_1} e^{- \int \frac{f_0(x)}{f_1(x)} dx} + C(x) \frac{f_0}{f_1} e^{- \int \frac{f_0(x)}{f_1(x)} dx} = \ln g(x)$$

$$C'(x) e^{- \int \frac{f_0(x)}{f_1(x)} dx} = \ln g(x)$$

$$C'(x)=e^{\int \frac{f_0(x)}{f_1(x)}\,dx}\cdot \ln g(x)$$

$$C(x)=\int e^{\int \frac{f_0(x)}{f_1(x)}\,dx}\cdot \ln g(x)\,dx+\tilde{C}$$

$$u=e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}\left(\int e^{\int \frac{f_0(x)}{f_1(x)}\,dx}\cdot \ln g(x)\,dx+\tilde{C}\right)$$

$$\ln y=e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}\int e^{\int \frac{f_0(x)}{f_1(x)}\,dx}\cdot \ln g(x)\,dx+\tilde{C}e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}$$

$$y=e^{\int e^{\int \frac{f_0(x)}{f_1(x)}\,dx}\cdot \ln g(x)\,dx}\cdot e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}\cdot \tilde{C}e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}$$

$$y=\left(\tilde{C}e^{\int e^{\int \frac{f_0(x)}{f_1(x)}\,dx}\cdot \ln g(x)\,dx}\right)e^{-\int \frac{f_0(x)}{f_1(x)}\,dx}$$

$$y_{\text{OH}}=\left(\tilde{C}\prod\left(g(x)^{\Pi e^{\frac{f_0(x)}{f_1(x)}dx}}\right)^{dx}\right)\Pi e^{-\frac{f_0(x)}{f_1(x)}dx}$$

Проверка

$$y^* = \left(g(x)^{\prod e^{\frac{f_0(x)}{f_1(x)}}}\right)^{\prod e^{-\frac{f_0(x)}{f_1(x)}}} \cdot \left(\tilde{C} \prod \left(g(x)^{\prod e^{\frac{f_0(x)}{f_1(x)}}}\right)^{dx}\right)^{-\frac{f_0(x)}{f_1(x)} \prod e^{-\frac{f_0(x)}{f_1(x)}}}$$

$$\left(g(x)^{\prod e^{\frac{f_0(x)}{f_1(x)}}}\right)^{\prod e^{-\frac{f_0(x)}{f_1(x)}}} = g(x)$$

$$g(x)^{\prod e^{\frac{f_0(x)}{f_1(x)}}} \cdot \prod e^{-\frac{f_0(x)}{f_1(x)}} = g(x)$$

$$g(x)^{\prod 1^{dx}} = g(x)$$

$$g(x)^C = g(x)$$

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$$g(x)^{e^{\int \frac{f_0(x)}{f_1(x)} dx} - \int \frac{f_0(x)}{f_1(x)} dx} = g(x)$$

$$g(x) = g(x)$$